

# An energy-conserving anelastic approximation for strongly stably-stratified fluids

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## 1 Introduction

Abstract: When acoustic oscillations are believed to be irrelevant to the dynamics of a fluid, it is useful to employ simplifying approximations to the equations of motion. The two most common of these (which are usually used to treat convection problems) are the Boussinesq approximation (when the background density does not significantly vary across the fluid layer) and the anelastic approximation (when the background density does vary significantly). There are many distinct forms of the anelastic approximation in the literature, and it has often been remarked that they do not properly conserve energy when the fluid is stable to convection. Here we show that the anelastic equations derived by Gough (1969) in fact do conserve energy for arbitrary motions of the fluid, even for strongly stratified background stratification. The key properties of these equations that allow them to conserve energy are (1) the absence of the Lantz-Braginsky-Roberts (LBR) approximation in the momentum equation and (2) the inclusion of a historically neglected term in the internal energy equation. These two properties allow the proper conversion between kinetic and internal energy at the correct order of the formal asymptotic expansion of the equations. We show that the scaling analysis of Gough (1969), which implicitly assumed a single typical value of the background entropy gradient, can be valid even for convective overshoot, where the entropy gradient changes from slightly unstable in the convecting region to stable (sometimes strongly so) in the overshoot region. The requirement for the anelastic equations to be valid for convective overshoot is that the buoyancy frequency be significantly less than the acoustic cutoff frequency.

The anelastic equations consist of an approximation to the continuity and momentum equations, originally derived by assuming small thermal perturbations about a nearly adiabatically stratified hydrostatic reference atmosphere (Batchelor, 1953; Charney & Ogura, 1960). The thermodynamics of the problem thus become “linear,” in the sense that products of thermodynamic variables reduce to linear expressions in the first-order perturbations. The two key consequences of linearized thermodynamics are divergenceless mass flux and the first-order buoyancy force (associated with the first-order perturbed density and pressure) being the primary driver of the flow. Ogura & Phillips (1962) formalized the approximation by expanding the fluid equations in a small parameter  $\epsilon$ , representing the relative variation

of potential temperature across the fluid layer, and hence the relative magnitude of the thermal perturbations. They recovered the equations of [Batchelor \(1953\)](#) and [Charney & Ogura \(1960\)](#) and showed an assumption about the *time scale* of the motion was necessary, in addition to the assumption of small thermal perturbations. Namely, the dynamical time scale of the buoyantly driven flows must be  $O(\epsilon^{-1/2})$  times *larger* than the sound crossing time of the region. Sound waves, which imply rapid temporal variations on the order of the sound crossing time, are thus absent from the anelastic equations, making them ideal for numerical integration, where large time steps are required to capture significant evolution of the system.

In the original asymptotic expansion of [Ogura & Phillips \(1962\)](#), the internal energy equation was replaced by a heat (or entropy) equation for the evolution of potential temperature, *before* non-dimensionalizing the equations. The approach of considering the entropy equation instead of the energy equation before nondimensionalization is repeated in all modern implementations of the anelastic approximation that we are aware of (e.g., [Gilman & Glatzmaier 1981](#); [Lipps & Hemler 1982](#); [Glatzmaier 1984](#); [Lantz 1992](#); [Braginsky & Roberts 1995](#); [Lantz & Fan 1999](#); [Clune et al. 1999](#); [Rogers & Glatzmaier 2005](#); [Brown et al. 2012](#); [Vasil et al. 2013](#); [Wilczyński et al. 2022](#)). The resulting energy equation is also used in all numerical codes we are aware of that utilize the anelastic equations, for example, the ASH code ([Brun et al., 2004](#)), the MagIC code ([Gastine & Wicht, 2012](#)), the Rayleigh code ([Featherstone & Hindman, 2016](#); [Featherstone et al., 2023](#)), the EULAG code ([Smolarkiewicz & Prusa, 2004](#)), and the Dedalus code ([Burns et al., 2020](#); [Brown et al., 2020](#)).

While nondimensionalizing the entropy equation instead of the internal energy equation may at first appear to be an arbitrary (and harmless) choice, we show in the present work that it leads to an asymptotically inconsistent set of equations that do not conserve energy when the background is stably stratified. [Gough \(1969\)](#), by contrast, took a different approach than [Ogura & Phillips \(1962\)](#) and performed a formal asymptotic expansion in  $\epsilon$  after nondimensionalizing the internal energy equation. We show that this equation set, which we dub the “Energy-conserving Generalized Gough” (AnEGG) anelastic equations, conserve energy for arbitrary fluid motions and for all hydrostatic background states (whether stably or unstably stratified).

## 2 Energy conservation under the fully compressible fluid equations

We begin by writing down the unapproximated fully compressible equations of motion for a nonrotating nonmagnetic fluid. These are [Gough \(1969\)](#)’s Equations (2.1)–(2.5), consisting of the continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u}) \quad (1)$$

the momentum equation,

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) = -\nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \nabla P + \rho \mathbf{g} + \nabla \cdot \overleftrightarrow{D}, \quad (2a)$$

$$\text{where} \quad D_{ij} = \mu \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}(\nabla \cdot \mathbf{u})\delta_{ij} \right], \quad (2b)$$

the internal energy equation,

$$\frac{\partial}{\partial t}(\rho U) + \nabla \cdot (\rho U \mathbf{u}) + P \nabla \cdot \mathbf{u} = D_{ij} \frac{\partial u_i}{\partial x_j} + Q - \nabla \cdot \mathbf{F}, \quad (3)$$

and a general equation of state,

$$U = U(P, T). \quad (4)$$

Here,  $t$  is the time, the  $x_i$  are Cartesian spatial coordinates,  $\rho$  is the density,  $P$  the pressure,  $T$  the temperature,  $U$  the internal energy per unit mass,  $\mu$  the dynamic viscosity,  $\mathbf{g} := -\nabla \Phi$  the gravitational acceleration field,  $\Phi$  the gravitational potential,  $Q$  an internal heat source, and  $\mathbf{F}$  the combined conductive and radiative heat flux. The subscripts  $i$  and  $j$  (taking on the values 1, 2, and 3) denote vector or tensor components in any of the Cartesian coordinate directions. The gravity  $\mathbf{g}$  is assumed to point in the vertical direction  $\hat{\mathbf{k}}$  (either the upward Cartesian direction for a plane-parallel fluid layer or the radial direction for spherical shell). Additionally,  $\mathbf{g}$  is assumed to depend only on the vertical coordinate  $q$  (either the upward Cartesian coordinate  $x_3$  or the radial coordinate  $r$ ) and to be time-independent (meaning that, among other consequences, self-gravity is ignored). The symbol “ $\leftrightarrow$ ” in the viscous stress tensor  $\overleftrightarrow{D}$  denotes a second-order tensor, as does the dyadic notation  $\mathbf{u} \mathbf{u}$ . We use the Einstein summation convention and  $\delta_{ij}$  denotes the Kronecker delta. These equations are not written in the exact form of Gough (1969) and use slightly different notation but are mathematically equivalent.

An equation for the evolution of kinetic energy can be formed from  $\mathbf{u}$  dotted into Equation (2),

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho u^2 \right) = -\nabla \cdot \left( \frac{1}{2} \rho u^2 \mathbf{u} \right) + \mathbf{u} \cdot \nabla P - \rho \mathbf{u} \cdot \nabla \Phi + u_i \frac{\partial D_{ij}}{\partial x_j}. \quad (5)$$

Equation (1) multiplied by  $\Phi$  yields an equation for the evolution of potential energy,

$$\frac{\partial}{\partial t}(\rho \Phi) = -\Phi \nabla \cdot (\rho \mathbf{u}). \quad (6)$$

Adding Equations (3), (5), and (6) yields an equation for the evolution of total energy,

$$\frac{\partial}{\partial t} \left[ \rho \left( \frac{1}{2} u^2 + U + \Phi \right) \right] = -\nabla \cdot \left\{ \left[ \rho \left( \frac{1}{2} u^2 + U + \Phi \right) + P \right] \mathbf{u} - \mathbf{u} \cdot \overleftrightarrow{D} + \mathbf{F} \right\} + Q. \quad (7)$$

Integrating Equation (7) over the volume  $V$  of the fluid layer, using the divergence theorem, and assuming that the sum of the surface-integrated fluxes balances the volume-integrated

heating yields<sup>1</sup>

$$E_{\text{tot}} := \int_V W_{\text{tot}} dV \equiv \text{constant}, \quad (8a)$$

$$\text{where } W_{\text{tot}} := W_{\text{KE}} + W_{\text{int}} + W_{\text{pot}}, \quad (8b)$$

$$W_{\text{KE}} := \frac{1}{2} \rho u^2, \quad (8c)$$

$$W_{\text{int}} := \rho U, \quad (8d)$$

$$\text{and } W_{\text{pot}} := \rho \Phi. \quad (8e)$$

We additionally define

$$W_{\text{NKE}} := W_{\text{tot}} - W_{\text{KE}} = W_{\text{int}} + W_{\text{pot}} \quad (9)$$

as the combined internal and potential energy density (i.e., the “nonkinetic” energy).

