Induction term in spherical coordinates

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1 The problem with the traditional induction terms

We consider the ideal (resistance-free) magnetohydrodynamic (MHD) induction equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \coloneqq \boldsymbol{I}, \tag{1}$$

$$= \boldsymbol{B} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{B} - (\nabla \cdot \boldsymbol{u})\boldsymbol{B}, \tag{2}$$

where $\boldsymbol{B}$ and $\boldsymbol{u}$ are the vector magnetic and velocity fields (respectively) and $\boldsymbol{I}$ is the vector induction. The three terms on the right-hand-side of Equation (2) are often interpreted as "shear," "advection," and "compression," respectively. However, this interpretation is problematic in general for two reasons:

1. The so-called shear and compression terms contain sub-terms that cancel; in particular, only velocity motions *transverse* to magnetic-field lines can shear or compress.
2. Solid-body rotation (which is a non-shearing motion that simply rotates the whole field configuration) shows up in the so-called shear and advection terms in a strange way.

Finally, even after these issues have been addressed, resolving the final terms into a particular curvilinear system (e.g., spherical coordinates) seems to present a major headache. Our goal here is to explain fully how these problems emerge and propose a tentative solution for the case of spherical coordinates.

2 Transverse shear and compression

To see how Problem 1 arises, we decompose the velocity field into components parallel and perpendicular to the local direction of $\boldsymbol{B}$:

$$\boldsymbol{u} \coloneqq u_{\parallel} \hat{\boldsymbol{e}}_{\parallel} + \boldsymbol{u}_{\perp}. \tag{3}$$

Obviously $\boldsymbol{B} = Bx_{\parallel}$, where $B = |\boldsymbol{B}|$. We denote the Cartesian distance along $\boldsymbol{B}$ by $x_{\parallel}$. We also decompose $\boldsymbol{u}$ into its parallel and perpendicular components:

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_{\parallel}}{\partial x_{\parallel}} + \nabla_{\perp} \cdot \boldsymbol{u}_{\perp}. \tag{4}$$

We then calculate

$$\begin{aligned}
\boldsymbol{B}\cdot\nabla\boldsymbol{u}-(\nabla\cdot\boldsymbol{u})\boldsymbol{B} &= B\frac{\partial}{\partial x_\parallel}(u_\parallel\hat{\boldsymbol{e}}_\parallel+\boldsymbol{u}_\perp)-\left(\frac{\partial u_\parallel}{\partial x_\parallel}+\nabla_\perp\cdot\boldsymbol{u}_\perp\right)B\hat{\boldsymbol{e}}_\parallel \\
&= \cancel{B\frac{\partial u_\parallel}{\partial x_\parallel}\hat{\boldsymbol{e}}_\parallel}+B\frac{\partial\boldsymbol{u}_\perp}{\partial x_\parallel}-\cancel{\frac{\partial u_\parallel}{\partial x_\parallel}B\hat{\boldsymbol{e}}_\parallel}-(\nabla_\perp\cdot\boldsymbol{u}_\perp)B\hat{\boldsymbol{e}}_\parallel \\
&= \boldsymbol{B}\cdot\nabla\boldsymbol{u}_\perp-(\nabla_\perp\cdot\boldsymbol{u}_\perp)\boldsymbol{B}.
\end{aligned} \tag{5}$$

Thus, only motions transverse to the local field line (i.e., $\boldsymbol{u}_\perp$) can shear or compress $\boldsymbol{B}$.

