# Viscous Force in Terms of Poloidal and Toroidal Stream Functions

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July 9, 2019

Under the anelastic approximation,

$$\nabla \cdot (\overline{\rho}\boldsymbol{v}) \equiv 0, \tag{1}$$

so we expand the mass flux in terms of stream functions:

$$\overline{\rho}\boldsymbol{v} = \nabla \times [\nabla \times (W\hat{\boldsymbol{e}}_r)] + \nabla \times (Z\hat{\boldsymbol{e}}_r). \tag{2}$$

Here $W$ and $Z$ are the poloidal and toroidal stream functions, respectively. Note that the velocity field from the toroidal stream function, $Z$ is purely horizontal; $v_r$ is thus purely determined by the poloidal stream function, $W$. In fact, we can show from the radial component of (2) that

$$\overline{\rho}v_r = -\frac{1}{r^2}(\boldsymbol{r} \times \nabla)^2 W. \tag{3}$$

The operator $\mathscr{L}^2 = -(\boldsymbol{r} \times \nabla)^2$ is just the total angular momentum operator, with the eigenvalues $L := l(l+1)$ ($l$ being an integer) and the eigenfunctions the spherical harmonics $Y_{lm}$. Thus, if we consider only one spherical harmonic component of $\overline{\rho}v_r$ at a time (or else all the spherical harmonics with a fixed $l$), we can write

$$\overline{\rho}v_r = \frac{L}{r^2}W. \tag{4}$$

We are interested in the Newtonian viscous forcing term:

$$\boldsymbol{f} := -\nabla \cdot \boldsymbol{D} = \nabla \cdot \left\{2\overline{\rho}\nu\left[\boldsymbol{S} - \frac{1}{3}(\nabla \cdot \boldsymbol{v})\boldsymbol{I}\right]\right\}, \tag{5}$$

where

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) \tag{6}$$

and $\boldsymbol{I}$ is the identity tensor, or Kronecker delta.

We define

$$\alpha(r) \coloneqq \frac{d \ln \nu}{dr}, \tag{7}$$

$$\beta(r) \coloneqq \frac{d \ln \overline{\rho}}{dr}, \tag{8}$$

and compute

$$\boldsymbol{f} = 2[\nabla(\overline{\rho}\nu)] \cdot \left[\boldsymbol{S} - \frac{1}{3}(\nabla \cdot \boldsymbol{v})\boldsymbol{I}\right] + 2\overline{\rho}\nu\nabla \cdot \left[\boldsymbol{S} - \frac{1}{3}(\nabla \cdot \boldsymbol{v})\boldsymbol{I}\right]$$

Now,

$$\nabla(\overline{\rho}\nu) = \frac{d(\overline{\rho}\nu)}{dr}\hat{\boldsymbol{e}}_r = [(\overline{\rho}\beta)\nu + \overline{\rho}(\nu\alpha)]\hat{\boldsymbol{e}}_r = \overline{\rho}\nu(\alpha + \beta)\hat{\boldsymbol{e}}_r$$

and

$$\begin{aligned}
\nabla \cdot \left[\boldsymbol{S} - \frac{1}{3}(\nabla \cdot \boldsymbol{v})\boldsymbol{I}\right] &= \hat{\boldsymbol{e}}_i \frac{\partial}{\partial x_j}\left[\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) - \frac{1}{3}(\nabla \cdot \boldsymbol{v})\delta_{ij}\right] \\
&= \frac{1}{2}\hat{\boldsymbol{e}}_i\left(\frac{\partial^2 u_j}{\partial x_i \partial x_j} + \frac{\partial^2 u_i}{\partial x_j^2}\right) - \frac{1}{3}\hat{\boldsymbol{e}}_i\frac{\partial}{\partial x_i}(\nabla \cdot \boldsymbol{v}) \\
&= \frac{1}{2}\nabla^2\boldsymbol{v} + \frac{1}{2}\nabla(\nabla \cdot \boldsymbol{v}) - \frac{1}{3}\nabla(\nabla \cdot \boldsymbol{v}) \\
&= \frac{1}{2}\nabla^2\boldsymbol{v} + \frac{1}{6}\nabla(\nabla \cdot \boldsymbol{v}).
\end{aligned}$$

Thus,

$$\boldsymbol{f} = 2\overline{\rho}\nu(\alpha + \beta)\left[S_{r\gamma}\hat{\boldsymbol{e}}_\gamma - \frac{1}{3}(\nabla \cdot \boldsymbol{v})\hat{\boldsymbol{e}}_r\right] + \overline{\rho}\nu\left[\nabla^2\boldsymbol{v} + \frac{1}{3}\nabla(\nabla \cdot \boldsymbol{v})\right], \tag{9}$$

where $\gamma$ runs over the spherical coordinates $r, \theta, \phi$.

