

Induction term in spherical coordinates

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1 The problem with the traditional induction terms

We consider the ideal (resistance-free) magnetohydrodynamic (MHD) induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) := \mathbf{I}, \quad (1)$$

$$= \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{B} - (\nabla \cdot \mathbf{u}) \mathbf{B}, \quad (2)$$

where \mathbf{B} and \mathbf{u} are the vector magnetic and velocity fields (respectively) and \mathbf{I} is the vector induction. The three terms on the right-hand-side of Equation (2) are often interpreted as “shear,” “advection,” and “compression,” respectively. However, this interpretation is problematic in general for two reasons:

1. The so-called shear and compression terms contain sub-terms that cancel; in particular, only velocity motions *perpendicular* to magnetic-field lines can shear or compress.
2. Solid-body rotation (which is a non-shearing motion that simply rotates the whole field configuration) shows up in the so-called shear and advection terms in a strange way.

Finally, even after these issues have been addressed, resolving the final terms into a particular curvilinear system (e.g., spherical coordinates) seems to present a major headache. Our goal here is to explain fully how these problems emerge and propose a tentative solution for the case of spherical coordinates.

2 Perpendicular shear and compression

To see how Problem 1 arises, we decompose the velocity field into components parallel and perpendicular to the local direction of \mathbf{B} :

$$\mathbf{u} := u_{\parallel} \hat{\mathbf{e}}_{\parallel} + \mathbf{u}_{\perp} \quad (3)$$

Obviously $\mathbf{B} = Bx_{\parallel}$, where $B = |\mathbf{B}|$. We denote the Cartesian distance along \mathbf{B} by x_{\parallel} . We also decompose \mathbf{u} into its parallel and perpendicular components:

$$\nabla \cdot \mathbf{u} = \frac{\partial u_{\parallel}}{\partial x_{\parallel}} + \nabla_{\perp} \cdot \mathbf{u}_{\perp} \quad (4)$$

We then calculate

$$\begin{aligned}
\mathbf{B} \cdot \nabla \mathbf{u} - (\nabla \cdot \mathbf{u}) \mathbf{B} &= B \frac{\partial}{\partial x_{\parallel}} (u_{\parallel} \hat{\mathbf{e}}_{\parallel} + \mathbf{u}_{\perp}) - \left(\frac{\partial u_{\parallel}}{\partial x_{\parallel}} + \nabla_{\perp} \cdot \mathbf{u}_{\perp} \right) B \hat{\mathbf{e}}_{\parallel} \\
&= B \cancel{\frac{\partial u_{\parallel}}{\partial x_{\parallel}} \hat{\mathbf{e}}_{\parallel}} + B \frac{\partial \mathbf{u}_{\perp}}{\partial x_{\parallel}} - \cancel{\frac{\partial u_{\parallel}}{\partial x_{\parallel}} B \hat{\mathbf{e}}_{\parallel}} - (\nabla_{\perp} \cdot \mathbf{u}_{\perp}) B \hat{\mathbf{e}}_{\parallel} \\
&= \mathbf{B} \cdot \nabla \mathbf{u}_{\perp} - (\nabla_{\perp} \cdot \mathbf{u}_{\perp}) \mathbf{B}.
\end{aligned} \tag{5}$$

Thus, only motions perpendicular to the local field line (i.e., \mathbf{u}_{\perp}) can shear or compress \mathbf{B} .

3 Rigid rotation

To see how Problem 2 arises, we consider a velocity field due to rigid rotation at constant angular velocity Ω about the z -axis in a cylindrical coordinate system:

$$\Omega = \Omega \hat{\mathbf{e}}_z = \text{constant} \tag{6a}$$

$$\mathbf{u} = \Omega \times \mathbf{r} = \Omega \lambda \hat{\mathbf{e}}_{\phi} \tag{6b}$$

Here, λ is the cylindrical radius, ϕ the longitude, and z the axial coordinate. In general, $\hat{\mathbf{e}}_{(\dots)}$ denotes a unit vector in the direction of its subscript. We calculate:

$$\begin{aligned}
\nabla \cdot \mathbf{u} &= \nabla \cdot (\Omega \times \mathbf{r}) \\
&= \Omega \cdot \nabla \times \mathbf{r} - \mathbf{r} \cdot \nabla \times \Omega \\
&= 0 \quad \text{no compression for rigid rotation (obviously)}.
\end{aligned} \tag{7}$$

