General Anelastic Non-Dimensionalization for Rayleigh

Loren Matilsky

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1 Dimensional Equations

We begin by writing down the full dimensional anelastic fluid equations, as implemented in Rayleigh.

$$\nabla\cdot[\overline{\rho}(r)\boldsymbol{u}]=0, \tag{1.1}$$
$$\nabla\cdot\boldsymbol{B}=0, \tag{1.2}$$

$$\begin{aligned}\overline{\rho}(r)\left[\frac{D\boldsymbol{u}}{Dt}+2\Omega_0\hat{\boldsymbol{e}}_z\times\boldsymbol{u}\right]=&-\overline{\rho}(r)\nabla\left[\frac{P}{\overline{\rho}(r)}\right]+\left[\frac{\overline{\rho}(r)g(r)}{c_{\rm p}}\right]S\hat{\boldsymbol{e}}_r,\\ &+\nabla\cdot\boldsymbol{D}+\frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B},\end{aligned} \tag{1.3a}$$

$$\text{where}\qquad D_{ij}\coloneqq 2\overline{\rho}(r)\nu(r)\left[e_{ij}-\frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{1.3b}$$

$$\text{and}\qquad e_{ij}\coloneqq\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{1.3c}$$

$$\begin{aligned}\overline{\rho}(r)\overline{T}(r)\frac{DS}{Dt}=&-\overline{\rho}(r)\overline{T}(r)\frac{d\overline{S}}{dr}u_r+\nabla\cdot[\overline{\rho}(r)\overline{T}(r)\kappa(r)\nabla S]\\ &+Q(r)+D_{ij}e_{ij}+\frac{\eta(r)}{4\pi}|\nabla\times\boldsymbol{B}|^2,\end{aligned} \tag{1.4}$$

$$\text{and}\qquad\frac{\partial\boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{u}\times\boldsymbol{B})-\nabla\times[\eta(r)\nabla\times\boldsymbol{B}]. \tag{1.5}$$

We note that

$$\frac{d\overline{S}}{dr}=c_{\rm p}\frac{N^2(r)}{g(r)}, \tag{1.6}$$

where $N^2(r)$ is the buoyancy frequency.

2 Non-Dimensional Scheme

We now non-dimensionalize Equations (1.1)–(1.5), according to the following scheme:

$$\nabla \to \frac{1}{L}\nabla, \tag{2.1a}$$
$$t \to \tau t, \tag{2.1b}$$
$$\boldsymbol{u} \to \frac{L}{\tau}\boldsymbol{u}, \tag{2.1c}$$
$$S \to \sigma S, \tag{2.1d}$$
$$P \to \tilde{\rho}\frac{L^2}{\tau^2}P, \tag{2.1e}$$
$$\boldsymbol{B} \to (4\pi\tilde{\rho})^{1/2}\frac{L}{\tau}\boldsymbol{B}, \tag{2.1f}$$
$$\overline{\rho}(r) = \tilde{\rho}\hat{\rho}(r), \tag{2.1g}$$
$$\overline{T}(r) = \tilde{T}\hat{T}(r), \tag{2.1h}$$
$$g(r) = \tilde{g}\hat{g}(r), \tag{2.1i}$$
$$N^2(r) = \widetilde{N^2}\widehat{N^2}(r), \tag{2.1j}$$
$$\nu(r) = \tilde{\nu}\hat{\nu}(r), \tag{2.1k}$$
$$\kappa(r) = \tilde{\kappa}\hat{\kappa}(r), \tag{2.1l}$$
$$\text{and} \qquad \eta(r) = \tilde{\eta}\hat{\eta}(r). \tag{2.1m}$$

Here, L is a typical length-scale, τ is a typical time-scale, and σ is a typical entropy scale. On the right-hand-sides of Equations (2.1a)–(2.1f) (and in the following non-dimensionalizations), ∇, t, $\boldsymbol{u}$, S, P, and $\boldsymbol{B}$ are all understood to be non-dimensional. In Equations (2.1g)–(2.1m), the tildes refer to"typical values" of the reference state functions and the hats refer to the radially-dependent non-dimensional versions of the reference-state functions.