3 Rigid rotation

To see how Problem 2 arises, we consider a velocity field due to rigid rotation at constant angular velocity Ω about the z-axis in a cylindrical coordinate system:

$$\boldsymbol{\Omega}=\Omega\hat{\boldsymbol{e}}_z=\text{constant} \tag{6a}$$

$$\text{and}\qquad \boldsymbol{u}=\boldsymbol{\Omega}\times\boldsymbol{r}=\Omega\lambda\hat{\boldsymbol{e}}_\phi. \tag{6b}$$

Here, λ is the cylindrical radius, ϕ the longitude, z the axial coordinate, and $\boldsymbol{r}:=r\hat{\boldsymbol{e}}_r$ the position vector (also see Figure 1). In general, $\hat{\boldsymbol{e}}_\alpha$ denotes a unit vector in the direction of the coordinate α. We calculate:

$$\begin{aligned}
\nabla\cdot\boldsymbol{u} &= \nabla\cdot(\boldsymbol{\Omega}\times\boldsymbol{r}) \\
&= \boldsymbol{\Omega}\cdot\nabla\times\boldsymbol{r}-\boldsymbol{r}\cdot\nabla\times\boldsymbol{\Omega} \\
&= 0 \qquad \text{no compression for rigid rotation (obviously).}
\end{aligned} \tag{7}$$

Then:

$$\begin{aligned}
\boldsymbol{B}\cdot\nabla\boldsymbol{u} &= (\boldsymbol{B}\cdot\nabla)(\boldsymbol{\Omega}\times\boldsymbol{r}) \\
&= \boldsymbol{\Omega}\times[(\boldsymbol{B}\cdot\nabla)\boldsymbol{r}] \\
&= \boldsymbol{\Omega}\times\boldsymbol{B} \qquad \text{“shear” for rigid rotation.}
\end{aligned} \tag{8}$$

Finally:

$$\begin{aligned}
-\boldsymbol{u}\cdot\nabla\boldsymbol{B} &= -\Omega\lambda\hat{\boldsymbol{e}}_\phi\cdot\nabla\boldsymbol{B} \\
&= -\Omega\frac{\partial}{\partial\phi}(B_\lambda\hat{\boldsymbol{e}}_\lambda+B_\phi\hat{\boldsymbol{e}}_\phi+B_z\hat{\boldsymbol{e}}_z) \\
&= -\Omega\sum_\alpha\left(\frac{\partial B_\alpha}{\partial\phi}\right)\hat{\boldsymbol{e}}_\alpha-\Omega\sum_\alpha B_\alpha\frac{\partial\hat{\boldsymbol{e}}_\alpha}{\partial\phi},
\end{aligned}$$

where the index α runs over the three cylindrical coordinates. Note that in the cylindrical coordinate system (or indeed any coordinate system with an axis of rotational symmetry),

$$\frac{\partial\hat{\boldsymbol{e}}_\alpha}{\partial\phi}=\hat{\boldsymbol{e}}_z\times\hat{\boldsymbol{e}}_\alpha \qquad \text{for each } \alpha. \tag{9}$$

Thus,

$$\begin{aligned}
-\Omega \sum_\alpha B_\alpha \frac{\partial \hat{\boldsymbol{e}}_\alpha}{\partial \phi} &= -\Omega \sum_\alpha B_\alpha \hat{\boldsymbol{e}}_z \times \hat{\boldsymbol{e}}_\alpha \\
&= -\Omega \hat{\boldsymbol{e}}_z \times \sum_\alpha B_\alpha \hat{\boldsymbol{e}}_\alpha \\
&= -\boldsymbol{\Omega} \times \boldsymbol{B},
\end{aligned}$$

and so

$$-\boldsymbol{u} \cdot \nabla \boldsymbol{B} = -\Omega \sum_\alpha \left(\frac{\partial B_\alpha}{\partial \phi} \right) \hat{\boldsymbol{e}}_\alpha - \boldsymbol{\Omega} \times \boldsymbol{B} \qquad \text{“advection” for rigid rotation.} \tag{10}$$

Mathematically, in any coordinate system with a z-axis of rotational symmetry, the action of rigid rotation is as follows:

$$\begin{aligned}
\frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times [(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{B}] \\
&= \sum_\alpha \left(-\Omega \frac{\partial B_\alpha}{\partial \phi} \right) \hat{\boldsymbol{e}}_\alpha
\end{aligned} \tag{11a}$$

$$\text{or} \qquad \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) B_\alpha = 0 \qquad \text{for each } \alpha. \tag{11b}$$

If you think about it, this makes sense: All the rigid rotation does is rotate the whole field configuration around the z-axis at the rate Ω. If you decide to also rotate at Ω (so your personal Eulerian time derivative is $\partial/\partial t + \Omega \partial/\partial \phi$), then each component of the magnetic-field configuration should remain the same in your frame. Note that the rotation does *not* advect the vector magnetic field (like the term $-\boldsymbol{u} \cdot \nabla \boldsymbol{B}$ viewed on its own would suggest), but rather it advects the field *components* (as if they were scalars) in any coordinate system with an axis of rotational symmetry.

