Non-Dimensionalization of an Anelastic Stable–Unstable Layer in `Rayleigh`

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1 General Equations Solved in `Rayleigh`

In general (with rotation and magnetism), `Rayleigh` time-evolves a set of coupled PDEs for the 3D vector velocity $\boldsymbol{u}$, vector magnetic field $\boldsymbol{B}$, pressure perturbation P' (perturbation away from the "reference-state" pressure $\tilde{P}$), and entropy perturbation S' (perturbation away from $\tilde{S}$). Note that S' can also be interpreted as a temperature perturbation in Boussinesq mode. For more details, see `Rayleigh`'s Documentation.

We use spherical coordinates: r (spherical radius), θ (colatitude), ϕ (azimuth angle), as well as cylindrical coordinates: $\lambda = r \sin\theta$ (cylindrical radius, or moment arm) and $z = \cos\theta$ (axial coordinate). In general, $\hat{\boldsymbol{e}}_q$ denotes a position-dependent unit vector in the direction of increasing q. With this notation, the full PDE-set solved by `Rayleigh` is:

$$\nabla \cdot (f_1 \boldsymbol{u}) = 0, \tag{1.1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{1.2}$$

$$f_1 \left(\frac{D\boldsymbol{u}}{Dt} + c_1 \hat{\boldsymbol{e}}_z \times \boldsymbol{u} \right) = c_2 f_2 S \hat{\boldsymbol{e}}_r - c_3 f_1 \nabla \left(\frac{P}{f_1} \right),$$
$$+ c_4 (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + c_5 \nabla \cdot \boldsymbol{D}, \tag{1.3a}$$

$$\text{where} \quad D_{ij} := 2 f_1 f_3 \left[e_{ij} - \frac{1}{3} (\nabla \cdot \boldsymbol{u}) \delta_{ij} \right] \tag{1.3b}$$

$$\text{and} \quad e_{ij} := \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \tag{1.3c}$$

$$f_1 f_4 \left(\frac{DS}{Dt} + c_{11} f_{14} u_r \right) = c_6 \nabla \cdot (f_1 f_4 f_5 \nabla S)$$
$$+ c_{10} f_6(r) + c_8 c_5 D_{ij} e_{ij} + c_9 c_7 f_7 |\nabla \times \boldsymbol{B}|^2, \tag{1.4}$$

$$\text{and} \quad \frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times [\boldsymbol{u} \times \boldsymbol{B} - c_7 f_7 \nabla \times \boldsymbol{B}], \tag{1.5}$$

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where $D/Dt := \partial/\partial t + \boldsymbol{u} \cdot \nabla$ denotes the material derivative.

The shell geometry (i.e., aspect ratio) and reference state fully define the problem. (I have always found that initial conditions seem to ultimately be irrelevant to the final state of the system, although this of course doesn't rule out hysteresis effects from occurring in future cases). The reference state is defined by the spherically-symmetric, time-independent functions $f_i = f_i(r)$ and constants c_j. By adjusting the f_i and c_j, the user can choose between a Boussinesq or anelastic approximation (the specific form of each approximation is described more fully in the Documentation), choose any arbitrary nondimensionalization of these two equation sets, and/or set the nondimensional parameters (Rayleigh number, Prandtl number, etc.) of the problem. Rayleigh has built-in modes to set the f_i and c_j for the single-layer (i.e., either convectively stable or unstable, but not both) Boussinesq approximation (nondimensional only) and anelastic approximation (either dimensional or two choices of nondimensional). These modes are chosen via the choices reference_type = 1,2,3 or 5.

More complex systems, which Rayleigh also supports, require the user to manually change the f_i and c_j. This can be done by editing an input binary file that Rayleigh reads upon initialization and setting reference_type = 4 . This document essentially describes how to produce such a "custom" input file (here called customfile) for a coupled stable–unstable anelastic system, using a variety of chosen time-scales in the nondimensionalization. I also unpack what the "energy flux" means when stable layers are present.

Note that the c_j can also be over-written at run-time by an ASCII text-file (i.e., the main_input file), allowing easy changes of the nondimensional numbers for a simulation suite that uses a common reference state—all without modifying the customfile.

2   Dimensional Anelastic Equations

We begin by writing down the full dimensional anelastic fluid equations, as they are usually implemented in Rayleigh (more precisely, this corresponds to reference_type = 2). This form of the anelastic approximation in a spherical shell is derived in, or more accurately, attributed to (since Rayleigh "updates" the reference state slightly differently than the cluge-y ASH implementation), two common sources: Gilman & Glatzmaier (1981) and Clune et al. (1999). Rayleigh's dimensional anelastic equation-set is:

$$\nabla \cdot (\tilde{\rho}\boldsymbol{u}) = 0, \tag{2.1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2.2}$$

$$\tilde{\rho}\left(\frac{D\boldsymbol{u}}{Dt} + 2\Omega_0\hat{\boldsymbol{e}}_z \times \boldsymbol{u}\right) = \left(\frac{\tilde{\rho}\tilde{g}}{C_{\mathrm{p}}}\right)S'\hat{\boldsymbol{e}}_r - \tilde{\rho}\nabla\left(\frac{P'}{\tilde{\rho}}\right),$$

$$+ \frac{1}{\mu}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nabla \cdot \boldsymbol{D}, \tag{2.3a}$$

$$\text{where} \quad D_{ij} := 2\tilde{\rho}\tilde{\nu}\left[e_{ij} - \frac{1}{3}(\nabla \cdot \boldsymbol{u})\delta_{ij}\right] \tag{2.3b}$$

$$\text{and} \quad e_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.3c}$$

$$\tilde{\rho}\tilde{T}\left(\frac{DS'}{Dt} + \frac{d\tilde{S}}{dr}u_r\right) = \nabla \cdot (\tilde{\rho}\tilde{T}\tilde{\kappa}\nabla S') + \tilde{Q}_{\texttt{Ra}} + D_{ij}e_{ij} + \frac{\tilde{\eta}}{\mu}|\nabla \times \boldsymbol{B}|^2, \qquad (2.4)$$

$$\text{and} \qquad \frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B} - \tilde{\eta}\nabla \times \boldsymbol{B}). \qquad (2.5)$$