# 1 Radial component of viscous force

We compute

$$f_r = \underbrace{2\overline{\rho}\nu(\alpha + \beta)\left(S_{rr} - \frac{1}{3}\nabla \cdot \boldsymbol{v}\right)}_{\coloneqq T_1} + \underbrace{\overline{\rho}\nu\left[(\nabla^2\boldsymbol{v})_r + \frac{1}{3}\frac{\partial}{\partial r}(\nabla \cdot \boldsymbol{v})\right]}_{\coloneqq T_2} \tag{10}$$

Now, $S_{rr} = \partial v_r/\partial r$, and from the anelastic assumption (1),

$$\nabla \cdot \boldsymbol{v} = -\beta v_r. \tag{11}$$

Thus, we compute

$$T_1 = 2\overline{\rho}\nu(\alpha+\beta)\left(\frac{\partial v_r}{\partial r} + \frac{1}{3}\beta v_r\right). \tag{12}$$

We also have

$$\begin{aligned}
\overline{\rho}\frac{\partial v_r}{\partial r} &= \frac{\partial}{\partial r}(\overline{\rho}v_r) - v_r\frac{d\overline{\rho}}{dr} \\
&= \frac{\partial}{\partial r}\left(\frac{LW}{r^2}\right) - (\beta\overline{\rho})v_r \\
&= \frac{L}{r^2}\frac{\partial W}{\partial r} - \frac{2L}{r^3}W - \beta\frac{LW}{r^2} \\
&= \frac{L}{r^2}\left[\frac{\partial W}{\partial r} - \left(\frac{2}{r}+\beta\right)W\right].
\end{aligned} \tag{13}$$

Plugging (4) and (13) into (12), we find

$$\begin{aligned}
T_1 &= \nu\left\{2(\alpha+\beta)\left[\frac{L}{r^2}\left(\frac{\partial W}{\partial r} - \left(\frac{2}{r}+\beta\right)W\right) + \frac{\beta}{3}\frac{LW}{r^2}\right]\right\} \\
&= \nu\frac{L}{r^2}\left[(2\alpha+2\beta)\frac{\partial W}{\partial r} + (2\alpha+2\beta)\left(-\frac{2}{r} - \frac{2}{3}\beta\right)W\right] \\
&= \nu\frac{L}{r^2}\left[(2\alpha+2\beta)\frac{\partial W}{\partial r} - \left(\frac{4\alpha}{r} + \frac{4\beta}{r} + \frac{4}{3}\alpha\beta + \frac{4}{3}\beta^2\right)W\right].
\end{aligned} \tag{14}$$

To calculate $T_2$, we note that

$$\begin{aligned}
(\nabla^2\boldsymbol{v})_r &= \nabla^2 v_r - \frac{2v_r}{r^2} - \frac{2}{r}\underbrace{\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\cot\theta v_\theta}{r} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)}_{\nabla\cdot\boldsymbol{v}-\ \frac{\partial v_r}{\partial r}-\ \frac{2}{r}\frac{v_r}{r}} \\
&= \nabla^2 v_r - \frac{2}{r}\nabla\cdot\boldsymbol{v} + \frac{2v_r}{r^2} + \frac{2}{r}\frac{\partial v_r}{\partial r} \\
&= \frac{\partial^2 v_r}{\partial r^2} + \frac{2}{r}\frac{\partial v_r}{\partial r} - \frac{L}{r^2}v_r + \frac{2}{r}\beta v_r + \frac{2v_r}{r^2} + \frac{2}{r}\frac{\partial v_r}{\partial r} \\
&= \frac{\partial^2 v_r}{\partial r^2} + \frac{4}{r}\frac{\partial v_r}{\partial r} + \left(\frac{2}{r^2} + \frac{2\beta}{r} - \frac{L}{r^2}\right)v_r.
\end{aligned} \tag{15}$$

Plugging (11) and (15) into the definition of $T_2$ in (10) yields

$$\begin{aligned}
T_2 &= \overline{\rho}\nu\left[\frac{\partial^2 v_r}{\partial r^2} + \frac{4}{r}\frac{\partial v_r}{\partial r} + \left(\frac{2}{r^2} + \frac{2\beta}{r} - \frac{L}{r^2}\right)v_r - \frac{1}{3}\frac{\partial}{\partial r}(\beta v_r)\right. \\
&= \nu\left[\overline{\rho}\frac{\partial^2 v_r}{\partial r^2} + \left(\frac{4}{r} - \frac{1}{3}\beta\right)\overline{\rho}\frac{\partial v_r}{\partial r} + \left(\frac{2}{r^2} + \frac{2\beta}{r} - \frac{L}{r^2} - \frac{1}{3}\frac{d\beta}{dr}\right)\overline{\rho}v_r\right].
\end{aligned} \tag{16}$$