Below, we will assume the time scale is either a viscous diffusion time (i.e., $\tau = L^2/\tilde{\nu}$) or a rotational time-scale (i.e., $\tau = \Omega_0^{-1}$). To describe the reference state, we will consider three cases for a given function's "typical value": Its value at the inner shell boundary, its value at the outer shell boundary, or its value volume-averaged over the shell.

3 Non-Dimensional Equations; $\tau = L^2/\tilde{\nu}$

In this case, Equations (1.1)–(1.5) become

$$\nabla \cdot [\hat{\rho}(r)\boldsymbol{u}] = 0, \tag{3.1}$$
$$\nabla \cdot \boldsymbol{B} = 0, \tag{3.2}$$

$$\hat{\rho}(r)\left[\frac{D\boldsymbol{u}}{Dt}+\frac{2}{\mathrm{Ek}}\hat{\boldsymbol{e}}_z\times\boldsymbol{u}\right]=-\,\hat{\rho}(r)\nabla\left[\frac{P}{\hat{\rho}(r)}\right]+\frac{\mathrm{Ra}}{\mathrm{Pr}}\hat{\rho}(r)\hat{g}(r)S\hat{\boldsymbol{e}}_r,$$
$$+\nabla\cdot\boldsymbol{D}+(\nabla\times\boldsymbol{B})\times\boldsymbol{B}, \tag{3.3a}$$
$$\text{where}\qquad D_{ij}\coloneqq 2\hat{\rho}(r)\hat{\nu}(r)\left[e_{ij}-\frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{3.3b}$$
$$\text{and}\qquad e_{ij}\coloneqq\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{3.3c}$$

$$\hat{\rho}(r)\hat{T}(r)\frac{DS}{Dt}=-\,\frac{\mathrm{Pr}}{\mathrm{Ra}}\mathrm{B}_{\mathrm{visc}}\hat{\rho}(r)\hat{T}(r)\frac{\widehat{N^2}(r)}{\hat{g}(r)}u_r+\frac{1}{\mathrm{Pr}}\nabla\cdot\left[\hat{\rho}(r)\hat{T}(r)\hat{\kappa}(r)\nabla S\right]$$
$$+\frac{1}{\mathrm{Pr}}\hat{Q}(r)+\frac{\mathrm{PrDi}}{\mathrm{Ra}}D_{ij}e_{ij}+\frac{\mathrm{PrDi}}{\mathrm{Pr_m Ra}}\hat{\eta}(r)|\nabla\times\boldsymbol{B}|^2, \tag{3.4}$$

$$\text{and}\qquad \frac{\partial\boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{u}\times\boldsymbol{B})-\frac{1}{\mathrm{Pr_m}}\nabla\times\left[\eta(r)\nabla\times\boldsymbol{B}\right]. \tag{3.5}$$

The non-dimensional numbers appearing are:

$$\mathrm{Ra}\coloneqq\frac{\tilde{g}L^3}{\tilde{\nu}\tilde{\kappa}}\frac{\sigma}{c_{\mathrm{p}}}, \tag{3.6a}$$
$$\mathrm{Pr}\coloneqq\frac{\tilde{\nu}}{\tilde{\kappa}}, \tag{3.6b}$$
$$\mathrm{Pr_m}\coloneqq\frac{\tilde{\nu}}{\tilde{\eta}}, \tag{3.6c}$$
$$\mathrm{Ek}\coloneqq\frac{\tilde{\nu}}{\Omega_0 H^2}, \tag{3.6d}$$
$$\mathrm{B_{visc}}\coloneqq\frac{\widetilde{N^2}L^4}{\tilde{\nu}^2}, \tag{3.6e}$$
$$\text{and}\qquad \mathrm{Di}=\frac{\tilde{g}L}{c_{\mathrm{p}}\tilde{T}}, \tag{3.6f}$$

along with the non-dimensional heating function

$$\hat{Q}(r)\coloneqq\frac{L^2}{\tilde{\rho}\tilde{T}\tilde{\kappa}\sigma}Q(r). \tag{3.7}$$

Note that the dissipation number is not an independent control parameter, but a function of the non-dimensional parameters characterizing the polytrope.