Here, the thermal variables ρ, T, P, and S refer to the density, temperature, pressure, and entropy (respectively). The tildes denote the reference state and the primes denote the (assumed small) perturbations from the reference state. More specifically, the essence of the anelastic approximation involves assuming all relative thermal perturbations ($\rho'/\rho_{\rm a}$, $T'/T_{\rm a}$, $P'/P_{\rm a}$, and $S'/C_{\rm p}$) are $O(\epsilon)$, where $\epsilon \ll 1$ (e.g., Ogura & Phillips 1962; Gough 1969 and Matilsky & Brummell 2024, coming soon!). Here, the "a" subscripts denote "typical reference-state values" and ϵ is also the same (small) size as the typical squared Mach number of the flow.

Other background quantities that appear are the full gravitational acceleration $\tilde{g}$, the momentum, thermal, and magnetic diffusivities ($\tilde{\nu}$, $\tilde{\kappa}$, and $\tilde{\eta}$, respectively; these all have c.g.s. units of cm^2 s^{-1}), Rayleigh's internal heating $\tilde{Q}_{\texttt{Ra}}$ (much more on this below), the frame rotation rate Ω_0, the specific heat at constant pressure $C_{\rm p}$, and the vacuum permeability μ ($= 4\pi$ in c.g.s. units). The equations are written in a frame rotating with angular velocity $\Omega_0\hat{\boldsymbol{e}}_z$ and the centrifugal force is neglected. Note that $C_{\rm p}$ is always assumed to be constant, while in a real stellar (or gas-giant) structure model, it should technically vary in radius as the "perfectness" of the gas varies.

3 My reference state: a flimsy approach to stellar structure

In the anelastic equations, the one-dimensional reference state should satisfy equilibrium conditions in the momentum and energy equations. These one-dimensional equations are then subtracted from the fully compressible equations and linearized in the primed thermodynamic quantities to produce the anelastic approximation. Ideally, the reference state would be chosen based on a state-of-the-art structure model, incorporating all known physics as best we can. In the Sun, for example, this structure model is fairly well agreed-upon (e.g., Christensen-Dalsgaard et al. 1996); but in the Jovian case, much less so (e.g., Guillot 2005). In our (or possibly just Loren's) philosophy, the form of the reference state itself matters far less than the fact that we cannot approach realistic astrophysical conditions for stars or gas giants. Therefore, we choose the reference state somewhat arbitrarily. Put another way, we choose the f_i to be slightly wrong and are consoled by the fact that the c_j are *incredibly* wrong.

When the reference state is fixed in time as it is in Rayleigh, the anelastic equations have no idea whether the one-dimensional equations subtracted out make any sense. It is up to us to be wise and enforce things like hydrostatic balance and thermal equilibrium. Rayleigh will happily chug along under any arbitrary reference state with no (obvious) poor consequences. It also helps to recall exactly what structure model we are "really"

subtracting off, since there can be a lot of confusion, especially when interpreting "flux through the system." Clarification of this energy flux is the main purpose of this and the following sections.

To specify our reference state, we demand that the gas be in hydrostatic balance,

$$\frac{d\tilde{P}}{dr} = -\tilde{\rho}\tilde{g}, \tag{3.1}$$

be perfect,

$$\tilde{P} = \tilde{\rho}\mathcal{R}\tilde{T}, \tag{3.2}$$

and be in local thermodynamic equilibrium (LTE),

$$\frac{1}{C_{\mathrm{p}}}\frac{d\tilde{S}}{dr} = \frac{1}{\gamma}\frac{d\ln\tilde{T}}{dr} - \frac{\gamma-1}{\gamma}\frac{d\ln\tilde{\rho}}{dr} \tag{3.3a}$$

$$= \frac{1}{\gamma}\frac{d\ln\tilde{P}}{dr} - \frac{d\ln\tilde{\rho}}{dr} \tag{3.3b}$$

$$= \frac{d\ln\tilde{T}}{dr} - \frac{\gamma-1}{\gamma}\frac{d\ln\tilde{P}}{dr} \tag{3.3c}$$

Here, $\mathcal{R} = C_{\mathrm{p}} - C_{\mathrm{v}}$ is the gas constant, C_{v} is the specific heat at constant volume (or equivalently, constant density), and $\gamma = C_{\mathrm{p}}/C_{\mathrm{v}}$ is the ratio of specific heats. All are assumed to be constant (that is the definition of a perfect, not just ideal, gas).

Really, Equation (3.3) is simply an expression of the first law of thermodynamics. It doesn't further constrain the structure problem, but rather introduces a new unknown, i.e., the entropy $\tilde{S}$. The approach I employ is simply to choose $d\tilde{S}/dr$ to enforce convective stability or not and also pick $\tilde{g} \propto 1/r^2$. Then the structure problem is easily solved!

