

General Anelastic Non-Dimensionalization for Rayleigh

Loren Matilsky

March 2, 2023

1 Dimensional Equations

We begin by writing down the full dimensional anelastic fluid equations, as implemented in Rayleigh.

$$\nabla \cdot [\bar{\rho}(r)\mathbf{u}] = 0, \quad (1.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (1.2)$$

$$\begin{aligned} \bar{\rho}(r) \left[\frac{D\mathbf{u}}{Dt} + 2\Omega_0 \hat{\mathbf{e}}_z \times \mathbf{u} \right] = & -\bar{\rho}(r) \nabla \left[\frac{P}{\bar{\rho}(r)} \right] + \left[\frac{\bar{\rho}(r)g(r)}{c_p} \right] S \hat{\mathbf{e}}_r, \\ & + \nabla \cdot \mathbf{D} + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}, \end{aligned} \quad (1.3a)$$

$$\text{where} \quad D_{ij} := 2\bar{\rho}(r)\nu(r) \left[e_{ij} - \frac{1}{3}(\nabla \cdot \mathbf{u})\delta_{ij} \right] \quad (1.3b)$$

$$\text{and} \quad e_{ij} := \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (1.3c)$$

$$\begin{aligned} \bar{\rho}(r)\bar{T}(r) \frac{DS}{Dt} = & -\bar{\rho}(r)\bar{T}(r) \frac{d\bar{S}}{dr} u_r + \nabla \cdot [\bar{\rho}(r)\bar{T}(r)\kappa(r)\nabla S] \\ & + Q(r) + D_{ij}e_{ij} + \frac{\eta(r)}{4\pi} |\nabla \times \mathbf{B}|^2, \end{aligned} \quad (1.4)$$

$$\text{and} \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times [\eta(r)\nabla \times \mathbf{B}]. \quad (1.5)$$

Here, the thermal variables ρ , T , P , and S refer to the density, temperature, pressure, and entropy (respectively). An overbar on a thermal variable indicates the time-independent, spherically symmetric reference state and the lack of an overbar indicates the (assumed small) deviation from the background (for the entropy, S/c_p is assumed small). The vector velocity is denoted by \mathbf{u} and the vector magnetic field by \mathbf{B} .

We use standard spherical coordinates (r, θ, ϕ) and cylindrical coordinates $(\lambda, \phi, z) = (r \sin \theta, \phi, r \cos \theta)$, and $\hat{\mathbf{e}}_q$ in general denotes a position-dependent unit vector in the direction

of increasing q . The equations are written in a frame rotating with angular velocity Ω_0 and the centrifugal force is neglected.

The additional reference state variables $\nu(r)$, $\kappa(r)$, and $\eta(r)$ (the momentum, thermal, and magnetic diffusivities, respectively), as well as the gravitational acceleration $g(r) = GM_*/r^2$ (where G is the gravitational constant and M_* the stellar mass), are also assumed to be spherically symmetric and fixed in time.

Note that the internal heating function $Q(r)$ is also assumed spherically symmetric and fixed in time, but should be interpreted as $-\nabla \cdot \mathcal{F}_{\text{rad}}$, where \mathcal{F}_{rad} is the radiative heat flux and properly should be proportional to the gradient of the total (background + perturbed) temperature. If the system is a convection zone, it is driven by a combination of internal heating and the thermal boundary conditions (conditions on S), that together ensure a stellar luminosity is transported throughout the layer in a steady state.

Note that Equation (1.3c) is only valid in a Cartesian coordinate system (x_1, x_2, x_3) (with i and j running over 1, 2, 3) and is translated into spherical coordinates before being used in Rayleigh.

Finally, we recall the relation,

$$\frac{d\bar{S}}{dr} = c_p \frac{N^2(r)}{g(r)}, \quad (1.6)$$

where $N^2(r)$ is the squared buoyancy frequency.

