Angular Momentum in Terms of Toroidal and Poloidal Stream Functions

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1 Stream Function Formalism

In anelastic approximations, we always have the condition of divergenceless mass flux,

$$\nabla \cdot (\overline{\rho}\boldsymbol{v}) \equiv 0, \tag{1}$$

where $\overline{\rho}$ is the reference state density (we assume $\overline{\rho} = \overline{\rho}(r)$ is spherically symmetric and time-independent) and $\boldsymbol{v}$ is the fluid velocity. Condition (1) admits a stream function representation for the mass flux,

$$\overline{\rho}\boldsymbol{v} = \nabla \times [\nabla \times (W\hat{\boldsymbol{e}}_r)] + \nabla \times (Z\hat{\boldsymbol{e}}_r), \tag{2}$$

where W and Z are the poloidal and toroidal stream functions, respectively. In the derivations that follow it will also be helpful to define the alternate stream functions $\tilde{W}$ and $\tilde{Z}$ through

$$\overline{\rho}\boldsymbol{v} = \nabla \times [\nabla \times (\tilde{W}\boldsymbol{r})] + \nabla \times (\tilde{Z}\boldsymbol{r}), \tag{3}$$

where $\boldsymbol{r} = r\hat{\boldsymbol{e}}_r$ is the position vector. Clearly $W = r\tilde{W}$ and $Z = r\tilde{Z}$.

We note that

$$\nabla \times (\tilde{Z}\boldsymbol{r}) = \nabla\tilde{Z} \times \boldsymbol{r} \tag{4}$$

and

$$\begin{aligned}\nabla \times [\nabla \times (\tilde{W}\boldsymbol{r})] &= \nabla \times [\nabla\tilde{W} \times \boldsymbol{r}] \\ &= \nabla\tilde{W}(\underbrace{\nabla \cdot \boldsymbol{r}}_{3}) - \boldsymbol{r}\nabla \cdot (\nabla\tilde{W}) - \underbrace{(\nabla\tilde{W}) \cdot \nabla\boldsymbol{r}}_{\nabla\tilde{W}} + (\boldsymbol{r} \cdot \nabla)(\nabla\tilde{W}) \\ &= 2\nabla\tilde{W} - \boldsymbol{r}\nabla^2\tilde{W} + (\boldsymbol{r} \cdot \nabla)(\nabla\tilde{W}) \end{aligned} \tag{5}$$

Equations (4) and (5) show that the radial velocity satisfies

$$\overline{\rho}v_r = 2\frac{\partial\tilde{W}}{\partial r} - r\nabla^2\tilde{W} + r\frac{\partial}{\partial r}\left(\frac{\partial\tilde{W}}{\partial r}\right)$$

$$
\begin{aligned}
&= r\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\tilde{W}}{\partial r}\right) - \nabla^2\tilde{W}\right] \\
&= -\frac{1}{r}(\boldsymbol{r}\times\nabla)^2\tilde{W},
\end{aligned}
\tag{6}
$$

where we have made use of the familiar identity involving the Laplacian in spherical coordinates and the quantum-mechanical total angular momentum operator (up to a constant coefficient) $\mathscr{L}^2 \coloneqq (\boldsymbol{r}\times\nabla)^2$. Since $\mathscr{L}^2$ is a purely "horizontal" operator, involving only derivatives with respect to θ and ϕ and not r, and since the eigenvalues of $\mathscr{L}^2$ are $l(l+1), l \in \mathbb{J}$ (all nonzero), the vanishing of v_r forces the vanishing of $\tilde{W}$ and W:

$$
v_r = 0 \Longleftrightarrow \tilde{W} = W = 0. \tag{7}
$$

We shall assume here that we work in a spherical shell of inner radius r_i and outer radius r_o that has impenetrable boundaries, i.e.,

$$
v_r = \tilde{W} = W \equiv 0 \qquad \text{at} \qquad r = r_\mathrm{i} \qquad \text{and} \qquad r = r_\mathrm{o}. \tag{8}
$$

We define the total angular momentum of the shell through

$$
\boldsymbol{L} = \int_V \boldsymbol{r}\times(\bar{\rho}\boldsymbol{v})dV, \tag{9}
$$

where V is the region occupied by the shell. For convenience, we define the poloidal and toroidal angular momentum densities

$$
\boldsymbol{\mathcal{L}}_W \coloneqq \boldsymbol{r}\times\{\nabla\times[\nabla\times(W\hat{\boldsymbol{e}}_r)]\} \qquad \text{and} \qquad \boldsymbol{\mathcal{L}}_Z \coloneqq \boldsymbol{r}\times[\nabla\times(Z\hat{\boldsymbol{e}}_r)], \tag{10}
$$

so that

$$
\boldsymbol{L} = \int_V \boldsymbol{\mathcal{L}}dV, \tag{11}
$$

where $\boldsymbol{\mathcal{L}} \coloneqq \boldsymbol{\mathcal{L}}_W + \boldsymbol{\mathcal{L}}_Z$ is the total angular momentum density.

