Induction term in spherical coordinates

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1 The problem with the traditional induction terms

We consider the ideal (resistance-free) magnetohydrodynamic (MHD) induction equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \coloneqq \boldsymbol{I}, \tag{1}$$

$$= \boldsymbol{B} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{B} - (\nabla \cdot \boldsymbol{u})\boldsymbol{B}, \tag{2}$$

where $\boldsymbol{B}$ and $\boldsymbol{u}$ are the vector magnetic and velocity fields (respectively) and $\boldsymbol{I}$ is the vector induction. The three terms on the right-hand-side of Equation (2) are often interpreted as "shear," "advection," and "compression," respectively. However, this interpretation is problematic in general for two reasons:

1. The so-called shear and compression terms contain sub-terms that cancel; in particular, only velocity motions *transverse* to magnetic-field lines can shear or compress.

2. Solid-body rotation (which is a non-shearing motion that simply rotates the whole field configuration) shows up in the so-called shear and advection terms in a strange way.

Finally, even after these issues have been addressed, resolving the final terms into a particular curvilinear system (e.g., spherical coordinates) seems to present a major headache. Our goal here is to explain fully how these problems emerge and propose a tentative solution for the case of spherical coordinates.

2 Transverse shear and compression

To see how Problem 1 arises, we decompose the velocity field into components parallel and perpendicular to the local direction of $\boldsymbol{B}$:

$$\boldsymbol{u} \coloneqq u_\parallel \hat{\boldsymbol{e}}_\parallel + \boldsymbol{u}_\perp \tag{3}$$

Obviously $\boldsymbol{B} = Bx_\parallel$, where $B = |\boldsymbol{B}|$. We denote the Cartesian distance along $\boldsymbol{B}$ by $x_\parallel$. We also decompose $\boldsymbol{u}$ into its parallel and perpendicular components:

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_\parallel}{\partial x_\parallel} + \nabla_\perp \cdot \boldsymbol{u}_\perp \tag{4}$$

We then calculate

$$
\begin{aligned}
\boldsymbol{B}\cdot\nabla\boldsymbol{u}-(\nabla\cdot\boldsymbol{u})\boldsymbol{B} &= B\frac{\partial}{\partial x_\parallel}(u_\parallel\hat{\boldsymbol{e}}_\parallel+\boldsymbol{u}_\perp)-\left(\frac{\partial u_\parallel}{\partial x_\parallel}+\nabla_\perp\cdot\boldsymbol{u}_\perp\right)B\hat{\boldsymbol{e}}_\parallel \\
&= \cancel{B\frac{\partial u_\parallel}{\partial x_\parallel}}\hat{\boldsymbol{e}}_\parallel+B\frac{\partial\boldsymbol{u}_\perp}{\partial x_\parallel}-\cancel{\frac{\partial u_\parallel}{\partial x_\parallel}}B\hat{\boldsymbol{e}}_\parallel-(\nabla_\perp\cdot\boldsymbol{u}_\perp)B\hat{\boldsymbol{e}}_\parallel \\
&= \boldsymbol{B}\cdot\nabla\boldsymbol{u}_\perp-(\nabla_\perp\cdot\boldsymbol{u}_\perp)\boldsymbol{B}. \qquad (5)
\end{aligned}
$$

Thus, only motions transverse to the local field line (i.e., $\boldsymbol{u}_\perp$) can shear or compress $\boldsymbol{B}$.

3 Rigid rotation

To see how Problem 2 arises, we consider a velocity field due to rigid rotation at constant angular velocity Ω about the z-axis in a cylindrical coordinate system:

$$\boldsymbol{\Omega}=\Omega\hat{\boldsymbol{e}}_z=\text{constant} \qquad (6a)$$

$$\boldsymbol{u}=\boldsymbol{\Omega}\times\boldsymbol{r}=\Omega\lambda\hat{\boldsymbol{e}}_\phi \qquad (6b)$$