2 Non-Dimensional Scheme

We now non-dimensionalize Equations (1.1)–(1.5), according to the following scheme:

$$\nabla \rightarrow \frac{1}{L} \nabla, \quad (2.1a)$$

$$t \rightarrow \tau t, \quad (2.1b)$$

$$\mathbf{u} \rightarrow \frac{L}{\tau} \mathbf{u}, \quad (2.1c)$$

$$S \rightarrow \sigma S, \quad (2.1d)$$

$$P \rightarrow \tilde{\rho} \frac{L^2}{\tau^2} P, \quad (2.1e)$$

$$\mathbf{B} \rightarrow (4\pi\tilde{\rho})^{1/2} \frac{L}{\tau} \mathbf{B}, \quad (2.1f)$$

$$\bar{\rho}(r) = \tilde{\rho} \hat{\rho}(r), \quad (2.1g)$$

$$\bar{T}(r) = \tilde{T} \hat{T}(r), \quad (2.1h)$$

$$g(r) = \tilde{g} \hat{g}(r), \quad (2.1i)$$

$$N^2(r) = \tilde{N}^2 \widehat{N}^2(r), \quad (2.1j)$$

$$\nu(r) = \tilde{\nu} \hat{\nu}(r), \quad (2.1k)$$

$$\kappa(r) = \tilde{\kappa} \hat{\kappa}(r), \quad (2.1l)$$

$$\text{and } \eta(r) = \tilde{\eta} \hat{\eta}(r). \quad (2.1m)$$

Here, L is a typical length-scale, τ is a typical time-scale, and σ is a typical entropy scale. On the right-hand-sides of Equations (2.1a)–(2.1f) (and in the following non-dimensionalizations), ∇ , t , \mathbf{u} , S , P , and \mathbf{B} are all understood to be non-dimensional. In Equations (2.1g)–(2.1m), the tildes refer to “typical values” of the reference state functions and the hats refer to the radially-dependent non-dimensional versions of the reference-state functions.

Below, we will assume the time scale is either a viscous diffusion time (i.e., $\tau = L^2/\tilde{\nu}$) or a rotational time-scale (i.e., $\tau = \Omega_0^{-1}$). To describe the reference state, we will consider three cases for a given function’s “typical value”: Its value at the inner shell boundary, its value at the outer shell boundary, or its value volume-averaged over the shell.

3 Non-Dimensional Equations; $\tau = L^2/\tilde{\nu}$

In this case, Equations (1.1)–(1.5) become

$$\nabla \cdot [\hat{\rho}(r)\mathbf{u}] = 0, \quad (3.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (3.2)$$

$$\begin{aligned} \hat{\rho}(r) \left[\frac{D\mathbf{u}}{Dt} + \frac{2}{\text{Ek}} \hat{\mathbf{e}}_z \times \mathbf{u} \right] = & -\hat{\rho}(r) \nabla \left[\frac{P}{\hat{\rho}(r)} \right] + \frac{\text{Ra}}{\text{Pr}} \hat{\rho}(r) \hat{g}(r) S \hat{\mathbf{e}}_r, \\ & + \nabla \cdot \mathbf{D} + (\nabla \times \mathbf{B}) \times \mathbf{B}, \end{aligned} \quad (3.3a)$$

$$\text{where} \quad D_{ij} := 2\hat{\rho}(r)\hat{\nu}(r) \left[e_{ij} - \frac{1}{3}(\nabla \cdot \mathbf{u})\delta_{ij} \right] \quad (3.3b)$$

$$\text{and} \quad e_{ij} := \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (3.3c)$$

$$\begin{aligned} \hat{\rho}(r)\hat{T}(r) \frac{DS}{Dt} = & -\frac{\text{Pr}}{\text{Ra}} \text{B}_{\text{visc}} \hat{\rho}(r)\hat{T}(r) \frac{\widehat{N^2}(r)}{\hat{g}(r)} u_r + \frac{1}{\text{Pr}} \nabla \cdot [\hat{\rho}(r)\hat{T}(r)\hat{\kappa}(r)\nabla S] \\ & + \frac{1}{\text{Pr}} \hat{Q}(r) + \frac{\text{PrDi}}{\text{Ra}} D_{ij} e_{ij} + \frac{\text{PrDi}}{\text{Pr}_m \text{Ra}} \hat{\eta}(r) |\nabla \times \mathbf{B}|^2, \end{aligned} \quad (3.4)$$

$$\text{and} \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{1}{\text{Pr}_m} \nabla \times [\eta(r) \nabla \times \mathbf{B}]. \quad (3.5)$$

