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THE METHOD OF CHARACTERISTICS IN COMPRESSIBLE FLOW

Part II
(Unsteady Flow)

prepared by
R. C. Roberts
under the supervision of
C. C. Lin
in the
Graduate Division of Applied Mathematics
Brown University

for the
Analysis Division, Intelligence Department
under contract
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December 1947

HEADQUARTERS AIR MATERIEL COMMAND
WRIGHT FIELD, DAYTON, OHIO

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JOEL S. ISENBERG

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H. M. McCoy, Colonel, USAF
Chief of Intelligence

FOREWORD

The purpose of Part II of this monograph is to present general methods of numerical and graphical integration of the equations of motion for one-dimensional unsteady compressible flow by means of the method of characteristics. The methods of integration presented here were for the most part developed by the Germans during the late war. For this reason it is hoped that this exposition will be of use in bringing such an important tool within the scope of the practicing American engineer.

Various numerical and graphical procedures are presented in some detail with the express purpose that the material may be lifted bodily from the text and applied to various compressible flow problems. Greater emphasis is placed on the numerical methods since they are more accurate than the graphical methods, and since the graphical techniques are applicable to only the simplest problems. The reader is reminded that, although the details of computation are given here, modifications may have to be made to fit the problem at hand. As in all numerical or graphical methods, the accuracy and speed of the computation depend for a large part on the ingenuity and resourcefulness of the computer.

R. C. Roberts

Brown University
December 1947

A9-M-II/2

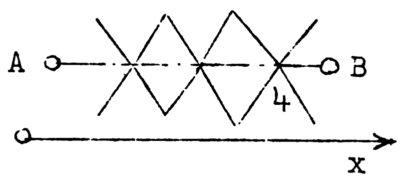
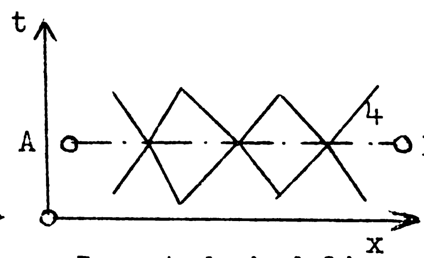
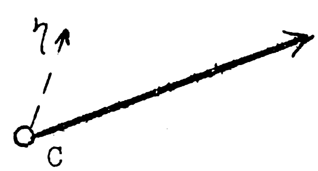
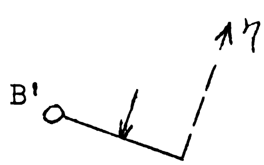
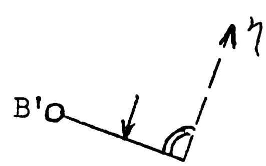
TABLE OF CONTENTS

TABLE OF CONTENTS (cont.)

TABLE OF CONTENTS (cont.)

A9-M-II/2

ERRATA

Page	Line	In place of	Read
2	1	$\frac{\rho_q}{F}$	$\frac{\rho q}{F}$
3	8		$(\partial F/\partial x)/F$
8	1*	$F = \frac{F}{F_0}$	$F' = \frac{F}{F_0}$
15	9	graphical method	graphical methods
	4*	$i = \frac{i}{c_v}$	$i' = \frac{i}{c_v}$
18	2*	Although methods	Although two methods
22	Fig. 3 LHS		
22	Fig. 3 RHS		Insert dashed line connecting 1',2',3',4'
25	5*	$5\Delta c \pm c$	$5\Delta c \pm \Delta c$
30	Fig. 9 LHS		
	Fig. 9 RHS		
31	3	$\Delta(\rho_q)$	$\Delta(\rho q)$

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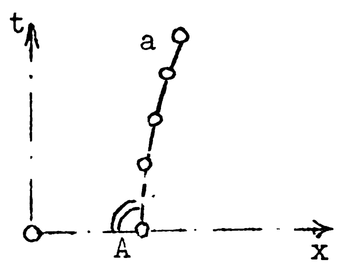
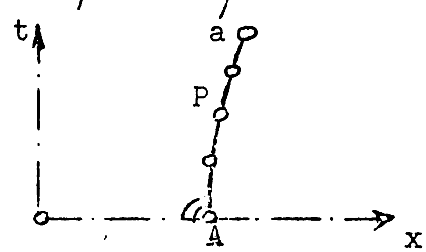
* Denotes lines counted from bottom of page.

Page	Line	In place of	Read
32	Fig. 11	$(\rho u)_2$	$(\rho q)_2$
33	11	of the problem.	of the problem into the procedure.
44	1	running[5]	running[5a]
	5*	$c_a = c_a(\lambda, q_b, c_b)$	$c_a = c_a(\lambda ; q_b, c_b)$
	footnote	5	5a
57	4*-1*	Number these four equations (2.4).	
58	6	$\gamma_a = \gamma_b = \gamma$	$\gamma_a = \gamma_b = \gamma$,
59			φ' for φ in all cases
65	Fig. 21		
73	11	1'2'	$\overline{1'2'}$
85	1*	Sect. 3	Sect. 3A

Chapter I

Statement of the Problem

1. Introductory remarks. The one-dimensional method of characteristics is concerned with the problem of plotting flows in which the velocity, density, pressure, etc., depend only on one distance coordinate x and the time t. The class of flows admitted in this category includes flow through pipes of straight and varying cross section, and spherically and cylindrically symmetrical flows.

In flow through pipes, it is assumed that the independent variables are constant across any cross section of the pipe. This assumption carries with it the requirement that the cross-sectional area should change very slowly along the length of the pipe. The cross-sectional area may then be expressed by the function $F = F(x)$, where x is the distance along the pipe.

In the following, external forces, friction, and heat conduction are neglected. The restriction to pure supersonic flow which was necessary for the two-dimensional steady flow problem is not present in this case. This is due to the fact that, in general, the equations for unsteady flows are hyperbolic.

2. Basic equations for one-dimensional motion. The Euler equation of motion and the equation of continuity for the most general case of one-dimensional motion are, respectively,

$$\frac{\partial q}{\partial t} + q \frac{\partial q}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial x}, \qquad (2.1)$$

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial q}{\partial x} + q \frac{\partial \rho}{\partial x} + \frac{\rho q}{F} \frac{\partial F}{\partial x} = 0. \qquad (2.2)$$

In the above equations $q(x,t)$, $p(x,t)$, and $\rho(x,t)$ represent the velocity, pressure, and density, respectively, at time t in the plane defined by the coordinate x, and $F(x)$ is the cross-sectional area of the pipe at x.

In addition to these equations, if the flow is non-isentropic, the entropy s obeys the equation

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} + q \frac{\partial s}{\partial x} = 0 \qquad (2.3)$$

along the path lines of the gas particles. This means that the entropy of any particle remains constant in time but varies from particle to particle.

Eqs. (2.1), (2.2), and (2.3) form a system of three equations with four s, ρ, q, p unknowns. However, one must also consider the equation of state, which relates the pressure, density and entropy. If it is assumed that the gas in question is a perfect gas, this equation takes the form

$$\frac{\rho}{\rho_0} = \left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}} \exp\left(\frac{s_0 - s}{c_p}\right), \quad \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \exp\left(\frac{s - s_0}{c_v}\right). \qquad (2.4)$$

The index zero refers to any arbitrary comparison state, and $\gamma = c_p/c_v$, where c_p and c_v are the specific heats at constant pressure and volume, respectively. The velocity of sound c in a gas is defined by

$$\frac{1}{c^2} = \left(\frac{\partial \rho}{\partial p}\right)_{s=\text{ const.}}, \qquad c^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s=\text{ const.}} = \frac{\gamma p}{\rho}. \quad (2.5)$$

The derivatives $\partial\rho/\partial t$ and $\partial\rho/\partial x$ may thus be eliminated from Eq. (2.2) by the use of Eqs. (2.4) and (2.5). The basic equations may then be written as

$$\frac{\partial q}{\partial t} + q\,\frac{\partial q}{\partial x} = -\,\frac{1}{\rho}\,\frac{\partial p}{\partial x},$$

$$\frac{1}{c^2}\,\frac{\partial p}{\partial t} + \frac{\partial \rho}{\partial s}\,\frac{\partial s}{\partial t} + \rho\,\frac{\partial q}{\partial x} + \frac{q}{c^2}\,\frac{\partial p}{\partial x} + q\,\frac{\partial \rho}{\partial s}\,\frac{\partial s}{\partial x} + \frac{\rho q}{F}\,\frac{\partial F}{\partial x} = 0, \qquad (2.6)$$

$$\frac{\partial s}{\partial t} + q\,\frac{\partial s}{\partial x} = 0.$$

It should be kept in mind that the Eqs. (2.6) are valid only in regions which are free from shocks. A treatment of the compression shock in one dimension is given in Chapt. III.

In the case of flow in cylindrical pipes and spherically and cylindrically symmetrical flow, the term $(\partial F/\partial x)/F$ takes the form

$$\frac{1}{F}\,\frac{\partial F}{\partial x} = \frac{\lambda}{x}, \text{ with } \begin{array}{l} \lambda = 0 \text{ for straight pipes,} \\ \lambda = 1 \text{ for cylindrically symmetrical flow,} \\ \lambda = 2 \text{ for spherically symmetrical flow.} \end{array}$$

For cylindrical and spherical waves, x designates the distance from the axis of the cylinder and the center of the sphere, respectively.

3. The analogy between one-dimensional and two-dimensional flow.

This analogy may be stated most conveniently by means of the following table:

Unsteady one-dimensional	Steady two-dimensional
Two space-time coordinates: x and t	Two space coordinates: x and y
Supersonic and subsonic velocities	Exclusively supersonic velocities
Flow through cylindrical pipes	Plane flow
Flow through pipes with variable cross section	Axially symmetrical flow

Unsteady one-dimensional	Steady two-dimensional
Path lines of the gas particles	Stream lines
Isentropic flow	Vortex-free isentropic flow
Non-isentropic flow	Non-isentropic flow (with vorticity)

The analogy is incomplete without the definitions of the stream function and the velocity potential for one-dimensional flow.

A. The stream function ψ of the non-isentropic flow.

A stream function $\psi\,(x,t)$ is defined, with the exception of a superfluous additive constant, by setting

$$\psi_x = \frac{F}{F_0}\frac{\rho}{\rho_0} \quad , \quad \psi_t = -\frac{F}{F_0}\frac{\rho}{\rho_0}\,q \;. \tag{3.1}$$

The condition that $\partial^2\psi/\partial x\partial t = \partial^2\psi/\partial t\partial x$ is satisfied on account of Eq. (2.2). The stream function satisfies the condition

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + q\,\frac{\partial\psi}{\partial x} = 0 \tag{3.2}$$

along the path lines of the gas particles. Since the entropy s is also constant along the path lines, the entropy may be inserted as a function $s(\psi)$ of the stream function ψ in any region which is free from compression shocks.

The physical meaning of the stream function can be seen by forming the difference in the values of the stream function for two path lines 1 and 2, namely,

$$\psi_2 - \psi_1 = \int_1^2 \left(\frac{\partial\psi}{\partial x}\,dx + \frac{\partial\psi}{\partial t}\,dt\right) \;. \tag{3.3}$$

Since it has already been shown that $\partial^2\psi/\partial x\partial t = \partial^2\psi/\partial t\partial x$, the condition that the line integral be independent of the path of integration is satisfied. If the flow is considered in the (x,t)-plane, the path of integration may be taken to be either the line parallel to the x-axis or the line parallel to the t-axis. If the line parallel to the x-axis is chosen, then Eq. (3.3) becomes

$$\psi_2 - \psi_1 = \int_1^2 \frac{\partial\psi}{\partial x}\, dx = \frac{1}{F_0 \rho_0}\int_1^2 F\rho dx,$$

i.e., the stream function ψ is proportional to the mass of gas enclosed at an arbitrary but fixed time between the sections x_1 and x_2. If the line parallel to the t-axis is chosen, then Eq. (3.3) becomes

$$\psi_2 - \psi_1 = \int_1^2 \frac{\partial\psi}{\partial t}\, dt = -\frac{1}{F_0 \rho_0}\int_1^2 F\rho q\, dt\ ;$$

i.e., the stream function ψ is proportional to the mass of gas which flows from the time t_1 to t_2 through an arbitrary but fixed cross section.

* From the physical meaning of the stream function it is clear that, on crossing a compression shock, the value of the stream function is preserved since the law of conservation of mass holds across a shock.

B. The potential φ of isentropic flow. In the case of isentropic flow, Eq. (2.3) is identically satisfied throughout the shock-free region of the flow. The gas law may now be written in the form $\rho = \rho(p)$. A potential $\varphi(x,t)$ may now be

defined, again with the exception of a superfluous additive constant, by the equations

$$\frac{\partial \varphi}{\partial x} = q \;, \qquad - \frac{\partial \varphi}{\partial t} = \frac{q^2}{2} + \int_{p_0}^{p} \frac{dp}{\rho(p)} \;, \qquad (3.4)$$

with

$$\frac{\partial^2 \varphi}{\partial x \partial t} + \frac{\partial \varphi}{\partial x}\frac{\partial^2 \varphi}{\partial x^2} = - \frac{c^2}{\rho}\frac{\partial \rho}{\partial x} \;, \quad \frac{\partial^2 \varphi}{\partial t^2} + \frac{\partial \varphi}{\partial x}\frac{\partial^2 \varphi}{\partial x \partial t} = - \frac{c^2}{\rho}\frac{\partial \rho}{\partial t} \;. \qquad (3.5)$$

Using the two equations immediately above, Eqs. (2.1) and (2.2) may be combined into the potential equation

$$\left[c^2 - \left(\frac{\partial \varphi}{\partial x}\right)^2 \right] \frac{\partial^2 \varphi}{\partial x^2} - 2 \frac{\partial \varphi}{\partial x}\frac{\partial^2 \varphi}{\partial x \partial t} - \frac{\partial^2 \varphi}{\partial t^2} + \frac{c^2}{F}\frac{\partial \varphi}{\partial x}\frac{\partial F}{\partial x} = 0, \qquad (3.6)$$

where the velocity of sound c is considered as a known function of $\partial\varphi/\partial x$ and $\partial\varphi/\partial t$.

4. <u>The derivation of the characteristic equations.</u> From the definition of the characteristics of a differential equation as lines across which there may be a discontinuity in the derivatives of the solution of that equation, the differential equations for these lines may be found by forming the matrix of the coefficients of the derivatives of Eqs. (2.6), together with the coefficients of the derivatives of the equations

$$dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial t} dt, \quad dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial t} dt,$$

$$ds = \frac{\partial s}{\partial x} dx + \frac{\partial s}{\partial t} dt. \qquad (4.1)$$

Thus, the seven by six matrix

$$\begin{array}{c l}
\begin{pmatrix}
q & 1 & \frac{1}{\rho} & 0 & 0 & 0 & 0 \\
\rho & 0 & \frac{q}{c^2} & \frac{1}{c^2} & q\frac{\partial \rho}{\partial s} & \frac{\partial \rho}{\partial s} & -\frac{\rho q}{F}\frac{\partial F}{\partial x} \\
0 & 0 & 0 & 0 & q & 1 & 0 \\
dx & dt & 0 & 0 & 0 & 0 & dq \\
0 & 0 & dx & dt & 0 & 0 & dp \\
0 & 0 & 0 & 0 & dx & dt & ds
\end{pmatrix} & \begin{array}{l} \text{Momentum} \\ \text{Continuity} \\ \\ \\ \\ \\ \end{array} \\
\begin{array}{ccccccc} u_x & u_t & \rho_x & \rho_t & x & t & C \end{array} &
\end{array} \tag{4.2}$$

may be formed. The vanishing of the six by six determinant, obtained by removing the seventh column, yields the differential equation for the characteristics in the (x,t)-plane. This determinant equated to zero gives

$$(dx - qdt)\left[(dx-qdt)^2 - c^2\,dt^2\right] = 0. \tag{4.3}$$

From this, three groups of characteristics are obtained, namely, the path lines of the gas particles,

$$\frac{dx}{dt} = q, \tag{4.4}$$

and the curves

$$\frac{dx}{dt} = q \pm c\,. \tag{4.5}$$

The latter curves will be called Mach lines in analogy with the steady two-dimensional supersonic flow. To complete the analogy, the Mach line to the left of a positively directed path line will be called the left running Mach line; the other,

the right running Mach line.

By setting any of the other six by six determinants equal to zero, the compatibility equations for the path lines

$$ds = 0, \text{ and therefore } s = \text{const.}, \tag{4.6}$$

and for the Mach lines

$$\frac{1}{c\rho}\frac{dp}{dt} \pm \frac{dq}{dt} + \frac{cq}{F}\frac{\partial F}{\partial x} = 0 \;, \tag{4.7}$$

are obtained. In the case of isentropic flow, Eq. (2.3) and the third equation of (4.1) are no longer needed since the entropy is constant throughout the shock-free region of flow. A five by four matrix of the coefficients of the derivatives may be formed. The vanishing of the four by four determinant gives Eqs. (4.5) and (4.7). Thus, for isentropic flow the path lines are no longer characteristic curves.

5. The dimensionless form of the characteristic equations.

In practice it is of great advantage to reduce the characteristic equations to dimensionless form. By this means several similar problems may be solved simultaneously. Moreover, the numbers used in the numerical work will then be of a size convenient for easy computation.

The following dimensionless quantities are defined:

$$x' = \frac{x}{\sqrt{F_0}}, \quad t' = t\,\frac{c_0}{\sqrt{F_0}}, \quad q' = \frac{q}{c_0}, \quad c' = \frac{c}{c_0}, \quad \rho' = \frac{\rho}{\rho_0}$$

$$p' = \frac{p}{c_0^2\,\rho_0}, \quad F' = \frac{F}{F_0}, \quad s' = \frac{s}{c_v}\,. \tag{5.1}$$

In the above definitions, F_0 has the dimensions of a length squared, for example, the initial cross section of the pipe. In this instance and thenceforth, the subscript zero refers to the state of the previously homogeneous, resting, or uniformly moving gas. The quantity F_0 may be taken as the cross section of the pipe at this state.

The dimensionless variables given in Eq. (5.1) are not in any way unique. The manner in which the variables are made dimensionless depends entirely on the problem under consideration. The dimensionless variables in Eq. (5.1) are merely an example of what may be done.

The characteristic Eqs. (4.4) and (4.5), and the compatibility Eqs. (4.6) and (4.7) take exactly the same form as before. It should, however, be remembered that the variables are now dimensionless. The characteristic equations are

$$\frac{dx}{dt} = q, \quad \frac{dx}{dt} = q \pm c, \tag{5.2}$$

and the compatibility equations are

$$ds = 0, \quad \frac{1}{c\rho}\frac{dp}{dt} \pm \frac{dq}{dt} + \frac{cq}{F}\frac{\partial F}{\partial x} = 0, \tag{5.3}$$

where, for convenience, the primes have been dropped.

The equations for the stream function may now be written as

$$\frac{\partial \psi}{\partial x} = F\rho, \quad \frac{\partial \psi}{\partial t} = -F\rho q. \tag{5.4}$$

The velocity potential takes the form

$$\frac{\partial \varphi}{\partial x} = q, \qquad \frac{\partial \varphi}{\partial t} = -\frac{q^2}{2} + \frac{1}{\gamma - 1}\left(1 - \rho^{\gamma - 1}\right), \qquad (5.5)$$

where the integration in Eq. (3.4) has now been carried out under the assumption that the adiabatic law holds.

In all of the following work, dimensionless variables will be used unless the contrary is explicitly stated.

6. Limiting lines. It is often the case in the solution of hyperbolic differential equations that one of the families of characteristics has an envelope. It is impossible to carry the solution past this envelope; hence it is commonly called a limiting line. The appearance of the limiting line means physically that the assumptions that the gas is inviscid and non-heat conducting are no longer true. In an actual flow, a shock arises before the limiting line is reached.

The mathematical treatment of the flow near a limiting line is quite difficult and will not be discussed here. References on this subject are given in the bibliography.

Chapter II

Approximate Methods of Solution

1. General procedure for the approximate solutions. The solution of a non-steady flow problem by the method of characteristics requires essentially finding the loci in the distance-time coordinate field which define the characteristic curves. This can be done approximately by replacing the differential equations governing the flow by difference equations. The values of the independent variables defining the characteristic curves may then be found at discrete points. Connecting these points by straight lines, a polygonal approximation to each characteristic curve is found. The basic assumption of the above method is that the points can be chosen sufficiently close together so that the values of the pressure, density, velocity, etc., may be taken to vary linearly between adjacent points without a prohibitive loss of accuracy.

For isentropic flow the solution of the difference equations can be carried out either numerically or graphically; for non-isentropic flow only the numerical method can be used.

Part I of this monograph contains two numerical methods of solution, the lattice point and mesh methods. Although either is applicable here, it appears that only the lattice point method has been used in practice.[1] Since this is the more accurate of the two methods, there is no disadvantage in omitting the procedure for the mesh method.

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1. This has been the case in the literature available to the author.

2. Numerical methods. The various numerical methods are presented for the most general type of flow, i.e., non-isentropic flow through a tube of varying cross section. The obvious simplifications to particular types of flow are made after these procedures have been formulated.

A. Method I. Method I consists of solving numerically the difference equations which are associated with the differential equations (5.2) and (5.3) of Chapt. I. The plus sign in these equations refers to the right-running Mach lines, (i.e., characteristic curves) and the minus sign to the left-running Mach lines. Using the so-called initial value method and the known values of the variables at two points i and j, one can obtain a first approximation[2] to the values of the coordinates and the variables at a point k (see Fig. 1). In this example, ik is the right-running Mach line and jk is the left-running Mach line. Hence, to find the values of x and t at the point k, we must solve the two difference equations

$$x_k - x_i = (q_i + c_i)(t_k - t_i),$$

$$x_k - x_j = (q_j - c_j)(t_k - t_j), \qquad (2.1)$$

where the subscripts characterize the points at which x, t, q, etc., should be evaluated.

In like manner, q_k and p_k can be determined from the

2. One wishes to find the coordinates of the point k, which is the intersection of the right-running Mach line through i and the left-running Mach line through j. Having obtained these coordinates, one then proceeds to find the values of the state variables at the point k.

difference equations

$$\frac{1}{c_i \rho_i}(p_k - p_i) + (q_k - q_i) + c_i q_i \left(\frac{1}{F}\frac{\partial F}{\partial x}\right)_i (t_k - t_i) = 0,$$
$$\frac{1}{c_j \rho_j}(p_k - p_j) - (q_k - q_j) + c_j q_j \left(\frac{1}{F}\frac{\partial F}{\partial x}\right)_j (t_k - t_j) = 0, \tag{2.2}$$

which are the finite difference approximations to the compatibility Eqs. (5.3) of Chapt. I. It is still necessary to calculate s_k and ρ_k. From the third equation of compatibility, $ds = 0$, one may calculate the entropy s_k. The path line through the point k can be found approximately by drawing the straight line from k with slope $dx/dt = q_k$. The path line intersects the line joining i and j at the point H, as shown in Fig. 1.

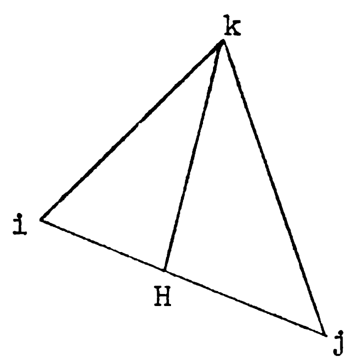


Fig. 1

The value of s at the point H is found by linear interpolation between the values of s_i and s_j. Since $s_H = s_k$, s_k may be found, and ρ_k may then be obtained from the equation of state

$$\rho = \left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}} \exp\left(\frac{s-s_0}{\gamma}\right).$$

A better approximation to the values of x, t, c, q, etc., at the point k, results when one uses average values instead of initial values, as in the above procedure. The equations for x_k and t_k then take the form

$$x_k - x_i = (q_{ik} + c_{ik})(t_k - t_i),$$
$$x_k - x_j = (q_{jk} - c_{jk})(t_k - t_j), \tag{2.3}$$

and the compatibility equations are

$$\frac{1}{c_{ik}\rho_{ik}}(p_k - p_i) + (q_k - q_i) + c_{ik}q_{ik}\left(\frac{1}{F}\frac{\partial F}{\partial x}\right)_{ik}(t_k - t_i) = 0$$

(2.4)

$$\frac{1}{c_{jk}\rho_{jk}}(p_k - p_j) - (q_k - q_j) + c_{jk}q_{jk}\left(\frac{1}{F}\frac{\partial F}{\partial x}\right)_{jk}(t_k - t_j) = 0$$

where the double subscript denotes an average value, e.g., $c_{ik} = (c_i+c_k)/2$. The previously found values for the variables at k are used to calculate these averages.

