Angular Momentum in Terms of Toroidal and Poloidal Stream Functions

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1 Prior Work Considered

In this set of notes, we consider 3D nonlinear dynamo simulations that explored magnetic-field amplitudes at a variety of rotation rates (i.e., enough to make scatter plots of various quantities versus Rossby number and thus theoretically address the activity-rotation relation). We aim to determine what region of parameter space has been explored by global models, and in particular, how the rotation-activity relation may or may not have been addressed. The works considered are:

Christensen & Aubert (2006),

Christensen et al. (2009),

Strugarek et al. (2017),

Guerrero et al. (2019),

Brun et al. (2022),

2 Observations of the Stellar Activity-Rotation Relation (ARR)

The stellar activity-rotation relation (hereafter ARR) has a long history of observation and informs astronomers' generally accepted picture of stellar spin-down. The positive correlation between rotational velocity ($v \sin i$) and chromospheric Ca^+ emission (more or less considered to be linearly proportional to strength of the surface magnetic field) was first noted by Kraft (1967). This correlation was made famous in the context of spin-down by the "Skumanich $t^{1/2}$" law. This law states that as a star ages (call its age t), its surface magnetic-field strength and rotation rate both decrease like $t^{-1/2}$ (Skumanich 1972). Since spin-down is believed to be caused by angular-momentum loss from a stellar wind, there thus appears to be a negative feedback loop between rotation (which produces magnetic field by a convective dynamo) and magnetic field (which drives the stellar wind and slows the rotation).

In addition to chromospheric emission, coronal emission (X-ray luminosity) is also found to be proportional to the rotation rate (e.g., Walter 1982). Furthermore, the X-ray data show that for a critical rotation rate (corresponding to a critical mixing-length Rossby number of $Ro \sim 0.1$), the ratio of X-ray luminosity to total luminosity saturated to a value of $L_X/L_* \sim 0.001$. It is still unclear whether this saturation regime corresponds a fundamental dynamo process (for example, the inability of the dynamo to convert additional kinetic energy to magnetic field) or simply geometric effects (for example, the stellar surface becoming so peppered with active regions that it cannot accommodate any more dynamo-produced flux) (Jardine & Unruh 1999). More recent work (Reiners et al. 2022) claims that the saturation regime indicates a fundamental dynamo process.

Since the detailed properties of the magnetic fields in stellar interiors are almost completely unconstrained (including for the Sun), it is reasonable to assume that the overall dynamo strength (defined here to be the rms magnetic-field strength, with the mean taken over the full volume of the star) is proportional to the surface magnetic-field strength, and thus to the proxies of Ca^+ and X-ray flux. With these assumptions, the ARR tells us about two stellar dynamo regimes: slow rotation (where the Sun lies), in which dynamo strength increases with more rapid rotation, and fast rotation, in which the dynamo strength is insensitive to the rotation rate (a saturation regime). Figure 2 shows the ARR as reported by Reiners et al. (2022). As the Rossby number Ro (ratio of well-measured rotation period $P_{\rm rot}$ to an ad hoc convective turnover time τ) decreases right to left (i.e., the rotation rate increases), the surface magnetic field strength increases up to a critical Rossby number $Ro = 0.13$. After that, faster rotation only leads to marginally increased field strength.

It should be noted that the "Rossby number" used in plots like Figure 2 is really an empirical parameter chosen to make the ARR collapse onto a single curve for all different stellar masses (e.g., Noyes et al. 1984; Pizzolato et al. 2003; Wright et al. 2011). In general, the empirical convective turnover time scales like $\tau \sim L_*^{1/2}$, which has some basis in mixing-length theory, but the true dependence of τ on stellar parameters could be far more complicated. In any case, simply plotting the normalized activity emission (e.g., L_X/L_*) with respect to rotation period yields an essentially similar ARR, possibly with different saturation breaks (all in the range of $P_{\rm rot} \sim$ 1–3 days) for different stellar masses (e.g., Pizzolato et al. 2003).

