Non-Dimensionalization of an Anelastic Stable–Unstable Layer in `Rayleigh`

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November 20, 2023

1 General Equations Solved in `Rayleigh`

In general (with rotation and magnetism), `Rayleigh` evolves in time a set of coupled PDEs for the 3D vector velocity $\boldsymbol{u}$, vector magnetic field $\boldsymbol{B}$, pressure perturbation P (perturbation away from the "reference" or "background" state), and entropy perturbation S. Note that S can also be interpreted as a temperature perturbation in Boussinesq mode. For more details, see `Rayleigh`'s Documentation.

We use standard spherical coordinates (r, θ, ϕ) and cylindrical coordinates $(\lambda, \phi, z) = (r\sin\theta, \phi, r\cos\theta)$, and $\hat{\boldsymbol{e}}_q$ in general denotes a position-dependent unit vector in the direction of increasing q. The full PDE-set is then:

$$\nabla \cdot [f_1(r)\boldsymbol{u}] = 0, \tag{1.1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{1.2}$$

$$\begin{aligned} f_1(r)\left[\frac{D\boldsymbol{u}}{Dt} + c_1\hat{\boldsymbol{e}}_z \times \boldsymbol{u}\right] &= c_2 f_2(r) S\hat{\boldsymbol{e}}_r - c_3 f_1(r)\nabla\left[\frac{P}{f_1(r)}\right], \\ &\quad + c_4(\nabla\times\boldsymbol{B})\times\boldsymbol{B} + c_5\nabla\cdot\boldsymbol{D}, \end{aligned} \tag{1.3a}$$

$$\text{where} \qquad D_{ij} \coloneqq 2f_1(r)f_3(r)\left[e_{ij} - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{1.3b}$$

$$\text{and} \qquad e_{ij} \coloneqq \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{1.3c}$$

$$\begin{aligned} f_1(r)f_4(r)\frac{DS}{Dt} =& - f_1(r)f_4(r)f_{14}(r)u_r + c_6\nabla\cdot[f_1(r)f_4(r)f_5(r)\nabla S] \\ &+ c_6 f_{10}(r) + c_8c_5 D_{ij}e_{ij} + \frac{\eta(r)}{4\pi}|\nabla\times\boldsymbol{B}|^2, \end{aligned} \tag{1.4}$$

$$\text{and} \qquad \frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - c_7\nabla\times[f_7(r)\nabla\times\boldsymbol{B}], \tag{1.5}$$

where $D/Dt \coloneqq \partial/\partial t + \boldsymbol{u}\cdot\nabla$ denotes the material derivative. The spherically-symmetric, time-independent reference (or background) functions $f_i(r)$ and constants c_j set the fluid approximation to be made. `Rayleigh` has built-in modes to set the f's and c's for single-layer (i.e., either convectively stable or unstable, but not both) Boussinesq or Anelastic spherical shells. More complex systems (coupled stable–unstable systems or alternative non-dimensionalizations) require the user to manually change the f's and c's. This can be done by editing an input binary file that `Rayleigh` reads upon initialization. The c's can also be changed in the ASCII text-file (i.e., the `main_input` file).

2 Dimensional Anelastic Equations

We begin by writing down the full dimensional anelastic fluid equations, as they are usually implemented in `Rayleigh` (`reference_type = 2`). We differ slightly from "tradition" by assuming at the outset that there is both volumetric heating (preferentially in the bottom of the layer), $\overline{Q}(r)$ and volumetric cooling (preferentially at the top of the layer), $\overline{C}(r)$ (our convention is that both $\overline{Q}$ and $\overline{C}$ are positive; the cooling gets subtracted).