This iteration procedure can be repeated until the desired accuracy is attained, i.e., until the difference between the values of the state variables for successive iterations is sufficiently small. Usually two iterations are sufficient.

The construction of the Mach lines is considerably simplified if the flow is isentropic, for in that case ds = 0 throughout the flow, and the equation of state becomes the

familiar isentropic law

$$\frac{p}{p_0} = \rho^{\gamma}.$$

The procedure required in this case is that given above with the exception that it is no longer necessary to find the value of the entropy from point to point. Eqs. (2.1) and (2.2), together with the equation of state, are sufficient to determine the flow. In the case of flow through a cylindrical pipe, where F = const., either Method II or one of the graphical method may be used to better advantage.

B. Method II. The numerical procedure may be somewhat simplified with the following change of variable:

$$q = \frac{\lambda - \mu}{2}, \quad c = 1 + \frac{\gamma - 1}{4}(\lambda + \mu). \tag{2.5}$$

The differential equations for the Mach lines become

$$\begin{aligned} \frac{dx}{dt} &= 1 + \frac{\gamma+1}{4}\lambda - \frac{3-\gamma}{4}\mu, \\ \frac{dx}{dt} &= -1 + \frac{3-\gamma}{4}\lambda - \frac{\gamma+1}{4}\mu. \end{aligned} \tag{2.6}$$

In order to transform the compatibility equations, the first law of thermodynamics is used. Before making this transformation, we define the following dimensionless quantities,

$$T' = T, \; R' = \frac{R}{c_0^{\,2}}, \quad i' = \frac{i}{c_v}, \quad A' = \frac{c_0^{\,2}}{c_v} A, \tag{2.7}$$

where T, R, i, and A are the temperature, gas constant, enthalpy, and the reciprocal of the mechanical equivalent of heat, respectively.

The first law of thermodynamics may now be stated in the form[3]

$$T\,ds = di - \frac{A}{\rho}\,dp \,. \qquad (2.8)$$

The term $dp/(c\rho)$ in the compatibility equations then takes the form

$$\frac{dp}{c\rho} = \frac{1}{A}\left(\frac{di}{c} - \frac{Tds}{c}\right) = \frac{1}{A}\left(\frac{\gamma dT}{c} - \frac{Tds}{c}\right) = \frac{2}{\gamma-1}\,dc - \frac{c}{\gamma(\gamma-1)}\,ds, \qquad (2.9)$$

where the following thermodynamic relations have been used:

$$di = \gamma dT, \quad p = \rho RT, \quad c^2 = \frac{\gamma p}{\rho}\,, \quad AR = \gamma - 1.$$

The compatibility equations now become

$$\frac{d\lambda}{dt} = -\frac{\lambda-\mu}{2}\left[1+\frac{\gamma-1}{4}(\lambda+\mu)\right]\frac{1}{F}\frac{\partial F}{\partial x} + \left[\frac{1}{\gamma(\gamma-1)} + \frac{1}{4\gamma}(\lambda+\mu)\right]\frac{ds}{dt}\,,$$

$$\frac{d\mu}{dt} = -\frac{\lambda-\mu}{2}\left[1+\frac{\gamma-1}{4}(\lambda+\mu)\right]\frac{1}{F}\frac{\partial F}{\partial x} + \left[\frac{1}{\gamma(\gamma-1)} + \frac{1}{4\gamma}(\lambda+\mu)\right]\frac{ds}{dt}\,. \qquad (2.10)$$

The first of each pair of Eqs. (2.6) and (2.10) is associated with the right-running Mach lines and the second with the left-running Mach lines.

The procedure is carried out in exactly the same manner as in Method I. For the first approximation to the values of x_k and t_k from the known values of the variables at the points i and j, the difference equations

---

3. The variables in Eq. (2.8) and in the following equations are dimensionless. The primes have been omitted for convenience.

$$x_k - x_i = \left[1 + \frac{\gamma+1}{4}\lambda_i - \frac{3-\gamma}{4}\mu_i\right](t_k - t_i),$$

and

$$x_k - x_j = \left[-1 + \frac{3-\gamma}{4}\lambda_j - \frac{\gamma+1}{4}\mu_j\right](t_k - t_j) \tag{2.11}$$

are used. We use the difference form of the compatibility equations

$$\lambda_k - \lambda_i = -\frac{\lambda_i - \mu_i}{2}\left[1 + \frac{\gamma-1}{4}(\lambda_i + \mu_i)\right]\left[\frac{1}{F}\frac{\partial F}{\partial x}\right]_i (t_k - t_i)$$

$$+ \left[\frac{1}{\gamma(\gamma-1)} + \frac{1}{4\gamma}(\lambda_i + \mu_i)\right](s_k - s_i),$$

$$\mu_k - \mu_i = -\frac{\lambda_j - \mu_j}{2}\left[1 + \frac{\gamma-1}{4}(\lambda_j + \mu_j)\right]\left[\frac{1}{F}\frac{\partial F}{\partial x}\right]_j (t_k - t_j) \tag{2.12}$$

$$+ \left[\frac{1}{\gamma(\gamma-1)} + \frac{1}{4\gamma}(\lambda_j + \mu_j)\right](s_k - s_j)$$

to obtain the first approximation to the values of λ and μ at k. The value for s_k in Eq. (2.12) is found as in Method I.

The iteration procedure may now be carried out using average instead of initial values. More exact values for x_k and t_k are obtained from the equations

$$x_k - x_i = \left[1 + \frac{\gamma+1}{4}\lambda_{ik} - \frac{3-\gamma}{4}\mu_{ik}\right](t_k - t_i)$$

$$x_k - x_j = \left[-1 + \frac{3-\gamma}{4}\lambda_{jk} - \frac{\gamma+1}{4}\mu_{jk}\right](t_k - t_j) \tag{2.13}$$

and better approximations to λ_k and μ_k may be found from the equations

$$\lambda_k - \lambda_i = -\frac{\lambda_{ik} - \mu_{ik}}{2}\left[1 + \frac{\gamma-1}{4}(\lambda_{ik} + \mu_{ik})\right]\left[\frac{1}{F}\frac{\partial F}{\partial x}\right]_{ik}(t_k - t_i)$$
$$+ \left[\frac{1}{\gamma(\gamma-1)} + \frac{1}{4\gamma}(\lambda_{ik} + \mu_{ik})\right](s_k - s_i),$$
$$\mu_k - \mu_j = -\frac{\lambda_{jk} - \mu_{jk}}{2}\left[1 + \frac{\gamma-1}{4}(\lambda_{jk} + \mu_{jk})\right]\left[\frac{1}{F}\frac{\partial F}{\partial x}\right]_{jk}(t_k - t_j) \tag{2.14}$$
$$+ \left[\frac{1}{\gamma(\gamma-1)} + \frac{1}{4\gamma}(\lambda_{jk} + \mu_{jk})\right](s_k - s_j).$$

The lines ik and jk are again the right and left running Mach lines, respectively. This iteration procedure is repeated until the desired accuracy is obtained.

Method II is preferable to Method I in that it requires the solution of fewer simultaneous linear equations. It is interesting to note the extreme simplification which takes place when isentropic flow through a cylindrical pipe is considered. In this case Eqs. (2.10) become

$$\lambda = \text{const.}, \quad \mu = \text{const.} \tag{2.15}$$

This is the simplest numerical procedure to follow when $\partial F/\partial x = 0$.

3. Graphical methods. The methods involving graphical procedures (which can be used only in the case of isentropic flow) are divided into two groups. The procedures for the first group are partly graphical and partly numerical, whereas the procedures for the second group are entirely graphical. The two methods considered first are of the latter group. Although methods belonging to the first group are outlined, they are

not to be recommended because of their inaccuracy. It has been found through experience that a strictly numerical or graphical method is superior to a combination of the two.

A. Method I. This method can be applied only to flows through cylindrical pipes, i.e., those for which $\partial F/\partial x = 0$. From Sect. 4, Chapt. I, the equations for the Mach lines in the case under consideration are

$$\frac{dx}{dt} = q \pm c. \tag{3.1}$$

The compatibility equations become

$$\frac{1}{\rho}\frac{dp}{dq} = \pm c . \tag{3.2}$$

If we let $n = \partial\varphi/\partial t$ and use the second of Eqs. (3.4) of Chapt. I, Eqs. (3.2) become

$$\frac{dn}{dq} = -(q \pm c) . \tag{3.3}$$

and in view of Eq. (3.1) it is clear that

$$\frac{dn}{dq}\frac{dt}{dx} = -1. \tag{3.4}$$

This means that at corresponding points in the (x,t)-and (q,n)-planes, corresponding families[4] of characteristic curves in the respective planes are perpendicular to one another provided the axes are properly oriented and the same scale

4. It should be noted that one family of characteristic curves results from the integration of Eqs. (3.1) or (3.3) when the plus sign is used, while another family is obtained when the minus sign is used.

is used in both planes.

The final step in formulating this graphical method is to show that the characteristics in the (q,n)-plane are fixed, irrespective of the boundary conditions of any particular problem. This can be seen from Eq. (3.3) if c is expressed in terms of n. Replacing ρ by the equivalent form of c in the second of Eqs. (5.5), the relation between c and n becomes

$$c^2 = 1 - \frac{\gamma-1}{2} q^2 - (\gamma-1) n . \tag{3.5}$$

Eq. (3.3) then can be written in the form

$$\frac{dn}{dq} = - q \pm \sqrt{1 - \frac{\gamma-1}{2} q^2 - (\gamma-1)n} , \tag{3.6}$$

which may be integrated to give

$$n = - \frac{\gamma+1}{4} q^2 + Cq + \frac{1-C^2}{\gamma-1} , \tag{3.7}$$

where C is an arbitrary constant of integration.

Accordingly, it is seen that the characteristics in the (q,n)-plane are congruent parabolas. The envelope of these curves is also a parabola whose equation is

$$n = - \frac{q^2}{2} + \frac{1}{\gamma-1} . \tag{3.8}$$

A discrete set of these parabolas in the (q,n)-plane is shown in Fig. 2.

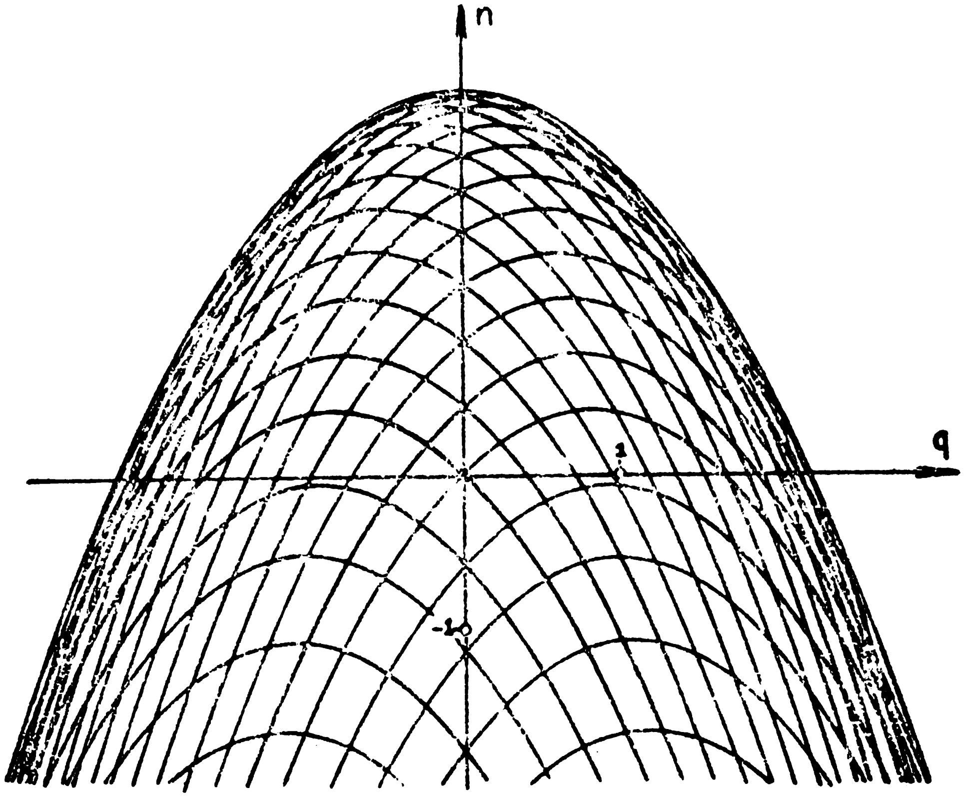


Fig. 2. Characteristic net for air (γ= 1.405) in (q,n)-plane.

Although the net of characteristic curves in the (q,n)-plane is fixed once and for all, the characteristic net in the (x,t)-plane depends on the initial and boundary conditions of the problem and must be determined from these conditions in each individual case. For the approximate construction of this net, the actual net of curvilinear quadrangles is replaced by one of rectilinear quadrangles, i.e., the smooth curves of the characteristic net are replaced by curves composed of straight line segments. This,

of course, is the graphical equivalent of replacing Eqs. (3.1) and (3.3) by difference equations. In each mesh of the (x,t)-plane the flow is assumed homogeneous; therefore each of these meshes corresponds to a definite point of the (q,n)-plane, i.e., the meshes of the (x,t)-plane are correlated with the points of intersection of the (q,n) net. Thus, the line segment connecting two points in the (q,n)-plane corresponds to the common boundary of the two correlated meshes in the (x,t)-plane. As a result, an intersection point in the (q,n)-plane corresponds to a mesh in the (x,t)-plane, and inversely, a mesh in the (q,n)-plane corresponds to an intersection point in the (x,t)-plane. The two nets, correlated in this way, are known as reciprocal nets (Fig. 3).

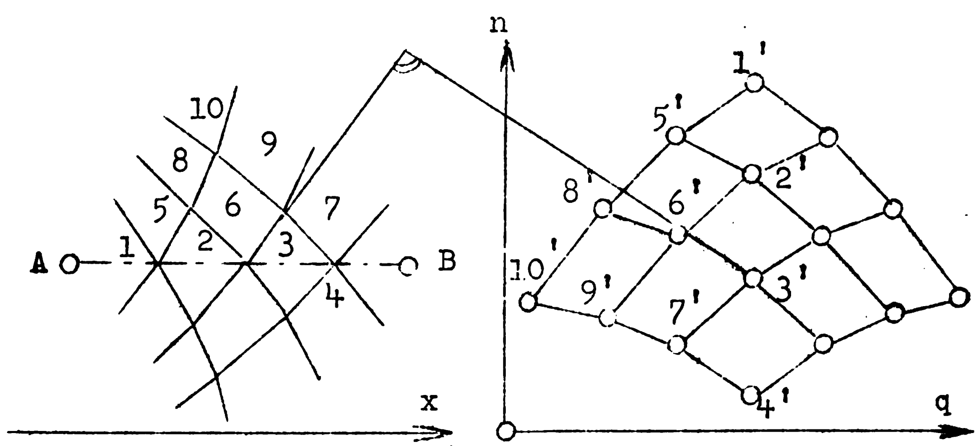


Fig. 3. Reciprocal nets.

The construction of the Mach lines in the (x,t)-plane is based on Eq. (3.4), which states that the two line segments

correlated in the previous paragraph are at right angles to one another (Fig. 3). With this in mind, the construction proceeds in the following manner.

a) As an initial condition it is assumed that the distribution of q and n is known along some non-characteristic curve AB in the (x,t)-plane. This smooth curve is replaced by straight line segments and the velocity distribution is taken to be piecewise constant over each of these segments, i.e., a definite point of the (q,n)-plane is coordinated with each segment of the line AB (see line AB and points 1' to 4' in Fig. 3).

b) The fixed characteristics proceeding from the points of 1'-4' form a parabolic net in the (q,n)-plane with the segment 1'-4' as diagonal and with the intersection points of the parabolas as nodal points (points 1' to 10' in Fig. 3).

c) The related reciprocal net of the (x,t)-plane proceeding from AB can be uniquely constructed, mesh for mesh, by prescribing the direction of each mesh side from the orthogonality relation between the sides of the meshes in the (q,n)-and (x,t)-planes.

It is seen that the initial values of q and n along a curve AB define the flow distribution for the (x,t) region bounded by the four characteristics proceeding from A and B. Later we shall show that the construction can be applied to other boundary conditions.

B. Method II. This method, like Method I, is restricted to the case in which $\partial F/\partial x = 0$. The equations for the characteristics in the (x,t)-plane are

$$\frac{dx}{dt} = q \pm c \ , \tag{3.9}$$

as before. By means of Eq. (3.7), n is eliminated from Eq. (3.3), yielding the differential equations of the characteristics in the (q,c)-plane:

$$dq = \pm \frac{2}{\gamma - 1} \, dc \ , \tag{3.10}$$

where the plus and minus signs refer to the right and left running Mach lines, respectively.

Integration of Eqs. (3.10) gives

$$q - q_1 = \pm \frac{2}{\gamma - 1} (c - c_1) \ . \tag{3.11}$$

Hence, the characteristics in the (q,c)-plane are straight lines and may be plotted once and for all, independently of the boundary conditions. This characteristic net is shown in Fig. 4.

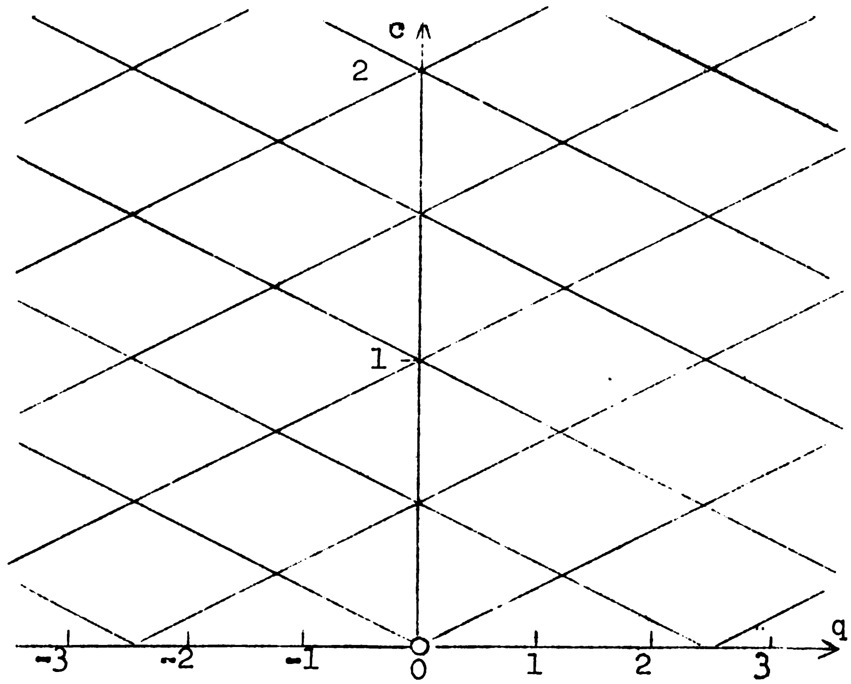


Fig. 4. Characteristic net for air (γ = 1.405) in (q,c)-plane.

As in the previous method the smooth characteristic curves are replaced by curves composed of straight line segments. The reciprocal relation between the points and meshes in the two planes is formed as in Method I. The disadvantage in the method now under discussion lies in the fact that the orthogonality relation between the characteristics in the two planes no longer exists.

This difficulty may be overcome in the case where $\gamma = 1.4$, (approximately true for air), for then Eq. (3.11) becomes

$$q - q_1 = \pm 5(c - c_1) \tag{3.12}$$

or

$$\Delta q = \pm 5 \Delta c. \tag{3.13}$$

The procedure for the construction of the characteristics is carried out entirely in the (x,t)-plane on the basis of the following considerations.

a) Using Eq. (3.9), the slope of a right-running Mach line is $q + c$. From Eq. (3.13) the change in slope of this Mach line when it crosses another Mach line is $5\Delta c \pm c$. Hence, the change in slope of the straight line segments approximating the Mach line is either $6\Delta c$ or $4\Delta c$. From physical considerations it may be found that along a Mach line the change in slope from one line segment to another is $4\Delta c$, whereas the

change in slope between two Mach lines of a single family in a mesh is $6\Delta c$. For a graphical illustration of this see Fig. 5.

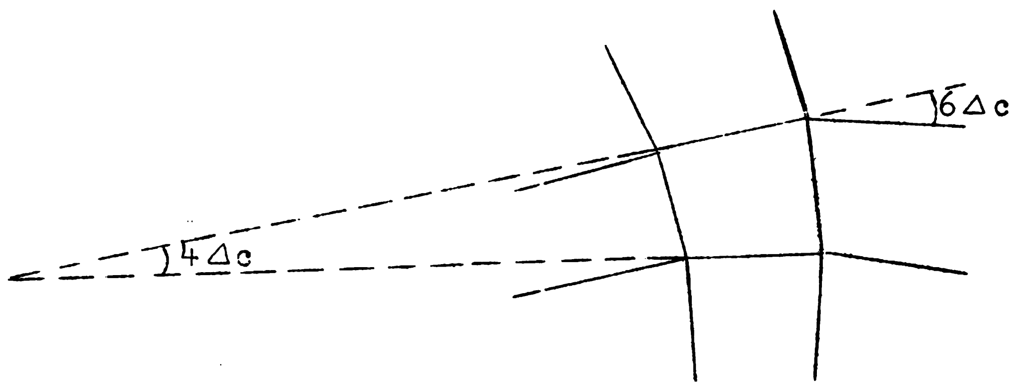


Fig. 5. Change in slope of the Mach lines in (x,t)-plane.

The sign of this change in slope is positive when crossing a compression region and negative when crossing an expansion region. The determination of whether the region is one of compression or expansion must be made from the physics of the particular problem. All the aspects of the situation are reversed in the case of the left-running Mach lines, i.e., the change of slope is positive when crossing an expansion region and negative when crossing a compression region.

b) The construction of the changes in slope of the Mach lines may be done graphically by marking off on the construction paper the various slopes in terms of c. This is shown in Fig. 6, with $\Delta c = 0.02$. The reason for this choice will be given in Chapt. III.

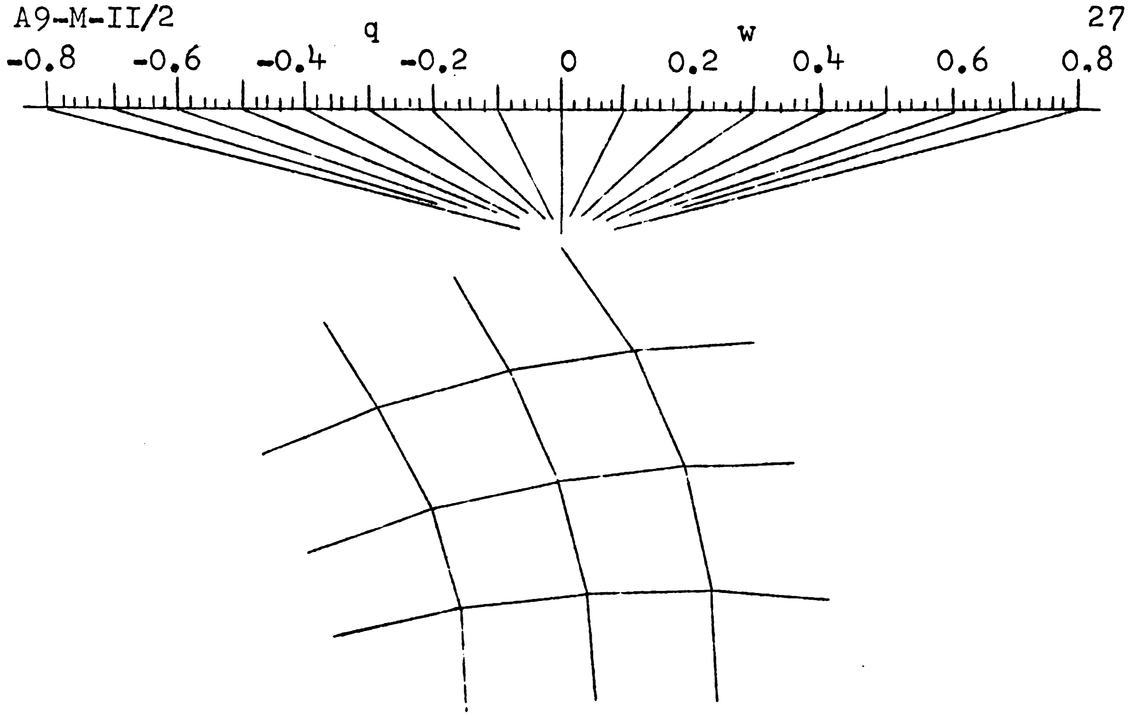


Fig. 6. Graphical construction of the Mach lines in (x,t)-plane.

