Induction term in spherical coordinates

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February 3, 2023

1 The problem with the traditional induction terms

We consider the ideal (resistance-free) magnetohydrodynamic (MHD) induction equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \coloneqq \boldsymbol{I}, \tag{1}$$

$$= \boldsymbol{B} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{B} - (\nabla \cdot \boldsymbol{u})\boldsymbol{B}, \tag{2}$$

where $\boldsymbol{B}$ and $\boldsymbol{u}$ are the vector magnetic and velocity fields (respectively) and $\boldsymbol{I}$ is the vector induction. The three terms on the right-hand-side of Equation (2) are often interpreted as "shear," "advection," and "compression," respectively. However, this interpretation is problematic in general for two reasons:

1. The so-called shear and compression terms contain sub-terms that cancel; in particular, only velocity motions *perpendicular* to magnetic-field lines can shear or compress.

2. Solid-body rotation (which is a non-shearing motion that simply rotates the whole field configuration) shows up in the so-called shear and advection terms in a strange way.

Finally, even after these issues have been addressed, resolving the final terms into a particular curvilinear system (e.g., spherical coordinates) seems to present a major headache. Our goal here is to explain fully how these problems emerge and propose a tentative solution for the case of spherical coordinates.

2 Perpendicular shear and compression

To see how Problem 1 arises, we decompose the velocity field into components parallel and perpendicular to the local direction of $\boldsymbol{B}$:

$$\boldsymbol{u} \coloneqq u_{\parallel}\hat{\boldsymbol{e}}_{\parallel} + \boldsymbol{u}_{\perp} \tag{3}$$

Obviously $\boldsymbol{B} = Bx_{\parallel}$, where $B = |\boldsymbol{B}|$. We denote the Cartesian distance along $\boldsymbol{B}$ by $x_{\parallel}$. We also decompose $\boldsymbol{u}$ into its parallel and perpendicular components:

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_{\parallel}}{\partial x_{\parallel}} + \nabla_{\perp} \cdot \boldsymbol{u}_{\perp} \tag{4}$$

We then calculate

$$\begin{aligned}\boldsymbol{B}\cdot\nabla\boldsymbol{u}-(\nabla\cdot\boldsymbol{u})\boldsymbol{B} &= B\frac{\partial}{\partial x_\parallel}(u_\parallel\hat{\boldsymbol{e}}_\parallel+\boldsymbol{u}_\perp)-\left(\frac{\partial u_\parallel}{\partial x_\parallel}+\nabla_\perp\cdot\boldsymbol{u}_\perp\right)B\hat{\boldsymbol{e}}_\parallel \\ &= \cancel{B\frac{\partial u_\parallel}{\partial x_\parallel}}\hat{\boldsymbol{e}}_\parallel+B\frac{\partial\boldsymbol{u}_\perp}{\partial x_\parallel}-\cancel{\frac{\partial u_\parallel}{\partial x_\parallel}B}\hat{\boldsymbol{e}}_\parallel-(\nabla_\perp\cdot\boldsymbol{u}_\perp)B\hat{\boldsymbol{e}}_\parallel \\ &= \boldsymbol{B}\cdot\nabla\boldsymbol{u}_\perp-(\nabla_\perp\cdot\boldsymbol{u}_\perp)\boldsymbol{B}. \end{aligned}\tag{5}$$

Thus, only motions perpendicular to the local field line (i.e., $\boldsymbol{u}_\perp$) can shear or compress $\boldsymbol{B}$.

3 Rigid rotation

To see how Problem 2 arises, we consider a velocity field due to rigid rotation at constant angular velocity Ω about the z-axis in a cylindrical coordinate system:

$$\boldsymbol{\Omega}=\Omega\hat{\boldsymbol{e}}_z=\text{constant} \tag{6a}$$

$$\boldsymbol{u}=\boldsymbol{\Omega}\times\boldsymbol{r}=\Omega\lambda\hat{\boldsymbol{e}}_\phi \tag{6b}$$

Here, λ is the cylindrical radius, ϕ the longitude, and z the axial coordinate. In general, $\hat{\boldsymbol{e}}_{(\dots)}$ denotes a unit vector in the direction of its subscript. We calculate:

$$\begin{aligned}\nabla\cdot\boldsymbol{u} &= \nabla\cdot(\boldsymbol{\Omega}\times\boldsymbol{r}) \\ &= \boldsymbol{\Omega}\cdot\nabla\times\boldsymbol{r}-\boldsymbol{r}\cdot\nabla\times\boldsymbol{\Omega} \\ &= 0 \qquad \text{no compression for rigid rotation (obviously).}\end{aligned}\tag{7}$$

Then:

$$\begin{aligned}\boldsymbol{B}\cdot\nabla\boldsymbol{u} &= (\boldsymbol{B}\cdot\nabla)(\boldsymbol{\Omega}\times\boldsymbol{r}) \\ &= \boldsymbol{\Omega}\times[(\boldsymbol{B}\cdot\nabla)(\boldsymbol{r})] \qquad (8) \\ &= \boldsymbol{\Omega}\times\boldsymbol{B} \qquad \text{“shear” for rigid rotation.} \qquad (9)\end{aligned}$$

