General Anelastic Non-Dimensionalization for Rayleigh

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March 8, 2023

1 Dimensional Equations

We begin by writing down the full dimensional anelastic fluid equations, as implemented in Rayleigh.

$$\nabla\cdot[\overline{\rho}(r)\boldsymbol{u}]=0, \tag{1.1}$$
$$\nabla\cdot\boldsymbol{B}=0, \tag{1.2}$$

$$\begin{aligned}\overline{\rho}(r)\left[\frac{D\boldsymbol{u}}{Dt}+2\Omega_0\hat{\boldsymbol{e}}_z\times\boldsymbol{u}\right]=&-\overline{\rho}(r)\nabla\left[\frac{P}{\overline{\rho}(r)}\right]+\left[\frac{\overline{\rho}(r)g(r)}{c_\mathrm{p}}\right]S\hat{\boldsymbol{e}}_r,\\ &+\nabla\cdot\boldsymbol{D}+\frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B},\end{aligned} \tag{1.3a}$$

$$\text{where}\qquad D_{ij}\coloneqq 2\overline{\rho}(r)\nu(r)\left[e_{ij}-\frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{1.3b}$$

$$\text{and}\qquad e_{ij}\coloneqq\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{1.3c}$$

$$\begin{aligned}\overline{\rho}(r)\overline{T}(r)\frac{DS}{Dt}=&-\overline{\rho}(r)\overline{T}(r)\frac{d\overline{S}}{dr}u_r+\nabla\cdot[\overline{\rho}(r)\overline{T}(r)\kappa(r)\nabla S]\\ &+Q(r)+D_{ij}e_{ij}+\frac{\eta(r)}{4\pi}|\nabla\times\boldsymbol{B}|^2,\end{aligned} \tag{1.4}$$

$$\text{and}\qquad\frac{\partial\boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{u}\times\boldsymbol{B})-\nabla\times[\eta(r)\nabla\times\boldsymbol{B}]. \tag{1.5}$$

Here, the thermal variables ρ, T, P, and S refer to the density, temperature, pressure, and entropy (respectively). An overbar on a thermal variable indicates the time-independent, spherically symmetric reference state and the lack of an overbar indicates the (assumed small) deviation from the background (for the entropy, S/c_p is assumed small). The vector velocity is denoted by $\boldsymbol{u}$ and the vector magnetic field by $\boldsymbol{B}$.

We use standard spherical coordinates (r,θ,ϕ) and cylindrical coordinates $(\lambda,\phi,z)=(r\sin\theta,\phi,r\cos\theta)$, and $\hat{\boldsymbol{e}}_q$ in general denotes a position-dependent unit vector in the direction of increasing q. The equations are written in a frame rotating with angular velocity Ω_0 and the centrifugal force is neglected.

The additional reference state variables $\nu(r)$, $\kappa(r)$, and $\eta(r)$ (the momentum, thermal, and magnetic diffusivities, respectively), as well as the gravitational acceleration $g(r) = GM_*/r^2$ (where G is the gravitational constant and M_* the stellar mass), are also assumed to be spherically symmetric and fixed in time.

Note that the internal heating function $Q(r)$ is also assumed spherically symmetric and fixed in time, but should be interpreted as $-\nabla \cdot \boldsymbol{\mathcal{F}}_{\rm rad}$, where $\boldsymbol{\mathcal{F}}_{\rm rad}$ is the radiative heat flux and properly should be proportional to the gradient of the total (background + perturbed) temperature. If the system is a convection zone, it is driven by a combination of internal heating and the thermal boundary conditions (conditions on S), that together ensure a stellar luminosity is transported throughout the layer in a steady state.

Note that Equation (1.3c) is only valid in a Cartesian coordinate system (x_1, x_2, x_3) (with i and j running over 1, 2, 3) and is translated into spherical coordinates before being used in Rayleigh.

Finally, we recall the relation,

$$\frac{d\overline{S}}{dr} = c_{\rm p}\frac{N^2(r)}{g(r)}, \tag{1.6}$$

where $N^2(r)$ is the squared buoyancy frequency.