C. Method III. The Mach lines in the (x,t)-plane are again defined by the equations

$$\frac{dx}{dt} = q \pm c \ . \tag{3.14}$$

A transformation from the (x,y)- to a (ξ, η)-system is now effected, where the new coordinate lines are the right and left running characteristics ξ= const. and η= const., respectively (see Fig. 7). The potential equation (3.6) of Chapt. I assumes a much simpler form under this transformation.

The equations of the transformation are

$$x = \xi \cos \alpha + \eta \cos \beta \ , \qquad t = \xi \sin \alpha + \eta \sin \beta \ ,$$

which yield the operators

$$\frac{\partial}{\partial \xi} = \cos \alpha \frac{\partial}{\partial x} + \sin \alpha \frac{\partial}{\partial t} \ , \quad \frac{\partial}{\partial \eta} = \cos \beta \frac{\partial}{\partial x} + \sin \beta \frac{\partial}{\partial t} \ .$$

Using these, the potential equation becomes

$$\frac{1}{\sin\alpha\sin\beta}\cdot\frac{\partial^2\varphi}{\partial\xi\partial\eta}=\cot\alpha\cot\beta\,\frac{\partial^2\varphi}{\partial x^2}+(\cot\alpha+\cot\beta)\frac{\partial^2\varphi}{\partial x\,\partial t}+\frac{\partial^2\varphi}{\partial t^2}$$

$$=(q^2-c^2)\,\frac{\partial^2\varphi}{\partial x^2}+2q\,\frac{\partial^2\varphi}{\partial x\,\partial t}+\frac{\partial^2\varphi}{\partial t^2},$$

or

$$\frac{\partial^2\varphi}{\partial\xi\partial\eta}=\frac{c^2\,q\,\sin\alpha\sin\beta}{F}\,\frac{\partial F}{\partial x}, \qquad (3.15)$$

where

$$\cot\alpha = q + c, \quad \cot\beta = q - c.$$

For the geometrical interpretation of Eq. (3.15) in the (q,n)-plane, the vector w, with components $q=\partial\varphi/\partial x$ and $n=\partial\varphi/\partial t$ is considered. The projections of this vector on the ξ and η axes are

$$l = q\cos\alpha + n\sin\alpha = \frac{\partial\varphi}{\partial\xi},$$

$$m = q\cos\beta + n\sin\beta = \frac{\partial\varphi}{\partial\eta},$$

as may be seen from Fig. 8.

The potential equation (3.15) is thus equivalent to the two relations

$$dl = c^2\,q\,\sin\alpha\sin\beta\,\frac{1}{F}\frac{\partial F}{\partial x}\,d\eta,$$

$$dm = c^2\,q\,\sin\alpha\sin\beta\,\frac{1}{F}\frac{\partial F}{\partial x}\,d\xi, \qquad (3.16)$$

The geometrical interpretations of dl and dm are shown in Fig. 9.

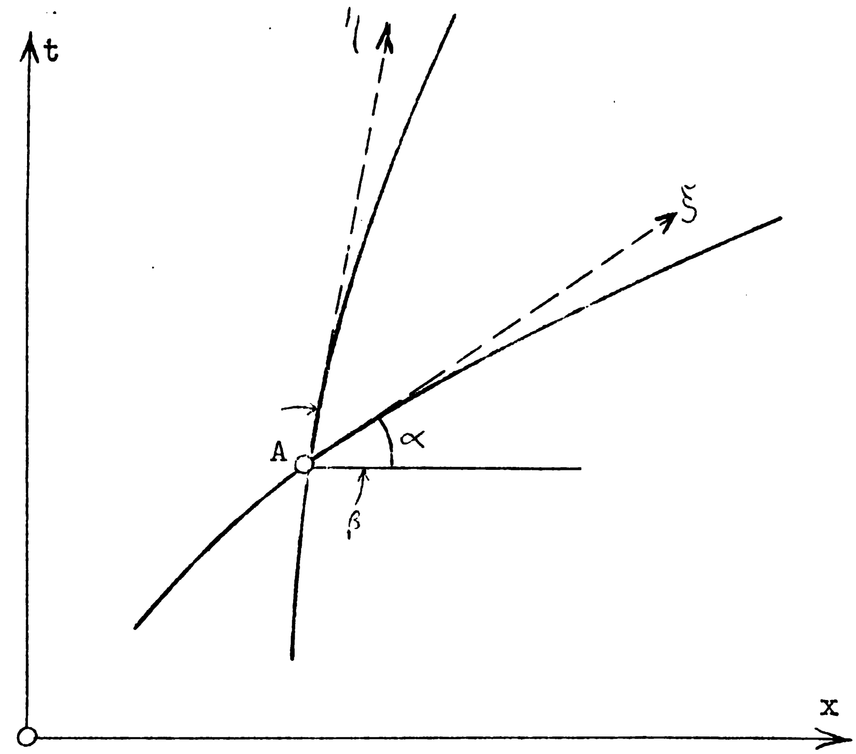


Fig. 7. Coordinate system for the characteristic net in (x,t)-plane.

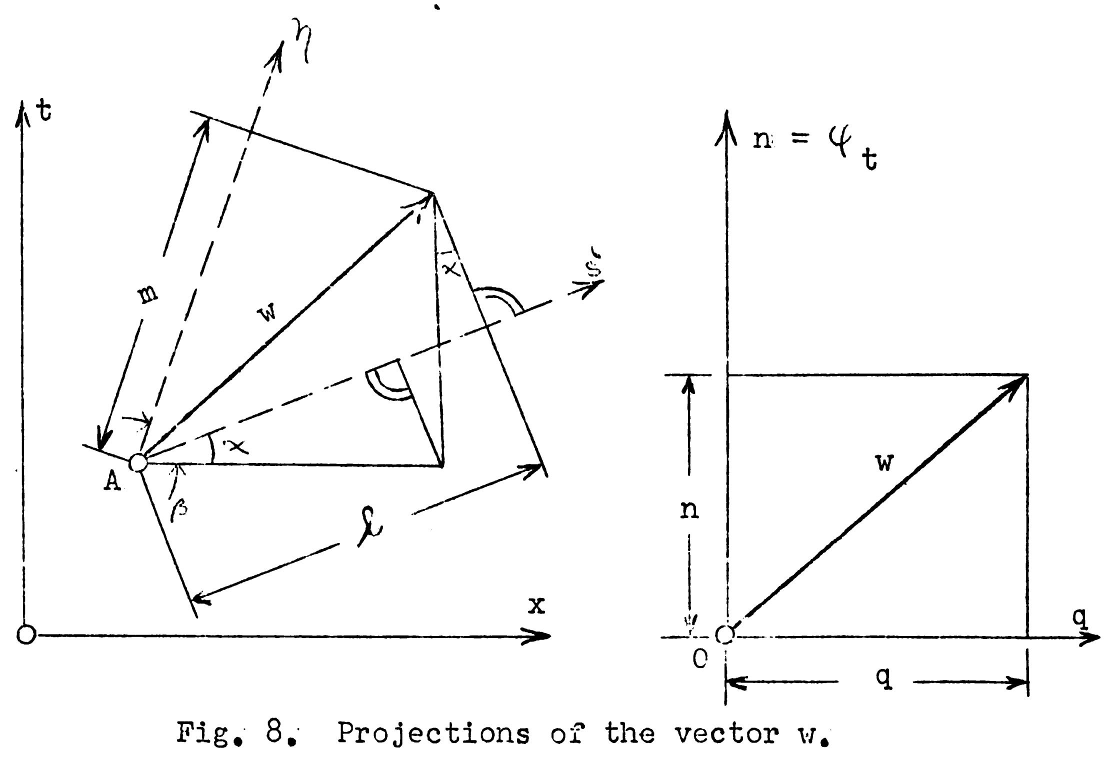


Fig. 8. Projections of the vector w.

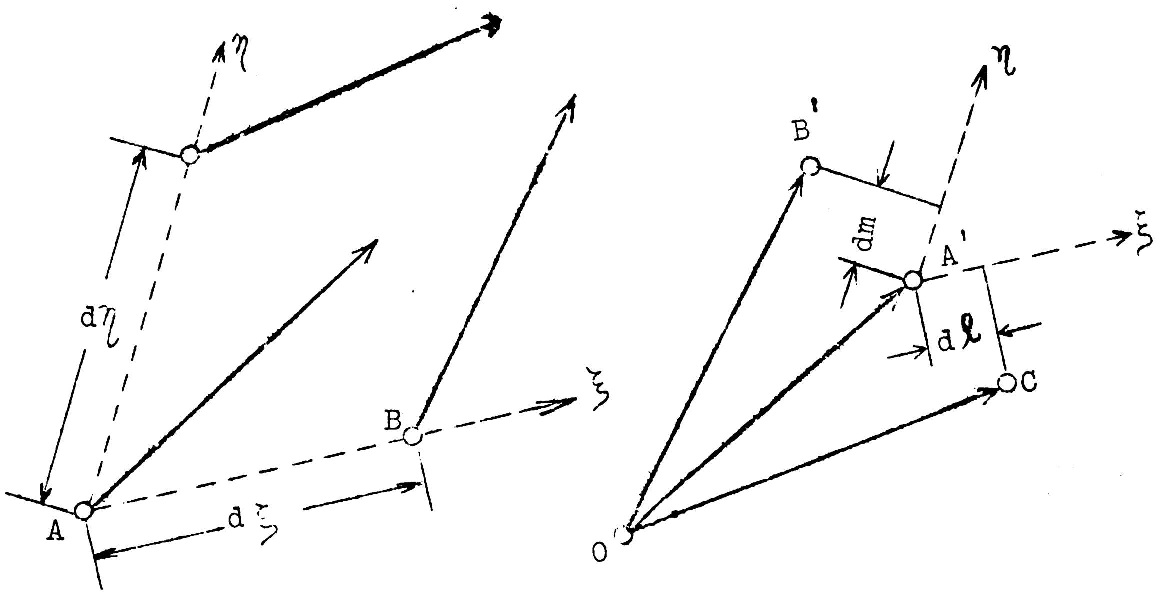


Fig. 9. Geometrical significance of the potential equation.

For the determination of the characteristics, the nets are replaced by straight line segments as in the previous methods. Eqs. (3.14) and (3.16) are replaced by difference equations. These equations are solved numerically but the values for $d\xi$ and $d\eta$ are measured from Fig. 9. The fact that these quantities must be obtained directly from the geometrical figure limits the accuracy of this method to a great extent. For this reason the above method is not recommended.

<u>D. Method IV</u>. This method is used in connection with Method II for the determination of flow through a tube of variable cross section. For this purpose, one begins by approximating the continuously changing cross section of the tube by straight cylindrical tubes, as shown in Fig. 10.

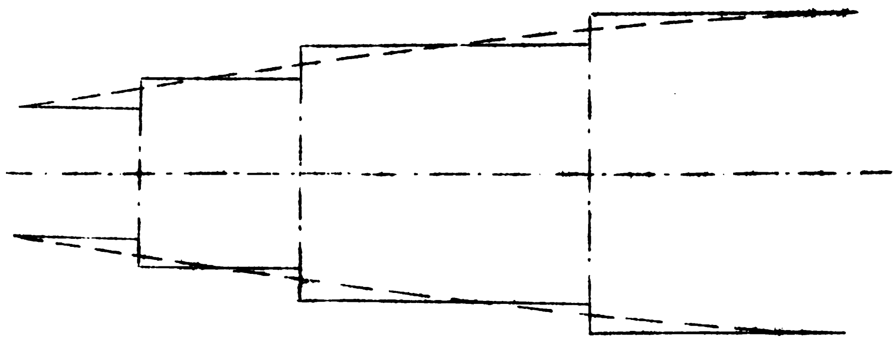


Fig. 10. Approximation of a tube with continuously varying cross-section by cylindrical tubes.

The following transition conditions must hold where a jump in cross section occurs. First, the continuity equation

$$\rho_1 q_1 F_1 = \rho_2 q_2 F_2 = \left[\rho_1 q_1 + \Delta(\rho q)\right]\left[F_1 + \Delta F\right]$$

must be satisfied, where the subscripts one and two refer to the quantities to the left and right, respectively, of the jump in cross section, and

$$\Delta(\rho q) = \rho_2 q_2 - \rho_1 q_1 , \; \Delta F = F_2 - F_1$$

Since the jump in cross-section is assumed small, only first order terms in the small quantities $\Delta(\rho q)$ and ΔF are considered. The continuity equation reduces to

$$\Delta(\rho q) = \rho_1 q_1 \frac{\Delta F}{F} \qquad (3.17)$$

We must also consider the energy equation

$$c_1^2 + \frac{\gamma-1}{2} q_1^2 = c_2^2 + \frac{\gamma-1}{2} q_2^2 = c_k, \qquad (3.18)$$

where, in general, c_k will change with each jump in cross section.

If the values of ρ_1, q_1, and c_1 on one side of the jump are known, then it is possible to find by a graphical procedure the corresponding values on the other side. Members of each of the two families of curves

$$\rho q = \text{const.}, \qquad c^2 + \frac{\gamma-1}{2} q^2 = \text{const.},$$

are plotted in the (q,c)-plane in close discrete sequence. The members of each family touch each other along the lines $q = \pm c$. It is now possible to proceed from the point 1 on the energy curve to the point 2 in such a way that the equation

$$\Delta(\rho q) = \rho_2 q_2 - \rho_1 q_1$$

is satisfied (see Fig. 11).

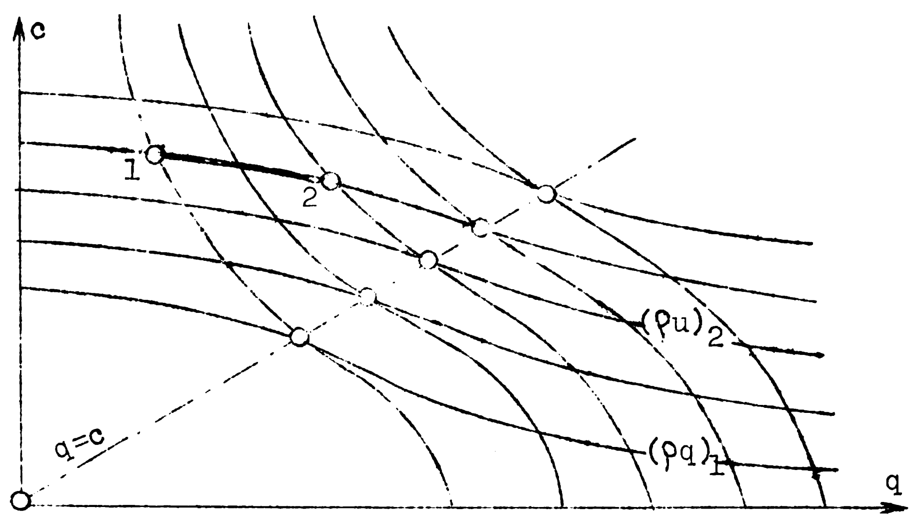


Fig. 11. Continuity condition and energy equation.

The flow through the cylindrical part of the tube is found by the use of Method II. This method, however, does not yield results of great accuracy since the discontinuities in the tube do not actually occur and, in addition, numerical values must be read directly from the graph. The method may be useful for rough approximations, but one of the numerical methods is recommended for accurate computation.

4. Boundary conditions. A few of the general procedures for the construction of a flow pattern have been given above. It is not possible, however, to complete the flow pattern without incorporating the boundary conditions of the problem. For this reason several admissible types of boundary conditions will now be given.

A. Initial value problem. The pure initial value problem, in which q, p, and s are known along a non-characteristic curve AB, has already been described in the above procedures. If no other boundary conditions exist, the flow is then determined throughout the rectangular region ABCD shown in Fig. 12. In the case of isentropic flow, since the path lines are no longer characteristic curves, they do not appear in the construction.

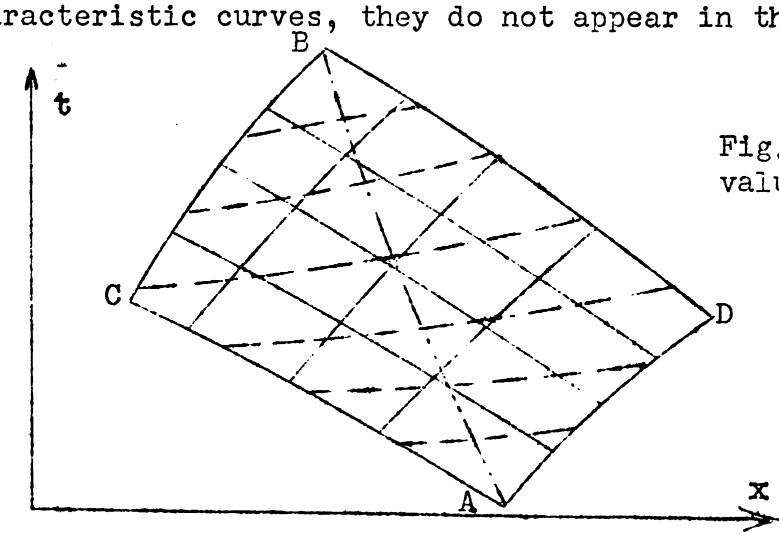


Fig. 12. Initial value problem.

B. Flow between fixed cross sections. Consider the gas enclosed in a pipe bounded at the ends by fixed cross sections (see Fig. 13). For $t = 0$, the values $q(x)$, and $s(x)$ are assumed known on AB, and at the points A and B the conditions $q_A = q_B = 0$ are fulfilled. In addition, the boundary conditions

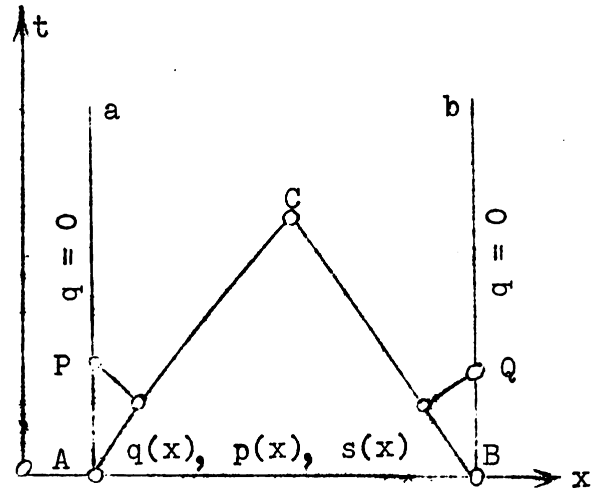
$$q_a = q_b = 0,$$

$$s_a = \text{const.} = s_A,$$

$$s_b = \text{const.} = s_B,$$

are satisfied along the lines a and b in the (x,t)-plane. The lines a and b, corresponding to the fixed sections, are also path lines of the gas particles and therefore characteristics. The construction proceeds in the triangle ABC as before. Finally, the boundary conditions on the lines a and b, together with the difference equations for the characteristic curves, make it possible to determine the Mach lines from AB and AC. These conditions are just sufficient to determine t and p at the points P and Q.

C. Flow between moving cross sections. When the gas is enclosed between two moving pistons, the boundary conditions are identical with those in the previous section except for the conditions on q. If the flow starts from rest, the condition $q_A = q_B = 0$ holds as before. However, the lines a and b will no longer be straight, and the velocities q_a and q_b will be the same as the velocities of the moving pistons. The velocities of the pistons are assumed known.



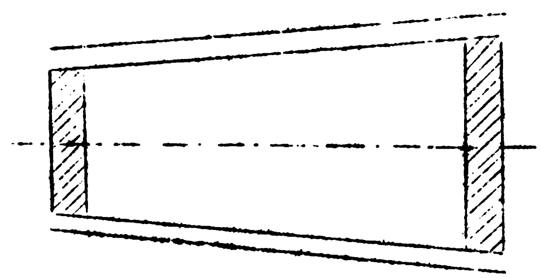


Fig. 13. Flow between fixed cross sections.

D. Discharge under constant pressure. Let it be assumed that the gas is constrained at the left by a fixed cross section, while at the right it flows into a tank at constant pressure $p = \bar{p}$ (Fig. 14). Along AB and a the same conditions exist as in section B, while along the line b of the exhaust cross section the new boundary condition $p = \bar{p}$ exists. The construction proceeds as before until the characteristic BC is reached. The continuation of the construction using one Mach line coming from BC and one path line furnishes two compatibility conditions[5] at each new intersection Q on b. The two compatibility conditions are sufficient for the determination of the unknown velocity q and entropy s at Q. It must be

5. See Chapt. I, Eqs. (4.6) and (4.7).

assumed that BC runs to the left of b, i.e., that the outflow velocity q is smaller than the velocity of sound c. For $q > c$, the condition $p = \bar{p}$ cannot be fulfilled. This is due to the fact that a gas discharging at supersonic velocity is not influenced by the gas pressure $\bar{p}$ at the exhaust cross section of the tube.

The above problem can also be solved if section a is bounded by a moving piston, the only difference being that boundary conditions of section C replace those of section B.

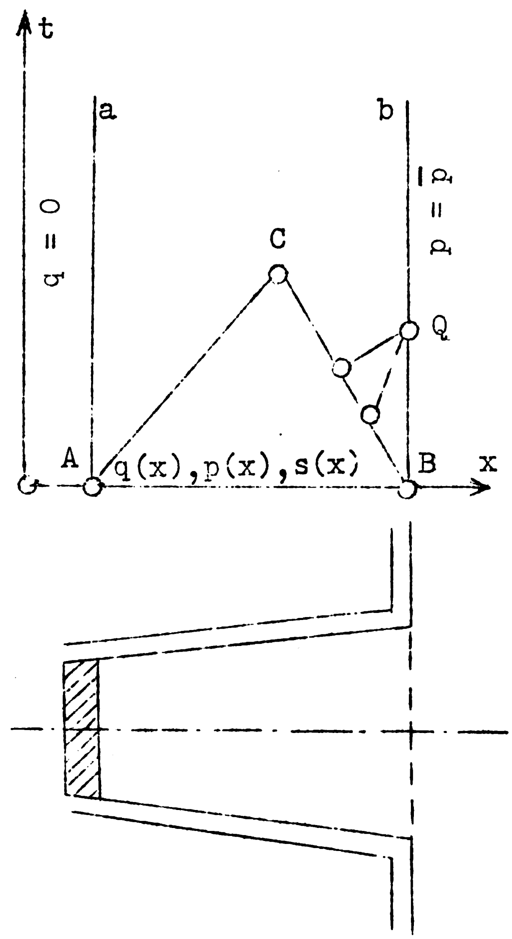


Fig. 14. Outflow into a tank with constant pressure $p = \bar{p}$ from a pipe closed on one side.

E. Spherically and cylindrically symmetrical flow. It is of great importance to formulate the boundary condition at the center of a spherically or cylindrically symmetrical flow which starts from a given initial pressure distribution. The condition at the center of the flow is that $q = 0$, which is precisely that for a fixed piston in a tube, but in the present instance the numerical difficulties are much greater. The boundary condition on the entropy is that along the line a, $s_A = s_a = \text{const.}$ (Fig. 15).