Equation (8) expresses the conservation of total energy that is implicit to the fully compressible fluid equations (1)–(4). The volume-integrated total energy is conserved, as long as the total surface fluxes balance the total internal heat sources. Locally by contrast, fluid parcels can exchange total energy density with one another, and these exchanges are further partitioned into kinetic, internal, and potential energy densities.

One of the main goals of this work is to elucidate how Equation (8) is modified under the anelastic approximation for a general equation of state and nonadiabatic stratification. It will be found that *only* exchange between internal energy (due to heating) and kinetic energy is possible for an energetically consistent anelastic approximation. Conversion to and from potential energy, by contrast, is fundamentally impossible. This is essentially a new result, which has been overlooked for several decades, mainly because of the pathological coincidence that for a perfect, adiabatically stratified gas, the internal energy due to heating *looks* exactly like a potential energy.<sup>2</sup> For non-adiabatic stratification, or for an imperfect gas, the coincidence no longer holds, and the more general law for conservation of total energy that is derived here should be considered.

The other new and related result is that total energy *should* be conserved under the anelastic approximation, for arbitrary fluid motions and equations of state and for all stratifications, even for strongly subadiabatic ones. The anelastic equations of Gough (1969) satisfy such a general conservation law, while modern implementations of the anelastic approximation do not. Thus, a major part of this paper is devoted to translating Gough (1969)’s equations into more conventional notation and elucidating the origin and form of the term (which should be present in the internal energy equation for asymptotic consistency) that is missing from modern anelastic implementations.

Finally, we note that Gough (1969) assumed zero net vertical transport of mass for the “true” fully compressible fluid whose motion the anelastic equations are intended to approximate. This may at first seem like an arbitrary choice to make the small-parameter expansion

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<sup>1</sup>Throughout this work, we use the symbol “:=” for definitions (i.e., for “is defined to be equal to”) and “≡” for homogeneous equivalence (i.e., for “is equal to [some constant value] everywhere and for all time”).

<sup>2</sup>We use the nomenclature that an “ideal gas” refers to a fluid for which the internal energy depends only on the temperature (i.e.,  $U = U(T)$ ), whereas a “perfect gas” is a fluid that is not only ideal, but has specific heats that are constants, independent of temperature.

simpler. It is concluded in this work, however, that the “zero mean mass flux” assumption is in fact fundamental to ensure the energetic consistency of the anelastic approximation.

### 3 The anelastic approximation of Gough (1969)

To set the stage, we define some new thermodynamic variables, namely, the specific enthalpy,

$$h := U + \frac{P}{\rho} \quad (10)$$

and the specific entropy,

$$S = S(P, T). \quad (11)$$

Equation (11) is also an equation of state, containing the equivalent information of Equation (4).

We define several first derivatives of thermodynamic quantities derivable from the generalized equation of state (4) or (11): the specific heat at constant pressure,

$$C_p = C_p(P, T) := T \left( \frac{\partial S}{\partial T} \right)_P, \quad (12)$$

the squared adiabatic sound speed,

$$c_s^2 = c_s^2(P, T) := \left( \frac{\partial P}{\partial \rho} \right)_S, \quad (13)$$

and the thermal expansion coefficient,

$$\delta = \delta(P, T) := - \left( \frac{\partial \ln \rho}{\partial \ln T} \right)_P. \quad (14)$$

The first law of thermodynamics<sup>3</sup> then takes the following forms:

$$TdS = dU - \frac{P}{\rho^2} d\rho \quad (15a)$$

$$= dh - \frac{dP}{\rho} \quad (15b)$$

$$= C_p dT - \frac{\delta}{\rho} dP \quad (15c)$$

$$= \frac{C_p T}{\rho \delta} \left[ \frac{dP}{c_s^2} - d\rho \right]. \quad (15d)$$

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<sup>3</sup>We assume local thermodynamic equilibrium holds for the fluid.

From Equations (1) and (15), the left-hand side of Equation (3) can be written in several equivalent forms:

$$\frac{\partial}{\partial t}(\rho U) + \nabla \cdot (\rho U \mathbf{u}) + P \nabla \cdot \mathbf{u} = \rho \frac{DU}{Dt} - \frac{P}{\rho} \frac{D\rho}{Dt} \quad (16a)$$

$$= \rho \frac{Dh}{Dt} - \frac{DP}{Dt} \quad (16b)$$

$$= \rho T \frac{DS}{Dt}, \quad (16c)$$

where

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \quad (17)$$

is the material (or Lagrangian) derivative,

### 3.1 Gough (1969)’s anelastic scale analysis for convection

We will not repeat the full asymptotic expansion in  $\epsilon$  performed by Gough (1969) of Equations (1), (2), (3), and (4) here. Instead, we reiterate the salient assumptions in the case where the horizontally averaged background atmosphere is time-independent and the layer depth is thicker than the typical pressure scale height.<sup>4</sup> The main assumption is that the thermodynamic perturbations from the horizontally averaged background state are small,

$$f = \bar{f}(q, t) + f_1(x_i, t) \quad \text{with} \quad f_1/\bar{f} = O(\epsilon) \ll 1, \quad (18)$$

where  $f$  denotes either  $\rho$ ,  $P$ ,  $T$ ,  $U$ ,  $h$ ,  $C_p$ ,  $\mu$ ,  $\delta$ ,  $c_s^2$ ,  $\mathbf{F}$ , or  $Q$ . Here, the overbars denote horizontal averages (taken at a particular instant in time) and the “1” subscripts denote the perturbations about this average. Note that it is *not* correct to write “ $S = \bar{S}(q, t) + S_1(x_i, t)$  with  $S_1/\bar{S} = O(\epsilon)$ .” The fully compressible equations of motion contain only differences in entropy and so no meaningful absolute value of  $\bar{S}$  can be defined. Instead, we must write

$$S = \bar{S}(q, t) + S_1(x_i, t) \quad \text{with} \quad S_1/\bar{C}_p = O(\epsilon) \ll 1. \quad (19)$$

The second central assumption is that the coordinate system can be chosen such that there is no mass flux across any horizontal surface, i.e.,

$$\overline{\rho u_i} \equiv 0. \quad (20)$$

In a spherical system (e.g., a spherical shell), the horizontal average would be a spherically symmetric average and the coordinates would point along the spatially varying curvilinear coordinate directions.

In writing  $\bar{f} = \bar{f}(q, t)$  in Equation (18), we make a third central assumption not made by Gough (1969), namely, that the mean state may be time-dependent.<sup>5</sup> In order to still

<sup>4</sup>Gough (1969) also considers thin layers, in which the anelastic equations become the Boussinesq equations.

<sup>5</sup>Gough (1969) also considered time-dependent atmospheres, but this was in the different context of pulsations, where the atmosphere itself was moving. In the present work, we consider “Eulerian-only” time-dependence of the mean state.

be consistent with Gough (1969)’s asymptotic expansion, we must assume that the time-variation of the mean state is sufficiently slow, i.e., comparable in magnitude to the time variation of the fluctuations:

$$\left| \frac{\partial \bar{f}}{\partial t} \right| = O \left( \left| \frac{\partial f_1}{\partial t} \right| \right). \quad (21)$$

The rest of the assumptions of Gough (1969) concern scale analysis, which is appropriate for convection in the presence of mean density stratification  $\partial \bar{\rho} / \partial q \neq 0$ . The characteristic length scale of variation of the fluid is assumed to be a typical value  $H_a$  for the pressure scale height. The flow is assumed to be buoyantly driven by the  $O(\epsilon)$  thermal perturbations, i.e.,

$$|\mathbf{u}| = O(\sqrt{\epsilon g_a H_a}) = O(\sqrt{\epsilon c_{sa}}), \quad (22)$$

where “a” subscripts denote typical atmospheric background-state values. Thus, the squared Mach number of the flow is assumed to be  $O(\epsilon)$ . The characteristic time scale of variation of the fluid is assumed to be advective, i.e.,

$$\left| \frac{\partial}{\partial t} \right| = O \left( \sqrt{\frac{\epsilon g_a}{H_a}} \right) = O \left( \sqrt{\epsilon} \frac{1}{H_a / c_{sa}} \right). \quad (23)$$

Thus, the characteristic time scale of variation for the convection is  $O(\epsilon^{-1/2})$  times longer than the time it takes a sound wave to cross a pressure scale height. From Equation (21), the characteristic time scale of variation for the mean state is  $O(\epsilon^{-3/2})$  times longer than the sound crossing time.

