Angular Momentum in Terms of Toroidal and Poloidal Stream Functions

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1   Stream Function Formalism

In anelastic approximations, we always have the condition of divergenceless mass flux,

$$\nabla \cdot (\overline{\rho} \boldsymbol{v}) \equiv 0, \tag{1}$$

where $\overline{\rho}$ is the reference state density (we assume $\overline{\rho} = \overline{\rho}(r)$ is spherically symmetric and time-independent) and $\boldsymbol{v}$ is the fluid velocity. Condition (1) admits a stream function representation for the mass flux,

$$\overline{\rho} \boldsymbol{v} = \nabla \times [\nabla \times (W \hat{\boldsymbol{e}}_r)] + \nabla \times (Z \hat{\boldsymbol{e}}_r), \tag{2}$$

where W and Z are the poloidal and toroidal stream functions, respectively. In the derivations that follow it will also be helpful to define the alternate stream functions $\tilde{W}$ and $\tilde{Z}$ through

$$\overline{\rho} \boldsymbol{v} = \nabla \times [\nabla \times (\tilde{W} \boldsymbol{r})] + \nabla \times (\tilde{Z} \boldsymbol{r}), \tag{3}$$

where $\boldsymbol{r} = r\hat{\boldsymbol{e}}_r$ is the position vector. Clearly $W = r\tilde{W}$ and $Z = r\tilde{Z}$.

We note that

$$\nabla \times (\tilde{Z} \boldsymbol{r}) = \nabla \tilde{Z} \times \boldsymbol{r} \tag{4}$$

and

$$
\begin{aligned}
\nabla \times [\nabla \times (\tilde{W} \boldsymbol{r})] &= \nabla \times [\nabla \tilde{W} \times \boldsymbol{r}] \\
&= \nabla \tilde{W} (\underbrace{\nabla \cdot \boldsymbol{r}}_{3}) - \boldsymbol{r} \nabla \cdot (\nabla \tilde{W}) - \underbrace{(\nabla \tilde{W}) \cdot \nabla \boldsymbol{r}}_{\nabla \tilde{W}} + (\boldsymbol{r} \cdot \nabla)(\nabla \tilde{W}) \\
&= 2\nabla \tilde{W} - \boldsymbol{r} \nabla^2 \tilde{W} + (\boldsymbol{r} \cdot \nabla)(\nabla \tilde{W}) \tag{5}
\end{aligned}
$$

Equations (4) and (5) show that the radial velocity satisfies

$$\overline{\rho} v_r = 2\frac{\partial \tilde{W}}{\partial r} - r\nabla^2 \tilde{W} + r\frac{\partial}{\partial r}\left(\frac{\partial \tilde{W}}{\partial r}\right)$$

$$= r\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \tilde{W}}{\partial r}\right) - \nabla^2\tilde{W}\right]$$

$$= -\frac{1}{r}(\boldsymbol{r} \times \nabla)^2\tilde{W}, \tag{6}$$

where we have made use of the familiar identity involving the Laplacian in spherical coordinates and the quantum-mechanical total angular momentum operator (up to a constant coefficient) $\mathscr{L}^2 := (\boldsymbol{r}\times\nabla)^2$. Since $\mathscr{L}^2$ is a purely "horizontal" operator, involving only derivatives with respect to θ and ϕ and not r, and since the eigenvalues of $\mathscr{L}^2$ are $l(l+1), l \in \mathbb{J}$ (all nonzero), the vanishing of v_r forces the vanishing of $\tilde{W}$ and W:

$$v_r = 0 \Longleftrightarrow \tilde{W} = W = 0. \tag{7}$$

We shall assume here that we work in a spherical shell of inner radius r_{i} and outer radius r_{o} that has impenetrable boundaries, i.e.,

$$v_r = \tilde{W} = W \equiv 0 \quad \text{at} \quad r = r_{\mathrm{i}} \quad \text{and} \quad r = r_{\mathrm{o}}. \tag{8}$$

We define the total angular momentum of the shell through

$$\boldsymbol{L} = \int_V \boldsymbol{r} \times (\overline{\rho}\boldsymbol{v})dV, \tag{9}$$

where V is the region occupied by the shell. For convenience, we define the poloidal and toroidal angular momentum densities

$$\boldsymbol{\mathcal{L}}_W := \boldsymbol{r} \times \{\nabla \times [\nabla \times (W\hat{\boldsymbol{e}}_r)]\} \quad \text{and} \quad \boldsymbol{\mathcal{L}}_Z := \boldsymbol{r} \times [\nabla \times (Z\hat{\boldsymbol{e}}_r)], \tag{10}$$

so that

$$\boldsymbol{L} = \int_V \boldsymbol{\mathcal{L}}dV, \tag{11}$$

where $\boldsymbol{\mathcal{L}} := \boldsymbol{\mathcal{L}}_W + \boldsymbol{\mathcal{L}}_Z$ is the total angular momentum density.