Difficulty arises because of the nature of the third term of the compatibility equations, viz.,

$$\frac{1}{\rho c}\frac{dp}{dt} \pm \frac{dq}{dt} + \frac{cq}{F}\frac{\partial F}{\partial x}$$

where $(\partial F/\partial x)/F = \lambda/x$ (see Sect. 2, Chapt. I). It will be noted that when the line a is approached, both x and q approach zero, and thus the third term in the compatibility equation becomes indeterminate. In order to continue the construction it is necessary to change the method of averaging in the iteration process and evaluate the indeterminate quantity.

Since q and x must be considered simultaneously near the origin, the average values in the iteration process are taken in such a way that Eqs. (2.4) are replaced by

$$\frac{1}{c_{ik}\rho_{ik}}(p_k-p_i) + (q_k-q_i) + \lambda c_{ik}\left(\frac{q}{x}\right)_{ik}(t_k-t_i) = 0,$$

$$\frac{1}{c_{jk}\rho_{jk}}(p_k-p_j) - (q_k-q_j) + \lambda c_{jk}\left(\frac{q}{x}\right)_{jk}(t_k-t_j) = 0. \tag{4.1}$$

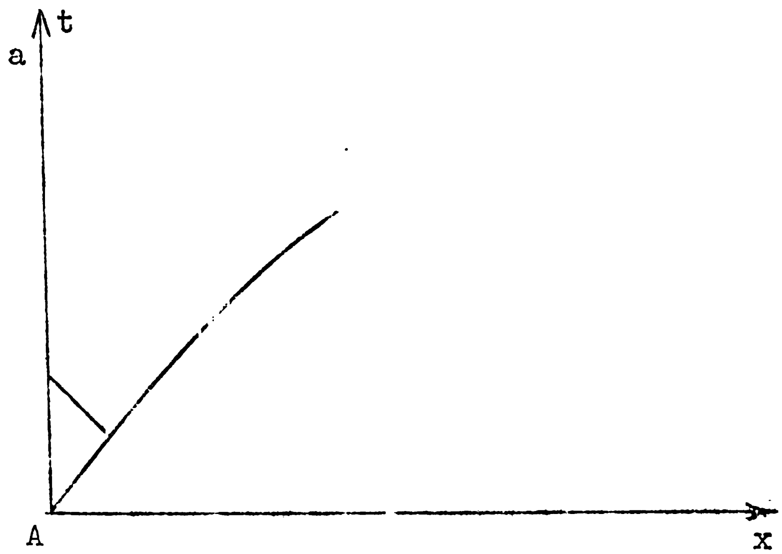


Fig. 15 Center condition for spherically or cylindrically symmetrical flow.

In order that q/x may be evaluated along the line a, use is made of the fact that

$$\left[\frac{q}{x}\right]_{x=0} = \left[\frac{\partial u}{\partial x}\right]_{x=0} .$$

This ratio can then be determined from the equations of motion and continuity (Eqs. (2.1) and (2.2) of Chapt. I). Since

$$\left[\frac{\partial q}{\partial t}\right]_{x=0} = 0,$$

we obtain

$$\left[\frac{q}{x}\right]_{x=0} = - \frac{1}{(1+\lambda)}\left[\frac{1}{\rho}\right]_{x=0} \left[\frac{\partial \rho}{\partial t}\right]_{x=0}. \tag{4.2}$$

The construction can then be carried out as in the case of the fixed piston.

5. Simple waves. In the preceding sections it has been assumed that to any region of the (x,t)-plane there corresponds a region in the (q,c)-plane. In certain cases the entire gas state in the (x,t)-plane corresponds to a single curve in the (q,n)- or (q,c)-plane. In order that the preceding statement be true, q and n or q and c must remain constant along each characteristic curve of one family of characteristics, while these quantities may vary along the curves of the other family of characteristics. Hence, from consideration of the characteristic equations

$$\frac{dx}{dt} = q \pm c \ ,$$

it may be seen that one of the families consists entirely of straight lines. To each of these straight lines there corresponds a single point in the (q,n)- or (q,c)-plane. The variations of these quantities along the other family of characteristics thus correspond to a single curve in the (q,n)- or (q,c)-plane, viz., a parabola or a straight line, respectively. It is only necessary to plot the straight family of characteristics and to know the values of q and c along them in order to determine completely the flow.

This so-called simple wave occurs when a plane disturbance is propagated into a homogeneous medium, e.g., a piston moving in one end of a semi-infinite cylindrical tube. The numerical or graphical calculation is carried out in essentially the same manner as before except that the curved family of characteristics is ignored. In Method I, Sect. 3, the straight

family of characteristics is seen to be perpendicular to the tangents to the parabola in the (q,n)-plane. The particular parabola to be used is determined from the initial conditions. The particular straight line used in the (q,c)-plane for Method II, Sect. 3, is determined in a similar manner.

Chapter III

Treatment of Shock Waves

1. Introduction. The final step in the treatment of the one-dimensional flow problem is the inclusion of the shock wave phenomenon. The shock wave is described in the usual idealized manner, i.e., as a discontinuity of zero thickness occurring in the fluid. The jump in entropy at the shock front can be shown to be of the third order in the difference of the sound velocities on either side of the shock. This fact enables one to compute the flow across weak shocks by a variation of the method of characteristics. A method for treating strong shocks is also given wherein the change in entropy is taken into account.

The basic shock equations are given, followed by the procedures for weak and strong shocks. The shock waves are divided into two types, the steady shock and the unsteady shock. The steady shock has a constant velocity of propagation and the unsteady shock has a propagation velocity which varies with the position of the shock. The former can be treated exactly while the latter can be calculated only by an approximate method.

2. Basic equations for the compression shock. Consider a gas flowing in an infinitely long cylindrical tube whose axis coincides with the x-axis. The positive x-axis is directed to the right. Let the velocity of the cross-section containing the shock be $\pm w$ with respect to the flow before the shock. The velocity, pressure, and density before the shock are denoted by

q_b, p_b, and ρ_b, respectively, while the same quantities after the shock are given by q_a, p_a, and ρ_a. Thus, the absolute velocity of the shock is given by $q_b \pm w$, where the plus or minus sign refers to a shock moving to the right or left, respectively. The gas state into which the shock is moving in both cases is denoted by b. In the case of the steady shock, $q_b \pm w = \text{const}$; if this quantity is not constant, one speaks of an unsteady shock. For tubes of varying cross-section the latter case always occurs.

It is convenient to refer the flow to a coordinate system moving with the shock. In this system of coordinates the shock is at rest and the velocities before and after the shock are $\mp w$ and $q_a - q_b \mp w$, respectively. This procedure of referring the shock to moving coordinates is valid also for the unsteady shock, since, as shall be seen, only the variables p, q, etc., and not their time derivatives, are involved in the shock conditions.

Using the above system of moving coordinates, the continuity equation becomes

$$\mp \rho_b w = \rho_a (q_a - q_b \mp w) \quad , \tag{2.1}$$

and the equation of the conservation of momentum may be written in the following manner:

$$p_b + \rho_b w^2 = p_a + \rho_a (q_a - q_b \mp w)^2 . \tag{2.2}$$

The equation of conservation of energy can be stated in the form

$$i_b + \frac{A}{2} w^2 = i_a + \frac{A}{2} (q_a - q_b \mp w)^2 , \tag{2.3}$$

in which it is assumed that there is no gain or loss of energy due to external action. Using the equation of state,

$$p = \rho RT,$$

and the relations for the enthalpy and specific heats,

$$i = \gamma T, \qquad AR = \gamma - 1,$$

respectively, the energy equation can be made to assume the form

$$c_a^2 - c_b^2 = -\frac{\gamma-1}{2}(q_a - q_b)(q_a - q_b \mp 2w). \tag{2.4}$$

It should again be noted that the above equations have been made dimensionless in the same manner as in the previous chapter.

From Eqs. (2.1), (2.2), (2.4), and the equation of state, one can derive the basic equations of the compression shock

$$\frac{q_a - q_b}{c_b} = \pm \frac{2}{\gamma-1}\left(\frac{w}{c_b} - \frac{c_b}{w}\right), \tag{2.5a}$$

$$\left(\frac{c_a}{c_b}\right)^2 = 1 + \frac{2(\gamma-1)}{(\gamma+1)^2}\left[\left(\frac{w}{c_b}\right)^2 - 1\right]\left[\gamma + \left(\frac{c_b}{w}\right)^2\right], \tag{2.5b}$$

$$\left(\frac{p_a}{p_b}\right) = 1 + \frac{2\gamma}{\gamma+1}\left[\left(\frac{w}{c_b}\right)^2 - 1\right], \tag{2.5c}$$

and

$$\frac{\rho_b}{\rho_a} = 1 - \frac{2}{\gamma+1}\left[1 - \left(\frac{c_b}{w}\right)^2\right]. \tag{2.5d}$$

The plus sign of the first equation refers to the right

running[5] shock and the minus sign to a left running shock. From these equations one can see that the quantities on the left hand side are functions of w/c_b. Thus, the state of flow behind the shock can be determined if one knows the state of flow before the shock and the shock velocity w.

From the second law of thermodynamics it is found that the change in entropy across a shock is positive. This means that the density after the shock is greater than that before the shock. Thus, since $\rho_a > \rho_b$, it is evident from Eq. (2.5d) that $w^2 > c_b^2$, i.e., the shock velocity is always greater than the velocity of sound before the shock.

Using the above facts, one can see from Eqs. (2.5b) and (2.5c) that the pressure and velocity of sound increase across the shock, while Eq. (2.1) indicates that the velocity of flow relative to the shock decreases across the shock.

3. Shock polar diagram. In analogy with two-dimensional flow, a shock polar diagram can be constructed in the (q,c)-plane for one-dimensional unsteady flow. Eqs. (2.5a) and (2.5b) give q_a and c_a as functions of the parameter $\lambda = \pm\, w/c_b$ and of the initial values q_b and c_b. Hence, to every point (q_b, c_b) in the (q,c)-plane, there corresponds a shock polar

$$q_a = q_a(\lambda; q_b, c_b), \quad c_a = c_a(\lambda, q_b, c_b),$$

which determines all possible final conditions for arbitrary values of the shock velocity.

The equation of the shock polars may be found by eliminating λ from the equations

5. This denotes a shock moving to the right.

$$q_a = q_b + \frac{2c_b}{\gamma+1}(\lambda - \lambda^{-1})$$
$$c_a^2 = c_b^2 + \frac{2(\gamma-1)}{(\gamma+1)^2} c_b^2 (\lambda^2-1)(\gamma+\lambda^{-2}), \tag{3.1}$$

which represent rational curves of the fourth order in q_a and c_a. The values of the parameter λ and, consequently, the values of p_a/p_b and ρ_a/ρ_b may be marked off along the shock polar, so that all of the variables in question may be read directly from the curve.

From Eqs. (3.1) one can see that a change in q_b merely shifts the curve along the q-axis, while a change in c_b magnifies the curve. With this in mind, it is necessary to consider only one of the shock polar curves in illustrating their behavior. For convenience, the curve with $q_b = 0$ and $c_b = 1$ is considered. Thus

$$q_a = \frac{2}{\gamma+1}(\lambda + \lambda^{-1}), \qquad c_a^2 = 1 + \frac{2(\gamma-1)}{(\gamma+1)^2}(\lambda^2-1)(\gamma+\lambda^{-2}). \tag{3.2}$$

To each parameter value in the range

$$\lambda^2 \geqq \lambda^2_{min} = \left| \sqrt{\frac{A^2}{(\gamma-1)^2} + \frac{1}{\gamma}} - \frac{A}{\gamma-1} \right|,$$

where

$$A = \frac{6\gamma - \gamma^2 - 1}{4\gamma},$$

there are two points of the curve symmetric about the q-axis, while the two points $\pm\lambda$ give two points symmetric about the c-axis. The coordinate axes are therefore the axes of symmetry

of the curve. The two points on the q-axis correspond to the values $\lambda = \pm \lambda_{min.}$, and the tangents at these points are perpendicular to the q-axis. The initial point (q_b, c_b), which is a double point of the curve, is obtained for $\lambda = \pm 1$. The curve for the upper half plane is shown in Fig. 16.

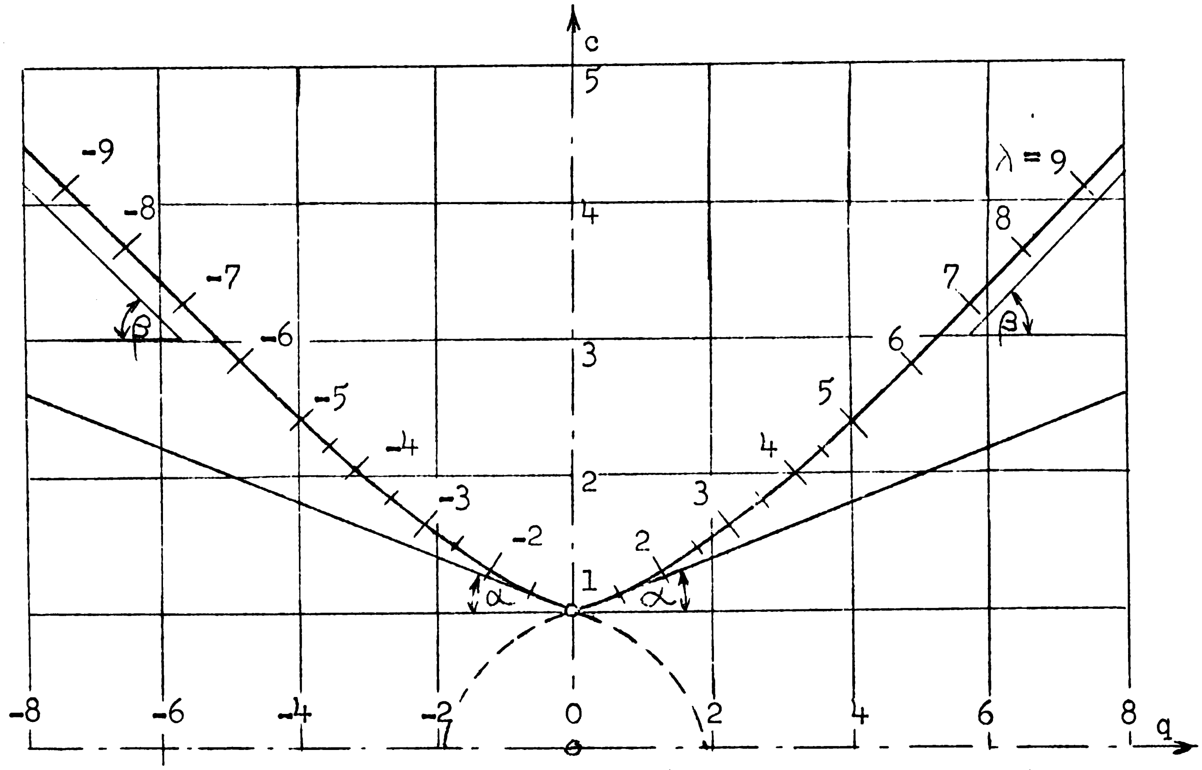


Fig.16: Shock polar at the point $q_b=0$, $c_b=1$ for air ($\gamma = 1.405$).

The continuation of the curve in the lower half-plane is merely the reflection about the q-axis of the portion in the upper half-plane.

It follows from Eqs. (3.2) that for $\lambda = \pm 1$

$$\frac{dc_a}{dq_a} = \pm \frac{\gamma-1}{2} = \pm \tan \alpha, \quad \frac{d^2c_a}{dq_a^2} = 0, \quad \frac{d^3c_a}{dq_a^3} \neq 0,$$

i.e., the initial point (q_b, c_b) is a point of inflection. For $\lambda \to \pm\infty$ it follows that

$$\frac{dc_a}{dq_a} \rightarrow \pm \sqrt{\frac{\gamma(\gamma-1)}{2}} = \pm \tan \beta \ ,$$

i.e., the slope of the curve approaches $\pm \tan \beta$ asymptotically.

Only the branches of the curve with $\lambda > +1$ (right running shock) and $\lambda < -1$ (left running shock) have any physical importance. The shock polar diagram in Figs. 17a and 17b consists of these branches and the analogous curves whose initial points are on the c-axis. All further shock polars may be obtained by suitable translations of the shock polars parallel to the q-axis. The scales for the ratios p_a/p_b and w/c_b are marked off on the curves. The locus of constant values of these ratios is a bundle of straight lines with its vertex at the origin of coordinates.

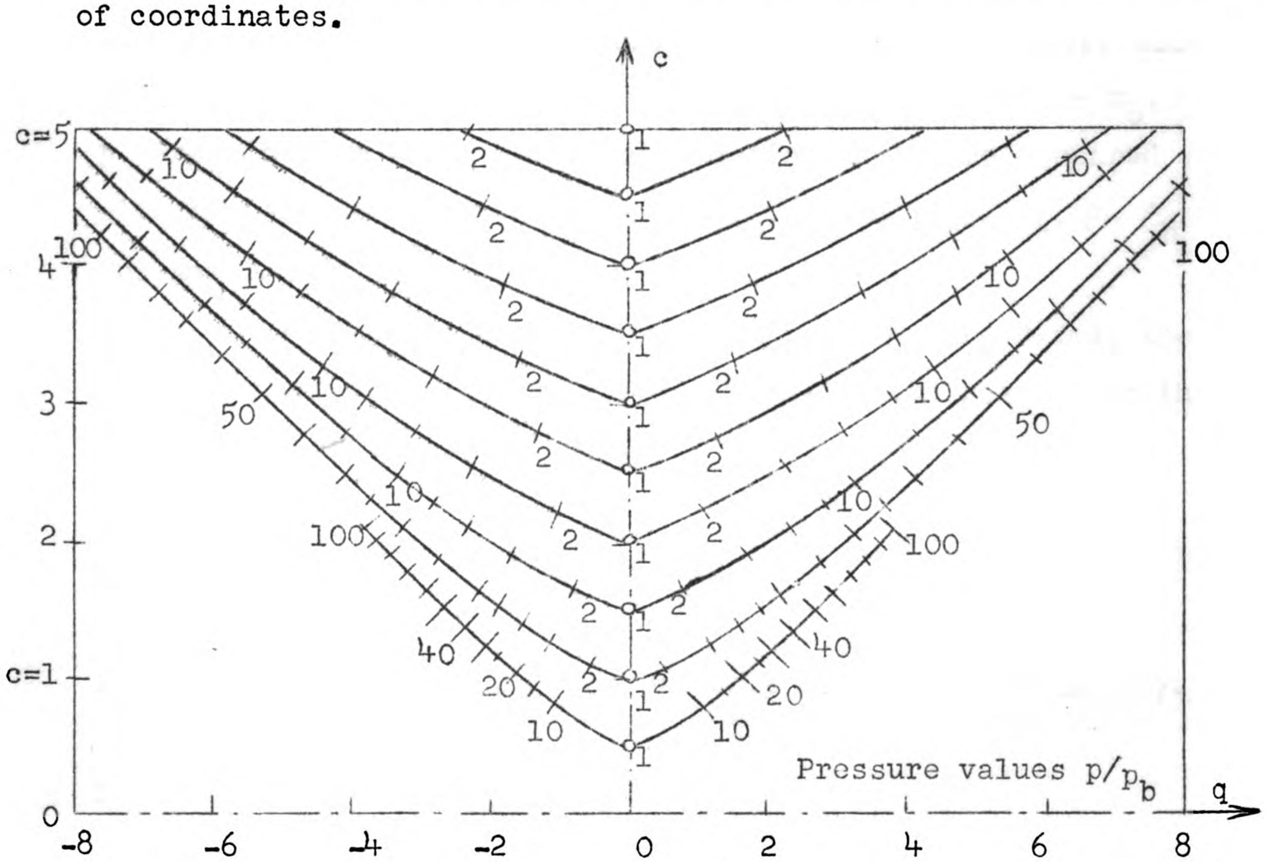


Fig. 17a. Shock polar diagram for air ($\gamma = 1.405$).

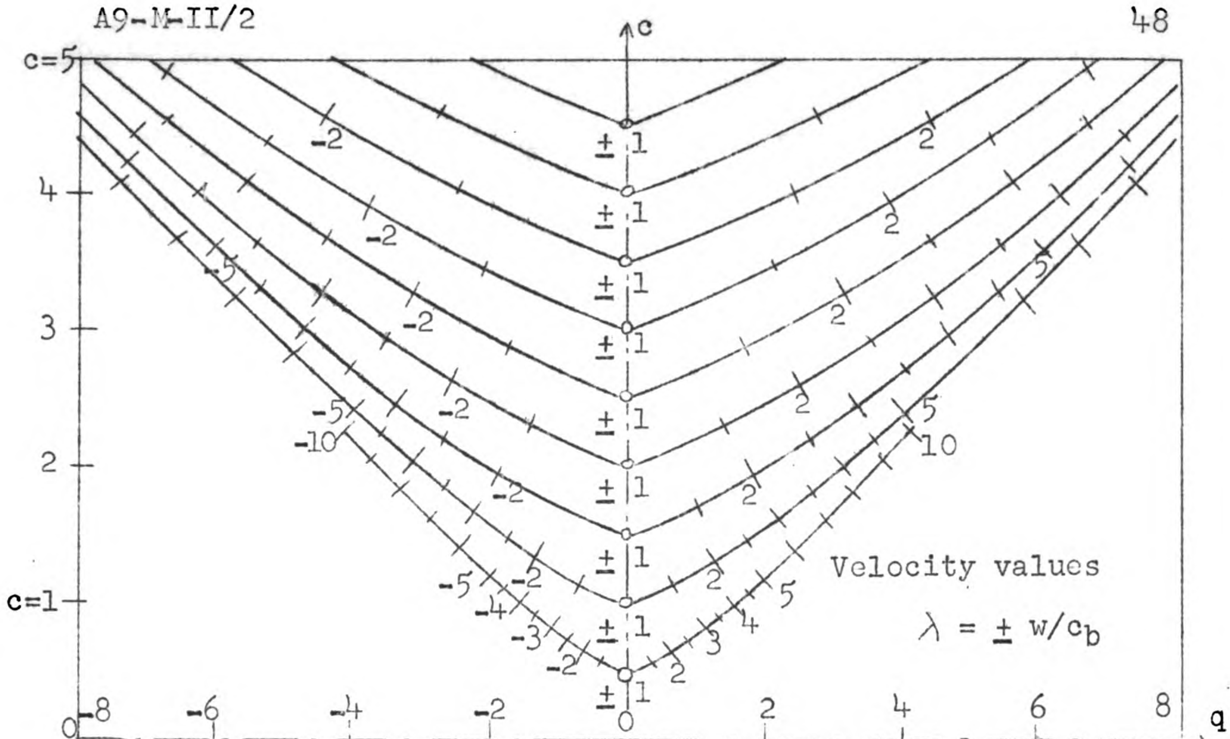


Fig. 17b. Shock polar diagram for air ($\gamma = 1.405$).

4. Transition from compression shock to adiabatic isentropic flow. The changes of the state variables $q_a - q_b$, $c_a - c_b$, $p_a - p_b$, and $\rho_a - \rho_b$ at the compression shock are compared with the corresponding changes $q - q_b$, $c - c_b$, $p - p_b$, $\rho - \rho_b$ for adiabatic isentropic flow.

