Induction term in spherical coordinates

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1 The problem with the traditional induction terms

We consider the ideal (resistance-free) magnetohydrodynamic (MHD) induction equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) := \boldsymbol{I}, \tag{1}$$

$$= \boldsymbol{B} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{B} - (\nabla \cdot \boldsymbol{u})\boldsymbol{B}, \tag{2}$$

where $\boldsymbol{B}$ and $\boldsymbol{u}$ are the vector magnetic and velocity fields (respectively) and $\boldsymbol{I}$ is the vector induction. The three terms on the right-hand-side of Equation (2) are often interpreted as "shear," "advection," and "compression," respectively. However, this interpretation is problematic in general for two reasons:

1. The so-called shear and compression terms contain sub-terms that cancel; in particular, only velocity motions *perpendicular* to magnetic-field lines can shear or compress.

2. Solid-body rotation (which is a non-shearing motion that simply rotates the whole field configuration) shows up in the so-called shear and advection terms in a strange way.

Finally, even after these issues have been addressed, resolving the final terms into a particular curvilinear system (e.g., spherical coordinates) seems to present a major headache. Our goal here is to explain fully how these problems emerge and propose a tentative solution for the case of spherical coordinates.

2 Perpendicular shear and compression

To see how Problem 1 arises, we decompose the velocity field into components parallel and perpendicular to the local direction of $\boldsymbol{B}$:

$$\boldsymbol{u} := u_\parallel \hat{\boldsymbol{e}}_\parallel + \boldsymbol{u}_\perp \tag{3}$$

Obviously $\boldsymbol{B} = Bx_\parallel$, where $B = |\boldsymbol{B}|$. We denote the Cartesian distance along $\boldsymbol{B}$ by $x_\parallel$. We also decompose $\boldsymbol{u}$ into its parallel and perpendicular components:

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_\parallel}{\partial x_\parallel} + \nabla_\perp \cdot \boldsymbol{u}_\perp \tag{4}$$

We then calculate

$$
\begin{aligned}
\boldsymbol{B} \cdot \nabla \boldsymbol{u} - (\nabla \cdot \boldsymbol{u})\boldsymbol{B} &= B\frac{\partial}{\partial x_\parallel}(u_\parallel \hat{\boldsymbol{e}}_\parallel + \boldsymbol{u}_\perp) - \left(\frac{\partial u_\parallel}{\partial x_\parallel} + \nabla_\perp \cdot \boldsymbol{u}_\perp\right)B\hat{\boldsymbol{e}}_\parallel \\
&= B\frac{\cancel{\partial u_\parallel}}{\partial x_\parallel}\hat{\boldsymbol{e}}_\parallel + B\frac{\partial \boldsymbol{u}_\perp}{\partial x_\parallel} - \frac{\cancel{\partial u_\parallel}}{\partial x_\parallel}B\hat{\boldsymbol{e}}_\parallel - (\nabla_\perp \cdot \boldsymbol{u}_\perp)B\hat{\boldsymbol{e}}_\parallel \\
&= \boldsymbol{B} \cdot \nabla \boldsymbol{u}_\perp - (\nabla_\perp \cdot \boldsymbol{u}_\perp)\boldsymbol{B}.
\end{aligned}
\tag{5}
$$

Thus, only motions perpendicular to the local field line (i.e., $\boldsymbol{u}_\perp$) can shear or compress $\boldsymbol{B}$.

3   Rigid rotation

To see how Problem 2 arises, we consider a velocity field due to rigid rotation at constant angular velocity Ω about the z-axis in a cylindrical coordinate system:

$$
\begin{aligned}
\boldsymbol{\Omega} &= \Omega\hat{\boldsymbol{e}}_z = \text{constant} \tag{6a} \\
\boldsymbol{u} &= \boldsymbol{\Omega} \times \boldsymbol{r} = \Omega\lambda\hat{\boldsymbol{e}}_\phi \tag{6b}
\end{aligned}
$$

Here, λ is the cylindrical radius, ϕ the longitude, and z the axial coordinate. In general, $\hat{\boldsymbol{e}}_{(\ldots)}$ denotes a unit vector in the direction of its subscript. We calculate:

$$
\begin{aligned}
\nabla \cdot \boldsymbol{u} &= \nabla \cdot (\boldsymbol{\Omega} \times \boldsymbol{r}) \\
&= \boldsymbol{\Omega} \cdot \nabla \times \boldsymbol{r} - \boldsymbol{r} \cdot \nabla \times \boldsymbol{\Omega} \\
&= 0 \quad \text{no compression for rigid rotation (obviously).} \tag{7}
\end{aligned}
$$

Then:

$$
\begin{aligned}
\boldsymbol{B} \cdot \nabla \boldsymbol{u} &= (\boldsymbol{B} \cdot \nabla)(\boldsymbol{\Omega} \times \boldsymbol{r}) \\
&= \boldsymbol{\Omega} \times [(\boldsymbol{B} \cdot \nabla)(\boldsymbol{r})] \tag{8} \\
&= \boldsymbol{\Omega} \times \boldsymbol{B} \quad \text{"shear" for rigid rotation.} \tag{9}
\end{aligned}
$$

