

# Induction term in spherical coordinates

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## 1 The problem with the traditional induction terms

We consider the ideal (resistance-free) magnetohydrodynamic (MHD) induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) := \mathbf{I}, \quad (1)$$

$$= \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{B} - (\nabla \cdot \mathbf{u}) \mathbf{B}, \quad (2)$$

where  $\mathbf{B}$  and  $\mathbf{u}$  are the vector magnetic and velocity fields (respectively) and  $\mathbf{I}$  is the vector induction. The three terms on the right-hand-side of Equation (2) are often interpreted as “shear,” “advection,” and “compression,” respectively. However, this interpretation is problematic in general for two reasons:

1. The so-called shear and compression terms contain sub-terms that cancel; in particular, only velocity motions *transverse* to magnetic-field lines can shear or compress.
2. Solid-body rotation (which is a non-shearing motion that simply rotates the whole field configuration) shows up in the so-called shear and advection terms in a strange way.

Finally, even after these issues have been addressed, resolving the final terms into a particular curvilinear system (e.g., spherical coordinates) seems to present a major headache. Our goal here is to explain fully how these problems emerge and propose a tentative solution for the case of spherical coordinates.

## 2 Transverse shear and compression

To see how Problem 1 arises, we decompose the velocity field into components parallel and perpendicular to the local direction of  $\mathbf{B}$ :

$$\mathbf{u} := u_{\parallel} \hat{\mathbf{e}}_{\parallel} + \mathbf{u}_{\perp} \quad (3)$$

Obviously  $\mathbf{B} = Bx_{\parallel}$ , where  $B = |\mathbf{B}|$ . We denote the Cartesian distance along  $\mathbf{B}$  by  $x_{\parallel}$ . We also decompose  $\mathbf{u}$  into its parallel and perpendicular components:

$$\nabla \cdot \mathbf{u} = \frac{\partial u_{\parallel}}{\partial x_{\parallel}} + \nabla_{\perp} \cdot \mathbf{u}_{\perp} \quad (4)$$

We then calculate

$$\begin{aligned}
\mathbf{B} \cdot \nabla \mathbf{u} - (\nabla \cdot \mathbf{u}) \mathbf{B} &= B \frac{\partial}{\partial x_{\parallel}} (u_{\parallel} \hat{\mathbf{e}}_{\parallel} + \mathbf{u}_{\perp}) - \left( \frac{\partial u_{\parallel}}{\partial x_{\parallel}} + \nabla_{\perp} \cdot \mathbf{u}_{\perp} \right) B \hat{\mathbf{e}}_{\parallel} \\
&= B \cancel{\frac{\partial u_{\parallel}}{\partial x_{\parallel}} \hat{\mathbf{e}}_{\parallel}} + B \frac{\partial \mathbf{u}_{\perp}}{\partial x_{\parallel}} - \cancel{\frac{\partial u_{\parallel}}{\partial x_{\parallel}} B \hat{\mathbf{e}}_{\parallel}} - (\nabla_{\perp} \cdot \mathbf{u}_{\perp}) B \hat{\mathbf{e}}_{\parallel} \\
&= \mathbf{B} \cdot \nabla \mathbf{u}_{\perp} - (\nabla_{\perp} \cdot \mathbf{u}_{\perp}) \mathbf{B}.
\end{aligned} \tag{5}$$

Thus, only motions transverse to the local field line (i.e.,  $\mathbf{u}_{\perp}$ ) can shear or compress  $\mathbf{B}$ .

### 3 Rigid rotation

To see how Problem 2 arises, we consider a velocity field due to rigid rotation at constant angular velocity  $\Omega$  about the  $z$ -axis in a cylindrical coordinate system:

$$\boldsymbol{\Omega} = \Omega \hat{\mathbf{e}}_z = \text{constant} \tag{6a}$$

$$\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{r} = \Omega \lambda \hat{\mathbf{e}}_{\phi} \tag{6b}$$

