Non-Dimensionalization of an Anelastic Stable–Unstable Layer in `Rayleigh`

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1 General Equations Solved in `Rayleigh`

In general (with rotation and magnetism), `Rayleigh` evolves in time a set of coupled PDEs for the 3D vector velocity $\boldsymbol{u}$, vector magnetic field $\boldsymbol{B}$, pressure perturbation P (perturbation away from the "reference" or "background" state), and entropy perturbation S. Note that S can also be interpreted as a temperature perturbation in Boussinesq mode. For more details, see `Rayleigh`'s Documentation.

We use standard spherical coordinates (r, θ, ϕ) and cylindrical coordinates $(\lambda, \phi, z) = (r\sin\theta, \phi, r\cos\theta)$, and $\hat{\boldsymbol{e}}_q$ in general denotes a position-dependent unit vector in the direction of increasing q. The full PDE-set is then:

$$\nabla\cdot[f_1(r)\boldsymbol{u}] = 0, \tag{1.1}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{1.2}$$

$$f_1(r)\left[\frac{D\boldsymbol{u}}{Dt} + c_1\hat{\boldsymbol{e}}_z\times\boldsymbol{u}\right] = c_2 f_2(r) S\hat{\boldsymbol{e}}_r - c_3 f_1(r)\nabla\left[\frac{P}{f_1(r)}\right], + c_4(\nabla\times\boldsymbol{B})\times\boldsymbol{B} + c_5\nabla\cdot\boldsymbol{D}, \tag{1.3a}$$

$$\text{where}\qquad D_{ij} \coloneqq 2f_1(r)f_3(r)\left[e_{ij} - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{1.3b}$$

$$\text{and}\qquad e_{ij} \coloneqq \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{1.3c}$$

$$f_1(r)f_4(r)\frac{DS}{Dt} = -f_1(r)f_4(r)f_{14}(r)u_r + c_6\nabla\cdot[f_1(r)f_4(r)f_5(r)\nabla S] + c_6 f_{10}(r) + c_8 c_5 D_{ij}e_{ij} + \frac{\eta(r)}{4\pi}|\nabla\times\boldsymbol{B}|^2, \tag{1.4}$$

$$\text{and}\qquad \frac{\partial\boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - c_7\nabla\times[f_7(r)\nabla\times\boldsymbol{B}], \tag{1.5}$$

where $D/Dt := \partial/\partial t + \boldsymbol{u}\cdot\nabla$ denotes the material derivative. The spherically-symmetric, time-independent reference (or background) functions $f_i(r)$ and constants c_j set the fluid approximation to be made. `Rayleigh` has built-in modes to set the f's and c's for single-layer (i.e., either convectively stable or unstable, but not both) Boussinesq or Anelastic spherical shells. More complex systems (coupled stable–unstable systems or alternative non-dimensionalizations) require the user to manually change the f's and c's. This can be done by editing an input binary file that `Rayleigh` reads upon initialization. The c's can also be changed in the ASCII text-file (i.e., the `main_input` file).

2 Dimensional Anelastic Equations

We begin by writing down the full dimensional anelastic fluid equations, as they are usually implemented in Rayleigh (`reference_type = 2`). This form of the anelastic approximation in a spherical shell is derived in, or more accurately, attributed to (since `Rayleigh` "updates" the background state slightly differently than the cluge-y `ASH` implementation), two common sources: Gilman & Glatzmaier (1981) and Clune et al. (1999). `Rayleigh`'s dimensional anelastic equation-set is:

$$\nabla\cdot[\overline{\rho}(r)\boldsymbol{u}] = 0, \tag{2.1}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{2.2}$$

$$\overline{\rho}(r)\left[\frac{D\boldsymbol{u}}{Dt} + 2\Omega_0\hat{\boldsymbol{e}}_z\times\boldsymbol{u}\right] = \left[\frac{\overline{\rho}(r)\overline{g}(r)}{c_{\mathrm{p}}}\right]S\hat{\boldsymbol{e}}_r - \overline{\rho}(r)\nabla\left[\frac{P}{\overline{\rho}(r)}\right],$$
$$+\frac{1}{\mu}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} + \nabla\cdot\boldsymbol{D}, \tag{2.3a}$$

$$\text{where}\qquad D_{ij} := 2\overline{\rho}(r)\overline{\nu}(r)\left[e_{ij} - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{2.3b}$$