4 Solution for spherical coordinates

Resolving these issues fully for the spherical coordinate system seems complicated and I am not fully sure how to do it! In particular (for “full” resolution) we should, separately at each point (r,θ,ϕ):

1. Form a local Cartesian coordinate system, say $(x_1,\ x_2,\ x_3)$, whose origin lies at the point (r,θ,ϕ). At the origin, $(\hat{\boldsymbol{e}}_1,\ \hat{\boldsymbol{e}}_2,\ \hat{\boldsymbol{e}}_3)$ will coincide with $(\hat{\boldsymbol{e}}_r,\ \hat{\boldsymbol{e}}_\theta,\ \hat{\boldsymbol{e}}_\phi)$. But slightly away from the origin, $(\hat{\boldsymbol{e}}_1,\ \hat{\boldsymbol{e}}_2,\ \hat{\boldsymbol{e}}_3)$ will stay fixed while $(\hat{\boldsymbol{e}}_r,\ \hat{\boldsymbol{e}}_\theta,\ \hat{\boldsymbol{e}}_\phi)$ curve away.

2. Calculate the velocity-gradient tensor: $\partial u_1/\partial x_1, \partial u_2/\partial x_1$, and $\partial u_3/\partial x_1$, etc. While calculating derivatives, be careful to differentiate along the Cartesian coordinates, *not* the curvilinear ones or along curving $\boldsymbol{B}$-lines. Express the final tensor components in spherical coordinates. (I have begun this process in Section 5).

3. Rotate "into $\boldsymbol{B}$" to form a new primed coordinate system (such that $\hat{\boldsymbol{e}}_1'$ points along $\boldsymbol{B}$), calculate the $\partial u_j'/\partial x_i'$, and thus form $\partial u_\parallel/\partial x_\parallel$ and $\partial \boldsymbol{u}_\perp/\partial x_\parallel$ (note that $\nabla_\perp \cdot \boldsymbol{u}_\perp = \nabla \cdot \boldsymbol{u} - \partial u_\parallel/\partial x_\parallel$). Note that the direction cosines $\hat{\boldsymbol{e}}_1'$ makes with ($\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$) are simply $B_r/|\boldsymbol{B}|, B_\theta/|\boldsymbol{B}|$, and $B_\phi/|\boldsymbol{B}|$, respectively. Note that both unit vectors and vector components transform like $x_1' = \sum_{j=1}^{3} R_{1j} x_j$, where R_{1j} is the j^{th} direction cosine.

4. Subtract the part of $\boldsymbol{u}_\perp$ corresponding to solid-body rotation from the new shear and advection terms, and put it in a term like the one in Equation (11a).

5. Exult!

Instead of addressing the full problem as just described, I have brute-forced my way into a quasi-solution, canceling obvious terms and expressing what seem to be "transverse shear with no solid-body rotation," "transverse compression," and "advection of $\boldsymbol{B}$-components as though they were scalars."