Finally, the vertical convective heat flux, which maximally could transport an energy flux of order  $\bar{\rho} \bar{T} w \Delta \bar{S}$ , where

$$w := \hat{\mathbf{k}} \cdot \mathbf{u} \quad (24)$$

is the vertical velocity and  $\Delta \bar{S}$  is the total drop in background entropy across the convecting layer, is assumed to be limited primarily by the thermal diffusion  $\mathbf{F}$ . This will be true if the conductive heating  $-\nabla \cdot \mathbf{F}$  in Equation (3) is at least as large as the viscous and internal heatings. In the case of negligible heatings (high Rayleigh number), one expects

$$\frac{|\Delta \bar{S}|}{C_{pa}} = O(\epsilon), \quad (25)$$

i.e., the convecting layer should be nearly adiabatically stratified for vigorous convection.

One consequence of Equation (20) is that the horizontally averaged velocity  $\bar{\mathbf{u}}$  is  $O(\epsilon)$  *smaller* than the perturbed velocity  $\mathbf{u}_1$ . To see this, we write

$$\begin{aligned} 0 &= \overline{(\bar{\rho} + \rho_1)(\bar{\mathbf{u}} + \mathbf{u}_1)} = \bar{\rho} \bar{\mathbf{u}} + \overline{\rho_1 \mathbf{u}_1}, \\ \text{or} \quad |\bar{\mathbf{u}}| &= \left| -\frac{\overline{\rho_1 \mathbf{u}_1}}{\bar{\rho}} \right| = O(\epsilon |\mathbf{u}_1|). \end{aligned} \quad (26)$$

For the total mass flux (or equivalently, momentum density),  $\mathbf{m} := \rho \mathbf{u} = (\bar{\rho} + \rho_1)(\bar{\mathbf{u}} + \mathbf{u}_1)$ , we can thus write

$$\mathbf{m} = \mathbf{m} - \bar{\mathbf{m}} = \bar{\rho} \mathbf{u}_1 + \rho_1 \mathbf{u}_1 - \bar{\rho}_1 \bar{\mathbf{u}}_1 + O(\epsilon^2). \quad (27)$$

Hence, at  $O(\epsilon)$ , only the perturbation velocity  $\mathbf{u}_1$  appears in the equations, so we subsequently use  $\mathbf{u}$  as shorthand for  $\mathbf{u}_1$  and drop the subscript “1.” Under this convention,

$$\bar{\mathbf{u}} \equiv 0 \quad (28)$$

and Equation (27) becomes

$$\mathbf{m} = \bar{\rho} \mathbf{u} + \rho_1 \mathbf{u} - \bar{\rho}_1 \bar{\mathbf{u}} + O(\epsilon^2). \quad (29)$$

Each of the two terms  $\rho_1 \mathbf{u}$  and  $-\bar{\rho}_1 \bar{\mathbf{u}}$  are  $O(\epsilon)$ . In most cases, we can thus write  $\mathbf{m} \approx \bar{\rho} \mathbf{u}$  to translate from Gough (1969) to the current notation (in which we use  $\mathbf{u}$  as the primary field variable), except when multiplying by potentially zeroth-order quantities.

One other change in notation is that Gough (1969) uses the superadiabatic mean background temperature gradient

$$\begin{aligned} \beta &:= -\frac{1}{C_p} \hat{\mathbf{k}} \cdot \left[ \nabla \bar{h} - \frac{1}{\bar{\rho}} \nabla \bar{P} \right] \\ &= -\frac{\bar{T}}{C_p} \hat{\mathbf{k}} \cdot \nabla \bar{S} + O(\epsilon^2), \end{aligned} \quad (30)$$

whereas we will use  $\nabla \bar{S}$ .

### 3.2 Gough (1969)’s anelastic equations

Once all of the above scaling assumptions have been made, Equations (1), (2), (3), and (4) are nondimensionalized, each term is expanded in powers of  $\epsilon$ , terms up to zeroth-order in the continuity equation and first-order in the other equations are retained, and redimensionalization then yields the anelastic equations. Specifically, we discuss Gough (1969)’s Equations (4.3)–(4.7) and (4.15)–(4.22). We translate these equations using the change of variables outlined in Equations (29) and (30).

Under Gough (1969)’s anelastic approximation (with the additional assumption (21)), the continuity equation (1) becomes

$$\nabla \cdot (\bar{\rho} \mathbf{u}) \equiv 0, \quad (31)$$

the momentum equation (2) becomes

$$\frac{\partial}{\partial t} (\bar{\rho} \mathbf{u}) = -\nabla \cdot (\bar{\rho} \mathbf{u} \mathbf{u}) - \nabla P_1 + \rho_1 \mathbf{g} + \nabla \cdot \overleftrightarrow{D} + [-\nabla \bar{P} + \bar{\rho} \mathbf{g}], \quad (32a)$$

$$\text{where now} \quad D_{ij} = \bar{\mu} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} (\nabla \cdot \mathbf{u}) \delta_{ij} \right), \quad (32b)$$



the energy equation (3) becomes

$$\begin{aligned} \bar{\rho}\bar{C}_p \left( \frac{\partial T_1}{\partial t} + \frac{\partial \bar{T}}{\partial t} \right) - \bar{\delta} \left( \frac{\partial P_1}{\partial t} + \frac{\partial \bar{P}}{\partial t} \right) = & -\bar{\rho}\mathbf{u} \cdot \left( \nabla h_1 - \frac{1}{\bar{\rho}} \nabla P_1 \right) - \bar{\rho}\bar{T}\mathbf{u} \cdot \nabla \bar{S} \\ & D_{ij} \frac{\partial u_i}{\partial x_j} + Q_1 - \nabla \cdot \mathbf{F}_1 - \rho_1 \mathbf{u} \cdot \mathbf{g} - \bar{T}(\rho_1 \mathbf{u} - \bar{\rho}_1 \bar{\mathbf{u}}) \cdot \nabla \bar{S} \\ & + [\bar{Q} - \nabla \cdot \bar{\mathbf{F}}], \end{aligned} \quad (33)$$

and the equation of state (4) or (11) becomes linearized via the first law of thermodynamics (15), with

$$\bar{T}S_1 = \bar{C}_p T_1 - \frac{\bar{\delta}}{\bar{\rho}} P_1 \quad (34a)$$

$$= U_1 - \frac{\bar{P}}{\bar{\rho}^2} \rho_1 \quad (34b)$$

$$= h_1 - \frac{P_1}{\bar{\rho}} \quad (34c)$$

$$= \frac{\bar{C}_p \bar{T}}{\bar{\delta} \bar{\rho}} \left[ \frac{P_1}{c_s^2} - \rho_1 \right]. \quad (34d)$$

We have used Equation (29) to yield the term  $-\bar{T}(\rho_1 \mathbf{u} - \bar{\rho}_1 \bar{\mathbf{u}}) \cdot \nabla \bar{S}$  in Equation (33). Note that the continuity equation (31) is unchanged from Gough (1969)'s Equation (4.16), despite  $\partial \bar{\rho} / \partial t$  being nonzero. This is because  $\partial \bar{\rho} / \partial t$  shows up only at  $O(\epsilon)$  by Equation (21). Similarly, the momentum equation (32) takes the same form regardless of whether the mean state is time-dependent, since we can replace  $\partial \mathbf{m} / \partial t$  with  $\partial(\bar{\rho} \mathbf{u}) / \partial t$  by virtue of Equations (21) and (29). The left-hand side of the internal energy equation (33) and the right-hand side of the mean internal energy equation (37) below, where we must keep the time-variation of the mean state, is the only place our equations differ materially from those of Gough (1969), specifically, his Equations (4.17) and (4.7).