3 Parameters Spaces Explored

Global, rotating, magnetized spherical shells are characterized by several non-dimensional parameters. These parameters (which completely characterize the Boussinesq system) are:

$$\text{shell aspect ratio} \equiv \beta \equiv \frac{r_{\rm i}}{r_{\rm o}}, \tag{1}$$

$$\text{Rayleigh number} \equiv \text{Ra} \equiv \frac{g_{\rm o}\alpha\Delta\overline{T}H^3}{\nu_{\rm o}\kappa_{\rm o}}, \tag{2}$$

$$\text{thermal Prandtl number} \equiv \text{Pr} = \frac{\nu_{\rm o}}{\kappa_{\rm o}}, \tag{3}$$

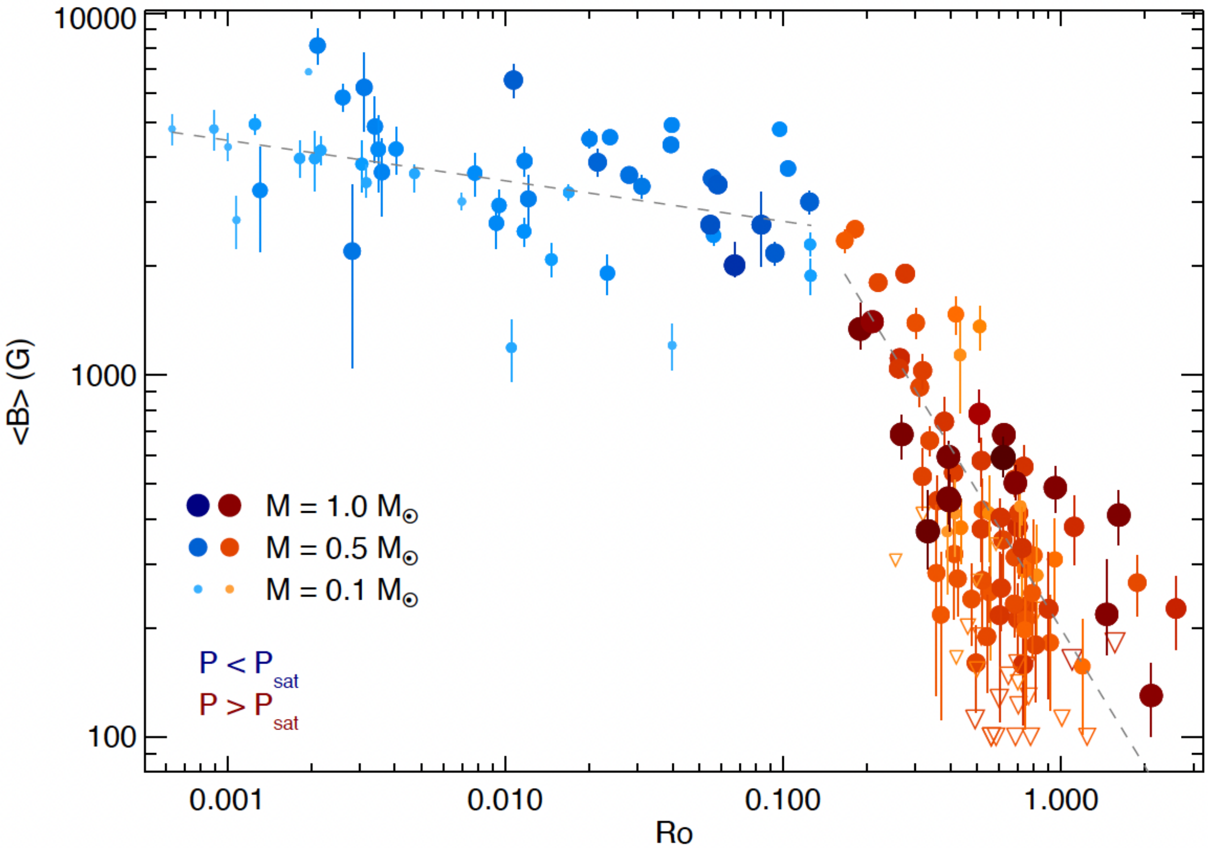

Figure 1: Copied from Reiners et al. (2022), Figure 5: Magnetic field–rotation relation for solar-like and low-mass stars. Symbols for stars rotating slower than $Ro = 0.13$ are colored red, while those of faster rotators are colored blue. Larger and darker symbols indicate higher stellar mass than smaller and lighter symbols. The gray dashed lines show linear fits separately for the slowly rotating stars ($Ro > 0.13$; $\langle B \rangle = 200G \times Ro^{-1.25}$) and the fast rotators ($Ro < 0.13$; $\langle B \rangle = 2050G \times Ro^{-0.11}$). Downward open triangles show upper limits for $\langle B \rangle$.