Here, λ is the cylindrical radius, ϕ the longitude, and z the axial coordinate. In general, $\hat{\boldsymbol{e}}_{(\cdots)}$ denotes a unit vector in the direction of its subscript. We calculate:

$$
\begin{aligned}
\nabla\cdot\boldsymbol{u} &= \nabla\cdot(\boldsymbol{\Omega}\times\boldsymbol{r}) \\
&= \boldsymbol{\Omega}\cdot\nabla\times\boldsymbol{r}-\boldsymbol{r}\cdot\nabla\times\boldsymbol{\Omega} \\
&= 0 \qquad \text{no compression for rigid rotation (obviously).} \qquad (7)
\end{aligned}
$$

Then:

$$
\begin{aligned}
\boldsymbol{B}\cdot\nabla\boldsymbol{u} &= (\boldsymbol{B}\cdot\nabla)(\boldsymbol{\Omega}\times\boldsymbol{r}) \\
&= \boldsymbol{\Omega}\times[(\boldsymbol{B}\cdot\nabla)(\boldsymbol{r})] \qquad (8) \\
&= \boldsymbol{\Omega}\times\boldsymbol{B} \qquad \text{“shear” for rigid rotation.} \qquad (9)
\end{aligned}
$$

Finally:

$$
\begin{aligned}
-\boldsymbol{u}\cdot\nabla\boldsymbol{B} &= -\Omega\lambda\hat{\boldsymbol{e}}_\phi\cdot\nabla\boldsymbol{B} \\
&= -\Omega\frac{\partial}{\partial\phi}(B_\lambda\hat{\boldsymbol{e}}_\lambda+B_\phi\hat{\boldsymbol{e}}_\phi+B_z\hat{\boldsymbol{e}}_z) \\
&= -\Omega\sum_\alpha\left(\frac{\partial B_\alpha}{\partial\phi}\right)\hat{\boldsymbol{e}}_\alpha-\Omega\sum_\alpha B_\alpha\frac{\partial\hat{\boldsymbol{e}}_\alpha}{\partial\phi},
\end{aligned}
$$

where the index α runs over the three cylindrical coordinates. Note that in the cylindrical coordinate system (or indeed any coordinate system with an axis of rotational symmetry), $\partial\hat{\boldsymbol{e}}_\alpha/\partial\phi=\hat{\boldsymbol{e}}_z\times\hat{\boldsymbol{e}}_\alpha$ for each α. Thus,

$$
\begin{aligned}
-\Omega\sum_\alpha B_\alpha\frac{\partial\hat{\boldsymbol{e}}_\alpha}{\partial\phi} &= -\Omega\sum_\alpha B_\alpha\hat{\boldsymbol{e}}_z\times\hat{\boldsymbol{e}}_\alpha \\
&= -\Omega\hat{\boldsymbol{e}}_z\times\sum_\alpha B_\alpha\hat{\boldsymbol{e}}_\alpha \\
&= -\boldsymbol{\Omega}\times\boldsymbol{B}
\end{aligned}
$$

and so

$$-\boldsymbol{u}\cdot\nabla\boldsymbol{B} = -\Omega\sum_{\alpha}\left(\frac{\partial B_\alpha}{\partial\phi}\right)\hat{\boldsymbol{e}}_\alpha - \boldsymbol{\Omega}\times\boldsymbol{B} \qquad \text{“advection” for rigid rotation.} \tag{10}$$

Mathematically, in any coordinate system with a z-axis of rotational symmetry, the action of rigid rotation is as follows:

$$\begin{aligned}\frac{\partial\boldsymbol{B}}{\partial t} &= \nabla\times[(\boldsymbol{\Omega}\times\boldsymbol{r})\times\boldsymbol{B}]\\ &= \sum_{\alpha}\left(-\Omega\frac{\partial B_\alpha}{\partial\phi}\right)\hat{\boldsymbol{e}}_\alpha \qquad (11\text{a})\\ \text{or}\qquad \left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right)B_\alpha &= 0 \qquad \text{for each } \alpha. \qquad (11\text{b})\end{aligned}$$

If you think about it, this makes sense: All the rigid rotation does is rotate the whole field configuration around the z-axis at the rate Ω. If you decide to also rotate at Ω (so your personal Eulerian time derivative is $\partial/\partial t + \Omega\partial/\partial\phi$), then each component of the magnetic-field configuration should remain the same in your frame. Note that rotation does *not* advect the vector magnetic field (like the term $-\boldsymbol{u}\cdot\nabla\boldsymbol{B}$ viewed on its own would suggest), but rather advects the field *components* (as if they were scalars) in any coordinate system with an axis of rotational symmetry.

