An energy-conserving anelastic approximation for strongly stably-stratified fluids

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1 Introduction

Abstract: When acoustic oscillations are believed to be irrelevant to the dynamics of an astrophysical fluid, it is useful to employ simplifying approximations to the equations of motion. The two most common of these (which are usually used to treat convection problem) are the Boussinesq approximation (when the background density does not significantly vary across the fluid layer) and the anelastic approximation (when the background density does vary significantly). There are many distinct forms of the anelastic approximation in the literature, and it has often been remarked that most of these do not properly conserve energy when the fluid is stable to convection. Here we show that the anelastic equations derived by Gough (1969) in fact do conserve energy for arbitrary motions of the fluid, even for strongly stratified background stratification. The key properties of these equations that allow them to conserve energy are (1) the absence of the Lantz-Braginsky-Roberts approximation in the momentum equation and (2) the inclusion of a historically neglected term in the energy equation, which allow the proper conversion between kinetic and potential energy at the correct order of the formal asymptotic expansion of the equations. We show that the scaling analysis of Gough (1969), which implicitly assumed a single typical value of the background entropy gradient, can be valid even for convective overshoot, where the entropy gradient changes from slightly unstable in the convecting region to stable (sometimes strongly so) in the overshoot region. The requirement for the anelastic equations to be valid for convective overshoot is that the buoyancy frequency be significantly less than the acoustic cutoff frequency.

The anelastic equations originally consisted of an approximation to the continuity and momentum equations, derived by assuming small thermal perturbations about a nearly adiabatically stratified hydrostatic reference atmosphere (Batchelor, 1953; Charney & Ogura, 1960). The thermodynamics of the problem thus become "linear," in the sense that products of thermodynamic variables reduced to linear expressions in the first-order perturbations. The two key consequences of linearized thermodynamics are divergenceless mass flux (i.e., $\nabla\cdot(\overline{\rho}\boldsymbol{u})\equiv 0$, where $\overline{\rho}$ is the background density and $\boldsymbol{u}$ the fluid velocity; this takes the place of the $\nabla\cdot\boldsymbol{u}=0$ condition from the Boussinesq approximation) and the first-order buoyancy force (associated with the first-order perturbed density and pressure) being the primary driver of the flow. (Ogura & Phillips, 1962) formalized the approximation by expanding the equations of motion in a small parameter ϵ, representing the relative variation of potential temperature across the fluid layer, and hence the relative magnitude of the thermal perturbations. They recovered the equations of Batchelor (1953); Charney & Ogura (1960) and showed an assumption about the *time scale* of the motion was necessary, in addition to the assumption of small thermal perturbations. Namely, the dynamical time scale of the buoyantly driven flows must be $O(\epsilon^{-1/2})$ times *larger* than the sound crossing time of the region. Sound waves, which imply rapid temporal variations on the order of the sound crossing time, are thus absent from the anelastic equations, making them ideal for numerical integration, where large time steps are required to capture significant evolution of the system.

In the original asymptotic expansion of Ogura & Phillips (1962), the energy equation was replaced by a heat (or entropy) equation for the evolution of potential temperature, *before* non-dimensionalizing the equations. The approach of considering the entropy equation instead of the energy equation before nondimensionalization is repeated in all modern implementations of the anelastic approximation that we are aware of (e.g., Gilman & Glatzmaier 1981; Lipps & Hemler 1982; Glatzmaier 1984; Lantz 1992; Braginsky & Roberts 1995; Lantz & Fan 1999; Clune et al. 1999; Rogers & Glatzmaier 2005; Brown et al. 2012; Vasil et al. 2013; Wilczyński et al. 2022). The resulting energy equation is also used in all numerical codes we are aware of that utilize the anelastic equations, for example, the `ASH` code (Brun et al., 2004), the `MagIC` code (Gastine & Wicht, 2012), the `Rayleigh` code (Featherstone & Hindman, 2016; Featherstone et al., 2023), the `EULAG` code (Smolarkiewicz & Prusa, 2004), and the `Dedalus` code (Burns et al., 2020; Brown et al., 2020).

