

# Angular Momentum in Terms of Toroidal and Poloidal Stream Functions

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## 1 Stream Function Formalism

In anelastic approximations, we always have the condition of divergenceless mass flux,

$$\nabla \cdot (\bar{\rho} \mathbf{v}) \equiv 0, \quad (1)$$

where  $\bar{\rho}$  is the reference state density (we assume  $\bar{\rho} = \bar{\rho}(r)$  is spherically symmetric and time-independent) and  $\mathbf{v}$  is the fluid velocity. Condition (1) admits a stream function representation for the mass flux,

$$\bar{\rho} \mathbf{v} = \nabla \times [\nabla \times (W \hat{\mathbf{e}}_r)] + \nabla \times (Z \hat{\mathbf{e}}_r), \quad (2)$$

where  $W$  and  $Z$  are the poloidal and toroidal stream functions, respectively. In the derivations that follow it will also be helpful to define the alternate stream functions  $\tilde{W}$  and  $\tilde{Z}$  through

$$\bar{\rho} \mathbf{v} = \nabla \times [\nabla \times (\tilde{W} \mathbf{r})] + \nabla \times (\tilde{Z} \mathbf{r}), \quad (3)$$

where  $\mathbf{r} = r \hat{\mathbf{e}}_r$  is the position vector. Clearly  $W = r \tilde{W}$  and  $Z = r \tilde{Z}$ .

We note that

$$\nabla \times (\tilde{Z} \mathbf{r}) = \nabla \tilde{Z} \times \mathbf{r} \quad (4)$$

and

$$\begin{aligned} \nabla \times [\nabla \times (\tilde{W} \mathbf{r})] &= \nabla \times [\nabla \tilde{W} \times \mathbf{r}] \\ &= \nabla \tilde{W} (\underbrace{\nabla \cdot \mathbf{r}}_3) - \mathbf{r} \nabla \cdot (\nabla \tilde{W}) - \underbrace{(\nabla \tilde{W}) \cdot \nabla \mathbf{r}}_{\nabla \tilde{W}} + (\mathbf{r} \cdot \nabla)(\nabla \tilde{W}) \\ &= 2 \nabla \tilde{W} - \mathbf{r} \nabla^2 \tilde{W} + (\mathbf{r} \cdot \nabla)(\nabla \tilde{W}) \end{aligned} \quad (5)$$

Equations (4) and (5) show that the radial velocity satisfies

$$\bar{\rho} v_r = 2 \frac{\partial \tilde{W}}{\partial r} - r \nabla^2 \tilde{W} + r \frac{\partial}{\partial r} \left( \frac{\partial \tilde{W}}{\partial r} \right)$$

$$\begin{aligned}
&= r \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \tilde{W}}{\partial r} \right) - \nabla^2 \tilde{W} \right] \\
&= -\frac{1}{r} (\mathbf{r} \times \nabla)^2 \tilde{W},
\end{aligned} \tag{6}$$

where we have made use of the familiar identity involving the Laplacian in spherical coordinates and the quantum-mechanical total angular momentum operator (up to a constant coefficient)  $\mathcal{L}^2 := (\mathbf{r} \times \nabla)^2$ . Since  $\mathcal{L}^2$  is a purely “horizontal” operator, involving only derivatives with respect to  $\theta$  and  $\phi$  and not  $r$ , and since the eigenvalues of  $\mathcal{L}^2$  are  $l(l+1)$ ,  $l \in \mathbb{J}$  (all nonzero), the vanishing of  $v_r$  forces the vanishing of  $\tilde{W}$  and  $W$ :

$$v_r = 0 \iff \tilde{W} = W = 0. \tag{7}$$

We shall assume here that we work in a spherical shell of inner radius  $r_i$  and outer radius  $r_o$  that has impenetrable boundaries, i.e.,

$$v_r = \tilde{W} = W \equiv 0 \quad \text{at} \quad r = r_i \quad \text{and} \quad r = r_o. \tag{8}$$

We define the total angular momentum of the shell through

$$\mathbf{L} = \int_V \mathbf{r} \times (\bar{\rho} \mathbf{v}) dV, \tag{9}$$

where  $V$  is the region occupied by the shell. For convenience, we define the poloidal and toroidal angular momentum densities

$$\mathcal{L}_W := \mathbf{r} \times \{\nabla \times [\nabla \times (W \hat{\mathbf{e}}_r)]\} \quad \text{and} \quad \mathcal{L}_Z := \mathbf{r} \times [\nabla \times (Z \hat{\mathbf{e}}_r)], \tag{10}$$

so that

$$\mathbf{L} = \int_V \mathcal{L} dV, \tag{11}$$

where  $\mathcal{L} := \mathcal{L}_W + \mathcal{L}_Z$  is the total angular momentum density.

