Non-Dimensionalization of an Anelastic Stable–Unstable Layer in `Rayleigh`

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1 General Equations Solved in `Rayleigh`

In general (with rotation and magnetism), `Rayleigh` evolves in time a set of coupled PDEs for the 3D vector velocity $\boldsymbol{u}$, vector magnetic field $\boldsymbol{B}$, pressure perturbation P (perturbation away from the "reference" or "background" state), and entropy perturbation S. Note that S can also be interpreted as a temperature perturbation in Boussinesq mode. For more details, see `Rayleigh`'s Documentation.

We use standard spherical coordinates (r, θ, ϕ) and cylindrical coordinates $(\lambda, \phi, z) = (r\sin\theta, \phi, r\cos\theta)$, and $\hat{\boldsymbol{e}}_q$ in general denotes a position-dependent unit vector in the direction of increasing q. The full PDE-set is then:

$$\nabla\cdot[f_1(r)\boldsymbol{u}] = 0, \tag{1.1}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{1.2}$$

$$f_1(r)\left[\frac{D\boldsymbol{u}}{Dt} + c_1\hat{\boldsymbol{e}}_z\times\boldsymbol{u}\right] = c_2 f_2(r)S\hat{\boldsymbol{e}}_r - c_3 f_1(r)\nabla\left[\frac{P}{f_1(r)}\right], \\ + c_4(\nabla\times\boldsymbol{B})\times\boldsymbol{B} + c_5\nabla\cdot\boldsymbol{D}, \tag{1.3a}$$

$$\text{where}\qquad D_{ij} \coloneqq 2f_1(r)f_3(r)\left[e_{ij} - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{1.3b}$$

$$\text{and}\qquad e_{ij} \coloneqq \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{1.3c}$$

$$f_1(r)f_4(r)\frac{DS}{Dt} = -f_1(r)f_4(r)f_{14}(r)u_r + c_6\nabla\cdot[f_1(r)f_4(r)f_5(r)\nabla S] \\ + c_6 f_{10}(r) + c_8 c_5 D_{ij}e_{ij} + \frac{\eta(r)}{4\pi}|\nabla\times\boldsymbol{B}|^2, \tag{1.4}$$

$$\text{and}\qquad \frac{\partial\boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - c_7\nabla\times[f_7(r)\nabla\times\boldsymbol{B}], \tag{1.5}$$

where $D/Dt \coloneqq \partial/\partial t + \boldsymbol{u}\cdot\nabla$ denotes the material derivative. The spherically-symmetric, time-independent reference (or background) functions $f_i(r)$ and constants c_j set the fluid approximation to be made. `Rayleigh` has built-in modes to set the f's and c's for single-layer (i.e., either convectively stable or unstable, but not both) Boussinesq or Anelastic spherical shells. More complex systems (coupled stable–unstable systems or alternative non-dimensionalizations) require the user to manually change the f's and c's. This can be done by editing an input binary file that `Rayleigh` reads upon initialization. The c's can also be changed in the ASCII text-file (i.e., the `main_input` file).

2 Dimensional Anelastic Equations

We begin by writing down the full dimensional anelastic fluid equations, as they are usually implemented in `Rayleigh` (`reference_type = 2`). This form of the anelastic approximation in a spherical shell is derived in, or more accurately, attributed to (since `Rayleigh` "updates" the background state slightly differently than the cluge-y `ASH` implementation), two common sources: Gilman & Glatzmaier (1981) and Clune et al. (1999). `Rayleigh`'s dimensional anelastic equation-set is:

$$\nabla\cdot[\overline{\rho}(r)\boldsymbol{u}] = 0, \tag{2.1}$$
$$\nabla\cdot\boldsymbol{B} = 0, \tag{2.2}$$

$$\begin{aligned}\overline{\rho}(r)\left[\frac{D\boldsymbol{u}}{Dt} + 2\Omega_0\hat{\boldsymbol{e}}_z\times\boldsymbol{u}\right] = \left[\frac{\overline{\rho}(r)\overline{g}(r)}{c_\mathrm{p}}\right]S\hat{\boldsymbol{e}}_r - \overline{\rho}(r)\nabla\left[\frac{P}{\overline{\rho}(r)}\right],\\ + \frac{1}{\mu}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} + \nabla\cdot\boldsymbol{D},\end{aligned} \tag{2.3a}$$

