Moffatt's Magnetic Field Generation in Electrically Conducting Fluids, Notes

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Ch 2: Magnetokinematic Preliminaries

1. Identity relating angular momentum operator L^2 and Laplacian operator ∇^2

We compute

$$\begin{aligned}(\boldsymbol{x}\wedge\nabla)^2\psi &= (\epsilon_{ijk}r_j\partial_k)(\epsilon_{ilm}r_l\partial_m)\psi\\ &= (\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})r_j(\delta_{lk}\partial_m\psi + r_l\partial_k\partial_m\psi)\\ &= r_k\partial_k\psi + r_jr_j\partial_m\partial_m\psi - 3r_m\partial_m\psi - r_jr_k\partial_k\partial_j\psi\\ &= r^2\nabla^2\psi - 2\boldsymbol{x}\cdot\nabla\psi - \boldsymbol{x}\cdot(\boldsymbol{x}\cdot\nabla)\nabla\psi,\end{aligned}$$

which verifies (2.23). Noting that $\boldsymbol{x}\cdot\nabla = r(\partial/\partial r)$, and (since all unit vectors are independent of r) that

$$\begin{aligned}\boldsymbol{x}\cdot(\boldsymbol{x}\cdot\nabla)\nabla\psi &= (r\hat{\boldsymbol{e}}_r)\cdot r\frac{\partial}{\partial r}\left(\frac{\partial\psi}{\partial r}\hat{\boldsymbol{e}}_r + \dots\right)\\ &= r^2\hat{\boldsymbol{e}}_r\cdot\left(\frac{\partial^2\psi}{\partial r^2}\hat{\boldsymbol{e}}_r + \dots\right) = r^2\frac{\partial^2\psi}{\partial r^2},\end{aligned}$$

we then compute

$$\begin{aligned}&r^2\nabla^2\psi - 2\boldsymbol{x}\cdot\nabla\psi - \boldsymbol{x}\cdot(\boldsymbol{x}\cdot\nabla)\nabla\psi =\\ &r^2\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\psi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right) - 2r\frac{\partial\psi}{\partial r} - r^2\frac{\partial^2\psi}{\partial r^2}\\ &\cancel{2r\frac{\partial\psi}{\partial r} + r^2\frac{\partial^2\psi}{\partial r^2}} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} - \cancel{2r\frac{\partial\psi}{\partial r} - r^2\frac{\partial^2\psi}{\partial r^2}}\\ &\qquad = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = L^2\psi.\end{aligned}$$

2. How to invert L^2

We now derive (2.32), which seems trivial, but really should require a bit of thought. From (2.25), we have

$$\begin{aligned} f(r,\theta,\phi) &= \sum_n f_n(r) S_n(\theta,\phi), \\ \text{where} \qquad S_n(\theta,\phi) &= \sum_m A_n^m Y_n^m(\theta,\phi) \\ \text{and} \qquad Y_n^m(\theta,\phi) &:= P_n^m(\cos\theta) e^{im\phi}. \end{aligned}$$

Now suppose

$$L^2 \psi = f(r,\theta,\phi)$$

We can similarly expand ψ in spherical harmonics:

$$\begin{aligned} \psi(r,\theta,\phi) &= \sum_n \psi_n(r) \tilde{S}_n(\theta,\phi), \\ \text{where} \qquad \tilde{S}_n(\theta,\phi) &= \sum_m \tilde{A}_n^m Y_n^m(\theta,\phi). \end{aligned}$$

and write

$$\begin{aligned} L^2\psi &= L^2 \sum_n \psi_n(r) \tilde{S}_n(\theta,\phi) \\ &= \sum_n \psi_n(r) L^2 \tilde{S}_n(\theta,\phi) \\ &= \sum_n \psi_n(r) n(n+1) \tilde{S}_n(\theta,\phi) \\ &= \sum_n n(n+1)\psi_n(r) \tilde{S}_n(\theta,\phi) = f(r,\theta,\phi) = \sum_n f_n(r) S_n(\theta,\phi). \end{aligned}$$

From this last equality, it is tempting to identify $f_n = n(n+1)\psi_n$, and be done with it, thereby arriving at (2.32). But this is not really allowed, since we have not shown yet that the $\tilde{S}_n$ are the same as the S_n. So we really need to decompose further with respect to m and derive the relationship between the A-coefficients:

$$\begin{aligned} \sum_n n(n+1)\psi_n(r) \sum_m \tilde{A}_n^m Y_n^m(\theta,\phi) &= \sum_n f_n(r) \sum_m A_n^m Y_n^m(\theta,\phi) \Longrightarrow \\ \sum_{n,m} n(n+1)\psi_n(r) \tilde{A}_n^m Y_n^m(\theta,\phi) &= \sum_{n,m} f_n(r) A_n^m Y_n^m(\theta,\phi) \end{aligned}$$