While nondimensionalizing the heat equation instead of the energy equation may at first appear to be an arbitrary and innocuous choice, we show in the present work that it leads to an asymptotically inconsistent set of equations that do not conserve energy when the background is stably stratified. Gough (1969), by contrast, took a different approach than Ogura & Phillips (1962) and performed a formal asymptotic expansion in ϵ while nondimensionalizing the energy equation. We now show that this equation set, which we dub the "Energy-conserving Generalized Gough" (EGG) anelastic equations, conserve energy for arbitrary fluid motions and for all hydrostatic background states (whether stably or unstably stratified).

2 The fully compressible equations

We begin by writing down the unapproximated fully compressible equations of motion for a nonrotating nonmagnetic fluid considered by Gough (1969). These are the continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{u}) \tag{1}$$

the momentum equation,

$$\frac{\partial}{\partial t}(\rho \boldsymbol{u}) = -\nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) - \nabla P + \rho \boldsymbol{g} + \nabla \cdot \overleftrightarrow{D}, \tag{2a}$$

$$\text{where} \qquad D_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}(\nabla \cdot \boldsymbol{u})\delta_{ij} \right), \tag{2b}$$

the energy equation,

$$\frac{\partial}{\partial t}(\rho U) + \nabla \cdot (\rho U \boldsymbol{u}) + P \nabla \cdot \boldsymbol{u} = D_{ij} \frac{\partial u_i}{\partial x_j} + Q - \nabla \cdot \boldsymbol{F}, \tag{3}$$

and a general equation of state,

$$U = U(P, T). \tag{4}$$

Here, t is the time, the x_i are Cartesian spatial coordinates, ρ is the density, P the pressure, T the temperature, U the internal energy per unit mass, μ the dynamic viscosity, $\boldsymbol{g} \coloneqq -\nabla\Phi$ the graviational acceleration field, Φ the graviational potential,Q an internal heat source, $\boldsymbol{F}$ the combined conductive and radiative heat flux. $\boldsymbol{g}$ is assumed to point in the vertical direction $\hat{\boldsymbol{k}}$ (either the upward Cartesian direction or the radial direction), depends only on the vertical coordinate q (either the upward Cartesian coordinate x_3 or the radial coordinate r), and is time-independent (self-gravity is ignored). The symbol "$\leftrightarrow$" in the viscous stress tensor $\overleftrightarrow{D}$ denotes a second-order tensor, as does the dyadic notation $\boldsymbol{uu}$. The subscripts i and j (taking the values 1, 2,3) denote vector or tensor components in any of the Cartesian spatial directions. We use the Einstein summation convention and δ_{ij} denotes the Kronecker delta.

These equations are not written in the exact form of Gough (1969) (and use slightly different notation) but are mathematically equivalent. Note that the left-hand side (LHS) of Equation (3) can be written in several other forms which will prove useful:

$$\frac{\partial}{\partial t}(\rho U) + \nabla \cdot (\rho U \boldsymbol{u}) + P \nabla \cdot \boldsymbol{u} = \rho \frac{DU}{Dt} - \frac{P}{\rho}\frac{D\rho}{Dt} \tag{5a}$$

$$= \rho \frac{Dh}{Dt} - \frac{DP}{Dt} \tag{5b}$$

$$= \rho T \frac{DS}{Dt}, \tag{5c}$$

where

$$\frac{D}{Dt} \coloneqq \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla \tag{6}$$

is the material (or Lagrangian) derivative,

$$h \coloneqq U + \frac{P}{\rho} \tag{7}$$

is the specific enthalpy, and

$$S = S(P, T) \tag{8}$$

is the specific entropy.

It will also be helpful to define the following fluid properties associated with the generalized equations of state (4) and (8): the specific heat at constant pressure,

$$C_\mathrm{p} = C_\mathrm{p}(P, T) \coloneqq T \left(\frac{\partial S}{\partial T}\right)_P, \tag{9}$$

the squared adiabatic sound speed,

$$c_{\mathrm{s}}^2 = c_{\mathrm{s}}^2(P,T) \coloneqq \left(\frac{\partial P}{\partial \rho}\right)_S, \tag{10}$$

and the thermal expansion coefficient,

$$\delta = \delta(P,T) \coloneqq -\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_P. \tag{11}$$

The first law of thermodynamics takes the following forms:

$$TdS = dU - \frac{P}{\rho^2}d\rho \tag{12a}$$

$$= dh - \frac{dP}{\rho} \tag{12b}$$

$$= C_{\mathrm{p}}dT - \frac{\delta}{\rho}dP \tag{12c}$$

$$= \frac{C_{\mathrm{p}}T}{\rho\delta}\left[\frac{dP}{c_{\mathrm{s}}^2} - d\rho\right]. \tag{12d}$$