$$\text{Ekman number} \quad \equiv \quad \text{Ek} \quad = \frac{\nu_\text{o}}{\Omega_0 H^2}, \tag{4}$$

$$\text{and} \quad \text{magnetic Prandtl number} \quad \equiv \quad \text{Pr}_\text{m} \quad = \quad \frac{\nu_\text{o}}{\eta_\text{o}}. \tag{5}$$

Here, r is the radius, the subscripts "o" and "i" refer to the outer and inner radii of the shell (respectively), $H = r_\text{o} - r_\text{i}$ is the shell depth, g is the gravitational acceleration, α is the coefficient of thermal expansion, $\Delta\overline{T}$ is the background (superadiabatic, or adverse) temperature difference between the bottom and top of the shell, ν is the kinematic viscosity, κ the thermal diffusivity, Ω_0 the rotation rate, and η the magnetic diffusivity. The "o" subscripts on the diffusivities and g indicate they are in general functions of r. Technically, this radial dependence makes the parameter space accessed by dynamo simulations infinite-dimensional, so hopefully it's not too important! Just kidding.

If the system is anelastic, there are additional parameters because of the stratification. For a polytrope describing a convection zone, these are:

$$\text{shell density contrast} \quad \equiv \quad \text{DC} \quad \equiv \frac{\overline{\rho}(r_\text{i})}{\overline{\rho}(r_\text{o})} \tag{6}$$

$$\text{or} \quad \text{number of density scale heights} \quad \equiv \quad N_\rho \quad \equiv \ln\text{(DC)}, \tag{7}$$

$$\text{polytropic index} \quad \equiv \quad n \quad \equiv \text{something near } \frac{3}{2}, \tag{8}$$

$$\text{or} \quad \text{ratio of specific heats} \quad \equiv \quad \gamma \quad \equiv \frac{c_\text{p}}{c_\text{v}} = \frac{n+1}{n} = \text{something near } \frac{5}{3}, \tag{9}$$

$$\text{and} \quad \text{dissipation number} \quad \equiv \quad \text{Di} \quad = \quad \frac{g_\text{o} H}{c_\text{p}\overline{T}_\text{o}}. \tag{10}$$

Here, $\overline{\rho}(r)$ is the background density and c_p is the constant-pressure specific heat. The dissipation number arises because viscous and joule heating are typically neglected in the Boussinesq approximation.

If there is also a stable layer in the system (say, below the convection zone), some other parameters are needed: the transition location and width, along with the polytropic index of the stable layer or (if the stable layer isn't polytropic) the full entropy gradient $d\overline{S}/dr$ in the stable layer. For polytrope, n determines $d\overline{S}/dr$, which will be a complicated function of r but with an order of magnitude of $[n/(3/2) - 1]c_\text{p}/r_\text{i}$. Regardless, the stable layer introduces an important, final dimensionless number:

$$\text{buoyancy parameter} \quad \equiv \quad \text{B} \quad \equiv \frac{\widetilde{N^2}}{\Omega_0^2}, \tag{11}$$

$$\text{where} \qquad N^2 \equiv \frac{g}{c_\text{p}}\frac{d\overline{S}}{dr} \tag{12}$$

is the squared buoyancy frequency and the tilde indicates a volume average over the stable layer.

We note also that some authors use a modified Rayleigh number that is independent of any diffusion coefficient, replacing the diffusion time combination $(H^2/\nu_\text{o})(H^2/\kappa_\text{o})$ with

$1/\Omega_0^2$:

$$\text{modified Rayleigh number} \quad \equiv \quad \mathrm{Ra}^* \quad \equiv \frac{g_\mathrm{o}\alpha\Delta\overline{T}}{\Omega_0^2 H} = \mathrm{Ra}\mathrm{Pr}^{-1}\mathrm{Ek}^2. \tag{13}$$