We know that the Y_n^m are all orthogonal, and so we identify

$$n(n+1)\psi_n(r)\tilde{A}_n^m = f_n(r) A_n^m.$$

Thus,

$$\begin{aligned}
\psi(r,\theta,\phi) &= \sum_n \psi_n(r) \sum_m \tilde{A}_n^m Y_n^m(\theta,\phi) \\
&= \sum_{n,m} (\psi_n(r)\tilde{A}_n^m) Y_n^m(\theta,\phi) \\
&= \sum_{n,m} \left(\frac{1}{n(n+1)} f_n(r) A_n^m \right) Y_n^m(\theta,\phi) \\
&= \sum_n \frac{1}{n(n+1)} f_n(r) \sum_m A_n^m Y_n^m(\theta,\phi) \\
&= \sum_n \frac{1}{n(n+1)} f_n(r) S_n(\theta,\phi).
\end{aligned}$$

3. Curl of a Poloidal Field is a Toroidal Field

A "poloidal" field is written

$$\boldsymbol{B}_P = \nabla \wedge [\nabla \wedge (P\boldsymbol{x})] = -\nabla \wedge (\boldsymbol{x} \wedge \nabla P) = -\nabla^2(P\boldsymbol{x}) + \nabla[\nabla \cdot (P\boldsymbol{x})].$$

We compute

$$\begin{aligned}
\nabla \wedge \boldsymbol{B}_P &= \nabla \wedge \{-\nabla^2(P\boldsymbol{x}) + \cancel{\nabla[\nabla \cdot (P\boldsymbol{x})]}\} \\
&= -\nabla \wedge [\nabla^2(P\boldsymbol{x})]
\end{aligned}$$

Now,

$$\begin{aligned}
\nabla^2(P\boldsymbol{x})_i &= \partial_j\partial_j(Pr_i) \\
&= \partial_j[(\partial_j P)r_i + P\delta_{ij}] \\
&= (\partial_j^2 P)r_i + (\partial_j P)\delta_{ij} + (\partial_j P)\delta_{ij} + 0 \\
&= r_i\partial_j^2 P + 2\partial_i P \implies \\
\nabla^2(P\boldsymbol{x}) &= (\nabla^2 P)\boldsymbol{x} + 2\nabla P.
\end{aligned}$$

Thus,

$$\begin{aligned}
\nabla \wedge \boldsymbol{B}_P &= -\nabla \wedge [(\nabla^2 P)\boldsymbol{x} + 2\cancel{\nabla P}] \\
&= \nabla \wedge [(-\nabla^2 P)\boldsymbol{x}],
\end{aligned}$$

and so is a toroidal field with toroidal streamfunction $-\nabla^2 P$.

Note that the same holds true even if we define the streamfunctions in the alternate way

$$\boldsymbol{B} = \boldsymbol{B}_P + \boldsymbol{B}_T = \nabla \wedge [\nabla \wedge (P\hat{\boldsymbol{e}}_r)] + \nabla \wedge (T\hat{\boldsymbol{e}}_r),$$

where $\hat{\boldsymbol{e}}_r = \boldsymbol{x}/r$. That the curl of the toroidal field is a poloidal field is trivial, and the curl of the poloidal field is

$$\nabla \wedge \boldsymbol{B}_P = -\nabla \wedge [\nabla^2(P\hat{\boldsymbol{e}}_r)].$$

Now, following the calculations from before,

$$\nabla^2(P\hat{\boldsymbol{e}}_r) = \nabla^2\left(\frac{P}{r}\boldsymbol{x}\right) = \nabla^2\left(\frac{P}{r}\right)\boldsymbol{x} + 2\nabla\left(\frac{P}{r}\right),$$

so

$$\begin{aligned}\nabla \wedge \boldsymbol{B}_P &= -\nabla \wedge \left[\nabla^2\left(\frac{P}{r}\right)\boldsymbol{x}\right] \\ &= \nabla \wedge \left\{\left[-r\nabla^2\left(\frac{P}{r}\right)\right]\hat{\boldsymbol{e}}_r\right\},\end{aligned}$$

which is by (the new) definition, a toroidal field.

