

# Non-Dimensionalization of an Anelastic Stable–Unstable Layer in **Rayleigh**

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## 1 General Equations Solved in **Rayleigh**

In general (with rotation and magnetism), **Rayleigh** evolves in time a set of coupled PDEs for the 3D vector velocity  $\mathbf{u}$ , vector magnetic field  $\mathbf{B}$ , pressure perturbation  $P$  (perturbation away from the “reference” or “background” state), and entropy perturbation  $S$ . Note that  $S$  can also be interpreted as a temperature perturbation in Boussinesq mode. For more details, see **Rayleigh**’s [Documentation](#).

We use standard spherical coordinates  $(r, \theta, \phi)$  and cylindrical coordinates  $(\lambda, \phi, z) = (r \sin \theta, \phi, r \cos \theta)$ , and  $\hat{\mathbf{e}}_q$  in general denotes a position-dependent unit vector in the direction of increasing  $q$ . The full PDE-set is then:

$$\nabla \cdot [f_1(r)\mathbf{u}] = 0, \quad (1.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (1.2)$$

$$f_1(r) \left[ \frac{D\mathbf{u}}{Dt} + c_1 \hat{\mathbf{e}}_z \times \mathbf{u} \right] = c_2 f_2(r) S \hat{\mathbf{e}}_r - c_3 f_1(r) \nabla \left[ \frac{P}{f_1(r)} \right],$$

$$+ c_4 (\nabla \times \mathbf{B}) \times \mathbf{B} + c_5 \nabla \cdot \mathbf{D}, \quad (1.3a)$$

$$\text{where} \quad D_{ij} := 2f_1(r)f_3(r) \left[ e_{ij} - \frac{1}{3}(\nabla \cdot \mathbf{u})\delta_{ij} \right] \quad (1.3b)$$

$$\text{and} \quad e_{ij} := \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (1.3c)$$

$$f_1(r)f_4(r) \frac{DS}{Dt} = -f_1(r)f_4(r)f_{14}(r)u_r + c_6 \nabla \cdot [f_1(r)f_4(r)f_5(r)\nabla S]$$

$$+ c_6 f_{10}(r) + c_8 c_5 D_{ij} e_{ij} + \frac{\eta(r)}{4\pi} |\nabla \times \mathbf{B}|^2, \quad (1.4)$$

$$\text{and} \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - c_7 \nabla \times [f_7(r)\nabla \times \mathbf{B}], \quad (1.5)$$

where  $D/Dt := \partial/\partial t + \mathbf{u} \cdot \nabla$  denotes the material derivative. The spherically-symmetric, time-independent reference (or background) functions  $f_i(r)$  and constants  $c_j$  set the fluid approximation to be made. **Rayleigh** has built-in modes to set the  $f$ 's and  $c$ 's for single-layer (i.e., either convectively stable or unstable, but not both) Boussinesq or Anelastic spherical shells. More complex systems (coupled stable-unstable systems or alternative non-dimensionalizations) require the user to manually change the  $f$ 's and  $c$ 's. This can be done by editing an input binary file that **Rayleigh** reads upon initialization. The  $c$ 's can also be changed in the ASCII text-file (i.e., the `main_input` file).

## 2 Dimensional Anelastic Equations

We begin by writing down the full dimensional anelastic fluid equations, as they are usually implemented in **Rayleigh** (`reference_type = 2`). This form of the anelastic approximation in a spherical shell is derived in, or more accurately, attributed to (since **Rayleigh** “updates” the background state slightly differently than the cluge-y **ASH** implementation), two common sources: [Gilman & Glatzmaier \(1981\)](#) and [Clune et al. \(1999\)](#). **Rayleigh**'s dimensional anelastic equation-set is:

$$\nabla \cdot [\bar{\rho}(r)\mathbf{u}] = 0, \quad (2.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (2.2)$$

$$\begin{aligned} \bar{\rho}(r) \left[ \frac{D\mathbf{u}}{Dt} + 2\Omega_0 \hat{\mathbf{e}}_z \times \mathbf{u} \right] &= \left[ \frac{\bar{\rho}(r)\bar{g}(r)}{c_p} \right] S \hat{\mathbf{e}}_r - \bar{\rho}(r) \nabla \left[ \frac{P}{\bar{\rho}(r)} \right], \\ &+ \frac{1}{\mu} (\nabla \times \mathbf{B}) \times \mathbf{B} + \nabla \cdot \mathbf{D}, \end{aligned} \quad (2.3a)$$