This form of the anelastic approximation in a spherical shell is derived in, or more accurately, attributed to (since `Rayleigh` "updates" the background state slightly differently than the cluge-y `ASH` implementation), two common sources: Gilman & Glatzmaier (1981) and Clune et al. (1999). `Rayleigh`'s dimensional anelastic equation-set is then:

$$\nabla\cdot[\overline{\rho}(r)\boldsymbol{u}] = 0, \tag{2.1}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{2.2}$$

$$\overline{\rho}(r)\left[\frac{D\boldsymbol{u}}{Dt} + 2\Omega_0\hat{\boldsymbol{e}}_z\times\boldsymbol{u}\right] = \left[\frac{\overline{\rho}(r)\overline{g}(r)}{c_{\rm p}}\right]S\hat{\boldsymbol{e}}_r - \overline{\rho}(r)\nabla\left[\frac{P}{\overline{\rho}(r)}\right], \\ + \frac{1}{\mu}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} + \nabla\cdot\boldsymbol{D}, \tag{2.3a}$$

$$\text{where}\qquad D_{ij} \coloneqq 2\overline{\rho}(r)\overline{\nu}(r)\left[e_{ij} - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{2.3b}$$

$$\text{and}\qquad e_{ij} \coloneqq \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.3c}$$

$$\overline{\rho}(r)\overline{T}(r)\frac{DS}{Dt} = -\,\overline{\rho}(r)\overline{T}(r)\frac{d\overline{S}}{dr}u_r + \nabla\cdot[\overline{\rho}(r)\overline{T}(r)\overline{\kappa}(r)\nabla S] \\ + \overline{Q}(r) - \overline{C}(r) + D_{ij}e_{ij} + \frac{\overline{\eta}(r)}{\mu}|\nabla\times\boldsymbol{B}|^2, \tag{2.4}$$

$$\text{and}\qquad \frac{\partial\boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - \nabla\times[\eta(r)\nabla\times\boldsymbol{B}]. \tag{2.5}$$

Here, the thermal variables ρ, T, P, and S refer to the density, temperature, pressure, and entropy (respectively). The overbars denote the spherically-symmetric, time-independent background state. The lack of an overbar on a thermal variable indicates the (assumed small) perturbation from the background (for the entropy, $S/c_{\rm p}$ is assumed small).

Other background quantities that appear are the gravity $\overline{g}(r)$, the momentum, thermal, and magnetic diffusivities [$\overline{\nu}(r)$, $\overline{\kappa}(r)$, and $\overline{\eta}(r)$, respectively], the internal heating or cooling $\overline{Q}(r)$, the frame rotation rate Ω_0, the specific heat at constant pressure $c_{\rm p}$, and the vacuum permeability μ ($= 4\pi$ in c.g.s. units). The equations are written in a frame rotating with angular velocity Ω_0 and the centrifugal force is neglected.

Note that the internal heating and cooling functions $\overline{Q}(r)$ and $\overline{C}(r)$ are reference-state quantities (and thus assumed spherically-symmetric and time-independent) but should be interpreted as $\overline{Q} - \overline{C} = -\nabla\cdot\boldsymbol{\mathcal{F}}_{\rm rad}$, where $\boldsymbol{\mathcal{F}}_{\rm rad}$ is the radiative heat flux. Properly, $\overline{Q}$ should be proportional to the radiative diffusivity $\kappa_{\rm rad}$ (which takes on a specific form in the radiative diffusion approximation, derivable from the opacity) and to the gradient of the total temperature $\overline{T}+T$; and $\overline{C}$ should be calculated using complicated near-surface physics.

In `Rayleigh`, a convective layer is usually driven by a combination of internal heating and the thermal boundary conditions (which are conditions on S), that together ensure that an imposed energy flux is transported throughout the layer in a steady state. (Note that energy could also be forced across the layer by fixing the entropy S at each boundary, such that an "adverse" (negative) radial entropy gradient is obtained in a steady state). **In the Jupiter models, which will have both internal heating and cooling, we will set $\partial S/\partial r \equiv 0$ at both the top and bottom boundary (no conduction in or out), and the flux of energy across the system will be imposed purely by the combination $\overline{Q} - \overline{C}$.**

Also, we recall the relation

$$\frac{d\overline{S}}{dr} = c_{\rm p}\frac{\overline{N^2}(r)}{\overline{g}(r)}, \tag{2.6}$$

where $\overline{N^2}(r)$ is the squared buoyancy frequency, which we will use in favor of $d\overline{S}/dr$ in subsequent equations.

Note that the original equations in Gilman & Glatzmaier (1981) and Clune et al. (1999) were derived assuming a nearly-adiabatic background state (i.e., $d\overline{S}/dr \approx 0$). Brown et al. (2012) and Vasil et al. (2013) have raised concerns about using various anelastic approximations in stable layers due to non-energy-conserving gravity waves. Should we be concerned?

