

Non-Dimensionalization of an Anelastic Stable–Unstable Layer in **Rayleigh**

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1 General Equations Solved in **Rayleigh**

In general (with rotation and magnetism), **Rayleigh** evolves in time a set of coupled PDEs for the 3D vector velocity \mathbf{u} , vector magnetic field \mathbf{B} , pressure perturbation P (perturbation away from the “reference” or “background” state), and entropy perturbation S . Note that S can also be interpreted as a temperature perturbation in Boussinesq mode. For more details, see **Rayleigh**’s [Documentation](#).

We use standard spherical coordinates (r, θ, ϕ) and cylindrical coordinates $(\lambda, \phi, z) = (r \sin \theta, \phi, r \cos \theta)$, and $\hat{\mathbf{e}}_q$ in general denotes a position-dependent unit vector in the direction of increasing q . The full PDE-set is then:

$$\nabla \cdot [f_1(r)\mathbf{u}] = 0, \quad (1.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (1.2)$$

$$f_1(r) \left[\frac{D\mathbf{u}}{Dt} + c_1 \hat{\mathbf{e}}_z \times \mathbf{u} \right] = c_2 f_2(r) S \hat{\mathbf{e}}_r - c_3 f_1(r) \nabla \left[\frac{P}{f_1(r)} \right], \\ + c_4 (\nabla \times \mathbf{B}) \times \mathbf{B} + c_5 \nabla \cdot \mathbf{D}, \quad (1.3a)$$

$$\text{where} \quad D_{ij} := 2f_1(r)f_3(r) \left[e_{ij} - \frac{1}{3}(\nabla \cdot \mathbf{u})\delta_{ij} \right] \quad (1.3b)$$

$$\text{and} \quad e_{ij} := \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (1.3c)$$

$$f_1(r)f_4(r) \frac{DS}{Dt} = -f_1(r)f_4(r)f_{14}(r)u_r + c_6 \nabla \cdot [f_1(r)f_4(r)f_5(r)\nabla S] \\ + c_6 f_{10}(r) + c_8 c_5 D_{ij} e_{ij} + \frac{\eta(r)}{4\pi} |\nabla \times \mathbf{B}|^2, \quad (1.4)$$

$$\text{and} \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - c_7 \nabla \times [f_7(r)\nabla \times \mathbf{B}], \quad (1.5)$$

where $D/Dt := \partial/\partial t + \mathbf{u} \cdot \nabla$ denotes the material derivative. The spherically-symmetric, time-independent reference (or background) functions $f_i(r)$ and constants c_j set the fluid approximation to be made. **Rayleigh** has built-in modes to set the f 's and c 's for single-layer (i.e., either convectively stable or unstable, but not both) Boussinesq or Anelastic spherical shells. More complex systems (coupled stable-unstable systems or alternative non-dimensionalizations) require the user to manually change the f 's and c 's. This can be done by editing an input binary file that **Rayleigh** reads upon initialization. The c 's can also be changed in the ASCII text-file (i.e., the `main_input` file).

2 Dimensional Anelastic Equations

We begin by writing down the full dimensional anelastic fluid equations, as they are usually implemented in **Rayleigh** (`reference_type = 2`). We differ slightly from “tradition” by assuming at the outset that there is both volumetric heating (preferentially in the bottom of the layer), $\bar{Q}(r)$ and volumetric cooling (preferentially at the top of the layer), $\bar{C}(r)$ (our convention is that both \bar{Q} and \bar{C} are positive; the cooling gets subtracted).

This form of the anelastic approximation in a spherical shell is derived in, or more accurately, attributed to (since **Rayleigh** “updates” the background state slightly differently than the cluge-y **ASH** implementation), two common sources: [Gilman & Glatzmaier \(1981\)](#) and [Clune et al. \(1999\)](#). **Rayleigh**'s dimensional anelastic equation-set is then:

$$\nabla \cdot [\bar{\rho}(r)\mathbf{u}] = 0, \quad (2.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (2.2)$$

$$\begin{aligned} \bar{\rho}(r) \left[\frac{D\mathbf{u}}{Dt} + 2\Omega_0 \hat{\mathbf{e}}_z \times \mathbf{u} \right] &= \left[\frac{\bar{\rho}(r)\bar{g}(r)}{c_p} \right] S \hat{\mathbf{e}}_r - \bar{\rho}(r) \nabla \left[\frac{P}{\bar{\rho}(r)} \right], \\ &+ \frac{1}{\mu} (\nabla \times \mathbf{B}) \times \mathbf{B} + \nabla \cdot \mathbf{D}, \end{aligned} \quad (2.3a)$$