$$\text{where} \quad D_{ij} := 2\bar{\rho}(r)\bar{\nu}(r) \left[ e_{ij} - \frac{1}{3}(\nabla \cdot \mathbf{u})\delta_{ij} \right] \quad (2.3b)$$

$$\text{and} \quad e_{ij} := \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (2.3c)$$

$$\begin{aligned} \bar{\rho}(r)\bar{T}(r) \frac{DS}{Dt} &= -\bar{\rho}(r)\bar{T}(r) \frac{d\bar{S}}{dr} u_r + \nabla \cdot [\bar{\rho}(r)\bar{T}(r)\bar{\kappa}(r)\nabla S] \\ &+ \bar{Q}(r) + D_{ij}e_{ij} + \frac{\bar{\eta}(r)}{\mu} |\nabla \times \mathbf{B}|^2, \end{aligned} \quad (2.4)$$

$$\text{and} \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times [\eta(r)\nabla \times \mathbf{B}]. \quad (2.5)$$

Here, the thermal variables  $\rho$ ,  $T$ ,  $P$ , and  $S$  refer to the density, temperature, pressure, and entropy (respectively). The overbars denote the spherically-symmetric, time-independent background state. The lack of an overbar on a thermal variable indicates the (assumed small) perturbation from the background (for the entropy,  $S/c_p$  is assumed small).

Other background quantities that appear are the gravity  $\bar{g}(r)$ , the momentum, thermal, and magnetic diffusivities  $[\bar{\nu}(r), \bar{\kappa}(r), \text{ and } \bar{\eta}(r)]$ , respectively, the internal heating  $\bar{Q}(r)$ , the frame rotation rate  $\Omega_0$ , the specific heat at constant pressure  $c_p$ , and the vacuum permeability  $\mu$  ( $= 4\pi$  in c.g.s. units). The equations are written in a frame rotating with angular velocity  $\Omega_0$  and the centrifugal force is neglected.

Note that the internal heating function  $\bar{Q}(r)$  is also assumed spherically symmetric and fixed in time, but should be interpreted as  $-\nabla \cdot \mathcal{F}_{\text{rad}}$ , where  $\mathcal{F}_{\text{rad}}$  is the radiative heat flux and properly should be proportional to the gradient of the total (background + perturbed) temperature. If the system is a convection zone, it is driven by a combination of internal heating and the thermal boundary conditions (which are conditions on  $S$ ), that together ensure the imposed energy flux is transported throughout the layer in a steady state.

Finally, we recall the relation

$$\frac{d\bar{S}}{dr} = c_p \frac{nsqref(r)}{\bar{g}(r)}, \quad (2.6)$$

where  $\bar{N}^2(r)$  is the squared buoyancy frequency, which we use in favor of  $d\bar{S}/dr$  in subsequent equations.

Note that the original equations in [Gilman & Glatzmaier \(1981\)](#) and [Clune et al. \(1999\)](#) were derived assuming a nearly-adiabatic background state (i.e.,  $d\bar{S}/dr \approx 0$ ). [Brown et al. \(2012\)](#) and [Vasil et al. \(2013\)](#) have raised concerns about using various anelastic approximations in stable layers due to non-energy-conserving gravity waves. Should we be concerned?

### 3 Non-Dimensional Scheme

We now non-dimensionalize Equations (2.1)–(2.5), according to the following scheme:

$$\nabla \rightarrow \frac{1}{H} \nabla, \quad (3.1a)$$

$$t \rightarrow \tau t, \quad (3.1b)$$

$$\mathbf{u} \rightarrow \frac{H}{\tau} \mathbf{u}, \quad (3.1c)$$

$$S \rightarrow \sigma S, \quad (3.1d)$$

$$P \rightarrow \tilde{\rho} \frac{H^2}{\tau^2} P, \quad (3.1e)$$

$$\mathbf{B} \rightarrow (4\pi\tilde{\rho})^{1/2} \frac{H}{\tau} \mathbf{B}, \quad (3.1f)$$

$$\bar{\rho}(r) \rightarrow \tilde{\rho} \bar{\rho}(r), \quad (3.1g)$$

$$\bar{T}(r) \rightarrow \tilde{T} \bar{T}(r), \quad (3.1h)$$

$$\bar{g}(r) \rightarrow \tilde{g} \bar{g}(r), \quad (3.1i)$$

$$\overline{N^2}(r) \rightarrow \widetilde{N^2} \overline{N^2}(r), \quad (3.1j)$$

$$\bar{\nu}(r) \rightarrow \tilde{\nu} \bar{\nu}(r), \quad (3.1k)$$

$$\bar{\kappa}(r) \rightarrow \tilde{\kappa} \bar{\kappa}(r), \quad (3.1l)$$

$$\text{and } \bar{\eta}(r) \rightarrow \tilde{\eta} \bar{\eta}(r). \quad (3.1m)$$