2 Non-Dimensional Scheme

We now non-dimensionalize Equations (1.1)–(1.5), according to the following scheme:

$$\nabla \rightarrow \frac{1}{L}\nabla, \tag{2.1a}$$

$$t \rightarrow \tau t, \tag{2.1b}$$

$$\boldsymbol{u} \rightarrow \frac{L}{\tau}\boldsymbol{u}, \tag{2.1c}$$

$$S \rightarrow \sigma S, \tag{2.1d}$$

$$P \rightarrow \tilde{\rho}\frac{L^2}{\tau^2}P, \tag{2.1e}$$

$$\boldsymbol{B} \rightarrow (4\pi\tilde{\rho})^{1/2}\frac{L}{\tau}\boldsymbol{B}, \tag{2.1f}$$

$$\overline{\rho}(r) = \tilde{\rho}\hat{\rho}(r), \tag{2.1g}$$

$$\overline{T}(r) = \tilde{T}\hat{T}(r), \tag{2.1h}$$

$$g(r) = \tilde{g}\hat{g}(r), \tag{2.1i}$$

$$N^2(r) = \widetilde{N^2}\widehat{N^2}(r), \tag{2.1j}$$

$$\nu(r) = \tilde{\nu}\hat{\nu}(r), \tag{2.1k}$$

$$\kappa(r) = \tilde{\kappa}\hat{\kappa}(r), \tag{2.1l}$$

$$\text{and} \qquad \eta(r) = \tilde{\eta}\hat{\eta}(r). \tag{2.1m}$$

Here, L is a typical length-scale, τ is a typical time-scale, and σ is a typical entropy scale. On the right-hand-sides of Equations (2.1a)–(2.1f) (and in the following non-dimensionalizations), ∇, t, $\boldsymbol{u}$, S, P, and $\boldsymbol{B}$ are all understood to be non-dimensional. In Equations (2.1g)–(2.1m), the tildes refer to"typical values" of the reference state functions and the hats refer to the radially-dependent non-dimensional versions of the reference-state functions.

Below, we will assume the time scale is either a viscous diffusion time (i.e., $\tau = L^2/\tilde{\nu}$) or a rotational time-scale (i.e., $\tau = \Omega_0^{-1}$). To describe the reference state, we will consider three cases for a given function's "typical value": Its value at the inner shell boundary, its value at the outer shell boundary, or its value volume-averaged over the shell.

3 Non-Dimensional Equations; $\tau = L^2/\tilde{\nu}$

In this case, Equations (1.1)–(1.5) become

$$\nabla \cdot [\hat{\rho}(r)\boldsymbol{u}] = 0, \tag{3.1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{3.2}$$

$$\begin{aligned}\hat{\rho}(r)\left[\frac{D\boldsymbol{u}}{Dt} + \frac{2}{\mathrm{Ek}}\hat{\boldsymbol{e}}_z \times \boldsymbol{u}\right] =& -\hat{\rho}(r)\nabla\left[\frac{P}{\hat{\rho}(r)}\right] + \frac{\mathrm{Ra}}{\mathrm{Pr}}\hat{\rho}(r)\hat{g}(r)S\hat{\boldsymbol{e}}_r,\\ &+ \nabla \cdot \boldsymbol{D} + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B},\end{aligned} \tag{3.3a}$$

$$\text{where} \qquad D_{ij} \coloneqq 2\hat{\rho}(r)\hat{\nu}(r)\left[e_{ij} - \frac{1}{3}(\nabla \cdot \boldsymbol{u})\delta_{ij}\right] \tag{3.3b}$$

$$\text{and} \qquad e_{ij} \coloneqq \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{3.3c}$$

$$\begin{aligned}\hat{\rho}(r)\hat{T}(r)\frac{DS}{Dt} =& -\frac{\mathrm{Pr}}{\mathrm{Ra}}\mathrm{B}_{\mathrm{visc}}\hat{\rho}(r)\hat{T}(r)\frac{\widehat{N^2}(r)}{\hat{g}(r)}u_r + \frac{1}{\mathrm{Pr}}\nabla \cdot [\hat{\rho}(r)\hat{T}(r)\hat{\kappa}(r)\nabla S]\\ &+ \frac{1}{\mathrm{Pr}}\hat{Q}(r) + \frac{\mathrm{PrDi}}{\mathrm{Ra}}D_{ij}e_{ij} + \frac{\mathrm{PrDi}}{\mathrm{Pr_m Ra}}\hat{\eta}(r)|\nabla \times \boldsymbol{B}|^2,\end{aligned} \tag{3.4}$$

$$\text{and} \qquad \frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \frac{1}{\mathrm{Pr_m}}\nabla \times [\eta(r)\nabla \times \boldsymbol{B}]. \tag{3.5}$$