The non-dimensional numbers appearing are:

$$\text{Ra} := \frac{\tilde{g}L^3}{\tilde{\nu}\tilde{\kappa}} \frac{\sigma}{c_p}, \quad (3.6a)$$

$$\text{Pr} := \frac{\tilde{\nu}}{\tilde{\kappa}}, \quad (3.6b)$$

$$\text{Pr}_m := \frac{\tilde{\nu}}{\tilde{\eta}}, \quad (3.6c)$$

$$\text{Ek} := \frac{\tilde{\nu}}{\Omega_0 H^2}, \quad (3.6d)$$

$$\text{B}_{\text{visc}} := \frac{\widetilde{N^2}L^4}{\tilde{\nu}^2}, \quad (3.6e)$$

$$\text{and} \quad \text{Di} = \frac{\tilde{g}L}{c_p \tilde{T}}, \quad (3.6f)$$

along with the non-dimensional heating function

$$\hat{Q}(r) := \frac{L^2}{\tilde{\rho}\tilde{T}\tilde{\kappa}\sigma} Q(r). \quad (3.7)$$

Note that the dissipation number is not an independent control parameter, but a function of the non-dimensional parameters characterizing the polytrope.

In general, $\hat{Q}(r)$ is simply an arbitrary—hopefully order unity—function. Assuming the thermal boundary conditions remove whatever $\hat{Q}(r)$ dumps in: If $\hat{Q}(r) \gg 1$, the user is dilating their Rayleigh number without saying so. If $\hat{Q}(r) \ll 1$, the user is contracting their Rayleigh number without saying so. If $\hat{Q}(r) \equiv 0$ (and the user wants to simulate a convection zone), the user should typically identify σ with $-\Delta S = S_{\text{in}} - S_{\text{out}}$ (the imposed entropy drop across the layer) and thus set $S \equiv 1$ at $r = r_{\text{in}}$ and $S \equiv 0$ at $r = r_{\text{out}}$.

The non-dimensional heating takes a specific form if we assume the Rayleigh number is a “flux” Rayleigh number. In that case we identify

$$\sigma = \frac{L \langle \mathcal{F}_{\text{nr}}(r) \rangle_v}{\tilde{\rho}\tilde{T}\tilde{\kappa}}, \quad (3.8)$$

$$\text{where} \quad \mathcal{F}_{\text{nr}}(r) := \frac{1}{r^2} \int_{r_{\text{in}}}^r Q(x) x^2 dx \quad (3.9)$$

is the flux not carried by radiation in a statistically steady state and $\langle \cdots \rangle_v$ refers to a volume average over the whole shell. We thus have

$$\hat{Q}(r) = \frac{L}{\langle \mathcal{F}_{\text{nr}}(r) \rangle_v} Q(r) \quad (3.10)$$

and whatever amplitude (luminosity) the user chooses for the dimensional $Q(r)$, the ultimate $\hat{Q}(r)$ will normalize that amplitude away in Equation (3.10).

The viscous buoyancy number B_{visc} is the ratio of the typical squared buoyancy frequency to the squared viscous diffusion time (it is essentially a kind of Richardson number). Although it has to do with background entropy stratification, B_{visc} is nominally independent

of the Rayleigh number (which derives from the entropy perturbations associated with the thermal boundary conditions and/or heating that force energy through the layer). However, the following notes are warranted:

(1) In the typical convection problem (polytropic index $n = n_{\text{ad}} := 1/(\gamma - 1)$), $d\bar{S}/dr \equiv 0$ and the value of B_{visc} is irrelevant.

(2) For an isolated stable polytrope ($n > n_{\text{ad}}$), it is unclear how the typical entropy perturbation σ is established. Even if energy is driven through the system by the thermal boundary conditions and/or heating, the energy will likely be carried by spherically symmetric conduction (depending on how stable the stratification is) and σ may not be set by the boundary conditions. Furthermore, σ cannot be set by the total entropy contrast across the layer (which is $\sim c_p$, and if $\sigma = c_p$, we would have $B_{\text{visc}}\text{Pr} = \text{Ra}$). That is because no plume can traverse a large portion of the stably stratified layer. Thus, although the user can choose a “Rayleigh number” (probably a misnomer) for the isolated stable layer, they have no way of implementing the background state, boundary conditions, or heating profile to be consistent with their choice.

