Solutions of

Nielson and Chuang's Quantum Computation and Quantum Information by

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Contents

1 Chapter 2

Solution 2.1, Page 63

Let

$$A = (1, -1) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$B = (1, 2) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$A = (2, 1) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Here, C-B = A
Therefore, we can say that A, B, and C are *Linear Dependent*

Solution 2.2, Page 64

We can write this as: $A|0\rangle = |1\rangle = 0.|0\rangle + 1.|1\rangle$
and, $A|1\rangle = |0\rangle = 1.|0\rangle + 0.|1\rangle$
Therefore, we can write the coefficients as,
$A_{11} = 0, A_{10} = 1, A_{21} = 1, A_{22} = 0$
We will get,
A = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

OR

We know, $V \rightarrow$ Vector space in $|0\rangle$ and $|1\rangle$ basis.
and $A : V \rightarrow$ V, $A|0\rangle$=$|1\rangle$ and $A|1\rangle = |0\rangle$
and also, $A|v_j\rangle = \Sigma_i A_{ij}|w_i\rangle$
as, $A|0\rangle = |1\rangle$, therefore, $A_{12} = 1$
and $A|1\rangle = |0\rangle$, therefore, $A_{21} = 1$
and, A = $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$
A_{11} and A_{22} equals to 0 as, we don't have $A|0\rangle = |0\rangle$ and $A|1\rangle = |1\rangle$ respectively.
Therefore, A = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Solution 2.3, Page 64

As given, $A : V \rightarrow W$ and $B : W \rightarrow X$
Now, $|v_i\rangle$ belongs to V, $|w_j\rangle$ belongs to W and, $|x_k\rangle$ belongs to X.
$A|v_i\rangle = \Sigma_j A_{ji}|w_j\rangle$ and,
$A|w_j\rangle = \Sigma_k B_{kj}|x_k\rangle$.
Now,
$\Rightarrow BA|v_i\rangle = B[A|v_i\rangle]$
$\Rightarrow \qquad = B[\Sigma_j A_{ji}|w_j\rangle]$
$\Rightarrow \qquad = \Sigma_j A_{ji}[B|w_j\rangle]$
$\Rightarrow \qquad = \Sigma_j A_{ji}\Sigma_k B_{kj}|x_k\rangle$
$\Rightarrow \qquad = \Sigma_k B_{kj}\Sigma_j A_{ji}|x_k\rangle$
$\Rightarrow BA|v_i\rangle = \Sigma_k [BA]_{ki}|x_k\rangle$

Solution 2.4, Page 65

Considering computational basis for input and output basis. Computational basis are nothing but the $\{|0\rangle, |1\rangle\}$ basis.
Now,
$I|0\rangle = |0\rangle = 1.|0\rangle + 0.|1\rangle$
$I|1\rangle = |1\rangle = 0.|0\rangle + 1.|1\rangle$
Therefore, $I_{11} = 1, I_{12} = 0, I_{21} = 0, I_{22} = 1$
$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution 2.6, Page 66

To Prove: $(\Sigma_i \lambda_i |w_i\rangle, |v\rangle) = \Sigma_i \lambda_i^* (|w_i\rangle, |v\rangle)$
LHS:
$\Rightarrow (\Sigma_i \lambda_i |w_i\rangle, |v\rangle) = (|v\rangle, \Sigma_i \lambda_i |w_i\rangle)^*$
$\Rightarrow \qquad = [\Sigma_i \lambda_i (|v\rangle, |w_i\rangle)]^*$
$\Rightarrow \qquad = [\Sigma_i \lambda_i^* (|v\rangle, |w_i\rangle)^*]$
$\Rightarrow \qquad = [\Sigma_i \lambda_i^* (|w_i\rangle, |v\rangle)]$
$\Rightarrow (\Sigma_i \lambda_i |w_i\rangle, |v\rangle) = [\Sigma_i \lambda_i^* |w_i\rangle, |v\rangle] =$ **RHS**

Solution 2.7, Page 66

Here, $|w\rangle = (1,1) = \begin{bmatrix}1\\1\end{bmatrix}$ and $|v\rangle = (1,-1) = \begin{bmatrix}1\\-1\end{bmatrix}$

Now, $\langle w|v\rangle = \begin{bmatrix}1 & 1\end{bmatrix} \cdot \begin{bmatrix}1\\-1\end{bmatrix} = 1 + (-1) = 0$

$\Rightarrow \langle w|v\rangle = 0,$ Therefore, Orthogonal.

and $\langle w|w\rangle = \begin{bmatrix}1 & 1\end{bmatrix} \cdot \begin{bmatrix}1\\1\end{bmatrix} = 2$

and $\langle w|w\rangle = \begin{bmatrix}1 & -1\end{bmatrix} \cdot \begin{bmatrix}1\\-1\end{bmatrix} = 2$

Therefore, $|w\rangle_{norm} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ and $|v\rangle_{norm} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

Solution 2.8, Page 66

Now, Let Un-normalized basis set be: $|w_1\rangle, ..., |w_d\rangle$.
and Let, Normalized basis set be: $|v_1\rangle, ..., |v_d\rangle$
Now, according to Gram-Schmidt Decomposition:

$\Rightarrow |v_{k+1}\rangle = \frac{|w_{k+1}\rangle - \sum_{i=1}^{k}\langle v_i|w_{k+1}\rangle|v_i\rangle}{\left|\left||w_{k+1}\rangle - \sum_{i=1}^{k}\langle v_i|w_{k+1}\rangle|v_i\rangle\right|\right|}$

$\Rightarrow \langle v_{k+1}| = \frac{\langle w_{k+1}| - \sum_{i=1}^{k}\langle v_i|\langle w_{k+1}|v_i\rangle}{\left|\left|\langle w_{k+1}| - \sum_{i=1}^{k}\langle v_i|\langle w_{k+1}|v_i\rangle\right|\right|}$

Now,

$\Rightarrow |v_1\rangle = \frac{|w_1\rangle}{|||w_1\rangle||}$ and $\langle v_1| = \frac{\langle w_1|}{||\langle w_1|||}$

Therefore,

$\Rightarrow \langle v_1|v_1\rangle = \frac{\langle w_1|w_1\rangle}{|||w_1\rangle||^2} = \frac{\langle w_1|w_1\rangle}{\langle w_1|w_1\rangle} = 1$

and,

$\Rightarrow |v_2\rangle = \frac{|w_2\rangle - \langle v_1|w_2\rangle|v_1\rangle}{|||w_2\rangle - \langle v_1|w_2\rangle|v_1\rangle||}$

Now, we know $\langle v_1|w_2\rangle = 0$

Therefore, $\Rightarrow |v_2\rangle = \frac{|w_2\rangle}{|||w_2\rangle||}$ and so,

$\Rightarrow \langle v_1|v_2\rangle = 0.$

Hence Verified!

Solution 2.9, Page 68

Here, Pauli Operators are: I, X, Y, Z.
First, look at the identity operator (I). Don't mind the gap in outer product. Sorry guys!

$\Rightarrow I = I_0 I I_1 = \sum_i |i\rangle\langle i| I (\sum_j |j\rangle\langle j|)$

$\Rightarrow \quad = \sum_{ij} |i\rangle\langle j| \langle i| I |j\rangle$

$\Rightarrow \quad = \sum_{ij} \langle i| I |j\rangle |i\rangle \langle j|$
$\Rightarrow \quad = \langle 0| I |0\rangle |0\rangle \langle 0| + \langle 0| I |1\rangle |0\rangle \langle 1| + \langle 1| I |0\rangle |1\rangle \langle 0| + \langle 1| I |1\rangle |1\rangle \langle 1|$
$\Rightarrow \quad = \langle 0|0\rangle |0\rangle \langle 0| + \langle 0|1\rangle |0\rangle \langle 1| + \langle 1|0\rangle |1\rangle \langle 0| + \langle 1|1\rangle |1\rangle \langle 1|$
As Computational Basis is an orthogonal basis, so $\langle i|j\rangle = \delta_{ij}$
Therefore,
$\Rightarrow \boxed{I = |0\rangle \langle 0| + |1\rangle \langle 1|}$

Similarly we can do the math for X, Y and Z.
I leave you with the hint and final answers.

For **X**:
X $|0\rangle = |1\rangle$ and $X |1\rangle = |0\rangle$
Answer: $\boxed{X = |0\rangle \langle 1| + |1\rangle \langle 0|}$

For **Y**:
Y $|0\rangle = \iota |1\rangle$ and $Y |1\rangle = -\iota |0\rangle$
Answer: $\boxed{Y = \iota |0\rangle \langle 1| - \iota |1\rangle \langle 0|}$

For **Z**:
Z $|0\rangle = |0\rangle$ and $Z |1\rangle = - |1\rangle$
Answer: $\boxed{Z = |0\rangle \langle 0| - |1\rangle \langle 1|}$

Solution 2.11, Page 69

For **X:**
$\Rightarrow X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$\Rightarrow |X - \lambda I| = 0$
$\Rightarrow \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0$
$\Rightarrow \lambda^2 - 1 = 0$
$\Rightarrow \lambda = \pm 1$
Now, $for \lambda = 1:$
$\Rightarrow (X - 1.I) |v_1\rangle = 0$
$\Rightarrow X |v_1\rangle = 1.I |v_1\rangle$

Let, $|v_1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

$\Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 1. \begin{bmatrix} a \\ b \end{bmatrix}$

$\Rightarrow \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

This can only be true if, a=b.

Therefore,

$\Rightarrow \boxed{|v_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

Similarly, for $\lambda = -1$:

$\Rightarrow \boxed{|v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

Diagonal Representation for X:

$\Rightarrow X = \sum_i \lambda_i |i\rangle \langle i| = 1. |v_1\rangle \langle v_1| + (-1). |v_2\rangle \langle v_2|$

$\Rightarrow X = \boxed{|v_1\rangle \langle v_1| - |v_2\rangle \langle v_2|}$

Now, as $\Rightarrow |v_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

Similarly, $\Rightarrow |v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Put these values in above equation

$\Rightarrow X = |0\rangle \langle 1| + |1\rangle \langle 0|$