Non-Dimensionalization of an Anelastic Stable–Unstable Layer in `Rayleigh`

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1 Purpose

This document describes how to implement, using the `Rayleigh` code, a coupled stable–unstable system. More precisely, the system consists of an unstable convection zone (CZ) that lies adjacent to a stable radiative zone (RZ) in a spherical shell. If the CZ is on top of the RZ we describe the system as "solar-like" or a "tachocline," etc. If the CZ is underneath the RZ (now considered a "weather layer") we describe the system as "Jovian" or "Jupiter," etc. It essentially the same problem (at least to specify) either away, making this document relevant to Nic & COFFIES (tachocline), and to Geoff, Nic, and Matt (Jupiter). In the following sections, we describe the nondimensionalization, governing equations, and reference state.

2 General Equation Set Solved by `Rayleigh`

In general (with rotation and magnetism), `Rayleigh` time-evolves a set of coupled PDEs for the 3D vector velocity $\boldsymbol{u}$, vector magnetic field $\boldsymbol{B}$, pressure perturbation P' (perturbation away from the "reference-state" pressure $\tilde{P}$), and entropy perturbation S' (perturbation away from $\tilde{S}$). Note that S' can also be interpreted as a temperature perturbation in Boussinesq mode. For more details, see `Rayleigh`'s Documentation.

We use spherical coordinates: r (spherical radius), θ (colatitude), ϕ (azimuth angle), as well as cylindrical coordinates: $\lambda = r\sin\theta$ (cylindrical radius, or moment arm) and $z = \cos\theta$ (axial coordinate). In general, $\hat{\boldsymbol{e}}_q$ denotes a position-dependent unit vector in the direction of increasing q. With this notation, the full PDE-set solved by `Rayleigh` is:

$$\nabla\cdot(f_1\boldsymbol{u}) = 0, \tag{2.1}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{2.2}$$

$$f_1\left(\frac{D\boldsymbol{u}}{Dt}+c_1\hat{\boldsymbol{e}}_z\times\boldsymbol{u}\right)=c_2f_2S\hat{\boldsymbol{e}}_r-c_3f_1\nabla\left(\frac{P}{f_1}\right),$$
$$+c_4(\nabla\times\boldsymbol{B})\times\boldsymbol{B}+c_5\nabla\cdot\boldsymbol{D}, \tag{2.3a}$$
$$\text{where}\qquad D_{ij}\coloneqq 2f_1f_3\left[e_{ij}-\frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{2.3b}$$
$$\text{and}\qquad e_{ij}\coloneqq\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{2.3c}$$

$$f_1f_4\left(\frac{DS}{Dt}+c_{11}f_{14}u_r\right)=c_6\nabla\cdot(f_1f_4f_5\nabla S)$$
$$+c_{10}f_6(r)+c_8c_5D_{ij}e_{ij}+c_9c_7f_7|\nabla\times\boldsymbol{B}|^2, \tag{2.4}$$

$$\text{and}\qquad\frac{\partial\boldsymbol{B}}{\partial t}=\nabla\times\left[\boldsymbol{u}\times\boldsymbol{B}-c_7f_7\nabla\times\boldsymbol{B}\right], \tag{2.5}$$

where $D/Dt\coloneqq\partial/\partial t+\boldsymbol{u}\cdot\nabla$ denotes the material derivative.

The shell geometry (i.e., aspect ratio) and reference state fully define the problem. (I have always found that initial conditions seem to ultimately be irrelevant to the final state of the system, although this of course doesn't rule out hysteresis effects from occurring in future cases). The reference state is defined by the spherically-symmetric, time-independent functions $f_i=f_i(r)$ and constants c_j. By adjusting the f_i and c_j, the user can choose between a Boussinesq or anelastic approximation (the specific form of each approximation is described more fully in the Documentation), choose any arbitrary nondimensionalization of these two equation sets, and/or set the nondimensional parameters (Rayleigh number, Prandtl number, etc.) of the problem. `Rayleigh` has built-in modes to set the f_i and c_j for the single-layer (i.e., either convectively stable or unstable, but not both) Boussinesq approximation (nondimensional only) and anelastic approximation (either dimensional or two choices of nondimensional). These modes are chosen via the choices `reference_type = 1,2,3` or `5`.

More complex systems, which `Rayleigh` also supports, require the user to manually change the f_i and c_j. This can be done by editing an input binary file that `Rayleigh` reads upon initialization and setting `reference_type = 4` . This document essentially describes how to produce such a "custom" input file (here called `customfile`) for a coupled stable–unstable anelastic system, using a variety of chosen time-scales in the nondimensionalization. I also unpack what the "energy flux" means when stable layers are present.

Note that the c_j can also be over-written at run-time by an ASCII text-file (i.e., the `main_input` file), allowing easy changes of the nondimensional numbers for a simulation suite that uses a common reference state—all without modifying the `customfile`.

3 Dimensional Anelastic Equations

We begin by writing down the full dimensional anelastic fluid equations, as they are usually implemented in `Rayleigh` (more precisely, this corresponds to `reference_type = 2`). This form of the anelastic approximation in a spherical shell is derived in, or more accurately, attributed to (since `Rayleigh` "updates" the reference state slightly differently than the cluge-y `ASH` implementation), two common sources: Gilman & Glatzmaier (1981) and Clune et al. (1999). `Rayleigh`'s dimensional anelastic equation-set is:

$$\nabla \cdot (\tilde{\rho} \boldsymbol{u}) = 0, \tag{3.1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{3.2}$$

$$\tilde{\rho}\left(\frac{D\boldsymbol{u}}{Dt} + 2\Omega_0 \hat{\boldsymbol{e}}_z \times \boldsymbol{u}\right) = \left(\frac{\tilde{\rho}\tilde{g}}{C_{\rm p}}\right) S' \hat{\boldsymbol{e}}_r - \tilde{\rho}\nabla\left(\frac{P'}{\tilde{\rho}}\right), \quad + \frac{1}{\mu}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} + \nabla\cdot\boldsymbol{D}, \tag{3.3a}$$

$$\text{where} \qquad D_{ij} \coloneqq 2\tilde{\rho}\tilde{\nu}\left[e_{ij} - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{3.3b}$$

$$\text{and} \qquad e_{ij} \coloneqq \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{3.3c}$$

$$\tilde{\rho}\tilde{T}\left(\frac{DS'}{Dt} + \frac{d\tilde{S}}{dr}u_r\right) = \nabla\cdot(\tilde{\rho}\tilde{T}\tilde{\kappa}\nabla S') + \tilde{Q}_{\texttt{Ra}} + D_{ij}e_{ij} + \frac{\tilde{\eta}}{\mu}|\nabla\times\boldsymbol{B}|^2, \tag{3.4}$$

$$\text{and} \qquad \frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B} - \tilde{\eta}\nabla\times\boldsymbol{B}). \tag{3.5}$$

Here, the thermal variables ρ, T, P, and S refer to the density, temperature, pressure, and entropy (respectively). The tildes denote the reference state and the primes denote the (assumed small) perturbations from the reference state. More specifically, the essence of the anelastic approximation involves assuming all relative thermal perturbations ($\rho'/\rho_{\rm a}$, $T'/T_{\rm a}$, $P'/P_{\rm a}$, and $S'/C_{\rm p}$) are $O(\epsilon)$, where $\epsilon \ll 1$ (e.g., Ogura & Phillips 1962; Gough 1969 and Matilsky & Brummell 2024, coming soon!). Here, the "a" subscripts denote "typical reference-state values" and ϵ is also the same (small) size as the typical squared Mach number of the flow.