2 Contribution to the Angular Momentum from the Rotation of the Shell

Physically, angular momentum is only conserved in the non-rotating (lab) frame, in which the velocity is

$$\boldsymbol{v}_{\mathrm{lab}} = \boldsymbol{v} + \Omega_0 r \sin\theta\hat{\boldsymbol{e}}_\phi \tag{12}$$

and the total angular momentum is

$$\boldsymbol{L}_{\mathrm{lab}} = \int_V \boldsymbol{r} \times (\overline{\rho}\boldsymbol{v}_{\mathrm{lab}})dV = \boldsymbol{L}_0 + \boldsymbol{L}, \tag{13}$$

where

$$\boldsymbol{L}_0 := \int_V \boldsymbol{r} \times (\overline{\rho}\Omega_0 r \sin\theta \hat{\boldsymbol{e}}_\phi)dV = \Omega_0 \left(\int_V \overline{\rho} r^2 \sin^2\theta dV \right) \hat{\boldsymbol{e}}_z \tag{14}$$

is the angular momentum due to the rotation of the shell. The angular integral in (14) is

$$2\pi \int_0^\pi \sin^3\theta d\theta = -2\pi \int_{\cos\theta=1}^{\cos\theta=-1} (1-\cos^2\theta)d\cos\theta$$

$$= 2\pi \left[\cos\theta - \left(\frac{1}{3}\right)\cos^3\theta \right]_{-1}^1 = \frac{8\pi}{3},$$

and so

$$\boldsymbol{L}_0 = \left[\frac{8\pi\Omega_0}{3} \int_{r_{\mathrm{i}}}^{r_{\mathrm{o}}} \overline{\rho}(r)r^4 dr \right] \hat{\boldsymbol{e}}_z. \tag{15}$$

For an adiabatically stratified solar-like convection zone, in which

$$r_{\mathrm{i}} = 5.0000000 \times 10^{10} \text{ cm}$$
$$r_{\mathrm{o}} = 6.5860209 \times 10^{10} \text{ cm}$$
$$\rho_{\mathrm{i}} = 0.18053428 \text{ g cm}^{-3}$$
$$\rho_{\mathrm{o}} = 0.0089882725 \text{ g cm}^{-3}$$
$$\Omega_0 = 8.61 \times 10^{-6} \text{ rad s}^{-1} \quad \text{(3 times solar Carrington)},$$

we compute

$$L_0 = 8.0719 \times 10^{47} \text{ g cm}^2 \text{ s}^{-1} \quad \text{and} \quad \mathcal{L}_0 = 1.1993 \times 10^{15} \text{ g cm}^{-1} \text{ s}^{-1}, \tag{16}$$

where $\mathcal{L}_0 := L_0/|V|$ and $|V| = 6.7302581 \times 10^{32}$ cm^3 is the volume of the shell.

3 Contribution to the Angular Momentum from the Toroidal Stream Function Z

As the name might suggest, the *toroidal* stream function Z gives the only non-vanishing contribution to the total angular momentum $\boldsymbol{L}$. We compute

$$\boldsymbol{\mathcal{L}}_Z = \boldsymbol{r} \times [\nabla\tilde{Z} \times \boldsymbol{r}]$$

$$= \nabla\tilde{Z}(\boldsymbol{r}\cdot\boldsymbol{r}) - \boldsymbol{r}(\boldsymbol{r}\cdot\nabla\tilde{Z})$$

$$= r^2 \left[\left(\cancel{\frac{\partial\tilde{Z}}{\partial r}}\right)\hat{\boldsymbol{e}}_r + \left(\frac{1}{r}\frac{\partial\tilde{Z}}{\partial\theta}\right)\hat{\boldsymbol{e}}_\theta + \left(\frac{1}{r\sin\theta}\frac{\partial\tilde{Z}}{\partial\phi}\right)\hat{\boldsymbol{e}}_\phi \right] - r^2 \left(\cancel{\frac{\partial\tilde{Z}}{\partial r}}\right)\hat{\boldsymbol{e}}_r$$

$$= \left(r\frac{\partial\tilde{Z}}{\partial\theta} \right)\hat{\boldsymbol{e}}_\theta + \left(\frac{r}{\sin\theta}\frac{\partial\tilde{Z}}{\partial\phi} \right)\hat{\boldsymbol{e}}_\phi.$$