We write:

$$
\begin{aligned}
I_r = {} & \cancel{B_r \frac{\partial u_r}{\partial r}} + \frac{B_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{B_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \cancel{\frac{B_\theta u_\theta + B_\phi u_\phi}{r}} \\
& - \boldsymbol{u}\cdot\nabla B_r + \cancel{\frac{u_\theta B_\theta + u_\phi B_\phi}{r}} \\
& - \left[\cancel{\frac{\partial u_r}{\partial r}} + \frac{2u_r}{r} + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + \cot\theta u_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right] B_r
\end{aligned}
$$

or

$$
\begin{aligned}
I_r = {} & \left(\frac{B_\theta}{r}\frac{\partial}{\partial \theta} + \frac{B_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\right) u_r && r\text{: transverse shear} \quad (12a)\\
& - \boldsymbol{u}\cdot\nabla B_r && r\text{: advection} \quad (12b)\\
& - \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{2u_r}{r} + \frac{\cot\theta u_\theta}{r}\right) B_r. && r\text{: transverse compression} \quad (12c)
\end{aligned}
$$

Then, for I_θ:

$$
\begin{aligned}
I_\theta = {} & B_r \frac{\partial u_\theta}{\partial r} + \cancel{\frac{B_\theta}{r}\frac{\partial u_\theta}{\partial \theta}} + \frac{B_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{B_\theta u_r}{r} - \cancel{\frac{\cot\theta B_\phi u_\phi}{r}} \\
& - \boldsymbol{u}\cdot\nabla B_\theta - \frac{u_\theta B_r}{r} + \cancel{\frac{\cot\theta u_\phi B_\phi}{r}} \\
& - \left[\frac{\partial u_r}{\partial r} + \frac{2u_r}{r} + \frac{1}{r}\left(\cancel{\frac{\partial u_\theta}{\partial \theta}} + \cot\theta u_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right] B_\theta
\end{aligned}
$$

or

$$
\begin{aligned}
I_\theta = {} & r\left(B_r\frac{\partial}{\partial r} + \frac{B_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\left(\frac{u_\theta}{r}\right) && \theta\text{: transverse shear} \quad (13a)\\
& - \boldsymbol{u}\cdot\nabla B_\theta && \theta\text{: advection} \quad (13b)\\
& - \left(\frac{\partial u_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{\cot\theta u_\theta}{r}\right) B_\theta. && \theta\text{: transverse compression} \quad (13c)
\end{aligned}
$$

Finally, for I_ϕ:

$$\begin{aligned}
I_\phi =& B_r \frac{\partial u_\phi}{\partial r} + \frac{B_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + \cancel{\frac{B_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}} + \frac{B_\phi u_r}{r} + \cancel{\frac{\cot\theta B_\phi u_\theta}{r}} \\
& - \boldsymbol{u}\cdot\nabla B_\phi - \frac{u_\phi B_r}{r} - \frac{\cot\theta u_\phi B_\theta}{r} \\
& - \left[\frac{\partial u_r}{\partial r} + \frac{2u_r}{r} + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + \cancel{\cot\theta u_\theta}\right) + \cancel{\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}}\right] B_\phi
\end{aligned}$$

or

$$I_\phi = r\sin\theta\left(B_r\frac{\partial}{\partial r} + \frac{B_\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{u_\phi}{r\sin\theta}\right) \qquad \phi\text{: transverse shear} \tag{14a}$$

$$- \ \boldsymbol{u}\cdot\nabla B_\phi \qquad \phi\text{: advection} \tag{14b}$$

$$- \left(\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right) B_\phi. \qquad \phi\text{: transverse compression} \tag{14c}$$

These equations have the following positive attributes:

1. The obvious "non-transverse" components of the shear and compression have canceled.

2. Solid-body rotation is eliminated from the ϕ shear term, since $u_\phi/r\sin\theta$ is differentiated instead of u_ϕ. I am less happy with the θ shear term, although $u_\theta = r\Omega$ does imply rigid rotation of a particular plane, so differentiating u_θ/r instead of u_θ seems to get rid of *some* of the solid-body rotation.

3. The new "advection" advects the vector components of $\boldsymbol{B}$ instead of $\boldsymbol{B}$ itself, something that was seen to be important to express the action of solid-body rotation.

4. The "transverse compression" terms have a satisfying interpretation, as shown in the following section.

5. The terms were found simply by rearranging to full induction equation expressed in spherical coordinates.

I am not sure how close these equations are to what we actually want (namely, shear and compression transverse to a field line), but hopefully they are a step in the right direction. If anyone can see a better way forward, please tell me!