4. $\boldsymbol{x} \cdot \boldsymbol{B}$ and $\boldsymbol{x} \cdot \nabla \wedge \boldsymbol{B}$ in Terms of Angular Momentum Operator

We write

$$\boldsymbol{B} = \boldsymbol{B}_P + \boldsymbol{B}_T = \nabla \wedge [\nabla \wedge (P\boldsymbol{x})] + \nabla \wedge (T\boldsymbol{x})$$

Since the Toroidal fields are orthogonal to $\boldsymbol{x}$, we compute

$$\begin{aligned}\boldsymbol{x} \cdot \boldsymbol{B} = \boldsymbol{x} \cdot \boldsymbol{B}_P &= \boldsymbol{x} \cdot \{\nabla \wedge [\nabla \wedge (P\boldsymbol{x})]\} \\ &= x_i\{\epsilon_{ijk}\partial_j[\epsilon_{klm}\partial_l(Pr_m)]\} \\ &= (\epsilon_{ijk}x_i\partial_j)\{\epsilon_{klm}[(\partial_l P)r_m + P\cancel{\delta_{lm}}]\} \\ &= -\underbrace{(\epsilon_{ijk}x_i\partial_j)}_{(\boldsymbol{x}\wedge\nabla)_k}\underbrace{(\epsilon_{kml}r_m\partial_l)}_{(\boldsymbol{x}\wedge\nabla)_k}P \\ &= -(\boldsymbol{x} \wedge \nabla)^2 P\end{aligned}$$

Since we have already shown that $\nabla \wedge \boldsymbol{B}_P$ is a toroidal field (and thus orthogonal to $\boldsymbol{x}$), we have

$$\begin{aligned}\boldsymbol{x} \cdot (\nabla \wedge \boldsymbol{B}) = \boldsymbol{x} \cdot (\nabla \wedge \boldsymbol{B}_T) &= \boldsymbol{x} \cdot \{\nabla \wedge [\nabla \wedge (T\boldsymbol{x})]\} \\ &= -(\boldsymbol{x} \wedge \nabla)^2 T,\end{aligned}$$

by an identical calculation to the one manipulating $\boldsymbol{x} \cdot \boldsymbol{B}$.

5. Purely poloidal form for the term $T\boldsymbol{x} + \nabla U$ in vector potential

We identify the vector potential $\boldsymbol{A}$ in terms of the streamfunctions in (2.37):

$$\boldsymbol{A} = \nabla \wedge (P\boldsymbol{x}) + T\boldsymbol{x} + \nabla U,$$

which is (2.40). If we pick the Coulomb gauge, then

$$\begin{aligned}0 = \nabla \cdot \boldsymbol{A} &= \nabla \cdot [\cancel{\nabla \wedge (P\boldsymbol{x})} + T\boldsymbol{x} + \nabla U] \\ &= \nabla \cdot (T\boldsymbol{x} + \nabla U).\end{aligned}$$

Thus, using the formalism just laid out, we can write $T\boldsymbol{x} + \nabla U$ as the sum of a yet another set of toroidal and poloidal fields. But note that

$$\boldsymbol{x} \cdot [\nabla \wedge (T\boldsymbol{x} + \cancel{\nabla U})] = \boldsymbol{x} \cdot [-\boldsymbol{x} \wedge \nabla T] = 0,$$

and so the toroidal part of $T\boldsymbol{x} + \nabla U$ is zero by virtue of 2.39(b).

We thus have (for the Coulomb gauge)

$$\begin{aligned} & \boldsymbol{A} = \boldsymbol{A}_P + \boldsymbol{A}_T \\ \text{where} \qquad & \boldsymbol{A}_P = T\boldsymbol{x} + \nabla U = \nabla \wedge [\nabla \wedge (S\boldsymbol{x})] \\ \text{and} \qquad & \boldsymbol{A}_T = \nabla \wedge (P\boldsymbol{x}) \end{aligned}$$

.