The spherical unit vectors can be translated into Cartesian unit vectors via

$$\hat{\boldsymbol{e}}_\theta = \cos\theta\cos\phi\hat{\boldsymbol{e}}_x + \cos\theta\sin\phi\hat{\boldsymbol{e}}_y - \sin\theta\hat{\boldsymbol{e}}_z$$

and $\quad \hat{\boldsymbol{e}}_\phi = -\sin\phi\hat{\boldsymbol{e}}_x + \cos\phi\hat{\boldsymbol{e}}_y.$

Thus,

$$(\boldsymbol{\mathcal{L}}_Z)_x = r\cos\phi\cos\theta\left(\frac{\partial\tilde{Z}}{\partial\theta}\right) - \left(\frac{r}{\sin\theta}\right)\sin\phi\left(\frac{\partial\tilde{Z}}{\partial\phi}\right), \tag{17a}$$

$$(\boldsymbol{\mathcal{L}}_Z)_y = r\sin\phi\cos\theta\left(\frac{\partial\tilde{Z}}{\partial\theta}\right) + \left(\frac{r}{\sin\theta}\right)\cos\phi\left(\frac{\partial\tilde{Z}}{\partial\phi}\right), \tag{17b}$$

and $\quad (\boldsymbol{\mathcal{L}}_Z)_z = -r\sin\theta\left(\frac{\partial\tilde{Z}}{\partial\theta}\right). \tag{17c}$

In anelastic spherical harmonic codes, we expand the stream functions in terms of the spherical harmonics $Y_{lm}(\theta,\phi) \sim P_{lm}(\cos\theta)e^{im\phi}$, where the P_{lm} are the associated Legendre functions. We thus write

$$\tilde{Z}(r,\theta,\phi) = \sum_{l,m}\tilde{Z}_{lm}(r)Y_{lm}(\theta,\phi). \tag{18}$$

We consider the contributions to the total angular momentum of each $\tilde{Z}_{lm}(r)$ separately, integrating the densities in (17) over the spherical shell and using orthogonality relations. We first note that the ϕ-dependence of each spherical harmonic component of $\tilde{Z}$ (and also $\partial\tilde{Z}/\partial\phi$) is $e^{im\phi}$. The $e^{im\phi}$ are orthogonal over the interval $(0,2\pi)$, and since $\cos\phi$ and $\sin\phi$ can be written as linear combinations of $e^{\pm i\phi}$, we see that the only nonzero spherical harmonics contributing to the total angular momentum have

$$m = \pm 1 \quad \text{for} \quad L_x \quad \text{and} \quad L_y \tag{19}$$

and $\quad m = 0 \quad \text{for} \quad L_z. \tag{20}$

It will be helpful to also define $x = \cos\theta$, so that $\partial/\partial\theta = -\sin\theta\partial/\partial x = -\sqrt{1-x^2}\partial/\partial x$. The latitudinal integral is then over $\int_{-1}^{1}dx$.

3.1 Equatorial Angular Momentum

For convenience we define

$$(\boldsymbol{\mathcal{L}}_Z)_{\text{eq}} := = (\boldsymbol{\mathcal{L}}_Z)_x + i(\boldsymbol{\mathcal{L}}_Z)_x$$

$$= re^{i\phi}\left(\cos\theta\frac{\partial\tilde{Z}}{\partial\theta} - \frac{1}{\sin\theta}\tilde{Z}\right) \tag{21}$$

The latitudinal integral for each spherical harmonic will then be

$$\int_0^\pi\left[\cos\theta\frac{\partial P_{l1}(\cos\theta)}{\partial\theta} - \frac{1}{\sin\theta}P_{l1}\right]\sin\theta d\theta = \int_{-1}^1\left[x(-\sin\theta)\frac{dP_{l1}}{dx} - \frac{1}{\sin\theta}P_{l1}\right]dx$$

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$$= -\int_{-1}^{1}\left[(x\sqrt{1-x^2})\frac{dP_{l1}}{dx} + \frac{1}{\sin\theta}P_{l1}\right]dx$$

This integral does not appear to vanish for any value of l, let alone all $l \neq 1$. Thus, it seems that the conclusion drawn in Jones et al. (2011), namely that only the $l = 1, m = 1$ components contribute to the equatorial angular momentum (their Equations A12 and A13), is incorrect.