$$\text{where}\qquad D_{ij} \coloneqq 2\overline{\rho}(r)\overline{\nu}(r)\left[e_{ij} - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{2.3b}$$

$$\text{and}\qquad e_{ij} \coloneqq \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.3c}$$

$$\begin{aligned}\overline{\rho}(r)\overline{T}(r)\frac{DS}{Dt} =& -\overline{\rho}(r)\overline{T}(r)\frac{d\overline{S}}{dr}u_r + \nabla\cdot[\overline{\rho}(r)\overline{T}(r)\overline{\kappa}(r)\nabla S]\\ &+ \overline{Q}(r) + D_{ij}e_{ij} + \frac{\overline{\eta}(r)}{\mu}|\nabla\times\boldsymbol{B}|^2,\end{aligned} \tag{2.4}$$

$$\text{and}\qquad \frac{\partial\boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - \nabla\times[\eta(r)\nabla\times\boldsymbol{B}]. \tag{2.5}$$

Here, the thermal variables ρ, T, P, and S refer to the density, temperature, pressure, and entropy (respectively). The overbars denote the spherically-symmetric, time-independent background state. The lack of an overbar on a thermal variable indicates the (assumed small) perturbation from the background (for the entropy, S/c_p is assumed small).

Other background quantities that appear are the gravity $\overline{g}(r)$, the momentum, thermal, and magnetic diffusivities [$\overline{\nu}(r)$, $\overline{\kappa}(r)$, and $\overline{\eta}(r)$, respectively], the internal heating or cooling $\overline{Q}(r)$, the frame rotation rate Ω_0, the specific heat at constant pressure $c_{\rm p}$, and the vacuum permeability μ ($= 4\pi$ in c.g.s. units). The equations are written in a frame rotating with angular velocity Ω_0 and the centrifugal force is neglected.

Note that the internal heating and cooling function $\overline{Q}(r)$ is a reference-state quantity (and thus assumed spherically-symmetric and time-independent) but should be interpreted as $-\nabla \cdot \boldsymbol{\mathcal{F}}_{\rm rad}$, where $\boldsymbol{\mathcal{F}}_{\rm rad}$ is the radiative heat flux and properly should be proportional to the radiative diffusivity $\kappa_{\rm rad}$ (which takes on a specific form in the radiative diffusion approximation, derivable from the opacity) and to the gradient of the total temperature $\overline{T} + T$.

In `Rayleigh`, a convective layer is usually driven by a combination of internal heating and the thermal boundary conditions (which are conditions on S), that together ensure that an imposed energy flux is transported throughout the layer in a steady state. (Note that energy could also be forced across the layer by fixing the entropy S at each boundary, such that an "adverse" (negative) radial entropy gradient is obtained in a steady state). **In the Jupiter models, we will allow $\overline{Q}(r)$ to also include a cooling term at the top of the convection zone. That is, we will set $\partial S/\partial r \equiv 0$ at both the top and bottom boundary (no conduction in or out), and the flux of energy across the system will be imposed purely by the function $\overline{Q}(r)$.**

Finally, we recall the relation

$$\frac{d\overline{S}}{dr} = c_{\rm p}\frac{\overline{N^2}(r)}{\overline{g}(r)}, \tag{2.6}$$

where $\overline{N^2}(r)$ is the squared buoyancy frequency, which we will use in favor of $d\overline{S}/dr$ in subsequent equations.

Note that the original equations in Gilman & Glatzmaier (1981) and Clune et al. (1999) were derived assuming a nearly-adiabatic background state (i.e., $d\overline{S}/dr \approx 0$). Brown et al. (2012) and Vasil et al. (2013) have raised concerns about using various anelastic approximations in stable layers due to non-energy-conserving gravity waves. Should we be concerned?