The differentials in Equation (15) can be converted into gradients (e.g.,  $T \nabla S = \nabla h - \nabla P / \rho$ ) and the horizontally averaged form of these relations yields

$$\bar{T} \nabla \bar{S} = \bar{C}_p \nabla \bar{T} - \frac{\bar{\delta}}{\bar{\rho}} \nabla \bar{P} + O(\epsilon^2) \quad (35a)$$

$$= \frac{\bar{C}_p \bar{T}}{\bar{\delta} \bar{\rho}} \left[ \frac{\nabla \bar{P}}{c_s^2} - \nabla \bar{\rho} \right] + O(\epsilon^2). \quad (35b)$$

The horizontal averages of Equations (32) and (33) satisfy

$$-\nabla \bar{P} + \bar{\rho} \mathbf{g} = \nabla(\bar{\rho} w^2) \quad (36)$$

and

$$\bar{Q} - \nabla \cdot \bar{\mathbf{F}} = \overline{\bar{\rho} \mathbf{u} \cdot \left( \nabla h_1 - \frac{1}{\bar{\rho}} \nabla P_1 \right)} + \mathbf{g} \cdot \bar{\rho}_1 \bar{\mathbf{u}} - \overline{D_{ij} \frac{\partial u_i}{\partial x_j}} + \bar{\rho} \bar{C}_p \frac{\partial \bar{T}}{\partial t} - \bar{\delta} \frac{\partial \bar{P}}{\partial t} \quad (37)$$

Equations (31)–(37) are mathematically equivalent to Gough (1969)’s Equations (4.3)–(4.7) and (4.15)–(4.22), with the minor exceptions of Equations (33) and (37). Together, they form a complete system that can be solved for the evolution of both the fluctuating thermal variables and velocity, as well as the mean thermal profiles.

Dotting  $\mathbf{u}$  into Equation (32a) and using Equation (36) yields the anelastic kinetic energy equation,

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \bar{\rho} u^2 \right) = -\nabla \cdot \left( \frac{1}{2} \bar{\rho} u^2 \mathbf{u} \right) - \mathbf{u} \cdot \nabla P_1 + \rho_1 \mathbf{u} \cdot \mathbf{g} + u_i \frac{\partial D_{ij}}{\partial x_j} + \mathbf{u} \cdot \nabla (\bar{\rho} w^2). \quad (38)$$

Converting the differentials in Equation (15) into Eulerian differentials in time, the left-hand side of Equation (33) can be written in terms of the total entropy  $S_1 + \bar{S}$ ,

$$\bar{\rho} \bar{C}_p \left( \frac{\partial T_1}{\partial t} + \frac{\partial \bar{T}}{\partial t} \right) - \bar{\delta} \left( \frac{\partial P_1}{\partial t} + \frac{\partial \bar{P}}{\partial t} \right) = \bar{\rho} \bar{T} \frac{\partial S_1}{\partial t} + \bar{\rho} \bar{T} \frac{\partial \bar{S}}{\partial t}. \quad (39)$$

Using Equations (31), (34c), and (39) and then adding Equations (33) and (38) yields Gough (1969)’s total energy equation,

$$\begin{aligned} \frac{\partial}{\partial t} \left\{ \bar{\rho} \left[ \frac{1}{2} u^2 + \bar{T}(S_1 + \bar{S}) \right] \right\} = & -\nabla \cdot \left\{ \left[ \bar{\rho} \left( \frac{1}{2} u^2 + \bar{T} S_1 \right) + P_1 \right] \mathbf{u} - \mathbf{u} \cdot \overleftrightarrow{D} + \bar{\mathbf{F}} + \mathbf{F}_1 \right\} + \bar{Q} + Q_1 \\ & + \left[ -\bar{\rho} \bar{T} \mathbf{u} \cdot \nabla \bar{S} + \mathbf{u} \cdot \nabla (\bar{\rho} w^2) - \bar{T}(\rho_1 \mathbf{u} - \bar{\rho}_1 \mathbf{u}) \cdot \nabla \bar{S} \right]. \end{aligned} \quad (40)$$

Because of Equation (28), or more specifically, the condition

$$\bar{w} \equiv 0, \quad (41)$$

each of the rightmost terms in brackets in Equation (40) vanishes after volume integration over  $V$  (i.e., cannot transport any net energy). Note that Equation (41) is also a consequence of integrating Equation (31) over volumes bounded by horizontal surfaces (and also assuming  $w$  vanishes on the boundaries). Thus, the vanishing of the horizontal components of  $\bar{\mathbf{u}}$  is not strictly necessary for the bracketed terms to conserve energy.

Integrating Equation (40) over  $V$ , using the divergence theorem, and assuming that the sum of the surface-integrated fluxes balances the volume-integrated heating yields

$$E_{\text{tot}}^{\text{an}} := \int_V W_{\text{tot}}^{\text{an}} dV \equiv \text{constant}, \quad (42a)$$

$$\text{where } W_{\text{tot}}^{\text{an}} := W_{\text{KE}}^{\text{an}} + W_{\text{heat}}^{\text{an}}, \quad (42b)$$

$$W_{\text{KE}}^{\text{an}} := \frac{1}{2} \bar{\rho} u^2, \quad (42c)$$

$$\text{and } W_{\text{heat}}^{\text{an}} := \bar{\rho} \bar{T} (S_1 + \bar{S}). \quad (42d)$$

We have used the superscript “an” to denote energies and densities under the anelastic approximation and defined  $W_{\text{heat}}^{\text{an}}$  to be internal energy density of fluid parcels associated with irreversible heating processes (entropy increases). The conservation law (42) holds for arbitrary fluid motions that obey Equations (31), (32), (33), and (34) and for arbitrary magnitudes of  $|\nabla \bar{S}|$ . Note that the partition of energy density is fundamentally different than in the fully compressible conservation law (8); most notably, the potential energy density is absent under the anelastic approximation. We return to this central point in Section 7.

## 4 Gough (1969)'s anelastic equations in terms of pressure and entropy

The anelastic equations are often written using the pressure and entropy ( $P_1$ ,  $S_1$ , and  $\bar{S}$ ) in place of the quantities  $T_1$ ,  $\bar{T}$ ,  $P_1$ ,  $\bar{P}$ ,  $\rho_1$ , and  $h_1$  that appear in Equations (32) and (36). This is more than just a matter of convention, since transforming the variables to pressure and entropy isolates terms in the equations that depend on the mean entropy gradient  $\nabla \bar{S}$ , which can be neglected if  $\nabla \bar{S}$  is small.

In each of Equations (36) and (37), each term on the right-hand side is  $O(\epsilon)$  compared to each term on the left-hand side. In particular, we can make the mean hydrostatic approximation,

$$\nabla \bar{P} \approx \bar{\rho} \mathbf{g}, \quad (43)$$

when multiplying by terms that are already of first order in  $\epsilon$ .

We can change variables using Equations (34) and (35) and the approximation (43). In the momentum equation, we find

$$-\nabla P_1 + \rho_1 \mathbf{g} = -\bar{\rho} \nabla \left( \frac{P_1}{\bar{\rho}} \right) - \bar{\delta} \bar{\rho} \left( \frac{S_1}{\bar{C}_p} \right) \mathbf{g} + \frac{\bar{\delta} P_1}{\bar{C}_p} \nabla \bar{S}. \quad (44)$$

In the internal energy equation, we find

$$-\bar{\rho} \mathbf{u} \cdot \left( \nabla h_1 - \frac{1}{\bar{\rho}} \nabla P_1 \right) = -\bar{\rho} \bar{T} \mathbf{u} \cdot \nabla S_1 - \bar{\rho} S_1 \mathbf{u} \cdot \nabla \bar{T} + \left( \frac{P_1}{\bar{\rho}} \right) \mathbf{u} \cdot \nabla \bar{\rho}. \quad (45)$$

We then compute, with effort,

$$-\bar{\rho} S_1 \mathbf{u} \cdot \nabla \bar{T} + \left( \frac{P_1}{\bar{\rho}} \right) \mathbf{u} \cdot \nabla S_1 - \rho_1 \mathbf{u} \cdot \mathbf{g} = -\bar{\rho} T_1 \mathbf{u} \cdot \nabla \bar{S} + O(\epsilon^2). \quad (46)$$