In general, $\hat{Q}(r)$ is simply an arbitrary—hopefully order unity—function. If $\hat{Q}(r)\gg 1$, the user is dilating their Rayleigh number without saying so. If $\hat{Q}\ll 1$ (and both boundaries are not fixed-entropy), the user is contracting their Rayleigh number without saying so. If $\hat{Q}(r)\equiv 0$ (and the user is running a convection simulation), the user should typically identify

σ with $-\Delta\overline{S}$ (the entropy drop across the layer) and thus set $S \equiv 1$ at $r = r_{\rm in}$ and $S \equiv 0$ at $r = r_{\rm out}$.

The non-dimensional heating takes a specific form if we assume the Rayleigh number is a "flux" Rayleigh number. In that case we identify

$$\sigma = \frac{L \left\langle \mathcal{F}_{\rm nr}(r) \right\rangle_{\rm v}}{\tilde{\rho}\tilde{T}\tilde{\kappa}}, \tag{3.8}$$

$$\text{where} \qquad \mathcal{F}_{\rm nr}(r) \coloneqq \frac{1}{r^2}\int_{r_{\rm in}}^{r} Q(x)x^2 dx \tag{3.9}$$

is the flux not carried by radiation in a statistically steady state and $\langle \cdots \rangle_{\rm v}$ refers to a volume average over the whole shell. We thus have

$$\hat{Q}(r) = \frac{L}{\left\langle \mathcal{F}_{\rm nr}(r) \right\rangle_{\rm v}} Q(r) \tag{3.10}$$

and no matter how the user chooses the dimensional $Q(r)$, the ultimate $\hat{Q}(r)$ will be normalized in the manner given by Equation (3.10).

The viscous buoyancy number $\mathrm{B_{visc}}$ is the ratio of the typical squared buoyancy frequency to the squared viscous diffusion time (it is essentially a kind of Richardson number). Although it has to do with background entropy stratification, $\mathrm{B_{visc}}$ is nominally independent of the Rayleigh number (which derives from the entropy perturbations associated with the thermal boundary conditions and/or heating that force energy through the layer). However, the following notes are warranted:

(1) In the typical convection problem (polytropic index $n = n_{\rm ad} \coloneqq 1/(\gamma - 1)$), $d\overline{S}/dr \equiv 0$ and the value of $\mathrm{B_{visc}}$ is irrelevant.

(2) For an isolated stable polytrope ($n > n_{\rm ad}$), it is unclear how the typical entropy perturbation σ is established. Even if energy is driven through the system by the thermal boundary conditions and/or heating, the energy will likely be carried by spherically symmetric conduction (depending on how stable the stratification is) and σ may not be set by the boundary conditions. It is thus unclear if the value of Ra is relevant to the system. Furthermore, the parameters $\mathrm{B_{visc}}$ and Ra are still independent, since σ cannot be set by the total entropy contrast across the layer (which is $\sim c_{\rm p}$, and if $\sigma = c_{\rm p}$, we would have $\mathrm{B_{visc}Pr} = \mathrm{Ra}$). That is because no plume can traverse a large portion of the stably stratified layer.

(3) For unstable polytropes ($n < n_{\rm ad}$), we first note that if n differs by a factor of unity from $n_{\rm ad}$, order-unity thermal perturbations $S/c_{\rm p}$ are forced and the anelastic approximation is invalid. In any case (if the simulation survives numerically), we expect that the vigorous convection will restratify the system toward adiabaticity. In Equation (1.4), a background $(dS/dr)_{\ell=0}$ will be established to be the negative of $d\overline{S}/dr$ and as in (1), the value of $\mathrm{B_{visc}}$ will be irrelevant.