4 The real structure problem

In an ideal world, we would not choose an arbitrary $d\tilde{S}/dr$, but instead demand a condition for thermal (in stars, "radiative") equilibrium. This would generate another equation constraining $\tilde{\rho}$ and $\tilde{T}$ in addition to Equations (3.1)–(3.3), and thus fully define the structure problem. Let's consider radiative equilibrium first, since this applies in many stellar interiors and (maybe) in the weather layer of Jupiter. The radiative diffusion approximation in stars makes the structure problem much easier than for the (relatively cold) gas giants, since we only need consider the opacity of a fully ionized gas (and even then, it's complicated). Under the radiative diffusion approximation (and making a mixing-length assumption), the

equation of radiative equilibrium is

$$\tilde{Q} - \nabla \cdot \tilde{\boldsymbol{F}}_{\text{rad}} = \nabla \cdot \tilde{\boldsymbol{F}}_{\text{conv}}, \tag{4.1a}$$

where
$$\tilde{\boldsymbol{F}}_{\text{rad}} := -\tilde{\rho} C_{\text{p}} \tilde{\kappa}_{\text{rad}} \nabla \tilde{T} = -\tilde{\rho} C_{\text{p}} \tilde{\kappa}_{\text{rad}} \frac{d\tilde{T}}{dr} \hat{\boldsymbol{e}}_r, \tag{4.1b}$$

$$\nabla \cdot \tilde{\boldsymbol{F}}_{\text{conv}} := \begin{cases} 0 & \text{for} \quad \nabla_{\text{rad}} < \nabla_{\text{ad}} \text{ (stable to convection)} \\ [\tilde{Q} - \nabla \cdot \tilde{\boldsymbol{F}}_{\text{rad}}]_{(\tilde{\rho}, \tilde{T}) = (\tilde{\rho}_{\text{ad}}, \tilde{T}_{\text{ad}})} & \text{for} \quad \nabla_{\text{rad}} > \nabla_{\text{ad}} \text{ (unstable to convection)} \end{cases}, \tag{4.1c}$$

$$\nabla_{\text{rad}} := \frac{d \ln \tilde{T}_{\text{rad}}/dr}{d \ln \tilde{P}/dr} := \frac{\gamma - 1}{\gamma} \frac{\tilde{L}(r)}{4\pi r^2 \tilde{g} \tilde{\rho} \tilde{\kappa}_{\text{rad}}}, \tag{4.1d}$$

and
$$\nabla_{\text{ad}} := \left(\frac{d \ln \tilde{T}}{d \ln \tilde{P}} \right)_{\text{ad}} = \frac{\gamma - 1}{\gamma}. \tag{4.1e}$$

Here, $\tilde{Q} = \tilde{Q}(\tilde{\rho}, \tilde{T})$ is the "real" reference-state internal heating per unit volume (likely due to nuclear burning in the core, or in Jupiter, Helmholtz contraction and solar irradiance), $\tilde{L}(r) \equiv 4\pi \int_0^r \tilde{Q}(x) x^2 dx$ is the total luminosity (power output) of the object interior to r, and $\tilde{\kappa}_{\text{rad}} = \tilde{\kappa}_{\text{rad}}(\tilde{\rho}, \tilde{T})$ is the reference-state radiative diffusivity (units cm^2 s^{-1}), which in turn depends on the opacity. The notation $(\tilde{\rho}_{\text{ad}}, \tilde{T}_{\text{ad}})$ means "the continuation of $\tilde{\rho}$ and $\tilde{T}$ assuming adiabatic stratification."

In practice, Equations (3.1)–(4.1) would be solved by integrating inward from an appropriate boundary condition at the photosphere. At each point, the relative values of ∇_{rad} and ∇_{ad} would be compared to determine which of the two forms of Equation (4.1c) to use. Doing so implicitly adheres to the mixing-length assumption, because it is implied that "convection is locally determined." Wherever $\nabla_{\text{rad}} < \nabla_{\text{ad}}$ there is no convection at all ($\tilde{\boldsymbol{F}}_{\text{conv}} = 0$) and wherever $\nabla_{\text{rad}} > \nabla_{\text{ad}}$ the convection is so vigorous that the layer is completely adiabatically stratified. No overshoot is allowed!

5 Energy fluxes: what belongs to the reference state?

The reference-state $\tilde{Q}$ and the $\tilde{Q}_{\text{Ra}}$ used by `Rayleigh` have physically quite different origins. Really, $\tilde{Q}_{\text{Ra}}$ is shorthand for $\nabla \cdot \tilde{\boldsymbol{F}}_{\text{conv}}$. We describe explicitly how this works here.

Subtracting Equation (4.1) from the fully compressible heat equation (see Matilsky & Brummell 2024) yields

$$\tilde{\rho} \tilde{T} \left(\frac{DS'}{Dt} + \frac{d\tilde{S}}{dr} u_r \right) = Q' - \nabla \cdot \boldsymbol{F}'_{\text{rad}} + \nabla \cdot \tilde{\boldsymbol{F}}_{\text{conv}} + D_{ij} e_{ij} + \frac{\tilde{\eta}}{\mu} |\nabla \times \boldsymbol{B}|^2, \tag{5.1a}$$

where
$$\boldsymbol{F}'_{\text{rad}} := -\tilde{\rho} C_{\text{p}} \tilde{\kappa}_{\text{rad}} \nabla T' - \left(\frac{\rho'}{\tilde{\rho}} + \frac{\kappa'_{\text{rad}}}{\tilde{\kappa}_{\text{rad}}} \right) \tilde{\rho} C_{\text{p}} \tilde{\kappa}_{\text{rad}} \nabla \tilde{T} \tag{5.1b}$$

in place of Equation (2.4).

In `Rayleigh` therefore, we have implicitly set $Q' \equiv 0$ (i.e., we ignore the "real" internal heating processes; so we also set $\tilde{Q} \equiv 0$) and we choose

$$\tilde{Q}_{\text{Ra}} := \nabla \cdot \tilde{\boldsymbol{F}}_{\text{conv}} = -\nabla \cdot \tilde{\boldsymbol{F}}_{\text{rad}} \quad \text{(since } \tilde{Q} \equiv 0\text{)}. \tag{5.2}$$

However, we don't pick $\tilde{\boldsymbol{F}}_{\rm rad}$ from a structure model, rather we set it proportional to $\tilde{P}$. In the Sun, this isn't too bad an assumption actually (see Featherstone & Hindman 2016 and the appendix of Matilsky et al. 2024).