Substituting Equation (44) into Equation (32), and Equations (39), (45), and (46) into Equation (33), we find

$$\begin{aligned} \frac{\partial}{\partial t}(\bar{\rho} \mathbf{u}) = & -\nabla \cdot (\bar{\rho} \mathbf{u} \mathbf{u}) - \bar{\rho} \nabla \left( \frac{P_1}{\bar{\rho}} \right) - \bar{\delta} \bar{\rho} \left( \frac{S_1}{\bar{C}_p} \right) \mathbf{g} + \underbrace{\frac{\bar{\delta} P_1}{\bar{C}_p} \nabla \bar{S}}_{:= \mathbf{f}_{\text{NLBR}}} + \nabla \cdot \overleftrightarrow{D} \\ & + [-\nabla \bar{P} + \bar{\rho} \mathbf{g}], \end{aligned} \quad (47)$$

and

$$\begin{aligned} \bar{\rho} \bar{T} \left( \frac{\partial S_1}{\partial t} + \frac{\partial \bar{S}}{\partial t} \right) = & -\bar{\rho} \bar{T} \mathbf{u} \cdot \nabla S_1 - \bar{\rho} \bar{T} \mathbf{u} \cdot \nabla \bar{S} - \underbrace{\bar{\rho} T_1 \mathbf{u} \cdot \nabla \bar{S}}_{:= Q_{\text{NLBR}}} + D_{ij} \frac{\partial u_i}{\partial x_j} + Q_1 - \nabla \cdot \mathbf{F}_1 \\ & + [\bar{Q} - \nabla \cdot \mathbf{F} - \bar{T}(\rho_1 \mathbf{u} - \bar{\rho}_1 \bar{\mathbf{u}}) \cdot \nabla \bar{S}], \end{aligned} \quad (48a)$$

$$\text{where} \quad \rho_1 = \frac{P_1}{c_s^2} - \frac{\bar{\delta} \bar{\rho} S_1}{\bar{C}_p} \quad (48b)$$

$$\text{and} \quad T_1 = \frac{\bar{T} S_1}{\bar{C}_p} + \frac{\bar{\delta} P_1}{\bar{\rho} \bar{C}_p}. \quad (48c)$$

Finally, the mean internal energy equation (37) becomes

$$\overline{Q} - \nabla \cdot \overline{\mathbf{F}} = \overline{\rho T \mathbf{u} \cdot \nabla S_1} - \overline{D_{ij} \frac{\partial u_i}{\partial x_j}} + \overline{\rho (T_1 \mathbf{u}) \cdot \nabla \overline{S}} + \overline{\rho T \frac{\partial \overline{S}}{\partial t}}. \quad (49)$$

The essential terms required for energy conservation when  $\nabla \overline{S} \neq 0$  are thus the “non-LBR” force density

$$\mathbf{f}_{\text{NLBR}} := \frac{\overline{\delta P_1}}{\overline{C_p}} \nabla \overline{S} \quad (50)$$

and the “non-LBR” heating

$$Q_{\text{NLBR}} := -\overline{\rho T_1 \mathbf{u} \cdot \nabla \overline{S}}. \quad (51)$$

Both these terms vanish for a nearly adiabatic background state (where  $\nabla \overline{S} = O(\epsilon)$ ), which is expected for a fully (and sufficiently vigorously) convecting fluid layer. Neglecting  $\mathbf{f}_{\text{NLBR}}$  was first done independently by Lantz (1992) and Braginsky & Roberts (1995) and is referred as the “Lantz-Braginsky-Roberts” (LBR) approximation. The term  $Q_{\text{NLBR}}$  in the internal energy equation, which was implicitly contained in the equations of Gough (1969), seems to be absent in the other forms of the anelastic equations currently in use. Its neglect seems to have not been explicitly considered.

## 5 The equivalence of horizontally averaged atmospheres to fixed reference atmospheres

In the formalism of Gough (1969), the horizontally averaged atmosphere, denoted by the overbars, is assumed to be time-independent. Furthermore, it cannot be specified a priori because it depends on the ultimate flow via Equations (36) and (50). Many anelastic numerical codes (e.g., the Rayleigh, EULAG, and MagIC codes, which simulate the anelastic equations in spherical shells) instead treat the background state as a fixed-in-time, spherically symmetric, hydrostatic “reference” state and let the thermal perturbations about this reference state develop small but nonzero horizontal means. Some codes (e.g., the ASH code) alternatively solve for the perturbations about the horizontal average directly. In the latter approach, horizontally averaged terms like the bracketed terms in Equations (32) and (33) are retained on the right-hand sides of the momentum and energy equations. As we now show, these two approaches are exactly equivalent to the order of the anelastic equations, provided that the horizontal means of the thermal variables wander on an advective timescale and by no more than  $O(\epsilon)$  away from their preordained reference state values.

We define the horizontally averaged profiles as the sum of the reference-state profile (denoted by a tilde) and a horizontally symmetric deviation (denoted by a hat):

$$\overline{P} = \overline{P}(q, t) = \tilde{P}(q) + \hat{P}(q, t) \quad \text{with} \quad \hat{P}/\tilde{P} = O(\epsilon), \quad (52a)$$

$$\overline{\rho} = \overline{\rho}(q, t) = \tilde{\rho}(q) + \hat{\rho}(q, t) \quad \text{with} \quad \hat{\rho}/\tilde{\rho} = O(\epsilon), \quad (52b)$$

$$\overline{S} = \overline{S}(q, t) = \tilde{S}(q) + \hat{S}(q, t) \quad \text{with} \quad \hat{S}/\tilde{C_p} = O(\epsilon), \quad (52c)$$

etc. Most reference-state profiles are chosen to be time-independent, except for  $\tilde{Q}$  and  $\tilde{\mathbf{F}}$  (see below).

In plain English, Equation (21) requires that the time variation of the mean state is slow (e.g., altered by the convection on an advective time scale), while Equation (53) requires that the total variation of the mean state (integrated over all time) remains small.

We denote the (temporally and horizontally dependent) deviations from the reference state by primes and note that

$$P' := P - \tilde{P} = \hat{P} + P_1, \quad (53a)$$

$$\rho' := \rho - \tilde{\rho} = \hat{\rho} + \rho_1, \quad (53b)$$

$$S' := S - \tilde{S} = \hat{S} + S_1, \quad (53c)$$

etc. The primed quantities, being the sum of two assumed-small quantities, thus remain small as long Equations (21) and (53) are satisfied.

Strictly, Gough (1969) assumes a time-independent mean state in deriving the continuity equation (31) and the left-hand side of the internal energy equation (48). However, these equations are essentially unaltered (to the order of the asymptotic expansion from which they are derived) as long as the slow-variation condition (21) holds. To see this, note that  $\partial\bar{\rho}/\partial t$  would be  $O(\epsilon)$  and thus the zeroth-order continuity equation remains unchanged. The left-hand side of the internal energy equation must now include the  $O(\epsilon)$  term  $\bar{\rho}\bar{T}\partial\tilde{S}/\partial t$  and is thus transformed from

$$\bar{\rho}\bar{T}\frac{\partial S_1}{\partial t} \quad \text{to} \quad \bar{\rho}\bar{T}\frac{\partial S'}{\partial t}. \quad (54)$$

The mean energy equation (50) also must be transformed to

We assume that the reference state has been chosen to be hydrostatic,

$$\nabla\tilde{P} = \tilde{\rho}\mathbf{g}, \quad (55)$$

to be in thermal equilibrium,

$$\tilde{Q} = \nabla \cdot \tilde{\mathbf{F}}, \quad (56)$$

and to satisfy the first law of thermodynamics for gradients (compare to Equation (35)),

$$\tilde{T}\nabla\tilde{S} = \tilde{C}_p\nabla\tilde{T} - \frac{\tilde{\delta}}{\tilde{\rho}}\nabla\tilde{P} \quad (57a)$$

$$= \frac{\tilde{C}_p\tilde{T}}{\tilde{\delta}\tilde{\rho}} \left[ \frac{\nabla\tilde{P}}{\tilde{c}_s^2} - \nabla\tilde{\rho} \right]. \quad (57b)$$

The linearized equation of state for the primed quantities is exactly analogous to Equation

(34),

$$\tilde{T}S' = \tilde{C}_p T' - \frac{\tilde{\delta}}{\tilde{\rho}} P', \quad (58a)$$

$$= U' - \frac{\tilde{P}}{\tilde{\rho}^2} \rho' \quad (58b)$$

$$= h' - \frac{P'}{\tilde{\rho}} \quad (58c)$$

$$= \frac{\tilde{C}_p \tilde{T}}{\tilde{\delta} \tilde{\rho}} \left[ \frac{P'}{\tilde{c}_s^2} - \rho' \right]. \quad (58d)$$

as is the linearized equation of state for the hatted quantities. Note that in Equation (34), the overbars may be replaced by tildes, making Equation (59) applicable to the “subscript 1” quantities as well.