## 2 Contribution to the Angular Momentum from the Rotation of the Shell

Physically, angular momentum is only conserved in the non-rotating (lab) frame, in which the velocity is

$$\mathbf{v}_{\text{lab}} = \mathbf{v} + \Omega_0 r \sin \theta \hat{\mathbf{e}}_\phi \tag{12}$$

and the total angular momentum is

$$\mathbf{L}_{\text{lab}} = \int_V \mathbf{r} \times (\bar{\rho} \mathbf{v}_{\text{lab}}) dV = \mathbf{L}_0 + \mathbf{L}, \tag{13}$$

where

$$\mathbf{L}_0 := \int_V \mathbf{r} \times (\bar{\rho} \Omega_0 r \sin \theta \hat{\mathbf{e}}_\phi) dV = \Omega_0 \left( \int_V \bar{\rho} r^2 \sin^2 \theta dV \right) \hat{\mathbf{e}}_z \quad (14)$$

is the angular momentum due to the rotation of the shell. The angular integral in (14) is

$$\begin{aligned} 2\pi \int_0^\pi \sin^3 \theta d\theta &= -2\pi \int_{\cos \theta=1}^{\cos \theta=-1} (1 - \cos^2 \theta) d \cos \theta \\ &= 2\pi \left[ \cos \theta - \left( \frac{1}{3} \right) \cos^3 \theta \right]_{-1}^1 = \frac{8\pi}{3}, \end{aligned}$$

and so

$$\mathbf{L}_0 = \left[ \frac{8\pi \Omega_0}{3} \int_{r_i}^{r_o} \bar{\rho}(r) r^4 dr \right] \hat{\mathbf{e}}_z. \quad (15)$$

For an adiabatically stratified solar-like convection zone, in which

$$\begin{aligned} r_i &= 5.0000000 \times 10^{10} \text{ cm} \\ r_o &= 6.5860209 \times 10^{10} \text{ cm} \\ \rho_i &= 0.18053428 \text{ g cm}^{-3} \\ \rho_o &= 0.0089882725 \text{ g cm}^{-3} \\ \Omega_0 &= 8.61 \times 10^{-6} \text{ rad s}^{-1} \quad (3 \text{ times solar Carrington}), \end{aligned}$$

we compute

$$L_0 = 8.0719 \times 10^{47} \text{ g cm}^2 \text{ s}^{-1} \quad \text{and} \quad \mathcal{L}_0 = 1.1993 \times 10^{15} \text{ g cm}^{-1} \text{ s}^{-1}, \quad (16)$$

where  $\mathcal{L}_0 := L_0/|V|$  and  $|V| = 6.7302581 \times 10^{32} \text{ cm}^3$  is the volume of the shell.

### 3 Contribution to the Angular Momentum from the Toroidal Stream Function $Z$

As the name might suggest, the *toroidal* stream function  $Z$  gives the only non-vanishing contribution to the total angular momentum  $\mathbf{L}$ . We compute

$$\begin{aligned} \mathcal{L}_Z &= \mathbf{r} \times [\nabla \tilde{Z} \times \mathbf{r}] \\ &= \nabla \tilde{Z} (\mathbf{r} \cdot \mathbf{r}) - \mathbf{r} (\mathbf{r} \cdot \nabla \tilde{Z}) \\ &= r^2 \left[ \left( \frac{\partial \tilde{Z}}{\partial r} \right) \hat{\mathbf{e}}_r + \left( \frac{1}{r} \frac{\partial \tilde{Z}}{\partial \theta} \right) \hat{\mathbf{e}}_\theta + \left( \frac{1}{r \sin \theta} \frac{\partial \tilde{Z}}{\partial \phi} \right) \hat{\mathbf{e}}_\phi \right] - r^2 \left( \frac{\partial \tilde{Z}}{\partial r} \right) \hat{\mathbf{e}}_r \\ &= \left( r \frac{\partial \tilde{Z}}{\partial \theta} \right) \hat{\mathbf{e}}_\theta + \left( \frac{r}{\sin \theta} \frac{\partial \tilde{Z}}{\partial \phi} \right) \hat{\mathbf{e}}_\phi. \end{aligned}$$