If the initial values q_b, c_b, p_b, ρ_b are fixed, then the first two derivatives of Eqs. (2.5a-c) with respect to the parameter λ are, respectively,

$$\frac{dq_a}{d\lambda} = \frac{2c_b}{\gamma+1}(1+\lambda^{-2}) \quad , \quad \frac{d^2q_a}{d\lambda^2} = -\frac{4c_b}{\gamma+1}\lambda^{-3}$$

$$\frac{dc_a}{d\lambda} = \frac{2(\gamma-1)}{(\gamma+1)^2}\frac{c_b^2}{c_a}(\gamma\lambda+\lambda^{-3}) \quad , \quad c_a\frac{d^2c_a}{d\lambda^2} + \left(\frac{dc_a}{d\lambda}\right)^2 = \frac{2(\gamma-1)}{(\gamma+1)^2}c_b^2(\gamma-3\lambda^{-4})$$

$$\frac{dp_a}{d\lambda} = \frac{4\gamma}{\gamma+1}p_b\lambda \quad , \quad \frac{d^2p_a}{d\lambda^2} = \frac{4\gamma}{\gamma+1}p_b \; .$$

For $\lambda = \pm 1$ and $w = c_b = c_a$ one finds that

$$\frac{dq_a}{dc_a} = \pm \frac{2}{\gamma-1} , \quad \frac{d^2c_a}{dc_a^2} = 0 ,$$

$$\frac{dp_a}{dc_a} = \frac{2\gamma}{\gamma-1}\frac{p_b}{c_b} , \quad \frac{d^2p_a}{dc_a^2} = \frac{2\gamma(\gamma+1)}{(\gamma-1)^2}\frac{p_b}{c_b^2} . \tag{4.1}$$

However, using the equation of state for adiabatic isentropic flow,

$$\frac{p}{p_b} = \left(\frac{c}{c_b}\right)^{\frac{2\gamma}{\gamma-1}} = \left(\frac{\rho}{\rho_b}\right)^{\gamma} ,$$

one can see that at the point $c = c_b$

$$\frac{dq}{dc} = \pm \frac{2}{\gamma-1} , \quad \frac{d^2q}{dc^2} = 0$$

$$\frac{dp}{dc} = \frac{2\gamma}{\gamma-1}\frac{p_b}{c_b} , \quad \frac{d^2p}{dc^2} = \frac{2\gamma(\gamma-1)}{(\gamma-1)^2}\frac{p_b}{c_b^2} \tag{4.2}$$

(cf. Sect.3B, Chapt.II).

The right hand sides of Eqs. (4.1) and (4.2) are identical, i.e., for equal initial values q_b, c_b, p_b and equal increase of the velocity of sound $\Delta c = c_a - c_b = c - c_b$, the final values q_a, p_a of the compression shock and the final values q, p of the adiabatic change of state coincide, except for deviations of the order of magnitude $(\Delta c)^3$. The same is true for the density ρ which is determined from p and c according to the equation of state for a perfect gas:

$$p = \rho RT = \frac{\rho}{\gamma} c^2 .$$

The shock polar and the adiabatic change of state are plotted in Fig. 18 for $q_b = 0$ and $c_b = 1$. It can be clearly seen that the two curves nearly coincide for weak shocks (i.e., in the neighborhood of the initial point) while for strong shocks there is a large deviation.

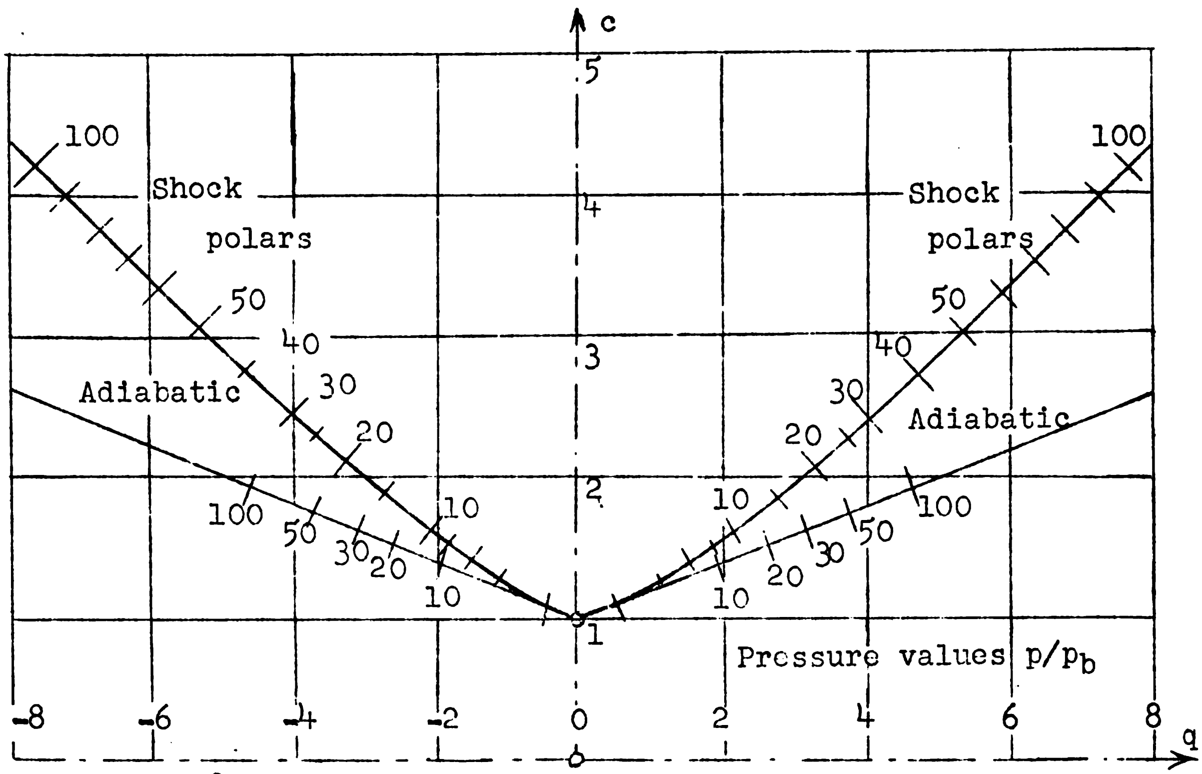


Fig. 18: Comparison of the compression shock with the adiabatic isentropic change of state.

Such close correspondence between the two curves in the neighborhood of the initial point justifies the use of the adiabatic relations for the approximation of weak shocks.

The propagation velocity $q_b \pm w$ of the compression shock is now compared with the sound velocities before and behind the shock. Let

$$c_a = c_b(1 + \varepsilon), \quad w = c_b(1 + \eta),$$

where the second and higher powers of ε and η may be neglected since only small variations in these quantities, i.e., weak shocks, are being considered. Using Eq. (3.11) of Chapt. II, one obtains

$$q_a = q_b \pm \frac{2}{\gamma-1}(c_a - c_b) = q_b \pm \frac{2}{\gamma-1} c_b \varepsilon ,$$

and the first shock condition of Eqs. (2.5) gives

$$\eta = \frac{1}{2}\frac{\gamma+1}{\gamma-1}\varepsilon .$$

Accordingly, the absolute velocity of propagation of the compression shock is

$$q_b \pm w = q_b \pm c_b \pm \eta c_b = q_b \pm c_b \pm \frac{1}{2}\frac{\gamma+1}{\gamma-1}\varepsilon c_b = q_b \pm c_b \pm \frac{1}{2}\frac{\gamma+1}{\gamma-1}(c_a - c_b).$$

If Eq. (3.11) of Chapt. II is used again, one obtains the expression

$$q_b \pm w = \frac{1}{2}\left[(q_b \pm c_b) + (q_a \pm c_a)\right] . \qquad (4.3)$$

5. Approximate methods for weak compression shocks. According to the calculations of the previous section, sufficiently weak compression shocks can be treated by the method of approximation given below.

The changes of the variables q, c, p, and ρ are calculated according to the adiabatic relation

$$\frac{p}{p_b} = \left(\frac{\rho}{\rho_b}\right)^{\gamma} ,$$

while the shock velocity is calculated either from the exact shock conditions of Eqs. (2.5) or from the approximate relation, Eq. (4.3).

Through the use of this rule one is able to extend the numerical and graphical methods of Chapt. II to include weak shocks.

A. Numerical methods. Numerical methods I and II of Chapt. II can be easily extended to the case of weak shocks by using either of the equations

$$\frac{dx}{dt} = q_b \pm w \tag{5.1}$$

in place of the equations for the characteristics in the (x,t)-plane. The compatibility equations remain the same as those used in the regular method of characteristics. Eqs. (5.1) are used only on the shock lines in the calculations. Otherwise the method outlined in the previous chapter is used.

Incorporating this small change into the general procedure, one is able to treat flows containing weak compression shocks as simply as flows without shocks. The boundary conditions are treated in the manner already described.

B. Graphical methods.

a. Method I. This method, an extension of Method II, Sect. 3 of Chapt. II, is used as before, with $\gamma = 1.4$. The change in slope of any characteristic is a multiple of Δc and holds for a weak shock when it crosses another characteristic. The slope of the shock lines is calculated from the equation

$$\frac{dx}{dt} = q_b \pm w. \tag{5.2}$$

In order to avoid the recalculation of the variable w, a table has been prepared giving w as a function of c_b and $n\Delta c$, where n is an integer.

TABLE I

PROPAGATION VELOCITY w OF WEAK COMPRESSION SHOCKS AS FUNCTION OF c_b and $n\Delta c$.

c_b	$n\Delta c$ = 0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
0.86	0.98	1.06	1.14	1.20	1.28	1.38	1.46	1.56	1.64
0.88	1.00	1.08	1.16	1.22	1.30	1.40	1.48	1.58	1.66
0.90	1.02	1.10	1.16	1.24	1.32	1.42	1.50	1.60	1.68
0.92	1.04	1.12	1.18	1.26	1.34	1.42	1.52	1.62	1.70
0.94	1.06	1.14	1.20	1.28	1.36	1.44	1.54	1.64	1.72
0.96	1.08	1.16	1.22	1.30	1.38	1.46	1.56	1.64	1.74
0.98	1.10	1.18	1.24	1.32	1.40	1.48	1.58	1.66	1.76
1.00	1.12	1.20	1.26	1.34	1.42	1.50	1.58	1.68	1.76
1.02	1.14	1.22	1.28	1.36	1.44	1.52	1.60	1.70	1.78
1.04	1.16	1.24	1.30	1.38	1.46	1.54	1.62	1.72	1.80
1.06	1.18	1.26	1.32	1.40	1.48	1.56	1.64	1.74	1.82
1.08	1.20	1.28	1.34	1.42	1.50	1.58	1.66	1.76	1.84
1.10	1.22	1.30	1.36	1.44	1.52	1.60	1.68	1.76	1.86
1.12	1.24	1.32	1.38	1.46	1.54	1.62	1.70	1.78	1.86

If an error of 0.02 in c_b is allowed in accordance with the step $\Delta c = 0.02$, then correct values are obtained from this method of approximation up to the ratios $p_a/p_b = 2.5$ or $c_a/c_b = 1.14$.

b. Method II. This is the extension of Method I, Sect. 3 of Chapt. II. If a shock occurs in the flow, the points (q_b, n_b), (q_a, n_a) in the (q,n)-plane must be correlated with the slope $dx/dt = q_b \pm w$ in the (x,t)-plane. Since the changes in q and n are assumed isentropic, the pair of points must be located on one of the parabolas in the (q,n)-plane, i.e., the compression shock is designated by a chord of the parabola in the (q,n)-plane. The inclination of this parabolic chord is equal to the mean value of the inclinations of the tangents at the end points of the chord. Hence, if the shock velocity is determined from the approximate relation Eq. (4.3), the slope of the shock is perpendicular to the parabolic chord. This means that the orthogonal relationship between the (q,n)-plane and the (x,t)-plane remains valid for the case of weak compression shocks. With this slight generalization, the procedure is carried out in the same manner as before.

6. Exact method for steady strong compression shocks. The gas state behind a steady compression shock may be either found from Eqs. (2.5) or read directly from the shock polar diagram. Since the shock has a constant velocity, the gas state is constant on either side of the shock so that both families of characteristics are straight in the two shock free regions. In this case the characteristic curves need not be plotted; only the shock curve will appear in the (x,t)-plane. This type of shock

will appear only in flow through a cylindrical tube.

7. <u>The unsteady strong compression shock.</u> The calculation of the flow field in which an unsteady strong compression shock appears is a very difficult problem. It is usually assumed that the smooth shock curve in the (x,t)-plane can be approximated by a series of straight steady shocks As far as the author is aware, no general numerical method of the type described here has as yet been developed to handle this problem. The difficulty lies in the fact that the position of the shock is not known beforehand and must be calculated point by point in such a way that the boundary conditions of the flow are satisfied. For further information on this subject consult the bibliography.

Chapter IV

Extensions and Additions to the General Theory

1. General remarks. A few extensions and additions to the general theory are possible, and a brief outline of these is necessary to complete the theory. In the first place the shock theory may be extended to cover steady detonation waves in which there is a chemical liberation of heat in the shock front itself. Much work is being done on this subject but as yet it has not reached the stage of development of the subjects previously discussed.

Some mention should also be made of possible analytical solutions. If the changes in the variables in some parts of the flow are very small, the well-known solutions for sound waves may be used. A few explicit solutions may also be found for waves of finite amplitude in special cases.

2. Detonation waves. Detonation waves may be described in a manner similar to ordinary shock waves. The equations expressing continuity and conservation of momentum remain the same and are, respectively,

$$\mp \rho_b w = \rho_a (q_a - q_b \mp w) \qquad (2.1)$$

$$p_a - p_b = \pm \rho_b w (q_a - q_b), \qquad (2.2)$$

where the flow is referred to coordinates moving with the detonation wave. The equation for the conservation of energy is now generalized to

$$E - (i_a - i_b) = \frac{A}{2}(q_a - q_b)(q_a - q_b \mp 2w), \qquad (2.3)$$

where E is the heat of reaction per unit mass liberated within the detonation wave. The various symbols and subscripts used have the same meaning as those in Chapt. III.

Since the gas on either side of the detonation wave has different physical and chemical properties due to the burning in the wave front, the specific heats before and after the detonation wave have different values. They are denoted by subscripts a and b in the usual manner.

If one uses the enthalpy equations

$$i_a = (c_p)_a T_a, \quad i_b = (c_p)_b T_b,$$

then from Eqs. (2.1), (2.2), and (2.3), together with the equation of state

$$p_a = R_a \rho_a T_a, \quad p_b = R_b \rho_b T_b,$$

one can derive the basic equations of the detonation wave

$$\frac{\rho_b}{\rho_a} = \frac{\frac{p_a}{p_b} + 1 + 2h_b}{\frac{\gamma_a+1}{\gamma_a-1}\frac{p_a}{p_b} + 1}$$

$$\gamma_b\left(\frac{w}{c_b}\right)^2 = \frac{1}{2}\,\frac{\left(\frac{p_a}{p_b} - 1\right)\left(\frac{\gamma_a+1}{\gamma_a-1}\frac{p_a}{p_b} + 1\right)}{\frac{1}{\gamma_a-1}\frac{p_a}{p_b} - h_b}$$

$$\gamma_b\left(\frac{q_a-q_b}{c_b}\right)^2 = 2\left(\frac{p_a}{p_b} - 1\right)\frac{\frac{1}{\gamma_a-1}\frac{p_a}{p_b} - h_b}{\frac{\gamma_a+1}{\gamma_a-1}\frac{p_a}{p_b} + 1}$$

$$\frac{R_aT_a}{R_bT_b} = \frac{\gamma_b}{\gamma_a}\left(\frac{c_a}{c_b}\right)^2 = \frac{p_a}{p_b}\,\frac{\frac{p_a}{p_b} + 1 + 2h_b}{\frac{\gamma_a+1}{\gamma_a-1}\frac{p_a}{p_b} + 1}\,.$$

For abbreviation, the dimensionless value

$$h_b = \frac{1}{\gamma_b - 1}\left[\frac{E}{(c_v)_b\, T_b} + 1\right] \qquad (2.5)$$

is introduced. The ratio of the enthalpic energies of the gas before and after the chemical reaction is represented by $(\gamma_b-1)h_b$, which is assumed to be a known quantity. If $E = 0$, $h_b = 1/(\gamma_b-1)$ and $\gamma_a = \gamma_b = \gamma$, the relations in Eqs. (2.4) become equivalent to the shock equations in Chapt. III.

It is clear from Eqs. (2.4) that if $E, \gamma_a, p_b, \rho_b, T_b, q_b$ and $\pm w$ are known, the gas state behind the detonation wave is completely determined.

From Eqs. (2.4), together with the method of characteristics, one can study flows containing detonation waves in much the same manner as the shock waves in Chapt. III. Additional work[6] done on this subject may be found in the bibliography.

3. Small amplitude waves. In acoustics one is confined to gas waves of small amplitude, i.e., small deviations from the state of rest or uniform motion. If the constant state quantities are denoted by the subscript zero and the small deviations by primes, one can write

$$q = q_0 + q', \quad \varphi = q_0 x + \varphi', \quad c = c_0 + c', \quad p = p_0 + p', \text{ etc.} \qquad (3.1)$$

It is assumed that the primed quantities and their derivatives are so small that their higher order combinations can be neglected.

If one considers small disturbances in a uniform flow through a cylindrical tube where F = const., the substitution of

6. The theory of detonation belongs to the general theory of combustion and should be treated in a separate monograph.

Eqs. (3.1) into the potential Eq. (3.6) of Chapt. I gives

$$(c_0^2 - q_0^2)\frac{\partial^2 \varphi'}{\partial x^2} - 2q_0\frac{\partial^2 \varphi'}{\partial x \partial t} - \frac{\partial^2 \varphi'}{\partial t^2} = 0, \qquad (3.2)$$

in which the second order terms have been neglected.

If the Galilean transformation is applied to Eq. (3.2), then

$$\xi = x - q_0 t, \qquad T = t,$$

and hence

$$\frac{\partial^2 \varphi'}{\partial x^2} = \frac{\partial^2 \varphi'}{\partial \xi^2}$$

$$\frac{\partial^2 \varphi'}{\partial x \partial t} = - q_0\frac{\partial \varphi'}{\partial \xi} + \frac{\partial \varphi'}{\partial T}$$

$$\frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi'}{\partial T^2} - 2c_0\frac{\partial^2 \varphi'}{\partial \xi \partial T} + q_0^2\frac{\partial^2 \varphi'}{\partial \xi^2}.$$

Eq. (3.2) then becomes the one-dimensional wave equation

$$\frac{\partial^2 \varphi'}{\partial \xi^2} = \frac{1}{c_0^2}\frac{\partial^2 \varphi'}{\partial T^2}. \qquad (3.3)$$

In the case of flow through tubes of varying cross-section, $q_0 = 0$ since there can be no uniform flow other than this. The substitution of Eqs. (3.1) into (3.6) of Chapt. I gives

$$c_0^2\frac{\partial^2 \varphi'}{\partial x^2} - \frac{\partial^2 \varphi'}{\partial t^2} + c_0^2\frac{1}{F}\frac{\partial F}{\partial x}\frac{\partial \varphi'}{\partial x} = 0. \qquad (3.4)$$

The solutions of Eqs. (3.2) and (3.3) are well-known

from the study of acoustics and apply to flows with very small disturbances. For a more thorough discussion of these equations one should consult a textbook on acoustics.

4. Exact analytical solutions. Exact analytical solutions of the original equations have been found by Riemann, Kobes, Beckert and others. These solutions are valid only in very special cases and are mainly of academic interest. The references to these papers are given in the bibliography.

Appendix to Chapter II

For purposes of illustration several examples taken from the "Theorie der Nichtstationaeren Gasstroemung I" by R. Sauer (see bibliography) are carried out using the graphical Method I. Four of these examples illustrate the method in the degenerate case of simple waves and the remainder the general case.

a) In Fig. 19 the flow has been plotted for an expansion wave in a semi-infinite tube closed at one end by a piston. The piston accelerates gradually from rest to finite velocities. As has been explained in the section on simple waves, the straight characteristics correspond to a single point on the parabola. The shaded area in the figure corresponds to the section of the gas at rest.

b) The flow in Fig. 20 is identical with that in Fig. 19 except that the piston is accelerated from rest to a finite velocity over an infinitesimal distance. This causes a fan of straight characteristics to emanate from point A in the figure.

c) Fig. 21 shows a compression flow plotted in a semi-infinite tube. The piston is accelerated gradually from rest to a finite velocity. At time $t = \bar{t}$ an envelope of the characteristics is formed. The construction of the flow cannot be carried beyond this envelope unless a shock is introduced into the flow.

d) Fig. 22 shows the expansion flow in a closed tube with a piston at one end. The straight characteristics emanating from the piston curve are intersected by their

reflections from the fixed end of the tube. The characteristics in the meshes of the (x,t)-plane are perpendicular to the corresponding chords of the parabolas in the (q,n)-plane as already described. The piston in this case is accelerated gradually from rest.

e) Fig. 23 shows the expansion flow caused by two moving pistons in a tube as well as the non-linear superposition of waves of finite amplitude. The construction of the characteristics is carried out in the manner already described for both straight and curved characteristics. The pistons are again accelerated gradually from rest.

f) In Fig. 24 the propagation of an infinitely thin disturbance through an infinite tube is shown. The flow on either side of the disturbance is assumed constant, as shown in the figure. This is another example of the degenerate case in which the characteristics are straight lines.

g) The last example in Fig. 25 illustrates propagation of a disturbed region of flow into a constant region. It is assumed that the state of flow in the disturbed region is known at time t = 0. This is another example of the superposition of waves of finite amplitude.

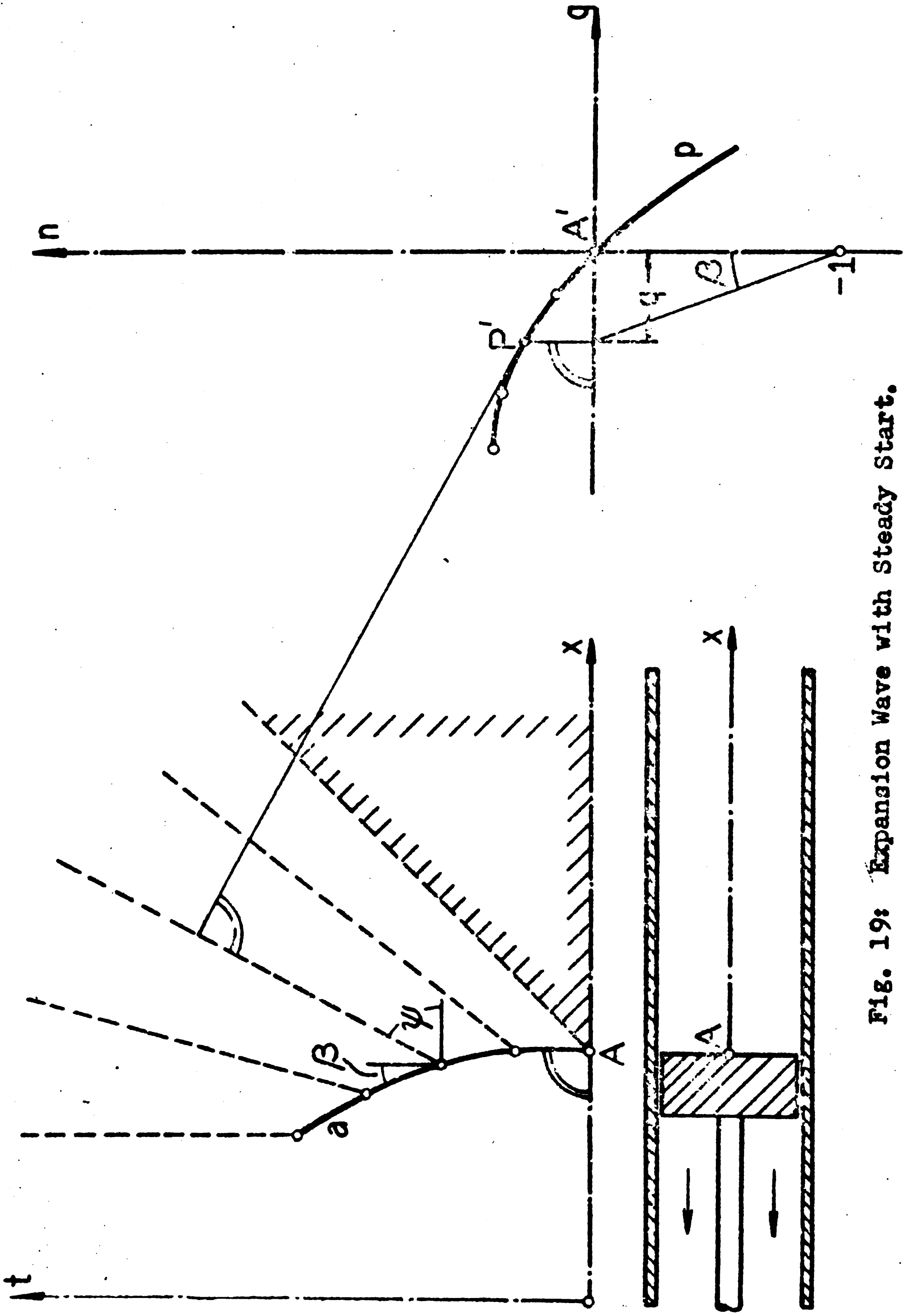


Fig. 19: Expansion Wave with Steady Start.