The non-dimensional numbers appearing are:

$$\mathrm{Ra} \coloneqq \frac{\tilde{g}L^3}{\tilde{\nu}\tilde{\kappa}}\frac{\sigma}{c_\mathrm{p}}, \tag{3.6a}$$

$$\mathrm{Pr} \coloneqq \frac{\tilde{\nu}}{\tilde{\kappa}}, \tag{3.6b}$$

$$\mathrm{Pr_m} \coloneqq \frac{\tilde{\nu}}{\tilde{\eta}}, \tag{3.6c}$$

$$\mathrm{Ek} \coloneqq \frac{\tilde{\nu}}{\Omega_0 H^2}, \tag{3.6d}$$

$$\mathrm{B_{visc}} \coloneqq \frac{\widetilde{N^2}L^4}{\tilde{\nu}^2}, \tag{3.6e}$$

$$\text{and} \qquad \mathrm{Di} = \frac{\tilde{g}L}{c_\mathrm{p}\tilde{T}}, \tag{3.6f}$$

along with the non-dimensional heating function

$$\hat{Q}(r) \coloneqq \frac{L^2}{\tilde{\rho}\tilde{T}\tilde{\kappa}\sigma}Q(r). \tag{3.7}$$

Note that the dissipation number is not an independent control parameter, but a function of the non-dimensional parameters characterizing the polytrope.

In general, $\hat{Q}(r)$ is simply an arbitrary—hopefully order unity—function. Assuming the thermal boundary conditions remove whatever $\hat{Q}(r)$ dumps in: If $\hat{Q}(r) \gg 1$, the user is dilating their Rayleigh number without saying so. If $\hat{Q} \ll 1$, the user is contracting their Rayleigh number without saying so. If $\hat{Q}(r) \equiv 0$ (and the user wants to simulate a convection zone), the user should typically identify σ with $-\Delta S = S_\mathrm{in} - S_\mathrm{out}$ (the imposed entropy drop across the layer) and thus set $S \equiv 1$ at $r = r_\mathrm{in}$ and $S \equiv 0$ at $r = r_\mathrm{out}$.

The non-dimensional heating takes a specific form if we assume the Rayleigh number is a "flux" Rayleigh number. In that case we identify

$$\sigma = \frac{L\left\langle \mathcal{F}_\mathrm{nr}(r)\right\rangle_\mathrm{v}}{\tilde{\rho}\tilde{T}\tilde{\kappa}}, \tag{3.8}$$

$$\text{where} \qquad \mathcal{F}_\mathrm{nr}(r) \coloneqq \frac{1}{r^2}\int_{r_\mathrm{in}}^{r} Q(x)x^2 dx \tag{3.9}$$

is the flux not carried by radiation in a statistically steady state and $\langle \cdots \rangle_\mathrm{v}$ refers to a volume average over the whole shell. We thus have

$$\hat{Q}(r) = \frac{L}{\left\langle \mathcal{F}_\mathrm{nr}(r)\right\rangle_\mathrm{v}}Q(r) \tag{3.10}$$

and whatever amplitude (luminosity) the user chooses for the dimensional $Q(r)$, the ultimate $\hat{Q}(r)$ will normalize that amplitude away in Equation (3.10).

The viscous buoyancy number $\mathrm{B_{visc}}$ is the ratio of the typical squared buoyancy frequency to the squared viscous diffusion time (it is essentially a kind of Richardson number). Although it has to do with background entropy stratification, $\mathrm{B_{visc}}$ is nominally independent of the Rayleigh number (which derives from the entropy perturbations associated with the thermal boundary conditions and/or heating that force energy through the layer). However, the following notes are warranted:

(1) In the typical convection problem (polytropic index $n = n_{\rm ad} \coloneqq 1/(\gamma-1)$), $d\overline{S}/dr \equiv 0$ and the value of $\mathrm{B_{visc}}$ is irrelevant.

(2) For an isolated stable polytrope ($n > n_{\rm ad}$), it is unclear how the typical entropy perturbation σ is established. Even if energy is driven through the system by the thermal boundary conditions and/or heating, the energy will likely be carried by spherically symmetric conduction (depending on how stable the stratification is) and σ may not be set by the boundary conditions. Furthermore, σ cannot be set by the total entropy contrast across the layer (which is $\sim c_{\rm p}$, and if $\sigma = c_{\rm p}$, we would have $\mathrm{B_{visc}Pr} = \mathrm{Ra}$). That is because no plume can traverse a large portion of the stably stratified layer. Thus, although the user can choose a "Rayleigh number" (probably a misnomer) for the isolated stable layer, they have no way of implementing the background state, boundary conditions, or heating profile to be consistent with their choice.