Other background quantities that appear are the full gravitational acceleration $\tilde{g}$, the momentum, thermal, and magnetic diffusivities ($\tilde{\nu}$, $\tilde{\kappa}$, and $\tilde{\eta}$, respectively; these all have c.g.s. units of $\text{cm}^2\ \text{s}^{-1}$), `Rayleigh`'s internal heating $\tilde{Q}_{\texttt{Ra}}$ (much more on this below), the frame rotation rate Ω_0, the specific heat at constant pressure $C_{\rm p}$, and the vacuum permeability μ ($= 4\pi$ in c.g.s. units). The equations are written in a frame rotating with angular velocity $\Omega_0\hat{\boldsymbol{e}}_z$ and the centrifugal force is neglected. Note that $C_{\rm p}$ is always assumed to be constant, while in a real stellar (or gas-giant) structure model, it should technically vary in radius as the "perfectness" of the gas varies.

4 My Reference State: a Flimsy Approach to Stellar Structure

In the anelastic equations, the one-dimensional reference state should satisfy equilibrium conditions in the momentum and energy equations. These one-dimensional equations are then subtracted from the fully compressible equations and linearized in the primed thermodynamic quantities to produce the anelastic approximation. Ideally, the reference state would be chosen based on a state-of-the-art structure model, incorporating all known physics as best we can. In the Sun, for example, this structure model is fairly well agreed-upon (e.g., Christensen-Dalsgaard et al. 1996); but in the Jovian case, much less so (e.g., Guillot 2005). In our (or possibly just Loren's) philosophy, the form of the reference state itself matters far less than the fact that we cannot approach realistic astrophysical conditions for stars or gas giants. Therefore, we choose the reference state somewhat arbitrarily. Put another way, we choose the f_i to be slightly wrong and are consoled by the fact that the c_j are *incredibly* wrong.

When the reference state is fixed in time as it is in `Rayleigh`, the anelastic equations have no idea whether the one-dimensional equations subtracted out make any sense. It is up to us to be wise and enforce things like hydrostatic balance and thermal equilibrium. `Rayleigh` will happily chug along under any arbitrary reference state with no (obvious) poor consequences. It also helps to recall exactly what structure model we are "really" subtracting off, since there can be a lot of confusion, especially when interpreting "flux through the system." Clarification of this energy flux is the main purpose of this and the following sections.

To specify our reference state, we demand that the gas be in hydrostatic balance,

$$\frac{d\tilde{P}}{dr} = -\tilde{\rho}\tilde{g}, \tag{4.1}$$

be perfect,

$$\tilde{P} = \tilde{\rho}\mathcal{R}\tilde{T}, \tag{4.2}$$

and be in local thermodynamic equilibrium (LTE),

$$\frac{1}{C_{\mathrm{p}}}\frac{d\tilde{S}}{dr} = \frac{1}{\gamma}\frac{d\ln\tilde{T}}{dr} - \frac{\gamma-1}{\gamma}\frac{d\ln\tilde{\rho}}{dr} \tag{4.3a}$$

$$= \frac{1}{\gamma}\frac{d\ln\tilde{P}}{dr} - \frac{d\ln\tilde{\rho}}{dr} \tag{4.3b}$$

$$= \frac{d\ln\tilde{T}}{dr} - \frac{\gamma-1}{\gamma}\frac{d\ln\tilde{P}}{dr} \tag{4.3c}$$

Here, $\mathcal{R} = C_{\mathrm{p}} - C_{\mathrm{v}}$ is the gas constant, C_{v} is the specific heat at constant volume (or equivalently, constant density), and $\gamma = C_{\mathrm{p}}/C_{\mathrm{v}}$ is the ratio of specific heats. All are assumed to be constant (that is the definition of a perfect, not just ideal, gas).

Really, Equation (4.3) is simply an expression of the first law of thermodynamics. It doesn't further constrain the structure problem, but rather introduces a new unknown, i.e., the entropy $\tilde{S}$. The approach I employ is simply to choose $d\tilde{S}/dr$ to enforce convective stability or not and also pick $\tilde{g} \propto 1/r^2$. Then the structure problem is easily solved (see Section 10).

5 The Real Structure Problem

In an ideal world, we would not choose an arbitrary $d\tilde{S}/dr$, but instead demand a condition for thermal (in stars, "radiative") equilibrium. This would generate another equation constraining $\tilde{\rho}$ and $\tilde{T}$ in addition to Equations (4.1)–(4.3), and thus fully define the structure problem. Let's consider radiative equilibrium first, since this applies in many stellar interiors and (maybe) in the weather layer of Jupiter. The radiative diffusion approximation in stars makes the structure problem much easier than for the (relatively cold) gas giants, since we only need consider the opacity of a fully ionized gas (and even then, it's complicated). Under the radiative diffusion approximation (and making a mixing-length assumption), the equation of radiative equilibrium is

$$\tilde{Q} - \nabla \cdot \tilde{\boldsymbol{F}}_{\text{rad}} = \nabla \cdot \tilde{\boldsymbol{F}}_{\text{conv}}, \tag{5.1a}$$

where $$\tilde{\boldsymbol{F}}_{\text{rad}} \coloneqq -\tilde{\rho} C_{\text{p}} \tilde{\kappa}_{\text{rad}} \nabla \tilde{T} = -\tilde{\rho} C_{\text{p}} \tilde{\kappa}_{\text{rad}} \frac{d\tilde{T}}{dr} \hat{\boldsymbol{e}}_r, \tag{5.1b}$$

$$\nabla \cdot \tilde{\boldsymbol{F}}_{\text{conv}} \coloneqq \begin{cases} 0 & \text{for} \quad \nabla_{\text{rad}} < \nabla_{\text{ad}} \text{ (stable to convection)} \\ [\tilde{Q} - \nabla \cdot \tilde{\boldsymbol{F}}_{\text{rad}}]_{(\tilde{\rho}, \tilde{T}) = (\tilde{\rho}_{\text{ad}}, \tilde{T}_{\text{ad}})} & \text{for} \quad \nabla_{\text{rad}} > \nabla_{\text{ad}} \text{ (unstable to convection)} \end{cases}, \tag{5.1c}$$

$$\nabla_{\text{rad}} \coloneqq \frac{d \ln \tilde{T}_{\text{rad}}/dr}{d \ln \tilde{P}/dr} \coloneqq \frac{\gamma - 1}{\gamma} \frac{\tilde{L}(r)}{4\pi r^2 \tilde{g} \tilde{\rho} \tilde{\kappa}_{\text{rad}}}, \tag{5.1d}$$

and $$\nabla_{\text{ad}} \coloneqq \left(\frac{d \ln \tilde{T}}{d \ln \tilde{P}} \right)_{\text{ad}} = \frac{\gamma - 1}{\gamma}. \tag{5.1e}$$