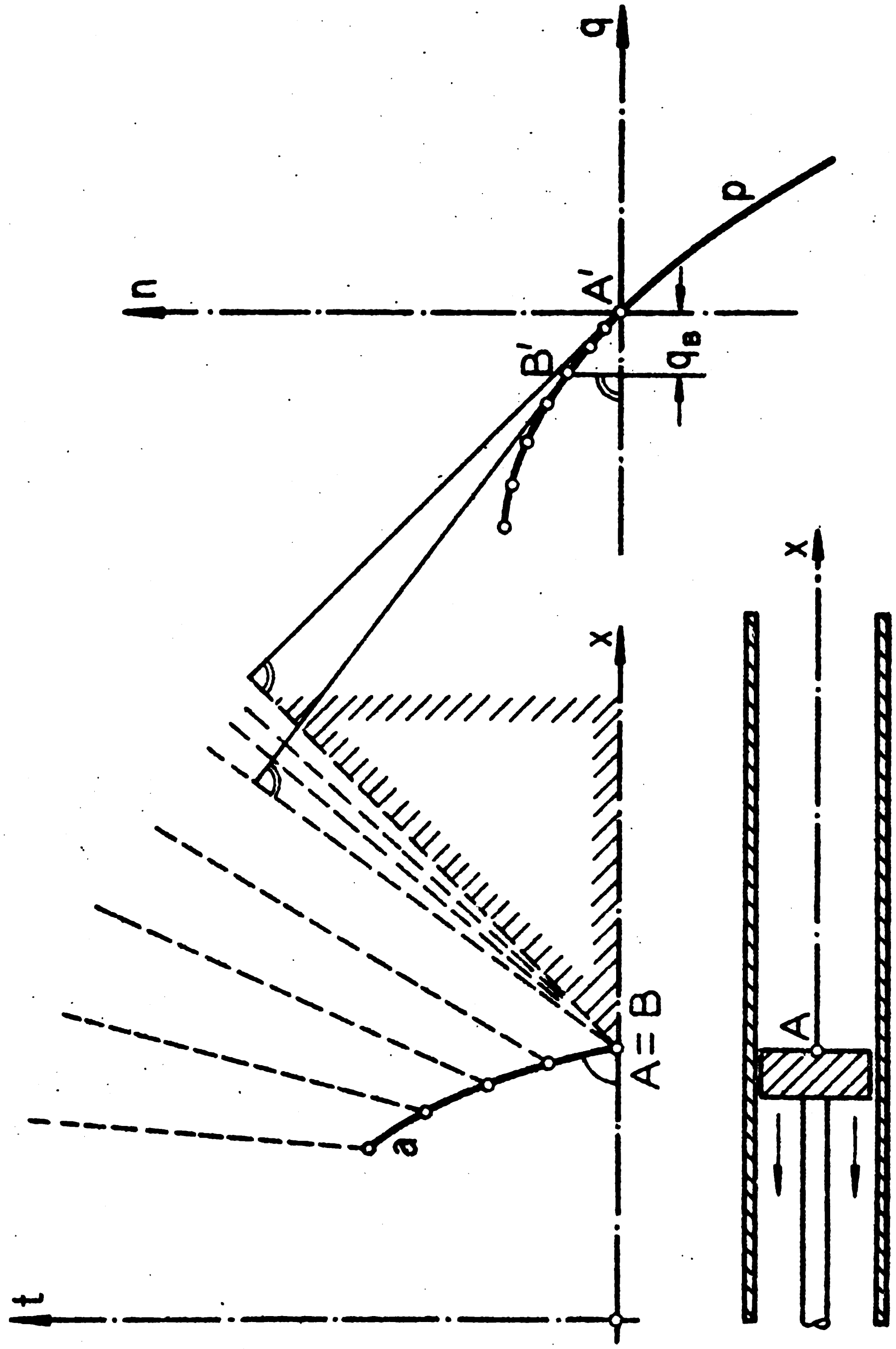


Fig. 20: Expansion Wave with Unsteady Start.

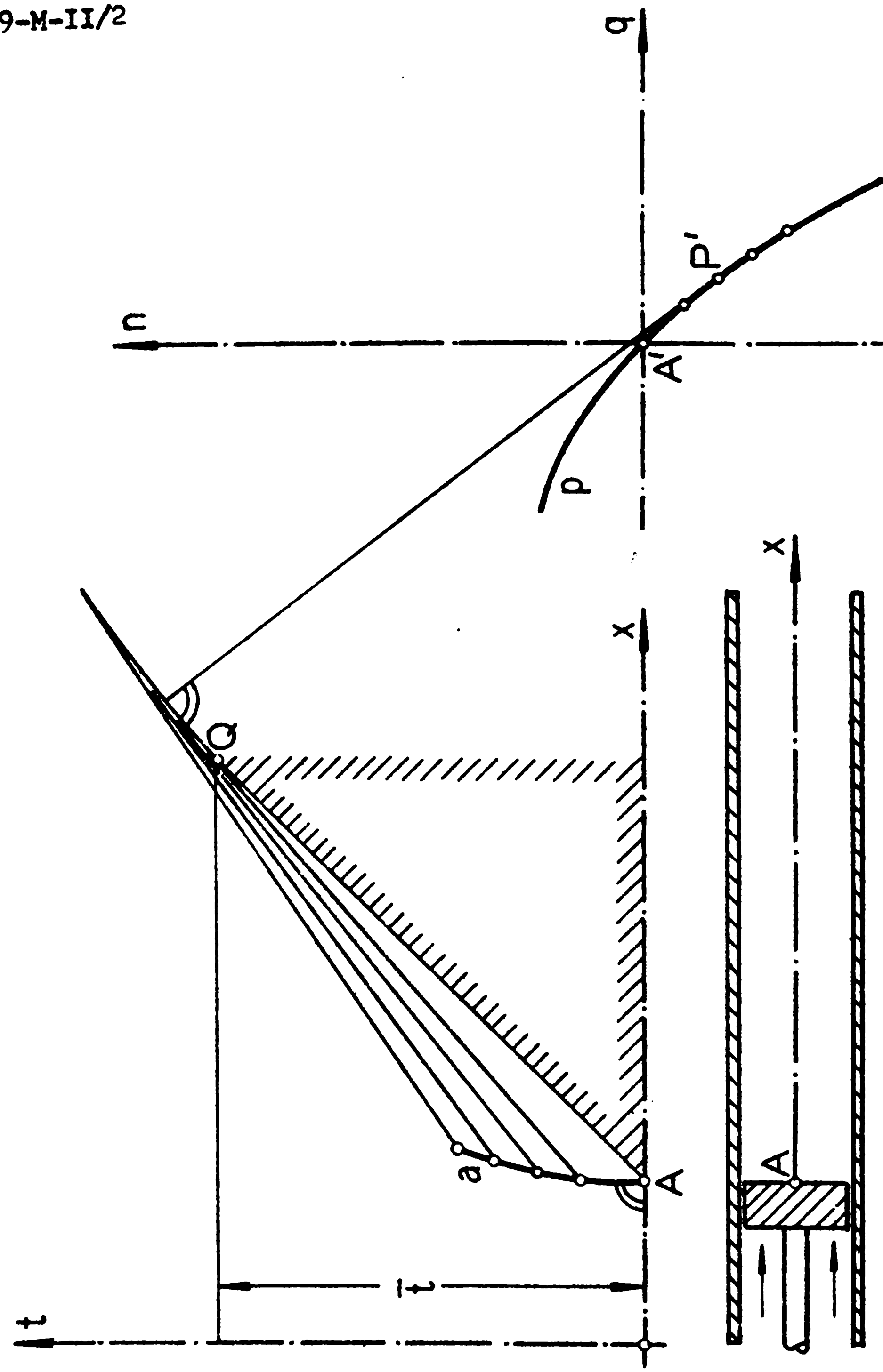


Fig. 21: Compression Wave with Steady Start.

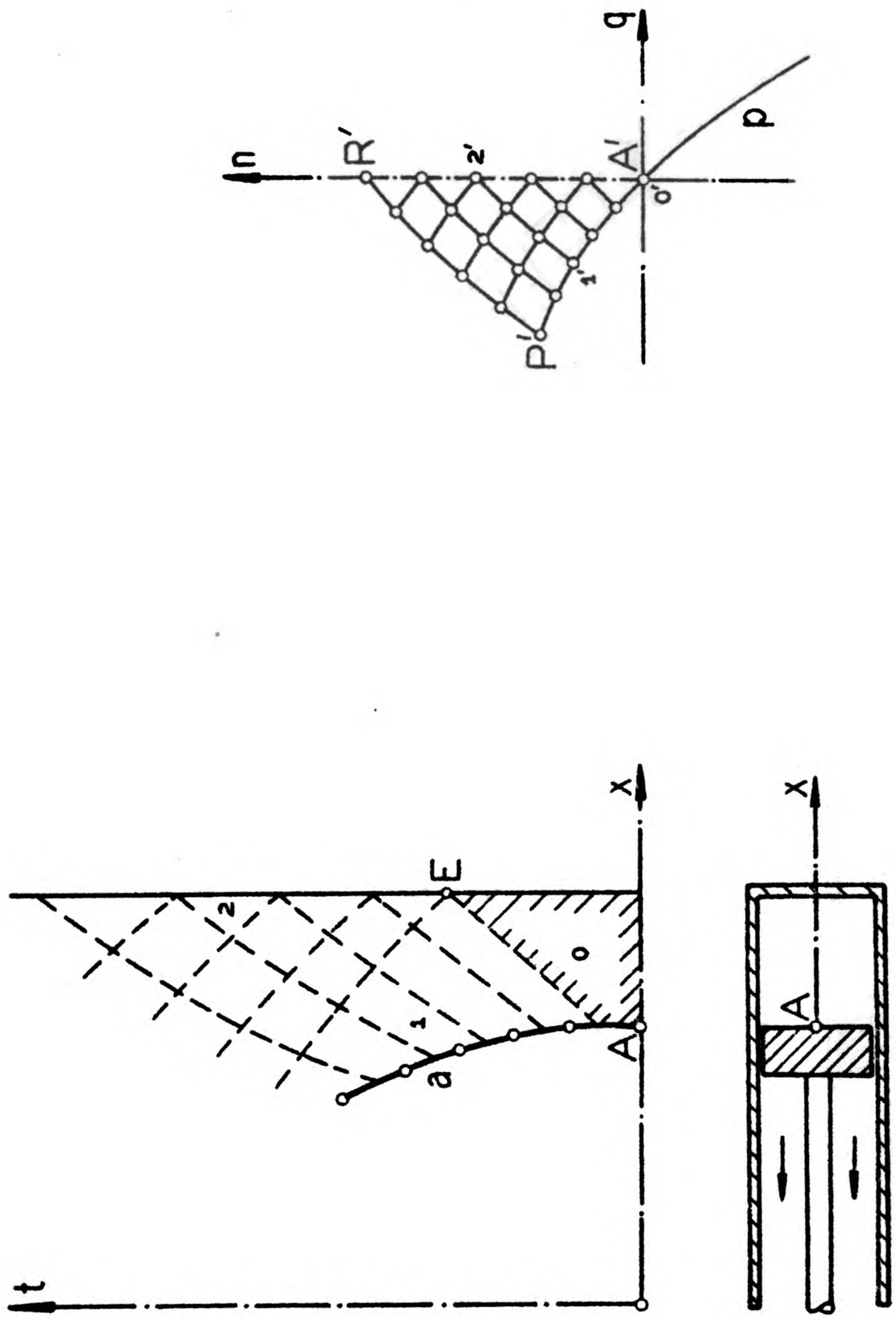


Fig. 22: Reflection on a Fixed Wall.

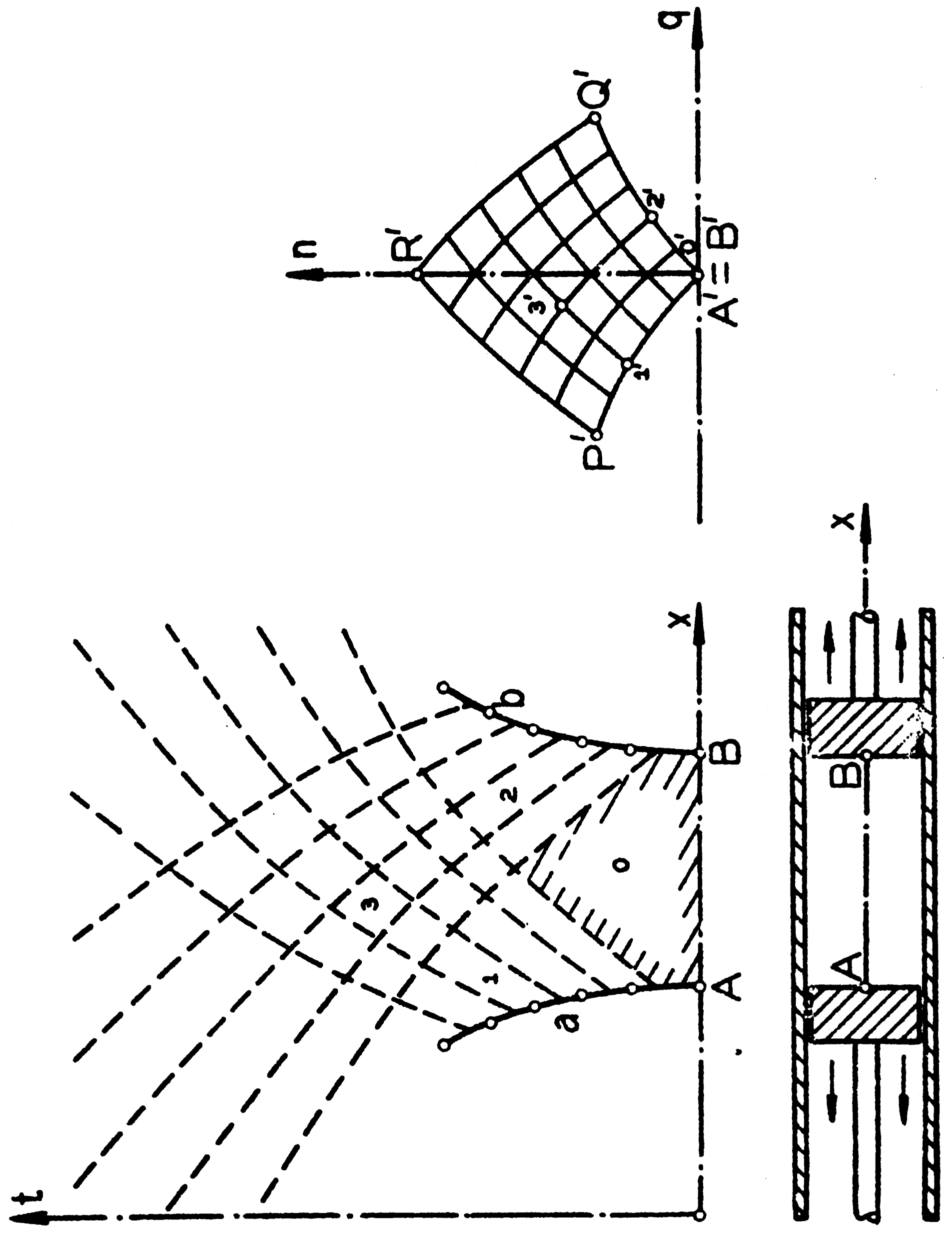


Fig. 23: : Reflection on a Movable Wall (Superposition of Single Waves of Large Amplitude.)

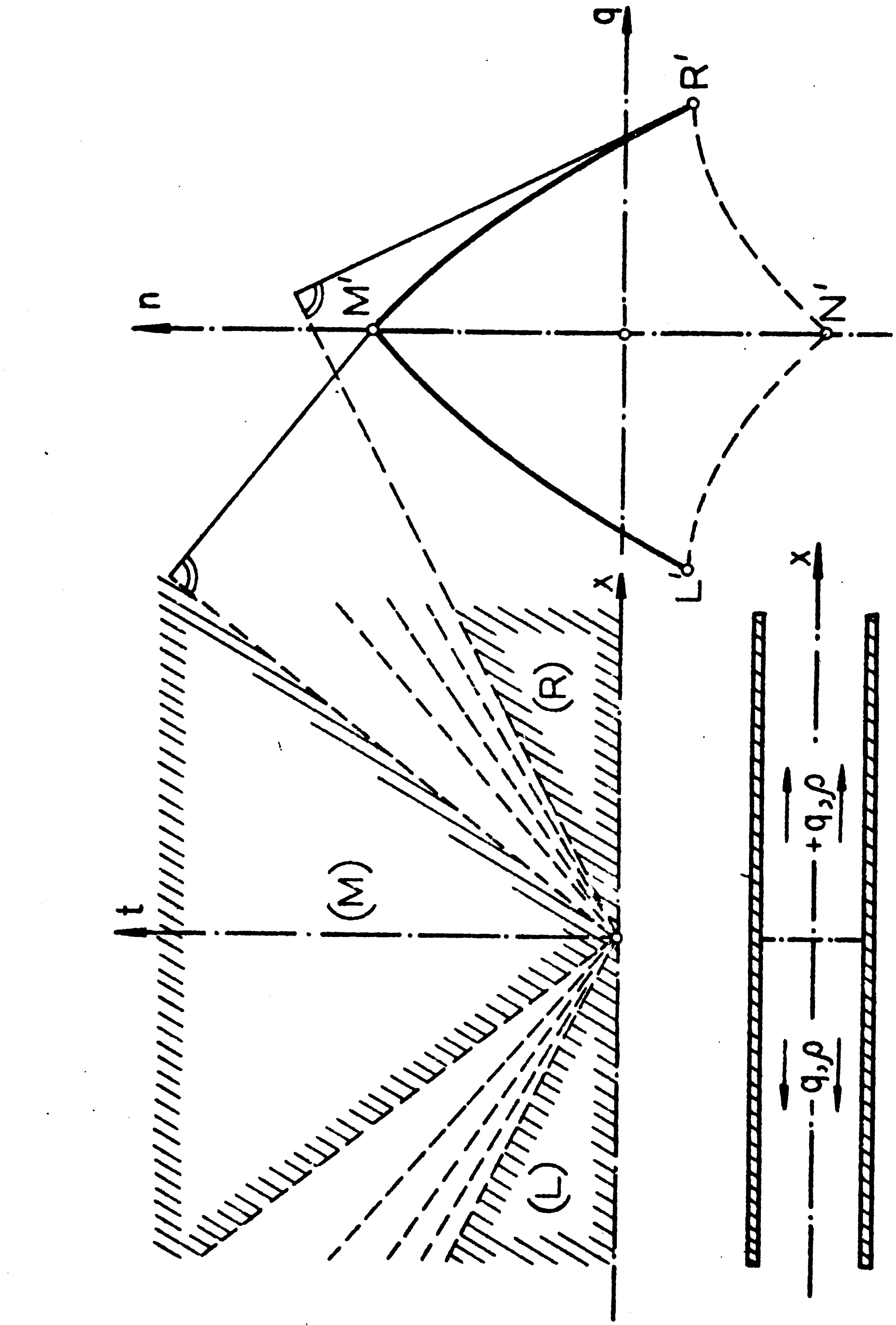


Fig. 24: Propagation of an Unsteady Interference.

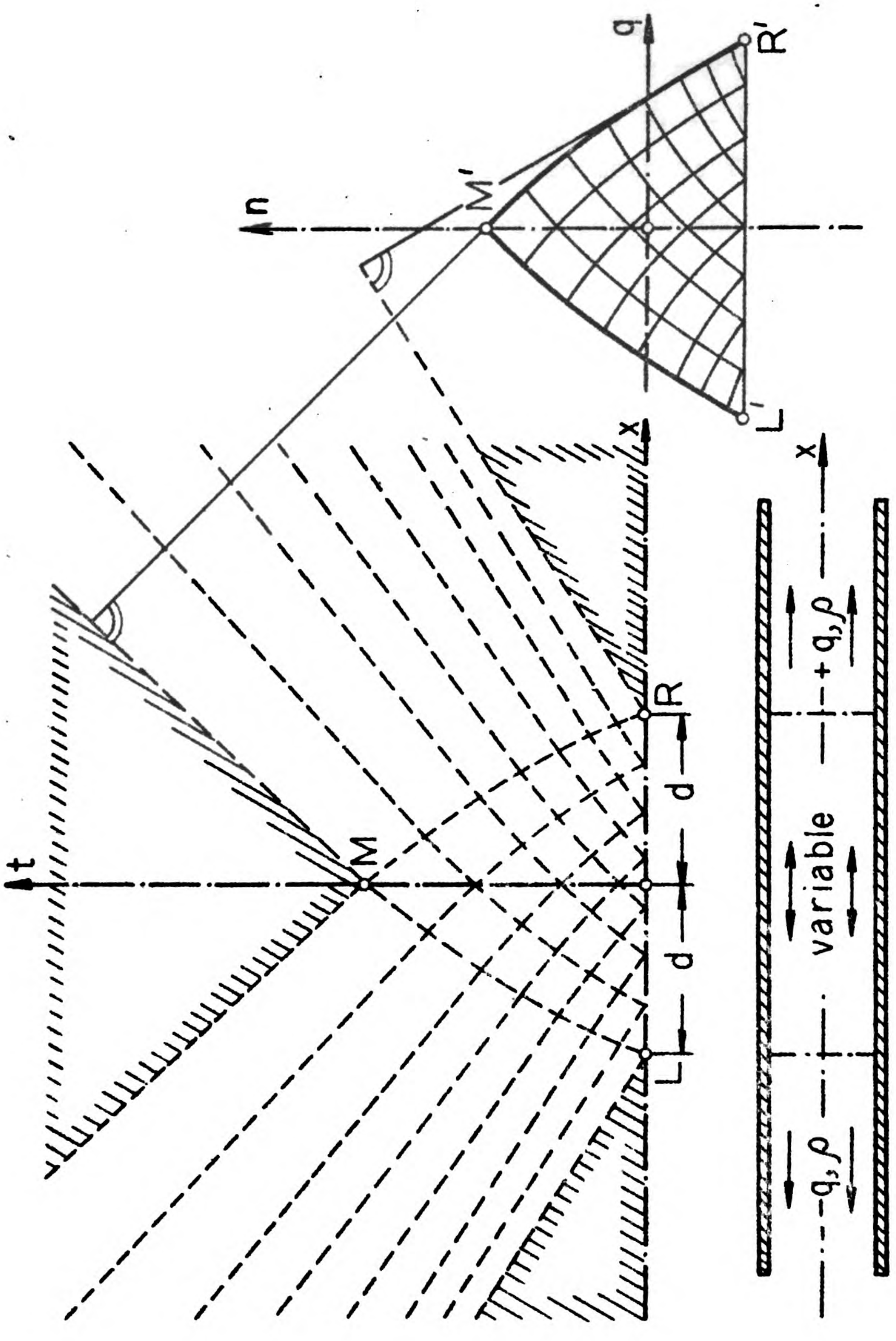


Fig. 25: Propagation of a Steady Interference.