$$\text{where} \quad D_{ij} := 2\bar{\rho}(r)\bar{\nu}(r) \left[e_{ij} - \frac{1}{3}(\nabla \cdot \mathbf{u})\delta_{ij} \right] \quad (2.3b)$$

$$\text{and} \quad e_{ij} := \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (2.3c)$$

$$\begin{aligned} \bar{\rho}(r)\bar{T}(r) \frac{DS}{Dt} &= -\bar{\rho}(r)\bar{T}(r) \frac{d\bar{S}}{dr} u_r + \nabla \cdot [\bar{\rho}(r)\bar{T}(r)\bar{\kappa}(r)\nabla S] \\ &+ \bar{Q}(r) - \bar{C}(r) + D_{ij}e_{ij} + \frac{\bar{\eta}(r)}{\mu} |\nabla \times \mathbf{B}|^2, \end{aligned} \quad (2.4)$$

$$\text{and} \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times [\eta(r)\nabla \times \mathbf{B}]. \quad (2.5)$$

Here, the thermal variables ρ , T , P , and S refer to the density, temperature, pressure, and entropy (respectively). The overbars denote the spherically-symmetric, time-independent

background state. The lack of an overbar on a thermal variable indicates the (assumed small) perturbation from the background (for the entropy, S/c_p is assumed small).

Other background quantities that appear are the gravity $\bar{g}(r)$, the momentum, thermal, and magnetic diffusivities $[\bar{\nu}(r), \bar{\kappa}(r), \text{ and } \bar{\eta}(r)]$, respectively, the internal heating or cooling $\bar{Q}(r)$, the frame rotation rate Ω_0 , the specific heat at constant pressure c_p , and the vacuum permeability μ ($= 4\pi$ in c.g.s. units). The equations are written in a frame rotating with angular velocity Ω_0 and the centrifugal force is neglected.

Note that the internal heating and cooling functions $\bar{Q}(r)$ and $\bar{C}(r)$ are reference-state quantities (and thus assumed spherically-symmetric and time-independent) but should be interpreted as $\bar{Q} - \bar{C} = -\nabla \cdot \mathcal{F}_{\text{rad}}$, where \mathcal{F}_{rad} is the radiative heat flux. Properly, \bar{Q} should be proportional to the radiative diffusivity κ_{rad} (which takes on a specific form in the radiative diffusion approximation, derivable from the opacity) and to the gradient of the total temperature $\bar{T} + T$; and \bar{C} should be calculated using complicated near-surface physics.

In Rayleigh, a convective layer is usually driven by a combination of internal heating and the thermal boundary conditions (which are conditions on S), that together ensure that an imposed energy flux is transported throughout the layer in a steady state. (Note that energy could also be forced across the layer by fixing the entropy S at each boundary, such that an “adverse” (negative) radial entropy gradient is obtained in a steady state). **In the Jupiter models, which will have both internal heating and cooling, we will set $\partial S/\partial r \equiv 0$ at both the top and bottom boundary (no conduction in or out), and the flux of energy across the system will be imposed purely by the combination $\bar{Q} - \bar{C}$.**

Also, we recall the relation

$$\frac{d\bar{S}}{dr} = c_p \frac{\bar{N}^2(r)}{\bar{g}(r)}, \quad (2.6)$$

where $\bar{N}^2(r)$ is the squared buoyancy frequency, which we will use in favor of $d\bar{S}/dr$ in subsequent equations.

Note that the original equations in [Gilman & Glatzmaier \(1981\)](#) and [Clune et al. \(1999\)](#) were derived assuming a nearly-adiabatic background state (i.e., $d\bar{S}/dr \approx 0$). [Brown et al. \(2012\)](#) and [Vasil et al. \(2013\)](#) have raised concerns about using various anelastic approximations in stable layers due to non-energy-conserving gravity waves. Should we be concerned?