An equation for the evolution of kinetic energy can be formed from $\rho\boldsymbol{u}$ dotted into Equation (2),

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) = -\nabla\cdot\left(\frac{1}{2}\rho u^2\boldsymbol{u}\right) + \boldsymbol{u}\cdot\nabla P - \rho\boldsymbol{u}\cdot\nabla\Phi + u_i\frac{\partial D_{ij}}{\partial x_j} \tag{13}$$

Equation (1) multiplied by Φ yields an equation for the evolution of potential energy,

$$\frac{\partial}{\partial t}(\rho\Phi) = -\Phi\nabla\cdot(\rho\boldsymbol{u}). \tag{14}$$

Adding Equations (3), (13), and (14) yields an equation for the evolution of total energy,

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}u^2 + U + \Phi\right)\right] = -\nabla\cdot\left\{\left[\rho\left(\frac{1}{2}u^2 + U + \Phi\right) + P\right]\boldsymbol{u} - \boldsymbol{u}\cdot\overleftrightarrow{D} + \boldsymbol{F}\right\} + Q. \tag{15}$$

3 The anelastic approximation of Gough (1969)

We will not repeat the full asymptotic expansion in ϵ of Equations (1), (2), (3), and (4) here. Instead, we reiterate the salient assumptions in the case where the horizontally averaged reference atmosphere is time-independent and the layer depth is thicker than the typical pressure scale height (Gough (1969) also considers thin layers, in which the anelastic equations become the Boussinesq equations, and time-dependent reference atmospheres). The main assumption is that the thermodynamic perturbations from the horizontally averaged state are small, e.g.,

$$\begin{aligned} \rho &= \overline{\rho}(q) + \rho_1(x_i,t) \qquad \text{with} \qquad \rho_1/\overline{\rho} = O(\epsilon) \ll 1, \\ P &= \overline{P}(q) + P_1(x_i,t) \qquad \text{with} \qquad P_1/\overline{P} = O(\epsilon) \ll 1, \end{aligned} \tag{16}$$

and similarly for T, U, h, C_p, μ, δ, c_s^2, $\boldsymbol{F}$, and Q. Here, the overbars denote horizontal averages and the "1" subscripts denote the perturbations about this average. Note that it is *not* correct to write "$S = \overline{S}(q) + S_1(x_i, t)$ with $S_1/\overline{S} = O(\epsilon)$." The fully compressible equations of motion contain only differences in entropy and so no meaningful absolute value of $\overline{S}$ can be defined. Instead, we must write

$$S = \overline{S}(q) + S_1(x_i, t) \qquad \text{with} \qquad S_1/\overline{C_\mathrm{p}} = O(\epsilon) \ll 1. \tag{17}$$

The second assumption is that the coordinate system can be chosen such that there is no mass flux across any horizontal surface, i.e.,

$$\overline{\rho u_i} = 0. \tag{18}$$

In a spherical system, the horizontal average would be an spherically symmetric average and the coordinates would point along the spatially varying curvilinear coordinate directions.

The characteristic length scale of variation of the fluid is assumed to be a typical value for the pressure scale height H. The flow is assumed to be buoyantly driven by the $O(\epsilon)$ thermal perturbations, i.e.,

$$|\boldsymbol{u}| = O(\sqrt{\epsilon \tilde{g} H}) = O(\sqrt{\epsilon}\tilde{c}_\mathrm{s}), \tag{19}$$

where the tildes denote typical reference-state values. Thus, the squared Mach number of the flow is $O(\epsilon)$. The characteristic time scale of variation of the fluid is assumed to be advective, i.e.,

$$\left|\frac{\partial}{\partial t}\right| = O\left(\sqrt{\frac{\epsilon \tilde{g}}{H}}\right) = O\left(\sqrt{\epsilon}\frac{1}{H/\tilde{c}_\mathrm{s}}\right). \tag{20}$$

Thus, the characteristic time scale is $O(\epsilon^(- 1/2)$ longer than the time it takes a sound wave to cross a pressure scale height.

