Writing the Induction Term in Spherical Coordinates

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1 The Induction Term

In MHD, the evolution of magnetic field $\boldsymbol{B}$ in a fluid with local velocity $\boldsymbol{u}$ is described by

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times [\boldsymbol{u} \times \boldsymbol{B} - \eta \nabla \times \boldsymbol{B}], \tag{1}$$

$$\text{and} \qquad \nabla \cdot \boldsymbol{B} = 0, \tag{2}$$

which is the induction equation and "no magnetic monopoles" condition, respectively. Here we ignore the diffusive piece (dependent on magnetic diffusivity η) and focus on the induction term,

$$\boldsymbol{I} = \nabla \times [\boldsymbol{u} \times \boldsymbol{B}]. \tag{3}$$

In this document, we show the various interpretations of $\boldsymbol{I}$, focusing on how it appears in spherical coordinates.

2 Differential Geoemetry Perspective

We first rewrite Equation (1) (ignoring the diffusive bit, as wel will for the rest of this document) as

$$\frac{D\boldsymbol{B}}{Dt} = \boldsymbol{B} \cdot \nabla \boldsymbol{u} - \boldsymbol{B} \nabla \cdot \boldsymbol{u}, \tag{4}$$

where we have expanded the product $\nabla \times [\boldsymbol{u} \times \boldsymbol{B}]$, made use of $\nabla \cdot \boldsymbol{B} = 0$, and defined the material derivative (Lagrangian derivative following a fluid parcel) as $D/Dt \equiv \partial/\partial t + \boldsymbol{u} \cdot \nabla$. We examine Equation (4) in the Frenet-Serret frame following magnetic field lines (orthogonal coordinates $\hat{\boldsymbol{t}}$, $\hat{\boldsymbol{n}}$, and $\hat{\boldsymbol{b}} = \hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}}$ as shown schematically in Figure 1).

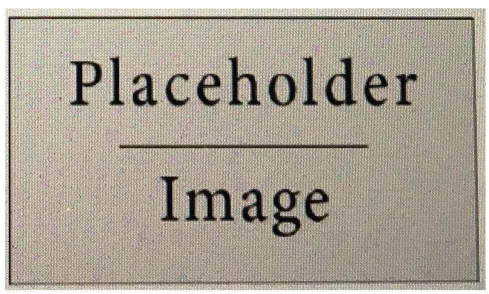

Figure 1: Temporally and spherically averaged production of poloidal magnetic energy in case M's radiative interior. The production by induction iThe y-axis is scaled logarithmically for positive and negative values away from zero. Values close to zero, in the range [-0.01, 0.01] marked by the dashed black lines, are scaled linearly.