$$\text{and}\qquad e_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.3c}$$

$$\overline{\rho}(r)\overline{T}(r)\frac{DS}{Dt} = -\,\overline{\rho}(r)\overline{T}(r)\frac{d\overline{S}}{dr}u_r + \nabla\cdot[\overline{\rho}(r)\overline{T}(r)\overline{\kappa}(r)\nabla S]$$
$$+\overline{Q}(r) + D_{ij}e_{ij} + \frac{\overline{\eta}(r)}{\mu}|\nabla\times\boldsymbol{B}|^2, \tag{2.4}$$

$$\text{and}\qquad \frac{\partial\boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - \nabla\times[\eta(r)\nabla\times\boldsymbol{B}]. \tag{2.5}$$

Here, the thermal variables ρ, T, P, and S refer to the density, temperature, pressure, and entropy (respectively). The overbars denote the spherically-symmetric, time-independent background state. The lack of an overbar on a thermal variable indicates the (assumed small) perturbation from the background (for the entropy, S/c_{p} is assumed small).

Other background quantities that appear are the gravity $\overline{g}(r)$, the momentum, thermal, and magnetic diffusivities [$\overline{\nu}(r)$, $\overline{\kappa}(r)$, and $\overline{\eta}(r)$, respectively], the internal heating $\overline{Q}(r)$, the frame rotation rate Ω_0, the specific heat at constant pressure c_{p}, and the vacuum permeability μ ($= 4\pi$ in c.g.s. units). The equations are written in a frame rotating with angular velocity Ω_0 and the centrifugal force is neglected.

Note that the internal heating function $\overline{Q}(r)$ is also assumed spherically symmetric and fixed in time, but should be interpreted as $-\nabla \cdot \boldsymbol{\mathcal{F}}_{\mathrm{rad}}$, where $\boldsymbol{\mathcal{F}}_{\mathrm{rad}}$ is the radiative heat flux and properly should be proportional to the gradient of the total (background + perturbed) temperature. If the system is a convection zone, it is driven by a combination of internal heating and the thermal boundary conditions (which are conditions on S), that together ensure the imposed energy flux is transported throughout the layer in a steady state.

Finally, we recall the relation

$$\frac{d\overline{S}}{dr} = c_{\mathrm{p}} \frac{\overline{N^2}(r)}{\overline{g}(r)}, \tag{2.6}$$

where $\overline{N^2}(r)$ is the squared buoyancy frequency, which we use in favor of $d\overline{S}/dr$ in subsequent equations.

Note that the original equations in Gilman & Glatzmaier (1981) and Clune et al. (1999) were derived assuming a nearly-adiabatic background state (i.e., $d\overline{S}/dr \approx 0$). Brown et al. (2012); Vasil et al. (2013) have raised concerns about using various anelastic approximations in stable layers due to non-energy-conserving gravity waves. Should we be concerned?

3 Non-Dimensional Scheme

We now non-dimensionalize Equations (2.1)–(2.5), according to the following scheme:

$$\nabla \rightarrow \frac{1}{L}\nabla, \tag{3.1a}$$
$$t \rightarrow \tau t, \tag{3.1b}$$
$$\boldsymbol{u} \rightarrow \frac{L}{\tau}\boldsymbol{u}, \tag{3.1c}$$
$$S \rightarrow \sigma S, \tag{3.1d}$$
$$P \rightarrow \tilde{\rho}\frac{L^2}{\tau^2}P, \tag{3.1e}$$
$$\boldsymbol{B} \rightarrow (4\pi\tilde{\rho})^{1/2}\frac{L}{\tau}\boldsymbol{B}, \tag{3.1f}$$
$$\overline{\rho}(r) = \tilde{\rho}\hat{\rho}(r), \tag{3.1g}$$
$$\overline{T}(r) = \tilde{T}\hat{T}(r), \tag{3.1h}$$
$$g(r) = \tilde{g}\hat{g}(r), \tag{3.1i}$$
$$N^2(r) = \widetilde{N^2}\widehat{N^2}(r), \tag{3.1j}$$
$$\nu(r) = \tilde{\nu}\hat{\nu}(r), \tag{3.1k}$$
$$\kappa(r) = \tilde{\kappa}\hat{\kappa}(r), \tag{3.1l}$$
$$\text{and} \qquad \eta(r) = \tilde{\eta}\hat{\eta}(r). \tag{3.1m}$$

