An energy-conserving anelastic approximation for strongly stably-stratified fluids

Loren Matilsky

February 24, 2024

1 Introduction

Abstract: When acoustic oscillations are believed to be irrelevant to the dynamics of an astrophysical fluid, it is useful to employ simplifying approximations to the equations of motion. The two most common of these (which are usually used to treat convection problem) are the Boussinesq approximation (when the background density does not significantly vary across the fluid layer) and the anelastic approximation (when the background density does vary significantly). There are many distinct forms of the anelastic approximation in the literature, and it has often been remarked that most of these do not properly conserve energy when the fluid is stable to convection. Here we show that the anelastic equations derived by Gough (1969) in fact do conserve energy for arbitrary motions of the fluid, even for strongly stratified background stratification. The key properties of these equations that allow them to conserve energy are (1) the absence of the Lantz-Braginsky-Roberts approximation in the momentum equation and (2) the inclusion of additional terms in the energy equation, which allow the proper conversion between kinetic and potential energy via compressive work. We show that the scaling analysis of Gough (1969), which implicitly assumed a single typical value of the background entropy gradient, can be valid even for convective overshoot, where the entropy gradient changes from slightly unstable in the convective region to stable (sometimes strongly so) in the overshoot region. The requirement for the anelastic equations to be valid for convective overshoot is that the buoyancy frequency be significantly less than the acoustic cutoff frequency.

The anelastic equations originally consisted of an approximation to the continuity and momentum equations, derived by assuming small thermal perturbations about a nearly adiabatically stratified hydrostatic reference atmosphere (Batchelor, 1953; CHARNEY & OGURA, 1960). The thermodynamics of the problem thus become "linear," in the sense that products of thermodynamic variables reduced to linear expressions in the first-order perturbations. The two key consequences of linearized thermodynamics are divergenceless mass flux (i.e., $\nabla\cdot(\overline{\rho}\boldsymbol{u})\equiv 0$, where $\overline{\rho}$ is the background density and $\boldsymbol{u}$ the fluid velocity; this takes the place of the $\nabla\cdot\boldsymbol{u}=0$ condition from the Boussinesq approximation) and the first-order buoyancy force (associated with the first-order perturbed density and pressure) being the primary driver of the flow. (Ogura & Phillips, 1962) formalized the approximation by expanding the equations of motion in a small parameter ϵ, representing the relative variation of potential temperature across the fluid layer, and hence the relative magnitude of the thermal perturbations. They recovered the equations of Batchelor (1953); CHARNEY & OGURA (1960) and showed an assumption about the *time scale* of the motion was necessary, in addition to the assumption of small thermal perturbations. Namely, the dynamical time scale of the buoyantly driven flows must be $O(\epsilon^{-1/2})$ times *larger* than the sound crossing time of the region. Sound waves, which imply rapid temporal variations on the order of the sound crossing time, are thus absent from the anelastic equations, making them ideal for numerical integration, where large time steps are required to capture significant evolution of the system.

We begin by writing down the

References

Batchelor, G. K. 1953, Quarterly Journal of the Royal Meteorological Society, 79, 224–235, doi: 10.1002/qj.49707934004

CHARNEY, J. G., & OGURA, Y. 1960, Journal of the Meteorological Society of Japan. Ser. II, 38, 19a–19a, doi: 10.2151/jmsj1923.38.6_19a

Ogura, Y., & Phillips, N. A. 1962, Journal of the Atmospheric Sciences, 19, 173, doi: 10.1175/1520-0469(1962)019<0173:saodas>2.0.co;2