Finally:

$$
\begin{aligned}
-\boldsymbol{u} \cdot \nabla \boldsymbol{B} &= -\Omega\lambda\hat{\boldsymbol{e}}_\phi \cdot \nabla \boldsymbol{B} \\
&= -\Omega\frac{\partial}{\partial\phi}(B_\lambda\hat{\boldsymbol{e}}_\lambda + B_\phi\hat{\boldsymbol{e}}_\phi + B_z\hat{\boldsymbol{e}}_z) \\
&= -\Omega\sum_\alpha\left(\frac{\partial B_\alpha}{\partial\phi}\right)\hat{\boldsymbol{e}}_\alpha - \Omega\sum_\alpha B_\alpha\frac{\partial\hat{\boldsymbol{e}}_\alpha}{\partial\phi},
\end{aligned}
$$

where the index α runs over the three cylindrical coordinates. Note that in the cylindrical coordinate system (or indeed any coordinate system with an axis of rotational symmetry), $\partial\hat{\boldsymbol{e}}_\alpha/\partial\phi = \hat{\boldsymbol{e}}_z \times \hat{\boldsymbol{e}}_\alpha$ for each α. Thus,

$$
\begin{aligned}
-\Omega\sum_\alpha B_\alpha\frac{\partial\hat{\boldsymbol{e}}_\alpha}{\partial\phi} &= -\Omega\sum_\alpha B_\alpha\hat{\boldsymbol{e}}_z \times \hat{\boldsymbol{e}}_\alpha \\
&= -\Omega\hat{\boldsymbol{e}}_z \times \sum_\alpha B_\alpha\hat{\boldsymbol{e}}_\alpha \\
&= -\boldsymbol{\Omega} \times \boldsymbol{B}
\end{aligned}
$$

and so

$$-\boldsymbol{u} \cdot \nabla \boldsymbol{B} = -\Omega \sum_\alpha \left(\frac{\partial B_\alpha}{\partial \phi}\right) \hat{\boldsymbol{e}}_\alpha - \boldsymbol{\Omega} \times \boldsymbol{B} \qquad \text{``advection'' for rigid rotation.} \qquad (10)$$

Mathematically, in any coordinate system with a z-axis of rotational symmetry, the action of rigid rotation is as follows:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times [(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{B}]$$

$$= \sum_\alpha \left(-\Omega \frac{\partial B_\alpha}{\partial \phi}\right) \hat{\boldsymbol{e}}_\alpha \qquad (11a)$$

$$\text{or} \qquad \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}\right) B_\alpha = 0 \qquad \text{for each } \alpha. \qquad (11b)$$

If you think about it, this makes sense: All the rigid rotation does is rotate the whole field configuration around the z-axis at the rate Ω. If you decide to also rotate at Ω (so your personal Eulerian time derivative is $\partial/\partial t + \Omega \partial/\partial \phi$), then each component of the magnetic-field configuration should remain the same in your frame. Note that rotation does *not* advect the vector magnetic field (like the term $-\boldsymbol{u} \cdot \nabla \boldsymbol{B}$ viewed on its own would suggest), but rather advects the field *components* (as if they were scalars) in any coordinate system with an axis of rotational symmetry.

4 Solution for spherical coordinates

Resolving these issues fully for the spherical coordinate system seems complicated and I am not fully sure how to do it! In particular (for "full" resolution) we should, separately at each point (r,θ,ϕ):

1. Form a local Cartesian coordinate system, say (x_1, x_2, x_3), whose origin lies at the point (r, θ, ϕ). At the origin, $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ will coincide with $(\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta, \hat{\boldsymbol{e}}_\phi)$. But slightly away from the origin, $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ will stay fixed while $(\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta, \hat{\boldsymbol{e}}_\phi)$ curve away.

2. Calculate the velocity-gradient tensor: $\partial u_1/\partial x_1, \partial u_2/\partial x_1, \partial u_3/\partial x_1$, etc. While calculating derivatives, be careful to differentiate along the Cartesian coordinates, *not* the curvilinear ones or along the actual $\boldsymbol{B}$-line). Express the final tensor components in spherical coordinates.

3. Rotate "into $\boldsymbol{B}$" to form a new primed coordinate system (such that $\hat{\boldsymbol{e}}'_1$ points along $\boldsymbol{B}$), calculate the $\partial u'_j/\partial x'_i$, and thus form $\partial u_\parallel/\partial x_\parallel$ and $\partial \boldsymbol{u}_\perp/\partial x_\parallel$ (note that $\nabla_\perp \cdot \boldsymbol{u}_\perp = \nabla \cdot \boldsymbol{u} - \partial u_\parallel/\partial x_\parallel$). Note that the direction cosines $\hat{\boldsymbol{e}}'_1$ makes with $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ are simply $B_r/|\boldsymbol{B}|, B_\theta/|\boldsymbol{B}|, B_\phi/|\boldsymbol{B}|$, respectively. Note that both unit vectors and vector components transform like $x'_1 = \sum_{j=1}^3 R_{1j} x_j$, where R_{1j} is the jth direction cosine.

