General Anelastic Non-Dimensionalization for Rayleigh

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1 Dimensional Equations

We begin by writing down the full dimensional anelastic fluid equations, as implemented in Rayleigh.

$$\nabla \cdot [\overline{\rho}(r)\boldsymbol{u}] = 0 \tag{1.1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{1.2}$$

$$\overline{\rho}(r)\left[\frac{D\boldsymbol{u}}{Dt} + 2\Omega_0\hat{\boldsymbol{e}}_z \times \boldsymbol{u}\right] = -\,\overline{\rho}(r)\nabla\left[\frac{P}{\overline{\rho}(r)}\right] + \left[\frac{\overline{\rho}(r)g(r)}{c_{\mathrm{p}}}\right]S\hat{\boldsymbol{e}}_r,$$

$$+ \nabla \cdot \boldsymbol{D} + \frac{1}{4\pi}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}, \tag{1.3a}$$

$$\text{where} \quad D_{ij} \coloneqq 2\overline{\rho}(r)\nu(r)\left[e_{ij} - \frac{1}{3}(\nabla \cdot \boldsymbol{u})\delta_{ij}\right] \tag{1.3b}$$

$$\text{and} \quad e_{ij} \coloneqq \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.3c}$$

$$\overline{\rho}(r)\overline{T}(r)\frac{DS}{Dt} = -\,\overline{\rho}(r)\overline{T}(r)\frac{d\overline{S}}{dr}u_r + \nabla \cdot [\overline{\rho}(r)\overline{T}(r)\kappa(r)\nabla S]$$

$$+ Q(r) + D_{ij}e_{ij} + \frac{\eta(r)}{4\pi}|\nabla \times \boldsymbol{B}|^2 \tag{1.4}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \nabla \times [\eta(r)\nabla \times \boldsymbol{B}] \tag{1.5}$$

2   Non-Dimensional Scheme

We now non-dimensionalize Equations (1.1)–(1.5), according to the following scheme:

$$\nabla \to \frac{1}{L}\nabla \tag{2.1a}$$

$$t \to \tau t \tag{2.1b}$$

$$\boldsymbol{u} \to \frac{L}{\tau}\boldsymbol{u} \tag{2.1c}$$

$$S \to \sigma S \tag{2.1d}$$

$$P \to \tilde{\rho}\frac{L^2}{\tau^2}P \tag{2.1e}$$

$$\boldsymbol{B} \to (4\pi\tilde{\rho})^{1/2}\frac{L}{\tau}\boldsymbol{B} \tag{2.1f}$$

$$\overline{\rho}(r) = \tilde{\rho}\hat{\rho}(r) \tag{2.1g}$$

$$\overline{T}(r) = \tilde{T}\hat{T}(r) \tag{2.1h}$$

$$g(r) = \tilde{g}\hat{g}(r) \tag{2.1i}$$

$$\nu(r) = \tilde{\nu}\hat{\nu}(r) \tag{2.1j}$$

$$\kappa(r) = \tilde{\kappa}\hat{\kappa}(r) \tag{2.1k}$$

$$\eta(r) = \tilde{\eta}\hat{\eta}(r) \tag{2.1l}$$

Here, L is a typical length-scale, τ is a typical time-scale, and σ is a typical entropy scale. On the right-hand-sides of Equations (2.1a)–(2.1f) (and in the following non-dimensionalizations), ∇, t, $\boldsymbol{u}$, S, P, and $\boldsymbol{B}$ are all understood to be non-dimensional. In Equations (2.1g)–(2.1l), the tildes refer to "typical values" of the reference state functions and the hats refer to the radially dependent non-dimensional versions of the reference-state functions.

Below, we will assume the time scale is either a viscous diffusion time (i.e., $\tau = L^2/\tilde{\nu}$) or a rotational time-scale (i.e., $\tau = \Omega_0^{-1}$). To describe the reference state, we will consider three cases for a given function's "typical value": Its value at the inner shell boundary, its value at the outer shell boundary, or its value volume-averaged over the shell.

3   Non-Dimensional Equations; $\tau = L^2/\tilde{\nu}$

In this case, Equations (1.1)–(1.5) become

$$\nabla \cdot [\hat{\rho}(r)\boldsymbol{u}] = 0 \tag{3.1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{3.2}$$

$$\hat{\rho}(r)\left[\frac{D\boldsymbol{u}}{Dt} + \frac{2}{\mathrm{Ek}}\hat{\boldsymbol{e}}_z \times \boldsymbol{u}\right] = -\,\hat{\rho}(r)\nabla\left[\frac{P}{\hat{\rho}(r)}\right] + \frac{\mathrm{Ra}}{\mathrm{Pr}}\hat{\rho}(r)\hat{g}(r)S\hat{\boldsymbol{e}}_r,$$