The spherical unit vectors can be translated into Cartesian unit vectors via

$$\begin{aligned} \hat{\mathbf{e}}_\theta &= \cos \theta \cos \phi \hat{\mathbf{e}}_x + \cos \theta \sin \phi \hat{\mathbf{e}}_y - \sin \theta \hat{\mathbf{e}}_z \\ \text{and} \quad \hat{\mathbf{e}}_\phi &= -\sin \phi \hat{\mathbf{e}}_x + \cos \phi \hat{\mathbf{e}}_y. \end{aligned}$$

Thus,

$$(\mathcal{L}_Z)_x = r \cos \phi \cos \theta \left( \frac{\partial \tilde{Z}}{\partial \theta} \right) - \left( \frac{r}{\sin \theta} \right) \sin \phi \left( \frac{\partial \tilde{Z}}{\partial \phi} \right), \quad (17a)$$

$$(\mathcal{L}_Z)_y = r \sin \phi \cos \theta \left( \frac{\partial \tilde{Z}}{\partial \theta} \right) + \left( \frac{r}{\sin \theta} \right) \cos \phi \left( \frac{\partial \tilde{Z}}{\partial \phi} \right), \quad (17b)$$

$$\text{and} \quad (\mathcal{L}_Z)_z = -r \sin \theta \left( \frac{\partial \tilde{Z}}{\partial \theta} \right). \quad (17c)$$

In anelastic spherical harmonic codes, we expand the stream functions in terms of the spherical harmonics  $Y_{lm}(\theta, \phi) \sim P_{lm}(\cos \theta)e^{im\phi}$ , where the  $P_{lm}$  are the associated Legendre functions. We thus write

$$\tilde{Z}(r, \theta, \phi) = \sum_{l,m} \tilde{Z}_{lm}(r) Y_{lm}(\theta, \phi). \quad (18)$$

We consider the contributions to the total angular momentum of each  $\tilde{Z}_{lm}(r)$  separately, integrating the densities in (17) over the spherical shell and using orthogonality relations. We first note that the  $\phi$ -dependence of each spherical harmonic component of  $\tilde{Z}$  (and also  $\partial \tilde{Z} / \partial \phi$ ) is  $e^{im\phi}$ . The  $e^{im\phi}$  are orthogonal over the interval  $(0, 2\pi)$ , and since  $\cos \phi$  and  $\sin \phi$  can be written as linear combinations of  $e^{\pm i\phi}$ , we see that the only nonzero spherical harmonics contributing to the total angular momentum have

$$m = \pm 1 \quad \text{for} \quad L_x \quad \text{and} \quad L_y \quad (19)$$

$$\text{and} \quad m = 0 \quad \text{for} \quad L_z. \quad (20)$$

It will be helpful to also define  $x = \cos \theta$ , so that  $\partial / \partial \theta = -\sin \theta \partial / \partial x = -\sqrt{1-x^2} \partial / \partial x$ . The latitudinal integral is then over  $\int_{-1}^1 dx$ .

### 3.1 Equatorial Angular Momentum

For convenience we define

$$\begin{aligned} (\mathcal{L}_Z)_{\text{eq}} &:= (\mathcal{L}_Z)_x + i(\mathcal{L}_Z)_y \\ &= r e^{i\phi} \left( \cos \theta \frac{\partial \tilde{Z}}{\partial \theta} - \frac{1}{\sin \theta} \tilde{Z} \right) \end{aligned} \quad (21)$$

The latitudinal integral for each spherical harmonic will then be

$$\int_0^\pi \left[ \cos \theta \frac{\partial P_{l1}(\cos \theta)}{\partial \theta} - \frac{1}{\sin \theta} P_{l1} \right] \sin \theta d\theta = \int_{-1}^1 \left[ x(-\sin \theta) \frac{dP_{l1}}{dx} - \frac{1}{\sin \theta} P_{l1} \right] dx$$

$$= - \int_{-1}^1 \left[ (x\sqrt{1-x^2}) \frac{dP_{l1}}{dx} + \frac{1}{\sin \theta} P_{l1} \right] dx$$

This integral does not appear to vanish for any value of  $l$ , let alone all  $l \neq 1$ . Thus, it seems that the conclusion drawn in [Jones et al. \(2011\)](#), namely that only the  $l = 1, m = 1$  components contribute to the equatorial angular momentum (their Equations A12 and A13), is incorrect.