(3) For unstable polytropes ($n < n_{\rm ad}$), we first note that if n differs by a factor of unity from $n_{\rm ad}$, order-unity thermal perturbations $S/c_{\rm p}$ are forced and the anelastic approximation is invalid. In any case (either if the simulation survives an order-unity $n_{\rm ad} - n$, or if the user chooses a well-posed small $n_{\rm ad} - n$ to help drive the convection), we expect that the convection will restratify the system toward adiabaticity. In Equation (1.4), a background $(dS/dr)_{\ell=0}$ will be established to be close to the negative of $d\overline{S}/dr$ and (similar to point (1)), the value of $\mathrm{B_{visc}}$ will not truly be an independent parameter, but will help to determine an ultimate "effective" Rayleigh number.

(4) In the more logical case of a stable layer being pummeled by a neighboring overshooting convection layer, both $\mathrm{B_{visc}}$ and Ra are truly independent (and relevant). The overshooting flows (driven by Ra, which is a property of the convection zone) establish the typical σ in both the convection zone and overshoot layer. Meanwhile, $\mathrm{B_{visc}}$ (which is a property of the stable layer) controls how strongly the overshoot is decelerated.

4 Non-Dimensional Equations; $\tau = \Omega_0^{-1}$

In the previous section, t (and things with time in the dimensions) was implied to mean $(\tilde{\nu}/H^2)t_{\rm dim}$, where $t_{\rm dim}$ was the dimensional time. We now want to use $t_{\rm new} = \Omega_0 t_{\rm dim} = t/\mathrm{Ek}$. We can thus find the new equations easily from Equations (3.1)–(3.5): Every place we see a time dimension, we recall $t = \mathrm{Ek}t_{\rm new}$, so we multiply the place where the dimension appears by Ek and drop the "new" subscript (e.g., $t \to \mathrm{Ek}\, t$, $\boldsymbol{u} \to \boldsymbol{u}/\mathrm{Ek}$, etc.). We thus find (after rearranging terms)

$$\nabla \cdot [\hat{\rho}(r)\boldsymbol{u}] = 0, \tag{4.1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{4.2}$$

$$\hat{\rho}(r)\left[\frac{D\boldsymbol{u}}{Dt}+2\hat{\boldsymbol{e}}_z\times\boldsymbol{u}\right]=-\,\hat{\rho}(r)\nabla\left[\frac{P}{\hat{\rho}(r)}\right]+\mathrm{Ra}^*\hat{\rho}(r)\hat{g}(r)S\hat{\boldsymbol{e}}_r,$$
$$+\,\mathrm{Ek}\nabla\cdot\boldsymbol{D}+(\nabla\times\boldsymbol{B})\times\boldsymbol{B}, \tag{4.3a}$$
$$\text{where}\qquad D_{ij}\coloneqq 2\hat{\rho}(r)\hat{\nu}(r)\left[e_{ij}-\frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{4.3b}$$
$$\text{and}\qquad e_{ij}\coloneqq\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{4.3c}$$

$$\hat{\rho}(r)\hat{T}(r)\frac{DS}{Dt}=-\,\frac{\mathrm{B_{rot}}}{\mathrm{Ra}^*}\hat{\rho}(r)\hat{T}(r)\frac{\widehat{N^2}(r)}{\hat{g}(r)}u_r+\frac{\mathrm{Ek}}{\mathrm{Pr}}\nabla\cdot[\hat{\rho}(r)\hat{T}(r)\hat{\kappa}(r)\nabla S]$$
$$+\,\frac{\mathrm{Ek}}{\mathrm{Pr}}\hat{Q}(r)+\frac{\mathrm{DiEk}}{\mathrm{Ra}^*}D_{ij}e_{ij}+\frac{\mathrm{DiEk}}{\mathrm{Pr_m}\mathrm{Ra}^*}\hat{\eta}(r)|\nabla\times\boldsymbol{B}|^2, \tag{4.4}$$

$$\text{and}\qquad \frac{\partial\boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{u}\times\boldsymbol{B})-\frac{\mathrm{Ek}}{\mathrm{Pr_m}}\nabla\times[\eta(r)\nabla\times\boldsymbol{B}]. \tag{4.5}$$

The new non-dimensional numbers appearing are:

$$\mathrm{Ra}^*\coloneqq\frac{\mathrm{Ek}^2}{\mathrm{Pr}}\mathrm{Ra}=\frac{\tilde{g}}{L\Omega_0^2}\frac{\sigma}{c_\mathrm{p}}, \tag{4.6a}$$
$$\text{and}\qquad \mathrm{B_{rot}}\coloneqq\mathrm{Ek}^2\mathrm{B_{visc}}=\frac{\widetilde{N^2}}{\Omega_0^2}\sim\frac{\tilde{g}}{L\Omega_0^2}=\frac{1}{\text{oblateness factor}}. \tag{4.6b}$$