3 Non-Dimensional Scheme

We now non-dimensionalize Equations (2.1)–(2.5), according to the following scheme:

$$\nabla \to \frac{1}{H}\nabla, \tag{3.1a}$$
$$t \to \tau t, \tag{3.1b}$$
$$\boldsymbol{u} \to \frac{H}{\tau}\boldsymbol{u}, \tag{3.1c}$$
$$S \to (\Delta S)S, \tag{3.1d}$$
$$P \to \tilde{\rho}\frac{H^2}{\tau^2}P, \tag{3.1e}$$
$$\boldsymbol{B} \to (\mu\tilde{\rho})^{1/2}\frac{H}{\tau}\boldsymbol{B}, \tag{3.1f}$$
$$\overline{\rho}(r) \to \tilde{\rho}\overline{\rho}(r), \tag{3.1g}$$
$$\overline{T}(r) \to \tilde{T}\overline{T}(r), \tag{3.1h}$$
$$\overline{g}(r) \to \tilde{g}\overline{g}(r), \tag{3.1i}$$
$$\overline{N^2}(r) \to \widetilde{N^2}\,\overline{N^2}(r), \tag{3.1j}$$
$$\overline{\nu}(r) \to \tilde{\nu}\overline{\nu}(r), \tag{3.1k}$$
$$\overline{\kappa}(r) \to \tilde{\kappa}\overline{\kappa}(r), \tag{3.1l}$$
$$\overline{\eta}(r) \to \tilde{\eta}\overline{\eta}(r), \tag{3.1m}$$
$$\overline{Q}(r) \to \tilde{C}\overline{Q}(r), \tag{3.1n}$$
$$\text{and} \qquad \overline{C}(r) \to \tilde{Q}\overline{C}(r). \tag{3.1o}$$

Here, H is a typical length-scale, τ a typical time-scale, and ΔS a typical (*estimated*) entropy scale (in Rayleigh-Bénard-type convection, the true entropy difference is imposed directly, but we will set the true value indirectly via heating and cooling functions). On the right-hand-sides of Equation (3.1) and in the following non-dimensionalizations, all fluid variables, coordinates, and background-state quantities are understood to be non-dimensional. The tildes refer to"typical values" of the (dimensional) reference-state functions. These typical values will be a volume-average over the convection zone (CZ) of the shell, except for $\widetilde{N^2}$, which will be a volume-average over the stably stratified weather layer (WL). Since cooling takes out what heating dumps in, we will normalize such that $\tilde{C} = \tilde{Q}$.

Below, we will assume the time-scale is either a thermal diffusion time (i.e., $\tau = H^2/\tilde{\nu}$) or a rotational time-scale [i.e., $\tau = (2\Omega_0)^{-1}$].

4 Non-Dimensional Equations, Non-Rotating ($\tau = H^2/\tilde{\kappa}$)

In this case, Equations (2.1)–(2.5) become

$$\nabla\cdot[\overline{\rho}(r)\boldsymbol{u}] = 0, \tag{4.1}$$
$$\nabla\cdot\boldsymbol{B} = 0, \tag{4.2}$$

$$\overline{\rho}(r)\left[\frac{D\boldsymbol{u}}{Dt}+\frac{\mathrm{Pr}}{\mathrm{Ek}}\hat{\boldsymbol{e}}_z\times\boldsymbol{u}\right]=\mathrm{Pr}\mathrm{Ra}\overline{\rho}(r)\overline{g}(r)S\hat{\boldsymbol{e}}_r-\overline{\rho}(r)\nabla\left[\frac{P}{\overline{\rho}(r)}\right],$$

$$+(\nabla\times\boldsymbol{B})\times\boldsymbol{B}+\mathrm{Pr}\nabla\cdot\boldsymbol{D}, \tag{4.3a}$$

$$\text{where}\qquad D_{ij}:=2\overline{\rho}(r)\overline{\nu}(r)\left[e_{ij}-\frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{4.3b}$$

$$\text{and}\qquad e_{ij}:=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{4.3c}$$