Appendix to Chapter III

In this appendix a few examples taken from "Theorie der Nichtstationaeren Gasstroemung II", by R. Sauer (see bibliography) are given to illustrate the methods for weak and strong shocks. Method II is used for weak shocks and the shock polar for strong shocks.

a) In Fig. 26 an example is given in which a weak shock is developed from a Hugoniot wave. The Hugoniot wave is defined to be the special compression flow gradually accelerated from rest in which all characteristics end in a fixed point ($t = \bar{t}$) in the (x,t)-plane. The steady compression flow is transformed at $t = \bar{t}$ into a steady compression shock with a straight shock line s. The areas 0 to 4 of the (x,t)-plane correspond to the parabolic points 0' to 4' in the (q,n)-plane. The shock line s is perpendicular to the parabolic chord $\overline{0'4'}$.

b) Fig. 27 shows the reflection of a weak compression shock in a homogeneous gas. A finite tube with a piston at one end and a fixed wall at the other is considered. If the piston is moved suddenly into the tube with a velocity q_1, the shock line s is produced. The straight line a represents the movement of the piston. At the wall the shock line s is reflected as the shock line p in such a way that the straight lines s and p are perpendicular to the parabolic chords $\overline{0'1'}$ and $\overline{1'2'}$, respectively. Note that the angle of incidence α and the angle of reflection β are not equal.

c) Fig. 28 illustrates the intersection of two weak compression shocks. Both pistons move toward each other after being suddenly accelerated from rest to the constant velocities q_1 and q_2. The piston curves a, b and the shock lines s, p are

straight lines. The shock lines are deflected at the point P and are perpendicular to the parabolic chords $\overline{0'1'}$, $\overline{2'3'}$, $\overline{0'2'}$, and $\overline{1'3'}$.

d) Fig. 29 shows the construction of a weak compression shock followed by a compression wave. The piston is accelerated suddenly from rest to the velocity q_1 and is gradually accelerated from the point $t = t_0$. A straight shock line s originates from the point L and is deflected to the right by the intersections of the compression lines of the adiabatic gas wave. The areas 0 to 5 in the (x,t)-plane again correspond to the points $0'$ to $5'$ in the (q,n)-plane. The boundary line between any two areas is perpendicular to the respective parabolic chord, e.g., the last line element of the shock line between 0 and 5 is perpendicular to the chord $\overline{0'5'}$.

e) In Fig. 30 the construction is carried out for a flow with a weak compression shock reflected into a non-homogeneous gas. The flow is started in the same manner as in example b, but at time $t = \bar{t}$ the piston is decelerated and an expansion wave is propagated into the reflected shock. The shock line s is deflected after the reflection when it meets the expansion wave.

f) The flow in Fig. 31 is similar to that in the previous example except that the left moving shock is produced by a piston moving to the left. An adiabatic wave is propagated to the right by the leftward movement of the piston, which accelerates gradually from rest.

g) Fig. 32 shows the development of a strong compression shock from a Hugoniot wave. The Hugoniot compression wave (assumed steady) between the areas 0 and 1 in the (x,t)-plane is made to correspond with the characteristic $\overline{0'1'}$ in the (q,c)-plane. The resulting compression shock has to be constructed so that the gas particles crossing the shock and those crossing the Hugoniot wave finally attain the same pressure and velocity. If the shock polar originating from 0' were interrupted at H', the point located perpendicularly above 1', then the same velocity $q = q_1$ would be reached at the terminal states 1' and H' but the pressures would be different. In this example the pressure increases are

$$(p/p_0)_H = 36, \qquad (p/p_0)_1 = 100 > (p/p_0)_H .$$

In order to obtain equalities not only in q, but also in p, one must begin at 1' and follow the characteristic down to the right while simultaneously proceeding along the shock polar from H' to the right. Since the pressure decreases along the characteristic and increases along the shock polar, a pair of points 2', 3' (situated perpendicularly above each other) is found at which the pressure and velocity have the values

$$q_2 = q_3 = 5.3, \qquad (p/p_0)_2 = (p/p_0)_3 = 50,$$

respectively.

The states 0' to 3' in the (q,c)-plane correspond to the areas 0 to 3 in the (x,t)-plane. The Hugoniot wave is situated between 0 and 1, the compression shock between 0 and 3,

and the adiabatic expansion wave between 1 and 2. The adiabatic expansion wave corresponds to the characteristic $\overline{1'2'}$ in the (q,c)-plane. The double line separating the areas 2 and 3 is a so-called contact surface across which there is a discontinuity in temperature and density but not in velocity and pressure. This discontinuity can be eliminated only by considering heat exchange, which has been neglected in this treatment.

h) Fig. 33 illustrates the reflection of a strong shock in a homogeneous gas. The approaching compression shock is given by the shock line s and by the shock polar $\overline{0'1'}$; the reflected shock by p and $1'2'$. The inclinations are determined by

$$\tan \sigma_s = q_0 + w_1, \qquad \tan \sigma_p = q_1 - w_2 .$$

As in the case of the weak compression shock, the angle of reflection β is smaller than the angle of incidence α.

i) Fig. 34 shows the intersection of two strong compression shocks. The shock polars $\overline{0'1'}$ and $\overline{0'2'}$ of the two compression shocks before the intersection are known. The shock polars $\overline{1'3'}$ and $\overline{1'4'}$ must be cut off at the points 3' and 4' in order that equal pressure and velocity values be obtained in the areas 3 and 4. The inclination of the shock lines before the intersection are

$$\tan \sigma_s = q_0 + w_1, \qquad \tan \sigma_p = q_0 - w_2,$$

and after the intersection they are

$$\tan \sigma_s = q_2 + w_4, \qquad \tan \sigma_p = q_1 - w_3.$$

As in example (g) there is a contact surface between the areas 3 and 4. In the case of symmetry ($\sigma_s = -\sigma_p$), the states $3'$ and $4'$ become identical and the contact surface disappears.

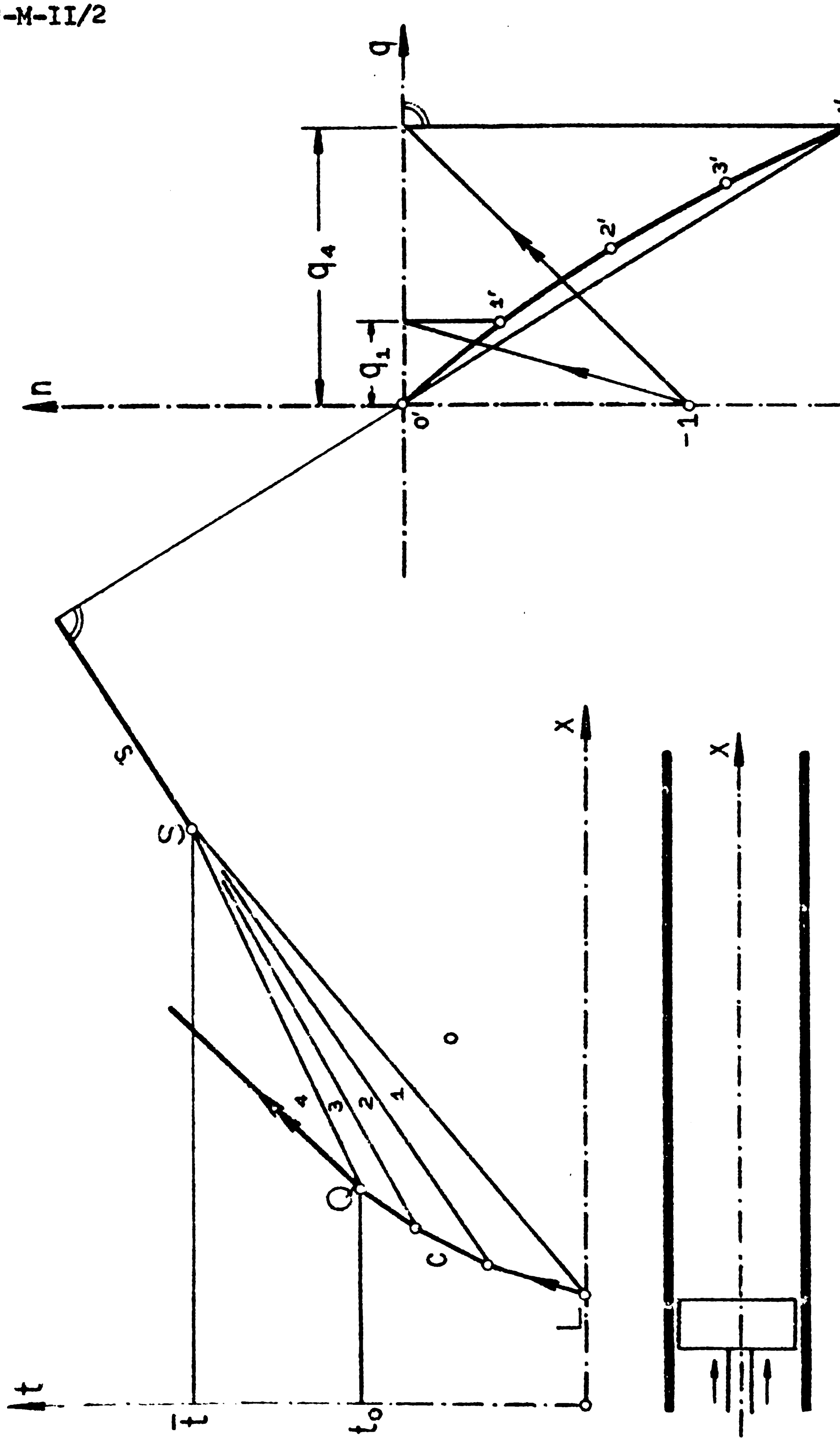


Fig.26: Development of a Weak Compression Shock from a Hugoniot Wave.

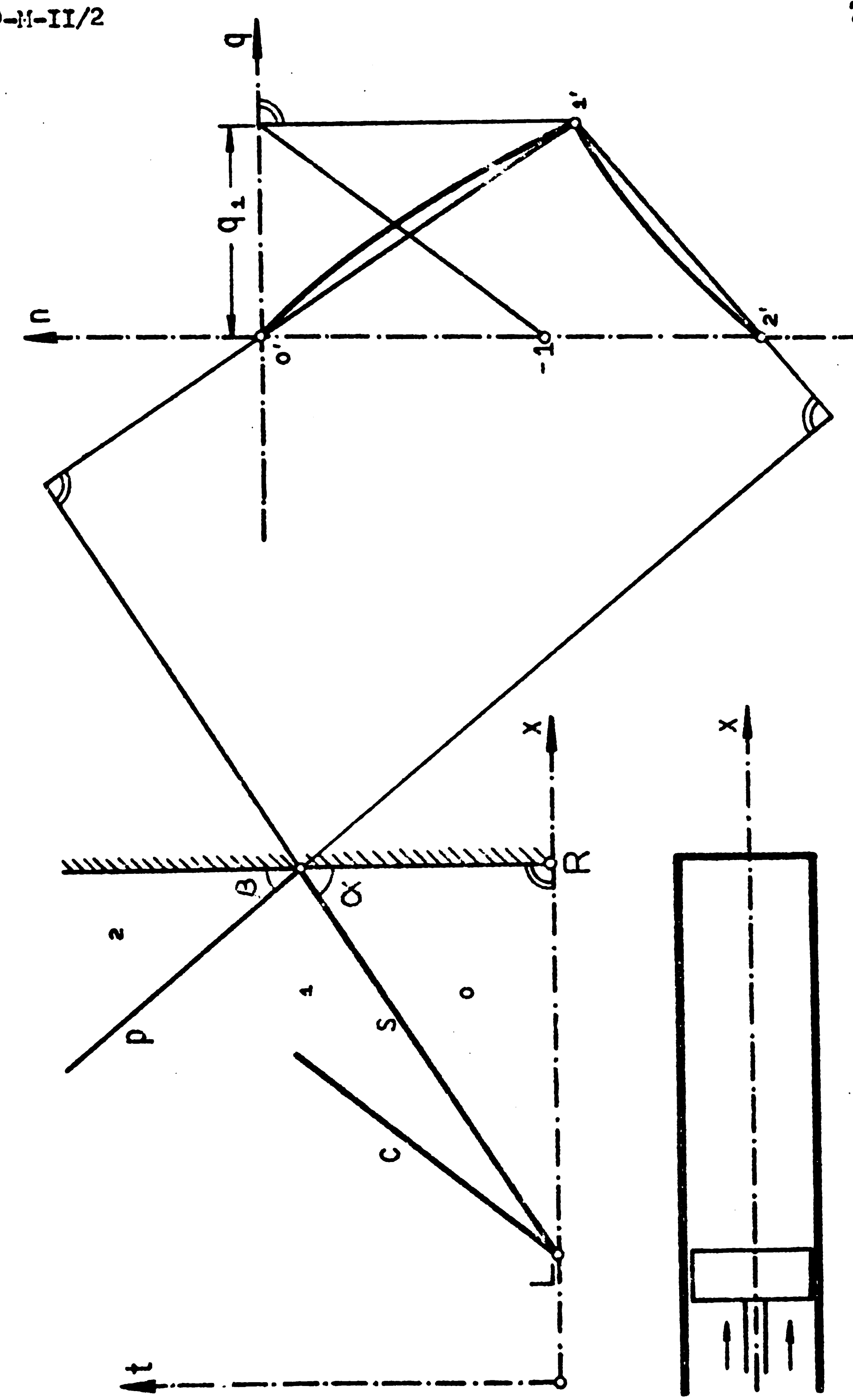


Fig. 27: Reflection of a Weak Compression Shock in Homogeneous Gas.

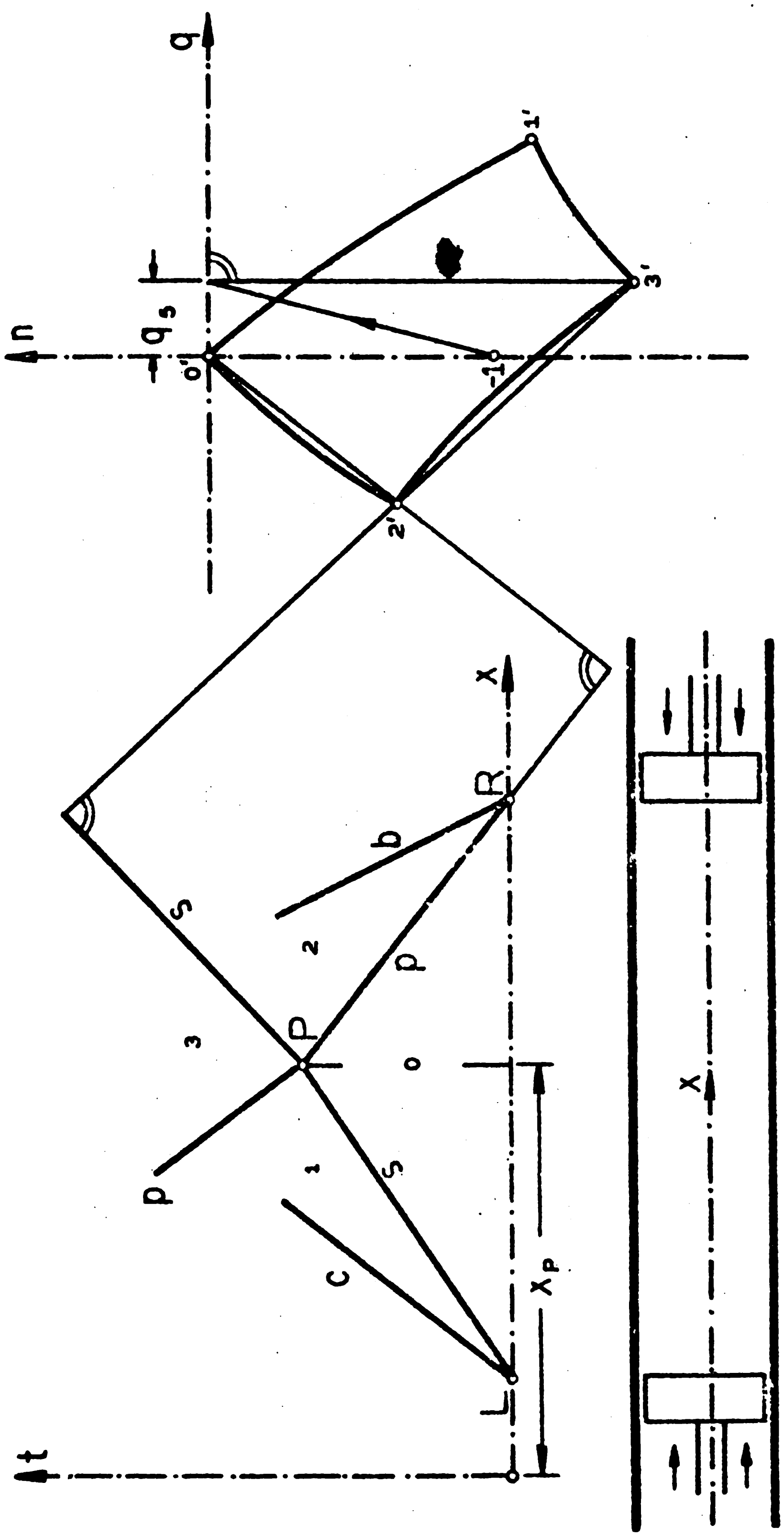


Fig. 28: Intersection of Two Weak Compression Shocks.

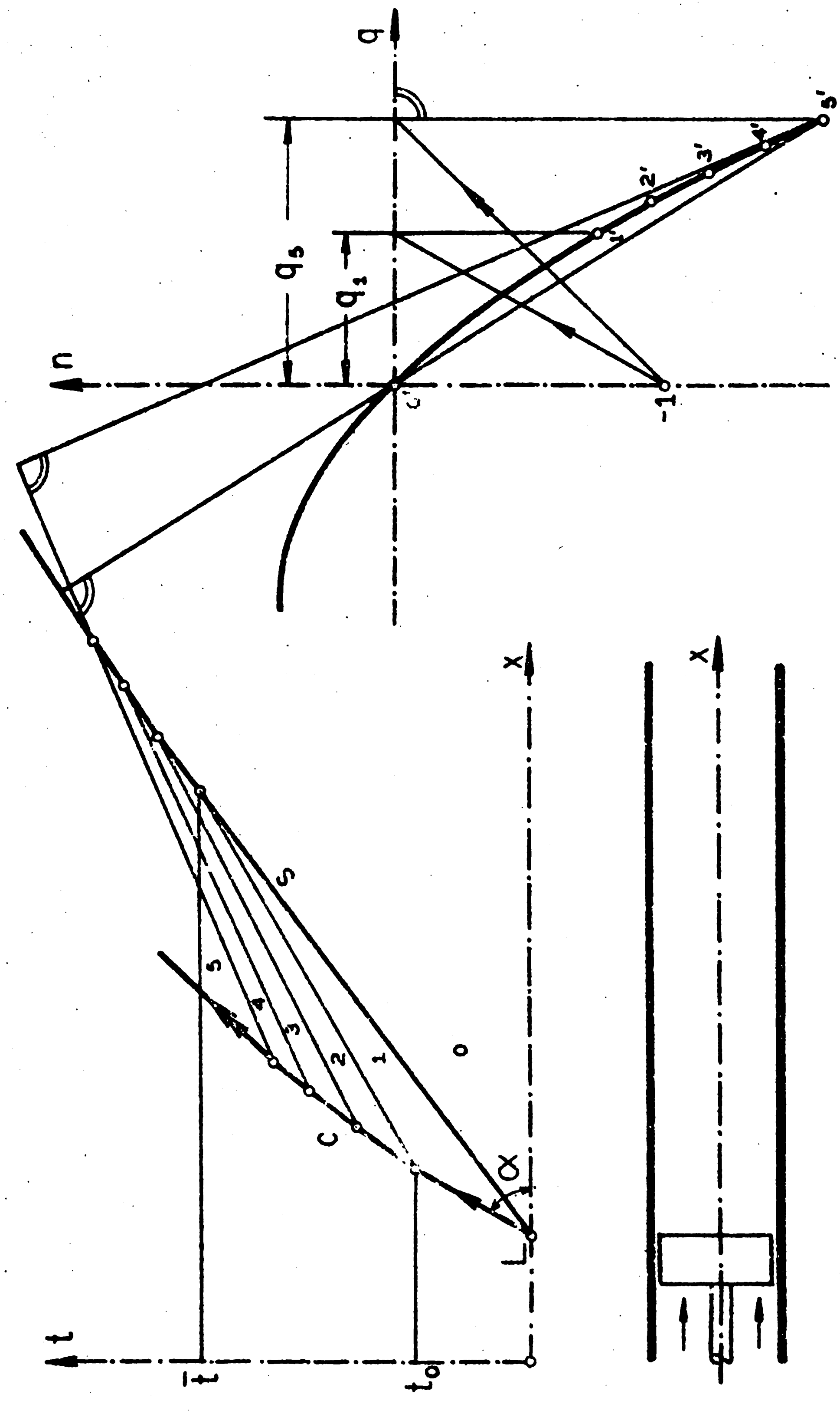


Fig. 29: Weak Compression Shock with Succeeding Compression Wave.

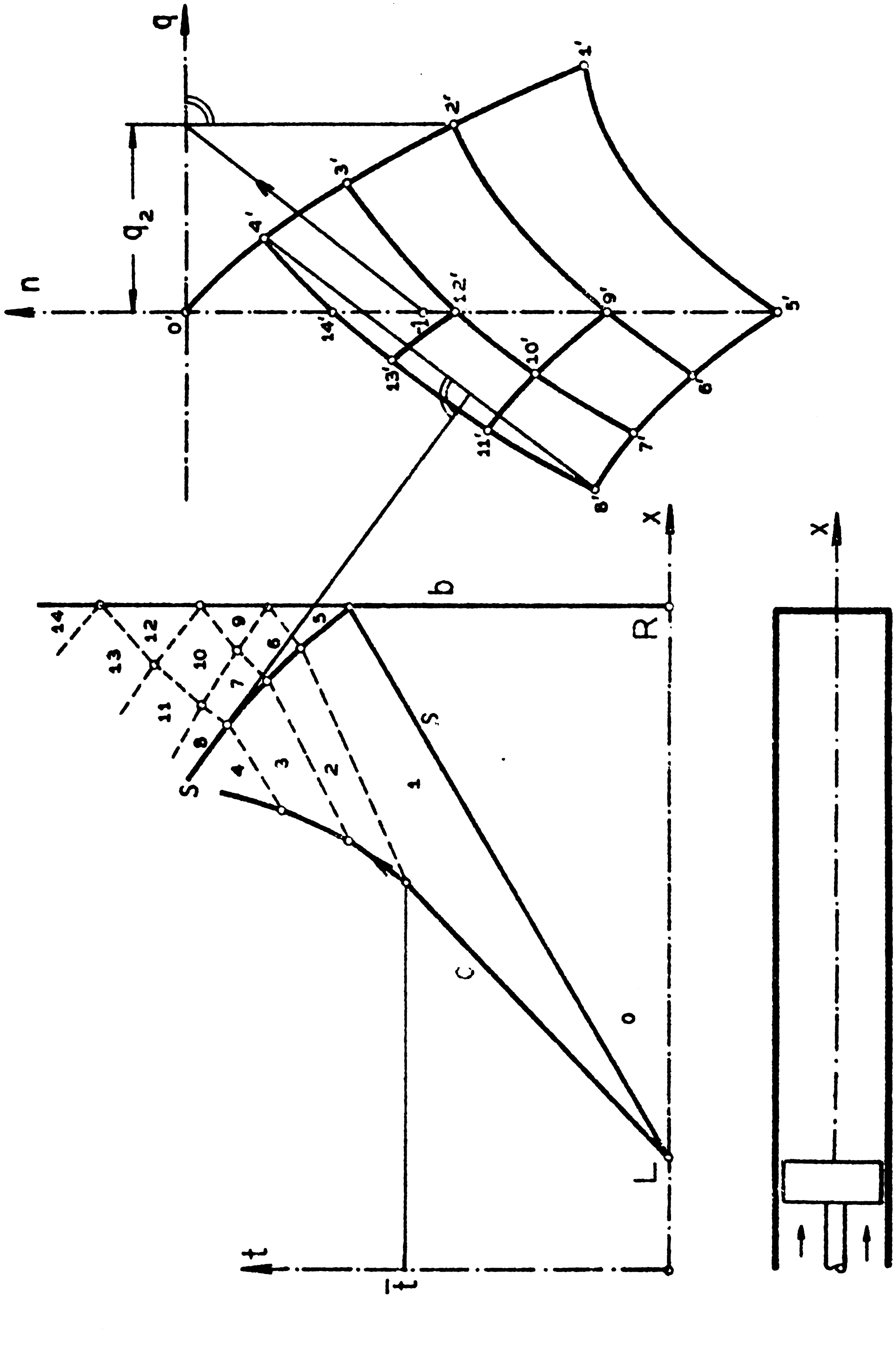


Fig. 30: Reflection of a Weak Compression Shock in Non-homogeneous Gas .