Finally, since the curl of a poloidal field is a toroidal field, the curl of a toroidal field is a poloidal field, and $\boldsymbol{B} = \nabla \wedge \boldsymbol{A}$, we have

$$\begin{aligned} \boldsymbol{B}_P &= \nabla \wedge \boldsymbol{A}_T, \\ \boldsymbol{B}_T &= \nabla \wedge \boldsymbol{A}_P. \end{aligned}$$

6. Axisymmetric Fields

For axisymmetric fields, $\partial_\phi \equiv 0$, so

$$\begin{aligned} \boldsymbol{B}_T &= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\boldsymbol{e}}_r & r\hat{\boldsymbol{e}}_\theta & r\sin\theta\hat{\boldsymbol{e}}_\phi \\ \partial_r & \partial_\theta & 0 \\ Tr & 0 & 0 \end{vmatrix} \\ &= \frac{1}{r^2 \sin\theta} \left[-\frac{\partial}{\partial\theta}(Tr) \right] (r\sin\theta\hat{\boldsymbol{e}}_\phi) \\ &= -\frac{\partial T}{\partial\theta}\hat{\boldsymbol{e}}_\phi. \end{aligned}$$

Similarly,

$$\boldsymbol{A}_T = -\frac{\partial P}{\partial\theta}\hat{\boldsymbol{e}}_\phi,$$

and thus

$$\begin{aligned} \boldsymbol{B}_P = \nabla \wedge \boldsymbol{A}_T &= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\boldsymbol{e}}_r & r\hat{\boldsymbol{e}}_\theta & r\sin\theta\hat{\boldsymbol{e}}_\phi \\ \partial_r & \partial_\theta & 0 \\ 0 & 0 & -r\sin\theta\frac{\partial P}{\partial\theta} \end{vmatrix} \\ &= \frac{1}{r^2 \sin\theta} \left[-r\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)\hat{\boldsymbol{e}}_\theta + \left(\frac{\partial}{\partial r} r\sin\theta\frac{\partial P}{\partial\theta}\right)(r\hat{\boldsymbol{e}}_\theta) + 0 \right] \\ &= -\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\Bigg(\underbrace{r\sin\theta\frac{\partial P}{\partial\theta}}_{:=\chi}\Bigg)\hat{\boldsymbol{e}}_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\Bigg(\underbrace{r\sin\theta\frac{\partial P}{\partial\theta}}_{:=\chi}\Bigg)\hat{\boldsymbol{e}}_\theta, \end{aligned}$$

from whence we recover what Moffatt meant by (2.47). Note however, that Moffatt's (2.47) is completely wrong. It should read

$$\chi = -r\sin\theta A_\phi, \qquad A_\phi = -\frac{\partial P}{\partial \theta},$$

so that

$$\chi = r\sin\theta\frac{\partial P}{\partial\theta}.$$

Now, the $\boldsymbol{B}_P$-lines satisfy

$$\begin{aligned} dr &= C(B_P)_r \\ rd\theta &= C(B_P)_\theta, \end{aligned}$$

where C is some constant of proportionality, possibly a function of position along the streamline, but the same between both the above equations. Dividing the two equations above thus yields

$$\begin{aligned} \frac{dr}{rd\theta} = \frac{(B_P)_r}{(B_P)_\theta} \Longrightarrow 0 = (B_P)_\theta dr - (B_P)_r rd\theta \quad &= \frac{1}{r\sin\theta}\frac{\partial\chi}{\partial r}dr - \left(-\frac{1}{r^2\sin\theta}\frac{\partial\chi}{\partial\theta}\right)(rd\theta) \\ = \frac{1}{r\sin\theta}\left(\frac{\partial\chi}{\partial r}dr + \frac{\partial\chi}{\partial\theta}d\theta\right) = \frac{1}{r\sin\theta}d\chi. \end{aligned}$$

The defining condition for $\boldsymbol{B}_P$-lines is thus

$$d\chi = 0 \qquad \text{or} \qquad \chi(r,\theta) = \text{constant}.$$

Consider the flux through a ring made by rotating the infinitesimal line element between (r,θ) and $(r+dr,\theta+d\theta)$. This line element as length $dl = \sqrt{dr^2 + (rd\theta)^2}$, and the ring has area $a = 2\pi r\sin\theta dl$. The normal to this area has associated unit vector $\hat{\boldsymbol{e}}_n = (-rd\theta, dr, 0)/dl$. Thus, the flux of $\boldsymbol{B}_P$ across the ring is

$$\begin{aligned} \boldsymbol{B}_P\cdot(a\hat{\boldsymbol{e}}_n) &= \frac{-(B_P)_r rd\theta + (B_P)_\theta dr}{dl}(2\pi r\sin\theta dl) \\ &= 2\pi r\sin\theta\left[\frac{1}{r^2\sin\theta}\frac{\partial\chi}{\partial\theta}rd\theta + \frac{1}{r\sin\theta}\frac{\partial\chi}{\partial r}dr\right] \\ &= 2\pi\left[\frac{\partial\chi}{\partial\theta}d\theta + \frac{\partial\chi}{\partial r}dr\right] = 2\pi d\chi. \end{aligned}$$