2 Contribution to the Angular Momentum from the Rotation of the Shell

Physically, angular momentum is only conserved in the non-rotating (lab) frame, in which the velocity is

$$
\boldsymbol{v}_\mathrm{lab} = \boldsymbol{v} + \Omega_0 r\sin\theta\hat{\boldsymbol{e}}_\phi \tag{12}
$$

and the total angular momentum is

$$
\boldsymbol{L}_\mathrm{lab} = \int_V \boldsymbol{r}\times(\bar{\rho}\boldsymbol{v}_\mathrm{lab})dV = \boldsymbol{L}_0 + \boldsymbol{L}, \tag{13}
$$

where

$$\boldsymbol{L}_0 \coloneqq \int_V \boldsymbol{r} \times (\overline{\rho}\Omega_0 r \sin\theta \hat{\boldsymbol{e}}_\phi) dV = \Omega_0 \left(\int_V \overline{\rho} r^2 \sin^2\theta dV \right) \hat{\boldsymbol{e}}_z \tag{14}$$

is the angular momentum due to the rotation of the shell. The angular integral in (14) is

$$\begin{aligned} 2\pi \int_0^\pi \sin^3\theta d\theta &= -2\pi \int_{\cos\theta=1}^{\cos\theta=-1} (1-\cos^2\theta) d\cos\theta \\ &= 2\pi \left[\cos\theta - \left(\frac{1}{3}\right)\cos^3\theta \right]_{-1}^{1} = \frac{8\pi}{3}, \end{aligned}$$

and so

$$\boldsymbol{L}_0 = \left[\frac{8\pi\Omega_0}{3} \int_{r_\mathrm{i}}^{r_\mathrm{o}} \overline{\rho}(r) r^4 dr \right] \hat{\boldsymbol{e}}_z. \tag{15}$$

For an adiabatically stratified solar-like convection zone, in which

$$\begin{aligned} r_\mathrm{i} &= 5.0000000 \times 10^{10} \text{ cm} \\ r_\mathrm{o} &= 6.5860209 \times 10^{10} \text{ cm} \\ \rho_\mathrm{i} &= 0.18053428 \text{ g cm}^{-3} \\ \rho_\mathrm{o} &= 0.0089882725 \text{ g cm}^{-3} \\ \Omega_0 &= 8.61 \times 10^{-6} \text{ rad s}^{-1} \qquad \text{(3 times solar Carrington)}, \end{aligned}$$

we compute

$$L_0 = 8.0719 \times 10^{47} \text{ g cm}^2 \text{ s}^{-1} \qquad \text{and} \qquad \mathcal{L}_0 = 1.1993 \times 10^{15} \text{ g cm}^{-1} \text{ s}^{-1}, \tag{16}$$

where $\mathcal{L}_0 \coloneqq L_0/|V|$ and $|V| = 6.7302581 \times 10^{32}$ cm^3 is the volume of the shell.

3 Contribution to the Angular Momentum from the Toroidal Stream Function Z

As the name might suggest, the *toroidal* stream function Z gives the only non-vanishing contribution to the total angular momentum $\boldsymbol{L}$. We compute

$$\begin{aligned} \boldsymbol{\mathcal{L}}_Z &= \boldsymbol{r} \times [\nabla\tilde{Z} \times \boldsymbol{r}] \\ &= \nabla\tilde{Z}(\boldsymbol{r}\cdot\boldsymbol{r}) - \boldsymbol{r}(\boldsymbol{r}\cdot\nabla\tilde{Z}) \\ &= r^2 \left[\cancel{\left(\frac{\partial\tilde{Z}}{\partial r}\right)} \hat{\boldsymbol{e}}_r + \left(\frac{1}{r}\frac{\partial\tilde{Z}}{\partial\theta}\right)\hat{\boldsymbol{e}}_\theta + \left(\frac{1}{r\sin\theta}\frac{\partial\tilde{Z}}{\partial\phi}\right)\hat{\boldsymbol{e}}_\phi \right] - \cancel{r^2\left(\frac{\partial\tilde{Z}}{\partial r}\right)}\hat{\boldsymbol{e}}_r \\ &= \left(r\frac{\partial\tilde{Z}}{\partial\theta}\right)\hat{\boldsymbol{e}}_\theta + \left(\frac{r}{\sin\theta}\frac{\partial\tilde{Z}}{\partial\phi}\right)\hat{\boldsymbol{e}}_\phi. \end{aligned}$$