Here,  $\lambda$  is the cylindrical radius,  $\phi$  the longitude, and  $z$  the axial coordinate. In general,  $\hat{\mathbf{e}}_{(\dots)}$  denotes a unit vector in the direction of its subscript. We calculate:

$$\begin{aligned}
\nabla \cdot \mathbf{u} &= \nabla \cdot (\boldsymbol{\Omega} \times \mathbf{r}) \\
&= \boldsymbol{\Omega} \cdot \nabla \times \mathbf{r} - \mathbf{r} \cdot \nabla \times \boldsymbol{\Omega} \\
&= 0 \quad \text{no compression for rigid rotation (obviously)}.
\end{aligned} \tag{7}$$

Then:

$$\begin{aligned}
\mathbf{B} \cdot \nabla \mathbf{u} &= (\mathbf{B} \cdot \nabla) (\boldsymbol{\Omega} \times \mathbf{r}) \\
&= \boldsymbol{\Omega} \times [(\mathbf{B} \cdot \nabla) (\mathbf{r})]
\end{aligned} \tag{8}$$

$$= \boldsymbol{\Omega} \times \mathbf{B} \quad \text{“shear” for rigid rotation.} \tag{9}$$

Finally:

$$\begin{aligned}
-\mathbf{u} \cdot \nabla \mathbf{B} &= -\Omega \lambda \hat{\mathbf{e}}_{\phi} \cdot \nabla \mathbf{B} \\
&= -\Omega \frac{\partial}{\partial \phi} (B_{\lambda} \hat{\mathbf{e}}_{\lambda} + B_{\phi} \hat{\mathbf{e}}_{\phi} + B_z \hat{\mathbf{e}}_z) \\
&= -\Omega \sum_{\alpha} \left( \frac{\partial B_{\alpha}}{\partial \phi} \right) \hat{\mathbf{e}}_{\alpha} - \Omega \sum_{\alpha} B_{\alpha} \frac{\partial \hat{\mathbf{e}}_{\alpha}}{\partial \phi},
\end{aligned}$$

where the index  $\alpha$  runs over the three cylindrical coordinates. Note that in the cylindrical coordinate system (or indeed any coordinate system with an axis of rotational symmetry),  $\partial \hat{\mathbf{e}}_{\alpha} / \partial \phi = \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_{\alpha}$  for each  $\alpha$ . Thus,

$$\begin{aligned}
-\Omega \sum_{\alpha} B_{\alpha} \frac{\partial \hat{\mathbf{e}}_{\alpha}}{\partial \phi} &= -\Omega \sum_{\alpha} B_{\alpha} \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_{\alpha} \\
&= -\Omega \hat{\mathbf{e}}_z \times \sum_{\alpha} B_{\alpha} \hat{\mathbf{e}}_{\alpha} \\
&= -\boldsymbol{\Omega} \times \mathbf{B}
\end{aligned}$$

and so

$$-\mathbf{u} \cdot \nabla \mathbf{B} = -\Omega \sum_{\alpha} \left( \frac{\partial B_{\alpha}}{\partial \phi} \right) \hat{\mathbf{e}}_{\alpha} - \Omega \times \mathbf{B} \quad \text{“advection” for rigid rotation.} \quad (10)$$

Mathematically, in any coordinate system with a  $z$ -axis of rotational symmetry, the action of rigid rotation is as follows:

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times [(\Omega \times \mathbf{r}) \times \mathbf{B}] \\ &= \sum_{\alpha} \left( -\Omega \frac{\partial B_{\alpha}}{\partial \phi} \right) \hat{\mathbf{e}}_{\alpha} \end{aligned} \quad (11a)$$

$$\text{or} \quad \left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) B_{\alpha} = 0 \quad \text{for each } \alpha. \quad (11b)$$

If you think about it, this makes sense: All the rigid rotation does is rotate the whole field configuration around the  $z$ -axis at the rate  $\Omega$ . If you decide to also rotate at  $\Omega$  (so your personal Eulerian time derivative is  $\partial/\partial t + \Omega \partial/\partial \phi$ ), then each component of the magnetic-field configuration should remain the same in your frame. Note that rotation does *not* advect the vector magnetic field (like the term  $-\mathbf{u} \cdot \nabla \mathbf{B}$  viewed on its own would suggest), but rather advects the field *components* (as if they were scalars) in any coordinate system with an axis of rotational symmetry.