Finally, in an astrophysical convective system (like a star or the Earth), the adiabatic temperature difference across the convection zone ($\Delta\overline{T}$) is typically unknown. Instead, we can measure the flux driven through the system, and need to define a "flux-based Rayleigh number" with respect to this flux. For example, we usually know the stellar luminosity L_* and radius r. Throughout the star, the different energy fluxes must add up to $\mathcal{F}(r) = L_*/4\pi r^2$. If the convection is vigorous, the convective heat transport, or enthalpy flux, should dominate the other fluxes, i.e., $\mathcal{F} \sim \overline{\rho} c_\mathrm{p} u_r T$, where u_r and T are typical radial velocity and temperature perturbations, respectively. If cold plumes dissipate their energy thermally, we might expect $u_r \sim \kappa_\mathrm{o}/H$, or

$$\text{first flux-based Rayleigh number} \quad \equiv \quad \mathrm{Ra}_{\mathrm{F1}} \quad \equiv \frac{g_\mathrm{o}\mathcal{F}_\mathrm{o}H^4}{c_\mathrm{p}\overline{\rho}_\mathrm{o}\overline{T}_\mathrm{o}\nu_\mathrm{o}\kappa_\mathrm{o}^2} \tag{14}$$

Note that the Boulder crew (Featherstone & Hindman 2016a,b; Hindman et al. 2020; Matilsky & Toomre 2020; Matilsky et al. 2020; Korre & Featherstone 2021; Camisassa & Featherstone 2022) use a different "volume-averaged" flux-based Rayleigh number:

$$\text{second flux-based Rayleigh number} \quad \equiv \quad \mathrm{Ra}_{\mathrm{F2}} \quad \equiv \frac{\tilde{g}\tilde{F}H^4}{c_\mathrm{p}\tilde{\overline{\rho}}\tilde{\overline{T}}\tilde{\nu}\tilde{\kappa}^2}, \tag{15}$$

where the tildes refer to volume averages over the convection zone only and $F(r) \equiv \mathcal{F}(r) - \mathcal{F}_\mathrm{rad}(r)$ is the flux convection and conduction must carry in equilibrium. In the Rayleigh models, F is determined by the volumetric heating $Q(r)$ (which is a stand-in for the radiation flux $\boldsymbol{\mathcal{F}}_\mathrm{rad}$), i.e., $F(r) = (1/r^2)\int_{r_\mathrm{i}}^r Q(x)x^2dx$.

There is also a modified flux-based Rayleigh number, used, for example, in Christensen & Aubert (2006):

$$\text{modified flux-based Rayleigh number} \quad \equiv \quad \mathrm{Ra}_\mathrm{F}^* \quad \equiv \frac{1}{4\pi r_\mathrm{i} r_\mathrm{o}}\frac{g_\mathrm{o}\alpha Q_\mathrm{adv}}{\rho_\mathrm{o}\Omega_0^3 H^2} \tag{16}$$

where Q_adv is the "total" (presumably the volume-averaged) spherical-surface-integrated convective heat flux (enthalpy flux). Physically, such a Rayleigh number would be appropriate if the rotation set the typical radial velocity: $u_r \sim H\Omega_0$. Note that this Rayleigh number is not an input parameter, but more related to the vigor of established convection (like a Nusselt number).

4 Christensen & Aubert (2006)

Christensen & Aubert (2006) explore Boussinesq dynamos driven by a fixed temperature difference $\Delta\overline{T}$ in the context of planetary magnetic fields. They cover at least two decades in

Table 1: Parameter space explored by prior global dynamo simulations. In Christensen & Aubert (2006), the system is Boussinesq with a fixed temperature difference. Only cases with dipolar dynamos ($f_{\rm dip} > 0.35$) and sufficiently vigorous convection (Nu > 2, where Nu is the Nusselt number) are considered. The min/max supercriticality was estimated from their Figure 4(a)'s x-axis. In Brun et al. (2022), the numbers are reported local to the middle of the convection zone.

Paper	Christensen & Aubert (2006)	Brun et al. (2022)
Ek min	10^{-6}	4×10^{-5}
Ek max	3×10^{-4}	9.0×10^{-3}
Ra/Ra$_{\rm crit}$ min	≈5	5.7
Ra/Ra$_{\rm crit}$ max	≈50	73
Pr min	0.1	0.25
Pr max	10	0.25
Pr$_{\rm m}$ min	0.06	1
Pr$_{\rm m}$ max	10	2

the relevant input parameters (Rayleigh, Ekman, and two Prandtl numbers) and characterize scaling relationships between diffusion-independent output numbers. They find that all these output numbers, to first approximation, have simple power-law dependencies on the modified Rayleigh number, with only weak dependencies on the diffusion-controlled parameters (Ekman and two Prandtl numbers). From these dependencies, they conclude (among other things) that magnetic-field strengths in dipole-dominated dynamos are independent of the rotation rate. For the most part, their scaling laws seem robust (the plots speak for themselves).