3 Non-Dimensional Scheme

We now non-dimensionalize Equations (2.1)–(2.5), according to the following scheme:

$$\nabla \to \frac{1}{H}\nabla, \tag{3.1a}$$
$$t \to \tau t, \tag{3.1b}$$
$$\boldsymbol{u} \to \frac{H}{\tau}\boldsymbol{u}, \tag{3.1c}$$
$$S \to \sigma S, \tag{3.1d}$$
$$P \to \tilde{\rho}\frac{H^2}{\tau^2}P, \tag{3.1e}$$
$$\boldsymbol{B} \to (\mu\tilde{\rho})^{1/2}\frac{H}{\tau}\boldsymbol{B}, \tag{3.1f}$$
$$\overline{\rho}(r) \to \tilde{\rho}\overline{\rho}(r), \tag{3.1g}$$
$$\overline{T}(r) \to \tilde{T}\overline{T}(r), \tag{3.1h}$$
$$\overline{g}(r) \to \tilde{g}\overline{g}(r), \tag{3.1i}$$
$$\overline{N^2}(r) \to \widetilde{N^2}\,\overline{N^2}(r), \tag{3.1j}$$
$$\overline{\nu}(r) \to \tilde{\nu}\overline{\nu}(r), \tag{3.1k}$$
$$\overline{\kappa}(r) \to \tilde{\kappa}\overline{\kappa}(r), \tag{3.1l}$$
$$\overline{\eta}(r) \to \tilde{\eta}\overline{\eta}(r), \tag{3.1m}$$
$$\text{and}\qquad \overline{Q}(r) \to \frac{H^2}{\tilde{\rho}\tilde{T}\tilde{\kappa}\sigma}\overline{Q}(r), \tag{3.1n}$$

Here, H is a typical length-scale, τ a typical time-scale, and σ a typical entropy scale. On the right-hand-sides of Equation (3.1) and in the following non-dimensionalizations, all fluid variables, coordinates, and background-state quantities are understood to be non-dimensional. The tildes refer to"typical values" of the (dimensional) reference-state functions.

Below, we will assume the time-scale is either a viscous diffusion time (i.e., $\tau = H^2/\tilde{\nu}$) or a rotational time-scale [i.e., $\tau = (2\Omega_0)^{-1}$].

4 Non-Dimensional Equations; $\tau = L^2/\tilde{\nu}$

In this case, Equations (2.1)–(2.5) become

$$\nabla\cdot[\overline{\rho}(r)\boldsymbol{u}] = 0, \tag{4.1}$$
$$\nabla\cdot\boldsymbol{B} = 0, \tag{4.2}$$

$$
\begin{aligned}
\overline{\rho}(r)\left[\frac{D\boldsymbol{u}}{Dt}+\frac{1}{\mathrm{Ek}}\hat{\boldsymbol{e}}_z\times\boldsymbol{u}\right]=&-\overline{\rho}(r)\nabla\left[\frac{P}{\overline{\rho}(r)}\right]+\frac{\mathrm{Ra}}{\mathrm{Pr}}\overline{\rho}(r)\overline{g}(r)S\hat{\boldsymbol{e}}_r,\\
&+\nabla\cdot\boldsymbol{D}+(\nabla\times\boldsymbol{B})\times\boldsymbol{B},
\end{aligned}
\tag{4.3a}
$$

$$
\text{where}\qquad D_{ij}:=2\overline{\rho}(r)\overline{\nu}(r)\left[e_{ij}-\frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right]
\tag{4.3b}
$$

$$
\text{and}\qquad e_{ij}:=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right),
\tag{4.3c}
$$

$$
\begin{aligned}
\overline{\rho}(r)\overline{T}(r)\frac{DS}{Dt}=&-\frac{\mathrm{Pr}}{\mathrm{Ra}}\mathrm{Bu}_{\mathrm{visc}}\overline{\rho}(r)\overline{T}(r)\frac{\overline{N^2}(r)}{\overline{g}(r)}u_r+\frac{1}{\mathrm{Pr}}\nabla\cdot[\overline{\rho}(r)\overline{T}(r)\overline{\kappa}(r)\nabla S]\\
&+\frac{1}{\mathrm{Pr}}\overline{Q}(r)+\frac{\mathrm{PrDi}}{\mathrm{Ra}}D_{ij}e_{ij}+\frac{\mathrm{PrDi}}{\mathrm{Pr_m Ra}}\overline{\eta}(r)|\nabla\times\boldsymbol{B}|^2,
\end{aligned}
\tag{4.4}
$$

$$
\text{and}\qquad \frac{\partial\boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{u}\times\boldsymbol{B})-\frac{1}{\mathrm{Pr_m}}\nabla\times[\overline{\eta}(r)\nabla\times\boldsymbol{B}].
\tag{4.5}
$$