We further implicitly assume

$$\boldsymbol{F}'_{\rm rad} = -\tilde{\rho}\tilde{T}\tilde{\kappa}\nabla S'. \tag{5.3}$$

This assumption appears to originate in Gilman & Glatzmaier (1981), which assumes a turbulent "eddy" convective heat flux, due to unresolved motions, of the form $-\rho T \kappa \nabla S$ in the fully compressible heat equation. Making the choice to diffuse entropy, we must therefore regard κ (and thus $\tilde{\kappa}$) as an "eddy" diffusivity. Strictly, we really shouldn't keep having $\boldsymbol{F}'_{\rm rad} \propto \nabla S'$ in radiative zones, where there are no "eddies" (although maybe there are two-dimensional eddies and also secondary vertical shear instabilities that can mix things—see Cope et al. 2020; Garaud 2020). Hopefully the baroclinic consequences of keeping $\boldsymbol{F}'_{\rm rad} \propto \nabla S'$ in radiative zones isn't so bad!

Why all this rigamarole? It clarifies several things. First, it really only makes sense to add "code internal heating" (like $\tilde{Q}_{\rm Ra}$) to the *convective* portions of the simulation. Really, $\tilde{Q}_{\rm Ra} = \nabla \cdot \tilde{\boldsymbol{F}}_{\rm conv}$ and it only happens to look like $-\nabla \cdot \tilde{\boldsymbol{F}}_{\rm rad}$ in our case because we exclude the "real" internal heating $\tilde{Q}$. Furthermore, nonzero conductive flux boundary conditions (i.e., $\partial S'/\partial r \neq 0$), which act like sources and sinks of heat in the same way as $\tilde{Q}_{\rm Ra}$, should not be added to the boundaries of radiative zones.

Second, the Jupiter irradiance, which heats a stable layer, really does *not* constitute "heating from above" in the same way that the solar radiation, which heats a convection zone, constitutes "heating from below." This is because a weather layer fully in radiative equilbrium would have $\tilde{Q}_{\rm Ra} = \nabla \cdot \tilde{\boldsymbol{F}}_{\rm rad} \equiv 0$, and there is no unbalanced heating. The only place "heating from above" comes in is in the boundary condition for $\tilde{\boldsymbol{F}}_{\rm rad}$, which is affected by the irradiance. This alters the reference state, but it doesn't heat or cool the fluid.

The situation could (philosophically) be different in the sense that absorption of all wavelengths of solar radiation in the upper layers could *act* like a heating from above (i.e, take the form of a volumetric heating $\tilde{Q}$, that occupied a particular outer layer independent of the temperature gradients there), while re-emission would occur (maybe in the infra-red) by altering the temperature gradient to enforce $\nabla \cdot \tilde{\boldsymbol{F}}_{\rm rad} = \tilde{Q}$. But regardless, this balance is subtracted off the heat equation and again leaves no unbalanced "code internal heating" (i.e., $\tilde{Q}_{\rm Ra} = \nabla \cdot \tilde{\boldsymbol{F}}_{\rm conv}$ still $= 0$).

Third and finally, when we think of anelastic energy fluxes, these fluxes are only responsible for carrying out $L_{\rm KH}$, i.e., the Kelvin-Helmholtz contraction luminosity that is internal to Jupiter. There is no flux necessary to account for $L_{\rm irrad}$ (the luminosity associated with energy absorbed through irradiance), because this flux has been subtracted out and should be associated with the reference state. We *do* need to set an outward conductive flux at the top of the weather layer (i.e., $\partial S'/\partial r < 0$) in order to maintain thermal equilibrium in the convecting state. This is unfortunate given the convective origin of entropy diffusion discussed above. Its baroclinic consequences (which now may be relevant, given our goal to investigate "shallow driving") have not been explored to my knowledge. Note that the top conductive flux will make the weather layer less stable than it otherwise would be. But as long as the weather layer is sufficiently stiff (i.e., has high buoyancy frequency), it should be a minimal effect.

This is all to say, I am now fully converted to the idea of setting stability via $d\tilde{S}/dr$, instead of using heating layers in different places. I also view Heimpel et al. (2022), who implements a stable layer above the convective layer, as effectively using heating layers also, since they set the conductive flux to be "into the layer" at both boundaries. This approach has the added disadvantage that the location between convectively stable and unstable cannot be set a priori.

Also, we recall the relation

$$\frac{d\overline{S}}{dr} = c_{\mathrm{p}}\frac{\tilde{N}^2}{\tilde{g}}, \tag{5.4}$$

where $\tilde{N}^2$ is the squared buoyancy frequency, which we will use in favor of $d\overline{S}/dr$ in subsequent equations.

Note that the original equations in Gilman & Glatzmaier (1981) and Clune et al. (1999) were derived assuming a nearly-adiabatic background state (i.e., $d\overline{S}/dr \approx 0$). Brown et al. (2012) and Vasil et al. (2013) have raised concerns about using various anelastic approximations in stable layers due to non-energy-conserving gravity waves. Should we be concerned?