Here,  $H$  is a typical length-scale,  $\tau$  is a typical time-scale, and  $\sigma$  is a typical entropy scale. On the right-hand-sides of Equations (3.1a)–(3.1f) and in the following non-dimensionalizations, all fluid variables, coordinates, and background-state quantities are understood to be non-dimensional. The tildes refer to “typical values” of the (dimensional) reference-state functions.

Below, we will assume the time-scale is either a viscous diffusion time (i.e.,  $\tau = H^2/\tilde{\nu}$ ) or a rotational time-scale [i.e.,  $\tau = (2\Omega_0)^{-1}$ ].

### 4 Non-Dimensional Equations; $\tau = L^2/\tilde{\nu}$

In this case, Equations (2.1)–(2.5) become

$$\nabla \cdot [\bar{\rho}(r) \mathbf{u}] = 0, \quad (4.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (4.2)$$

$$\begin{aligned} \bar{\rho}(r) \left[ \frac{D\mathbf{u}}{Dt} + \frac{1}{\text{Ek}} \hat{\mathbf{e}}_z \times \mathbf{u} \right] = & -\bar{\rho}(r) \nabla \left[ \frac{P}{\bar{\rho}(r)} \right] + \frac{\text{Ra}}{\text{Pr}} \bar{\rho}(r) \bar{g}(r) S \hat{\mathbf{e}}_r, \\ & + \nabla \cdot \mathbf{D} + (\nabla \times \mathbf{B}) \times \mathbf{B}, \end{aligned} \quad (4.3a)$$

$$\text{where} \quad D_{ij} := 2\bar{\rho}(r) \bar{\nu}(r) \left[ e_{ij} - \frac{1}{3} (\nabla \cdot \mathbf{u}) \delta_{ij} \right] \quad (4.3b)$$

$$\text{and} \quad e_{ij} := \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (4.3c)$$

$$\begin{aligned} \bar{\rho}(r) \bar{T}(r) \frac{DS}{Dt} = & -\frac{\text{Pr}}{\text{Ra}} \text{Bu}_{\text{visc}} \bar{\rho}(r) \bar{T}(r) \frac{\widehat{N^2}(r)}{\hat{g}(r)} u_r + \frac{1}{\text{Pr}} \nabla \cdot [\bar{\rho}(r) \bar{T}(r) \bar{\kappa}(r) \nabla S] \\ & + \frac{1}{\text{Pr}} \bar{Q}(r) + \frac{\text{PrDi}}{\text{Ra}} D_{ij} e_{ij} + \frac{\text{PrDi}}{\text{Pr}_m \text{Ra}} \bar{\eta}(r) |\nabla \times \mathbf{B}|^2, \end{aligned} \quad (4.4)$$

$$\text{and} \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{1}{\text{Pr}_m} \nabla \times [\eta(r) \nabla \times \mathbf{B}]. \quad (4.5)$$

The non-dimensional numbers appearing are:

$$\text{Ra} := \frac{\tilde{g} H^3}{\tilde{\nu} \tilde{\kappa}} \frac{\sigma}{c_p}, \quad (4.6a)$$

$$\text{Pr} := \frac{\tilde{\nu}}{\tilde{\kappa}}, \quad (4.6b)$$

$$\text{Pr}_m := \frac{\tilde{\nu}}{\tilde{\eta}}, \quad (4.6c)$$

$$\text{Ek} := \frac{\tilde{\nu}}{2\Omega_0 H^2}, \quad (4.6d)$$

$$\text{Bu}_{\text{visc}} := \frac{\widehat{N^2} H^4}{\tilde{\nu}^2}, \quad (4.6e)$$

$$\text{and} \quad \text{Di} = \frac{\tilde{g} H}{c_p \tilde{T}}, \quad (4.6f)$$

along with the non-dimensional heating function

$$\bar{Q}(r) := \frac{L^2}{\tilde{\rho} \tilde{T} \tilde{\kappa} \sigma} \bar{Q}_{\text{dim}}(r) \quad (4.7)$$

Note that the dissipation number is not an independent control parameter, but a function of the non-dimensional parameters characterizing the polytrope.

