Induction term in spherical coordinates

Loren Matilsky

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1 The problem with the traditional induction terms

We consider the ideal (resistance-free) magnetohydrodynamic (MHD) induction equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}), \tag{1}$$

$$= \boldsymbol{B} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{B} - (\nabla \cdot \boldsymbol{u})\boldsymbol{B}, \tag{2}$$

where $\boldsymbol{B}$ and $\boldsymbol{u}$ are the vector magnetic and velocity fields, respectively. The three terms on the right-hand-side of Equation (2) are often interpreted as "shear," "advection," and "compression," respectively. However, this interpretation is problematic in general for two reasons:

1. The so-called shear and compression terms contain sub-terms that cancel; in particular, only velocity motions *perpendicular* to magnetic-field lines can shear or compress.

2. Solid-body rotation (which is a non-shearing motion that simply rotates the whole field configuration) shows up in the so-called shear and advection terms in a strange way.

When resolving the induction equation into a particular curvilinear system (e.g., spherical coordinates), another problem arises:

3. Large curvature terms appear, which are difficult to interpret and occasionally cancel.

Our goal here is to explain fully how these problems emerge and propose a solution for the case of spherical coordinates.

2 Perpendicular shear and compression

To see how Problem 1 arises, we decompose the velocity field into components parallel and perpendicular to the local direction of $\boldsymbol{B}$:

$$\boldsymbol{u} \coloneqq u_{\parallel}\hat{\boldsymbol{e}}_{\parallel} + \boldsymbol{u}_{\perp} \tag{3}$$

Obviously $\boldsymbol{B} = Bx_\parallel$, where $B = |\boldsymbol{B}|$. We denote the Cartesian distance along $\boldsymbol{B}$ by $x_\parallel$. We also decompose $\boldsymbol{u}$ into its parallel and perpendicular components:

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_\parallel}{\partial x_\parallel} + \nabla_\perp \cdot \boldsymbol{u}_\perp \tag{4}$$

We then calculate

$$\begin{aligned}
\boldsymbol{B} \cdot \nabla \boldsymbol{u} - (\nabla \cdot \boldsymbol{u})\boldsymbol{B} &= B\frac{\partial}{\partial x_\parallel}(u_\parallel \hat{\boldsymbol{e}}_\parallel + \boldsymbol{u}_\perp) - \left(\frac{\partial u_\parallel}{\partial x_\parallel} + \nabla_\perp \cdot \boldsymbol{u}_\perp\right) B\hat{\boldsymbol{e}}_\parallel \\
&= \cancel{B\frac{\partial u_\parallel}{\partial x_\parallel}\hat{\boldsymbol{e}}_\parallel} + B\frac{\partial \boldsymbol{u}_\perp}{\partial x_\parallel} - \cancel{\frac{\partial u_\parallel}{\partial x_\parallel} B\hat{\boldsymbol{e}}_\parallel} - (\nabla_\perp \cdot \boldsymbol{u}_\perp) B\hat{\boldsymbol{e}}_\parallel \\
&= \boldsymbol{B} \cdot \nabla \boldsymbol{u}_\perp - (\nabla_\perp \cdot \boldsymbol{u}_\perp)\boldsymbol{B}.
\end{aligned} \tag{5}$$

Thus, only motions perpendicular to the local field line (i.e., $\boldsymbol{u}_\perp$) can shear or compress $\boldsymbol{B}$.

3 Rigid rotation

To see how Problem 2 arises, we consider a velocity field due to rigid rotation at constant angular velocity Ω about the z-axis in a cylindrical coordinate system:

$$\boldsymbol{\Omega} = \Omega \hat{\boldsymbol{e}}_z = \text{constant} \tag{6a}$$

$$\boldsymbol{u} = \boldsymbol{\Omega} \times \boldsymbol{r} = \Omega \lambda \hat{\boldsymbol{e}}_\phi \tag{6b}$$

Here, λ is the cylindrical radius, ϕ the longitude, and z the axial coordinate. In general, $\hat{\boldsymbol{e}}_{(\dots)}$ denotes a unit vector in the direction of its subscript. We calculate:

$$\begin{aligned}
\nabla \cdot \boldsymbol{u} &= \nabla \cdot (\boldsymbol{\Omega} \times \boldsymbol{r}) \\
&= \boldsymbol{\Omega} \cdot \nabla \times \boldsymbol{r} - \boldsymbol{r} \cdot \nabla \times \boldsymbol{\Omega} \\
&= 0 \qquad \text{no compression for rigid rotation (obviously).}
\end{aligned} \tag{7}$$