Here, $\tilde{Q} = \tilde{Q}(\tilde{\rho}, \tilde{T})$ is the "real" reference-state internal heating per unit volume (likely due to nuclear burning in the core, or in Jupiter, Helmholtz contraction and solar irradiance), $\tilde{L}(r) \equiv 4\pi \int_0^r \tilde{Q}(x) x^2 dx$ is the total luminosity (power output) of the object interior to r, and $\tilde{\kappa}_{\text{rad}} = \tilde{\kappa}_{\text{rad}}(\tilde{\rho}, \tilde{T})$ is the reference-state radiative diffusivity (units cm^2 s^{-1}), which in turn depends on the opacity. The notation $(\tilde{\rho}_{\text{ad}}, \tilde{T}_{\text{ad}})$ means "the continuation of $\tilde{\rho}$ and $\tilde{T}$ assuming adiabatic stratification."

In practice, Equations (4.1)–(5.1) would be solved by integrating inward from an appropriate boundary condition at the photosphere. At each point, the relative values of ∇_{rad} and ∇_{ad} would be compared to determine which of the two forms of Equation (5.1c) to use. Doing so implicitly adheres to the mixing-length assumption, because it is implied that "convection is locally determined." Wherever $\nabla_{\text{rad}} < \nabla_{\text{ad}}$ there is no convection at all ($\tilde{\boldsymbol{F}}_{\text{conv}} = 0$) and wherever $\nabla_{\text{rad}} > \nabla_{\text{ad}}$ the convection is so vigorous that the layer is completely adiabatically stratified. No overshoot is allowed!

6 Energy Fluxes: What Belongs to the Reference State?

The reference-state $\tilde{Q}$ and the $\tilde{Q}_{\texttt{Ra}}$ used by `Rayleigh` have physically quite different origins. Really, $\tilde{Q}_{\texttt{Ra}}$ is shorthand for $\nabla \cdot \tilde{\boldsymbol{F}}_{\text{conv}}$. We describe explicitly how this works here.

Subtracting Equation (5.1) from the fully compressible heat equation (see Matilsky & Brummell 2024) yields

$$\tilde{\rho}\tilde{T}\left(\frac{DS'}{Dt} + \frac{d\tilde{S}}{dr}u_r\right) = Q' - \nabla \cdot \boldsymbol{F}'_{\text{rad}} + \nabla \cdot \tilde{\boldsymbol{F}}_{\text{conv}} + D_{ij}e_{ij} + \frac{\tilde{\eta}}{\mu}|\nabla \times \boldsymbol{B}|^2, \tag{6.1a}$$

$$\text{where} \qquad \boldsymbol{F}'_{\text{rad}} \coloneqq -\,\tilde{\rho}C_{\text{p}}\tilde{\kappa}_{\text{rad}}\nabla T' - \left(\frac{\rho'}{\tilde{\rho}} + \frac{\kappa'_{\text{rad}}}{\tilde{\kappa}_{\text{rad}}}\right)\tilde{\rho}C_{\text{p}}\tilde{\kappa}_{\text{rad}}\nabla\tilde{T} \tag{6.1b}$$

in place of Equation (3.4).

In `Rayleigh` therefore, we have implicitly set $Q' \equiv 0$ (i.e., we ignore the "real" internal heating processes; so we also set $\tilde{Q} \equiv 0$) and we choose

$$\tilde{Q}_{\texttt{Ra}} \coloneqq \nabla \cdot \tilde{\boldsymbol{F}}_{\text{conv}} = -\nabla \cdot \tilde{\boldsymbol{F}}_{\text{rad}} \qquad (\text{since } \tilde{Q} \equiv 0). \tag{6.2}$$

However, we don't pick $\tilde{\boldsymbol{F}}_{\text{rad}}$ from a structure model, rather we set it proportional to $\tilde{P}$. In the Sun, this isn't too bad an assumption actually (see Featherstone & Hindman 2016 and the appendix of Matilsky et al. 2024).

We further implicitly assume

$$\boldsymbol{F}'_{\text{rad}} = -\tilde{\rho}\tilde{T}\tilde{\kappa}\nabla S'. \tag{6.3}$$

This assumption appears to originate in Gilman & Glatzmaier (1981), which assumes a turbulent "eddy" convective heat flux, due to unresolved motions, of the form $-\rho T\kappa\nabla S$ in the fully compressible heat equation. Making the choice to diffuse entropy, we must therefore regard κ (and thus $\tilde{\kappa}$) as an "eddy" diffusivity. Strictly, we really shouldn't keep having $\boldsymbol{F}'_{\text{rad}} \propto \nabla S'$ in radiative zones, where there are no "eddies" (although maybe there are two-dimensional eddies and also secondary vertical shear instabilities that can mix things—see Cope et al. 2020; Garaud 2020). Hopefully the baroclinic consequences of keeping $\boldsymbol{F}'_{\text{rad}} \propto \nabla S'$ in radiative zones isn't so bad!

Why all this rigamarole? It clarifies several things. First, it really only makes sense to add "code internal heating" (like $\tilde{Q}_{\texttt{Ra}}$) to the *convective* portions of the simulation. Really, $\tilde{Q}_{\texttt{Ra}} = \nabla \cdot \tilde{\boldsymbol{F}}_{\text{conv}}$ and it only happens to look like $-\nabla \cdot \tilde{\boldsymbol{F}}_{\text{rad}}$ in our case because we exclude the "real" internal heating $\tilde{Q}$. Furthermore, nonzero conductive flux boundary conditions (i.e., $\partial S'/\partial r \neq 0$), which act like sources and sinks of heat in the same way as $\tilde{Q}_{\texttt{Ra}}$, should not be added to the boundaries of radiative zones.