6 Non-Dimensional Scheme

We now non-dimensionalize Equations (2.1)–(2.5), according to the following scheme:

$$\nabla \to \frac{1}{H}\nabla, \tag{6.1a}$$

$$t \to \tau t, \tag{6.1b}$$

$$\boldsymbol{u} \to \frac{H}{\tau}\boldsymbol{u}, \tag{6.1c}$$

$$S \to (\Delta S)S, \tag{6.1d}$$

$$P \to \tilde{\rho}\frac{H^2}{\tau^2}P, \tag{6.1e}$$

$$\boldsymbol{B} \to (\mu\tilde{\rho})^{1/2}\frac{H}{\tau}\boldsymbol{B}, \tag{6.1f}$$

$$\tilde{\rho} \to \tilde{\rho}\tilde{\rho}, \tag{6.1g}$$

$$\tilde{T} \to \tilde{T}\tilde{T}, \tag{6.1h}$$

$$\tilde{g} \to \tilde{g}\tilde{g}, \tag{6.1i}$$

$$\tilde{N}^2 \to \widetilde{N^2}\tilde{N}^2, \tag{6.1j}$$

$$\tilde{\nu} \to \tilde{\nu}\tilde{\nu}, \tag{6.1k}$$

$$\tilde{\kappa} \to \tilde{\kappa}\tilde{\kappa}, \tag{6.1l}$$

$$\tilde{\eta} \to \tilde{\eta}\tilde{\eta}, \tag{6.1m}$$

$$Q \to \tilde{C}Q, \tag{6.1n}$$

$$\text{and} \quad C \to \tilde{Q}C. \tag{6.1o}$$

Here, H is a typical length-scale, τ a typical time-scale, and ΔS a typical (*estimated*) entropy scale (in Rayleigh-Bénard-type convection, the true entropy difference is imposed directly, but we will set the true value indirectly via heating and cooling functions). On the right-hand-sides of Equation (6.1) and in the following non-dimensionalizations, all fluid variables, coordinates, and background-state quantities are understood to be non-dimensional. The tildes refer to "typical values" of the (dimensional) reference-state functions. These typical values will be a volume-average over the convection zone (CZ) of the shell, except for $\widetilde{N^2}$, which will be a volume-average over the stably stratified weather layer (WL). Since cooling takes out what heating dumps in, we will normalize such that $\tilde{C} = \tilde{Q}$.

Below, we will assume the time-scale is either a thermal diffusion time (i.e., $\tau = H^2/\tilde{\nu}$) or a rotational time-scale [i.e., $\tau = (2\Omega_0)^{-1}$].

7 Non-Dimensional Equations, Non-Rotating ($\tau = H^2/\tilde{\kappa}$)

In this case, Equations (2.1)–(2.5) become

$$\nabla \cdot [\tilde{\rho}\boldsymbol{u}] = 0, \tag{7.1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{7.2}$$

$$\tilde{\rho}\left[\frac{D\boldsymbol{u}}{Dt} + \frac{\mathrm{Pr}}{\mathrm{Ek}}\hat{\boldsymbol{e}}_z \times \boldsymbol{u}\right] = \mathrm{Pr}\mathrm{Ra}\tilde{\rho}\tilde{g}S\hat{\boldsymbol{e}}_r - \tilde{\rho}\nabla\left[\frac{P}{\tilde{\rho}}\right],$$
$$+ (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \mathrm{Pr}\nabla \cdot \boldsymbol{D}, \tag{7.3a}$$

$$\text{where} \quad D_{ij} := 2\tilde{\rho}\tilde{\nu}\left[e_{ij} - \frac{1}{3}(\nabla \cdot \boldsymbol{u})\delta_{ij}\right] \tag{7.3b}$$

$$\text{and} \quad e_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{7.3c}$$

$$\tilde{\rho}\tilde{T}\left[\frac{DS}{Dt} + \frac{\mathrm{Bu}}{\mathrm{Ra}}\frac{\tilde{N}^2}{\tilde{g}}u_r\right] = \nabla \cdot [\tilde{\rho}\tilde{T}\tilde{\kappa}\nabla S]$$
$$+ Q(r) - C + \frac{\mathrm{Di}}{\mathrm{Ra}}D_{ij}e_{ij} + \frac{\mathrm{Di}}{\mathrm{Pr_m}\mathrm{Ra}}\tilde{\eta}|\nabla \times \boldsymbol{B}|^2, \tag{7.4}$$

$$\text{and} \quad \frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \left[\boldsymbol{u} \times \boldsymbol{B} - \frac{\mathrm{Pr}}{\mathrm{Pr_m}}\tilde{\eta}\nabla \times \boldsymbol{B}\right]. \tag{7.5}$$

The non-dimensional numbers appearing are:

$$\mathrm{Ra} := \frac{\tilde{g} H^3}{\tilde{\nu}\tilde{\kappa}} \frac{\Delta S}{c_\mathrm{p}} \qquad \text{(Rayleigh number)}, \tag{7.6a}$$

$$\mathrm{Pr} := \frac{\tilde{\nu}}{\tilde{\kappa}} \qquad \text{(Prandtl number)}, \tag{7.6b}$$

$$\mathrm{Pr_m} := \frac{\tilde{\nu}}{\tilde{\eta}} \qquad \text{(magnetic Prandtl number)}, \tag{7.6c}$$

$$\mathrm{Ek} := \frac{\tilde{\nu}}{2\Omega_0 H^2} \qquad \text{(Ekman number)}, \tag{7.6d}$$

$$\mathrm{Bu} := \frac{\widetilde{N^2} H^4}{\tilde{\nu}\tilde{\kappa}} \qquad \text{(buoyancy number)}, \tag{7.6e}$$

$$\text{and} \quad \mathrm{Di} = \frac{\tilde{g} H}{c_\mathrm{p}\tilde{T}} \qquad \text{(dissipation number)}, \tag{7.6f}$$

Note that in our convention, the dissipation number is not an independent control parameter, but a function of the non-dimensional parameters characterizing the reference state (this will be seen in Section 3).

Note that we have chosen the entropy-scale (and thus the Rayleigh number) based on the internal heating:

$$\Delta S := \frac{\tilde{Q}\tau}{\tilde{\rho}\tilde{T}} = \frac{\tilde{Q} H^2}{\tilde{\rho}\tilde{T}\tilde{\kappa}} \tag{7.7a}$$

$$\text{and} \quad \mathrm{Ra} := \frac{\tilde{g}\tilde{Q} H^3 \tau}{\tilde{\rho}\tilde{T} c_\mathrm{p}\tilde{\nu}\tilde{\kappa}} = \frac{\tilde{g}\tilde{Q} H^5}{\tilde{\rho}\tilde{T} c_\mathrm{p}\tilde{\nu}\tilde{\kappa}^2}, \tag{7.7b}$$

$$\tag{7.7c}$$

where the first equality in each equation is general (it holds for any choice of time-scale τ) and the second equality is specific to the non-rotating case. Essentially, we have assumed that the heating (or cooling) operates on the time-scale τ before the fluid parcel buoyantly moves to another part of the shell, carrying with it an entropy perturbation ΔS (in the non-rotating case, this should happen on the thermal dissipation time-scale $\tau = H^2/\tilde{\kappa}$).