Then:

$$\begin{aligned}
\mathbf{B} \cdot \nabla \mathbf{u} &= (\mathbf{B} \cdot \nabla)(\Omega \times \mathbf{r}) \\
&= \Omega \times [(\mathbf{B} \cdot \nabla)(\mathbf{r})]
\end{aligned} \tag{8}$$

$$= \Omega \times \mathbf{B} \quad \text{“shear” for rigid rotation.} \tag{9}$$

Finally:

$$\begin{aligned}
-\mathbf{u} \cdot \nabla \mathbf{B} &= -\Omega \lambda \hat{\mathbf{e}}_{\phi} \cdot \nabla \mathbf{B} \\
&= -\Omega \frac{\partial}{\partial \phi} (B_{\lambda} \hat{\mathbf{e}}_{\lambda} + B_{\phi} \hat{\mathbf{e}}_{\phi} + B_z \hat{\mathbf{e}}_z) \\
&= -\Omega \sum_{\alpha} \left(\frac{\partial B_{\alpha}}{\partial \phi} \right) \hat{\mathbf{e}}_{\alpha} - \Omega \sum_{\alpha} B_{\alpha} \frac{\partial \hat{\mathbf{e}}_{\alpha}}{\partial \phi},
\end{aligned}$$

where the index α runs over the three cylindrical coordinates. Note that in the cylindrical coordinate system (or indeed any coordinate system with an axis of rotational symmetry), $\partial \hat{\mathbf{e}}_{\alpha} / \partial \phi = \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_{\alpha}$ for each α . Thus,

$$\begin{aligned}
-\Omega \sum_{\alpha} B_{\alpha} \frac{\partial \hat{\mathbf{e}}_{\alpha}}{\partial \phi} &= -\Omega \sum_{\alpha} B_{\alpha} \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_{\alpha} \\
&= -\Omega \hat{\mathbf{e}}_z \times \sum_{\alpha} B_{\alpha} \hat{\mathbf{e}}_{\alpha} \\
&= -\Omega \times \mathbf{B}
\end{aligned}$$

and so

$$-\mathbf{u} \cdot \nabla \mathbf{B} = -\Omega \sum_{\alpha} \left(\frac{\partial B_{\alpha}}{\partial \phi} \right) \hat{\mathbf{e}}_{\alpha} - \Omega \times \mathbf{B} \quad \text{“advection” for rigid rotation.} \quad (10)$$

Mathematically, in any coordinate system with a z -axis of rotational symmetry, the action of rigid rotation is as follows:

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times [(\Omega \times \mathbf{r}) \times \mathbf{B}] \\ &= \sum_{\alpha} \left(-\Omega \frac{\partial B_{\alpha}}{\partial \phi} \right) \hat{\mathbf{e}}_{\alpha} \end{aligned} \quad (11a)$$

$$\text{or} \quad \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) B_{\alpha} = 0 \quad \text{for each } \alpha. \quad (11b)$$

If you think about it, this makes sense: All the rigid rotation does is rotate the whole field configuration around the z -axis at the rate Ω . If you decide to also rotate at Ω (so your personal Eulerian time derivative is $\partial/\partial t + \Omega\partial/\partial\phi$), then each component of the magnetic-field configuration should remain the same in your frame. Note that rotation does *not* advect the vector magnetic field (like the term $-\mathbf{u} \cdot \nabla \mathbf{B}$ viewed on its own would suggest), but rather advects the field *components* (as if they were scalars) in any coordinate system with an axis of rotational symmetry.

4 Solution for spherical coordinates

Resolving these issues fully for the spherical coordinate system seems complicated and I am not fully sure how to do it! In particular (for “full” resolution) we should, separately at each point (r, θ, ϕ) :