### 3.2 Axial Angular Momentum

Since only the  $m = 0$  harmonics contribute to  $(\mathcal{L}_Z)_z$ , only the functions  $P_{l0}(\cos \theta) = P_l(x)$  (the non-associated Legendre polynomials) appear in the  $\theta$ -integral. The  $\theta$ -integral (excluding constant factors) can be transformed as

$$\begin{aligned} \int_0^\pi \sin^2 \theta \left( \frac{\partial P_l}{\partial \theta} \right) d\theta &\sim \int_{-1}^1 \sin \theta \left( \sin \theta \frac{\partial P_l}{\partial x} \right) dx \\ &= \int_{-1}^1 (1-x^2) \left( \frac{dP_l}{dx} \right) dx \\ &\sim \int_{-1}^1 x P_l(x) dx. \end{aligned}$$

In the final manipulation we have used integration by parts and thrown away the boundary term due to the vanishing of  $1-x^2$  at  $x = \pm 1$ . We recall that  $x = P_1(x)$ , and since the Legendre polynomials are orthogonal over the interval  $(-1, 1)$ , only the  $l = 1$  (and  $m = 0$ ) spherical harmonic contributes to  $(\mathcal{L}_Z)_z$ . Under unit normalization, we have  $Y_{10} = (1/2)\sqrt{3/\pi} \cos \theta$ , from which

$$(\mathcal{L}_Z)_z = -r \sin \theta \left( -\frac{1}{2} \sqrt{\frac{3}{\pi}} \sin \theta \right) \tilde{Z}_{10}(r) = \frac{1}{2} \sqrt{\frac{3}{\pi}} \sin^2 \theta r \tilde{Z}_{10}(r)$$

and

$$(L_Z)_z = \frac{4}{3} \sqrt{3\pi} \int_{r_i}^{r_o} r^3 \tilde{Z}_{10}(r) dr = \frac{4}{3} \sqrt{3\pi} \int_{r_i}^{r_o} r^2 Z_{10}(r) dr \quad (22)$$

The constant on the RHS of (22) will depend on the convention for the normalization of the spherical harmonics. For example, compare to [Jones et al. \(2011\)](#), Equations A11–A14.

## 4 Contribution to the Angular Momentum from the Poloidal Stream Function $W$

We shall now show that the contribution to the total angular momentum from  $\mathcal{L}_W$  mathematically vanishes for a spherical shell. We use (5) to compute

$$\mathcal{L}_W = \mathbf{r} \times [2\nabla \tilde{W} + (\mathbf{r} \cdot \nabla)(\nabla \tilde{W})] \implies$$

$$(\mathcal{L}_W)_i = \epsilon_{ijk} r_j [2\partial_k \tilde{W} + (\mathbf{r} \cdot \nabla)(\partial_k \tilde{W})].$$

We note that

$$\epsilon_{ijk} r_j \partial_k \tilde{W} = \epsilon_{ijk} [\partial_k (r_j \tilde{W}) - \underbrace{\tilde{W} \partial_k r_j}_{\delta_{kj}}] = \partial_k (\epsilon_{ijk} r_j \tilde{W}),$$

$$\partial_k \mathbf{r} = \partial_k (r_i \hat{\mathbf{e}}_i) = \delta_{ik} \hat{\mathbf{e}}_i = \hat{\mathbf{e}}_k,$$

and

$$\begin{aligned} \epsilon_{ijk} r_j (\mathbf{r} \cdot \nabla)(\partial_k \tilde{W}) &= \epsilon_{ijk} \{ \partial_k [r_j (\mathbf{r} \cdot \nabla \tilde{W})] - \underbrace{(\partial_k r_j)(\mathbf{r} \cdot \nabla \tilde{W})}_{\delta_{kj}} - r_j \underbrace{(\partial_k \mathbf{r}) \cdot \nabla \tilde{W}}_{\hat{\mathbf{e}}_k \cdot \nabla \tilde{W} = \partial_k \tilde{W}} \} \\ &= \partial_k [\epsilon_{ijk} r_j (\mathbf{r} \cdot \nabla \tilde{W})] - \underbrace{\epsilon_{ijk} r_j \partial_k \tilde{W}}_{\partial_k (\epsilon_{ijk} r_j \tilde{W})} \\ &= \partial_k [\epsilon_{ijk} r_j (\mathbf{r} \cdot \nabla \tilde{W} - \tilde{W})] \end{aligned}$$