The user is thus free to choose the shapes of Q and C, but not their amplitude, since they must have unity volume-averages over the CZ.

The buoyancy number Bu is the ratio of the typical squared buoyancy frequency to the thermal and viscous diffusion times. It is essentially a "second (stable) Rayleigh number", and measures the stiffness of the stable layer (recall $\widetilde{N^2}$ refers to the typical value of $\tilde{N}^2$ in the WL). The buoyancy number is independent of the Rayleigh number, which estimates the ultimate instability of the CZ.

8 Non-Dimensional Equations, Rotating $[\tau = (2\Omega_0)^{-1}]$

In the previous section, t (and things with time in the dimensions) was implied to mean $(\tilde{\kappa}/H^2) t_\mathrm{dim}$, where t_dim was the dimensional time. We now want to use a new non-dimensional

time, $t_{\text{new}} = \Omega_0 t_{\text{dim}} = (\text{Pr}/\text{Ek})t$. We can thus find the new equations easily from Equations (7.1)–(7.5). Every place we see a time dimension, we recall $t = (\text{Ek}/\text{Pr})t_{\text{new}}$, so we multiply the place where the time-dimension appears by (Ek/Pr) and drop the "new" subscript [e.g., $t \to (\text{Ek}/\text{Pr})\, t$, $\boldsymbol{u} \to (\text{Pr}/\text{Ek})\boldsymbol{u}$, etc.]

Note that we should now choose a different typical entropy-scale and corresponding Rayleigh number:

$$\Delta S^* := \frac{\tilde{Q}\tau}{\tilde{\rho}\tilde{T}} = \frac{\tilde{Q}}{2\Omega_0\tilde{\rho}\tilde{T}} \tag{8.1a}$$

$$\text{and} \quad \text{Ra} := \frac{\tilde{g}\tilde{Q}H^3\tau}{\tilde{\rho}\tilde{T}c_{\text{p}}\tilde{\nu}\tilde{\kappa}} = \frac{\tilde{g}\tilde{Q}H^3}{2\Omega_0\tilde{\rho}\tilde{T}c_{\text{p}}\tilde{\nu}\tilde{\kappa}}. \tag{8.1b}$$

The reasoning here is that under the influence of rapid rotation, the life-times of upflows or downflows are no longer set by the thermal dissipation time, but by the rotation period. Thus, the heating or cooling of a fluid parcel occurs on a shorter time-scale, leading to a smaller entropy difference across the shell than in the non-rotating case with the same amount of heating. Of course, neither Equations (7.7a) or (8.1a) are particularly convincing estimates and there is a large degree of uncertainty in the actual magnitude of typical entropy perturbations. We can only see how good these estimates are (after the fact) by checking if the achieved (non-dimensional) entropy difference across the shell winds up being close to unity.

We thus find (after rearranging terms),

$$\nabla \cdot [\tilde{\rho}\boldsymbol{u}] = 0, \tag{8.2}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{8.3}$$

$$\tilde{\rho}\left[\frac{D\boldsymbol{u}}{Dt} + \hat{\boldsymbol{e}}_z \times \boldsymbol{u}\right] = \text{Ra}^*\tilde{\rho}\tilde{g}S\hat{\boldsymbol{e}}_r - \tilde{\rho}\nabla\left[\frac{P}{\tilde{\rho}}\right],$$

$$+ (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \text{Ek}\nabla \cdot \boldsymbol{D}, \tag{8.4a}$$

$$\text{where} \quad D_{ij} := 2\tilde{\rho}\tilde{\nu}\left[e_{ij} - \frac{1}{3}(\nabla \cdot \boldsymbol{u})\delta_{ij}\right] \tag{8.4b}$$

$$\text{and} \quad e_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{8.4c}$$

$$\tilde{\rho}\tilde{T}\left[\frac{DS}{Dt} + \frac{\text{Bu}^*}{\text{Ra}^*}\frac{\tilde{N}^2}{\tilde{g}}u_r\right] = \frac{\text{Ek}}{\text{Pr}}\nabla \cdot [\tilde{\rho}\tilde{T}\tilde{\kappa}\nabla S]$$

$$+ Q - C + \frac{\text{DiEk}}{\text{Ra}^*}D_{ij}e_{ij} + \frac{\text{DiEk}}{\text{Pr}_{\text{m}}\text{Ra}^*}\tilde{\eta}|\nabla \times \boldsymbol{B}|^2, \tag{8.5}$$

$$\text{and} \quad \frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \left[\boldsymbol{u} \times \boldsymbol{B} - \frac{\text{Ek}}{\text{Pr}_{\text{m}}}\eta\nabla \times \boldsymbol{B}\right], \tag{8.6}$$

where we have introduced two modified non-dimensional numbers:

$$\mathrm{Ra}^* := \frac{\mathrm{Ek}^2}{\mathrm{Pr}}\mathrm{Ra} = \frac{\tilde{g}}{H\Omega_0^2}\frac{\Delta S}{c_\mathrm{p}} = \frac{\tilde{g}\tilde{Q}}{(2\Omega_0)^3\tilde{\rho}\tilde{T}c_\mathrm{p}H}, \tag{8.7a}$$

$$\text{and} \quad \mathrm{Bu}^* := \frac{\mathrm{Ek}^2}{\mathrm{Pr}}\mathrm{Bu} = \frac{\widetilde{N^2}}{4\Omega_0^2} \sim \frac{\tilde{g}}{H\Omega_0^2} = \frac{1}{\text{geometric oblateness}}. \tag{8.7b}$$

Note that although the "$d\overline{S}/dr$-terms" in the non-dimensionalizations have seemingly different definitions, they are similar, since

$$\frac{\mathrm{Bu}}{\mathrm{Ra}} \sim \frac{c_\mathrm{p}}{\Delta S} \quad \text{and} \quad \frac{\mathrm{Bu}^*}{\mathrm{Ra}^*} \sim \frac{c_\mathrm{p}}{\Delta S^*}. \tag{8.8}$$

The only difference is in the different estimates ΔS and ΔS^*.