In general,  $\bar{Q}(r)$  is simply an arbitrary—hopefully order unity—function. Assuming the thermal boundary conditions remove whatever  $\bar{Q}(r)$  dumps in: If  $\bar{Q}(r) \gg 1$ , the user is dilating their Rayleigh number without saying so. If  $\bar{Q} \ll 1$ , the user is contracting their Rayleigh number without saying so. If  $\bar{Q}(r) \equiv 0$  (and the user wants to simulate a convection

zone), the user should typically identify  $\sigma$  with  $-\Delta S = S_{\text{in}} - S_{\text{out}}$  (the imposed entropy drop across the layer) and thus set  $S \equiv 1$  at  $r = r_{\text{in}}$  and  $S \equiv 0$  at  $r = r_{\text{out}}$ .

The non-dimensional heating takes a specific form if we assume the Rayleigh number is a “flux” Rayleigh number. In that case we identify

$$\sigma = \frac{L \langle \mathcal{F}_{\text{nr}}(r) \rangle_{\text{v}}}{\tilde{\rho} \tilde{T} \tilde{\kappa}}, \quad (4.8)$$

$$\text{where} \quad \mathcal{F}_{\text{nr}}(r) := \frac{1}{r^2} \int_{r_{\text{in}}}^r Q(x) x^2 dx \quad (4.9)$$

is the flux not carried by radiation in a statistically steady state and  $\langle \cdots \rangle_{\text{v}}$  refers to a volume average over the whole shell. We thus have

$$\overline{Q}(r) = \frac{L}{\langle \mathcal{F}_{\text{nr}}(r) \rangle_{\text{v}}} Q(r) \quad (4.10)$$

and whatever amplitude (luminosity) the user chooses for the dimensional  $Q(r)$ , the ultimate  $\overline{Q}(r)$  will normalize that amplitude away in Equation (4.10).

The viscous buoyancy number  $\text{Bu}_{\text{visc}}$  is the ratio of the typical squared buoyancy frequency to the squared viscous diffusion time (it is essentially a kind of Richardson number). Although it has to do with background entropy stratification,  $\text{Bu}_{\text{visc}}$  is nominally independent of the Rayleigh number (which derives from the entropy perturbations associated with the thermal boundary conditions and/or heating that force energy through the layer). However, the following notes are warranted:

(1) In the typical convection problem (polytropic index  $n = n_{\text{ad}} := 1/(\gamma - 1)$ ),  $d\overline{S}/dr \equiv 0$  and the value of  $\text{Bu}_{\text{visc}}$  is irrelevant.

(2) For an isolated stable polytrope ( $n > n_{\text{ad}}$ ), it is unclear how the typical entropy perturbation  $\sigma$  is established. Even if energy is driven through the system by the thermal boundary conditions and/or heating, the energy will likely be carried by spherically symmetric conduction (depending on how stable the stratification is) and  $\sigma$  may not be set by the boundary conditions. Furthermore,  $\sigma$  cannot be set by the total entropy contrast across the layer (which is  $\sim c_{\text{p}}$ , and if  $\sigma = c_{\text{p}}$ , we would have  $\text{Bu}_{\text{visc}} \text{Pr} = \text{Ra}$ ). That is because no plume can traverse a large portion of the stably stratified layer. Thus, although the user can choose a “Rayleigh number” (probably a misnomer) for the isolated stable layer, they have no way of implementing the background state, boundary conditions, or heating profile to be consistent with their choice.

(3) For unstable polytropes ( $n < n_{\text{ad}}$ ), we first note that if  $n$  differs by a factor of unity from  $n_{\text{ad}}$ , order-unity thermal perturbations  $S/c_{\text{p}}$  are forced and the anelastic approximation is invalid. In any case (either if the simulation survives an order-unity  $n_{\text{ad}} - n$ , or if the user chooses a well-posed small  $n_{\text{ad}} - n$  to help drive the convection), we expect that the convection will restratify the system toward adiabaticity. In Equation (2.4), a background  $(dS/dr)_{\ell=0}$  will be established to be close to the negative of  $d\overline{S}/dr$  and (similar to point (1)), the value of  $\text{Bu}_{\text{visc}}$  will not truly be an independent parameter, but will help to determine an ultimate “effective” Rayleigh number.