3.2 Axial Angular Momentum

Since only the $m = 0$ harmonics contribute to $(\boldsymbol{\mathcal{L}}_Z)_z$, only the functions $P_{l0}(\cos\theta) = P_l(x)$ (the non-associated Legendre polynomials) appear in the θ-integral. The θ-integral (excluding constant factors) can be transformed as

$$\int_0^{\pi}\sin^2\theta\left(\frac{\partial P_l}{\partial \theta}\right)d\theta \sim \int_{-1}^{1}\sin\theta\left(\sin\theta\frac{\partial P_l}{\partial x}\right)dx$$
$$= \int_{-1}^{1}(1-x^2)\left(\frac{dP_l}{dx}\right)dx$$
$$\sim \int_{-1}^{1}xP_l(x)dx.$$

In the final manipulation we have used integration by parts and thrown away the boundary term due to the vanishing of $1 - x^2$ at $x = \pm 1$. We recall that $x = P_1(x)$, and since the Legendre polynomials are orthogonal over the interval $(-1, 1)$, only the $l = 1$ (and $m = 0$) spherical harmonic contributes to $(\boldsymbol{\mathcal{L}}_Z)_z$. Under unit normalization, we have $Y_{10} = (1/2)\sqrt{3/\pi}\cos\theta$, from which

$$(\boldsymbol{\mathcal{L}}_Z)_z = -r\sin\theta\left(-\frac{1}{2}\sqrt{\frac{3}{\pi}}\sin\theta\right)\tilde{Z}_{10}(r) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\sin^2\theta r\tilde{Z}_{10}(r)$$

and

$$(L_Z)_z = \frac{4}{3}\sqrt{3\pi}\int_{r_{\mathrm{i}}}^{r_{\mathrm{o}}}r^3\tilde{Z}_{10}(r)dr = \frac{4}{3}\sqrt{3\pi}\int_{r_{\mathrm{i}}}^{r_{\mathrm{o}}}r^2 Z_{10}(r)dr \qquad (22)$$

The constant on the RHS of (22) will depend on the convention for the normalization of the spherical harmonics. For example, compare to Jones et al. (2011), Equations A11–A14.

4 Contribution to the Angular Momentum from the Poloidal Stream Function W

We shall now show that the contribution to the total angular momentum from $\boldsymbol{\mathcal{L}}_W$ mathematically vanishes for a spherical shell. We use (5) to compute

$$\boldsymbol{\mathcal{L}}_W = \boldsymbol{r} \times [2\nabla\tilde{W} + (\boldsymbol{r}\cdot\nabla)(\nabla\tilde{W})] \Longrightarrow$$

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$$(\mathcal{L}_W)_i = \epsilon_{ijk}r_j[2\partial_k\tilde{W} + (\boldsymbol{r}\cdot\nabla)(\partial_k\tilde{W})].$$

We note that

$$\epsilon_{ijk}r_j\partial_k\tilde{W} = \epsilon_{ijk}[\partial_k(r_j\tilde{W}) - \tilde{W}\underbrace{\partial_k r_j}_{\delta_{kj}}] = \partial_k(\epsilon_{ijk}r_j\tilde{W}),$$

$$\partial_k\boldsymbol{r} = \partial_k(r_i\hat{\boldsymbol{e}}_i) = \delta_{ik}\hat{\boldsymbol{e}}_i = \hat{\boldsymbol{e}}_k,$$

and

$$\epsilon_{ijk}r_j(\boldsymbol{r}\cdot\nabla)(\partial_k\tilde{W}) = \epsilon_{ijk}\{\partial_k[r_j(\boldsymbol{r}\cdot\nabla\tilde{W})] - \underbrace{(\partial_k r_j)}_{\delta_{kj}}(\boldsymbol{r}\cdot\nabla\tilde{W}) - r_j\underbrace{(\partial_k\boldsymbol{r})\cdot\nabla\tilde{W}}_{\hat{\boldsymbol{e}}_k\cdot\nabla\tilde{W} = \partial_k\tilde{W}}\}$$