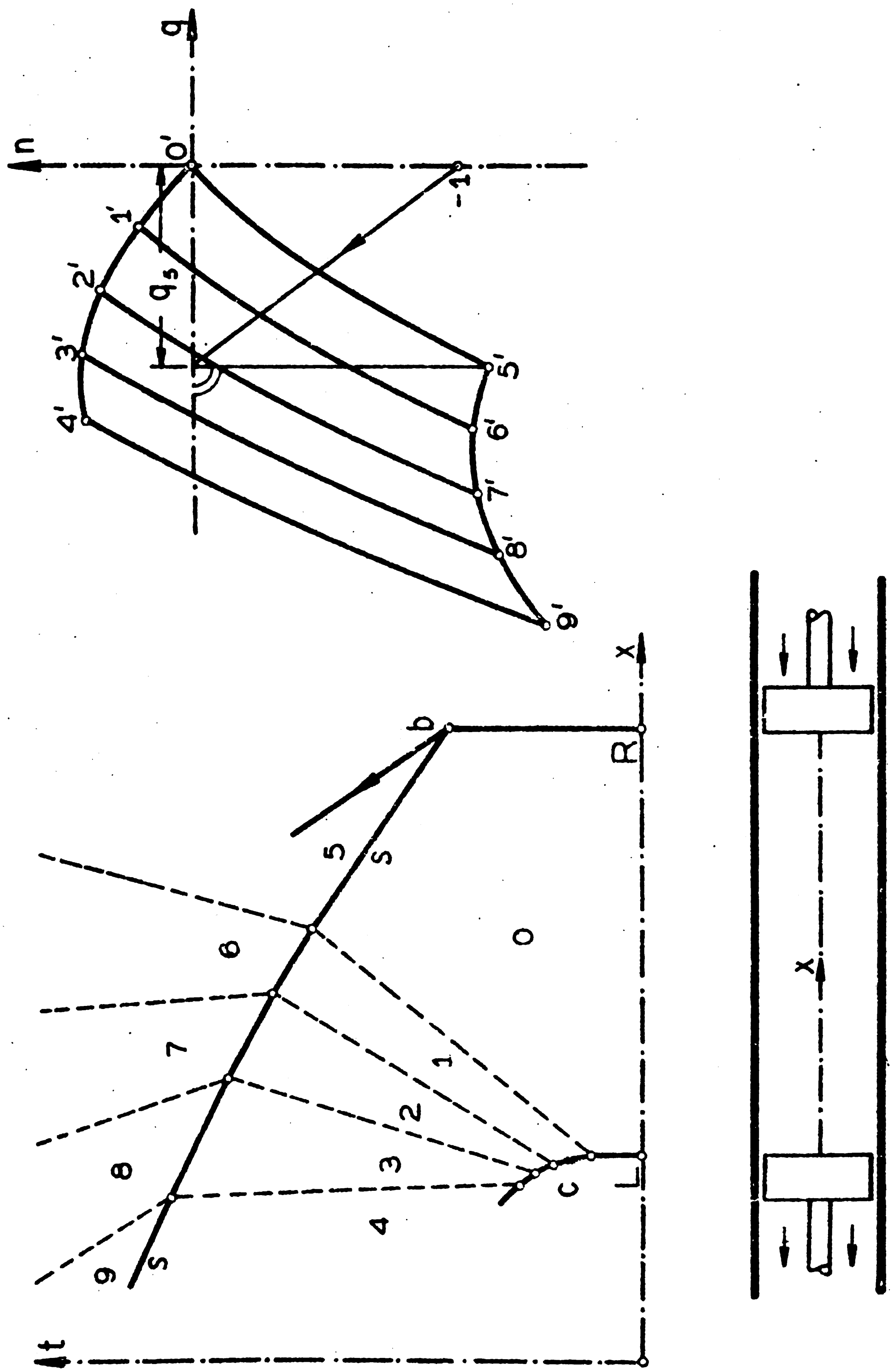


Fig. 31: Intersection of a Weak Compression Shock and an Adiabatic Wave.

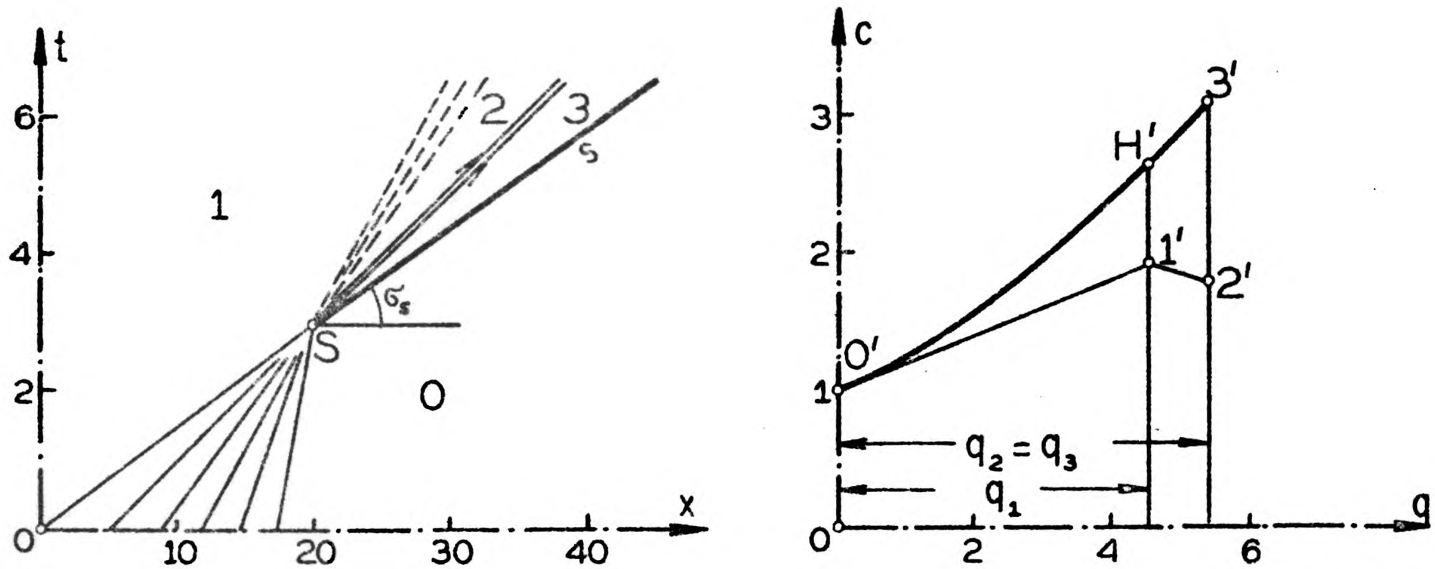


Fig. 32: Development of a Strong Compression Shock from a Hugoniot Wave.

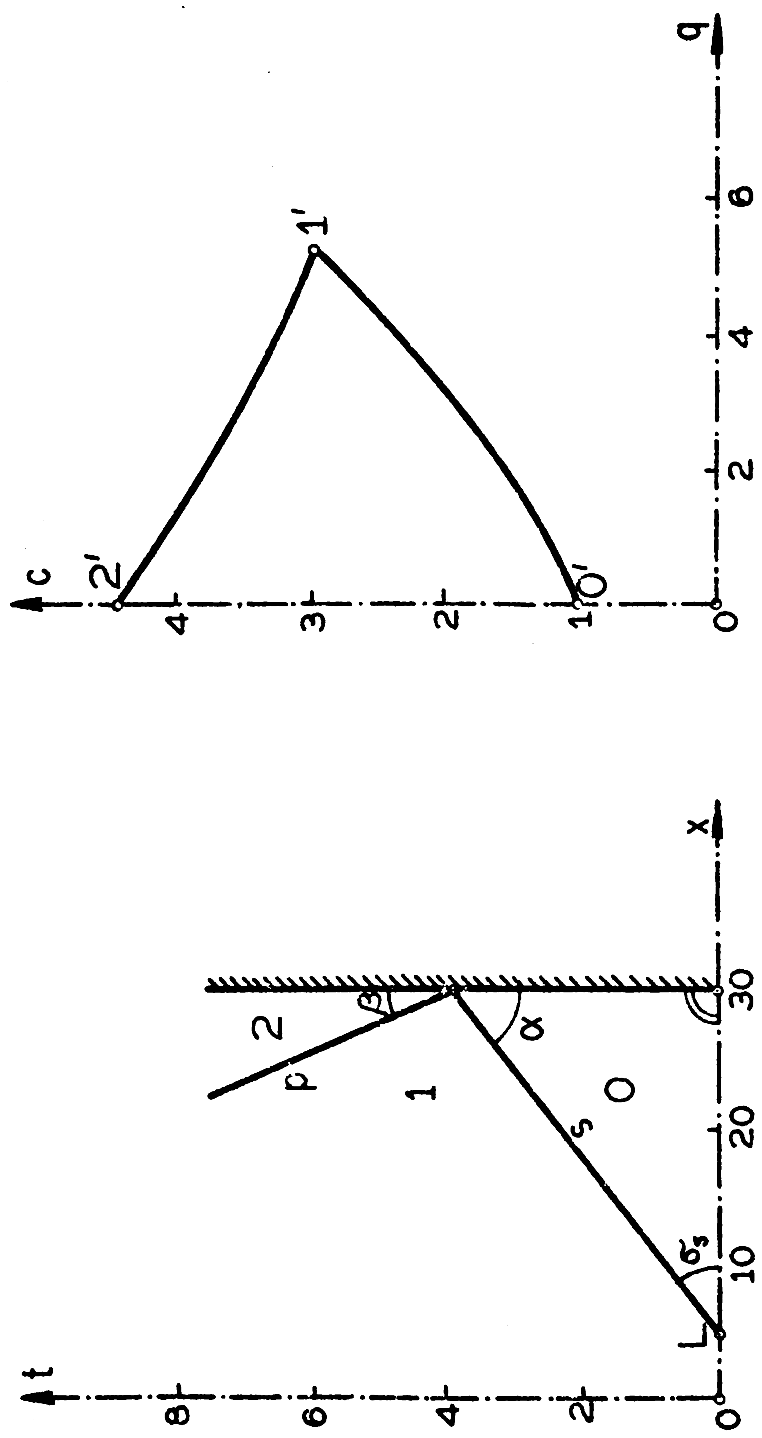


Fig. 33: Reflection of a Strong Compression Shock in Homogeneous Gas..

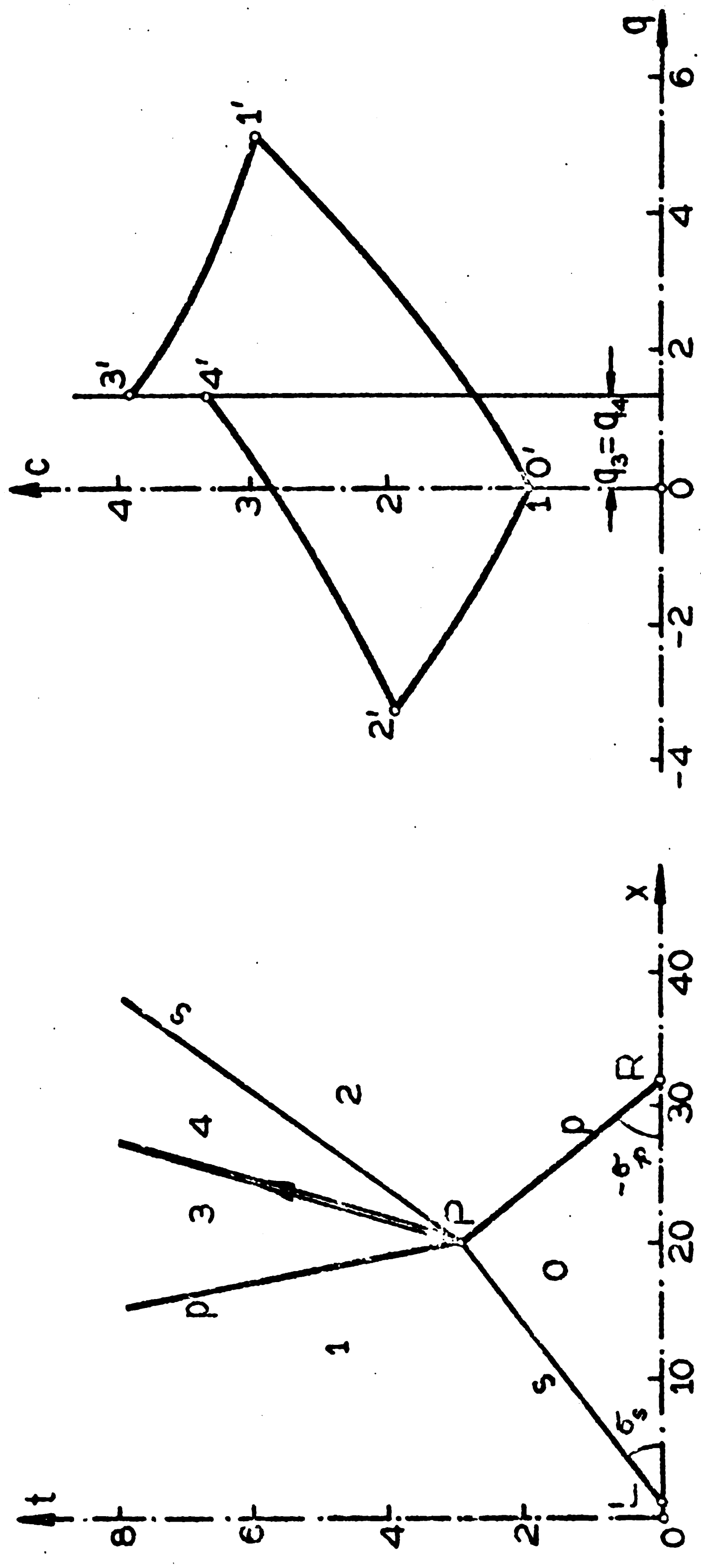


Fig. 34: Intersection of Two Strong Compression Shocks.

BIBLIOGRAPHY

CHAPTER I Statement of the Problem

1. Introductory remarks

2. Basic equations for one-dimensional motion

Sauer, R. Theorie der nichtstationaeren Gasstroemung I, ZWB FB 1675/1 (1942). Translation by AAF, No. F-TS-758-RE, Aug. 1946. Summary by Brown University, No. A-9039.

Sauer, R. Theorie der nichtstationaeren Gasstroemung III, ZWB FB 1675/3 (1943). Translation by AAF, No. F-TS-770-RE, Feb. 1947. Summary by Brown University, No. A-9054.

3. The analogy between one-dimensional and two-dimensional flow

Sauer, R. Theorie der nichtstationaeren Gasstroemung III (see bibliography for Chapt. I, Sect. 2).

A. The stream function ψ of the non-isentropic flow

Sauer, R. ibid.

Guderley, G. Nichtstationaere Stroemungen in duennen Rohren veraenderlichen Querschnitts, FB 1744 (1942).

B. The potential φ of isentropic flow

Sauer, R. loc. cit.

Hantzsche, W. and Wendt, H. Zum Verdichtungsstoss bei Zylinder und Kugelwellen, Jahrb. der Deut. Luftfahrtforschung, 1940, pp. 536-538. Summary by Brown University, No. A-9046.

4. The derivation of the characteristic equations

Sauer, R. Theorie der nichtstationaeren Gasstroemung I and III (see bibliography for Chapt. I, Sect.2).

Guderley, G. Nichtstationaere Stroemungen in duennen Rohren veraenderlichen Querschnitts (see bibliography for Chapt. I, Sect. 3A).

Courant, R. and Hilbert, D. Methoden der mathematischen Physik, vol. II, p. 290, Springer, Berlin, 1937. Reprinted by Interscience Publishers, Inc., New York, 1943.

5. The dimensionless form of the characteristic equations

6. Limiting lines

Tollmien, W. Z.A.M.M. 17, 117-136, (1937).

Tollmien, W. Z.A.M.M. 21, 140-152, (1940).

Ringleb, F. Z.A.M.M. 20, 185-198, (1940).

Ringleb, F. Deutsche Mathematik 5, 377-384, (1940).

Sauer, R. Theoretische Einfuehrung in die Gasdynamik, p. 96, Springer, Berlin (1943). (Reprinted by Edwards Brothers Inc., Ann Arbor, Mich.).

CHAPTER II Approximate Methods of Solution

1. General procedure for the approximate solutions

Sauer, R. Theorie der nichtstationaeren Gasstroemung I and III (see bibliography for Chapt. I, Sect.2).

2. Numerical methods

A. Method I

Sauer, R. Theorie der nichtstationaeren Gasstroemung III, p. 15 (see bibliography for Chapt. I, Sect.2).

B. Method II

Sauer, R. Theorie der nichtstationaeren Gasstroemung III, p. 19 (see bibliography for Chapt. I, Sect. 2).

Sauer, R. Zur Theorie des Stahlrohrs, Deutsche Lufo. FB 1864/1. Summary by Brown University, No. A-9055.

Guderley, G. Nichtstationaere Stroemungen in duennen Rohren veraenderlichen Querschnitts (see bibliography for Chapt. I, Sect.3).

3. Graphical methods

A. Method I

Sauer, R. Theorie der nichtstationaeren Gasstroemung I, p. 12 (see bibliography for Chapt. I, Sect. 2).

Sauer, R. Characteristikenverfahren fuer die eindimensionale instationaere Gasstroemung, Ingen Archiv 13, 79-89, (1942). Translation by Brown University. Summary by Brown University, No. A-9001.

B. Method II

Sauer, R. Theorie der nichtstationaeren Gasstroemung I, p. 10 (see bibliography for Chapt. I, Sect. 2).

Schultz-Grunow, F. Nichtstationaere eindimensionale Gasbewegung, Forsch, Ing. Wesen, 3, 125-234, (1942). Summary by Brown University, No. A-9040.

C. Method III

Sauer, R. Theorie der nichtstationaeren Gasstroemung III, p. 22 (see bibliography for Chapt. I, Sect. 2).

Sauer, R. Characteristikenverfahren fuer Kugel und Zylinderwellen reibungsloser Gase, Z.A.M.M. 23, p. 29, (1943). Summary by Brown University, A-9065.

D. Method IV

Sauer, R. Theorie der nichtstationaeren Gasstroemung III, p. 25 (see bibliography for Chapt. I, Sect. 2).

Schultz-Grunow, F. Nichtstationaere Gasstroemung in Duesen und Diffusoren, Ingen. Archiv 14, (1943). Summary by Brown University, No. A-9069.

Schultz-Grunow, F. Gasdynamische Untersuchungen am Verpuffungstrahlrohr, FB 2015/1,2. Translation by NACA, TM 1131. Summary by Brown University, No. A-9038.

4. Boundary conditions

A. Initial value problem

Sauer, R. Theorie der nichtstationaeren Gasstroemung III, p. 17 (see bibliography for Chapt. I, Sect. 2).

Courant, R. and Hilbert, D. Methoden der mathematischen Physik, vol. II, Chapt. V. (see bibliography for Chapt. I, Sect. 4).

B. Flow between fixed cross sections

Sauer, R. loc. cit.

C. Flow between moving cross sections

D. Discharge under constant pressure

Sauer, R. loc. cit.

Schultz-Grunow, F. Nichtstationaere eindimensionale Gasbewegung (see bibliography for Chapt. II, Sect. 3B).

E. Spherically and cylindrically symmetrical flow

5. Simple waves

Sauer, R. Theorie der nichtstationaeren Gasstroemung I, p. 21 (see bibliography for Chapt. I, Sect. 2).

Lin, C.C. An Introduction to the Dynamics of Compressible Fluids, Chapt. II. Notes prepared from lectures given at Brown University.

CHAPTER III Treatment of Shock Waves.

1. Introduction

2. Basic equations for the compression shock

Sauer, R. Theorie der nichtstationaeren Gasstroemung II, ZWB FB 1675/2 (1942). Translation by AAF, No. F-TS-758-RE, Sept. 1946. Summary by Brown University, No. A-9041.

Guderley, G. Nichtstationaere Stroemungen in duennen Rohren veraenderlichen Querschnitts (see bibliography for Chapt. I, Sect. 3A).

Liepmann, H.W. and Puckett, A.E. Introduction to the Aerodynamics of a Compressible Fluid, John Wiley and Sons, New York, (1947).

3. Shock polar diagram

Sauer, R. Theorie der nichtstationaeren Gasstroemung II (see bibliography for Chapt. III, Sect. 2).

4. Transition from compression shock to adiabatic isentropic flow

Sauer, R. Theorie der nichtstationaeren Gasstroemung II (see bibliography for Chapt. III, Sect. 2).

Sauer, R. Zur Theorie des nichtstationaeren ebenen Verdichtungsstosse, Ingen. Archiv 14, (1943). Summary by Brown University, No. A-9033.

5. Approximate methods for weak compression shocks

A. Numerical methods

Guderley, G. Nichtstationaere Stroemungen in duennen Rohren veraenderlichen Querschnitts (see bibliography for Chapt. I, Sect. 3A).

B. Graphical methods

a. Method I

Sauer, R. Theorie der nichtstationaeren Gasstroemung II, p. 13 (bibliography for Chapt. III, Sect. 2).

Sauer, R. Zur Theorie des nichtstationaeren ebenen Verdichtungsstosse (see bibliography for Chapt. III, Sect. 4).

b. Method II

Sauer, R. Theorie der nichtstationaere Gasstroemung II, p. 10 (see bibliography for Chapt. III, Sect. 2).

Schultz-Grunow, F. Nichtstationaere eindimensionale Gasbewegung (see bibliography for Chapt. II, Sect. 3B).

Schultz-Grunow, F. Nichtstationaere Gasstroemung in Duesen und Diffusoren (see bibliography for Chapt. II, Sect. 3D).

Schultz-Grunow, F. Gasdynamische Untersuchungen am Verpuffungstrahlrohr (see bibliography for Chapt. II, Sect. 3D).

6. Exact method for steady strong compression shocks

Sauer, R. Theorie der nichtstationaeren Gasstroemung II, p. 17 (see bibliography for Chapt. III, Sect. 2).

Pfriem, H. Zur theorie ebener Druckwellen mit steiler Front, Akust. Zeitschr, 6, 222-244, (1941).

7. The unsteady strong compression shock

Sauer, R. Theorie der nichtstationaeren Gasstroemung II, p. 20 (see bibliography for Chapt. III, Sect. 2).

CHAPTER IV Extensions and Additions to the General Theory

1. General remarks

2. Detonation waves

Sauer, R. Theorie der nichtstationaeren Gasstroemung IV, ZWB FB 1675/4 (1943). Translation by AAF, No. F-TS-949-RE, March 1947. Summary by Brown University, No. A-9042.

Becker, R. Stosswelle und Detonation, Zeitschr. fuer Phys. 8, 321-362, (1921).

Pfriem, H. Die stationaere Detonationswelle in Gasen, Forsch. Ing. Wesen 12, 143-158, (1941). Summary by Brown University, No. A-9092.

Damkoehler, G. and Schmidt, A. Gasdynamische Beitrage zur Auswertung von Flammenversuchen in Rohrstrecken, Zeitsch. fuer Elektrochemie, 47, 547-567, (1941). Summary by Brown University, No. A-9093.

Jost, W. Explosion and Combustion Processes in Gases, McGraw-Hill, New York (1946).

Doering, W. Grundlegende Betrachtungen ueber den Detonationsvorgang, Bericht fuer die Forschungsabteilung der RLM, (1940).

Langweiler, F. Beitrag zur hydrodynamischen Detonationstheorie, Zeitschr. fuer techn. Phys. 19, 271-283, (1938).

3. Small amplitude waves

Rayleigh, Lord. Theory of Sound, Dover (1945).

Hadamard, J. Leçons sur la propagation des ondes, A. Hermann, Paris (1903).

4. Exact analytical solutions

Riemann, B. Gesammelte Werke, 2nd ed., p. 156, Teubner, Leipzig (1892).

Beckert, K. Zur Theorie ebener Stroemungen in reibungsfreien Gasen II, Ann. Phys. 5, 37, 89-123, (1940).

Kobes, K. Z. ost. Ing.-und Archiv.- Ver. 62,553-578 (1910).

Beckert, K. Zur Theorie ebener Stroemungen in reibungsfreien Gasen II, Ann. Phys. 5, 38, 1-25, (1940).

Pfriem, H. Zur gegenseitigen Ueberlagerung ungedampfter ebener Gaswellen mit grosser Schwingungsweite, Akust. Zeitschr. 64, 222-244, (1941).