(4) In the more logical case of a stable layer being pummeled by a neighboring overshooting convection layer, both $\mathrm{B_{visc}}$ and Ra are truly independent (and relevant). The overshooting flows (driven by Ra, which is a property of the convection zone) establish the typical σ in both the convection zone and overshoot layer. Meanwhile, $\mathrm{B_{visc}}$ (which is a property of the stable layer) controls how strongly the overshoot is decelerated.

4 Non-Dimensional Equations; $\tau = \Omega_0^{-1}$

In the last section, t (and things with time in the dimensions) was implied to mean $(\tilde{\nu}/H^2)t_{\text{dim}}$, where t_{dim} was the dimensional time. We now want to use $t_{\text{new}} = \Omega_0 t_{\text{dim}} = t/\text{Ek}$. We can thus find the new equations easily from Equations (3.1)–(3.5): Every place we see a time dimension, we recall $t = \text{Ek}t_{\text{new}}$, so we multiply the place where the dimension appears by Ek and drop the "new" subscript (e.g., $t \to \text{Ek}\ t$, $\boldsymbol{u} \to \boldsymbol{u}/\text{Ek}$, etc.). We thus find (after rearranging terms)

$$\nabla \cdot [\hat{\rho}(r)\boldsymbol{u}] = 0, \tag{4.1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{4.2}$$

$$\begin{aligned} \hat{\rho}(r)\left[\frac{D\boldsymbol{u}}{Dt} + 2\hat{\boldsymbol{e}}_z \times \boldsymbol{u}\right] = &- \hat{\rho}(r)\nabla\left[\frac{P}{\hat{\rho}(r)}\right] + \text{Ra}^*\hat{\rho}(r)\hat{g}(r)S\hat{\boldsymbol{e}}_r, \\ &+ \text{Ek}\nabla\cdot\boldsymbol{D} + (\nabla\times\boldsymbol{B})\times\boldsymbol{B}, \end{aligned} \tag{4.3a}$$

$$\text{where} \qquad D_{ij} := 2\hat{\rho}(r)\hat{\nu}(r)\left[e_{ij} - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{4.3b}$$

$$\text{and} \qquad e_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{4.3c}$$

$$\begin{aligned} \hat{\rho}(r)\hat{T}(r)\frac{DS}{Dt} = &- \frac{\text{B}_{\text{rot}}}{\text{Ra}^*}\hat{\rho}(r)\hat{T}(r)\frac{\widehat{N^2}(r)}{\hat{g}(r)}u_r + \frac{\text{Ek}}{\text{Pr}}\nabla\cdot[\hat{\rho}(r)\hat{T}(r)\hat{\kappa}(r)\nabla S] \\ &+ \frac{\text{Ek}}{\text{Pr}}\hat{Q}(r) + \frac{\text{DiEk}}{\text{Ra}^*}D_{ij}e_{ij} + \frac{\text{DiEk}}{\text{Pr}_{\text{m}}\text{Ra}^*}\hat{\eta}(r)|\nabla\times\boldsymbol{B}|^2, \end{aligned} \tag{4.4}$$

$$\text{and} \qquad \frac{\partial\boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - \frac{\text{Ek}}{\text{Pr}_{\text{m}}}\nabla\times[\eta(r)\nabla\times\boldsymbol{B}]. \tag{4.5}$$

The new non-dimensional numbers appearing are:

$$\text{Ra}^* := \frac{\text{Ek}^2}{\text{Pr}}\text{Ra} = \frac{\tilde{g}}{L\Omega_0^2}\frac{\sigma}{c_{\text{p}}}, \tag{4.6a}$$

$$\text{and} \qquad \text{B}_{\text{rot}} := \text{Ek}^2\text{B}_{\text{visc}} = \frac{\widetilde{N^2}}{\Omega_0^2} \sim \frac{\tilde{g}}{L\Omega_0^2} = \frac{1}{\text{oblateness factor}}. \tag{4.6b}$$