7. Two-dimensional analogues

Moffatt seems to switch here to the second convection for streamfunctions (e.g., $P\boldsymbol{x} \to P\hat{\boldsymbol{e}}_r$, etc.) He then zooms in so that the "sphere is flat" and takes $(r,\theta,\phi) \to (z,x,y)$, leaving

$$\boldsymbol{B} = \boldsymbol{B}_P + \boldsymbol{B}_T = \nabla\wedge[\nabla\wedge(P\hat{\boldsymbol{e}}_z)] + \nabla\wedge(T\hat{\boldsymbol{e}}_z).$$

The vector (field?) $\hat{\boldsymbol{e}}_z$ is just a constant, which makes things easier, e.g., proving that the curl of the poloidal field is a toroidal field, so no need to do it again!

Now, clearly $\boldsymbol{B}_T$ is orthogonal to $\hat{\boldsymbol{e}}_z$, and so

$$\begin{aligned}
\hat{\boldsymbol{e}}_z \cdot \boldsymbol{B} = \hat{\boldsymbol{e}}_z \cdot \boldsymbol{B}_P &= \hat{\boldsymbol{e}}_z \cdot \{-\nabla^2(P\hat{\boldsymbol{e}}_z) + \nabla[\underbrace{\nabla \cdot (P\hat{\boldsymbol{e}}_z)}_{\nabla P \cdot \hat{\boldsymbol{e}}_z = \frac{\partial P}{\partial z}}]\} \\
&= \hat{\boldsymbol{e}}_z \cdot \left[-(\nabla^2 P)\hat{\boldsymbol{e}}_z + \nabla\left(\frac{\partial P}{\partial z}\right)\right] \\
&= -\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \cancel{\frac{\partial^2 P}{\partial z^2}}\right) + \cancel{\frac{\partial^2 P}{\partial z^2}} \\
&= -\nabla_2^2 P.
\end{aligned}$$

Thus, the two-dimensional Laplacian operator

$$\nabla_2^2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the analogue of the angular momentum operator.

The fact that the curl of $\boldsymbol{B}_P$ is toroidal (and thus $\perp \hat{\boldsymbol{e}}_z$) and an identical calculation to the preceding one immediately yields

$$\hat{\boldsymbol{e}}_z \cdot \nabla \wedge \boldsymbol{B} = \hat{\boldsymbol{e}}_z \cdot \nabla \wedge \boldsymbol{B}_T = -\nabla_2^2 T.$$

For "axisymmetric fields" $P = P(x, z)$, $T = T(x, z)$, $\boldsymbol{B}_T = \nabla T \wedge \hat{\boldsymbol{e}}_z = (\partial_x T\hat{\boldsymbol{e}}_x + \partial_z \cancel{\hat{\boldsymbol{e}}_z}) \wedge \hat{\boldsymbol{e}}_z = -\partial_x T\hat{\boldsymbol{e}}_y$ (eq. (2.53)). Similar arguments to the ones before show that $\boldsymbol{A}$ has a toroidal part given purely by $\nabla \wedge (P\hat{\boldsymbol{e}}_z)$, so that $\boldsymbol{A}_T = -\partial_x P\hat{\boldsymbol{e}}_y$ and $\boldsymbol{B}_P = \nabla \wedge \boldsymbol{A}_T = \nabla \wedge (A\hat{\boldsymbol{e}}_y)$, where $A = -\partial_x P$.

Thus,

$$\boldsymbol{B}_P = \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ \partial_x & 0 & \partial_z \\ 0 & A & 0 \end{vmatrix} = -(\partial_z A)\hat{\boldsymbol{e}}_x + (\partial_x A)\hat{\boldsymbol{e}}_z.$$

The $\boldsymbol{B}_P$-lines thus satisfy

$$\frac{dx}{dy} = \frac{-\partial_z A}{\partial_x A} \implies \partial_x A dx + \partial_z A dz = dA = 0.$$

Finally, the flux per unit length through a strip parallel to y, i.e., one obtained by sliding the line segment joining (x, z) and $(x + dx, z + dz)$ along y, is

$$\boldsymbol{B}_P \cdot (dz\hat{\boldsymbol{e}}_x - dx\hat{\boldsymbol{e}}_z) = (B_P)_x dz - (B_P)_z dx = \frac{\partial A}{\partial z}dz - \frac{\partial A}{\partial x}dx = -dA,$$

making A $(-A?)$ the flux-function for $\boldsymbol{B}_P$.