Then:

$$\begin{aligned}
\boldsymbol{B} \cdot \nabla \boldsymbol{u} &= (\boldsymbol{B} \cdot \nabla)(\boldsymbol{\Omega} \times \boldsymbol{r}) \\
&= \boldsymbol{\Omega} \times [(\boldsymbol{B} \cdot \nabla)(\boldsymbol{r})] \qquad (8) \\
&= \boldsymbol{\Omega} \times \boldsymbol{B} \qquad \text{“shear” for rigid rotation.} \qquad (9)
\end{aligned}$$

Finally:

$$\begin{aligned}
-\boldsymbol{u} \cdot \nabla \boldsymbol{B} &= -\Omega \lambda \hat{\boldsymbol{e}}_\phi \cdot \nabla \boldsymbol{B} \\
&= -\Omega \frac{\partial}{\partial \phi}(B_\lambda \hat{\boldsymbol{e}}_\lambda + B_\phi \hat{\boldsymbol{e}}_\phi + B_z \hat{\boldsymbol{e}}_z) \\
&= -\Omega \sum_\alpha \left(\frac{\partial B_\alpha}{\partial \phi}\right) \hat{\boldsymbol{e}}_\alpha - \Omega \sum_\alpha B_\alpha \frac{\partial \hat{\boldsymbol{e}}_\alpha}{\partial \phi},
\end{aligned}$$

where the index α runs over the three cylindrical coordinates. Note that in the cylindrical coordinate system (or indeed any coordinate system with an axis of rotational symmetry), $\partial\hat{\boldsymbol{e}}_\alpha/\partial\phi = \hat{\boldsymbol{e}}_z \times \hat{\boldsymbol{e}}_\alpha$ for each α. Thus,

$$
\begin{aligned}
-\Omega \sum_\alpha B_\alpha \frac{\partial \hat{\boldsymbol{e}}_\alpha}{\partial \phi} &= -\Omega \sum_\alpha B_\alpha \hat{\boldsymbol{e}}_z \times \hat{\boldsymbol{e}}_\alpha \\
&= -\Omega \hat{\boldsymbol{e}}_z \times \sum_\alpha B_\alpha \hat{\boldsymbol{e}}_\alpha \\
&= -\boldsymbol{\Omega} \times \boldsymbol{B}
\end{aligned}
$$

and so

$$
-\boldsymbol{u}\cdot\nabla\boldsymbol{B} = -\Omega \sum_\alpha \left(\frac{\partial B_\alpha}{\partial \phi}\right) \hat{\boldsymbol{e}}_\alpha - \boldsymbol{\Omega}\times\boldsymbol{B} \qquad \text{“advection” for rigid rotation.} \tag{10}
$$

Mathematically, in any coordinate system with an axis of rotational symmetry about z, the action of rigid rotation is as follows:

$$
\begin{aligned}
\frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times [(\boldsymbol{\Omega}\times\boldsymbol{r})\times\boldsymbol{B}] \\
&= \sum_\alpha \left(-\Omega \frac{\partial B_\alpha}{\partial \phi}\right) \hat{\boldsymbol{e}}_\alpha \\
\text{or} \qquad \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}\right) B_\alpha &= 0 \qquad \text{for each } \alpha.
\end{aligned} \tag{11}
$$

If you think about it, this makes sense: All the rigid rotation does is rotate the whole field configuration around the z-axis at the rate Ω. If you decide to also rotate at Ω (so your personal Eulerian time derivative is $\partial/\partial t + \Omega\partial/\partial\phi$), then each component if the magnetic-field configuration should remain the same in your frame.

4 Solution for spherical coordinates

Resolving these issues fully for the spherical coordinate system seems complicated and I am not fully sure how to do it! In particular (for “full” resolution) we should, separately at each point (r,θ,ϕ):

1. Form a local Cartesian coordinate system, say (x_1, x_2, x_3). At the origin of this system (which lies at the point (r,θ,ϕ)), ($\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$) will coincide with ($\hat{\boldsymbol{e}}_r$, $\hat{\boldsymbol{e}}_\theta$, $\hat{\boldsymbol{e}}_\phi$). But slightly away from the origin, the Cartesian coordinates will remain the same while ($\hat{\boldsymbol{e}}_r$, $\hat{\boldsymbol{e}}_\theta$, $\hat{\boldsymbol{e}}_\phi$) curve away.

2. Calculate the velocity-gradient tensor (being careful to differentiate along the Cartesian coordinates, *not* the curvilinear ones or along the actual $\boldsymbol{B}$-line): $\partial u_1/\partial x_1, \partial u_2/\partial x_1$, etc. in spherical coordinates.

3. Rotate this velocity-gradient tensor “into $\boldsymbol{B}$” to form $\partial u_\parallel/\partial x_\parallel$, $\partial\boldsymbol{u}_\perp/\partial x_\parallel$, and $\nabla_\perp\cdot\boldsymbol{u}_\perp$