Finally:

$$\begin{aligned}-\boldsymbol{u}\cdot\nabla\boldsymbol{B} &= -\Omega\lambda\hat{\boldsymbol{e}}_\phi\cdot\nabla\boldsymbol{B} \\ &= -\Omega\frac{\partial}{\partial\phi}(B_\lambda\hat{\boldsymbol{e}}_\lambda+B_\phi\hat{\boldsymbol{e}}_\phi+B_z\hat{\boldsymbol{e}}_z) \\ &= -\Omega\sum_\alpha\left(\frac{\partial B_\alpha}{\partial\phi}\right)\hat{\boldsymbol{e}}_\alpha-\Omega\sum_\alpha B_\alpha\frac{\partial\hat{\boldsymbol{e}}_\alpha}{\partial\phi},\end{aligned}$$

where the index α runs over the three cylindrical coordinates. Note that in the cylindrical coordinate system (or indeed any coordinate system with an axis of rotational symmetry), $\partial\hat{\boldsymbol{e}}_\alpha/\partial\phi=\hat{\boldsymbol{e}}_z\times\hat{\boldsymbol{e}}_\alpha$ for each α. Thus,

$$\begin{aligned}-\Omega\sum_\alpha B_\alpha\frac{\partial\hat{\boldsymbol{e}}_\alpha}{\partial\phi} &= -\Omega\sum_\alpha B_\alpha\hat{\boldsymbol{e}}_z\times\hat{\boldsymbol{e}}_\alpha \\ &= -\Omega\hat{\boldsymbol{e}}_z\times\sum_\alpha B_\alpha\hat{\boldsymbol{e}}_\alpha \\ &= -\boldsymbol{\Omega}\times\boldsymbol{B}\end{aligned}$$

and so

$$-\boldsymbol{u}\cdot\nabla\boldsymbol{B} = -\Omega\sum_\alpha\left(\frac{\partial B_\alpha}{\partial\phi}\right)\hat{\boldsymbol{e}}_\alpha - \boldsymbol{\Omega}\times\boldsymbol{B} \qquad \text{“advection” for rigid rotation.} \tag{10}$$

Mathematically, in any coordinate system with a z-axis of rotational symmetry, the action of rigid rotation is as follows:

$$\begin{aligned}\frac{\partial\boldsymbol{B}}{\partial t} &= \nabla\times[(\boldsymbol{\Omega}\times\boldsymbol{r})\times\boldsymbol{B}]\\ &= \sum_\alpha\left(-\Omega\frac{\partial B_\alpha}{\partial\phi}\right)\hat{\boldsymbol{e}}_\alpha \end{aligned}\tag{11a}$$

$$\text{or} \qquad \left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right)B_\alpha = 0 \qquad \text{for each } \alpha. \tag{11b}$$

If you think about it, this makes sense: All the rigid rotation does is rotate the whole field configuration around the z-axis at the rate Ω. If you decide to also rotate at Ω (so your personal Eulerian time derivative is $\partial/\partial t + \Omega\partial/\partial\phi$), then each component of the magnetic-field configuration should remain the same in your frame. Note that rotation does *not* advect the vector magnetic field (like the term $-\boldsymbol{u}\cdot\nabla\boldsymbol{B}$ viewed on its own would suggest), but rather advects the field *components* (as if they were scalars) in any coordinate system with an axis of rotational symmetry.

4 Solution for spherical coordinates

Resolving these issues fully for the spherical coordinate system seems complicated and I am not fully sure how to do it! In particular (for “full” resolution) we should, separately at each point (r,θ,ϕ):

1. Form a local Cartesian coordinate system, say (x_1, x_2, x_3), whose origin lies at the point (r,θ,ϕ). At the origin, ($\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$) will coincide with ($\hat{\boldsymbol{e}}_r$, $\hat{\boldsymbol{e}}_\theta$, $\hat{\boldsymbol{e}}_\phi$). But slightly away from the origin, ($\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$) will stay fixed while ($\hat{\boldsymbol{e}}_r$, $\hat{\boldsymbol{e}}_\theta$, $\hat{\boldsymbol{e}}_\phi$) curve away.

2. Calculate the velocity-gradient tensor: $\partial u_1/\partial x_1, \partial u_2/\partial x_1, \partial u_3/\partial x_1$, etc. While calculating derivatives, be careful to differentiate along the Cartesian coordinates, *not* the curvilinear ones or along the actual $\boldsymbol{B}$-line). Express the final tensor components in spherical coordinates.