$$\overline{\rho}(r)\overline{T}(r)\frac{DS}{Dt}=-\frac{\mathrm{Bu}}{\mathrm{Ra}}\overline{\rho}(r)\overline{T}(r)\frac{\overline{N^2}(r)}{\overline{g}(r)}u_r+\nabla\cdot[\overline{\rho}(r)\overline{T}(r)\overline{\kappa}(r)\nabla S]$$

$$+\overline{Q}(r)-\overline{C}(r)+\frac{\mathrm{Di}}{\mathrm{Ra}}D_{ij}e_{ij}+\frac{\mathrm{Di}}{\mathrm{Pr_m}\mathrm{Ra}}\overline{\eta}(r)|\nabla\times\boldsymbol{B}|^2, \tag{4.4}$$

$$\text{and}\qquad \frac{\partial\boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{u}\times\boldsymbol{B})-\frac{\mathrm{Pr}}{\mathrm{Pr_m}}\nabla\times[\overline{\eta}(r)\nabla\times\boldsymbol{B}]. \tag{4.5}$$

The non-dimensional numbers appearing are:

$$\mathrm{Ra}:=\frac{\tilde{g}H^3}{\tilde{\nu}\tilde{\kappa}}\frac{\Delta S}{c_\mathrm{p}}\qquad\text{(Rayleigh number)}, \tag{4.6a}$$

$$\mathrm{Pr}:=\frac{\tilde{\nu}}{\tilde{\kappa}}\qquad\text{(Prandtl number)}, \tag{4.6b}$$

$$\mathrm{Pr_m}:=\frac{\tilde{\nu}}{\tilde{\eta}}\qquad\text{(magnetic Prandtl number)}, \tag{4.6c}$$

$$\mathrm{Ek}:=\frac{\tilde{\nu}}{2\Omega_0H^2}\qquad\text{(Ekman number)}, \tag{4.6d}$$

$$\mathrm{Bu}:=\frac{\widetilde{N^2}H^4}{\tilde{\nu}\tilde{\kappa}}\qquad\text{(buoyancy number)}, \tag{4.6e}$$

$$\text{and}\qquad \mathrm{Di}=\frac{\tilde{g}H}{c_\mathrm{p}\tilde{T}}\qquad\text{(dissipation number)}, \tag{4.6f}$$

Note that in our convention, the dissipation number is not an independent control parameter, but a function of the non-dimensional parameters characterizing the reference state (this will be seen in Section **??**).

Note that we have chosen the entropy-scale (and thus the Rayleigh number) based on the internal heating:

$$\Delta S:=\frac{\tilde{Q}\tau}{\tilde{\rho}\tilde{T}}=\frac{\tilde{Q}H^2}{\tilde{\rho}\tilde{T}\tilde{\kappa}} \tag{4.7a}$$

$$\text{and}\qquad \mathrm{Ra}:=\frac{\tilde{g}\tilde{Q}H^3\tau}{\tilde{\rho}\tilde{T}c_\mathrm{p}\tilde{\nu}\tilde{\kappa}}=\frac{\tilde{g}\tilde{Q}H^5}{\tilde{\rho}\tilde{T}c_\mathrm{p}\tilde{\nu}\tilde{\kappa}^2}, \tag{4.7b}$$

$$\tag{4.7c}$$

where the first equality in each equation is general (it holds for any choice of time-scale τ) and the second equality is specific to the non-rotating case. Essentially, we have assumed that the heating (or cooling) operates on the time-scale τ before the fluid parcel buoyantly moves to another part of the shell, carrying with it an entropy perturbation ΔS (in the non-rotating case, this should happen on the thermal dissipation time-scale $\tau = H^2/\tilde{\kappa}$).

The user is thus free to choose the shapes of $\overline{Q}(r)$ and $\overline{C}(r)$, but not their amplitude, since they must have unity volume-averages over the CZ.

The buoyancy number Bu is the ratio of the typical squared buoyancy frequency to the thermal and viscous diffusion times. It is essentially a "second (stable) Rayleigh number", and measures the stiffness of the stable layer (recall $\widetilde{N^2}$ refers to the typical value of $\overline{N^2}(r)$ in the WL). The buoyancy number is independent of the Rayleigh number, which estimates the ultimate instability of the CZ.