We also need

$$\overline{\rho}\frac{\partial^2 v_r}{\partial r^2} = \frac{\partial}{\partial r}\left(\overline{\rho}\frac{\partial v_r}{\partial r}\right) - \beta\overline{\rho}\frac{\partial v_r}{\partial r}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial r}\left\{\frac{L}{r^2}\left[\frac{\partial W}{\partial r} - \left(\frac{2}{r}+\beta\right)W\right]\right\} - \beta\frac{L}{r^2}\left[\frac{\partial W}{\partial r} - \left(\frac{2}{r}+\beta\right)W\right] \\
&= \frac{L}{r^2}\left[\frac{\partial^2 W}{\partial r^2} + \left(\frac{2}{r^2} - \frac{d\beta}{dr}\right)W - \left(\frac{2}{r}+\beta\right)\frac{\partial W}{\partial r}\right] + \left(-\frac{2L}{r^3} - \beta\frac{L}{r^2}\right)\left[\frac{\partial W}{\partial r} - \left(\frac{2}{r}+\beta\right)W\right] \\
&= \frac{L}{r^2}\left\{\frac{\partial^2 W}{\partial r^2} - \left(\frac{4}{r}+2\beta\right)\frac{\partial W}{\partial r} + \left[\frac{2}{r^2} - \frac{d\beta}{dr} + \left(\frac{2}{r}+\beta\right)^2\right]W\right\} \\
&= \frac{L}{r^2}\left[\frac{\partial^2 W}{\partial r^2} - \left(\frac{4}{r}+2\beta\right)\frac{\partial W}{\partial r} + \left(\frac{6}{r^2} + \frac{4\beta}{r} + \beta^2 - \frac{d\beta}{dr}\right)W\right].
\end{aligned} \tag{17}
$$

Plugging (4), (13), and (17) into (16) finally yields $T_2$ in terms of the poloidal stream function $W$:

$$
\begin{aligned}
T_2 = \frac{L}{r^2}\nu\Bigg\{&\frac{\partial^2 W}{\partial r^2} - \left(\frac{4}{r}+2\beta\right)\frac{\partial W}{\partial r} + \left(\frac{6}{r^2} + \frac{4\beta}{r} + \beta^2 - \underline{\frac{d\beta}{dr}}\right)W + \\
&\left(\frac{4}{r} - \frac{1}{3}\beta\right)\left[\frac{\partial W}{\partial r} - \left(\frac{2}{r}+\beta\right)W\right] + \left(\frac{2}{r^2} + \frac{2\beta}{r} - \frac{L}{r^2} - \underline{\frac{1}{3}\frac{d\beta}{dr}}\right)W\Bigg\} \\
= \frac{L}{r^2}\nu\Bigg\{&\frac{\partial^2 W}{\partial r^2} + \left[-\cancel{\frac{4}{r}} - 2\beta + \cancel{\frac{4}{r}} - \frac{1}{3}\beta\right]\frac{\partial W}{\partial r} \\
&+ \left[\cancel{\frac{6}{r^2}} + \cancel{\frac{4\beta}{r}} + \underline{\beta^2} - \cancel{\frac{8}{r^2}} - \cancel{\frac{4\beta}{r}} + \frac{2}{3}\frac{\beta}{r} + \underline{\frac{1}{3}\beta^2} + \cancel{\frac{2}{r^2}} + \frac{2\beta}{r} - \frac{L}{r^2} - \frac{4}{3}\frac{d\beta}{dr}\right]W\Bigg\} \\
= \frac{L}{r^2}\nu\Bigg[&\frac{\partial^2 W}{\partial r^2} - \frac{7}{3}\beta\frac{\partial W}{\partial r} + \left(\frac{4}{3}\beta^2 + \frac{8}{3}\frac{\beta}{r} - \frac{L}{r^2} - \frac{4}{3}\frac{d\beta}{dr}\right)W\Bigg].
\end{aligned} \tag{18}
$$

At last, we use the expressions for $T_1$ and $T_2$ in terms of $W$ (equations (14) and (18), respectively) in (10) to find $f_r$:

$$
\begin{aligned}
f_r = T_1 + T_2 = \frac{L}{r^2}\nu\Bigg[&\frac{\partial^2 W}{\partial r^2} + \left(2\alpha + 2\beta - \frac{7}{3}\beta\right)\frac{\partial W}{\partial r} \\
&+ \left[-\frac{4\alpha}{r} - \underline{\frac{4\beta}{r}} - \frac{4}{3}\alpha\beta - \cancel{\frac{4}{3}\beta^2} + \cancel{\frac{4}{3}\beta^2} + \underline{\frac{8}{3}\frac{\beta}{r}} - \frac{L}{r^2} - \frac{4}{3}\frac{d\beta}{dr}\right]W\Bigg\} \\
= \frac{L}{r^2}\nu\Bigg[&\frac{\partial^2 W}{\partial r^2} + \left(2\alpha - \frac{1}{3}\beta\right)\frac{\partial W}{\partial r} + \left(-\frac{4\alpha}{r} - \frac{4}{3}\frac{\beta}{r} - \frac{4}{3}\alpha\beta - \frac{L}{r^2} - \frac{4}{3}\frac{d\beta}{dr}\right)W\Bigg] \\
= \frac{L}{r^2}\nu\Bigg\{&\frac{\partial^2 W}{\partial r^2} + \left(\frac{6\alpha - \beta}{3}\right)\frac{\partial W}{\partial r} - \left[\frac{4}{3}\left(\alpha\beta + \frac{d\beta}{dr} + \frac{3\alpha+\beta}{r}\right) + \frac{L}{r^2}\right]W\Bigg\}.
\end{aligned} \tag{19}
$$

## 2 Radial component of curl of viscous force

ASH and Rayleigh don't solve the momentum equation directly, but rather the radial component of the momentum equation and radial component of the curl of the momentum equation, to get things in terms of the stream function W and Z. Thus, we wish to write

$$
h_r \coloneqq (\nabla \times \boldsymbol{f})_r = [\nabla \times (-\nabla \cdot \boldsymbol{D})]_r \tag{20}
$$

in terms of the stream functions (in the end it will only be in terms of the toroidal stream function, $Z$). We have here defined $\boldsymbol{h} \coloneqq \nabla \times (-\nabla \cdot \boldsymbol{D})$ to be the "curl-of-force" field from the viscosity (kind of like a torque–or, more appropriately, a vorticity driving field).

From (9), we have

$$h_r = \{2\overline{\rho}\nu(\alpha+\beta)[\nabla \times (S_{r\gamma}\hat{\boldsymbol{e}}_\gamma)] + \overline{\rho}\nu\nabla^2\boldsymbol{\omega}\}_r,$$

where $\boldsymbol{\omega} \coloneqq \nabla \times \boldsymbol{v}$ is the vorticity field. In the preceding equation, we have noted several times that $\nabla \times \hat{\boldsymbol{e}}_r \equiv 0$, $[\nabla(\text{function of radius})] \times \hat{\boldsymbol{e}}_r \equiv 0$, and $\nabla \times (\nabla\psi) \equiv 0$ for any scalar $\psi$. We have also noted that on vectors, the operators $\nabla\times$ and $\nabla^2$ commute.

To go further, we compute

$$\begin{aligned}
[\nabla \times (S_{r\gamma}\hat{\boldsymbol{e}}_\gamma)]_r &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta S_{r\phi}) - \frac{1}{r\sin\theta}\frac{\partial S_{r\theta}}{\partial\phi} \\
&= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left\{\frac{\sin\theta}{2}\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]\right\} \\
&\quad - \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left\{\frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]\right\} \\
&= \frac{1}{2}\left\{\cancel{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial v_r}{\partial\phi}\right)} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]\right. \\
&\quad \left. - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right] - \cancel{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial v_r}{\partial\theta}\right)}\right\} \\
&= \frac{1}{2}\frac{\partial}{\partial r}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\phi) - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right] && (21) \\
&= \frac{1}{2}\frac{\partial\omega_r}{\partial r}. && (22)
\end{aligned}$$

The identity (15) holds for $(\nabla^2\boldsymbol{\omega})_r$, but with no term $2\beta\omega_r/r$ since $\nabla\cdot\boldsymbol{\omega} \equiv 0$. Plugging (22) and (a modified) (15) into (21) yields

$$\begin{aligned}
h_r &= 2\overline{\rho}\nu(\alpha+\beta)\left(\frac{1}{2}\frac{\partial\omega_r}{\partial r}\right) + \overline{\rho}\nu(\nabla^2\boldsymbol{\omega})_r && (23) \\
&= \nu\left\{(\alpha+\beta)\overline{\rho}\frac{\partial\omega_r}{\partial r} + \overline{\rho}\left[\frac{\partial^2\omega_r}{\partial r^2} + \frac{4}{r}\frac{\partial\omega_r}{\partial r} + \left(\frac{2}{r^2} - \frac{L}{r^2}\right)\omega_r\right]\right\} \\
&= \nu\left[\overline{\rho}\frac{\partial^2\omega_r}{\partial r^2} + \left(\alpha+\beta+\frac{4}{r}\right)\overline{\rho}\frac{\partial\omega_r}{\partial r} + \left(\frac{2}{r^2} - \frac{L}{r^2}\right)\overline{\rho}\omega_r\right]. && (24)
\end{aligned}$$