Second, the Jupiter irradiance, which heats a stable layer, really does *not* constitute "heating from above" in the same way that the solar radiation, which heats a convection zone, constitutes "heating from below." This is because a weather layer fully in radiative equilbrium would have $\tilde{Q}_{\texttt{Ra}} = \nabla \cdot \tilde{\boldsymbol{F}}_{\text{rad}} \equiv 0$, and there is no unbalanced heating. The only place "heating from above" comes in is in the boundary condition for $\tilde{\boldsymbol{F}}_{\text{rad}}$, which is affected by the irradiance. This alters the reference state, but it doesn't heat or cool the fluid.

The situation could (philosophically) be different in the sense that absorption of all wavelengths of solar radiation in the upper layers could *act* like a heating from above (i.e, take the form of a volumetric heating $\tilde{Q}$, that occupied a particular outer layer independent of the temperature gradients there), while re-emission would occur (maybe in the infra-red) by altering the temperature gradient to enforce $\nabla \cdot \tilde{\boldsymbol{F}}_{\rm rad} = \tilde{Q}$. But regardless, this balance is subtracted off the heat equation and again leaves no unbalanced "code internal heating" (i.e., $\tilde{Q}_{\tt Ra} = \nabla \cdot \tilde{\boldsymbol{F}}_{\rm conv}$ still $= 0$).

Third and finally, when we think of anelastic energy fluxes, these fluxes are only responsible for carrying out $L_{\rm KH}$, i.e., the Kelvin-Helmholtz contraction luminosity that is internal to Jupiter. There is no flux necessary to account for $L_{\rm irrad}$ (the luminosity associated with energy absorbed through irradiance), because this flux has been subtracted out and should be associated with the reference state. We *do* need to set an outward conductive flux at the top of the weather layer (i.e., $\partial S'/\partial r < 0$) in order to maintain thermal equilibrium in the convecting state. This is unfortunate given the convective origin of entropy diffusion discussed above. Its baroclinic consequences (which now may be relevant, given our goal to investigate "shallow driving") have not been explored to my knowledge. Note that the top conductive flux will make the weather layer less stable than it otherwise would be. But as long as the weather layer is sufficiently stiff (i.e., has high buoyancy frequency), it should be a minimal effect.

This is all to say, I am now fully converted to the idea of setting stability via $d\tilde{S}/dr$, instead of using heating layers in different places. I also view Heimpel et al. (2022), who implements a stable layer above the convective layer, as effectively using heating layers also, since they set the conductive flux to be "into the layer" at both boundaries. This approach has the added disadvantage that the location between convectively stable and unstable cannot be set a priori.

7 Non-Dimensional Scheme

In this section, H denotes the typical length-scale (chosen to be the CZ thickness) and τ the typical time-scale (left general for now).

We recall the relation,

$$\widetilde{N^2} = \frac{\tilde{g}}{C_{\rm p}} \frac{d\tilde{S}}{dr}, \tag{7.1}$$

where $\widetilde{N^2}$ is the squared buoyancy frequency, which we will use in favor of $d\tilde{S}/dr$ in subsequent equations.

We define the "energy flux not carried by radiation" via

$$\tilde{F}_{\rm nrad} \coloneqq \frac{1}{r^2} \int_{r_{\rm bcz}}^{r} \tilde{Q}_{\tt Ra}(x) x^2 dx, \tag{7.2}$$

where $r_{\rm bcz}$ is the radius of the base of the CZ. We also denote the top of the CZ by $r_{\rm tcz}$, the base of the RZ by $r_{\rm brz}$, and the top of the RZ by $r_{\rm trz}$. Note that $\tilde{F}_{\rm nrad}$ is basically similar to $|\tilde{\boldsymbol{F}}_{\rm conv}|$, however in the context of stellar structure theory, $\tilde{\boldsymbol{F}}_{\rm conv}$ is computed according to a definite logic, whereas we choose our $\tilde{Q}_{\tt Ra}$ arbitrarily.

ΔS denotes the (*estimated*) entropy difference across the CZ, and thus a typical value of S' in the CZ. In Rayleigh-Bénard-type convection, the true entropy difference is imposed directly. We could do this as well, by fixing S' itself at the inner and outer boundaries. However, instead we typically set the entropy difference indirectly by adding an internal heating $\tilde{Q}_{\texttt{Ra}}$ and balancing it with the conductive energy flux at the boundaries (which we set by fixing $\partial S'/\partial r$ at the boundaries in place of S' itself). We estimate the resultant entropy difference via

$$\Delta S := \frac{F_{\rm nrad,a}H}{\rho_{\rm a}T_{\rm a}\kappa_{\rm a}}, \tag{7.3}$$

The "a" subscripts again refer to "typical values" of the (dimensional) atmospheric reference-state functions. Equation (7.3) essentially assumes that there is a conductive boundary layer (at the top of the CZ, since $\tilde{Q}_{\texttt{Ra}}$ must be concentrated more toward the base of the CZ in order to drive convection) that carries the full internal luminosity $L := 4\pi\int_{r_{\rm bcz}}^{r_{\rm tcz}}\tilde{Q}_{\texttt{Ra}}(x)x^2dx$ out of the CZ. Note that this is a crude estimate, since the ultimate boundary layer thickness is unknown (and will be substantially less than H). Furthermore, we will choose CZ volume-averages for the "typical values" of $\tilde{\rho}$, $\tilde{F}_{\rm nrad}$, etc., which will be higher than the boundary-layer appropriate values. We can easily see how bad our assumption is, by calculating the entropy difference across the CZ post-equilibration and comparing to 1.

We now nondimensionalize Equations (3.1)–(3.5), according to the following scheme:

$$\nabla \rightarrow \frac{1}{H}\nabla, \tag{7.4a}$$

$$t \rightarrow \tau t, \tag{7.4b}$$

$$\boldsymbol{u} \rightarrow \frac{H}{\tau}\boldsymbol{u}, \tag{7.4c}$$

$$S' \rightarrow (\Delta S)S', \tag{7.4d}$$

$$P' \rightarrow \tilde{\rho}\frac{H^2}{\tau^2}P', \tag{7.4e}$$

$$\boldsymbol{B} \rightarrow (\mu\tilde{\rho})^{1/2}\frac{H}{\tau}\boldsymbol{B}, \tag{7.4f}$$

$$\tilde{\rho} \rightarrow \rho_{\rm a}\tilde{\rho}, \tag{7.4g}$$

$$\tilde{T} \rightarrow T_{\rm a}\tilde{T}, \tag{7.4h}$$

$$\tilde{g} \rightarrow g_{\rm a}\tilde{g}, \tag{7.4i}$$

$$\widetilde{N^2} \rightarrow N_{\rm a}^2\widetilde{N^2}, \tag{7.4j}$$

$$\tilde{S} \rightarrow C_{\rm p}\tilde{S}, \tag{7.4k}$$

$$\tilde{\nu} \rightarrow \nu_{\rm a}\tilde{\nu}, \tag{7.4l}$$

$$\tilde{\kappa} \rightarrow \kappa_{\rm a}\tilde{\kappa}, \tag{7.4m}$$

$$\tilde{\eta} \rightarrow \eta_{\rm a}\tilde{\eta}, \tag{7.4n}$$

$$\text{and}\qquad \tilde{Q}_{\texttt{Ra}} \rightarrow \frac{F_{\rm nrad,a}}{H}\tilde{Q}_{\texttt{Ra}} \tag{7.4o}$$

