☆ Relation (or not) between Ro & Ro_{GPR}

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Ro_{GPR}: write $\vec{u} = (u_\lambda, u_\phi, u_z)$ (1)

$$(\vec{u}\cdot\nabla\vec{u})_\lambda = u_\lambda \frac{\partial u_\lambda}{\partial \lambda} + \frac{u_\phi}{\lambda}\frac{\partial u_\lambda}{\partial \phi} + u_z \frac{\partial u_\lambda}{\partial z} - \frac{u_\phi^2}{\lambda} \quad (2)$$

$$(\vec{u}\cdot\nabla\vec{u})_z = u_\lambda \frac{\partial u_z}{\partial \lambda} + \frac{u_\phi}{\lambda}\frac{\partial u_z}{\partial \phi} + u_z \frac{\partial u_z}{\partial z} \quad (3)$$

write $\vec{u} = (\overline{u_\lambda} + u_\lambda', \overline{u_\phi} + u_\phi', \overline{u_z} + u_z')$ (4)

with $\overline{u_\phi} = \lambda \Omega_*$ (5)

overbar $\rightarrow$ zonal + temporal mean (6)

$$\Rightarrow \overline{(\vec{u}\cdot\nabla\vec{u})_\lambda} = \overline{u_\lambda}\frac{\partial \overline{u_\lambda}}{\partial \lambda} + \overline{u_\lambda' \frac{\partial u_\lambda'}{\partial \lambda}} + \Omega_* \underbrace{\cancel{\frac{\partial \overline{u_\lambda}}{\partial \phi}}}_{0} + \overline{\frac{u_\phi'}{\lambda}\frac{\partial u_\lambda'}{\partial \phi}}$$

$$+ \overline{u_z}\frac{\partial \overline{u_\lambda}}{\partial z} + \overline{u_z' \frac{\partial u_\lambda'}{\partial z}} - \lambda \Omega_*^2 - \overline{\frac{u_\phi'^2}{\lambda}} \quad (7)$$

$$\overline{(\vec{u}\cdot\nabla\vec{u})_z} = \overline{u_\lambda}\frac{\partial \overline{u_z}}{\partial \lambda} + \overline{u_\lambda' \frac{\partial u_z'}{\partial \lambda}} + \Omega_* \underbrace{\cancel{\frac{\partial \overline{u_z}}{\partial \phi}}}_{0} + \overline{\frac{u_\phi'}{\lambda}\frac{\partial u_z'}{\partial \phi}}$$

$$+ \overline{u_z}\frac{\partial \overline{u_z}}{\partial z} + \overline{u_z' \frac{\partial u_z'}{\partial z}} \quad (8)$$

Now, $(\overline{u_\lambda}, \overline{u_z}) \Longleftrightarrow$ meridional circulation (9)

$\boxed{\{\cdots\} = \text{"typical value of"}}$

$(u_\lambda', u_\phi', u_z') \Longleftrightarrow$ convection + waves (10)

Might suspect $\{|\overline{u_\lambda}|, |\overline{u_z}|\} \ll \{|u_\lambda'|, |u_\phi'|, |u_z'|\}$ (11)

(7), (8) $\Rightarrow$

$$\overline{(\vec{u}\cdot\nabla\vec{u})_\lambda} \approx -\lambda\Omega_*^2 + \overline{\vec{u}'\cdot\nabla u_\lambda'} - \overline{u_\phi'^2}/\lambda \quad (12)$$

$$\overline{(\vec{u}\cdot\nabla\vec{u})_z} \approx \overline{\vec{u}'\cdot\nabla u_z'} \quad (13)$$

Define $\quad U \equiv \{ |u_\lambda'|, |u_\phi'|, |u_z'| \} = \{ |\vec{u}' - \lambda\Omega_* \hat{e}_\phi| \}$ ③ (14)

$$H = \text{"scale of variation in } \vec{u}' \text{"}$$

Might suspect $\quad H >> \lambda \quad$ (ignore $\overline{u_\phi'^2}/\lambda$ in (13)) (15)

$$\text{Then: } \overline{(\vec{u}\cdot\nabla\vec{u})}_m = \underbrace{-\lambda\Omega_*^2 \hat{e}_\lambda}_{\lambda\Omega_*^2} + \underbrace{(\overline{\vec{u}'\cdot\nabla u_\lambda'})\hat{e}_\lambda}_{U^2/H} + \underbrace{(\overline{\vec{u}'\cdot\nabla u_z'})\hat{e}_z}_{U^2/H} \quad (16)$$

magnitudes:

"m" = "meridional part": $\vec{A}_m \equiv A_\lambda \hat{e}_\lambda + A_z \hat{e}_z$ (17)

so define

$$Ro_{GPR} \equiv \frac{U^2}{\lambda H \Omega_*^2} \quad ✓ \quad (18)$$

<u>Ro</u>: pick rotating frame (Ω_0). Ignore centrifugal force

i.e., ignore $+\lambda\Omega_0^2 \hat{e}_\lambda$ on RHS of momentum equation.