3. Rotate “into $\boldsymbol{B}$” to form a new primed coordinate system (such that $\hat{\boldsymbol{e}}_1'$ points along $\boldsymbol{B}$), calculate the $\partial u_j'/\partial x_i'$, and thus form $\partial u_\parallel/\partial x_\parallel$ and $\partial\boldsymbol{u}_\perp/\partial x_\parallel$ (note that $\nabla_\perp\cdot\boldsymbol{u}_\perp = \nabla\cdot\boldsymbol{u} - \partial u_\parallel/\partial x_\parallel$). Note that the direction cosines $\hat{\boldsymbol{e}}_1'$ makes with ($\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$) are simply $B_r/|\boldsymbol{B}|, B_\theta/|\boldsymbol{B}|, B_\phi/|\boldsymbol{B}|$, respectively. Note that both unit vectors and vector components transform like $x_1' = \sum_{j=1}^3 R_{1j}x_j$, where R_{1j} is the j^{th} direction cosine.

4. Subtract the part of $\boldsymbol{u}_\perp$ corresponding to solid-body rotation and put it in the form of Equation (11a).

5. Exult, because I haven't been able to do this!

Instead of addressing the full problem as just described, I have brute-forced my way into a quasi-solution, canceling obvious terms and expressing what seem to be "perpendicular shear with no solid-body rotation," "perpendicular compression," and advection.

As a shorthand, we write the following for the part of each vector perpendicular to the local coordinate lines:

$$\boldsymbol{u}_{\text{hor}} \coloneqq u_\theta \hat{\boldsymbol{e}}_\theta + u_\phi \hat{\boldsymbol{e}}_\phi \qquad \text{horizontal part } (\perp\ \hat{\boldsymbol{e}}_r), \tag{12a}$$

$$\boldsymbol{u}_{\text{tor}} \coloneqq u_r \hat{\boldsymbol{e}}_r + u_\phi \hat{\boldsymbol{e}}_\phi \qquad \text{toroidal part } (\perp\ \hat{\boldsymbol{e}}_\theta), \tag{12b}$$

$$\text{and} \qquad \boldsymbol{u}_{\text{pol}} \coloneqq u_r \hat{\boldsymbol{e}}_r + u_\theta \hat{\boldsymbol{e}}_\theta \qquad \text{poloidal part } (\perp\ \hat{\boldsymbol{e}}_\phi) \tag{12c}$$

...and similarly for $\boldsymbol{B}$.

We now write:

$$\begin{aligned}
I_r &= \cancel{B_r \frac{\partial u_r}{\partial r}} + \frac{B_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{B_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \cancel{\frac{B_\theta u_\theta + B_\phi u_\phi}{r}} \\
&\quad - \left[u_r \frac{\partial B_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial B_r}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial B_r}{\partial \phi} - \cancel{\frac{u_\theta B_\theta + u_\phi B_\phi}{r}}\right] \\
&\quad - B_r \left[\cancel{\frac{\partial u_r}{\partial r}} + \frac{\partial 2u_r}{\partial r} + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + \cot\theta u_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right] \\
&= \underbrace{\left(\frac{B_\theta}{r}\frac{\partial}{\partial \theta} + \frac{B_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\right)}_{\boldsymbol{B}_{\text{hor}}\cdot\nabla} u_r - B_r \left[\underbrace{\frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + \cot\theta u_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}}_{\nabla\cdot\boldsymbol{u}_{\text{hor}}}\right] \\
&\quad - \boldsymbol{u}\cdot\nabla B_r - \frac{2B_r u_r}{r}
\end{aligned}$$

or

$$I_r = \underbrace{\boldsymbol{B}_{\text{hor}}\cdot\nabla u_r}_{\text{perpendicular shear}} \underbrace{- \left(\nabla\cdot\boldsymbol{u}_{\text{hor}} + \frac{2u_r}{r}\right) B_r}_{\text{perpendicular compression}} \underbrace{- \boldsymbol{u}\cdot\nabla B_r}_{\text{advection}} \tag{13}$$

Our justification for including the term $-2B_r u_r/r$ in the compression will be given in the next section.

Now that we know a bit more how things fall into place, we calculate

$$\begin{aligned}
I_\theta &= \cancel{\frac{B_\theta}{r}\frac{\partial u_\theta}{\partial \theta}} + \boldsymbol{B}_{\text{tor}}\cdot\nabla u_\theta + \frac{B_\theta u_r}{r} - \cancel{\frac{\cot\theta B_\phi u_\phi}{r}} \\
&\quad - \boldsymbol{u}\cdot\nabla B_\theta - \frac{u_\theta B_r}{r} + \cancel{\frac{\cot\theta u_\phi B_\phi}{r}} \\
&\quad - B_\theta \left(\cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}} + \frac{\cot\theta u_\theta}{r} + \nabla\cdot\boldsymbol{u}_{\text{tor}}\right)
\end{aligned}$$

or

$$I_\theta = \underbrace{r\boldsymbol{B}_{\text{tor}}\cdot\nabla\left(\frac{u_\theta}{r}\right)}_{\text{perpendicular shear}} \underbrace{- \left(\nabla\cdot\boldsymbol{u}_{\text{tor}} + \frac{\cot\theta u_\theta}{r} - \frac{u_r}{r}\right) B_\theta}_{\text{perpendicular compression}} \underbrace{- \boldsymbol{u}\cdot\nabla B_r}_{\text{advection}} \tag{14}$$