On the right-hand-sides of Equation (7.4) and in the following equations, all fluid variables, coordinates, and background-state quantities are understood to be nondimensional. The "'typical values" denoted by the "a" subscripts are chosen to be volume-averages over the CZ, except for N_{a}^2, which is chosen to be the volume-average over the stably stratified RZ (really "weather layer" in the case of Jupiter).

We define the relative thicknesses of the RZ and CZ via the "first aspect ratio"

$$\alpha \coloneqq \frac{r_{\mathrm{trz}} - r_{\mathrm{brz}}}{r_{\mathrm{tcz}} - r_{\mathrm{bcz}}} \tag{7.5}$$

and the aspect ratio of the CZ (the "second aspect ratio") via

$$\beta = \frac{r_{\mathrm{bcz}}}{r_{\mathrm{tcz}}}. \tag{7.6}$$

Since H is the thickness of the CZ, we have

$$r_{\mathrm{bcz}} = \frac{\beta}{1-\beta}, \tag{7.7a}$$

$$r_{\mathrm{tcz}} = \frac{1}{1-\beta}, \tag{7.7b}$$

and

$$r_{\mathrm{brz}} = r_{\mathrm{trz}} - \alpha, \tag{7.8a}$$

$$\text{with} \quad r_{\mathrm{trz}} = r_{\mathrm{bcz}} \qquad \text{for the solar tachocline} \tag{7.8b}$$

and

$$r_{\mathrm{brz}} = r_{\mathrm{tcz}}, \tag{7.9a}$$

$$\text{and} \quad r_{\mathrm{trz}} = r_{\mathrm{brz}} + \alpha \qquad \text{for Jupiter.} \tag{7.9b}$$

The only difference between Jupiter and the tachocline is which of the CZ or RZ is on top.

We leave the time-scale general for now, but ultimately assume it is a rotational time-scale when running the simulations [i.e., $\tau = (2\Omega_0)^{-1}$]. Leaving τ general allows easy translation between different nondimensionalizations, if they are needed later.

8 Non-Dimensional Equations (General τ)

There are several time-scales relevant to the problem. Putting the nondimensional parameters in terms of these more clearly elucidates their meaning. Practically, it also makes translating between different nondimensionalizations easier. The relevant (dimensional) timescales are:

$$\tau_\nu := \frac{H^2}{\nu_{\rm a}} \qquad \text{viscous diffusion time (across the CZ),} \tag{8.1a}$$

$$\tau_\kappa := \frac{H^2}{\kappa_{\rm a}} \qquad \text{thermal diffusion time (across the CZ)} \tag{8.1b}$$

$$\tau_\eta := \frac{H^2}{\eta_{\rm a}} \qquad \text{magnetic diffusion time (across the CZ)} \tag{8.1c}$$

$$\tau_{\rm ff} := \sqrt{\frac{C_{\rm p} H}{\Delta S g_{\rm a}}} \qquad \text{“effective” free-fall time,} \tag{8.1d}$$

$$\tau_\Omega := \frac{1}{2\Omega_0} \qquad \text{rotational (Coriolis) time-scale,} \tag{8.1e}$$

$$\tau_N := \frac{1}{N_{\rm a}} \qquad \text{buoyancy (gravity-wave) time-scale,} \tag{8.1f}$$

$$\tau_{\rm ES} := \frac{N_{\rm a}^2}{4\Omega_0^2}\frac{H^2}{\kappa_{\rm a}} = \left(\frac{\tau_\Omega^2}{\tau_N^2}\right)\tau_\kappa \qquad \text{Eddington-sweet time} \tag{8.1g}$$

The relevant nondimensional parameters are:

$$\mathrm{Ra} := \frac{\tau_\nu \tau_\kappa}{\tau_{\rm ff}^2} = \frac{g_{\rm a} H^3}{\nu_{\rm a}\kappa_{\rm a}}\frac{\Delta S}{C_{\rm p}} \qquad \text{(Rayleigh number),} \tag{8.2a}$$

$$\mathrm{Pr} := \frac{\tau_\kappa}{\tau_\nu} = \frac{\nu_{\rm a}}{\kappa_{\rm a}} \qquad \text{(Prandtl number),} \tag{8.2b}$$

$$\mathrm{Pr_m} := \frac{\tau_\eta}{\tau_\nu} = \frac{\nu_{\rm a}}{\tilde{\eta}} \qquad \text{(magnetic Prandtl number),} \tag{8.2c}$$

$$\mathrm{Ek} := \frac{\tau_\Omega}{\tau_\nu} = \frac{\nu_{\rm a}}{2\Omega_0 H^2} \qquad \text{(Ekman number),} \tag{8.2d}$$

$$\mathrm{Bu} := \frac{\tau_\nu \tau_\kappa}{\tau_N^2} = \frac{N_{\rm a}^2 H^4}{\nu_{\rm a}\kappa_{\rm a}} \qquad \text{(buoyancy number),} \tag{8.2e}$$

$$\text{and} \qquad \mathrm{Di} = \frac{g_{\rm a} H}{C_{\rm p}\tilde{T}} \qquad \text{(dissipation number),} \tag{8.2f}$$

Note that in our convention, the dissipation number is not an independent control parameter, but a function of the non-dimensional parameters characterizing the reference state (this will be seen in Section 10). Assuming rotation is present, we also define some derivative nondimensional parameters that can be formed from the parameters in Equation (8.2):

$$\mathrm{Ra}^* := \frac{\tau_\Omega^2}{\tau_{\rm ff}^2} = \frac{\mathrm{Ra}\mathrm{Ek}^2}{\mathrm{Pr}} = \frac{g_{\rm a}}{4\Omega_0^2 H}\frac{\Delta S}{C_{\rm p}} \qquad \text{(modified Rayleigh number),} \tag{8.3a}$$

$$\mathrm{Ro_c} := \frac{\tau_\Omega}{\tau_{\rm ff}} = \mathrm{Ek}\sqrt{\frac{\mathrm{Ra}}{\mathrm{Pr}}} \qquad \text{(convective Rossby number),} \tag{8.3b}$$

$$\mathrm{Bu}^* := \frac{\tau_\Omega^2}{\tau_N^2} = \frac{\tau_{\rm ES}}{\tau_\kappa} = \frac{\mathrm{Bu}\mathrm{Ek}^2}{\mathrm{Pr}} = \frac{N_{\rm a}^2}{4\Omega_0^2} \qquad \text{(modified buoyancy number),} \tag{8.3c}$$

$$\sigma := \sqrt{\frac{\tau_{\rm ES}}{\tau_\nu}} = \sqrt{\mathrm{Bu}^*\mathrm{Pr}} = \frac{N_{\rm a}}{2\Omega_0}\sqrt{\mathrm{Pr}} \qquad \text{(“}\sigma\text{ parameter”)} \tag{8.3d}$$

