Hydrostatic, Geostrophic Balance in the Meridional Plane

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Once a fluid dynamical system has reached a steady state, the Navier-Stokes equations often lead to a balance of fluxes for a given quantity, in which the sum of the fluxes has zero divergence. In spherical shells (inner radius $r_{\rm i}$ and outer radius $r_{\rm o}$), we usually consider the zonally and temporally averaged fluxes $\mathcal{F}_r(r,\theta)$ and $\mathcal{F}_\theta(r,\theta)$ in the meridional plane:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\mathcal{F}_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\mathcal{F}_\theta) = 0. \tag{1}$$

If we apply $2\pi r^2\int_0^\pi d\theta\sin\theta$ to (1), we find

$$2\pi\int_0^\pi d\theta\sin\theta\frac{\partial}{\partial r}(r^2\mathcal{F}_r) + 2\pi r\int_0^\pi d\theta\frac{\partial}{\partial\theta}(\sin\theta\mathcal{F}_\theta) = 0,$$

or

$$\frac{\partial}{\partial r}\left[2\pi r^2\int_0^\pi \mathcal{F}_r(r,\theta)\sin\theta d\theta\right] = -2\pi r\sin\theta\mathcal{F}_\theta(r,\theta)\Big|_{\theta=0}^{\theta=\pi} = 0.$$

Thus, the spherically integrated radial flux is a constant in a steady state:

$$\mathscr{I}_r(r) \coloneqq 2\pi r^2\int_0^\pi \mathcal{F}_r(r,\theta)\sin\theta d\theta = \text{constant}. \tag{2}$$

Similarly, we can apply $2\pi\sin\theta\int_{r_{\rm i}}^{r_{\rm o}} drr^2$ to (1), to find

$$2\pi\sin\theta\int_{r_{\rm i}}^{r_{\rm o}} dr\frac{\partial}{\partial r}(r^2\mathcal{F}_r) + 2\pi\int_{r_{\rm i}}^{r_{\rm o}} r\sin\theta\frac{\partial}{\partial\theta}(\sin\theta\mathcal{F}_\theta) = 0,$$

or

$$\begin{aligned}\frac{\partial}{\partial\theta}\underbrace{\left[2\pi\int_{r_{\rm i}}^{r_{\rm o}} drr\sin\theta\mathcal{F}_\theta(r,\theta)\right]}_{\coloneqq\mathscr{I}_\theta(\theta)} &= -2\pi\sin\theta[r^2\mathcal{F}_r(r,\theta)]\Big|_{r=r_{\rm i}}^{r=r_{\rm o}} \\ &= -2\pi\sin\theta(r_{\rm o}^2\mathcal{F}_{\rm o}(\theta) - r_{\rm i}^2\mathcal{F}_{\rm i}(\theta)),\end{aligned} \tag{3}$$

where $\mathcal{F}_{\{\rm i,\ o\}}(\theta) \coloneqq \mathcal{F}_r(\{r_{\rm i},\ r_{\rm o}\},\theta)$ are the radial fluxes in through the inner boundary and out through the outer boundary.

For any vector field $\boldsymbol{A}$ that is continuous across the poles, the θ- and ϕ-components must vanish at the poles under a zonal average: $\langle A_\theta \rangle = \langle A_\phi \rangle = 0$ at $\theta = 0, \pi$. Thus, we can integrate (3) to find

$$\mathscr{I}_\theta(\theta) = -2\pi \int_0^\theta \sin\theta' [r_\mathrm{o}^2 \mathcal{F}_\mathrm{o}(\theta') - r_\mathrm{i}^2 \mathcal{F}_\mathrm{i}(\theta')] d\theta'$$
$$= 2\pi \int_\theta^\pi \sin\theta' [r_\mathrm{o}^2 \mathcal{F}_\mathrm{o}(\theta') - r_\mathrm{i}^2 \mathcal{F}_\mathrm{i}(\theta')] d\theta', \tag{4}$$

the analogue of Equation (2), but for the conically integrated latitudinal flux.