4. Subtract the part of $\boldsymbol{u}_\perp$ corresponding to solid-body rotation and put it in the form of Equation (11a).

3

5. Exult, because I haven't been able to do this!

Instead of addressing the full problem as just described, I have brute-forced my way into a quasi-solution, canceling obvious terms and expressing what seem to be "perpendicular shear with no solid-body rotation," "perpendicular compression," and advection.

As a shorthand, we write the following for the part of each vector perpendicular to the local coordinate lines:

$$\boldsymbol{u}_{\mathrm{hor}} := u_\theta \hat{\boldsymbol{e}}_\theta + u_\phi \hat{\boldsymbol{e}}_\phi \qquad \text{horizontal part } (\perp \ \hat{\boldsymbol{e}}_r), \tag{12a}$$

$$\boldsymbol{u}_{\mathrm{tor}} := u_r \hat{\boldsymbol{e}}_r + u_\phi \hat{\boldsymbol{e}}_\phi \qquad \text{toroidal part } (\perp \ \hat{\boldsymbol{e}}_\theta), \tag{12b}$$

$$\text{and} \qquad \boldsymbol{u}_{\mathrm{pol}} := u_r \hat{\boldsymbol{e}}_r + u_\theta \hat{\boldsymbol{e}}_\theta \qquad \text{poloidal part } (\perp \ \hat{\boldsymbol{e}}_\phi) \tag{12c}$$

...and similarly for $\boldsymbol{B}$.

We now write:

$$
\begin{aligned}
I_r &= \cancel{B_r \frac{\partial u_r}{\partial r}} + \frac{B_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{B_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \cancel{\frac{B_\theta u_\theta + B_\phi u_\phi}{r}} \\
&\quad - \left[u_r \frac{\partial B_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial B_r}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial B_r}{\partial \phi} - \cancel{\frac{u_\theta B_\theta + u_\phi B_\phi}{r}} \right] \\
&\quad - B_r \left[\cancel{\frac{\partial u_r}{\partial r}} + \frac{\partial 2 u_r}{\partial r} + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + \cot\theta\, u_\theta \right) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} \right] \\
&= \underbrace{\left(\frac{B_\theta}{r}\frac{\partial}{\partial \theta} + \frac{B_\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \right) u_r}_{\boldsymbol{B}_{\mathrm{hor}}\cdot\nabla} - B_r \underbrace{\left[\frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + \cot\theta\, u_\theta \right) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} \right]}_{\nabla\cdot\boldsymbol{u}_{\mathrm{hor}}} \\
&\quad - \boldsymbol{u}\cdot\nabla B_r - \frac{2 B_r u_r}{r}
\end{aligned}
$$

or

$$I_r = \underbrace{\boldsymbol{B}_{\mathrm{hor}}\cdot\nabla u_r}_{\text{perpendicular shear}} - \underbrace{\left(\nabla\cdot\boldsymbol{u}_{\mathrm{hor}} + \frac{2 u_r}{r} \right) B_r}_{\text{perpendicular compression}} \underbrace{- \boldsymbol{u}\cdot\nabla B_r}_{\text{advection}} \tag{13}$$

Our justification for including the term $-2 B_r u_r / r$ in the compression will be given in the next section.

Now that we know a bit more how things fall into place, we calculate

$$
\begin{aligned}
I_\theta &= \cancel{\frac{B_\theta}{r}\frac{\partial u_\theta}{\partial \theta}} + \boldsymbol{B}_{\mathrm{tor}}\cdot\nabla u_\theta + \frac{B_\theta u_r}{r} - \cancel{\frac{\cot\theta\, B_\phi u_\phi}{r}} \\
&\quad - \boldsymbol{u}\cdot\nabla B_\theta - \frac{u_\theta B_r}{r} + \cancel{\frac{\cot\theta\, u_\phi B_\phi}{r}} \\
&\quad - B_\theta \left(\cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}} + \frac{\cot\theta\, u_\theta}{r} + \nabla\cdot\boldsymbol{u}_{\mathrm{tor}} \right)
\end{aligned}
$$

or

$$I_\theta = \underbrace{r\boldsymbol{B}_{\mathrm{tor}}\cdot\nabla\left(\frac{u_\theta}{r} \right)}_{\text{perpendicular shear}} - \underbrace{\left(\nabla\cdot\boldsymbol{u}_{\mathrm{tor}} + \frac{\cot\theta\, u_\theta}{r} - \frac{u_r}{r} \right) B_r}_{\text{perpendicular compression}} \underbrace{- \boldsymbol{u}\cdot\nabla B_r}_{\text{advection}} \tag{14}$$

4