Thus,

$$(\mathcal{L}_W)_i = \partial_k [\epsilon_{ijk} r_j (\tilde{W} + \mathbf{r} \cdot \nabla \tilde{W})]$$

and (using the divergence theorem)

$$(\mathbf{L}_W)_i := \int_V (\mathcal{L}_W)_i dV = \oint_{\partial V} [\epsilon_{ijk} r_j (\tilde{W} + \mathbf{r} \cdot \nabla \tilde{W})] n_k dS,$$

where  $\mathbf{n}$  is the unit normal to  $\partial V$  and  $dS$  is an area element on  $\partial V$ . The boundary  $\partial V$  consists of the two spheres  $r = r_i$  and  $r = r_o$ , so  $n_k = \pm r_k/r$ , and the identity  $\epsilon_{ijk} r_j r_k \equiv 0$  immediately yields

$$(\mathbf{L}_W)_i = 0. \tag{23}$$

So actually the impenetrability condition has nothing to do with the vanishing of the angular momentum from the poloidal stream function! We only require that the integration region is a spherical shell. Note that ? derive (in their Equation A10)

$$(\mathbf{L}_W)_i = \oint_{\partial V} \epsilon_{ijk} r_j \left[ \frac{\partial(r_m \tilde{W})}{\partial x_k} - \frac{\partial(r_k \tilde{W})}{\partial x_m} \right] n_m dS \tag{24}$$

(note that they define different stream functions  $P = \tilde{W}/\bar{\rho}$  and  $T = \tilde{Z}/\bar{\rho}$ ). They attribute the vanishing of the integral due to the fact that  $\tilde{W}$  (and by extension,  $\mathbf{r} \times \nabla \tilde{W}$ ) vanishes on  $\partial V$  due to the impenetrability condition (8). This is unnecessary, however. One can immediately see that the second term in the integrand is zero:

$$\epsilon_{ijk} r_j \frac{\partial(r_k \tilde{W})}{\partial x_m} n_m = \epsilon_{ijk} \left[ \delta_{km} \tilde{W} + r_k \frac{\partial \tilde{W}}{\partial x_m} \right] \frac{r_j (\pm r_m)}{r} \equiv 0,$$

where we have used the asymmetry of  $\epsilon_{ijk}$  in  $j$  and  $k$  and the identity  $\delta_{km}r_m = r_k$ .  
The remaining part of the integral may be written

$$\begin{aligned}
\oint \epsilon_{ijk} r_j \left[ \frac{\partial(r_m \tilde{W})}{\partial x_k} \right] n_m dS &= \oint \left[ \frac{\partial}{\partial x_k} (\epsilon_{ijk} r_j r_m \tilde{W}) \right] n_m dS \\
&= \int_V \frac{\partial}{\partial x_m} \left[ \frac{\partial}{\partial x_k} (\epsilon_{ijk} r_j r_m \tilde{W}) \right] dV \\
&= \int_V \frac{\partial}{\partial x_k} \left[ \frac{\partial}{\partial x_m} (\epsilon_{ijk} r_j r_m \tilde{W}) \right] dV \\
&= \oint \epsilon_{ijk} \left[ \frac{\partial}{\partial x_m} (\epsilon_{ijk} r_j r_m \tilde{W}) \right] n_k dS \\
&= \oint \epsilon_{ijk} [\delta_{jm} r_m \tilde{W} + r_j \partial_m (r_m \tilde{W})] \left( \frac{\pm r_k}{r} \right) dS,
\end{aligned}$$

whose integrand can be shown to vanish using previous arguments on the asymmetry of  $\epsilon_{ijk}$ .

## References

Jones, C., Boronski, P., Brun, A., Glatzmaier, G., Gastine, T., Miesch, M., & Wicht, J.  
2011, “Anelastic convection-driven dynamo benchmarks”, *Icar.*, 216, 120, doi: [10.1016/j.icarus.2011.08.014](https://doi.org/10.1016/j.icarus.2011.08.014)