4 Solution for spherical coordinates

Resolving these issues fully for the spherical coordinate system seems complicated and I am not fully sure how to do it! In particular (for “full” resolution) we should, separately at each point (r,θ,ϕ):

1. Form a local Cartesian coordinate system, say (x_1, x_2, x_3), whose origin lies at the point (r,θ,ϕ). At the origin, ($\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$) will coincide with ($\hat{\boldsymbol{e}}_r$, $\hat{\boldsymbol{e}}_\theta$, $\hat{\boldsymbol{e}}_\phi$). But slightly away from the origin, ($\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$) will stay fixed while ($\hat{\boldsymbol{e}}_r$, $\hat{\boldsymbol{e}}_\theta$, $\hat{\boldsymbol{e}}_\phi$) curve away.

2. Calculate the velocity-gradient tensor: $\partial u_1/\partial x_1, \partial u_2/\partial x_1, \partial u_3/\partial x_1$, etc. While calculating derivatives, be careful to differentiate along the Cartesian coordinates, *not* the curvilinear ones or along the actual $\boldsymbol{B}$-line). Express the final tensor components in spherical coordinates.

3. Rotate “into $\boldsymbol{B}$” to form a new primed coordinate system (such that $\hat{\boldsymbol{e}}_1'$ points along $\boldsymbol{B}$), calculate the $\partial u_j'/\partial x_i'$, and thus form $\partial u_\parallel/\partial x_\parallel$ and $\partial\boldsymbol{u}_\perp/\partial x_\parallel$ (note that $\nabla_\perp\cdot\boldsymbol{u}_\perp = \nabla\cdot\boldsymbol{u} - \partial u_\parallel/\partial x_\parallel$). Note that the direction cosines $\hat{\boldsymbol{e}}_1'$ makes with ($\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$) are simply $B_r/|\boldsymbol{B}|, B_\theta/|\boldsymbol{B}|, B_\phi/|\boldsymbol{B}|$, respectively. Note that both unit vectors and vector components transform like $x_1' = \sum_{j=1}^{3} R_{1j}x_j$, where R_{1j} is the j^{th} direction cosine.

4. Subtract the part of $\boldsymbol{u}_\perp$ corresponding to solid-body rotation and put it in the form of Equation (11a).

5. Exult, because I haven't been able to do this!

Instead of addressing the full problem as just described, I have brute-forced my way into a quasi-solution, canceling obvious terms and expressing what seem to be "transverse shear with no solid-body rotation," "transverse compression," and advection.

We write:

$$\begin{aligned} I_r = {} & \cancel{B_r \frac{\partial u_r}{\partial r}} + \frac{B_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{B_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \cancel{\frac{B_\theta u_\theta + B_\phi u_\phi}{r}} \\ & - \boldsymbol{u}\cdot\nabla B_r + \cancel{\frac{u_\theta B_\theta + u_\phi B_\phi}{r}} \\ & - \left[\cancel{\frac{\partial u_r}{\partial r}} + \frac{2u_r}{r} + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + \cot\theta\, u_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right] B_r \end{aligned}$$

or

$$I_r = \left(\frac{B_\theta}{r}\frac{\partial}{\partial\theta} + \frac{B_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right) u_r \qquad r\text{: transverse shear} \tag{12a}$$

$$- \ \boldsymbol{u}\cdot\nabla B_r \qquad r\text{: advection} \tag{12b}$$

$$- \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{2u_r}{r} + \frac{\cot\theta\, u_\theta}{r}\right) B_r \qquad r\text{: transverse compression} \tag{12c}$$