Then "all inertial terms" $= \vec{V}\cdot\nabla\vec{V}^* + 2\Omega_0 \hat{e}_z \times \vec{V}$ (19)

where $\quad \vec{V} \equiv \vec{u} - \lambda\Omega_0 \hat{e}_\phi$ (20)

Note: $(\overline{V_\lambda}, \overline{V_z}) = (\overline{u_\lambda}, \overline{u_z})$ = meridional circulation is same in both frames (21)

$$\text{compute: } \left[\overline{2\Omega_0 \hat{e}_z \times \vec{V}}\right]_m = -\left[2\Omega_0 (\overline{V_\lambda}\hat{e}_\lambda + \overline{V_\phi}\hat{e}_\phi + \overline{V_z}\hat{e}_z) \times \hat{e}_z\right]_m$$

$\hat{e}_\lambda \times \hat{e}_z \perp m$

$$= -2\Omega_0 \overline{V_\phi} \hat{e}_\lambda = -2\lambda\Omega_0(\Omega_* - \Omega_0)\hat{e}_\lambda \quad (22)$$

$$\text{(since } \overline{V_\phi} = \underbrace{\overline{u_\phi}}_{\lambda\Omega_*} - \lambda\Omega_0 = \lambda(\Omega_* - \Omega_0)\text{)} \quad (23)$$

* This term must be evaluated <u>in co-rotating frame</u>. i.e. the $\partial/\partial\phi$ in $\vec{V}\cdot\nabla\vec{V}$ is w.r.t. ϕ in rotating frame.

Ignoring [meridional circulation]^2 terms at outset:

$$\overline{(\vec{V}\cdot\nabla\vec{V})_\lambda} = \overline{V_\lambda \frac{\partial V_\lambda}{\partial \lambda} + \frac{V_\phi}{\lambda}\frac{\partial V_\lambda}{\partial \phi} + V_z \frac{\partial V_\lambda}{\partial z} - \frac{V_\phi^2}{\lambda}}$$

$$= \overline{V_\lambda' \frac{\partial V_\lambda'}{\partial \lambda}} + (\Omega_* - \Omega_0)\cancel{\frac{\partial \overline{V_\lambda}}{\partial \phi}}^{\,0} + \overline{\frac{V_\phi'}{\lambda}\frac{\partial V_\lambda'}{\partial \phi}} + \overline{V_z' \frac{\partial V_\lambda'}{\partial z}}$$

$$- \lambda(\Omega_* - \Omega_0)^2 - \frac{\overline{(V_\phi')^2}}{\lambda}$$

$$= -\lambda(\Omega_* - \Omega_0)^2 + \overline{\vec{V}'\cdot\nabla V_\lambda'} - \overline{V_\phi'^2}/\lambda \qquad (24)$$

$$\overline{(\vec{V}\cdot\nabla\vec{V})_z} = \overline{\vec{V}'\cdot\nabla V_z'} \qquad \text{(convince yourself!)} \qquad (25)$$

I can convince myself:

$$\overline{\vec{V}'\cdot\nabla\vec{V}'} = \overline{\vec{u}'\cdot\nabla\vec{u}'} \qquad (26)$$

Thus, in rotating frame, (also assume $H \ll \lambda$)

$$[\text{all inertial terms}]_m \approx \Big[\underbrace{-2\lambda\Omega_0(\Omega_* - \Omega_0)}_{\text{coriolis}} \underbrace{- \lambda(\Omega_* - \Omega_0)^2}_{\text{"extra" centrifugal}}\Big]\hat{e}_\lambda$$

magnitudes: $2\lambda\Omega_0|\Omega_* - \Omega_0|$ $\qquad$ $\lambda|\Omega_* - \Omega|^2$

$$+ \overline{(\vec{u}'\cdot\nabla u_\lambda')}\,\hat{e}_\lambda + \overline{\vec{u}'\cdot\nabla u_z'}\,\hat{e}_z \qquad (27)$$

magnitudes: U^2/H $\qquad$ U^2/H

If coriolis force dominates everything (weak DR), define:

$$\widetilde{Ro}_{DR} \equiv \frac{|\Omega_* - \Omega_0|}{2\Omega_0} = \frac{1}{2} Ro_{DR}^{*} \ll 1 \qquad (28)$$

$$Ro \equiv \frac{U^2}{2\lambda H \Omega_0 |\Omega_* - \Omega_0|} \ll 1 \qquad (29)$$

*As defined in the paper: $Ro_{DR} \equiv |\Omega_* - \Omega_0|/\Omega_0$

If furthermore

$$V \approx |\overline{V_\phi}| = \lambda |\Omega_* - \Omega_0| \qquad (30)$$

(technically a separate assumption), then

$$\widetilde{Ro}_{DR} \approx \frac{V}{2\lambda\Omega_0} \qquad (31)$$

$$Ro \approx \frac{V}{2\Omega_0 H} \quad ✓ \qquad (32)$$

If coriolis is comparable to "extra" centrifugal, then:
(i.e., the Sun, <u>strong</u> DR)

$$-2\lambda\Omega_0(\Omega_* - \Omega_0) - \lambda(\Omega_* - \Omega_0)^2$$

$$= -\lambda\left[\cancel{2\Omega_0\Omega_*} - 2\Omega_0^2 + \Omega_*^2 - \cancel{2\Omega_0\Omega_*} + \Omega_0^2\right]$$

$$= -\lambda(\Omega_*^2 - \Omega_0^2) \qquad (33)$$

This is obvious: we must get back everything we had in the inertial frame, except the ignored "frame" centrifugal force. But should we now define

$$\widetilde{Ro}_{GPR} \equiv \frac{V^2}{\lambda H |\Omega_*^2 - \Omega_0^2|} \qquad (34) \qquad ???$$

seems weird. Shouldn't have ignored frame centrifugal force to begin with.

Note: in limiting case (weak DR + $V \approx \lambda|\Omega_* - \Omega_0|$):

$$Ro_{GPR} \approx \frac{V^2}{\lambda H \Omega_0^2} = \left(\frac{4H}{\lambda}\right) Ro^2 \qquad (35)$$

So under no circumstances does $Ro_{GPR} = Ro$.
$Ro \ll 1$ (geostrophy) $\Longrightarrow$ $Ro_{GPR} \ll 1$ (GPR balance) but not vice versa.