(4) In the more logical case of a stable layer being pummeled by a neighboring overshooting convection layer, both  $\text{Bu}_{\text{visc}}$  and  $\text{Ra}$  are truly independent (and relevant). The

overshooting flows (driven by  $Ra$ , which is a property of the convection zone) establish the typical  $\sigma$  in both the convection zone and overshoot layer. Meanwhile,  $Bu_{\text{visc}}$  (which is a property of the stable layer) controls how strongly the overshoot is decelerated.

## 5 Non-Dimensional Equations; $\tau = \Omega_0^{-1}$

In the previous section,  $t$  (and things with time in the dimensions) was implied to mean  $(\tilde{\nu}/H^2)t_{\text{dim}}$ , where  $t_{\text{dim}}$  was the dimensional time. We now want to use  $t_{\text{new}} = \Omega_0 t_{\text{dim}} = t/\text{Ek}$ . We can thus find the new equations easily from Equations (4.1)–(4.5): Every place we see a time dimension, we recall  $t = \text{Ek} t_{\text{new}}$ , so we multiply the place where the dimension appears by  $\text{Ek}$  and drop the “new” subscript (e.g.,  $t \rightarrow \text{Ek } t$ ,  $\mathbf{u} \rightarrow \mathbf{u}/\text{Ek}$ , etc.). We thus find (after rearranging terms)

$$\nabla \cdot [\bar{\rho}(r)\mathbf{u}] = 0, \quad (5.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (5.2)$$

$$\begin{aligned} \bar{\rho}(r) \left[ \frac{D\mathbf{u}}{Dt} + 2\hat{\mathbf{e}}_z \times \mathbf{u} \right] = & -\bar{\rho}(r) \nabla \left[ \frac{P}{\bar{\rho}(r)} \right] + \text{Ra}^* \bar{\rho}(r) \bar{g}(r) S \hat{\mathbf{e}}_r, \\ & + \text{Ek} \nabla \cdot \mathbf{D} + (\nabla \times \mathbf{B}) \times \mathbf{B}, \end{aligned} \quad (5.3a)$$

$$\text{where} \quad D_{ij} := 2\bar{\rho}(r)\bar{\nu}(r) \left[ e_{ij} - \frac{1}{3}(\nabla \cdot \mathbf{u})\delta_{ij} \right] \quad (5.3b)$$

$$\text{and} \quad e_{ij} := \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (5.3c)$$

$$\begin{aligned} \bar{\rho}(r)\bar{T}(r) \frac{DS}{Dt} = & -\frac{Bu_{\text{rot}}}{\text{Ra}^*} \bar{\rho}(r)\bar{T}(r) \frac{\widehat{N}^2(r)}{\hat{g}(r)} u_r + \frac{\text{Ek}}{\text{Pr}} \nabla \cdot [\bar{\rho}(r)\bar{T}(r)\bar{\kappa}(r)\nabla S] \\ & + \frac{\text{Ek}}{\text{Pr}} \bar{Q}(r) + \frac{\text{DiEk}}{\text{Ra}^*} D_{ij} e_{ij} + \frac{\text{DiEk}}{\text{Pr}_m \text{Ra}^*} \bar{\eta}(r) |\nabla \times \mathbf{B}|^2, \end{aligned} \quad (5.4)$$

$$\text{and} \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{\text{Ek}}{\text{Pr}_m} \nabla \times [\eta(r) \nabla \times \mathbf{B}]. \quad (5.5)$$

The new non-dimensional numbers appearing are:

$$\text{Ra}^* := \frac{\text{Ek}^2}{\text{Pr}} \text{Ra} = \frac{\tilde{g}}{L\Omega_0^2} \frac{\sigma}{c_p}, \quad (5.6a)$$

$$\text{and} \quad Bu_{\text{rot}} := \text{Ek}^2 Bu_{\text{visc}} = \frac{\widehat{N}^2}{\Omega_0^2} \sim \frac{\tilde{g}}{L\Omega_0^2} = \frac{1}{\text{oblateness factor}}. \quad (5.6b)$$

Note that although the “ $d\bar{S}/dr$ -terms” in the non-dimensionalizations have seemingly different definitions, they are the same, since:

$$\frac{\text{Pr}}{\text{Ra}} Bu_{\text{visc}} = \frac{Bu_{\text{rot}}}{\text{Ra}^*} \sim \frac{c_p}{\sigma}. \quad (5.6c)$$

## References

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