5 Non-Dimensional Equations, Rotating $[\tau = (2\Omega_0)^{-1}]$

In the previous section, t (and things with time in the dimensions) was implied to mean $(\tilde{\kappa}/H^2)t_{\text{dim}}$, where t_{dim} was the dimensional time. We now want to use a new non-dimensional time, $t_{\text{new}} = \Omega_0 t_{\text{dim}} = (\text{Pr}/\text{Ek})t$. We can thus find the new equations easily from Equations (4.1)–(4.5). Every place we see a time dimension, we recall $t = (\text{Ek}/\text{Pr})t_{\text{new}}$, so we multiply the place where the time-dimension appears by (Ek/Pr) and drop the "new" subscript [e.g., $t \to (\text{Ek}/\text{Pr})\ t$, $\boldsymbol{u} \to (\text{Pr}/\text{Ek})\boldsymbol{u}$, etc.]

Note that we should now choose a different typical entropy-scale and corresponding Rayleigh number:

$$\Delta S^* := \frac{\tilde{Q}\tau}{\tilde{\rho}\tilde{T}} = \frac{\tilde{Q}}{2\Omega_0\tilde{\rho}\tilde{T}} \tag{5.1a}$$

$$\text{and} \qquad \text{Ra} := \frac{\tilde{g}\tilde{Q}H^3\tau}{\tilde{\rho}\tilde{T}c_{\text{p}}\tilde{\nu}\tilde{\kappa}} = \frac{\tilde{g}\tilde{Q}H^3}{2\Omega_0\tilde{\rho}\tilde{T}c_{\text{p}}\tilde{\nu}\tilde{\kappa}}. \tag{5.1b}$$

The reasoning here is that under the influence of rapid rotation, the life-times of upflows or downflows are no longer set by the thermal dissipation time, but by the rotation period. Thus, the heating or cooling of a fluid parcel occurs on a shorter time-scale, leading to a smaller entropy difference across the shell than in the non-rotating case with the same amount of heating. Of course, neither Equations (4.7a) or (5.1a) are particularly convincing estimates and there is a large degree of uncertainty in the actual magnitude of typical entropy perturbations. We can only see how good these estimates are (after the fact) by checking if the achieved (non-dimensional) entropy difference across the shell winds up being close to unity.

We thus find (after rearranging terms),

$$\nabla \cdot [\overline{\rho}(r)\boldsymbol{u}] = 0, \tag{5.2}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{5.3}$$

$$\overline{\rho}(r)\left[\frac{D\boldsymbol{u}}{Dt}+\hat{\boldsymbol{e}}_z\times\boldsymbol{u}\right]=\mathrm{Ra}^*\overline{\rho}(r)\overline{g}(r)S\hat{\boldsymbol{e}}_r-\overline{\rho}(r)\nabla\left[\frac{P}{\overline{\rho}(r)}\right],$$
$$+(\nabla\times\boldsymbol{B})\times\boldsymbol{B}+\mathrm{Ek}\nabla\cdot\boldsymbol{D}, \tag{5.4a}$$
$$\text{where}\qquad D_{ij}\coloneqq 2\overline{\rho}(r)\overline{\nu}(r)\left[e_{ij}-\frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{5.4b}$$
$$\text{and}\qquad e_{ij}\coloneqq\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{5.4c}$$

$$\overline{\rho}(r)\overline{T}(r)\frac{DS}{Dt}=-\frac{\mathrm{Bu}^*}{\mathrm{Ra}^*}\overline{\rho}(r)\overline{T}(r)\frac{\overline{N^2}(r)}{\overline{g}(r)}u_r+\frac{\mathrm{Ek}}{\mathrm{Pr}}\nabla\cdot[\overline{\rho}(r)\overline{T}(r)\overline{\kappa}(r)\nabla S]$$
$$+\frac{\mathrm{Ek}}{\mathrm{Pr}}\overline{Q}(r)+\frac{\mathrm{DiEk}}{\mathrm{Ra}^*}D_{ij}e_{ij}+\frac{\mathrm{DiEk}}{\mathrm{Pr_m Ra}^*}\overline{\eta}(r)|\nabla\times\boldsymbol{B}|^2, \tag{5.5}$$