$$= \partial_k[\epsilon_{ijk}r_j(\boldsymbol{r}\cdot\nabla\tilde{W})] - \underbrace{\epsilon_{ijk}r_j\partial_k\tilde{W}}_{\partial_k(\epsilon_{ijk}r_j\tilde{W})}$$

$$= \partial_k[\epsilon_{ijk}r_j(\boldsymbol{r}\cdot\nabla\tilde{W} - \tilde{W})]$$

Thus,

$$(\mathcal{L}_W)_i = \partial_k[\epsilon_{ijk}r_j(\tilde{W} + \boldsymbol{r}\cdot\nabla\tilde{W})]$$

and (using the divergence theorem)

$$(\boldsymbol{L}_W)_i := \int_V (\mathcal{L}_W)_i dV = \oint_{\partial V}[\epsilon_{ijk}r_j(\tilde{W} + \boldsymbol{r}\cdot\nabla\tilde{W})]n_k dS,$$

where $\boldsymbol{n}$ is the unit normal to ∂V and dS is an area element on ∂V. The boundary ∂V consists of the two spheres $r = r_{\mathrm{i}}$ and $r = r_{\mathrm{o}}$, so $n_k = \pm r_k/r$, and the identity $\epsilon_{ijk}r_j r_k \equiv 0$ immediately yields

$$(\boldsymbol{L}_W)_i = 0. \tag{23}$$

So actually the impenetrability condition has nothing to do with the vanishing of the angular momentum from the poloidal stream function! We only require that the integration region is a spherical shell. Note that **?** derive (in their Equation A10)

$$(\boldsymbol{L}_W)_i = \oint_{\partial V}\epsilon_{ijk}r_j\left[\frac{\partial(r_m\tilde{W})}{\partial x_k} - \frac{\partial(r_k\tilde{W})}{\partial x_m}\right]n_m dS \tag{24}$$

(note that they define different stream functions $P = \tilde{W}/\bar{\rho}$ and $T = \tilde{Z}/\bar{\rho}$). They attribute the vanishing of the integral due to the fact that $\tilde{W}$ (and by extension, $\boldsymbol{r}\times\nabla\tilde{W}$) vanishes on ∂V due to the impenetrability condition (8). This is unnecessary, however. One can immediately see that the second term in the integrand is zero:

$$\epsilon_{ijk}r_j\frac{\partial(r_k\tilde{W})}{\partial x_m}n_m = \epsilon_{ijk}\left[\delta_{km}\tilde{W} + r_k\frac{\partial\tilde{W}}{\partial x_m}\right]\frac{r_j(\pm r_m)}{r} \equiv 0,$$

where we have used the asymmetry of ϵ_{ijk} in j and k and the identity $\delta_{km}r_m = r_k$.

The remaining part of the integral may be written

$$\oint \epsilon_{ijk}r_j \left[\frac{\partial(r_m\tilde{W})}{\partial x_k}\right]n_m dS = \oint \left[\frac{\partial}{\partial x_k}(\epsilon_{ijk}r_j r_m\tilde{W})\right]n_m dS$$

$$= \int_V \frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_k}(\epsilon_{ijk}r_j r_m\tilde{W})\right]dV$$

$$= \int_V \frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_m}(\epsilon_{ijk}r_j r_m\tilde{W})\right]dV$$

$$= \oint \epsilon_{ijk}\left[\frac{\partial}{\partial x_m}(\epsilon_{ijk}r_j r_m\tilde{W})\right]n_k dS$$

$$= \oint \epsilon_{ijk}[\delta_{jm}r_m\tilde{W} + r_j\partial_m(r_m\tilde{W})]\left(\frac{\pm r_k}{r}\right)dS,$$

whose integrand can be shown to vanish using previous arguments on the asymmetry of ϵ_{ijk}.

References

Jones, C., Boronski, P., Brun, A., Glatzmaier, G., Gastine, T., Miesch, M., & Wicht, J. 2011, "Anelastic convection-driven dynamo benchmarks", Icar., 216, 120, doi: 10.1016/j.icarus.2011.08.014