## 4 Solution for spherical coordinates

Resolving these issues fully for the spherical coordinate system seems complicated and I am not fully sure how to do it! In particular (for “full” resolution) we should, separately at each point  $(r, \theta, \phi)$ :

1. Form a local Cartesian coordinate system, say  $(x_1, x_2, x_3)$ , whose origin lies at the point  $(r, \theta, \phi)$ . At the origin,  $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$  will coincide with  $(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_{\theta}, \hat{\mathbf{e}}_{\phi})$ . But slightly away from the origin,  $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$  will stay fixed while  $(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_{\theta}, \hat{\mathbf{e}}_{\phi})$  curve away.
2. Calculate the velocity-gradient tensor:  $\partial u_1/\partial x_1, \partial u_2/\partial x_1, \partial u_3/\partial x_1$ , etc. While calculating derivatives, be careful to differentiate along the Cartesian coordinates, *not* the curvilinear ones or along the actual  $\mathbf{B}$ -line). Express the final tensor components in spherical coordinates.
3. Rotate “into  $\mathbf{B}$ ” to form a new primed coordinate system (such that  $\hat{\mathbf{e}}'_1$  points along  $\mathbf{B}$ ), calculate the  $\partial u'_j/\partial x'_i$ , and thus form  $\partial u_{\parallel}/\partial x_{\parallel}$  and  $\partial \mathbf{u}_{\perp}/\partial x_{\parallel}$  (note that  $\nabla_{\perp} \cdot \mathbf{u}_{\perp} = \nabla \cdot \mathbf{u} - \partial u_{\parallel}/\partial x_{\parallel}$ ). Note that the direction cosines  $\hat{\mathbf{e}}'_1$  makes with  $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$  are simply  $B_r/|\mathbf{B}|, B_{\theta}/|\mathbf{B}|, B_{\phi}/|\mathbf{B}|$ , respectively. Note that both unit vectors and vector components transform like  $x'_1 = \sum_{j=1}^3 R_{1j} x_j$ , where  $R_{1j}$  is the  $j^{\text{th}}$  direction cosine.
4. Subtract the part of  $\mathbf{u}_{\perp}$  corresponding to solid-body rotation and put it in the form of Equation (11a).

5. Exult, because I haven't been able to do this!

Instead of addressing the full problem as just described, I have brute-forced my way into a quasi-solution, canceling obvious terms and expressing what seem to be “transverse shear with no solid-body rotation,” “transverse compression,” and advection.

We write:

$$\begin{aligned}
I_r = & \cancel{B_r \frac{\partial u_r}{\partial r}} + \frac{B_\theta}{r} \frac{\partial u_r}{\partial \theta} + \frac{B_\phi}{r \sin \theta} \frac{\partial u_r}{\partial \phi} - \cancel{\frac{B_\theta u_\theta + B_\phi u_\phi}{r}} \\
& - \mathbf{u} \cdot \nabla B_r + \cancel{\frac{u_\theta B_\theta + u_\phi B_\phi}{r}} \\
& - \left[ \cancel{\frac{\partial u_r}{\partial r}} + \frac{2u_r}{r} + \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} + \cot \theta u_\theta \right) + \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} \right] B_r
\end{aligned}$$

or

$$I_r = \left( \frac{B_\theta}{r} \frac{\partial}{\partial \theta} + \frac{B_\phi}{r \sin \theta} \frac{\partial}{\partial \phi} \right) u_r \quad r: \text{ transverse shear} \quad (12a)$$

$$- \mathbf{u} \cdot \nabla B_r \quad r: \text{ advection} \quad (12b)$$

$$- \left( \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{2u_r}{r} + \frac{\cot \theta u_\theta}{r} \right) B_r \quad r: \text{ transverse compression} \quad (12c)$$

Then, for  $I_\theta$ :

$$\begin{aligned}
I_\theta = & B_r \frac{\partial u_\theta}{\partial r} + \cancel{\frac{B_\theta}{r} \frac{\partial u_\theta}{\partial \theta}} + \frac{B_\phi}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} + \frac{B_\theta u_r}{r} - \cancel{\frac{\cot \theta B_\phi u_\phi}{r}} \\
& - \mathbf{u} \cdot \nabla B_\theta - \frac{u_\theta B_r}{r} + \cancel{\frac{\cot \theta u_\phi B_\phi}{r}} \\
& - \left[ \frac{\partial u_r}{\partial r} + \frac{2u_r}{r} + \frac{1}{r} \left( \cancel{\frac{\partial u_\theta}{\partial \theta}} + \cot \theta u_\theta \right) + \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} \right] B_\theta
\end{aligned}$$