3 Non-Dimensional Scheme

We now non-dimensionalize Equations (2.1)–(2.5), according to the following scheme:

$$\nabla \rightarrow \frac{1}{H} \nabla, \quad (3.1a)$$

$$t \rightarrow \tau t, \quad (3.1b)$$

$$\mathbf{u} \rightarrow \frac{H}{\tau} \mathbf{u}, \quad (3.1c)$$

$$S \rightarrow (\Delta S) S, \quad (3.1d)$$

$$P \rightarrow \tilde{\rho} \frac{H^2}{\tau^2} P, \quad (3.1e)$$

$$\mathbf{B} \rightarrow (\mu \tilde{\rho})^{1/2} \frac{H}{\tau} \mathbf{B}, \quad (3.1f)$$

$$\bar{\rho}(r) \rightarrow \tilde{\rho} \bar{\rho}(r), \quad (3.1g)$$

$$\bar{T}(r) \rightarrow \tilde{T} \bar{T}(r), \quad (3.1h)$$

$$\bar{g}(r) \rightarrow \tilde{g} \bar{g}(r), \quad (3.1i)$$

$$\overline{N^2}(r) \rightarrow \widetilde{N^2} \overline{N^2}(r), \quad (3.1j)$$

$$\bar{\nu}(r) \rightarrow \tilde{\nu} \bar{\nu}(r), \quad (3.1k)$$

$$\bar{\kappa}(r) \rightarrow \tilde{\kappa} \bar{\kappa}(r), \quad (3.1l)$$

$$\bar{\eta}(r) \rightarrow \tilde{\eta} \bar{\eta}(r), \quad (3.1m)$$

$$\bar{Q}(r) \rightarrow \tilde{C} \bar{Q}(r), \quad (3.1n)$$

$$\text{and } \bar{C}(r) \rightarrow \tilde{Q} \bar{C}(r). \quad (3.1o)$$

Here, H is a typical length-scale, τ a typical time-scale, and ΔS a typical (*estimated*) entropy scale (in Rayleigh-Bénard-type convection, the true entropy difference is imposed directly, but we will set the true value indirectly via heating and cooling functions). On the right-hand-sides of Equation (3.1) and in the following non-dimensionalizations, all fluid variables, coordinates, and background-state quantities are understood to be non-dimensional. The tildes refer to “typical values” of the (dimensional) reference-state functions. These typical values will be a volume-average over the convection zone (CZ) of the shell, except for $\widetilde{N^2}$, which will be a volume-average over the stably stratified weather layer (WL). Since cooling takes out what heating dumps in, we will normalize such that $\tilde{C} = \tilde{Q}$.

Below, we will assume the time-scale is either a thermal diffusion time (i.e., $\tau = H^2/\tilde{\nu}$) or a rotational time-scale [i.e., $\tau = (2\Omega_0)^{-1}$].

4 Non-Dimensional Equations, Non-Rotating ($\tau = H^2/\tilde{\kappa}$)

In this case, Equations (2.1)–(2.5) become

$$\nabla \cdot [\bar{\rho}(r) \mathbf{u}] = 0, \quad (4.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (4.2)$$

$$\begin{aligned} \bar{\rho}(r) \left[\frac{D\mathbf{u}}{Dt} + \frac{\text{Pr}}{\text{Ek}} \hat{\mathbf{e}}_z \times \mathbf{u} \right] = & -\bar{\rho}(r) \nabla \left[\frac{P}{\bar{\rho}(r)} \right] + \text{PrRa}\bar{\rho}(r)\bar{g}(r)S\hat{\mathbf{e}}_r, \\ & + \text{Pr} \nabla \cdot \mathbf{D} + (\nabla \times \mathbf{B}) \times \mathbf{B}, \end{aligned} \quad (4.3a)$$

$$\text{where} \quad D_{ij} := 2\bar{\rho}(r)\bar{\nu}(r) \left[e_{ij} - \frac{1}{3}(\nabla \cdot \mathbf{u})\delta_{ij} \right] \quad (4.3b)$$

$$\text{and} \quad e_{ij} := \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (4.3c)$$

$$\begin{aligned} \bar{\rho}(r)\bar{T}(r)\frac{DS}{Dt} = & -\frac{\text{Bu}}{\text{Ra}}\bar{\rho}(r)\bar{T}(r)\frac{\bar{N}^2(r)}{\bar{g}(r)}u_r + \nabla \cdot [\bar{\rho}(r)\bar{T}(r)\bar{\kappa}(r)\nabla S] \\ & + \bar{Q}(r) - \bar{C}(r) + \frac{\text{Di}}{\text{Ra}}D_{ij}e_{ij} + \frac{\text{Di}}{\text{Pr}_m\text{Ra}}\bar{\eta}(r)|\nabla \times \mathbf{B}|^2, \end{aligned} \quad (4.4)$$

$$\text{and} \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{\text{Pr}}{\text{Pr}_m} \nabla \times [\bar{\eta}(r)\nabla \times \mathbf{B}]. \quad (4.5)$$