The spherical unit vectors can be translated into Cartesian unit vectors via

$$\begin{aligned} & \hat{\boldsymbol{e}}_\theta = \cos\theta\cos\phi\hat{\boldsymbol{e}}_x + \cos\theta\sin\phi\hat{\boldsymbol{e}}_y - \sin\theta\hat{\boldsymbol{e}}_z \\ \text{and} \qquad & \hat{\boldsymbol{e}}_\phi = -\sin\phi\hat{\boldsymbol{e}}_x + \cos\phi\hat{\boldsymbol{e}}_y. \end{aligned}$$

Thus,

$$(\boldsymbol{\mathcal{L}}_Z)_x = r\cos\phi\cos\theta\left(\frac{\partial\tilde{Z}}{\partial\theta}\right) - \left(\frac{r}{\sin\theta}\right)\sin\phi\left(\frac{\partial\tilde{Z}}{\partial\phi}\right), \tag{17a}$$

$$(\boldsymbol{\mathcal{L}}_Z)_y = r\sin\phi\cos\theta\left(\frac{\partial\tilde{Z}}{\partial\theta}\right) + \left(\frac{r}{\sin\theta}\right)\cos\phi\left(\frac{\partial\tilde{Z}}{\partial\phi}\right), \tag{17b}$$

$$\text{and} \qquad (\boldsymbol{\mathcal{L}}_Z)_z = -r\sin\theta\left(\frac{\partial\tilde{Z}}{\partial\theta}\right). \tag{17c}$$

In anelastic spherical harmonic codes, we expand the stream functions in terms of the spherical harmonics $Y_{lm}(\theta,\phi) \sim P_{lm}(\cos\theta)e^{im\phi}$, where the P_{lm} are the associated Legendre functions. We thus write

$$\tilde{Z}(r,\theta,\phi) = \sum_{l,m}\tilde{Z}_{lm}(r)Y_{lm}(\theta,\phi). \tag{18}$$

We consider the contributions to the total angular momentum of each $\tilde{Z}_{lm}(r)$ separately, integrating the densities in (17) over the spherical shell and using orthogonality relations. We first note that the ϕ-dependence of each spherical harmonic component of $\tilde{Z}$ (and also $\partial\tilde{Z}/\partial\phi$) is $e^{im\phi}$. The $e^{im\phi}$ are orthogonal over the interval $(0, 2\pi)$, and since $\cos\phi$ and $\sin\phi$ can be written as linear combinations of $e^{\pm i\phi}$, we see that the only nonzero spherical harmonics contributing to the total angular momentum have

$$m = \pm 1 \qquad \text{for} \qquad L_x \qquad \text{and} \qquad L_y \tag{19}$$

$$\text{and} \qquad m = 0 \qquad \text{for} \qquad L_z. \tag{20}$$

It will be helpful to also define $x = \cos\theta$, so that $\partial/\partial\theta = -\sin\theta\partial/\partial x = -\sqrt{1-x^2}\partial/\partial x$. The latitudinal integral is then over $\int_{-1}^{1} dx$.

3.1 Equatorial Angular Momentum

For convenience we define

$$\begin{aligned} (\boldsymbol{\mathcal{L}}_Z)_{\text{eq}} &\coloneqq = (\boldsymbol{\mathcal{L}}_Z)_x + i(\boldsymbol{\mathcal{L}}_Z)_x \\ &= re^{i\phi}\left(\cos\theta\frac{\partial\tilde{Z}}{\partial\theta} - \frac{1}{\sin\theta}\tilde{Z}\right) \end{aligned} \tag{21}$$

The latitudinal integral for each spherical harmonic will then be

$$\int_0^\pi \left[\cos\theta\frac{\partial P_{l1}(\cos\theta)}{\partial\theta} - \frac{1}{\sin\theta}P_{l1}\right]\sin\theta d\theta = \int_{-1}^{1}\left[x(-\sin\theta)\frac{dP_{l1}}{dx} - \frac{1}{\sin\theta}P_{l1}\right]dx$$

$$= -\int_{-1}^{1}\left[(x\sqrt{1-x^2})\frac{dP_{l1}}{dx} + \frac{1}{\sin\theta}P_{l1}\right]dx$$

This integral does not appear to vanish for any value of l, let alone all $l \neq 1$. Thus, it seems that the conclusion drawn in Jones et al. (2011), namely that only the $l = 1, m = 1$ components contribute to the equatorial angular momentum (their Equations A12 and A13), is incorrect.

