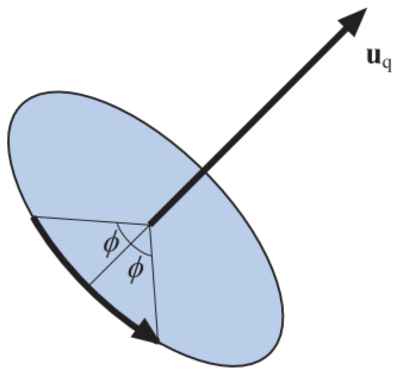
In-class Worksheet: Quaternions

Quaternions



A rotation transform represented by a unit quaternion,
$q = (\cos \varphi, \sin \varphi\ u_q)$.
The transform rotates 2φ radians around the axis u_q.
from *Real-Time Rendering*, 4th Edition

1. What rotations are performed by the following quaternions:

a. (0, (1,0,0))
Here $\cos^{-1}(0)=90$, so the angle of rotation is 2x90=180 degrees
And $\sin(90)=1$, so u_q = (1,0,0) and the axis of rotation is the x-axis

b. (0, (0,1,0))
180 degrees around the y-axis

c. (0, (0,0,1))
180 degrees around the z-axis

2. Compute a quaternion that performs twice the rotation of the quaternion (0.965, (0.149, -0.149, 0.149)). You can use a calculator….

The vector part of the quaternion must be scalar multiple of a unit length vector so we can solve for that vector (a, -a, a) by seeing that $\text{sqrt}(3a^2)=1$ and a=1/sqrt(3).

$\cos^{-1}(0.965)$= =15.20360409 so cos(2 x 15.20360409)=0.8624500001
and sin(2 x 15.20360409)=0.5061422699

So the quaternion we seek is: (0.862, (0.293,-0.293,0.293))

3. Let v_1 and v_2 be nonparallel 3D unit vectors with an angle of θ between them. Find the unit quaternion $(c, (s\ \mathrm{a}))$ where $\mathrm{a} = \frac{v_1 \times v_2}{\sin\theta}$ that rotates v_1 onto v_2

The magnitude of $v_1 \times v_2$ will be $|v_1||v_2|\sin\theta$
Since these are unit vectors, the magnitude of the vector a will be 1.
This means we just need to set:
c = cos (θ/2) and s= sin (θ/2)

4. What are the comparative computational costs of generating a rotation matrix from Euler Angles versus a quaternion? Let's assume that multiplication and addition each are 1 FLOP and that evaluating a sine or cosine is 5 FLOPs. *Note: the only way to really know the comparative cost of a trig function on a system is to profile it…but the weighting in this question should be approximately correct.*

For Euler angles we will need 2 multiplications of 3x3 matrices.
Each multiplication will have 9 entries each requiring 3 multiplications and 2 additions.

So, 2x9x5=90 operations plus each Euler angle matrix requires us to compute the sine and cosine of an angle so thats another 3x10 flops...so approximately 120 flops for Euler angles

For the quaternion matrix below we have: 3x6+6x5=48 flops.

For reference, here is a rotation matrix constructed from a quaternion (q0, (q1,q2,q3)):

$$\begin{bmatrix} 1-2q_2^2-2q_3^2 & 2q_1q_2+2q_0q_3 & 2q_1q_3-2q_0q_2 \\ 2q_1q_2-2q_0q_3 & 1-2q_1^2-2q_3^2 & 2q_2q_3+2q_0q_1 \\ 2q_1q_3+2q_0q_2 & 2q_2q_3-2q_0q_1 & 1-2q_1^2-2q_2^2 \end{bmatrix}$$