Here, L is a typical length-scale, τ is a typical time-scale, and σ is a typical entropy scale. On the right-hand-sides of Equations (3.1a)–(3.1f) (and in the following non-dimensionalizations), ∇, t, $\boldsymbol{u}$, S, P, and $\boldsymbol{B}$ are all understood to be non-dimensional. In Equations (3.1g)–(3.1m), the tildes refer to"typical values" of the reference state functions and the hats refer to the radially-dependent non-dimensional versions of the reference-state functions.

Below, we will assume the time scale is either a viscous diffusion time (i.e., $\tau = L^2/\tilde{\nu}$) or a rotational time-scale (i.e., $\tau = \Omega_0^{-1}$). To describe the reference state, we will consider three cases for a given function's "typical value": Its value at the inner shell boundary, its value at the outer shell boundary, or its value volume-averaged over the shell.

4 Non-Dimensional Equations; $\tau = L^2/\tilde{\nu}$

In this case, Equations (2.1)–(2.5) become

$$\nabla \cdot [\hat{\rho}(r)\boldsymbol{u}] = 0, \tag{4.1}$$
$$\nabla \cdot \boldsymbol{B} = 0, \tag{4.2}$$

$$\hat{\rho}(r)\left[\frac{D\boldsymbol{u}}{Dt}+\frac{2}{\mathrm{Ek}}\hat{\boldsymbol{e}}_z\times\boldsymbol{u}\right]=-\,\hat{\rho}(r)\nabla\left[\frac{P}{\hat{\rho}(r)}\right]+\frac{\mathrm{Ra}}{\mathrm{Pr}}\hat{\rho}(r)\hat{g}(r)S\hat{\boldsymbol{e}}_r,$$
$$+\nabla\cdot\boldsymbol{D}+(\nabla\times\boldsymbol{B})\times\boldsymbol{B}, \tag{4.3a}$$
$$\text{where}\qquad D_{ij}:=2\hat{\rho}(r)\hat{\nu}(r)\left[e_{ij}-\frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{4.3b}$$
$$\text{and}\qquad e_{ij}:=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{4.3c}$$

$$\hat{\rho}(r)\hat{T}(r)\frac{DS}{Dt}=-\,\frac{\mathrm{Pr}}{\mathrm{Ra}}\mathrm{B}_{\mathrm{visc}}\hat{\rho}(r)\hat{T}(r)\frac{\widehat{N^2}(r)}{\hat{g}(r)}u_r+\frac{1}{\mathrm{Pr}}\nabla\cdot[\hat{\rho}(r)\hat{T}(r)\hat{\kappa}(r)\nabla S]$$
$$+\frac{1}{\mathrm{Pr}}\hat{Q}(r)+\frac{\mathrm{PrDi}}{\mathrm{Ra}}D_{ij}e_{ij}+\frac{\mathrm{PrDi}}{\mathrm{Pr_m Ra}}\hat{\eta}(r)|\nabla\times\boldsymbol{B}|^2, \tag{4.4}$$

$$\text{and}\qquad \frac{\partial\boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{u}\times\boldsymbol{B})-\frac{1}{\mathrm{Pr_m}}\nabla\times[\eta(r)\nabla\times\boldsymbol{B}]. \tag{4.5}$$

The non-dimensional numbers appearing are:

$$\mathrm{Ra}:=\frac{\tilde{g}L^3}{\tilde{\nu}\tilde{\kappa}}\frac{\sigma}{c_{\mathrm{p}}}, \tag{4.6a}$$
$$\mathrm{Pr}:=\frac{\tilde{\nu}}{\tilde{\kappa}}, \tag{4.6b}$$
$$\mathrm{Pr_m}:=\frac{\tilde{\nu}}{\tilde{\eta}}, \tag{4.6c}$$
$$\mathrm{Ek}:=\frac{\tilde{\nu}}{\Omega_0 H^2}, \tag{4.6d}$$
$$\mathrm{B_{visc}}:=\frac{\widetilde{N^2}L^4}{\tilde{\nu}^2}, \tag{4.6e}$$
$$\text{and}\qquad \mathrm{Di}=\frac{\tilde{g}L}{c_{\mathrm{p}}\tilde{T}}, \tag{4.6f}$$

along with the non-dimensional heating function

$$\hat{Q}(r):=\frac{L^2}{\tilde{\rho}\tilde{T}\tilde{\kappa}\sigma}Q(r). \tag{4.7}$$

Note that the dissipation number is not an independent control parameter, but a function of the non-dimensional parameters characterizing the polytrope.