To zeroth order in  $\epsilon$ , Equation (31) becomes simply

$$\nabla \cdot (\tilde{\rho} \mathbf{u}) \equiv 0. \quad (59)$$

Because the right-hand side of Equations (32) is  $O(\epsilon)$  compared to the left-hand side we can write, using Equation (56),

$$\begin{aligned} [-\nabla \bar{P} + \bar{\rho} \mathbf{g}] &= -\nabla \hat{P} + \hat{\rho} \mathbf{g} \\ &= -\tilde{\rho} \nabla \left( \frac{\hat{P}}{\tilde{\rho}} \right) - \tilde{\delta} \tilde{\rho} \left( \frac{\hat{S}}{\tilde{C}_p} \right) \mathbf{g} + \frac{\tilde{\delta} \tilde{\rho}}{\tilde{C}_p} \left( \frac{\hat{P}}{\tilde{\rho}} \right) \nabla \tilde{S}. \end{aligned} \quad (60)$$

Note that the non-LBR force density from Equation (51) is only significant when  $\nabla \bar{S}$  is large (in which case,  $\nabla \bar{S} = \nabla \tilde{S} + O(\epsilon)$  by Equations (21) and (53)), otherwise it is  $O(\epsilon^2)$ . For all magnitudes of  $|\nabla \bar{S}|$ , we can thus write

$$\mathbf{f}_{\text{NLBR}} = \frac{\bar{\delta} P_1}{\tilde{C}_p} \nabla \tilde{S} + O(\epsilon^2) \quad (61)$$

Plugging Equations (61) and (62) into Equation (47) and noting that all terms are of  $O(\epsilon)$  (so that we can replace overbars with tildes), we find

$$\frac{\partial}{\partial t}(\tilde{\rho} \mathbf{u}) = -\nabla \cdot (\tilde{\rho} \mathbf{u} \mathbf{u}) - \tilde{\rho} \nabla \left( \frac{P'}{\tilde{\rho}} \right) - \tilde{\delta} \tilde{\rho} \left( \frac{S'}{\tilde{C}_p} \right) \mathbf{g} + \frac{\tilde{\delta} P'}{\tilde{C}_p} \nabla \tilde{S} + \nabla \cdot \overleftrightarrow{D}, \quad (62a)$$

$$\text{where now} \quad D_{ij} = \tilde{\mu} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} (\nabla \cdot \mathbf{u}) \delta_{ij} \right), \quad (62b)$$

Using Equation (28), Equation (29) can be additionally written

$$\bar{\rho} \mathbf{u} + \rho_1 \mathbf{u} - \bar{\rho}_1 \mathbf{u} = \tilde{\rho} \mathbf{u} + \rho' \mathbf{u} - \bar{\rho}' \mathbf{u} + O(\epsilon^2). \quad (63)$$

We finally write

$$Q_{\text{NLBR}} = -\tilde{\rho}T_1\mathbf{u} \cdot \nabla \tilde{S} + O(\epsilon^2) \quad (64)$$

and

$$\begin{aligned} -\bar{\rho}\bar{T}\mathbf{u} \cdot \nabla S_1 - \bar{\rho}\bar{T}\mathbf{u} \cdot \nabla \bar{S} &= -\bar{\rho}\bar{T}\mathbf{u} \cdot \nabla S' - \bar{\rho}\bar{T}\mathbf{u} \cdot \nabla \tilde{S} \\ &= -\tilde{\rho}\tilde{T}\mathbf{u} \cdot \nabla S' - \tilde{T}(\bar{\rho}\mathbf{u}) \cdot \nabla \tilde{S} - \tilde{\rho}\hat{T}\mathbf{u} \cdot \nabla \tilde{S} + O(\epsilon^2). \end{aligned} \quad (65)$$

Substituting Equation (57) and Equations (64)–(66) into Equation (48) (with the modified left-hand side given by Equation (55)) yields

$$\begin{aligned} \tilde{\rho}\tilde{T}\frac{\partial S'}{\partial t} &= -\tilde{\rho}\tilde{T}\mathbf{u} \cdot \nabla S' - \tilde{\rho}\tilde{T}\mathbf{u} \cdot \nabla \tilde{S} - \tilde{\rho}T'\mathbf{u} \cdot \nabla \tilde{S} + D_{ij}\frac{\partial u_i}{\partial x_j} + Q' - \nabla \cdot \mathbf{F}' \\ &\quad - \tilde{T}(\rho'\mathbf{u} - \overline{\rho'\mathbf{u}}) \cdot \nabla \tilde{S}, \end{aligned} \quad (66a)$$

$$\text{where} \quad \rho' = \frac{P'}{\tilde{c}_s^2} - \frac{\tilde{\delta}\tilde{\rho}S'}{\tilde{C}_p}, \quad (66b)$$

$$T' = \frac{\tilde{T}S'}{\tilde{C}_p} + \frac{\tilde{\delta}P'}{\tilde{\rho}\tilde{C}_p}, \quad (66c)$$

and  $D_{ij}$  is now given by Equation (63b).

Taking the horizontal means of Equations (63) and (67) and then using Equations (56), (57), and (61) recovers the mean momentum and energy equations (36) and (?). The anelastic formulation with fixed reference states (Equations (60), (63) and (67)) is thus seen to be asymptotically equivalent to the formulation with horizontally averaged background states (Equations (31), (47), and (48)—with the modified left-hand side given in Equation (55)—combined with Equations (36) and (?), provided the restrictions (21) and (53).

Intuitively, none of the results of this section should seem too surprising and may even seem obvious. However, formalizing the equivalence between mean-state expansions of the anelastic equations and reference-state expansions via (21) and (53) is important. The mean-state anelastic approximation of (e.g.) Gough (1969) can consistently allow for time-varying mean states, subject only to the slow-variation constraint (21).

## 6 The Anelastic Energy-conserving Generalized Gough (AnEGG) approximation

To arrive at the final form of an energy-conserving set of anelastic equations, we make one final argument: that the last term in Equation (67a) is negligible. We do so because this term has zero horizontal mean and therefore cannot affect the net transport of energy. Furthermore, pointwise it should always be much smaller than the background advection term  $-\tilde{\rho}\tilde{T}\mathbf{u} \cdot \nabla \tilde{S}$ , provided  $\rho'$  remains  $\ll \tilde{\rho}$ .

The final equations, representing what we call the Anelastic Energy-conserving Generalized Gough (AnEGG) approximation, are thus

$$\nabla \cdot (\tilde{\rho} \mathbf{u}) \equiv 0, \quad (67a)$$

$$\frac{\partial}{\partial t}(\tilde{\rho} \mathbf{u}) = -\nabla \cdot (\tilde{\rho} \mathbf{u} \mathbf{u}) - \tilde{\rho} \nabla \left( \frac{P'}{\tilde{\rho}} \right) - \tilde{\rho} \left( \frac{S'}{\tilde{C}_p} \right) \mathbf{g} + \underbrace{\frac{\tilde{\delta} P'}{\tilde{C}_p} \nabla \tilde{S}}_{:= \tilde{\mathbf{f}}_{\text{NLBR}}} + \nabla \cdot \overleftrightarrow{\mathbf{D}}, \quad (67b)$$

$$\begin{aligned} \text{and} \quad \tilde{\rho} \tilde{T} \frac{\partial S'}{\partial t} = & -\tilde{\rho} \tilde{T} \mathbf{u} \cdot \nabla S' - \tilde{\rho} \tilde{T} \mathbf{u} \cdot \nabla \tilde{S} - \underbrace{\left( \frac{\tilde{T} S'}{\tilde{C}_p} + \frac{\tilde{\delta} P'}{\tilde{\rho} \tilde{C}_p} \right) \tilde{\rho} \mathbf{u} \cdot \nabla \tilde{S}}_{:= \tilde{Q}_{\text{NLBR}}} \\ & + D_{ij} \frac{\partial u_i}{\partial x_j} + Q' - \nabla \cdot \mathbf{F}', \end{aligned} \quad (67c)$$

where  $\overleftrightarrow{\mathbf{D}}$  is defined in Equation (63b).