3.2 Axial Angular Momentum

Since only the $m = 0$ harmonics contribute to $(\boldsymbol{\mathcal{L}}_Z)_z$, only the functions $P_{l0}(\cos\theta) = P_l(x)$ (the non-associated Legendre polynomials) appear in the θ-integral. The θ-integral (excluding constant factors) can be transformed as

$$\begin{aligned}\int_0^{\pi}\sin^2\theta\left(\frac{\partial P_l}{\partial\theta}\right)d\theta &\sim \int_{-1}^{1}\sin\theta\left(\sin\theta\frac{\partial P_l}{\partial x}\right)dx\\ &= \int_{-1}^{1}(1-x^2)\left(\frac{dP_l}{dx}\right)dx\\ &\sim \int_{-1}^{1}xP_l(x)dx.\end{aligned}$$

In the final manipulation we have used integration by parts and thrown away the boundary term due to the vanishing of $1 - x^2$ at $x = \pm 1$. We recall that $x = P_1(x)$, and since the Legendre polynomials are orthogonal over the interval $(-1, 1)$, only the $l = 1$ (and $m = 0$) spherical harmonic contributes to $(\boldsymbol{\mathcal{L}}_Z)_z$. Under unit normalization, we have $Y_{10} = (1/2)\sqrt{3/\pi}\cos\theta$, from which

$$(\boldsymbol{\mathcal{L}}_Z)_z = -r\sin\theta\left(-\frac{1}{2}\sqrt{\frac{3}{\pi}}\sin\theta\right)\tilde{Z}_{10}(r) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\sin^2\theta r\tilde{Z}_{10}(r)$$

and

$$(L_Z)_z = \frac{4}{3}\sqrt{3\pi}\int_{r_\mathrm{i}}^{r_\mathrm{o}} r^3\tilde{Z}_{10}(r)dr = \frac{4}{3}\sqrt{3\pi}\int_{r_\mathrm{i}}^{r_\mathrm{o}} r^2 Z_{10}(r)dr \tag{22}$$

The constant on the RHS of (22) will depend on the convention for the normalization of the spherical harmonics. For example, compare to Jones et al. (2011), Equations A11–A14.

4 Contribution to the Angular Momentum from the Poloidal Stream Function W

We shall now show that the contribution to the total angular momentum from $\boldsymbol{\mathcal{L}}_W$ mathematically vanishes for a spherical shell. We use (5) to compute

$$\boldsymbol{\mathcal{L}}_W = \boldsymbol{r}\times[2\nabla\tilde{W} + (\boldsymbol{r}\cdot\nabla)(\nabla\tilde{W})] \Longrightarrow$$

$$(\boldsymbol{\mathcal{L}}_W)_i = \epsilon_{ijk} r_j [2\partial_k \tilde{W} + (\boldsymbol{r}\cdot\nabla)(\partial_k \tilde{W})].$$

We note that

$$\epsilon_{ijk} r_j \partial_k \tilde{W} = \epsilon_{ijk}[\partial_k (r_j \tilde{W}) - \tilde{W}\underbrace{\partial_k r_j}_{\delta_{kj}}] = \partial_k(\epsilon_{ijk} r_j \tilde{W}),$$

$$\partial_k \boldsymbol{r} = \partial_k (r_i \hat{\boldsymbol{e}}_i) = \delta_{ik}\hat{\boldsymbol{e}}_i = \hat{\boldsymbol{e}}_k,$$

