Induction term in spherical coordinates

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1 The problem

We consider the ideal (resistance-free) magnetohydrodynamic (MHD) induction equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}), \tag{1}$$

$$= \boldsymbol{B} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{B} - (\nabla \cdot \boldsymbol{u})\boldsymbol{B}, \tag{2}$$

where $\boldsymbol{B}$ and $\boldsymbol{u}$ are the vector magnetic and velocity fields, respectively. The three terms on the right-hand-side of Equation (2) are often interpreted as "shear," "advection," and "compression," respectively. However, this interpretation is problematic in general for two reasons:

1. The so-called shear and compression terms contain sub-terms that cancel; in particular, only velocity motions *perpendicular* to magnetic-field lines can shear or compress.

2. Solid-body rotation (which is a rigid, non-shearing motion) shows up in the so-called shear term.

When resolving the induction equation into a particular curvilinear system (e.g., spherical coordinates), another problem arises:

3. Large curvature terms appear, which are difficult to interpret and occasionally cancel.

To see how problem 1 arises, we decompose the velocity field into components parallel and perpendicular to the local direction of $\boldsymbol{B}$:

$$\boldsymbol{u} \coloneqq u_\parallel \hat{\boldsymbol{e}}_\parallel + \boldsymbol{u}_\perp \tag{3}$$

Obviously $\boldsymbol{B} = B x_\parallel$, where $B = |\boldsymbol{B}|$. We denote the Cartesian distance along $\boldsymbol{B}$ by $x_\parallel$. We also decompose $\boldsymbol{u}$ into its parallel and perpendicular components:

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_\parallel}{\partial x_\parallel} + \nabla_\perp \cdot \boldsymbol{u}_\perp \tag{4}$$

We then calculate

$$
\begin{aligned}
\boldsymbol{B} \cdot \nabla \boldsymbol{u} - (\nabla \cdot \boldsymbol{u})\boldsymbol{B} &= B \frac{\partial}{\partial x_\parallel}(u_\parallel \hat{\boldsymbol{e}}_\parallel + \boldsymbol{u}_\perp) - \left(\frac{\partial u_\parallel}{\partial x_\parallel} + \nabla_\perp \cdot \boldsymbol{u}_\perp \right) B \hat{\boldsymbol{e}}_\parallel \\
&= B \cancel{\frac{\partial u_\parallel}{\partial x_\parallel}} \hat{\boldsymbol{e}}_\parallel + B \frac{\partial \boldsymbol{u}_\perp}{\partial x_\parallel} - \cancel{\frac{\partial u_\parallel}{\partial x_\parallel}} B \hat{\boldsymbol{e}}_\parallel - (\nabla_\perp \cdot \boldsymbol{u}_\perp) B \hat{\boldsymbol{e}}_\parallel \\
&= \boldsymbol{B} \cdot \nabla \boldsymbol{u}_\perp - (\nabla_\perp \cdot \boldsymbol{u}_\perp) \boldsymbol{B}. \qquad (5)
\end{aligned}
$$

Thus, only motions perpendicular to the local field line (i.e., $\boldsymbol{u}_\perp$) can shear or compress $\boldsymbol{B}$.

To see how Problem 2 arises, we consider a velocity field due to rigid rotation at constant angular velocity Ω about the z-axis in spherical coordinate system:

$$\boldsymbol{\Omega} = \Omega \hat{\boldsymbol{e}}_z = \text{constant} \qquad (6a)$$

$$\boldsymbol{u} = \boldsymbol{\Omega} \times \boldsymbol{r} = r \sin\theta \hat{\boldsymbol{e}}_\phi \qquad (6b)$$