The AnEGG kinetic energy equation, derived from  $\mathbf{u}$  dotted into Equation (68b), is

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \tilde{\rho} u^2 \right) = -\nabla \cdot \left( \frac{1}{2} \tilde{\rho} u^2 \mathbf{u} \right) - \nabla \cdot (P' \mathbf{u}) - \tilde{\rho} \left( \frac{S'}{\tilde{C}_p} \right) \mathbf{u} \cdot \mathbf{g} + \frac{\tilde{\delta} P'}{\tilde{C}_p} \mathbf{u} \cdot \nabla \tilde{S} + u_i \frac{\partial D_{ij}}{\partial x_j}, \quad (68)$$

Adding Equations (68c) and (69) yields the AnEGG total energy equation,

$$\begin{aligned} \frac{\partial}{\partial t} \left[ \tilde{\rho} \left( \frac{1}{2} u^2 + \tilde{T} S' \right) \right] = & -\nabla \cdot \left\{ \left[ \tilde{\rho} \left( \frac{1}{2} u^2 + \tilde{T} S' \right) + P' \right] \mathbf{u} - \mathbf{u} \cdot \overleftrightarrow{\mathbf{D}} + \mathbf{F}' \right\} + Q' \\ & - \tilde{\rho} \tilde{T} \mathbf{u} \cdot \nabla \tilde{S}. \end{aligned} \quad (69)$$

Again using condition (28) (or equivalently, Equation (60)), the integration of Equation (70) yields conservation of total energy,

$$E_{\text{tot}} := \int_V W_{\text{tot}} dV = \text{constant}, \quad (70a)$$

$$\text{where} \quad W_{\text{tot}} := \tilde{\rho} \left( \frac{1}{2} u^2 + \tilde{T} S' \right) \quad (70b)$$

is the total energy density per unit volume. This conservation law holds for arbitrary fluid motions obeying Equations (68) and for all magnitudes of  $|\nabla \tilde{S}|$ .

Note that in practice when simulating stiff systems (large  $|\nabla \tilde{S}|$ ) numerically (e.g., Guerrero et al. 2016; Matilsky et al. 2022, 2024), the term

$$\tilde{Q}_{\text{adv}} := -\tilde{\rho} \tilde{T} \mathbf{u} \cdot \nabla \tilde{S} \quad (71)$$

may pointwise be quite large. The degree to which energy is conserved numerically may thus be limited by the precision of the condition (41). In the streamfunction formulation



of ASH and Rayleigh for example (e.g., [Clune et al. 1999](#); [Featherstone & Hindman 2016](#)), Equation (41) holds to near machine precision.

To summarize, necessary conditions for anelastic codes implementing background stable layers to conserve energy are (1) including the non-LBR force density,

$$\mathbf{f}_{\text{NLBR}} := \frac{\tilde{\delta}P'}{\tilde{C}_p} \mathbf{u} \cdot \nabla \tilde{S} \quad (72)$$

in the momentum equation (i.e., not making the LBR approximation) and (2) including the non-LBR heating term,

$$\tilde{Q}_{\text{NLBR}} := - \left( \frac{\tilde{T}S'}{\tilde{C}_p} + \frac{\tilde{\delta}P'}{\tilde{\rho}\tilde{C}_p} \right) \tilde{\rho} \mathbf{u} \cdot \nabla \tilde{S} \quad (73)$$

in the internal energy equation. Alternatively, if the LBR approximation is made, (setting  $\mathbf{f}_{\text{NLBR}} \equiv 0$  in Equation (63)), energy can still be conserved if  $\tilde{Q}_{\text{NLBR}}$  in Equation (67) is replaced by

$$\tilde{Q}_{\text{LBR}} := - \left( \frac{\tilde{\rho}\tilde{T}}{\tilde{C}_p} \right) S' \mathbf{u} \cdot \nabla \tilde{S} \quad (74)$$

However, doing this would still yield an asymptotically inconsistent equation set if  $|\nabla \tilde{S}|$  is large.

One slightly surprising result concerns the second law of thermodynamics. In the absence of heat sources or sinks, when the true motion should be adiabatic, Equation (68c) becomes

$$\frac{D}{Dt}(\tilde{S} + S') = \frac{Q_{\text{NLBR}}}{\tilde{\rho}\tilde{T}}. \quad (75)$$

Thus, the AnEGG equations do not conserve total entropy for adiabatic motion, violating the second law of thermodynamics if  $S'$  is strictly interpreted as the entropy. Apparently making the anelastic approximation allows for the conservation of energy or the second law of thermodynamics to be satisfied, but not both.

## 7 The meaning of total energy under the anelastic approximation

From Equation (59b), the primed (i.e., perturbed from the reference state) internal plus potential energy per unit volume is

$$\begin{aligned} W' &= \tilde{U}\rho' + \tilde{\rho}U' + \Phi\rho' \\ &= \underbrace{\tilde{\rho}\tilde{T}S'}_{:=W'_{\text{heat}}} + \underbrace{\left( \tilde{U} + \frac{\tilde{P}}{\tilde{\rho}} + \Phi \right)}_{:=W'_{\text{comp}}} \rho'. \end{aligned} \quad (76)$$

Equation (59b) is really an expression of the first law of thermodynamics, which states that changes in the internal energy of fluid parcels come from the sum of external heating and pressure work done by the environment during compression. Comparing to Equation (8), it is clear that *only* changes in internal energy from heat sources can effect the total energy. The changes in internal energy from density variations ( $\tilde{U}\rho'$ ) the pressure work,  $\tilde{P}(\rho'/\tilde{\rho})$ ; and the changes in potential energy,  $\Phi\rho'$ ; are all absent. Since all three of these latter effects are due to density variations (i.e., compression or expansion of the fluid), we write

$$W' = W'_{\text{heat}} + W'_{\text{comp}}, \quad (77a)$$

$$\text{where} \quad W'_{\text{heat}} := \tilde{\rho}\tilde{T}S' \quad (77b)$$

$$\text{and} \quad W'_{\text{comp}} := \left( \tilde{U} + \frac{\tilde{P}}{\tilde{\rho}} + \Phi \right) \rho' \quad (77c)$$

We note that

$$W'_{\text{comp}} \equiv 0 \quad \text{under the anelastic approximation,} \quad (78)$$

i.e., *no* compression effects can alter the total energy of the fluid at a particular vertical level.

A possibly deeper interpretation of Equation (79) is that the assumption of zero mean mass flux, Equation (20), is *fundamental* if the anelastic approximation is to be energetically consistent. For the “true” (i.e., fully compressible) fluid, Equation (20) applied to the continuity equation (1) yields

$$\frac{\partial \bar{\rho}}{\partial t} \equiv 0, \quad (79)$$

i.e., there can be no Evidently from Equation (??), the internal energy (as far as energy transport is concerned) becomes simply  $\bar{\rho}\bar{T}S'$  and the potential energy is eliminated entirely. Both effects are consequences of the assumption of zero mean mass flux, Equation (20). This also has the consequence that  $S'$  should be interpreted as a deviation of internal energy from the mean, rather than as an entropy.

Ogura & Phillips (1962) write

$$\begin{aligned} W_1 &= -\bar{\rho}\Phi \left( \frac{S'}{C_p} \right) + \bar{\rho}C_v\bar{T}_0 \left( \frac{P'}{\bar{P}} \right) \\ &= -\bar{\rho}\Phi \left( \frac{S'}{C_p} \right) + \frac{P'}{\gamma - 1} \end{aligned} \quad (80)$$

where the gas here is assumed to be perfect (i.e.,  $\bar{C}_p \equiv C_p$  and  $\bar{C}_v \equiv C_v$ , where  $C_p$  and  $C_v$  are constants and  $\gamma := C_p/C_v$ ) and adiabatically stratified ( $\nabla \bar{S} \equiv 0$ ), and the gravitational potential  $\Phi$  is measured from the location  $q_0$  where  $\bar{T}(q = q_0) := \bar{T}_0$  is equal to the assumed-constant background potential temperature.<sup>6</sup>

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<sup>6</sup>See the equation immediately following Equation (31) from Ogura & Phillips (1962). To see the equivalence to our Equation (81), note that Ogura & Phillips (1962) use  $\theta$  for the non-dimensional order-unity perturbed potential temperature,  $gz'$  for the gravitational potential, and  $\Theta$  for the constant (dimensional) potential temperature (equal to  $\bar{T}_0$ ) of the adiabatically stratified background. Also note the relation  $S' = \epsilon C_p \theta$ .

Eckart 1960 (p. 54) discussed the energy density associated with small perturbations about a background static atmosphere and identified  $P_1^2/(2\bar{\rho}c_s^2)$  as the “elastic” energy density from acoustics (Morse 1948, p. 237). Ogura & Phillips 1962 note that the disappearance of the compressive term, which was called the “elastic energy” in Eckart & Ferris (1956), is the origin of the term “anelastic,” coined by Jule Charney (see ). The potential energy term also disappears, which is a direct consequence of the assumption of zero mean mass flux, Equation (20).