Now, $\omega_r$ is related (purely) to the toroidal stream function $Z$. One can show, taking the radial component of the curl of (2), that (assuming the various spherical harmonic components are considered separately)

$$\overline{\rho}\omega_r = \frac{L}{r^2}Z. \tag{25}$$

Furthermore, the analogs of both (13) and (17) both hold exactly with the substitutions $v_r \to \omega_r$ and $W \to Z$. Plugging all this into (24) thus yields

$$h_r = \frac{L\nu}{r^2}\left\{\frac{\partial^2 Z}{\partial r^2} - \left(\frac{4}{r} + 2\beta\right)\frac{\partial Z}{\partial r} + \left(\frac{6}{r^2} + \frac{4\beta}{r} + \beta^2 - \frac{d\beta}{dr}\right)Z\right.$$

$$
\begin{aligned}
&+\left(\alpha+\beta+\frac{4}{r}\right)\left[\frac{\partial Z}{\partial r}-\left(\frac{2}{r}+\beta\right)Z\right]+\left(\frac{2}{r^2}-\frac{L}{r^2}\right)Z\Bigg\}\\
&=\frac{L\nu}{r^2}\Bigg[\frac{\partial^2 Z}{\partial r^2}+\left(-\cancel{\frac{4}{r}}-2\beta+\alpha+\beta+\cancel{\frac{4}{r}}\right)\frac{\partial Z}{\partial r}\\
&+\left(\cancel{\frac{6}{r^2}}+\cancel{\frac{4\beta}{r}}+\cancel{\beta^2}-\frac{d\beta}{dr}-\frac{2\alpha}{r}-\frac{2\beta}{r}-\cancel{\frac{8}{r^2}}-\alpha\beta-\cancel{\beta^2}-\cancel{\frac{4\beta}{r}}+\cancel{\frac{2}{r}}-\frac{L}{r^2}\right)Z\Bigg]\\
&=\frac{L\nu}{r^2}\left[\frac{\partial^2 Z}{\partial r^2}+(\alpha-\beta)\frac{\partial Z}{\partial r}-\left(\frac{2\alpha+2\beta}{r}+\alpha\beta+\frac{d\beta}{dr}+\frac{L}{r^2}\right)Z\right]. \qquad (26)
\end{aligned}
$$

# 3 Horizontal divergence of viscous force

For a vector field $\boldsymbol{A}$, we define the *radial* and *horizontal* divergences through

$$\nabla\cdot\boldsymbol{A}\coloneqq\nabla_r\cdot\boldsymbol{A}+\nabla_h\cdot\boldsymbol{A}=\left[\frac{\partial A_r}{\partial r}+\frac{2A_r}{r}\right]+\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta)+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right]. \qquad (27)$$

Clearly the operator $\nabla_h=\hat{\boldsymbol{e}}_\theta(1/r\sin\theta)(\partial/\partial\theta)\sin\theta+\hat{\boldsymbol{e}}_\phi(1/r\sin\theta)\partial/\partial\phi$ is insensitive to functions of radius. Furthermore, for a scalar $\psi$, it is easy to see that

$$\nabla_h\cdot\nabla\psi=-\frac{\mathscr{L}^2\psi}{r^2}=-\frac{L\psi}{r^2}, \qquad (28)$$

making it appropriate to call the total angular momentum operator the "horizontal Laplacian."

From (9), we calculate

$$\nabla_h\cdot\boldsymbol{f}=2\overline{\rho}\nu(\alpha+\beta)[\nabla_h\cdot(S_{r\gamma}\hat{\boldsymbol{e}}_\gamma)]+\overline{\rho}\nu\left[\nabla_h\cdot(\nabla^2\boldsymbol{v})+\frac{1}{3}\mathscr{L}^2(\nabla\cdot\boldsymbol{v})\right]. \qquad (29)$$

We compute

$$
\begin{aligned}
\nabla_h\cdot(S_{r\gamma}\hat{\boldsymbol{e}}_\gamma)&=\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta S_{r\theta})+\frac{1}{r\sin\theta}\frac{\partial S_{r\phi}}{\partial\phi}\\
&=\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\frac{1}{2}\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}+r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]\\
&=\frac{1}{2}\left\{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial v_r}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right\}\\
&=\frac{1}{2}\Bigg\{-\frac{\mathscr{L}^2 v_r}{r^2}+\frac{\partial}{\partial r}\underbrace{\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta v_\theta+\frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right]}_{\substack{\nabla_h\cdot\boldsymbol{v}\,=\,\nabla\cdot\boldsymbol{v}\,-\,\partial v_r/\partial r\,-\,2v_r/r\,=\\ -\beta v_r\,-\,\partial v_r/\partial r\,-\,2v_r/r}}\Bigg\}\\
&=\frac{1}{2}\left[-\frac{Lv_r}{r^2}-\beta\frac{\partial v_r}{\partial r}-\beta' v_r-\frac{\partial^2 v_r}{\partial r^2}-\frac{2}{r}\frac{\partial v_r}{\partial r}+\frac{2v_r}{r^2}\right]\\
&=-\frac{1}{2}\frac{\partial^2 v_r}{\partial r^2}-\left(\frac{\beta}{2}+\frac{1}{r}\right)\frac{\partial v_r}{\partial r}+\left(-\frac{L}{2r^2}-\beta'+\frac{1}{r^2}\right)v_r, \qquad (30)
\end{aligned}
$$

where $\beta' \coloneqq d\beta/dr$.