$$+ \nabla \cdot \boldsymbol{D} + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}, \tag{3.3a}$$

$$\text{where} \quad D_{ij} := 2\hat{\rho}(r)\hat{\nu}(r)\left[e_{ij} - \frac{1}{3}(\nabla \cdot \boldsymbol{u})\delta_{ij}\right] \tag{3.3b}$$

$$\text{and} \quad e_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3.3c}$$

$$\hat{\rho}(r)\hat{T}(r)\frac{DS}{Dt} = -\frac{\mathrm{Pr}}{\mathrm{Ra}}\mathrm{B}_{\mathrm{visc}}\hat{\rho}(r)\hat{T}(r)\left(\frac{L}{c_{\mathrm{p}}}\frac{d\overline{S}}{dr}\right)u_r + \frac{1}{\mathrm{Pr}}\nabla \cdot [\hat{\rho}(r)\hat{T}(r)\hat{\kappa}(r)\nabla S]$$

$$+ \frac{1}{\mathrm{Pr}}\hat{Q}(r) + \frac{\mathrm{PrDi}}{\mathrm{Ra}}D_{ij}e_{ij} + \frac{\mathrm{PrDi}}{\mathrm{Pr_m Ra}}\hat{\eta}(r)|\nabla \times \boldsymbol{B}|^2 \tag{3.4}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \frac{1}{\mathrm{Pr_m}}\nabla \times [\eta(r)\nabla \times \boldsymbol{B}] \tag{3.5}$$

The non-dimensional numbers appearing are:

$$\mathrm{Ra} := \frac{\tilde{g}L^3}{\tilde{\nu}\tilde{\kappa}}\frac{\sigma}{c_{\mathrm{p}}} \tag{3.6a}$$

$$\mathrm{Pr} := \frac{\tilde{\nu}}{\tilde{\kappa}} \tag{3.6b}$$

$$\mathrm{Pr_m} := \frac{\tilde{\nu}}{\tilde{\eta}} \tag{3.6c}$$

$$\mathrm{Ek} := \frac{\tilde{\nu}}{\Omega_0 H^2} \tag{3.6d}$$

$$\mathrm{B_{visc}} := \frac{\tilde{g}L^3}{\tilde{\nu}^2} \tag{3.6e}$$

$$\mathrm{Di} = \frac{\tilde{g}L}{c_{\mathrm{p}}\tilde{T}} \tag{3.6f}$$

Note that the viscous buoyancy number $\mathrm{B_{visc}}$ is the ratio of a typical squared buoyancy frequency to the squared viscous diffusion time (it is essentially a kind of Richardson number). Although it has to do with background entropy stratification, $\mathrm{B_{visc}}$ is nominally independent from the Rayleigh number (which derives from the entropy perturbations associated with the thermal boundary conditions and/or heating that force energy through the layer). However, the following notes are warranted:

(1) In the typical convection problem (polytropic index $n = n_{\mathrm{ad}} := 1/(\gamma - 1)$), $d\overline{S}/dr \equiv 0$ and the value of $\mathrm{B_{visc}}$ is irrelevant.

(2) For an isolated stable polytrope ($n > n_{\mathrm{ad}}$), it is unclear how the typical entropy perturbation σ is established. Even if energy is driven through the system by the thermal

boundary conditions and/or heating, the energy will likely be carried by spherically symmetric conduction (depending on how stable the stratification is) and σ may not be set by the boundary conditions. It is thus unclear if the value of Ra is relevant to the system. Furthermore, the parameters B_{visc} and Ra are still independent, since σ cannot be set by the total entropy contrast across the layer (which is $\sim c_{\mathrm{p}}$, and if $\sigma = c_{\mathrm{p}}$, we would have $B_{\mathrm{visc}}\mathrm{Pr} = \mathrm{Ra}$). That is because no plume can traverse a large portion of the stably stratified layer.

(3) For unstable polytropes ($n < n_{\mathrm{ad}}$), we first note that if n differs by a factor of unity from n_{ad}, order-unity thermal perturbations S/c_{p} are forced and the anelastic approximation is invalid. In any case (if the simulation survives numerically), we expect that the vigorous convection will restratify the system toward adiabaticity. In Equation (1.4), a background $(dS/dr)_{\ell=0}$ will be established to be the negative of $d\overline{S}/dr$ and as in (1), the value of B_{visc} will be irrelevant.

(4) In the more logical case of a stable layer being pummeled by a neighboring overshooting convection layer, both B_{visc} and Ra are truly independent (and relevant). The overshooting flows (driven by Ra, which is a property of the convection zone) establish the typical σ in both the convection zone and overshoot layer. Meanwhile, B_{visc} (which is a property of the stable layer) controls how strongly the overshoot is decelerated.

References