However, their main conclusion that field strength is independent of rotation rate (which would be particularly relevant to the saturation regime of the ARR) I don't follow. They define several output numbers:

$$\text{Lorentz number} \quad \equiv \quad \text{Lo} \quad \equiv \frac{B_{\rm rms}}{(\overline{\rho}\mu)^{1/2}\Omega_0 H}, \tag{17}$$

$$\text{fractional Ohmic dissipation} \quad \equiv f_{\rm ohm} = \frac{D_\lambda}{P_{\rm buoy}}, \tag{18}$$

$$\text{where} \quad \text{total Ohmic dissipation} \quad \equiv D_\eta \equiv \eta \int |\nabla \times \boldsymbol{B}|^2 dV \tag{19}$$

$$\text{and} \quad \text{total buoyant work rate} \quad \equiv P_{\rm buoy} \equiv \alpha\overline{\rho} \int g(r) u_r T dV. \tag{20}$$

Here, $B_{\rm rms}$ refers to the rms magnetic-field strength (mean taken over time and the full shell volume), $\boldsymbol{B}$ is the vector magnetic field, the integrals are taken over the full shell volume, and I've changed some of the notation from Christensen & Aubert (2006) to be more consistent with what we do for Rayleigh (e.g., $\lambda \rightarrow \eta$ and the total powers (last two equations) are defined dimensionally). The main relevant scaling the authors find is

$$\frac{\text{Lo}}{f_{\rm ohm}^{1/2}} \sim (\text{Ra}_{\rm F}^*)^{1/3}, \tag{21}$$

which yields

$$B_{\rm rms} \sim f_{\rm ohm}^{1/2} \overline{\rho}^{1/6} \mu^{1/2} \left[\frac{(\alpha Q_{\rm adv}/c_{\rm p}) H g_{\rm o}}{4\pi r_{\rm i} r_{\rm o}} \right]^{1/3} \tag{22}$$

The quantity

$$\text{buoyancy flux} \quad \equiv Q_B = \frac{\alpha Q_{\rm adv}}{c_{\rm p}}. \tag{23}$$

(also apparently called the "mass anomaly flux") evidently represents the flux of fractional density perturbations (integrated over spherical surfaces and volumetrically averaged). I'm not sure about why Q_B is the more relevant parameter...is it more easily measurable for the Earth? Or do they use it because there is compositional buoyancy (in addition to thermal buoyancy) in the Earth's core that wouldn't be properly accounted for in $Q_{\rm adv}$? I'm missing some relevant background here.

Anyway, the authors then state: "The fraction of Ohmic dissipationin most of our models is in the range of 0.3–0.8. For the Earth's core $f_{\rm ohm} \approx 1$ is usually assumed, based on a ratio of magnetic energy to kinetic energy much larger than one and the high magnetic diffusivity. However, if the kinetic energy is allowed to cascade to much smaller length scales than the magnetic energy, viscous dissipation may still be significant. From our model results we did not find a simple rule of how $f_{\rm ohm}$ varies with the control parameters, but for simplicity we will make the usual assumption that viscous dissipation becomes negligible under core conditions."

In other words, the authors ignore the factor of $f_{\rm ohm}^{1/2}$ in Equation (22) (see their Equation (48)) and from there conclude the "remarkable result" that the magnetic-field strength is independent of rotation rate. But that only happens if $f_{\rm ohm}$ is independent of rotation rate! And they state themselves that they do not see any obvious way $f_{\rm ohm}$ scales with their parameters. They also more or less state that there doesn't seem to be any good theoretical reason $f_{\rm ohm}$ should be unity, either. As they note, the KE cascade could be to a much smaller scale than the ME cascade, and both viscous and ohmic dissipation might be important.

So in my view, the scaling of $f_{\rm ohm}$ with rotation rate seems unclear, and no inference regarding the scaling of $B_{\rm rms}$ with Ω_0 can be made. To be fair to the authors, they offer the independence of $B_{\rm rms}$ from Ω_0 as a surprising "prediction" or a "suggestion," which is perfectly valid. However, it seems their own models do not support such an independence (their $f_{\rm ohm}$ are not unity and vary un-systematically with their input parameters) and theoretically, the idea that $f_{\rm ohm} \approx 1$ is not supported either. So...am I missing something critical?

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