Finally, the vertical convective heat flux (which maximally could transport an energy flux of order $\overline{\rho}\overline{T}w\Delta\overline{S}$, where $w = \hat{\boldsymbol{k}} \cdot \boldsymbol{u}$ is the vertical velocity and $\Delta\overline{S}$ is the total drop in background entropy across the convecting layer) is assumed to be limited primarily by the thermal diffusion $\boldsymbol{F}$ (this will be true if the conductive heating $-\nabla \cdot \boldsymbol{F}$ in Equation (3) is at at least as large as the viscous and internal heatings). In the case of negligible heatings (high Rayleigh number), one expects

$$\frac{\Delta\overline{S}}{\tilde{C}_\mathrm{p}} = O(\epsilon), \tag{21}$$

i.e., the convecting layer is nearly adiabatically stratified for vigorous convection.

Once all of these scaling assumptions have been made, Equations (1), (2), (3), and (4) are nondimensionalized, each term is expanded in powers of ϵ, terms up to zeroth-order in the continuity equation and first-order in the other equations yields the anelastic equations. Note that one consequence of Equation (18) is that the horizontally averaged velocity $\overline{\boldsymbol{u}}$ is $O(\epsilon)$ *smaller* than the perturbed velocity $\boldsymbol{u}_1$. Hence, only $\boldsymbol{u}_1$ appears in the equations, and we subsequently drop the subscripts on $\boldsymbol{u}$.

Once the equations have been redimensionalized, the continuity equation becomes

$$\nabla \cdot (\overline{\rho}\boldsymbol{u}) = 0, \tag{22}$$

the momentum equation becomes

$$\frac{\partial}{\partial t}(\overline{\rho}\boldsymbol{u}) = -\nabla \cdot (\overline{\rho}\boldsymbol{u}\boldsymbol{u}) - \nabla P_1 + \rho_1 \boldsymbol{g} + \nabla \cdot \overleftrightarrow{D} + [-\nabla\overline{P} + \overline{\rho}\boldsymbol{g}], \tag{23a}$$

$$\text{where now} \qquad D_{ij} = \overline{\mu}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}(\nabla \cdot \boldsymbol{u})\delta_{ij}\right), \tag{23b}$$

the energy equation becomes

$$\begin{aligned}\overline{\rho}\overline{C_\mathrm{p}}\frac{\partial T_1}{\partial t} - \overline{\delta}\frac{\partial P_1}{\partial t} = &- \overline{\rho}\boldsymbol{u} \cdot \left(\nabla h_1 - \frac{1}{\overline{\rho}}\nabla P_1\right) - \overline{\rho}\overline{T}\boldsymbol{u} \cdot \nabla\overline{S} \\ &D_{ij}\frac{\partial u_i}{\partial x_j} + Q_1 - \nabla \cdot \boldsymbol{F}_1 - \rho_1 \boldsymbol{u} \cdot \boldsymbol{g} - \overline{T}(\rho_1\boldsymbol{u} - \overline{\rho_1\boldsymbol{u}}) \cdot \nabla\overline{S} \\ &+ [\overline{Q} - \nabla \cdot \overline{\boldsymbol{F}}],\end{aligned} \tag{24}$$

and the linearized equation of state becomes

$$\overline{T}S_1 = \overline{C_\mathrm{p}}T_1 - \frac{\overline{\delta}}{\overline{\rho}}P_1 \tag{25a}$$

$$= h_1 - \frac{P_1}{\rho} \tag{25b}$$

$$= \frac{\overline{C_\mathrm{p}T}}{\overline{\delta}\overline{\rho}}\left[\frac{P_1}{\overline{c_\mathrm{s}^2}} - \rho_1\right]. \tag{25c}$$

Again, these equations are not in the identical form of Gough (1969) but are mathematically equivalent. Note that Gough (1969) expands the momentum density $\boldsymbol{m} = \rho\boldsymbol{u}$ in ϵ rather than the velocity $\boldsymbol{u}$. Because of the zero mass flux condition (18), we can write

$$\boldsymbol{m} = \overline{\rho}\boldsymbol{u} + \rho_1\boldsymbol{u} - \overline{\rho_1\boldsymbol{u}} + O(\epsilon^2). \tag{26}$$

The latter two terms are $O(\epsilon)$. In most cases, we can thus write $\boldsymbol{m} \approx \overline{\rho}\boldsymbol{u}$ to translate from Gough (1969) to the current notation, except when multiplying by potentially $O(0)$ quantities like $\nabla\overline{S}$.