Next, we compute

$$
\begin{aligned}
\nabla_h \cdot \nabla^2 \boldsymbol{v} &= \nabla \cdot (\nabla^2 \boldsymbol{v}) - \nabla_r \cdot (\nabla^2 \boldsymbol{v}) = \nabla^2 (\nabla \cdot \boldsymbol{v}) - \nabla_r \cdot (\nabla^2 \boldsymbol{v}), \qquad (31)\\
\nabla^2 (\nabla \cdot \boldsymbol{v}) &= \left( \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{\mathscr{L}^2}{r^2} \right) (-\beta v_r) \qquad \text{using (11)}\\
&= -\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \left( \beta \frac{\partial v_r}{\partial r} + \beta' v_r \right) + \beta \frac{\mathscr{L}^2}{r^2} v_r\\
&= -\left( \beta \frac{\partial^2 v_r}{\partial r^2} + \beta' \frac{\partial v_r}{\partial r} + \beta' \frac{\partial v_r}{\partial r} + \beta'' v_r \right) - \frac{2}{r} \left( \beta \frac{\partial v_r}{\partial r} + \beta' v_r \right) + \beta \frac{L}{r^2} v_r\\
&= -\beta \frac{\partial^2 v_r}{\partial r^2} - \left( 2\beta' + \frac{2\beta}{r} \right) \frac{\partial v_r}{\partial r} + \left( -\beta'' - \frac{2\beta'}{r} + \frac{\beta L}{r^2} \right) v_r, \qquad (32)
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_r \cdot (\nabla^2 \boldsymbol{v}) &= \frac{1}{r^2} \frac{\partial}{\partial r} r^2 [(\nabla^2 \boldsymbol{v})_r]\\
&= \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \left[ \frac{\partial^2 v_r}{\partial r^2} + \frac{4}{r} \frac{\partial v_r}{\partial r} + \left( \frac{2}{r^2} + \frac{2\beta}{r} - \frac{L}{r^2} \right) v_r \right] \qquad \text{using (15)}\\
&= \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \frac{\partial^2 v_r}{\partial r^2} + 4r \frac{\partial v_r}{\partial r} + (2 + 2\beta r - L) v_r \right]\\
&= \frac{\partial^3 v_r}{\partial r^3} + \frac{2}{r} \frac{\partial^2 v_r}{\partial r^2} + \frac{4}{r} \frac{\partial^2 v_r}{\partial r^2} + \frac{4}{r^2} \frac{\partial v_r}{\partial r} + \frac{2 + 2\beta r - L}{r^2} \frac{\partial v_r}{\partial r} + \frac{2(\beta + \beta' r) v_r}{r^2}\\
&= \frac{\partial^3 v_r}{\partial r^3} + \frac{6}{r^2} \frac{\partial^2 v_r}{\partial r^2} + \left( \frac{6}{r^2} + \frac{2\beta}{r} - \frac{L}{r^2} \right) \frac{\partial v_r}{\partial r} + \left( \frac{2\beta}{r^2} + \frac{2\beta'}{r} \right) v_r. \qquad (33)
\end{aligned}
$$

Combining (32) and (33) into (31) then yields

$$
\begin{aligned}
\nabla_h \cdot (\nabla^2 \boldsymbol{v}) &= -\beta \frac{\partial^2 v_r}{\partial r^2} - \left( 2\beta' + \frac{2\beta}{r} \right) \frac{\partial v_r}{\partial r} + \left( -\beta'' - \frac{2\beta'}{r} + \frac{\beta L}{r^2} \right) v_r\\
&\quad - \frac{\partial^3 v_r}{\partial r^3} - \frac{6}{r^2} \frac{\partial^2 v_r}{\partial r^2} + \left( -\frac{6}{r^2} - \frac{2\beta}{r} + \frac{L}{r^2} \right) \frac{\partial v_r}{\partial r} - \left( \frac{2\beta}{r^2} + \frac{2\beta'}{r} \right) v_r\\
&= -\frac{\partial^3 v_r}{\partial r^3} - \left( \frac{6}{r} + \beta \right) \frac{\partial^2 v_r}{\partial r^2} + \left( -2\beta' - \frac{4\beta}{r} - \frac{6}{r^2} + \frac{L}{r^2} \right) \frac{\partial v_r}{\partial r}\\
&\quad + \left( -\beta'' - \frac{4\beta'}{r} + \frac{\beta L}{r^2} - \frac{2\beta}{r^2} \right) v_r \qquad (34)
\end{aligned}
$$