The non-dimensional numbers appearing are:

$$\text{Ra} := \frac{\tilde{g}H^3}{\tilde{\nu}\tilde{\kappa}} \frac{\Delta S}{c_p} \quad (\text{Rayleigh number}), \quad (4.6a)$$

$$\text{Pr} := \frac{\tilde{\nu}}{\tilde{\kappa}} \quad (\text{Prandtl number}), \quad (4.6b)$$

$$\text{Pr}_m := \frac{\tilde{\nu}}{\tilde{\eta}} \quad (\text{magnetic Prandtl number}), \quad (4.6c)$$

$$\text{Ek} := \frac{\tilde{\nu}}{2\Omega_0 H^2} \quad (\text{Ekman number}), \quad (4.6d)$$

$$\text{Bu} := \frac{\tilde{N}^2 H^4}{\tilde{\nu}\tilde{\kappa}} \quad (\text{buoyancy number}), \quad (4.6e)$$

$$\text{and} \quad \text{Di} = \frac{\tilde{g}H}{c_p\tilde{T}} \quad (\text{dissipation number}), \quad (4.6f)$$

Note that in our convention, the dissipation number is not an independent control parameter, but a function of the non-dimensional parameters characterizing the reference state (this will be seen in Section ??).

Note that we have chosen the entropy-scale (and thus the Rayleigh number) based on the internal heating:

$$\Delta S := \frac{\tilde{Q}\tau}{\tilde{\rho}\tilde{T}} = \frac{\tilde{Q}H^2}{\tilde{\rho}\tilde{T}\tilde{\kappa}} \quad (4.7a)$$

$$\text{and} \quad \text{Ra} := \frac{\tilde{g}\tilde{Q}H^3\tau}{\tilde{\rho}\tilde{T}c_p\tilde{\nu}\tilde{\kappa}} = \frac{\tilde{g}\tilde{Q}H^5}{\tilde{\rho}\tilde{T}c_p\tilde{\nu}\tilde{\kappa}^2}, \quad (4.7b)$$

$$(4.7c)$$

where the first equality in each equation is general (it holds for any choice of time-scale τ) and the second equality is specific to the non-rotating case. Essentially, we have assumed that the heating (or cooling) operates on the time-scale τ before the fluid parcel buoyantly moves to another part of the shell, carrying with it an entropy perturbation ΔS (in the non-rotating case, this should happen on the thermal dissipation time-scale $\tau = H^2/\tilde{\kappa}$).

The user is thus free to choose the shapes of $\overline{Q}(r)$ and $\overline{C}(r)$, but not their amplitude, since they must have unity volume-averages over the CZ.

The buoyancy number Bu is the ratio of the typical squared buoyancy frequency to the thermal and viscous diffusion times. It is essentially a “second (stable) Rayleigh number”, and measures the stiffness of the stable layer (recall $\widetilde{N^2}$ refers to the typical value of $\overline{N^2}(r)$ in the WL). The buoyancy number is independent of the Rayleigh number, which estimates the ultimate instability of the CZ.

5 Non-Dimensional Equations, Rotating [$\tau = (2\Omega_0)^{-1}$]

In the previous section, t (and things with time in the dimensions) was implied to mean $(\tilde{\kappa}/H^2)t_{\text{dim}}$, where t_{dim} was the dimensional time. We now want to use a new non-dimensional time, $t_{\text{new}} = \Omega_0 t_{\text{dim}} = (\text{Pr}/\text{Ek})t$. We can thus find the new equations easily from Equations (4.1)–(4.5). Every place we see a time dimension, we recall $t = (\text{Ek}/\text{Pr})t_{\text{new}}$, so we multiply the place where the time-dimension appears by (Ek/Pr) and drop the “new” subscript [e.g., $t \rightarrow (\text{Ek}/\text{Pr}) t$, $\mathbf{u} \rightarrow (\text{Pr}/\text{Ek})\mathbf{u}$, etc.]