(3) For unstable polytropes ($n < n_{\text{ad}}$), we first note that if n differs by a factor of unity from n_{ad} , order-unity thermal perturbations S/c_p are forced and the anelastic approximation is invalid. In any case (either if the simulation survives an order-unity $n_{\text{ad}} - n$, or if the user chooses a well-posed small $n_{\text{ad}} - n$ to help drive the convection), we expect that the convection will restratify the system toward adiabaticity. In Equation (1.4), a background $(dS/dr)_{\ell=0}$ will be established to be close to the negative of $d\bar{S}/dr$ and (similar to point (1)), the value of B_{visc} will not truly be an independent parameter, but will help to determine an ultimate “effective” Rayleigh number.

(4) In the more logical case of a stable layer being pummeled by a neighboring overshooting convection layer, both B_{visc} and Ra are truly independent (and relevant). The overshooting flows (driven by Ra , which is a property of the convection zone) establish the typical σ in both the convection zone and overshoot layer. Meanwhile, B_{visc} (which is a property of the stable layer) controls how strongly the overshoot is decelerated.

4 Non-Dimensional Equations; $\tau = \Omega_0^{-1}$

In the previous section, t (and things with time in the dimensions) was implied to mean $(\tilde{\nu}/H^2)t_{\text{dim}}$, where t_{dim} was the dimensional time. We now want to use $t_{\text{new}} = \Omega_0 t_{\text{dim}} = t/\text{Ek}$. We can thus find the new equations easily from Equations (3.1)–(3.5): Every place we see a time dimension, we recall $t = \text{Ek}t_{\text{new}}$, so we multiply the place where the dimension appears by Ek and drop the “new” subscript (e.g., $t \rightarrow \text{Ek } t$, $\mathbf{u} \rightarrow \mathbf{u}/\text{Ek}$, etc.). We thus find (after rearranging terms)

$$\nabla \cdot [\hat{\rho}(r)\mathbf{u}] = 0, \tag{4.1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4.2}$$

$$\begin{aligned} \hat{\rho}(r) \left[\frac{D\mathbf{u}}{Dt} + 2\hat{\mathbf{e}}_z \times \mathbf{u} \right] = & -\hat{\rho}(r) \nabla \left[\frac{P}{\hat{\rho}(r)} \right] + \text{Ra}^* \hat{\rho}(r) \hat{g}(r) S \hat{\mathbf{e}}_r, \\ & + \text{Ek} \nabla \cdot \mathbf{D} + (\nabla \times \mathbf{B}) \times \mathbf{B}, \end{aligned} \quad (4.3a)$$

$$\text{where} \quad D_{ij} := 2\hat{\rho}(r)\hat{\nu}(r) \left[e_{ij} - \frac{1}{3}(\nabla \cdot \mathbf{u})\delta_{ij} \right] \quad (4.3b)$$

$$\text{and} \quad e_{ij} := \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (4.3c)$$

$$\begin{aligned} \hat{\rho}(r)\hat{T}(r)\frac{DS}{Dt} = & -\frac{\text{B}_{\text{rot}}}{\text{Ra}^*} \hat{\rho}(r)\hat{T}(r) \frac{\widehat{N^2}(r)}{\hat{g}(r)} u_r + \frac{\text{Ek}}{\text{Pr}} \nabla \cdot [\hat{\rho}(r)\hat{T}(r)\hat{\kappa}(r)\nabla S] \\ & + \frac{\text{Ek}}{\text{Pr}} \hat{Q}(r) + \frac{\text{DiEk}}{\text{Ra}^*} D_{ij} e_{ij} + \frac{\text{DiEk}}{\text{Pr}_m \text{Ra}^*} \hat{\eta}(r) |\nabla \times \mathbf{B}|^2, \end{aligned} \quad (4.4)$$

$$\text{and} \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{\text{Ek}}{\text{Pr}_m} \nabla \times [\eta(r) \nabla \times \mathbf{B}]. \quad (4.5)$$

The new non-dimensional numbers appearing are:

$$\text{Ra}^* := \frac{\text{Ek}^2}{\text{Pr}} \text{Ra} = \frac{\tilde{g}}{L\Omega_0^2} \frac{\sigma}{c_p}, \quad (4.6a)$$

$$\text{and} \quad \text{B}_{\text{rot}} := \text{Ek}^2 \text{B}_{\text{visc}} = \frac{\widetilde{N^2}}{\Omega_0^2} \sim \frac{\tilde{g}}{L\Omega_0^2} = \frac{1}{\text{oblateness factor}}. \quad (4.6b)$$