The non-dimensional numbers appearing are:

$$
\mathrm{Ra}:=\frac{\tilde{g}H^3}{\tilde{\nu}\tilde{\kappa}}\frac{\sigma}{c_{\mathrm{p}}}\qquad\text{(Rayleigh number)},
\tag{4.6a}
$$

$$
\mathrm{Pr}:=\frac{\tilde{\nu}}{\tilde{\kappa}}\qquad\text{(Prandtl number)},
\tag{4.6b}
$$

$$
\mathrm{Pr_m}:=\frac{\tilde{\nu}}{\tilde{\eta}}\qquad\text{(magnetic Prandtl number)},
\tag{4.6c}
$$

$$
\mathrm{Ek}:=\frac{\tilde{\nu}}{2\Omega_0H^2}\qquad\text{(Ekman number)},
\tag{4.6d}
$$

$$
\mathrm{Bu_{visc}}:=\frac{\widetilde{N^2}H^4}{\tilde{\nu}^2}\qquad\text{(buoyancy number)},
\tag{4.6e}
$$

$$
\text{and}\qquad \mathrm{Di}=\frac{\tilde{g}H}{c_{\mathrm{p}}\tilde{T}}\qquad\text{(dissipation number)},
\tag{4.6f}
$$

Note that the dissipation number is not an independent control parameter, but a function of the non-dimensional parameters characterizing the reference state (this will be seen in Section **??**).

Note the form of the non-dimensional heating-and-cooling function:

$$
\overline{Q}(r):=\frac{H^2}{\tilde{\rho}\tilde{T}\tilde{\kappa}\sigma}\overline{Q}_{\mathrm{dim}}(r),
\tag{4.7}
$$

where the "dim" subscript explicitly denotes the dimensional version of a quantity. In general, $\overline{Q}(r)$ is simply an arbitrary—hopefully order unity—function. If $|\overline{Q}(r)|\gg 1$, the user is dilating their Rayleigh number without saying so. If $|\overline{Q}(r)| \ll 1$, the user is contracting their Rayleigh number without saying so.

The function $\overline{Q}(r)$ takes a specific form if we assume the Rayleigh number is a "flux" Rayleigh number. In that case, we identify the entropy scale σ via

$$\sigma = \frac{H\widetilde{F_{\rm nr}}}{\tilde{\rho}\tilde{T}\tilde{\kappa}}, \tag{4.8}$$

$$\text{where} \qquad \mathcal{F}_{\rm nr}(r) \coloneqq \frac{H}{r^2}\int_{r_{\rm in}}^{r} Q_{\rm dim}(x)x^2 dx \tag{4.9}$$

is the (dimensional) flux not carried by radiation in a statistically steady state, $\widetilde{F_{\rm nr}}$ refers to a volume-average of $\overline{F_{\rm nr}}(r)$ over the convection zone (CZ) of the shell, and $r_{\rm in}$ is the inner shell boundary. In general, we will ensure that $\overline{Q}(r)$ is nonzero only in the CZ and is normalized to have a total volume integral over the CZ of zero (i.e., heating and cooling balance). Hence, $\overline{F_{\rm nr}}(r)$ will zero outside of the CZ.

From Equation (4.7), we thus have

$$\overline{Q}(r) = \frac{H}{\widetilde{F_{\rm nr}}}\overline{Q}_{\rm dim}(r). \tag{4.10}$$

The user is thus free to choose the shape of $\overline{Q}(r)$, but not its amplitude, since it will have to be renormalized according to Equation (4.10), to be consistent with the definition of the Rayleigh number.

The viscous buoyancy number $\mathrm{Bu}_{\rm visc}$ is the ratio of the typical squared buoyancy frequency to the squared viscous diffusion time. It is essentially a "second (stable) Rayleigh number", and will measure the stiffness of the stable layer. In other words, $\widetilde{N^2}$ will refer to the typical value of $\overline{N^2}(r)$ over the stable weather layer (WL) in the shell. The buoyancy number is independent of the Rayleigh number, which measures the ultimate instability of the CZ.