and

$$\begin{aligned}
\epsilon_{ijk} r_j (\boldsymbol{r}\cdot\nabla)(\partial_k \tilde{W}) &= \epsilon_{ijk}\{\partial_k[r_j(\boldsymbol{r}\cdot\nabla\tilde{W})] - \underbrace{(\partial_k r_j)}_{\delta_{kj}}(\boldsymbol{r}\cdot\nabla\tilde{W}) - r_j \underbrace{(\partial_k \boldsymbol{r})\cdot\nabla\tilde{W}\}}_{\hat{\boldsymbol{e}}_k\cdot\nabla\tilde{W} \;=\; \partial_k \tilde{W}} \\
&= \partial_k[\epsilon_{ijk} r_j (\boldsymbol{r}\cdot\nabla\tilde{W})] - \underbrace{\epsilon_{ijk} r_j \partial_k \tilde{W}}_{\partial_k(\epsilon_{ijk} r_j \tilde{W})} \\
&= \partial_k[\epsilon_{ijk} r_j (\boldsymbol{r}\cdot\nabla\tilde{W} - \tilde{W})]
\end{aligned}$$

Thus,

$$(\boldsymbol{\mathcal{L}}_W)_i = \partial_k[\epsilon_{ijk} r_j (\tilde{W} + \boldsymbol{r}\cdot\nabla\tilde{W})]$$

and (using the divergence theorem)

$$(\boldsymbol{L}_W)_i \coloneqq \int_V (\boldsymbol{\mathcal{L}}_W)_i dV = \oint_{\partial V} [\epsilon_{ijk} r_j (\tilde{W} + \boldsymbol{r}\cdot\nabla\tilde{W})] n_k dS,$$

where $\boldsymbol{n}$ is the unit normal to ∂V and dS is an area element on ∂V. The boundary ∂V consists of the two spheres $r = r_{\rm i}$ and $r = r_{\rm o}$, so $n_k = \pm r_k/r$, and the identity $\epsilon_{ijk} r_j r_k \equiv 0$ immediately yields

$$(\boldsymbol{L}_W)_i = 0. \tag{23}$$

So actually the impenetrability condition has nothing to do with the vanishing of the angular momentum from the poloidal stream function! We only require that the integration region is a spherical shell. Note that Jones et al. (2011) derive (in their Equation A10)

$$(\boldsymbol{L}_W)_i = \oint_{\partial V} \epsilon_{ijk} r_j \left[\frac{\partial(r_m \tilde{W})}{\partial x_k} - \frac{\partial(r_k \tilde{W})}{\partial x_m}\right] n_m dS \tag{24}$$

(note that they define different stream functions $P = \tilde{W}/\overline{\rho}$ and $T = \tilde{Z}/\overline{\rho}$). They attribute the vanishing of the integral due to the fact that $\tilde{W}$ (and by extension, $\boldsymbol{r}\times\nabla\tilde{W}$) vanishes on ∂V due to the impenetrability condition (8). This is unnecessary, however. One can immediately see that the second term in the integrand is zero:

$$\epsilon_{ijk} r_j \frac{\partial(r_k \tilde{W})}{\partial x_m} n_m = \epsilon_{ijk}\left[\delta_{km}\tilde{W} + r_k \frac{\partial \tilde{W}}{\partial x_m}\right]\frac{r_j(\pm r_m)}{r} \equiv 0,$$

where we have used the asymmetry of ϵ_{ijk} in j and k and the identity $\delta_{km} r_m = r_k$.

The remaining part of the integral may be written

$$\begin{aligned}
\oint \epsilon_{ijk} r_j \left[\frac{\partial (r_m \tilde{W})}{\partial x_k} \right] n_m dS &= \oint \left[\frac{\partial}{\partial x_k} (\epsilon_{ijk} r_j r_m \tilde{W}) \right] n_m dS \\
&= \int_V \frac{\partial}{\partial x_m} \left[\frac{\partial}{\partial x_k} (\epsilon_{ijk} r_j r_m \tilde{W}) \right] dV \\
&= \int_V \frac{\partial}{\partial x_k} \left[\frac{\partial}{\partial x_m} (\epsilon_{ijk} r_j r_m \tilde{W}) \right] dV \\
&= \oint \epsilon_{ijk} \left[\frac{\partial}{\partial x_m} (\epsilon_{ijk} r_j r_m \tilde{W}) \right] n_k dS \\
&= \oint \epsilon_{ijk} [\delta_{jm} r_m \tilde{W} + r_j \partial_m (r_m \tilde{W})] \left(\frac{\pm r_k}{r} \right) dS,
\end{aligned}$$

whose integrand can be shown to vanish using previous arguments on the asymmetry of ϵ_{ijk}.

References

Jones, C.A., Boronski, P., Brun, A.S., et al., 2011, Icarus, 216, 120