

Induction term in spherical coordinates

Loren Matilsky

February 2, 2023

1 The problem with the traditional induction terms

We consider the ideal (resistance-free) magnetohydrodynamic (MHD) induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}), \quad (1)$$

$$= \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{B} - (\nabla \cdot \mathbf{u}) \mathbf{B}, \quad (2)$$

where \mathbf{B} and \mathbf{u} are the vector magnetic and velocity fields, respectively. The three terms on the right-hand-side of Equation (2) are often interpreted as “shear,” “advection,” and “compression,” respectively. However, this interpretation is problematic in general for two reasons:

1. The so-called shear and compression terms contain sub-terms that cancel; in particular, only velocity motions *perpendicular* to magnetic-field lines can shear or compress.
2. Solid-body rotation (which is a non-shearing motion that simply rotates the whole field configuration) shows up in the so-called shear and advection terms in a strange way.

When resolving the induction equation into a particular curvilinear system (e.g., spherical coordinates), another problem arises:

3. Large curvature terms appear, which are difficult to interpret and occasionally cancel.

Our goal here is to explain fully how these problems emerge and propose a solution for the case of spherical coordinates.

2 Perpendicular shear and compression

To see how Problem 1 arises, we decompose the velocity field into components parallel and perpendicular to the local direction of \mathbf{B} :

$$\mathbf{u} := u_{\parallel} \hat{\mathbf{e}}_{\parallel} + \mathbf{u}_{\perp} \quad (3)$$

Obviously $\mathbf{B} = Bx_{\parallel}$, where $B = |\mathbf{B}|$. We denote the Cartesian distance along \mathbf{B} by x_{\parallel} . We also decompose \mathbf{u} into its parallel and perpendicular components:

$$\nabla \cdot \mathbf{u} = \frac{\partial u_{\parallel}}{\partial x_{\parallel}} + \nabla_{\perp} \cdot \mathbf{u}_{\perp} \quad (4)$$

We then calculate

$$\begin{aligned} \mathbf{B} \cdot \nabla \mathbf{u} - (\nabla \cdot \mathbf{u}) \mathbf{B} &= B \frac{\partial}{\partial x_{\parallel}} (u_{\parallel} \hat{\mathbf{e}}_{\parallel} + \mathbf{u}_{\perp}) - \left(\frac{\partial u_{\parallel}}{\partial x_{\parallel}} + \nabla_{\perp} \cdot \mathbf{u}_{\perp} \right) B \hat{\mathbf{e}}_{\parallel} \\ &= B \cancel{\frac{\partial u_{\parallel}}{\partial x_{\parallel}} \hat{\mathbf{e}}_{\parallel}} + B \frac{\partial \mathbf{u}_{\perp}}{\partial x_{\parallel}} - \cancel{\frac{\partial u_{\parallel}}{\partial x_{\parallel}} B \hat{\mathbf{e}}_{\parallel}} - (\nabla_{\perp} \cdot \mathbf{u}_{\perp}) B \hat{\mathbf{e}}_{\parallel} \\ &= \mathbf{B} \cdot \nabla \mathbf{u}_{\perp} - (\nabla_{\perp} \cdot \mathbf{u}_{\perp}) \mathbf{B}. \end{aligned} \quad (5)$$

Thus, only motions perpendicular to the local field line (i.e., \mathbf{u}_{\perp}) can shear or compress \mathbf{B} .

To see how Problem 2 arises, we consider a velocity field due to rigid rotation at constant angular velocity Ω about the z -axis in a cylindrical coordinate system:

$$\Omega = \Omega \hat{\mathbf{e}}_z = \text{constant} \quad (6a)$$

$$\mathbf{u} = \Omega \times \mathbf{r} = \Omega \lambda \hat{\mathbf{e}}_{\phi} \quad (6b)$$

Here, λ is the cylindrical radius, ϕ the longitude, and z the axial coordinate. In general, $\hat{\mathbf{e}}_{(\dots)}$ denotes a unit vector in the direction of its subscript. We calculate:

$$\begin{aligned} \nabla \cdot \mathbf{u} &= \nabla \cdot (\Omega \times \mathbf{r}) \\ &= \Omega \cdot \nabla \times \mathbf{r} - \mathbf{r} \cdot \nabla \times \Omega \\ &= 0 \quad \text{no compression for rigid rotation (obviously)}. \end{aligned} \quad (7)$$

Then:

$$\begin{aligned} \mathbf{B} \cdot \nabla \mathbf{u} &= (\mathbf{B} \cdot \nabla)(\Omega \times \mathbf{r}) \\ &= \Omega \times [(\mathbf{B} \cdot \nabla)(\mathbf{r})] \end{aligned} \quad (8)$$

$$= \Omega \times \mathbf{B} \quad \text{“shear” for rigid rotation.} \quad (9)$$

Finally:

$$\begin{aligned} -\mathbf{u} \cdot \nabla \mathbf{B} &= -\Omega \lambda \hat{\mathbf{e}}_{\phi} \cdot \nabla \mathbf{B} \\ &= -\Omega \frac{\partial}{\partial \phi} (B_{\lambda} \hat{\mathbf{e}}_{\lambda} + B_{\phi} \hat{\mathbf{e}}_{\phi} + B_z \hat{\mathbf{e}}_z) \\ &= -\Omega \sum_{\alpha} \left(\frac{\partial B_{\alpha}}{\partial \phi} \right) \hat{\mathbf{e}}_{\alpha} - \Omega \sum_{\alpha} B_{\alpha} \frac{\partial \hat{\mathbf{e}}_{\alpha}}{\partial \phi}, \end{aligned}$$

where the index α runs over the three cylindrical coordinates. Note that in the cylindrical coordinate system (or indeed any coordinate system with an axis of rotational symmetry),

$\partial \hat{\mathbf{e}}_\alpha / \partial \phi = \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_\alpha$ for each α . Thus,

$$\begin{aligned} -\Omega \sum_{\alpha} B_{\alpha} \frac{\partial \hat{\mathbf{e}}_{\alpha}}{\partial \phi} &= -\Omega \sum_{\alpha} B_{\alpha} \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_{\alpha} \\ &= -\Omega \hat{\mathbf{e}}_z \times \sum_{\alpha} B_{\alpha} \hat{\mathbf{e}}_{\alpha} \\ &= -\mathbf{\Omega} \times \mathbf{B} \end{aligned}$$

and so

$$-\mathbf{u} \cdot \nabla \mathbf{B} = -\Omega \sum_{\alpha} \left(\frac{\partial B_{\alpha}}{\partial \phi} \right) \hat{\mathbf{e}}_{\alpha} - \mathbf{\Omega} \times \mathbf{B} \quad \text{“advection” for rigid rotation.} \quad (10)$$

Mathematically, in any coordinate system with an axis of rotational symmetry about z , the action of rigid rotation is as follows:

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times [(\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{B}] \\ &= \sum_{\alpha} \left(-\Omega \frac{\partial B_{\alpha}}{\partial \phi} \right) \hat{\mathbf{e}}_{\alpha} \\ \text{or} \quad \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) B_{\alpha} &= 0 \quad \text{for each } \alpha. \end{aligned} \quad (11)$$

If you think about it, this makes sense: All the rigid rotation does is rotate the whole field configuration around the z -axis at the rate Ω . If you decide to also rotate at Ω (so your personal Eulerian time derivative is $\partial/\partial t + \Omega \partial/\partial \phi$), then each component of the magnetic-field configuration should remain the same in your frame.