Then, for I_θ:

$$\begin{aligned} I_\theta = {} & B_r \frac{\partial u_\theta}{\partial r} + \cancel{\frac{B_\theta}{r}\frac{\partial u_\theta}{\partial \theta}} + \frac{B_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{B_\theta u_r}{r} - \cancel{\frac{\cot\theta B_\phi u_\phi}{r}} \\ & - \boldsymbol{u}\cdot\nabla B_\theta - \frac{u_\theta B_r}{r} + \cancel{\frac{\cot\theta u_\phi B_\phi}{r}} \\ & - \left[\frac{\partial u_r}{\partial r} + \frac{2u_r}{r} + \frac{1}{r}\left(\cancel{\frac{\partial u_\theta}{\partial \theta}} + \cot\theta\, u_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right] B_\theta \end{aligned}$$

or

$$I_\theta = r\left(B_r\frac{\partial}{\partial r} + \frac{B_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\left(\frac{u_\theta}{r}\right) \qquad \theta\text{: transverse shear} \tag{13a}$$

$$- \ \boldsymbol{u}\cdot\nabla B_\theta \qquad \theta\text{: advection} \tag{13b}$$

$$- \left(\frac{\partial u_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{\cot\theta\, u_\theta}{r}\right) B_\theta \qquad \theta\text{: transverse compression} \tag{13c}$$

Finally, for I_ϕ:

$$\begin{aligned} I_\phi = {} & B_r \frac{\partial u_\phi}{\partial r} + \frac{B_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + \cancel{\frac{B_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}} + \frac{B_\phi u_r}{r} + \cancel{\frac{\cot\theta B_\phi u_\theta}{r}} \\ & - \boldsymbol{u}\cdot\nabla B_\phi - \frac{u_\phi B_r}{r} - \frac{\cot\theta\, u_\phi B_\theta}{r} \\ & - \left[\frac{\partial u_r}{\partial r} + \frac{2u_r}{r} + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + \cancel{\cot\theta\, u_\theta}\right) + \cancel{\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}}\right] B_\phi \end{aligned}$$