After applying the nondimensional scheme described by Equation (7.4), Equations (3.1)–(3.5) become

$$\nabla\cdot(\tilde{\rho}\boldsymbol{u}) = 0, \tag{8.4}$$
$$\nabla\cdot\boldsymbol{B} = 0, \tag{8.5}$$

$$\begin{aligned}\tilde{\rho}\left(\frac{D\boldsymbol{u}}{Dt} + \frac{\tau}{\tau_\Omega}\hat{\boldsymbol{e}}_z\times\boldsymbol{u}\right) = \frac{\mathrm{Ra}}{\mathrm{Pr}}\left(\frac{\tau}{\tau_\nu}\right)^2\tilde{\rho}\tilde{g}S'\hat{\boldsymbol{e}}_r - \tilde{\rho}\nabla\left[\frac{P}{\tilde{\rho}}\right],\\ +(\nabla\times\boldsymbol{B})\times\boldsymbol{B} + \left(\frac{\tau}{\tau_\nu}\right)\nabla\cdot\boldsymbol{D},\end{aligned} \tag{8.6a}$$

$$\text{where}\qquad D_{ij} := 2\tilde{\rho}\tilde{\nu}\left[e_{ij} - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{8.6b}$$

$$\text{and}\qquad e_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{8.6c}$$

$$\begin{aligned}\tilde{\rho}\tilde{T}\left(\frac{DS'}{Dt} + \frac{\mathrm{Bu}}{\mathrm{Ra}}\frac{\widetilde{N^2}}{\tilde{g}}u_r\right) = \frac{1}{\mathrm{Pr}}\left(\frac{\tau}{\tau_\nu}\right)\left[\nabla\cdot(\tilde{\rho}\tilde{T}\tilde{\kappa}\nabla S') + \tilde{Q}_{\texttt{Ra}}\right]\\ + \frac{\mathrm{PrDi}}{\mathrm{Ra}}\left(\frac{\tau_\nu}{\tau}\right)\left(D_{ij}e_{ij} + \frac{1}{\mathrm{Pr_m}}\tilde{\eta}|\nabla\times\boldsymbol{B}|^2\right),\end{aligned} \tag{8.7}$$

$$\text{and}\qquad \frac{\partial\boldsymbol{B}}{\partial t} = \nabla\times\left[\boldsymbol{u}\times\boldsymbol{B} - \frac{1}{\mathrm{Pr_m}}\left(\frac{\tau}{\tau_\nu}\right)\tilde{\eta}\nabla\times\boldsymbol{B}\right]. \tag{8.8}$$

It is now clear how to reach any desired nondimensionalization. Use Equations (8.2) and (8.3) to compute the ratio τ/τ_ν in terms of the nondimensional parameters. The easiest choice (given our derivation) is $\tau = \tau_\nu$.

9 Nondimensional equations, functions, and constants for $\tau = \tau_\Omega$

We now reach the end of the main part of the document, where we write down the nondimensional equations to be used, and how to input these equations into `Rayleigh` by setting the f_i and c_j.

For $\tau = \tau_\Omega$, we have $\tau/\tau_\nu = \mathrm{Ek}$. Equations (8.4)–(8.8), after simplification, become

$$\nabla\cdot(\tilde{\rho}\boldsymbol{u}) = 0, \tag{9.1}$$
$$\nabla\cdot\boldsymbol{B} = 0, \tag{9.2}$$

$$\tilde{\rho}\left(\frac{D\boldsymbol{u}}{Dt}+\hat{\boldsymbol{e}}_z\times\boldsymbol{u}\right)=\mathrm{Ro}_\mathrm{c}^2\tilde{\rho}\tilde{g}S\hat{\boldsymbol{e}}_r-\tilde{\rho}\nabla\left[\frac{P}{\tilde{\rho}}\right],$$

$$+(\nabla\times\boldsymbol{B})\times\boldsymbol{B}+\mathrm{Ek}\nabla\cdot\boldsymbol{D}, \tag{9.3a}$$

$$\text{where}\qquad D_{ij}:=2\tilde{\rho}\tilde{\nu}\left[e_{ij}-\frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{9.3b}$$

$$\text{and}\qquad e_{ij}:=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{9.3c}$$

$$\tilde{\rho}\tilde{T}\left(\frac{DS'}{Dt}+\frac{\sigma^2}{\mathrm{Ro}_\mathrm{c}^2\mathrm{Pr}}\frac{\widetilde{N^2}}{\tilde{g}}u_r\right)=\frac{\mathrm{Ek}}{\mathrm{Pr}}[\nabla\cdot(\tilde{\rho}\tilde{T}\tilde{\kappa}\nabla\tilde{S})+\tilde{Q}_{\texttt{Ra}}]$$

$$+\sqrt{\frac{\mathrm{Pr}}{\mathrm{Ra}}}\left(\frac{\mathrm{Di}}{\mathrm{Ro}_\mathrm{c}}\right)\left(D_{ij}e_{ij}+\frac{1}{\mathrm{Pr}_\mathrm{m}}\tilde{\eta}|\nabla\times\boldsymbol{B}|^2\right), \tag{9.4}$$

$$\text{and}\qquad \frac{\partial\boldsymbol{B}}{\partial t}=\nabla\times\left(\boldsymbol{u}\times\boldsymbol{B}-\frac{\mathrm{Ek}}{\mathrm{Pr}_\mathrm{m}}\tilde{\eta}\nabla\times\boldsymbol{B}\right), \tag{9.5}$$