The differentials in Equation (12) can be converted into gradients (e.g., $T\nabla S = \nabla h - \nabla P/\rho$) and the horizontally averaged form of these relations yields

$$\overline{T}\nabla\overline{S} = \overline{C_\mathrm{p}}\nabla\overline{T} - \frac{\overline{\delta}}{\overline{\rho}}\nabla\overline{P} \tag{27a}$$

$$= \frac{\overline{C_\mathrm{p}T}}{\overline{\delta}\overline{\rho}}\left[\frac{\nabla\overline{T}}{\overline{c_\mathrm{s}^2}} - \nabla\overline{\rho}\right]. \tag{27b}$$

Note that Gough (1969) uses the superadiabatic temperature gradient

$$\beta := -\frac{\overline{T}}{\overline{C_\mathrm{p}}}\hat{\boldsymbol{k}} \cdot \nabla\overline{S} \tag{28}$$

in place of $\nabla\overline{S}$.

The zeroth order (horizontally averaged) parts of Equations (23) and (24) satisfy

$$-\nabla\overline{P} + \overline{\rho}\boldsymbol{g} = \nabla(\overline{\overline{\rho}w^2}) \tag{29}$$

and

$$\overline{Q} - \nabla\cdot\overline{\boldsymbol{F}} = \overline{\overline{\rho}\boldsymbol{u}\cdot\left(\nabla h_1 - \frac{1}{\overline{\rho}}\nabla P_1\right)} + \boldsymbol{g}\cdot\overline{\rho_1\boldsymbol{u}} - \overline{D_{ij}\frac{\partial u_i}{\partial x_j}} \tag{30}$$

In each of Equations (29) and (30), each term on the right-hand side (RHS) is $O(\epsilon)$ compared to each term on the LHS. In particular, we can approximate

$$\nabla\overline{P} \approx \overline{\rho}\boldsymbol{g} \tag{31}$$

in nonlinear terms. Dotting $\boldsymbol{u}$ into Equation (23a) yields the anelastic kinetic energy equation,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{\rho}u^2\right) = -\nabla\cdot\left(\frac{1}{2}\overline{\rho}u^2\boldsymbol{u}\right) - \boldsymbol{u}\cdot\nabla P_1 + \rho_1\boldsymbol{u}\cdot\boldsymbol{g} + u_i\frac{\partial D_{ij}}{\partial x_j} + \boldsymbol{u}\cdot\nabla(\overline{\overline{\rho}w^2}) \tag{32a}$$

Adding Equations (24) and (32) yields the anelastic total energy equation,

$$\begin{aligned}\frac{\partial}{\partial t}\left[\overline{\rho}\left(\frac{1}{2}u^2 + \overline{T}S_1\right)\right] =& -\nabla\cdot\left\{\left[\overline{\rho}\left(\frac{1}{2}u^2 + \overline{T}S_1\right) + P_1\right]\boldsymbol{u} - \boldsymbol{u}\cdot\overleftrightarrow{D} + \overline{\boldsymbol{F}} + \boldsymbol{F}_1\right\} \\ &+ \overline{Q} + Q_1 - \boldsymbol{u}\cdot\left[\overline{\rho}\overline{T}\nabla\overline{S} + \overline{T}(\rho_1\boldsymbol{u} - \overline{\rho_1\boldsymbol{u}})\cdot\nabla\overline{S} - \nabla(\overline{\overline{\rho}w^2})\right]\end{aligned} \tag{33}$$

Note that by definition,

$$\overline{\boldsymbol{u}} = 0, \tag{34}$$

so that the rightmost terms in brackets in Equation (33) cannot transport any net energy across the layer. The internal heating terms $\overline{Q} + Q_1$ are assumed "accounted for" (usually, they are inputs to drive convection) and so Equation (33) shows that the total energy integrated over the volume V of the layer,

$$E_{\text{tot}} := \int_V \overline{\rho}\left(\frac{1}{2}u^2 + \overline{T}S_1\right) dV \tag{35}$$

is conserved if the fluxes vanish on the boundaries. This conservation holds for arbitrary fluid motions that obey the anelastic equations ((22), (23), (24), and (25))

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