In general, $\hat{Q}(r)$ is simply an arbitrary—hopefully order unity—function. Assuming the thermal boundary conditions remove whatever $\hat{Q}(r)$ dumps in: If $\hat{Q}(r)\gg 1$, the user is dilating their Rayleigh number without saying so. If $\hat{Q}\ll 1$, the user is contracting their Rayleigh number without saying so. If $\hat{Q}(r)\equiv 0$ (and the user wants to simulate a convection zone), the user should typically identify σ with $-\Delta S = S_{\text{in}} - S_{\text{out}}$ (the imposed entropy drop across the layer) and thus set $S \equiv 1$ at $r = r_{\text{in}}$ and $S \equiv 0$ at $r = r_{\text{out}}$.

The non-dimensional heating takes a specific form if we assume the Rayleigh number is a "flux" Rayleigh number. In that case we identify

$$\sigma = \frac{L \left\langle \mathcal{F}_{\text{nr}}(r) \right\rangle_{\text{v}}}{\tilde{\rho}\tilde{T}\tilde{\kappa}}, \tag{4.8}$$

$$\text{where} \qquad \mathcal{F}_{\text{nr}}(r) \coloneqq \frac{1}{r^2}\int_{r_{\text{in}}}^{r} Q(x)x^2 dx \tag{4.9}$$

is the flux not carried by radiation in a statistically steady state and $\langle\cdots\rangle_{\text{v}}$ refers to a volume average over the whole shell. We thus have

$$\hat{Q}(r) = \frac{L}{\left\langle \mathcal{F}_{\text{nr}}(r) \right\rangle_{\text{v}}} Q(r) \tag{4.10}$$

and whatever amplitude (luminosity) the user chooses for the dimensional $Q(r)$, the ultimate $\hat{Q}(r)$ will normalize that amplitude away in Equation (4.10).

The viscous buoyancy number B_{visc} is the ratio of the typical squared buoyancy frequency to the squared viscous diffusion time (it is essentially a kind of Richardson number). Although it has to do with background entropy stratification, B_{visc} is nominally independent of the Rayleigh number (which derives from the entropy perturbations associated with the thermal boundary conditions and/or heating that force energy through the layer). However, the following notes are warranted:

(1) In the typical convection problem (polytropic index $n = n_{\text{ad}} \coloneqq 1/(\gamma - 1)$), $d\overline{S}/dr \equiv 0$ and the value of B_{visc} is irrelevant.

(2) For an isolated stable polytrope ($n > n_{\text{ad}}$), it is unclear how the typical entropy perturbation σ is established. Even if energy is driven through the system by the thermal boundary conditions and/or heating, the energy will likely be carried by spherically symmetric conduction (depending on how stable the stratification is) and σ may not be set by the boundary conditions. Furthermore, σ cannot be set by the total entropy contrast across the layer (which is $\sim c_{\text{p}}$, and if $\sigma = c_{\text{p}}$, we would have $\text{B}_{\text{visc}}\text{Pr} = \text{Ra}$). That is because no plume can traverse a large portion of the stably stratified layer. Thus, although the user can choose a "Rayleigh number" (probably a misnomer) for the isolated stable layer, they have no way of implementing the background state, boundary conditions, or heating profile to be consistent with their choice.

(3) For unstable polytropes ($n < n_{\text{ad}}$), we first note that if n differs by a factor of unity from n_{ad}, order-unity thermal perturbations S/c_{p} are forced and the anelastic approximation is invalid. In any case (either if the simulation survives an order-unity $n_{\text{ad}} - n$, or if the user chooses a well-posed small $n_{\text{ad}} - n$ to help drive the convection), we expect that the convection will restratify the system toward adiabaticity. In Equation (2.4), a background $(dS/dr)_{\ell=0}$ will be established to be close to the negative of $d\overline{S}/dr$ and (similar to point (1)), the value of B_{visc} will not truly be an independent parameter, but will help to determine an ultimate "effective" Rayleigh number.