Also,

$$
-\frac{\mathscr{L}^2}{r^2} (\nabla \cdot \boldsymbol{v}) = -\frac{\mathscr{L}^2}{r^2} (-\beta v_r) = \frac{\beta L}{r^2} v_r. \qquad (35)
$$

Combining (30), (34), and (35) into (29) then gives

$$
\begin{aligned}
\nabla_h \cdot \boldsymbol{f} = 2\overline{\rho}\nu(\alpha + \beta) &\left[ -\frac{1}{2} \frac{\partial^2 v_r}{\partial r^2} - \left( \frac{\beta}{2} + \frac{1}{r} \right) \frac{\partial v_r}{\partial r} + \left( -\frac{L}{2r^2} - \beta' + \frac{1}{r^2} \right) v_r \right]\\
+ \overline{\rho}\nu &\left[ -\frac{\partial^3 v_r}{\partial r^3} - \left( \frac{6}{r} + \beta \right) \frac{\partial^2 v_r}{\partial r^2} + \left( -2\beta' - \frac{4\beta}{r} - \frac{6}{r^2} + \frac{L}{r^2} \right) \frac{\partial v_r}{\partial r} \right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \left( -\beta'' - \frac{4\beta'}{r} + \frac{\beta L}{r^2} - \frac{2\beta}{r^2} \right) v_r + \frac{1}{3}\frac{\beta L}{r^2} v_r \Bigg] \\
&= \nu \Bigg[ -\overline{\rho}\frac{\partial^3 v_r}{\partial r^3} - \left( \alpha + 2\beta + \frac{6}{r} \right) \overline{\rho}\frac{\partial^2 v_r}{\partial r^2} \\
&+ \left( -\alpha\beta - \beta^2 - \frac{2\alpha}{r} - \frac{2\beta}{r} - 2\beta' - \frac{4\beta}{r} - \frac{6}{r^2} + \frac{L}{r^2} \right) \overline{\rho}\frac{\partial v_r}{\partial r} \\
&+ \left( -\frac{\alpha L}{r^2} - \cancel{\frac{\beta L}{r^2}} - 2\alpha\beta' - 2\beta\beta' + \frac{2\alpha}{r^2} + \cancel{\frac{2\beta}{r^2}} - \beta'' - \frac{4\beta'}{r} + \cancel{\frac{\beta L}{r^2}} - \cancel{\frac{2\beta}{r^2}} + \frac{1}{3}\frac{\beta L}{r^2} \right) v_r \Bigg] \\
&= \nu \Bigg[ -\overline{\rho}\frac{\partial^3 v_r}{\partial r^3} - \left( \alpha + 2\beta + \frac{6}{r} \right) \overline{\rho}\frac{\partial^2 v_r}{\partial r^2} + \left( -\alpha\beta - \beta^2 - \frac{2\alpha}{r} - \frac{6\beta}{r} - 2\beta' - \frac{6}{r^2} + \frac{L}{r^2} \right) \overline{\rho}\frac{\partial v_r}{\partial r} \\
&+ \left( -\frac{\alpha L}{r^2} + \frac{1}{3}\frac{\beta L}{r^2} - 2\alpha\beta' - 2\beta\beta' + \frac{2\alpha}{r^2} - \beta'' - \frac{4\beta'}{r} \right) v_r \Bigg]. \qquad (36)
\end{aligned}
$$

Unhappily, we must derive (using (17))