1. Form a local Cartesian coordinate system, say (x_1, x_2, x_3) , whose origin lies at the point (r, θ, ϕ) . At the origin, $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ will coincide with $(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_{\theta}, \hat{\mathbf{e}}_{\phi})$. But slightly away from the origin, $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ will stay fixed while $(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_{\theta}, \hat{\mathbf{e}}_{\phi})$ curve away.
2. Calculate the velocity-gradient tensor: $\partial u_1/\partial x_1, \partial u_2/\partial x_1, \partial u_3/\partial x_1$, etc. While calculating derivatives, be careful to differentiate along the Cartesian coordinates, *not* the curvilinear ones or along the actual \mathbf{B} -line). Express the final tensor components in spherical coordinates.
3. Rotate “into \mathbf{B} ” to form a new primed coordinate system (such that $\hat{\mathbf{e}}'_1$ points along \mathbf{B}), calculate the $\partial u'_j/\partial x'_i$, and thus form $\partial u_{\parallel}/\partial x_{\parallel}$ and $\partial \mathbf{u}_{\perp}/\partial x_{\parallel}$ (note that $\nabla_{\perp} \cdot \mathbf{u}_{\perp} = \nabla \cdot \mathbf{u} - \partial u_{\parallel}/\partial x_{\parallel}$). Note that the direction cosines $\hat{\mathbf{e}}'_1$ makes with $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ are simply $B_r/|\mathbf{B}|, B_{\theta}/|\mathbf{B}|, B_{\phi}/|\mathbf{B}|$, respectively. Note that both unit vectors and vector components transform like $x'_1 = \sum_{j=1}^3 R_{1j} x_j$, where R_{1j} is the j^{th} direction cosine.
4. Subtract the part of \mathbf{u}_{\perp} corresponding to solid-body rotation and put it in the form of Equation (11a).

5. Exult, because I haven't been able to do this!

Instead of addressing the full problem as just described, I have brute-forced my way into a quasi-solution, canceling obvious terms and expressing what seem to be “perpendicular shear with no solid-body rotation,” “perpendicular compression,” and advection.

We write:

$$\begin{aligned}
I_r &= \cancel{B_r \frac{\partial u_r}{\partial r}} + \frac{B_\theta}{r} \frac{\partial u_r}{\partial \theta} + \frac{B_\phi}{r \sin \theta} \frac{\partial u_r}{\partial \phi} - \frac{\cancel{B_\theta u_\theta} + B_\phi u_\phi}{r} \\
&\quad - \left[\mathbf{u} \cdot \nabla B_r - \frac{\cancel{u_\theta B_\theta} + u_\phi B_\phi}{r} \right] \\
&\quad - \left[\cancel{\frac{\partial u_r}{\partial r}} + \frac{2u_r}{r} + \frac{1}{r} \left(\frac{\partial u_\theta}{\partial \theta} + \cot \theta u_\theta \right) + \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} \right] B_r \\
&= \left(\frac{B_\theta}{r} \frac{\partial}{\partial \theta} + \frac{B_\phi}{r \sin \theta} \frac{\partial}{\partial \phi} \right) u_r \\
&\quad - \mathbf{u} \cdot \nabla B_r \\
&\quad - \left[\frac{1}{r} \left(\frac{\partial u_\theta}{\partial \theta} + \cot \theta u_\theta \right) + \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} \right] B_r - \frac{2B_r u_r}{r}
\end{aligned}$$

or

$$I_r = \underbrace{\mathbf{B}_{\text{hor}} \cdot \nabla u_r}_{\text{perpendicular shear}} - \underbrace{\left(\nabla \cdot \mathbf{u}_{\text{hor}} + \frac{2u_r}{r} \right) B_r}_{\text{perpendicular compression}} - \underbrace{\mathbf{u} \cdot \nabla B_r}_{\text{advection}} \quad (12)$$

Our justification for including the term $-2B_r u_r/r$ in the compression will be given in the next section.

Now that we know a bit more how things fall into place, we calculate

$$\begin{aligned}
I_\theta &= \cancel{\frac{B_\theta}{r} \frac{\partial u_\theta}{\partial \theta}} + \mathbf{B}_{\text{tor}} \cdot \nabla u_\theta + \frac{B_\theta u_r}{r} - \frac{\cancel{\cot \theta B_\phi u_\phi}}{r} \\
&\quad - \mathbf{u} \cdot \nabla B_\theta - \frac{u_\theta B_r}{r} + \frac{\cancel{\cot \theta u_\phi B_\phi}}{r} \\
&\quad - B_\theta \left(\cancel{\frac{1}{r} \frac{\partial u_\theta}{\partial \theta}} + \frac{\cot \theta u_\theta}{r} + \nabla \cdot \mathbf{u}_{\text{tor}} \right)
\end{aligned}$$

or

$$I_\theta = \underbrace{r \mathbf{B}_{\text{tor}} \cdot \nabla \left(\frac{u_\theta}{r} \right)}_{\text{perpendicular shear}} - \underbrace{\left(\nabla \cdot \mathbf{u}_{\text{tor}} + \frac{\cot \theta u_\theta}{r} - \frac{u_r}{r} \right) B_\theta}_{\text{perpendicular compression}} - \underbrace{\mathbf{u} \cdot \nabla B_r}_{\text{advection}} \quad (13)$$