Note that although the "$d\overline{S}/dr$-terms" in the non-dimensionalizations have seemingly different definitions, they are the same, since:

$$\frac{\text{Pr}}{\text{Ra}}\text{B}_{\text{visc}} = \frac{\text{B}_{\text{rot}}}{\text{Ra}^*} \sim \frac{c_{\text{p}}}{\sigma}. \tag{4.6c}$$

5 Non-Dimensional Polytrope

A polytrope depends on the following four non-dimensional parameters:

$$\gamma := \frac{c_\mathrm{p}}{c_\mathrm{v}} \qquad \text{specific-heat ratio,} \tag{5.1a}$$

$$0 \le n \le \infty \qquad \text{polytropic index,} \tag{5.1b}$$

$$N_\rho := \ln\left(\frac{\overline{\rho}_\mathrm{in}}{\overline{\rho}_\mathrm{out}}\right) \qquad \text{number of density scale-heights,} \tag{5.1c}$$

$$\text{and} \qquad \beta = \frac{r_\mathrm{in}}{r_\mathrm{out}} \qquad \text{aspect ratio.} \tag{5.1d}$$

If the typical values of the polytrope are taken at the inner boundary, we have

$$\hat{T}(r) = \frac{\overline{T}(r)}{\overline{T}_\mathrm{in}} = \left[\frac{\beta(1-e^{-N_\rho/n})}{(1-\beta)^2}\right]\left(\frac{H}{r}\right) - \left(\frac{\beta - e^{-N_\rho/n}}{1-\beta}\right) \tag{5.2a}$$

$$\hat{\rho}(r) = \frac{\overline{\rho}(r)}{\overline{\rho}_\mathrm{in}} = \left\{\left[\frac{\beta(1-e^{-N_\rho/n})}{(1-\beta)^2}\right]\left(\frac{H}{r}\right) - \left(\frac{\beta - e^{-N_\rho/n}}{1-\beta}\right)\right\}^n, \tag{5.2b}$$

$$\widehat{N^2}(r) = \frac{N^2(r)}{\widetilde{N^2}} = \left(\frac{r_\mathrm{in}}{r}\right)^3 \left[\frac{1-e^{-N_\rho/n}}{1-\beta} - \left(\frac{\beta - e^{-N_\rho/n}}{\beta}\right)\frac{r}{H}\right]^{-1}, \tag{5.2c}$$

$$\hat{g}(r) = \frac{g(r)}{g_\mathrm{in}} = \frac{r_\mathrm{in}^2}{r^2}, \tag{5.2d}$$

$$\text{and} \qquad \mathrm{Di} = \frac{L}{H}\left(\frac{n+1}{\tilde{n}+1}\right)\left(\frac{1}{\beta}\right)(1-e^{-N_\rho/n}), \tag{5.2e}$$

where $H = r_\mathrm{out} - r_\mathrm{in}$ is the shell depth.

Note that the range on r/H is (by definition)

$$\frac{\beta}{1-\beta} \le \frac{r}{H} \le \frac{1}{1-\beta}. \tag{5.3}$$

If instead we take the typical values at the outer boundary, it is simple to compute the ratios from outer to inner (e.g., $\overline{T}_\mathrm{out}/\overline{T}_\mathrm{in}$) directly from Equations (5.2) and thus change the non-dimensionalization of (e.g.) $\hat{T}(r)$.

If we instead take the typical values as volume-averages, the density profile in Equations (5.2) (because of the n exponent) must be integrated numerically but it is again straightforward to re-scale. We note, however, the analytic formula for Di when volume averages are used:

$$\begin{aligned}\mathrm{Di}_\mathrm{v} &:= \frac{\langle g\rangle_\mathrm{v}/g_\mathrm{in}}{\langle \overline{T}\rangle_\mathrm{v}/\overline{T}_\mathrm{in}}\mathrm{Di} \\ &= \left(\frac{n+1}{\tilde{n}+1}\right)\frac{3\beta(1-\beta)^2(1-e^{-N_\rho/n})}{(3\beta/2)(1-\beta^2)(1-e^{-N_\rho/n}) - (1-\beta^3)(\beta - e^{-N_\rho/n})}.\end{aligned} \tag{5.4}$$