In terms of `Rayleigh`'s f's and c's, we compare Equations (9.1)–(9.5) to (2.1)–(2.5) and find:

$$\begin{aligned}
f_1&\to\tilde{\rho} & c_1&\to 1\\
f_2&\to\tilde{\rho}\tilde{g} & c_2&\to \mathrm{Ro}_\mathrm{c}^2\\
f_3&\to\tilde{\nu} & c_3&\to 1\\
f_4&\to\tilde{T} & c_4&\to 1\\
f_5&\to\tilde{\kappa} & c_5&\to \mathrm{Ro}_\mathrm{c}\sqrt{\mathrm{Pr}/\mathrm{Ra}}\\
f_6&\to\tilde{Q}_{\texttt{Ra}} & c_6&\to \mathrm{Ro}_\mathrm{c}/\sqrt{\mathrm{PrRa}}\\
f_7&\to\tilde{\eta} & c_7&\to \mathrm{Ro}_\mathrm{c}\sqrt{\mathrm{Pr}/({\mathrm{Pr}_\mathrm{m}}^2\mathrm{Ra})}\\
&\ \vdots & c_8&\to \mathrm{Di}/{\mathrm{Ro}_\mathrm{c}}^2\\
& & c_9&\to \mathrm{Di}/{\mathrm{Ro}_\mathrm{c}}^2\\
f_{14}&\to\frac{\widetilde{N^2}}{\tilde{g}} & c_{10}&\to \mathrm{Ro}_\mathrm{c}/\sqrt{\mathrm{PrRa}}\\
& & c_{11}&\to \sigma^2/(\mathrm{PrRo}_\mathrm{c}^2)
\end{aligned}$$

To set our simulation, we choose the following five independent nondimensional parameters: Ra, Pr, Pr_m, Ro_c, and σ.

10 The Reference State (for Real, This Time)

Here we solve the "flimsy" stellar structure problem laid out in Section 4. We consider a spherical shell of gas (adjacent CZ and RZ) extending between inner radius r_in and outer radius $r_{\rm out}$. For the Sun, $(r_{\rm in}, r_{\rm out}) = (r_{\rm brz}, r_{\rm tcz})$, while for Jupiter, $(r_{\rm in}, r_{\rm out}) = (r_{\rm bcz}, r_{\rm trz})$. We model the stability transition using a quartic matching function in the chosen $d\tilde{S}/dr$. More specifically, the assumed transition in convective stability occurs near an intermediate radius r_0, over a width δ. We thus define the function:

$$\psi(r; r_0, \delta) \coloneqq \begin{cases} 0 & r \leq r_0 \\ 1 - \left[1 - \left(\frac{r-r_0}{\delta}\right)^2\right]^2 & r_0 < r < r_0 + \delta \\ 1 & r \geq r_0 + \delta. \end{cases} \tag{10.1}$$

We then choose

$$\frac{d\tilde{S}}{dr} \coloneqq \begin{cases} \Sigma\psi(r, r_0, \delta) \quad \text{with} \quad r_0 = r_{\rm tcz} & \text{for Jupiter,} \\ \Sigma[1 - \psi(r, r_0 - \delta, \delta)] \quad \text{with} \quad r_0 = r_{\rm bcz} & \text{for the tachocline,} \end{cases} \tag{10.2}$$

where Σ is the nondimensional amplitude of $d\tilde{S}/dr$ in the RZ. Contrary to what one might expect, Σ is actually independent of σ. The uppercase Σ affects the radial structure of the reference state, while σ affects the dynamics of the fluid. Each of these is a (different) aspect of the "stiffness" of the RZ. More on this (since Nic will probably need convincing) below.

With this formulation, the CZ is *strictly* unstable (really, marginally stable, but becomes unstable because of the heating). This ensures that none of the stable gradient "leaks" into the CZ, as happens with (e.g.) tanh matching (see Matilsky et al. 2024).

We also define the nondimensional entropy itself via

$$\tilde{S} = \int_{r_0}^{r} \frac{d\tilde{S}}{dx} dx, \tag{10.3}$$

where (without loss of generality, since only differences in $\tilde{S}$ matter in the end) we have chosen the zero-point of $\tilde{S}$ to be r_0.

We assume a centrally-concentrated mass so that $\tilde{g} \propto 1/r^2$. With the requirement that the volume-average of $\tilde{g}$ over the CZ be unity, we find

$$\tilde{g} = \left[\frac{1-\beta^3}{3(1-\beta)^3}\right] \frac{1}{r^2}. \tag{10.4}$$

Under the scaling of Equation (7.4), Equations (4.1), (4.2), and (4.3) become (respectively)

$$\frac{d\tilde{P}}{dr} = -\mathrm{Di}\left(\frac{\gamma}{\gamma - 1}\right)\tilde{\rho}\tilde{g}, \tag{10.5}$$

$$\tilde{P} = \tilde{\rho}\tilde{T}, \tag{10.6}$$

and

$$\frac{d\tilde{S}}{dr} = \frac{1}{\gamma}\frac{d\ln\tilde{T}}{dr} - \frac{\gamma-1}{\gamma}\frac{d\ln\tilde{\rho}}{dr} \tag{10.7a}$$

$$= \frac{1}{\gamma}\frac{d\ln\tilde{P}}{dr} - \frac{d\ln\tilde{\rho}}{dr} \tag{10.7b}$$

$$= \frac{d\ln\tilde{T}}{dr} - \frac{\gamma-1}{\gamma}\frac{d\ln\tilde{P}}{dr} \tag{10.7c}$$

We can then find an ODE for the temperature alone:

$$\frac{d\tilde{T}}{dr} - \left(\frac{d\tilde{S}}{dr}\right)\tilde{T} = -\mathrm{Di}\tilde{g} \tag{10.8}$$

This is integrated to find (e.g., Matilsky et al. 2024)

$$\tilde{T} = e^{\tilde{S}}\left[\tilde{T}(r_{\mathrm{bcz}}) - \mathrm{Di}\int_{r_{\mathrm{bcz}}}^{r}\tilde{g}(x)e^{-\tilde{S}(x)}dx\right]. \tag{10.9}$$

and

$$\tilde{\rho} = \tilde{\rho}(r_{\mathrm{bcz}})\exp\left[-\left(\frac{\gamma}{\gamma-1}\right)\tilde{S}\right]\tilde{T}^{1/(\gamma-1)}, \tag{10.10}$$

leaving the structure problem nearly solved.

Various derivatives of the thermal variables can also be computed:

$$\frac{d\ln\tilde{T}}{dr} = \frac{d\tilde{S}}{dr} - \mathrm{Di}\frac{\tilde{g}}{\tilde{T}}, \tag{10.11}$$

$$\frac{d\ln\tilde{\rho}}{dr} = -\left(\frac{d\tilde{S}}{dr} + \frac{\mathrm{Di}}{\gamma-1}\frac{\tilde{g}}{\tilde{T}}\right), \tag{10.12}$$

$$\text{and} \qquad \widetilde{H_\rho} = \frac{1}{\frac{d\tilde{S}}{dr} + \frac{\mathrm{Di}}{\gamma-1}\frac{\tilde{g}}{\tilde{T}}}, \tag{10.13}$$

where $H_\rho \coloneqq -(d\ln\tilde{\rho}/dr)^{-1}$ is the local density scale-height. (We can now start to see why the amplitude Σ is unrelated to σ and why it is also important; it partially sets the local density scale-height in the RZ).