In terms of `Rayleigh`'s f's and c's, we compare Equations (8.2)–(8.6) to (1.1)–(1.5) and find:

$$
\begin{aligned}
f_1 &\to \tilde{\rho} & c_1 &\to 1 \\
f_2 &\to \tilde{\rho}\tilde{g} & c_2 &\to \mathrm{Ra}^* \\
f_3 &\to \tilde{\nu} & c_3 &\to 1 \\
f_4 &\to \tilde{T} & c_4 &\to 4 \\
f_5 &\to \tilde{\kappa} & c_5 &\to \mathrm{Ek} \\
f_6 &\to Q - C & c_6 &\to \mathrm{Ek}/\mathrm{Pr} \\
f_7 &\to \tilde{\eta} & c_7 &\to \mathrm{Ek}/\mathrm{Pr_m} \\
&\;\;\vdots & c_8 &\to \mathrm{Di}/\mathrm{Ra}^* \\
& & c_9 &\to \mathrm{Di}/\mathrm{Ra}^* \\
f_{14} &\to \frac{\tilde{N}^2}{\tilde{g}} & c_{10} &\to 1 \\
& & c_{11} &\to \mathrm{Bu}^*/\mathrm{Ra}^*
\end{aligned}
$$

9 Hydrostatic, Ideal-Gas, Jovian Stable–Unstable Layer

To model a background CZ and WL in Jupiter, we consider a spherical shell composed of an ideal, hydrostatic gas extending between inner radius r_in and outer radius r_out. An assumed transition in convective stability occurs near an intermediate radius r_0, over width δ. More specifically, we choose quartic matching of the entropy gradient between the two layers:

$$\frac{d\tilde{S}}{dr} = \psi_\mathrm{WL}(r; r_0, \delta), \tag{9.1}$$

where

$$\psi_\mathrm{WL}(r; r_0, \delta) := \begin{cases} 0 & r \leq r_0 \\ 1 - \left[1 - \left(\frac{r-r_0}{\delta}\right)^2\right]^2 & r_0 < r < r_0 + \delta \\ 1 & r \geq r_0 + \delta. \end{cases} \tag{9.2}$$

We also define

$$\psi_{\text{CZ}}(r; r_0, \delta) := 1 - \psi_{\text{WL}}(r; r_0 - \delta, \delta) = \begin{cases} 1 & r \leq r_0 \\ 1 - \left[1 - \left(\frac{r-r_0}{\delta}\right)^2\right]^2 & r_0 - \delta < r < r_0 \\ 0 & r \geq r_0. \end{cases} \qquad (9.3)$$

ψ_{WL} thus "senses" only the WL, and ψ_{CZ} "senses" only the CZ.

With this formulation, the CZ is strictly unstable (really, marginally stable, but becomes unstable from the heating and cooling). This ensures that none of the stable gradient "leaks" into the CZ, as happens with (e.g.) tanh matching. We assume a centrally-concentrated mass so that $\tilde{g} \propto 1/r^2$.

It can then be shown that five non-dimensional parameters fully characterize the shell geometry, $\tilde{\rho}$, and $\tilde{T}$:

$$\alpha := \frac{r_{\text{out}} - r_0}{r_0 - r_{\text{in}}} \qquad \text{(WL-to-CZ aspect ratio)}, \qquad (9.4\text{a})$$

$$\beta := \frac{r_{\text{in}}}{r_0} \qquad \text{(CZ aspect ratio)}, \qquad (9.4\text{b})$$

$$\gamma := \frac{c_{\text{p}}}{c_{\text{v}}} \qquad \text{(specific-heat ratio)}, \qquad (9.4\text{c})$$

$$\delta \qquad \text{(stability transition width)}, \qquad (9.4\text{d})$$

$$N_\rho := \ln\left[\frac{\tilde{\rho}(r_0)}{\tilde{\rho}(r_{\text{out}})}\right] \qquad \text{(number of density scale-heights across CZ)}, \qquad (9.4\text{e})$$

where c_{v} is the specific heat at constant volume.

We choose H to be the thickness of the CZ ($r_0 - r_{\text{in}} = 1$ and $r_{\text{out}} - r_0 = \alpha$). Thus,

$$r_{\text{in}} = \frac{1 - \beta}{\beta}, \qquad (9.5\text{a})$$

$$r_0 = \frac{1}{1 - \beta}, \qquad (9.5\text{b})$$

$$\text{and} \qquad r_{\text{out}} = \frac{1}{1 - \beta} + \alpha \qquad (9.5\text{c})$$

If we choose $\alpha = 0.25$ and $\beta = 0.9$, then $(r_{\text{in}}, r_0, r_{\text{out}}) = (9, 10, 10.25)$.