5 Non-Dimensional Equations; $\tau = \Omega_0^{-1}$

In the previous section, t (and things with time in the dimensions) was implied to mean $(\tilde{\nu}/H^2)t_{\rm dim}$, where $t_{\rm dim}$ was the dimensional time. We now want to use a new non-dimensional time, $t_{\rm new} = \Omega_0 t_{\rm dim} = t/\mathrm{Ek}$. We can thus find the new equations easily from Equations (4.1)–(4.5). Every place we see a time dimension, we recall $t = \mathrm{Ek}t_{\rm new}$, so we multiply the place where the dimension appears by Ek and drop the "new" subscript (e.g., $t \to \mathrm{Ek}\, t$, $\boldsymbol{u} \to \boldsymbol{u}/\mathrm{Ek}$, etc.). We thus find (after rearranging terms)

$$\nabla \cdot [\overline{\rho}(r)\boldsymbol{u}] = 0, \tag{5.1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{5.2}$$

$$\overline{\rho}(r)\left[\frac{D\boldsymbol{u}}{Dt}+\hat{\boldsymbol{e}}_z\times\boldsymbol{u}\right]=-\,\overline{\rho}(r)\nabla\left[\frac{P}{\overline{\rho}(r)}\right]+\mathrm{Ra}^*\overline{\rho}(r)\overline{g}(r)S\hat{\boldsymbol{e}}_r,$$
$$+\,\mathrm{Ek}\nabla\cdot\boldsymbol{D}+(\nabla\times\boldsymbol{B})\times\boldsymbol{B}, \tag{5.3a}$$
$$\text{where}\qquad D_{ij}\coloneqq 2\overline{\rho}(r)\overline{\nu}(r)\left[e_{ij}-\frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{5.3b}$$
$$\text{and}\qquad e_{ij}\coloneqq\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{5.3c}$$

$$\overline{\rho}(r)\overline{T}(r)\frac{DS}{Dt}=-\,\frac{\mathrm{Bu_{rot}}}{\mathrm{Ra}^*}\overline{\rho}(r)\overline{T}(r)\frac{\overline{N^2}(r)}{\overline{g}(r)}u_r+\frac{\mathrm{Ek}}{\mathrm{Pr}}\nabla\cdot\left[\overline{\rho}(r)\overline{T}(r)\overline{\kappa}(r)\nabla S\right]$$
$$+\,\frac{\mathrm{Ek}}{\mathrm{Pr}}\overline{Q}(r)+\frac{\mathrm{DiEk}}{\mathrm{Ra}^*}D_{ij}e_{ij}+\frac{\mathrm{DiEk}}{\mathrm{Pr_m}\mathrm{Ra}^*}\overline{\eta}(r)|\nabla\times\boldsymbol{B}|^2, \tag{5.4}$$

$$\text{and}\qquad \frac{\partial\boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{u}\times\boldsymbol{B})-\frac{\mathrm{Ek}}{\mathrm{Pr_m}}\nabla\times[\eta(r)\nabla\times\boldsymbol{B}]. \tag{5.5}$$

The new non-dimensional numbers appearing are:

$$\mathrm{Ra}^*\coloneqq\frac{\mathrm{Ek}^2}{\mathrm{Pr}}\mathrm{Ra}=\frac{\tilde{g}}{H\Omega_0^2}\frac{\sigma}{c_\mathrm{p}}, \tag{5.6a}$$
$$\text{and}\qquad \mathrm{Bu_{rot}}\coloneqq\mathrm{Ek}^2\mathrm{Bu_{visc}}=\frac{\widetilde{N}^2}{4\Omega_0^2}\sim\frac{\tilde{g}}{H\Omega_0^2}=\frac{1}{\text{geometric oblateness}}. \tag{5.6b}$$

Note that although the "$d\overline{S}/dr$-terms" in the non-dimensionalizations have seemingly different definitions, they are the same, since:

$$\frac{\mathrm{Pr}}{\mathrm{Ra}}\mathrm{Bu_{visc}}=\frac{\mathrm{Bu_{rot}}}{\mathrm{Ra}^*}\sim\frac{c_\mathrm{p}}{\sigma}. \tag{5.6c}$$

References

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