Note that we should now choose a different typical entropy-scale and corresponding Rayleigh number:

$$\Delta S^* := \frac{\tilde{Q}\tau}{\tilde{\rho}\tilde{T}} = \frac{\tilde{Q}}{2\Omega_0\tilde{\rho}\tilde{T}} \quad (5.1a)$$

$$\text{and} \quad \text{Ra} := \frac{\tilde{g}\tilde{Q}H^3\tau}{\tilde{\rho}\tilde{T}c_p\tilde{\nu}\tilde{\kappa}} = \frac{\tilde{g}\tilde{Q}H^3}{2\Omega_0\tilde{\rho}\tilde{T}c_p\tilde{\nu}\tilde{\kappa}}. \quad (5.1b)$$

The reasoning here is that under the influence of rapid rotation, the life-times of upflows or downflows are no longer set by the thermal dissipation time, but by the rotation period. Thus, the heating or cooling of a fluid parcel occurs on a shorter time-scale, leading to a smaller entropy difference across the shell than in the non-rotating case with the same amount of heating. Of course, neither Equations (4.7a) or (5.1a) are particularly convincing estimates and there is a large degree of uncertainty in the actual magnitude of typical entropy perturbations. We can only see how good these estimates are (after the fact) by checking if the achieved (non-dimensional) entropy difference across the shell winds up being close to unity.

We thus find (after rearranging terms),

$$\nabla \cdot [\bar{\rho}(r)\mathbf{u}] = 0, \quad (5.2)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (5.3)$$

$$\begin{aligned} \bar{\rho}(r) \left[\frac{D\mathbf{u}}{Dt} + \hat{\mathbf{e}}_z \times \mathbf{u} \right] = & -\bar{\rho}(r) \nabla \left[\frac{P}{\bar{\rho}(r)} \right] + \text{Ra}^* \bar{\rho}(r) \bar{g}(r) S \hat{\mathbf{e}}_r, \\ & + \text{Ek} \nabla \cdot \mathbf{D} + (\nabla \times \mathbf{B}) \times \mathbf{B}, \end{aligned} \quad (5.4a)$$

$$\text{where} \quad D_{ij} := 2\bar{\rho}(r)\bar{\nu}(r) \left[e_{ij} - \frac{1}{3}(\nabla \cdot \mathbf{u})\delta_{ij} \right] \quad (5.4b)$$

$$\text{and} \quad e_{ij} := \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (5.4c)$$

$$\begin{aligned} \bar{\rho}(r)\bar{T}(r)\frac{DS}{Dt} = & -\frac{\text{Bu}^*}{\text{Ra}^*} \bar{\rho}(r)\bar{T}(r) \frac{\overline{N^2}(r)}{\bar{g}(r)} u_r + \frac{\text{Ek}}{\text{Pr}} \nabla \cdot [\bar{\rho}(r)\bar{T}(r)\bar{\kappa}(r)\nabla S] \\ & + \frac{\text{Ek}}{\text{Pr}} \bar{Q}(r) + \frac{\text{DiEk}}{\text{Ra}^*} D_{ij} e_{ij} + \frac{\text{DiEk}}{\text{Pr}_m \text{Ra}^*} \bar{\eta}(r) |\nabla \times \mathbf{B}|^2, \end{aligned} \quad (5.5)$$

$$\text{and} \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{\text{Ek}}{\text{Pr}_m} \nabla \times [\eta(r) \nabla \times \mathbf{B}], \quad (5.6)$$

where we have introduced two modified non-dimensional numbers:

$$\text{Ra}^* := \frac{\text{Ek}^2}{\text{Pr}} \text{Ra} = \frac{\tilde{g}}{H\Omega_0^2} \frac{\Delta S}{c_p} = \frac{\tilde{g}\tilde{Q}}{(2\Omega_0)^3 \tilde{\rho}\tilde{T}c_p H}, \quad (5.7a)$$

$$\text{and} \quad \text{Bu}^* := \frac{\text{Ek}^2}{\text{Pr}} \text{Bu} = \frac{\widetilde{N^2}}{4\Omega_0^2} \sim \frac{\tilde{g}}{H\Omega_0^2} = \frac{1}{\text{geometric oblateness}}. \quad (5.7b)$$

Note that although the “ $d\bar{S}/dr$ -terms” in the non-dimensionalizations have seemingly different definitions, they are similar, since

$$\frac{\text{Bu}}{\text{Ra}} \sim \frac{c_p}{\Delta S} \quad \text{and} \quad \frac{\text{Bu}^*}{\text{Ra}^*} \sim \frac{c_p}{\Delta S^*}. \quad (5.8)$$

The only difference is in the different estimates ΔS and ΔS^* .

References

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