or

$$I_\theta = r \left( B_r \frac{\partial}{\partial r} + \frac{B_\phi}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \left( \frac{u_\theta}{r} \right) \quad \theta: \text{ transverse shear} \quad (13a)$$

$$- \mathbf{u} \cdot \nabla B_\theta \quad \theta: \text{ advection} \quad (13b)$$

$$- \left( \frac{\partial u_r}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{\cot \theta u_\theta}{r} \right) B_\theta \quad \theta: \text{ transverse compression} \quad (13c)$$

Finally, for  $I_\phi$ :

$$\begin{aligned}
I_\phi = & B_r \frac{\partial u_\phi}{\partial r} + \frac{B_\theta}{r} \frac{\partial u_\phi}{\partial \theta} + \cancel{\frac{B_\phi}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi}} + \frac{B_\phi u_r}{r} + \cancel{\frac{\cot \theta B_\theta u_\theta}{r}} \\
& - \mathbf{u} \cdot \nabla B_\phi - \frac{u_\phi B_r}{r} - \cancel{\frac{\cot \theta u_\theta B_\theta}{r}} \\
& - \left[ \frac{\partial u_r}{\partial r} + \frac{2u_r}{r} + \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} + \cancel{\cot \theta u_\theta} \right) + \cancel{\frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi}} \right] B_\phi
\end{aligned}$$

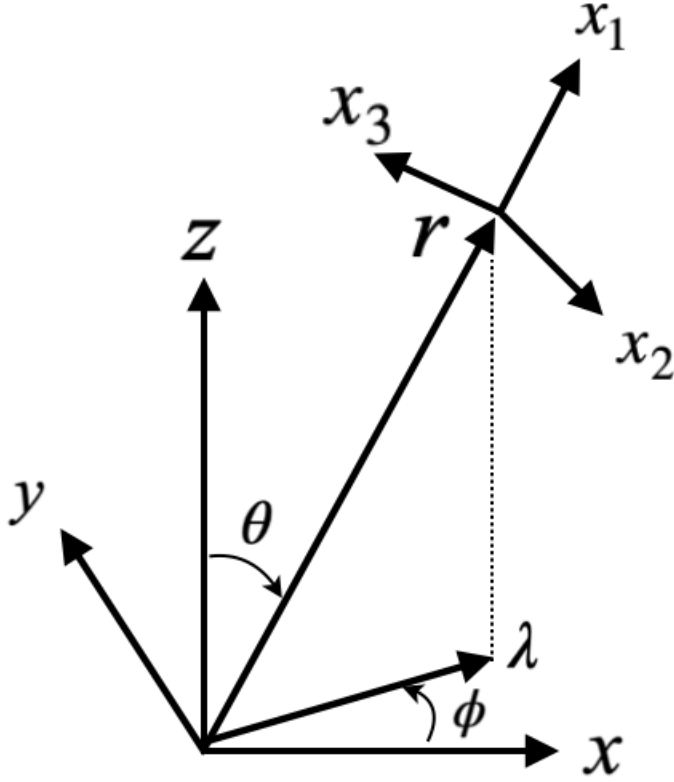


Figure 1: Schematic of coordinate systems used here, their relation to one another, and each coordinate's meaning. Note that  $(x, y, z) = (\lambda \cos \phi, \lambda \sin \phi, z) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$ . The local Cartesian system  $(x_1, x_2, x_3)$  has its origin at the point  $(r, \theta, \phi)$ , where  $(\hat{e}_1, \hat{e}_2, \hat{e}_3) = (\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$ .

or

$$I_\phi = r \sin \theta \left( B_r \frac{\partial}{\partial r} + \frac{B_\theta}{r} \frac{\partial}{\partial \theta} \right) \left( \frac{u_\phi}{r \sin \theta} \right) \quad \phi: \text{transverse shear} \quad (14a)$$

$$- \mathbf{u} \cdot \nabla B_\phi \quad \phi: \text{advection} \quad (14b)$$

$$- \left( \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) B_\phi \quad \phi: \text{transverse compression} \quad (14c)$$

## 5 Transverse compression