$$\text{and}\qquad\frac{\partial\boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{u}\times\boldsymbol{B})-\frac{\mathrm{Ek}}{\mathrm{Pr_m}}\nabla\times[\eta(r)\nabla\times\boldsymbol{B}], \tag{5.6}$$

where we have introduced two modified non-dimensional numbers:

$$\mathrm{Ra}^*\coloneqq\frac{\mathrm{Ek}^2}{\mathrm{Pr}}\mathrm{Ra}=\frac{\tilde{g}}{H\Omega_0^2}\frac{\Delta S}{c_\mathrm{p}}=\frac{\tilde{g}\tilde{Q}}{(2\Omega_0)^3\tilde{\rho}\tilde{T}c_\mathrm{p}H}, \tag{5.7a}$$
$$\text{and}\qquad\mathrm{Bu}^*\coloneqq\frac{\mathrm{Ek}^2}{\mathrm{Pr}}\mathrm{Bu}=\frac{\widetilde{N^2}}{4\Omega_0^2}\sim\frac{\tilde{g}}{H\Omega_0^2}=\frac{1}{\text{geometric oblateness}}. \tag{5.7b}$$

Note that although the "$d\overline{S}/dr$-terms" in the non-dimensionalizations have seemingly different definitions, they are similar, since

$$\frac{\mathrm{Bu}}{\mathrm{Ra}}\sim\frac{c_\mathrm{p}}{\Delta S}\qquad\text{and}\qquad\frac{\mathrm{Bu}^*}{\mathrm{Ra}^*}\sim\frac{c_\mathrm{p}}{\Delta S^*}. \tag{5.8}$$

The only difference is in the different estimates ΔS and ΔS^*.

In terms of `Rayleigh`'s f's and c's, we compare Equations (5.2)–(5.6) to (1.1)–(1.5) and find:

6 Hydrostatic, Ideal-Gas, Jovian Stable–Unstable Layer

To model a background CZ and WL in Jupiter, we consider a spherical shell composed of an ideal, hydrostatic gas extending between inner radius r_in and outer radius r_out. An assumed transition in convective stability occurs near an intermediate radius r_0, over width δ. More specifically, we choose quartic matching of the entropy gradient between the two layers:

$$\frac{d\overline{S}}{dr}=\psi_\mathrm{WL}(r;r_0,\delta), \tag{6.1}$$

where

$$\psi_{\mathrm{WL}}(r; r_0, \delta) \coloneqq \begin{cases} 0 & r \le r_0 \\ 1 - \left[1 - \left(\frac{r-r_0}{\delta}\right)^2\right]^2 & r_0 < r < r_0 + \delta \\ 1 & r \ge r_0 + \delta. \end{cases} \tag{6.2}$$

We also define

$$\psi_{\mathrm{CZ}}(r; r_0, \delta) \coloneqq 1 - \psi_{\mathrm{WL}}(r, r_0 - \delta, \delta) \begin{cases} 1 & r \le r_0 \\ 1 - \left[1 - \left(\frac{r-r_0}{\delta}\right)^2\right]^2 & r_0 - \delta < r < r_0 \\ 0 & r \ge r_0. \end{cases} \tag{6.3}$$

ψ_{WL} thus "senses" only the WL, and ψ_{CZ} "senses" only the CZ.

With this formulation, the CZ is strictly unstable (really, marginally stable, but becomes unstable from the heating and cooling). This ensures that none of the stable gradient "leaks" into the CZ, as happens with (e.g.) tanh matching. We assume a centrally-concentrated mass so that $\overline{g}(r) \propto 1/r^2$.

It can then be shown that five non-dimensional parameters fully characterize the shell geometry, $\overline{\rho}(r)$, and $\overline{T}(r)$:

$$\alpha \coloneqq \frac{r_{\mathrm{out}} - r_0}{r_0 - r_{\mathrm{in}}} \qquad \text{(WL-to-CZ aspect ratio)}, \tag{6.4a}$$

$$\beta \coloneqq \frac{r_{\mathrm{in}}}{r_0} \qquad \text{(CZ aspect ratio)}, \tag{6.4b}$$

$$\gamma \coloneqq \frac{c_{\mathrm{p}}}{c_{\mathrm{v}}} \qquad \text{(specific-heat ratio)}, \tag{6.4c}$$

$$\delta \qquad \text{(stability transition width)}, \tag{6.4d}$$

$$N_\rho \coloneqq \ln\left[\frac{\overline{\rho}(r_{\mathrm{in}})}{\overline{\rho}(r_{\mathrm{out}})}\right] \qquad \text{(number of density scale-heights across CZ)}, \tag{6.4e}$$

where c_{v} is the specific heat at constant volume.