Note that although the “ $d\bar{S}/dr$ -terms” in the non-dimensionalizations have seemingly different definitions, they are the same, since:

$$\frac{\text{Pr}}{\text{Ra}} \text{B}_{\text{visc}} = \frac{\text{B}_{\text{rot}}}{\text{Ra}^*} \sim \frac{c_p}{\sigma}. \quad (4.6c)$$

5 Non-Dimensional Polytrope

A polytrope depends on the following four non-dimensional parameters:

$$\gamma := \frac{c_p}{c_v} \quad \text{specific-heat ratio}, \quad (5.1a)$$

$$0 \leq n \leq \infty \quad \text{polytropic index}, \quad (5.1b)$$

$$N_\rho := \ln \left(\frac{\bar{\rho}_{\text{in}}}{\bar{\rho}_{\text{out}}} \right) \quad \text{number of density scale-heights}, \quad (5.1c)$$

$$\text{and} \quad \beta = \frac{r_{\text{in}}}{r_{\text{out}}} \quad \text{aspect ratio}. \quad (5.1d)$$

If the typical values of the polytrope are taken at the inner boundary, we have

$$\hat{T}(r) = \frac{\bar{T}(r)}{\bar{T}_{\text{in}}} = \left[\frac{\beta(1 - e^{-N_\rho/n})}{(1 - \beta)^2} \right] \left(\frac{H}{r} \right) - \left(\frac{\beta - e^{-N_\rho/n}}{1 - \beta} \right) \quad (5.2a)$$

$$\hat{\rho}(r) = \frac{\bar{\rho}(r)}{\bar{\rho}_{\text{in}}} = \left\{ \left[\frac{\beta(1 - e^{-N_\rho/n})}{(1 - \beta)^2} \right] \left(\frac{H}{r} \right) - \left(\frac{\beta - e^{-N_\rho/n}}{1 - \beta} \right) \right\}^n, \quad (5.2b)$$

$$\widehat{N^2}(r) = \frac{N^2(r)}{\widehat{N^2}} = \left(\frac{r_{\text{in}}}{r} \right)^3 \left[\frac{1 - e^{-N_\rho/n}}{1 - \beta} - \left(\frac{\beta - e^{-N_\rho/n}}{\beta} \right) \frac{r}{H} \right]^{-1}, \quad (5.2c)$$

$$\hat{g}(r) = \frac{g(r)}{g_{\text{in}}} = \frac{r_{\text{in}}^2}{r^2}, \quad (5.2d)$$

$$\text{and} \quad \text{Di} = \frac{L}{H} \left(\frac{n+1}{\tilde{n}+1} \right) \left(\frac{1}{\beta} \right) (1 - e^{-N_\rho/n}), \quad (5.2e)$$

where $H = r_{\text{out}} - r_{\text{in}}$ is the shell depth.

Note that the range on r/H is (by definition)

$$\frac{\beta}{1 - \beta} \leq \frac{r}{H} \leq \frac{1}{1 - \beta}. \quad (5.3)$$

If instead we take the typical values at the outer boundary, it is simple to compute the ratios from outer to inner directly from Equations (5.2) and thus change the non-dimensionalization of the polytrope.

If we instead take the typical values as volume-averages, the density profile in Equations (5.2) (because of the n exponent) must be integrated numerically (or else computed from the hypergeometric function) but it is again straightforward to re-scale. Because the formulas are complicated (and I am error-prone), it is easiest to numerically integrate the other functions as well. We note, however, the analytic formula for Di when volume averages are used:

$$\begin{aligned} \text{Di}_v &:= \frac{\langle g \rangle_v / g_{\text{in}}}{\langle \bar{T} \rangle_v / \bar{T}_{\text{in}}} \text{Di} \\ &= \frac{L}{H} \left(\frac{n+1}{\tilde{n}+1} \right) \frac{3\beta(1 - \beta)^2(1 - e^{-N_\rho/n})}{(3\beta/2)(1 - \beta^2)(1 - e^{-N_\rho/n}) - (1 - \beta^3)(\beta - e^{-N_\rho/n})}. \end{aligned} \quad (5.4)$$

This emphasises the fact that $\text{Di} = \text{Di}(\gamma, n, N_\rho, \beta, L/H)$ is not an independent control parameter of the system.