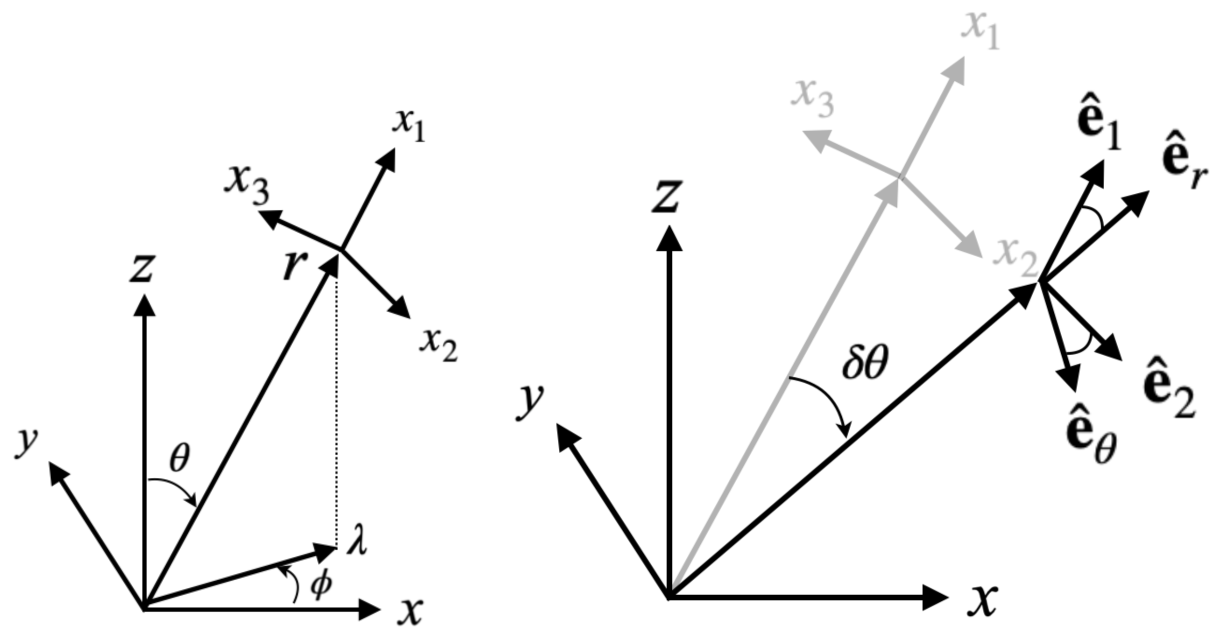

Figure 1: *Left*: Schematic of coordinate systems used here, their relation to one another, and each coordinate's meaning. Note that $(x, y, z) = (\lambda \cos\phi, \lambda \sin\phi, z) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$. The local Cartesian system (x_1, x_2, x_3) has its origin at the point (r, θ, ϕ), where $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3) = (\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta, \hat{\boldsymbol{e}}_\phi)$. *Right*: Away from the origin, the unit vectors no longer coincide. We show the relationship between the spherical-coordinate unit vectors and the Cartesian unit vectors when $\theta \to \theta + \delta\theta$ (or equivalently, $x_2 = 0 \to \delta x_2$).

or

$$
\begin{aligned}
I_\phi = {} & r \sin\theta \left(B_r \frac{\partial}{\partial r} + \frac{B_\theta}{r} \frac{\partial}{\partial \theta} \right) \left(\frac{u_\phi}{r \sin\theta} \right) && \phi\text{: transverse shear} && \text{(14a)} \\
& - \; \boldsymbol{u} \cdot \nabla B_\phi && \phi\text{: advection} && \text{(14b)} \\
& - \left(\frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) B_\phi && \phi\text{: transverse compression} && \text{(14c)}
\end{aligned}
$$

5 Transverse compression

Here we show that the "transverse compression" terms defined in Equations (12)–(14) make sense. The key point is that we must be careful (in calculating $\nabla_\perp \cdot \boldsymbol{u}_\perp$) when differentiating along θ or ϕ, because the local spherical unit vectors will all be changing. For example the divergence "transverse to $\hat{\boldsymbol{e}}_r$" is not simply $\nabla \cdot (u_\theta \hat{\boldsymbol{e}}_\theta + u_\phi \hat{\boldsymbol{e}}_\phi)$; there are also curvature terms.

Figure 1 shows schematically how this works. We set up a local Cartesian coordinate system (with origin at the point (r, θ, ϕ)) denoted by (x_1, x_2, x_3). As we differentiate $\boldsymbol{u}$, the spherical-coordinate unit vectors change, while the Cartesian ones stay fixed. In particular, if we move from $(r, \theta, \phi) \to (r + \delta r, \theta + \delta\theta, \phi + \delta\phi)$ (or equivalently in the Cartesian system,

$(0,0,0) \to (\delta x_1, \delta x_2, \delta x_3)$), we find the following (to first order in the δ's):

$$u_1 = u_r - (\delta\theta)u_\theta - (\sin\theta\delta\phi)u_\phi, \tag{15a}$$

$$u_2 = u_\theta + (\delta\theta)u_r - (\cos\theta\delta\phi)u_\phi, \tag{15b}$$

$$\text{and} \qquad u_3 = u_\phi + (\sin\theta\delta\phi)u_r + (\cos\theta\delta\phi)u_\theta. \tag{15c}$$

And trivially:

$$\delta x_1 = \delta r \tag{16a}$$

$$\delta x_2 = r\delta\theta \tag{16b}$$

$$\delta x_3 = r\sin\theta\delta\phi \tag{16c}$$

We thus compute:

$$\begin{aligned}
\frac{\partial u_1}{\partial x_1} &= \lim_{\delta x_1 \to 0} \frac{u_1(\delta x_1, 0, 0) - u_1(0,0,0)}{\delta x_1} \\
&= \lim_{\delta r \to 0} \frac{u_r(r + \delta r, \theta, \phi) - u_r(r, \theta, \phi)}{\delta r} \\
&= \frac{\partial u_r}{\partial r}
\end{aligned} \tag{17}$$