To fully specify the reference-state, we need the constants Di, $\tilde{T}(r_{\mathrm{bcz}})$, and $\tilde{\rho}(r_{\mathrm{bcz}})$. We define the number of density scale-heights across the CZ,

$$N_\rho \coloneqq \ln\left[\frac{\tilde{\rho}(r_{\mathrm{bcz}})}{\tilde{\rho}(r_{\mathrm{tcz}})}\right]. \tag{10.14}$$

Using Equation (10.14), the fact that $\tilde{S} \equiv 0$ in the CZ, and the requirement that $\tilde{T}$ integrates to unity over the CZ, Equations (10.10) and (10.9) yield

$$\mathrm{Di} := \frac{g_{\mathrm{a}} H}{C_{\mathrm{p}} T_{\mathrm{a}}} = \frac{3\beta(1-\beta)^2(1-e^{-N_\rho/n})}{(3\beta/2)(1-\beta^2)(1-e^{-N_\rho/n}) - (1-\beta^3)(\beta - e^{-N_\rho/n})} \tag{10.15}$$

$$\text{and} \qquad \tilde{T}(r_{\mathrm{bcz}}) = \frac{(1-\beta^3)(1-\beta)}{(3\beta/2)(1-\beta^2)(1-e^{-N_\rho/n}) - (1-\beta^3)(\beta - e^{-N_\rho/n})}, \tag{10.16}$$

$$\text{where} \qquad n := \frac{1}{1-\gamma} \tag{10.17}$$

is the "polytropic index" of the CZ. Since $d\tilde{S}/dr \equiv 0$ in the CZ, the stratification of the CZ winds up being an adiabatic polytrope; the full system, however, is *not* a polytrope, nor is it equivalent to matching multiple polytropes.

The final constant $\tilde{\rho}(r_{\mathrm{bcz}})$ ends up not having an analytical expression but can be found numerically by integration over the CZ of Equation (10.10).

The diffusivity profiles $\tilde{\nu}$, $\tilde{\kappa}$, and $\tilde{\eta}$ are arbitrary and do not affect the rest of the structure model.

Given the choice for ΔS and the definition of $\tilde{F}_{\mathrm{nrad}}$, $\tilde{Q}_{\mathtt{Ra}}$ must be normalized in a particular way for consistency. Following the solar case, where $|\tilde{\boldsymbol{F}}_{\mathrm{conv}}|$ is roughly proportional to $\tilde{P}$, we choose

$$\tilde{Q}_{\mathtt{Ra}} = \begin{cases} c\tilde{\rho}\tilde{T} & \text{inside the CZ} \\ 0 & \text{outside the CZ} \end{cases} \tag{10.18}$$

Normalization requires

$$\frac{1}{c} := \frac{3(1-\beta)^3}{1-\beta^3} \int_{r_{\mathrm{bcz}}}^{r_{\mathrm{tcz}}} \int_{r_{\mathrm{bcz}}}^{r} \tilde{\rho}(x)\tilde{T}(x) x^2 dx dr. \tag{10.19}$$

10.1 The Relationship between Σ and σ

What is the nondimensional parameter Σ appearing in Equation (10.2)? Recall our choice in Equation (7.4) to scale the dimensional background entropy with C_{p} instead of ΔS. Thus, Σ really measures the magnitude of the dimensional background entropy gradient relative to C_{p}/H. Thus, we write

$$\Sigma \approx \frac{H}{C_{\mathrm{p}}} \left(\frac{d\tilde{S}}{dr} \right)_{\mathrm{a}}, \tag{10.20}$$

with the "≈" here meaning "to within a knowable constant of order unity." Similarly, from Equation (**??**), we write

$$N_{\mathrm{a}}^2 \approx \frac{g_{\mathrm{a}}}{C_{\mathrm{p}}} \left(\frac{d\tilde{S}}{dr} \right)_{\mathrm{a}} \tag{10.21}$$

We thus find

$$\sigma^2 \approx \mathrm{Pr}\frac{g_\mathrm{a}}{4H\Omega_0^2}\Sigma \approx \frac{\mathrm{Pr}}{f}\Sigma, \tag{10.22a}$$

$$\text{where} \qquad f := \frac{R_\mathrm{eq} - R_\mathrm{pol}}{R_\mathrm{eq}} \tag{10.22b}$$

is the geometric oblateness of the object (R_eq and R_pol refer to the equatorial and polar radius of the object, respectively). We can also estimate

$$\frac{1}{f} \approx \frac{C_\mathrm{p}}{\Delta S}\mathrm{Ro}_\mathrm{c}^2 \tag{10.23}$$

and thus

$$\Sigma \approx \frac{\Delta S}{C_\mathrm{p}}\frac{1}{\mathrm{Pr}}\frac{\sigma^2}{\mathrm{Ro}_\mathrm{c}^2} = \frac{\Delta S}{C_\mathrm{p}}\frac{\mathrm{Bu}^*}{\mathrm{Ra}^*} = \frac{\Delta S}{C_\mathrm{p}}\frac{\mathrm{Bu}}{\mathrm{Ra}} = \frac{\Delta S}{C_\mathrm{p}}\frac{\tau_\mathrm{ff}^2}{\tau_N^2}. \tag{10.24}$$

Thus, however we manipulate the expressions, Σ cannot be reduced to a function of σ. That is because the relative entropy fluctuation $\Delta S/C_\mathrm{p}$ is ordinarily not set a priori. Without a stable layer, only the combination $\mathrm{Ro}_\mathrm{c}^2 = [\tilde{g}/(4H\Omega_0^2)](\Delta S/C_\mathrm{p})$ (or equivalent) is relevant, and only this may be set. In dimensional calculations, one can choose ΔS and C_p separately (and thus pick a definite $\Delta S/C_\mathrm{p}$), but this doesn't matter since still only the resultant Ro_c^2 is of significance. With a stable layer, we implicitly *must* choose a value for $\Delta S/C_\mathrm{p}$ (which we do by picking Σ).

If we believe $\Delta S/C_\mathrm{p}$ sets the typical entropy fluctuations in the convection zone (and presumably the overshoot layer), then it must be chosen to be consistent with the anelastic approximation. This places an upper bound on Σ. Physically, this makes sense: if Σ is too high, the entropy gradient will be too high (relative to C_p/H) and overshooting plumes will develop $O(1)$ entropy fluctuations, at odds with the anelastic approximation.

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