5 Transverse compression

Here we show that the "transverse compression" terms defined in Equations (12)–(14) make sense. The key point is that we must be careful when calculating the "transverse divergence." When we differentiate along θ or ϕ, the local spherical unit vectors will all change. For example, the divergence "transverse to $\hat{\boldsymbol{e}}_r$" is not simply $\nabla\cdot(u_\theta\hat{\boldsymbol{e}}_\theta + u_\phi\hat{\boldsymbol{e}}_\phi)$; there are also curvature terms.

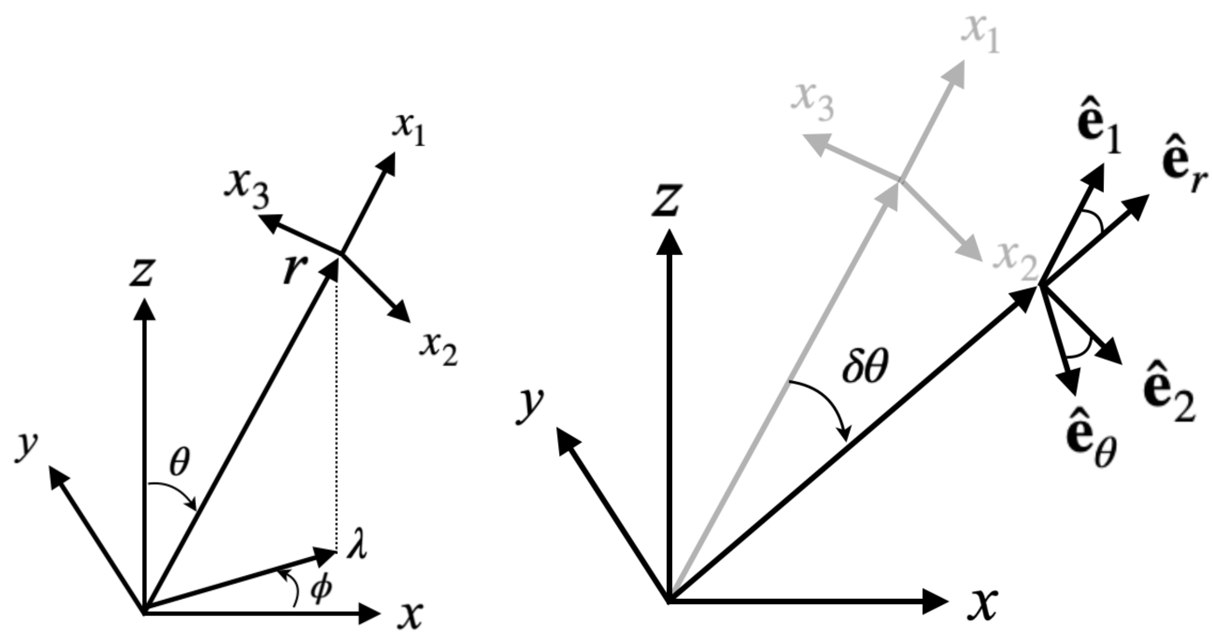

Figure 1: *Left*: Schematic of coordinate systems used here, their relation to one another, and each coordinate's meaning. Note that $(x, y, z) = (\lambda\cos\phi, \lambda\sin\phi, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$. The local Cartesian system (x_1, x_2, x_3) has its origin at the point (r, θ, ϕ), where $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3) = (\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta, \hat{\boldsymbol{e}}_\phi)$. *Right*: Away from the origin, the unit vectors no longer coincide. We show the relationship between the spherical-coordinate unit vectors and the Cartesian unit vectors when $\theta \to \theta + \delta\theta$ (or equivalently, $x_2 = 0 \to \delta x_2$). By viewing the system from above, a similar diagram can be made for what happens when $\phi \to \phi + \delta\phi$.