$$
\begin{aligned}
\overline{\rho}\frac{\partial^3 v_r}{\partial r^3} &= \frac{\partial}{\partial r}\left( \overline{\rho}\frac{\partial^2 v_r}{\partial r^2} \right) - \beta\overline{\rho}\frac{\partial^2 v_r}{\partial r^2} \\
&= \frac{\partial}{\partial r}\left\{ \frac{L}{r^2}\left[ \frac{\partial^2 W}{\partial r^2} - \left( \frac{4}{r} + 2\beta \right)\frac{\partial W}{\partial r} + \left( \frac{6}{r^2} + \frac{4\beta}{r} + \beta^2 - \beta' \right) W \right] \right\} \\
&\quad - \beta\frac{L}{r^2}\left[ \frac{\partial^2 W}{\partial r^2} - \left( \frac{4}{r} + 2\beta \right)\frac{\partial W}{\partial r} + \left( \frac{6}{r^2} + \frac{4\beta}{r} + \beta^2 - \beta' \right) W \right] \\
&= \frac{L}{r^2}\Bigg[ \frac{\partial^3 W}{\partial r^3} - \left( \frac{4}{r} + 2\beta \right)\frac{\partial^2 W}{\partial r^2} + \left( \frac{4}{r^2} - 2\beta' \right)\frac{\partial W}{\partial r} + \left( \frac{6}{r^2} + \frac{4\beta}{r} + \beta^2 - \beta' \right)\frac{\partial W}{\partial r} \\
&\quad + \left( -\frac{12}{r^3} - \frac{4\beta}{r^2} + \frac{4\beta'}{r} + 2\beta\beta' - \beta'' \right) W \Bigg] \\
&\quad - \frac{2L}{r^3}\frac{\partial^2 W}{\partial r^2} + \frac{2L}{r^3}\left( \frac{4}{r} + 2\beta \right)\frac{\partial W}{\partial r} + \frac{2L}{r^3}\left( -\frac{6}{r^2} - \frac{4\beta}{r} - \beta^2 + \beta' \right) W \\
&\quad + \beta\frac{L}{r^2}\left[ -\frac{\partial^2 W}{\partial r^2} + \left( \frac{4}{r} + 2\beta \right)\frac{\partial W}{\partial r} + \left( -\frac{6}{r^2} - \frac{4\beta}{r} - \beta^2 + \beta' \right) W \right] \\
&= \frac{L}{r^2}\Bigg[ \frac{\partial^3 W}{\partial r^3} + \left( -\frac{4}{r} - 2\beta - \frac{2}{r} - \beta \right)\frac{\partial^2 W}{\partial r^2} \\
&\quad + \left( \frac{4}{r^2} - 2\beta' + \frac{6}{r^2} + \frac{4\beta}{r} + \beta^2 - \beta' + \frac{8}{r^2} + \frac{4\beta}{r} + \frac{4\beta}{r} + 2\beta^2 \right)\frac{\partial W}{\partial r} \\
&\quad + \left( -\frac{12}{r^3} - \frac{4\beta}{r^2} + \frac{4\beta'}{r} + 2\beta\beta' - \beta'' - \frac{12}{r^3} - \frac{8\beta}{r^2} - \frac{2\beta^2}{r} + \frac{2\beta'}{r} - \frac{6\beta}{r^2} - \frac{4\beta^2}{r} - \beta^3 + \beta\beta' \right) W \Bigg] \\
&= \frac{L}{r^2}\Bigg[ \frac{\partial^3 W}{\partial r^3} - \left( \frac{6}{r} + 3\beta \right)\frac{\partial^2 W}{\partial r^2} + \left( \frac{18}{r^2} - 3\beta' + \frac{12\beta}{r} + 3\beta^2 \right)\frac{\partial W}{\partial r} \\
&\quad + \left( -\frac{24}{r^3} - \frac{18\beta}{r^2} + \frac{6\beta'}{r} + 3\beta\beta' - \beta'' - \frac{6\beta^2}{r} - \beta^3 \right) W \Bigg] \qquad (37)
\end{aligned}
$$

Finally, we plug (4), (13), (17), and (37) into (36) and compute

$$
\begin{aligned}
\nabla_h \cdot \boldsymbol{f} = \frac{L\nu}{r^2}\Bigg\{ & -\Bigg[\frac{\partial^3 W}{\partial r^3} - \left(\frac{6}{r} + 3\beta\right)\frac{\partial^2 W}{\partial r^2} + \left(\frac{18}{r^2} - 3\beta' + \frac{12\beta}{r} + 3\beta^2\right)\frac{\partial W}{\partial r} \\
& + \left(-\frac{24}{r^3} - \frac{18\beta}{r^2} + \frac{6\beta'}{r} + 3\beta\beta' - \beta'' - \frac{6\beta^2}{r} - \beta^3\right)W\Bigg] \\
& - \left(\alpha + 2\beta + \frac{6}{r}\right)\left[\frac{\partial^2 W}{\partial r^2} - \left(\frac{4}{r} + 2\beta\right)\frac{\partial W}{\partial r} + \left(\frac{6}{r^2} + \frac{4\beta}{r} + \beta^2 - \beta'\right)W\right] \\
& + \left(-\alpha\beta - \beta^2 - \frac{2\alpha}{r} - \frac{6\beta}{r} - 2\beta' - \frac{6}{r^2} + \frac{L}{r^2}\right)\left[\frac{\partial W}{\partial r} - \left(\frac{2}{r} + \beta\right)W\right] \\
& + \left(-\frac{\alpha L}{r^2} + \frac{1}{3}\frac{\beta L}{r^2} - 2\alpha\beta' - 2\beta\beta' + \frac{2\alpha}{r^2} - \beta'' - \frac{4\beta'}{r}\right)W\Bigg]\Bigg\}. \qquad (38)
\end{aligned}
$$