With the requirement that the volume-average of $\tilde{g}$ over the CZ be unity, we require

$$\tilde{g} = \left[\frac{1 - \beta^3}{3(1 - \beta)^3}\right]\frac{1}{r^2}. \qquad (9.6)$$

It can then be shown from the ideal-gas and hydrostatic conditions (e.g., Matilsky et al. 2023) that

$$\tilde{T} = e^{\tilde{S}}\left[\tilde{T}(r_0) - \text{Di}\int_{r_0}^{r} \tilde{g}(x)e^{-\tilde{S}(x)}dx\right]. \qquad (9.7)$$

and

$$\tilde{\rho} = \tilde{\rho}(r_0) \exp\left[-\left(\frac{\gamma}{\gamma - 1}\right)\tilde{S}\right]\tilde{T}^{1/(\gamma-1)}, \tag{9.8}$$

where (nastily)

$$\mathrm{Di} := \frac{\tilde{g}H}{c_\mathrm{p}\tilde{T}} = \frac{3\beta(1-\beta)^2(1-e^{-N_\rho/n})}{(3\beta/2)(1-\beta^2)(1-e^{-N_\rho/n}) - (1-\beta^3)(\beta - e^{-N_\rho/n})}, \tag{9.9}$$

$$\tilde{T}(r_0) = \frac{(1-\beta^3)(1-\beta)}{(3\beta/2)(1-\beta^2)(e^{N_\rho/n} - 1) - (1-\beta^3)(\beta e^{N_\rho/n} - 1)}, \tag{9.10}$$

$$\text{and} \quad n := \frac{1}{1-\gamma} \tag{9.11}$$

is the polytropic index of the CZ (note that since $d\overline{S}/dr \equiv 0$ in the CZ, the stratification of the CZ winds up being an adiabatic polytrope). We have also assumed (without loss of generality) that $\tilde{S}(r_0) = 0$, so that $\tilde{S} \equiv 0$ in the CZ). The constant $\tilde{\rho}(r_0)$ ends up not having an analytical expression but can be easily found by integration over the CZ of Equation (9.8) (and I should maybe just find Di and $\tilde{T}(r_0)$ from integration, too, since I am error-prone).

To follow Jones et al. (2011) Heimpel et al. (2022) somewhat, we assume $n = 2$. Apparently this is a common choice for Jupiter that better approximates its weird equation of state without breaking the ideal gas law. It results in the somewhat strange result $\gamma = 3/2$.

We also simplify our lives and assume constant diffusivities:

$$\tilde{\nu} = \tilde{\kappa} = \tilde{\eta} \equiv 1. \tag{9.12}$$

For the heating, we use the typical `Rayleigh` profile $Q \propto \tilde{\rho}\tilde{T}$, but ensure that the heating is fully contained in the CZ:

$$Q \propto \tilde{\rho}\tilde{T}\psi_\mathrm{CZ}(r; r_0, \delta) \tag{9.13}$$

For the cooling, we choose a cooling width δ_cool:

$$C \propto \exp\left[\frac{r - r_0}{\delta_\mathrm{cool}}\right]\psi_\mathrm{CZ}(r; r_0, \delta) \tag{9.14}$$

We thus fully define the following geometry and Jovian-ish reference state:

$$\alpha = 0.25, \tag{9.15a}$$

$$\beta = 0.9, \tag{9.15b}$$

$$\gamma = 3/2, \tag{9.15c}$$

$$\delta = \delta_\mathrm{cool} = 0.1 \quad \text{(may mess with these)}, \tag{9.15d}$$

$$\text{and} \quad N_\rho = 3. \tag{9.15e}$$

References

Brown, B. P., Vasil, G. M., & Zweibel, E. G. 2012, The Astrophysical Journal, 756, 109, doi: 10.1088/0004-637x/756/2/109

Christensen-Dalsgaard, J., Däppen, W., Ajukov, S. V., et al. 1996, Science, 272, 1286, doi: 10.1126/science.272.5266.1286

Clune, T., Elliott, J., Miesch, M., Toomre, J., & Glatzmaier, G. 1999, Parallel Comp., 25, 361, doi: 10.1016/s0167-8191(99)00009-5

Cope, L., Garaud, P., & Caulfield, C. P. 2020, Journal of Fluid Mechanics, 903, A1, doi: 10.1017/jfm.2020.600

Featherstone, N. A., & Hindman, B. W. 2016, The Astrophysical Journal, 818, 32, doi: 10.3847/0004-637x/818/1/32

Garaud, P. 2020, The Astrophysical Journal, 901, 146, doi: 10.3847/1538-4357/ab9c99

Gilman, P. A., & Glatzmaier, G. A. 1981, The Astrophysical Journal Supplement Series, 45, 335, doi: 10.1086/190714

Gough, D. O. 1969, J. Atmos. Soc., 26, 448, doi: 10.1175/1520-0469(1969)026<0448:taaftc>2.0.co;2

Guillot, T. 2005, Annual Review of Earth and Planetary Sciences, 33, 493–530, doi: 10.1146/annurev.earth.32.101802.120325

Heimpel, M. H., Yadav, R. K., Featherstone, N. A., & Aurnou, J. M. 2022, Icarus, 379, 114942, doi: 10.1016/j.icarus.2022.114942

Jones, C., Boronski, P., Brun, A., et al. 2011, Icar., 216, 120, doi: 10.1016/j.icarus.2011.08.014

Matilsky, L. I., & Brummell, N. H. 2024, Journal of Fluid Mechanics

Matilsky, L. I., Brummell, N. H., Hindman, B. W., & Toomre, J. 2023, The Astrophysical Journal, doi: 10.48550/ARXIV.2311.10202

—. 2024, The Astrophysical Journal, 962, 189, doi: 10.3847/1538-4357/ad18b2

Ogura, Y., & Phillips, N. A. 1962, Journal of the Atmospheric Sciences, 19, 173, doi: 10.1175/1520-0469(1962)019<0173:saodas>2.0.co;2

Vasil, G. M., Lecoanet, D., Brown, B. P., Wood, T. S., & Zweibel, E. G. 2013, The Astrophysical Journal, 773, 169, doi: 10.1088/0004-637x/773/2/169