(4) In the more logical case of a stable layer being pummeled by a neighboring overshooting convection layer, both B_{visc} and Ra are truly independent (and relevant). The

overshooting flows (driven by Ra, which is a property of the convection zone) establish the typical σ in both the convection zone and overshoot layer. Meanwhile, $\mathrm{B_{visc}}$ (which is a property of the stable layer) controls how strongly the overshoot is decelerated.

5 Non-Dimensional Equations; $\tau = \Omega_0^{-1}$

In the previous section, t (and things with time in the dimensions) was implied to mean $(\tilde{\nu}/H^2)t_{\mathrm{dim}}$, where t_{dim} was the dimensional time. We now want to use $t_{\mathrm{new}} = \Omega_0 t_{\mathrm{dim}} = t/\mathrm{Ek}$. We can thus find the new equations easily from Equations (4.1)–(4.5): Every place we see a time dimension, we recall $t = \mathrm{Ek}t_{\mathrm{new}}$, so we multiply the place where the dimension appears by Ek and drop the "new" subscript (e.g., $t \to \mathrm{Ek}\ t$, $\boldsymbol{u} \to \boldsymbol{u}/\mathrm{Ek}$, etc.). We thus find (after rearranging terms)

$$\nabla \cdot [\hat{\rho}(r)\boldsymbol{u}] = 0, \tag{5.1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{5.2}$$

$$\begin{aligned} \hat{\rho}(r)\left[\frac{D\boldsymbol{u}}{Dt} + 2\hat{\boldsymbol{e}}_z \times \boldsymbol{u}\right] = &- \hat{\rho}(r)\nabla\left[\frac{P}{\hat{\rho}(r)}\right] + \mathrm{Ra}^*\hat{\rho}(r)\hat{g}(r)S\hat{\boldsymbol{e}}_r, \\ &+ \mathrm{Ek}\nabla\cdot\boldsymbol{D} + (\nabla\times\boldsymbol{B})\times\boldsymbol{B}, \end{aligned} \tag{5.3a}$$

$$\text{where} \qquad D_{ij} := 2\hat{\rho}(r)\hat{\nu}(r)\left[e_{ij} - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right] \tag{5.3b}$$

$$\text{and} \qquad e_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{5.3c}$$

$$\begin{aligned} \hat{\rho}(r)\hat{T}(r)\frac{DS}{Dt} = &- \frac{\mathrm{B_{rot}}}{\mathrm{Ra}^*}\hat{\rho}(r)\hat{T}(r)\frac{\widehat{N^2}(r)}{\hat{g}(r)}u_r + \frac{\mathrm{Ek}}{\mathrm{Pr}}\nabla\cdot[\hat{\rho}(r)\hat{T}(r)\hat{\kappa}(r)\nabla S] \\ &+ \frac{\mathrm{Ek}}{\mathrm{Pr}}\hat{Q}(r) + \frac{\mathrm{DiEk}}{\mathrm{Ra}^*}D_{ij}e_{ij} + \frac{\mathrm{DiEk}}{\mathrm{Pr_m Ra}^*}\hat{\eta}(r)|\nabla\times\boldsymbol{B}|^2, \end{aligned} \tag{5.4}$$

$$\text{and} \qquad \frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - \frac{\mathrm{Ek}}{\mathrm{Pr_m}}\nabla\times[\eta(r)\nabla\times\boldsymbol{B}]. \tag{5.5}$$

The new non-dimensional numbers appearing are:

$$\mathrm{Ra}^* := \frac{\mathrm{Ek}^2}{\mathrm{Pr}}\mathrm{Ra} = \frac{\tilde{g}}{L\Omega_0^2}\frac{\sigma}{c_{\mathrm{p}}}, \tag{5.6a}$$

$$\text{and} \qquad \mathrm{B_{rot}} := \mathrm{Ek}^2\mathrm{B_{visc}} = \frac{\widetilde{N^2}}{\Omega_0^2} \sim \frac{\tilde{g}}{L\Omega_0^2} = \frac{1}{\text{oblateness factor}}. \tag{5.6b}$$