We choose H to be the thickness of the CZ ($r_0 - r_{\mathrm{in}} = 1$ and $r_{\mathrm{out}} - r_0 = \alpha$). Thus,

$$r_{\mathrm{in}} = \frac{1-\beta}{\beta}, \tag{6.5a}$$

$$r_0 = \frac{1}{1-\beta}, \tag{6.5b}$$

$$\text{and} \qquad r_{\mathrm{out}} = \frac{1}{1-\beta} + \alpha \tag{6.5c}$$

If we choose $\alpha = 0.25$ and $\beta = 0.9$, then $(r_{\mathrm{in}}, r_0, r_{\mathrm{out}}) = (9, 10, 10.25)$.

With the requirement that the volume-average of $\overline{g}(r)$ over the CZ be unity, we require

$$\overline{g}(r) = \left[\frac{1-\beta^3}{3(1-\beta)^3}\right]\frac{1}{r^2}. \tag{6.6}$$

It can then be shown from the ideal-gas and hydrostatic conditions (e.g., **?**) that

$$\overline{T} = e^{\overline{S}}\left[\overline{T}(r_0) - \mathrm{Di}\int_{r_0}^{r}\overline{g}(x)e^{-\overline{S}(x)}dx\right]. \tag{6.7}$$

and

$$\overline{\rho} = \overline{\rho}(r_0)\exp\left[-\left(\frac{\gamma}{\gamma-1}\right)\overline{S}\right]\overline{T}^{1/(\gamma-1)}, \tag{6.8}$$

where (nastily)

$$\mathrm{Di} := \frac{\tilde{g}H}{c_{\mathrm{p}}\tilde{T}} = \frac{3\beta(1-\beta)^2(1-e^{-N_\rho/n})}{(3\beta/2)(1-\beta^2)(1-e^{-N_\rho/n}) - (1-\beta^3)(\beta - e^{-N_\rho/n})} \tag{6.9}$$

$$\text{and} \qquad n := \frac{1}{1-\gamma} \tag{6.10}$$

is the polytropic index of the CZ (note that since $d\overline{S}/dr \equiv 0$ in the CZ, the stratification of the CZ winds up being an adiabatic polytrope). We have also assumed (without loss of generality) that $\overline{S}(r_0) = 0$, so that $\overline{S} \equiv 0$ in the CZ).

To follow Jones et al. (2011) Heimpel et al. (2022) somewhat, we assume $n = 2$. Apparently this is a common choice for Jupiter that better approximates its weird equation of state without breaking the ideal gas law. It results in the somewhat strange result $\gamma = 3/2$.

We thus choose the following geometry and reference state:

$$\alpha = 0.25, \tag{6.11a}$$

$$\beta = 0.9, \tag{6.11b}$$

$$\gamma = 3/2, \tag{6.11c}$$

$$\delta = 0.05 \qquad \text{(may mess with this one)}, \tag{6.11d}$$

$$\text{and} \qquad N_\rho = 3. \tag{6.11e}$$

We also simplify our lives and assume constant diffusivities:

$$\overline{\nu}(r) = \overline{\kappa}(r) = \overline{\eta}(r) \equiv 1. \tag{6.12}$$

For the heating, we use the typical `Rayleigh` profile $\overline{Q} \propto \overline{\rho}\overline{T}$, but ensure that the heating is fully contained in the CZ:

$$\overline{Q}(r) \propto \overline{\rho}(r)\overline{T}(r)\psi_{\mathrm{CZ}}(r; r_0, \delta) \tag{6.13}$$

For the cooling, we choose a cooling width w_{BL}:

$$\overline{C}(r) \propto \exp\left[\frac{r-r_0}{w_{\mathrm{BL}}}\right]\psi_{\mathrm{CZ}}(r; r_0, \delta) \tag{6.14}$$

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