## 8 Conditions for anelastic overshoot

[THIS SECTION IS A STUB]. What I have so far is that a downflowing plume should have kinetic energy

$$w^2 \sim g_a H_a \left( \frac{S'}{C_{pa}} \right) \sim c_{sa}^2 \epsilon \quad (81)$$

If it descends into a stable stratification with buoyancy frequency

$$N_a^2 = \frac{g_a}{C_{pa}} |\nabla S_a|, \quad (82)$$

where  $|\nabla S_a|$  is a typical value for  $|\nabla \tilde{S}|$  in the stable layer. At most, the plume should reach a depth  $d$  before it decelerates to  $w = 0$ , with  $d$  given by

$$w^2 \sim \left( \frac{S'}{C_{pa}} \right) g d \sim \frac{g_a}{C_{pa}} |\nabla S_a| d^2 \sim N_a^2 d^2 \quad (83)$$

or

$$d \sim \frac{w}{N_a}. \quad (84)$$

Thus,  $|S'|$  should reach a maximum during overshoot of

$$\begin{aligned} \max \left( \frac{S'}{C_{pa}} \right)_{\text{stable}} &\sim \left( \frac{|\nabla S_a|}{C_{pa}} \right) d \sim \left( \frac{N_a^2}{g_a} \right) \frac{w}{N_a} \\ &\sim \left( \frac{N_a}{g_a} \right) c_{sa} \epsilon^{1/2} \sim \left( \frac{N_a}{c_{sa}^2/H_a} \right) c_{sa} \epsilon^{1/2} = \left( \frac{N_a}{c_{sa}/H_a} \right) \epsilon^{1/2}. \end{aligned} \quad (85)$$

We thus expect the thermal perturbations to remain small in the stable layer, provided

$$\frac{N_a^2}{\omega_{ac}^2} \lesssim \epsilon, \quad (86a)$$

$$\text{where } \omega_{ac} := \frac{c_{sa}}{2H_a} \quad (86b)$$

is the acoustic cutoff frequency.

In the solar radiative zone for example,  $\omega_{ac} \sim 3 \times 10^{-2} \text{ rad s}^{-1}$  and  $N_a \sim 1.4 \times 10^{-3} \text{ rad s}^{-1}$ , so condition (87a) should be satisfied if  $\epsilon \sim 10^{-3}$ . On the other hand, maybe the scale analysis of Gough (1969) breaks down, since buoyantly decelerated flows (i.e., overshoot and g-modes) may have very different dynamical balances than buoyantly accelerated flows (i.e., convection)? So maybe another expansion is necessary that takes into account not only small thermal perturbations, but anisotropic velocity and length scales.

## References

- Batchelor, G. K. 1953, Quarterly Journal of the Royal Meteorological Society, 79, 224–235, doi: [10.1002/qj.49707934004](https://doi.org/10.1002/qj.49707934004)
- Braginsky, S. I., & Roberts, P. H. 1995, Geophysical & Astrophysical Fluid Dynamics, 79, 1, doi: [10.1080/03091929508228992](https://doi.org/10.1080/03091929508228992)
- Brown, B. P., Oishi, J. S., Vasil, G. M., Lecoanet, D., & Burns, K. J. 2020, The Astrophysical Journal, 902, L3, doi: [10.3847/2041-8213/abb9a4](https://doi.org/10.3847/2041-8213/abb9a4)
- Brown, B. P., Vasil, G. M., & Zweibel, E. G. 2012, The Astrophysical Journal, 756, 109, doi: [10.1088/0004-637x/756/2/109](https://doi.org/10.1088/0004-637x/756/2/109)
- Brun, A. S., Miesch, M. S., & Toomre, J. 2004, The Astrophysical Journal, 614, 1073, doi: [10.1086/423835](https://doi.org/10.1086/423835)
- Burns, K. J., Vasil, G. M., Oishi, J. S., Lecoanet, D., & Brown, B. P. 2020, Physical Review Research, 2, doi: [10.1103/physrevresearch.2.023068](https://doi.org/10.1103/physrevresearch.2.023068)
- Charney, J. G., & Ogura, Y. 1960, Journal of the Meteorological Society of Japan. Ser. II, 38, 19a–19a, doi: [10.2151/jmsj1923.38.6\\_19a](https://doi.org/10.2151/jmsj1923.38.6_19a)
- Clune, T., Elliott, J., Miesch, M., Toomre, J., & Glatzmaier, G. 1999, Parallel Comp., 25, 361, doi: [10.1016/s0167-8191\(99\)00009-5](https://doi.org/10.1016/s0167-8191(99)00009-5)
- Eckart, C. 1960, Hydrodynamics of Oceans and Atmospheres (Oxford: Pergamon Press)
- Eckart, C., & Ferris, H. G. 1956, Reviews of Modern Physics, 28, 48–52, doi: [10.1103/revmodphys.28.48](https://doi.org/10.1103/revmodphys.28.48)
- Featherstone, N. A., & Hindman, B. W. 2016, The Astrophysical Journal, 818, 32, doi: [10.3847/0004-637x/818/1/32](https://doi.org/10.3847/0004-637x/818/1/32)
- Featherstone, N. A., Augustson, K. C., Aurnou, J. M., et al. 2023, The Puzzling Structure of Solar Convection: Window into the Dynamo, arXiv, doi: [10.48550/ARXIV.2305.08823](https://doi.org/10.48550/ARXIV.2305.08823)
- Gastine, T., & Wicht, J. 2012, Icar., 219, 428, doi: [10.1016/j.icarus.2012.03.018](https://doi.org/10.1016/j.icarus.2012.03.018)
- Gilman, P. A., & Glatzmaier, G. A. 1981, The Astrophysical Journal Supplement Series, 45, 335, doi: [10.1086/190714](https://doi.org/10.1086/190714)
- Glatzmaier, G. A. 1984, J. Comp. Phys., 55, 461, doi: [10.1016/0021-9991\(84\)90033-0](https://doi.org/10.1016/0021-9991(84)90033-0)
- Gough, D. O. 1969, J. Atmos. Soc., 26, 448, doi: [10.1175/1520-0469\(1969\)026<0448:taaftc>2.0.co;2](https://doi.org/10.1175/1520-0469(1969)026<0448:taaftc>2.0.co;2)
- Guerrero, G., Smolarkiewicz, P. K., de Gouveia Dal Pino, E. M., Kosovichev, A. G., & Mansour, N. N. 2016, The Astrophysical Journal, 819, 104, doi: [10.3847/0004-637x/819/2/104](https://doi.org/10.3847/0004-637x/819/2/104)

- Lantz, S. R. 1992, PhD thesis, Cornell University, United States
- Lantz, S. R., & Fan, Y. 1999, *The Astrophysical Journal Supplement Series*, 121, 247, doi: [10.1086/313187](https://doi.org/10.1086/313187)
- Lipps, F. B., & Hemler, R. S. 1982, *Journal of the Atmospheric Sciences*, 39, 2192–2210, doi: [10.1175/1520-0469\(1982\)039<2192:asaodm>2.0.co;2](https://doi.org/10.1175/1520-0469(1982)039<2192:asaodm>2.0.co;2)
- Matilsky, L. I., Brummell, N. H., Hindman, B. W., & Toomre, J. 2024, *The Astrophysical Journal*, 962, 189, doi: [10.3847/1538-4357/ad18b2](https://doi.org/10.3847/1538-4357/ad18b2)
- Matilsky, L. I., Hindman, B. W., Featherstone, N. A., Blume, C. C., & Toomre, J. 2022, *The Astrophysical Journal Letters*, 940, L50, doi: [10.3847/2041-8213/ac93ef](https://doi.org/10.3847/2041-8213/ac93ef)
- Morse, P. M. 1948, *Vibration and Sound* (New York: McGraw-Hill)
- Ogura, Y., & Phillips, N. A. 1962, *Journal of the Atmospheric Sciences*, 19, 173, doi: [10.1175/1520-0469\(1962\)019<0173:saodas>2.0.co;2](https://doi.org/10.1175/1520-0469(1962)019<0173:saodas>2.0.co;2)
- Rogers, T. M., & Glatzmaier, G. A. 2005, *The Astrophysical Journal*, 620, 432–441, doi: [10.1086/423415](https://doi.org/10.1086/423415)
- Smolarkiewicz, P. K., & Prusa, J. M. 2004, in *Turbulent Flow Computation* (Kluwer Academic Publishers), 279–312, doi: [10.1007/0-306-48421-8\\_8](https://doi.org/10.1007/0-306-48421-8_8)
- Vasil, G. M., Lecoanet, D., Brown, B. P., Wood, T. S., & Zweibel, E. G. 2013, *The Astrophysical Journal*, 773, 169, doi: [10.1088/0004-637x/773/2/169](https://doi.org/10.1088/0004-637x/773/2/169)
- Wilczyński, F., Hughes, D. W., & Kersalé, E. 2022, *Journal of Fluid Mechanics*, 942, doi: [10.1017/jfm.2022.325](https://doi.org/10.1017/jfm.2022.325)