Note that although the "$d\overline{S}/dr$-terms" in the non-dimensionalizations have seemingly different definitions, they are the same, since:

$$\frac{\mathrm{Pr}}{\mathrm{Ra}}\mathrm{B_{visc}} = \frac{\mathrm{B_{rot}}}{\mathrm{Ra}^*} \sim \frac{c_{\mathrm{p}}}{\sigma}. \tag{5.6c}$$

6 Non-Dimensional Polytrope

A polytrope depends on the following four non-dimensional parameters:

$$\gamma \coloneqq \frac{c_\mathrm{p}}{c_\mathrm{v}} \qquad \text{specific-heat ratio,} \tag{6.1a}$$

$$0 \le n \le \infty \qquad \text{polytropic index,} \tag{6.1b}$$

$$N_\rho \coloneqq \ln\left(\frac{\overline{\rho}_\mathrm{in}}{\overline{\rho}_\mathrm{out}}\right) \qquad \text{number of density scale-heights,} \tag{6.1c}$$

$$\text{and} \qquad \beta = \frac{r_\mathrm{in}}{r_\mathrm{out}} \qquad \text{aspect ratio.} \tag{6.1d}$$

If the typical values of the polytrope are taken at the inner boundary, we have

$$\hat{T}(r) = \frac{\overline{T}(r)}{\overline{T}_\mathrm{in}} = \left[\frac{\beta(1-e^{-N_\rho/n})}{(1-\beta)^2}\right]\left(\frac{H}{r}\right) - \left(\frac{\beta - e^{-N_\rho/n}}{1-\beta}\right) \tag{6.2a}$$

$$\hat{\rho}(r) = \frac{\overline{\rho}(r)}{\overline{\rho}_\mathrm{in}} = \left\{\left[\frac{\beta(1-e^{-N_\rho/n})}{(1-\beta)^2}\right]\left(\frac{H}{r}\right) - \left(\frac{\beta - e^{-N_\rho/n}}{1-\beta}\right)\right\}^n, \tag{6.2b}$$

$$\widehat{N^2}(r) = \frac{N^2(r)}{\widetilde{N^2}} = \left(\frac{r_\mathrm{in}}{r}\right)^3 \left[\frac{1-e^{-N_\rho/n}}{1-\beta} - \left(\frac{\beta - e^{-N_\rho/n}}{\beta}\right)\frac{r}{H}\right]^{-1}, \tag{6.2c}$$

$$\hat{g}(r) = \frac{g(r)}{g_\mathrm{in}} = \frac{r_\mathrm{in}^2}{r^2}, \tag{6.2d}$$

$$\text{and} \qquad \mathrm{Di} = \frac{L}{H}\left(\frac{n+1}{\tilde{n}+1}\right)\left(\frac{1}{\beta}\right)(1-e^{-N_\rho/n}), \tag{6.2e}$$

where $H = r_\mathrm{out} - r_\mathrm{in}$ is the shell depth.

Note that the range on r/H is (by definition)

$$\frac{\beta}{1-\beta} \le \frac{r}{H} \le \frac{1}{1-\beta}. \tag{6.3}$$

If instead we take the typical values at the outer boundary, it is simple to compute the ratios from outer to inner directly from Equations (6.2) and thus change the non-dimensionalization of the polytrope.

If we instead take the typical values as volume-averages, the density profile in Equations (6.2) (because of the n exponent) must be integrated numerically (or else computed from the hypergeometric function) but it is again straightforward to re-scale. Because the formulas are complicated (and I am error-prone), it is easiest to numerically integrate the other functions as well. We note, however, the analytic formula for Di when volume averages are used:

$$\begin{aligned}\mathrm{Di}_\mathrm{v} &\coloneqq \frac{\langle g\rangle_\mathrm{v}/g_\mathrm{in}}{\langle \overline{T}\rangle_\mathrm{v}/\overline{T}_\mathrm{in}}\mathrm{Di} \\ &= \frac{L}{H}\left(\frac{n+1}{\tilde{n}+1}\right)\frac{3\beta(1-\beta)^2(1-e^{-N_\rho/n})}{(3\beta/2)(1-\beta^2)(1-e^{-N_\rho/n}) - (1-\beta^3)(\beta - e^{-N_\rho/n})}.\end{aligned} \tag{6.4}$$

This emphases the fact that $\mathrm{Di} = \mathrm{Di}(\gamma, n, N_\rho, \beta, L/H)$ is not an independent control parameter of the system.

References

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