Figure 1 shows schematically how this works. We set up a local Cartesian coordinate system (with origin at the point (r,θ,ϕ)) denoted by (x_1,x_2,x_3). As we differentiate $\boldsymbol{u}$, the spherical-coordinate unit vectors change, while the Cartesian ones stay fixed. In particular, if we move from $(r,\theta,\phi)\to(r+\delta r,\theta+\delta\theta,\phi+\delta\phi)$ (or equivalently in the Cartesian system, $(0,0,0)\to(\delta x_1,\delta x_2,\delta x_3)$), we find the following (to first order in the δ's):

$$u_1 = u_r - (\delta\theta)u_\theta - (\sin\theta\delta\phi)u_\phi, \tag{15a}$$

$$u_2 = u_\theta + (\delta\theta)u_r - (\cos\theta\delta\phi)u_\phi, \tag{15b}$$

$$\text{and} \qquad u_3 = u_\phi + (\sin\theta\delta\phi)u_r + (\cos\theta\delta\phi)u_\theta. \tag{15c}$$

And trivially:

$$\delta x_1 = \delta r \tag{16a}$$

$$\delta x_2 = r\delta\theta \tag{16b}$$

$$\delta x_3 = r\sin\theta\delta\phi. \tag{16c}$$

We thus compute:

$$\begin{aligned}
\frac{\partial u_1}{\partial x_1} &= \lim_{\delta x_1\to 0}\frac{u_1(\delta x_1,0,0)-u_1(0,0,0)}{\delta x_1}\\
&= \lim_{\delta r\to 0}\frac{u_r(r+\delta r,\theta,\phi)-u_r(r,\theta,\phi)}{\delta r}\\
&= \frac{\partial u_r}{\partial r}.
\end{aligned} \tag{17}$$

That was the easy one! Now for the curvy ones:

$$\begin{aligned}
\frac{\partial u_2}{\partial x_2} &= \lim_{\delta x_2\to 0}\frac{u_2(0,\delta x_2,0)-u_2(0,0,0)}{\delta x_2}\\
&= \lim_{\delta\theta\to 0}\frac{u_\theta(r,\theta+\delta\theta,\phi)+u_r(r,\theta+\delta\theta,\phi)\delta\theta-u_\theta(r,\theta,\phi)}{r\delta\theta}\\
&= \lim_{\delta\theta\to 0}\frac{u_\theta(r,\theta+\delta\theta,\phi)-u_\theta(r,\theta,\phi)}{r\delta\theta}+\lim_{\delta\theta\to 0}\frac{u_r(r,\theta+\delta\theta,\phi)}{r}\\
&= \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}.
\end{aligned} \tag{18}$$

And finally:

$$\begin{aligned}
\frac{\partial u_3}{\partial x_3} &= \lim_{\delta x_3\to 0}\frac{u_3(0,0,\delta x_3)-u_3(0,0,0)}{\delta x_3}\\
&= \lim_{\delta\phi\to 0}\frac{u_\phi(r,\theta,\phi+\delta\phi)+(\sin\theta u_r+\cos\theta u_\theta)\delta\phi-u_\phi(r,\theta,\phi)}{r\sin\theta\delta\phi}\\
&= \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}+\frac{u_r}{r}+\frac{\cot\theta u_\theta}{r}.
\end{aligned} \tag{19}$$

We thus find the various components of transverse divergence:

$$
\begin{aligned}
(\nabla \cdot \boldsymbol{u})_{\perp \text{ to } r} &= \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \\
&= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{2u_r}{r} + \frac{\cot\theta u_\theta}{r}, \qquad (20)\\
(\nabla \cdot \boldsymbol{u})_{\perp \text{ to } \theta} &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3} \\
&= \frac{\partial u_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{\cot\theta u_\theta}{r}, \qquad (21)\\
\text{and} \qquad (\nabla \cdot \boldsymbol{u})_{\perp \text{ to } \phi} &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \\
&= \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}. \qquad (22)
\end{aligned}
$$

Comparing with Equations (12)–(14), we see that these are the exact same divergence terms showing up in the "transverse compression."