Note that although the "$d\overline{S}/dr$-terms" in the non-dimensionalizations have seemingly different definitions, they are the same, since:

$$\frac{\mathrm{Pr}}{\mathrm{Ra}}\mathrm{B_{visc}}=\frac{\mathrm{B_{rot}}}{\mathrm{Ra}^*}\sim\frac{c_\mathrm{p}}{\sigma}. \tag{4.6c}$$

5 Non-Dimensional Polytrope

A polytrope depends on the following four non-dimensional parameters:

$$\gamma\coloneqq\frac{c_\mathrm{p}}{c_\mathrm{v}}\qquad\text{specific-heat ratio,} \tag{5.1a}$$
$$0\le n\le\infty\qquad\text{polytropic index,} \tag{5.1b}$$
$$N_\rho\coloneqq\ln\left(\frac{\overline{\rho}_\mathrm{in}}{\overline{\rho}_\mathrm{out}}\right)\qquad\text{number of density scale-heights,} \tag{5.1c}$$
$$\text{and}\qquad \beta=\frac{r_\mathrm{in}}{r_\mathrm{out}}\qquad\text{aspect ratio.} \tag{5.1d}$$

If the typical values of the polytrope are taken at the inner boundary, we have

$$\hat{T}(r) = \frac{\overline{T}(r)}{\overline{T}_{\text{in}}} = \left[\frac{\beta(1-e^{-N_\rho/n})}{(1-\beta)^2}\right]\left(\frac{H}{r}\right) - \left(\frac{\beta - e^{-N_\rho/n}}{1-\beta}\right) \tag{5.2a}$$

$$\hat{\rho}(r) = \frac{\overline{\rho}(r)}{\overline{\rho}_{\text{in}}} = \left\{\left[\frac{\beta(1-e^{-N_\rho/n})}{(1-\beta)^2}\right]\left(\frac{H}{r}\right) - \left(\frac{\beta - e^{-N_\rho/n}}{1-\beta}\right)\right\}^n, \tag{5.2b}$$

$$\widehat{N^2}(r) = \frac{N^2(r)}{\widetilde{N^2}} = \left(\frac{r_{\text{in}}}{r}\right)^3\left[\frac{1-e^{-N_\rho/n}}{1-\beta} - \left(\frac{\beta - e^{-N_\rho/n}}{\beta}\right)\frac{r}{H}\right]^{-1}, \tag{5.2c}$$

$$\hat{g}(r) = \frac{g(r)}{g_{\text{in}}} = \frac{r_{\text{in}}^2}{r^2}, \tag{5.2d}$$

$$\text{and} \qquad \mathrm{Di} = \frac{L}{H}\left(\frac{n+1}{\tilde{n}+1}\right)\left(\frac{1}{\beta}\right)(1-e^{-N_\rho/n}), \tag{5.2e}$$

where $H = r_{\text{out}} - r_{\text{in}}$ is the shell depth.

Note that the range on r/H is (by definition)

$$\frac{\beta}{1-\beta} \leq \frac{r}{H} \leq \frac{1}{1-\beta}. \tag{5.3}$$

If instead we take the typical values at the outer boundary, it is simple to compute the ratios from outer to inner directly from Equations (5.2) and thus change the non-dimensionalization of the polytrope.

If we instead take the typical values as volume-averages, the density profile in Equations (5.2) (because of the n exponent) must be integrated numerically (or else computed from the hypergeometric function) but it is again straightforward to re-scale. Because the formulas are complicated (and I am error-prone), it is easiest to numerically integrate the other functions as well. We note, however, the analytic formula for Di when volume averages are used:

$$\begin{aligned}\mathrm{Di}_{\mathrm{v}} &\coloneqq \frac{\langle g\rangle_{\mathrm{v}}/g_{\text{in}}}{\langle\overline{T}\rangle_{\mathrm{v}}/\overline{T}_{\text{in}}}\mathrm{Di} \\ &= \frac{L}{H}\left(\frac{n+1}{\tilde{n}+1}\right)\frac{3\beta(1-\beta)^2(1-e^{-N_\rho/n})}{(3\beta/2)(1-\beta^2)(1-e^{-N_\rho/n}) - (1-\beta^3)(\beta - e^{-N_\rho/n})}.\end{aligned} \tag{5.4}$$

This emphases the fact that $\mathrm{Di} = \mathrm{Di}(\gamma, n, N_\rho, \beta, L/H)$ is not an independent control parameter of the system.