An energy-conserving anelastic approximation for strongly stably-stratified fluids

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Abstract

When acoustic oscillations are believed to decouple from the dynamics of a fluid, it is useful to employ simplifying approximations to the equations of motion. The two most common of these, often used to treat convection problems, are the Boussinesq approximation (when the background density does not significantly vary across the fluid layer) and the anelastic approximation (when the background density does vary significantly). There are many distinct forms of the anelastic approximation in the literature, and it has often been remarked that they do not properly conserve energy when the fluid is stable to convection. Here we show that the anelastic equations derived by Gough (1969) in fact do conserve energy for arbitrary motions of the fluid and arbitrary equations of state, even in the presence of strong subadiabatic or superadiabatic background temperature gradients. We rewrite these equations in terms of the background entropy gradient and perturbed pressure and entropy. We further reduce the equation set to a form appropriate to fixed-in-time reference states (whereas Gough (1969) considered horizontally averaged background states), dubbing the final equation set the Anelastic Energy-conserving Generalized Gough (AnEGG) approximation. The key properties the AnEGG approximation that allows it to conserve energy are (1) the absence of the Lantz-Braginsky-Roberts (LBR) approximation in the momentum equation and (2) the inclusion of a historically neglected term in the internal energy equation that properly accounts for the conversion of internal energy to kinetic energy via buoyancy work.

We also show that the total conserved energy is partitioned into two types only: internal and kinetic. Changes in potential energy, as well as other changes in energy due to compressive effects, are entirely absent from the anelastic energy budget. This is shown to be a direct consequence of Gough (1969)'s assumption of zero mean mass flux in the fully compressible flow that the anelastic equations are meant to approximate. We argue that this "no mean mass flux" assumption for the fully compressible flow is fundamental for the energetic consistency of the anelastic equations (whatever the value of the background entropy gradient). We speculate that the scaling analysis of Gough (1969), which implicitly assumed a single typical value of the background entropy gradient, may be valid even for convective overshoot, where the entropy gradient changes from slightly unstable in the convecting region to stable (sometimes strongly so) in the overshoot region. In particular, the smallness of the relative thermodynamic perturbations, which is a key assumption for the anelastic approximation, is

ensured, provided that the buoyancy frequency is significantly less than the acoustic cutoff frequency. Finally, we argue that the thermodynamic linearization of the fully compressible equations that accurately describes internal gravity waves (which assumes an advective turnover time much longer than the inverse buoyancy frequency) is fundamentally different than the linearization inherent to the anelastic approximation (which assumes an advective turnover time on par with the inverse buoyancy frequency). This is the reason that the AnEGG approximation (or other anelastic approximations when the background state is close to adiabatic) conserve energy for general fluid motions, but not internal gravity waves.

1   Introduction

The anelastic equations consist of an approximation to the continuity and momentum equations, originally derived by assuming small thermal perturbations about a nearly adiabatically stratified hydrostatic reference atmosphere (Batchelor, 1953; Charney & Ogura, 1960). The thermodynamics of the problem thus become "linear," in the sense that products of thermodynamic variables reduce to linear expressions in the first-order perturbations. The two key consequences of linearized thermodynamics are divergenceless mass flux and the first-order buoyancy force (associated with the first-order perturbed density and pressure) being the primary driver of the flow. Ogura & Phillips (1962) formalized the approximation by expanding the fluid equations in a small parameter ϵ, representing the relative variation of potential temperature across the fluid layer, and hence the relative magnitude of the thermal perturbations. They recovered the equations of Batchelor (1953) and Charney & Ogura (1960) and showed an assumption about the *time scale* of the motion was necessary, in addition to the assumption of small thermal perturbations. Namely, the dynamical time scale of the buoyantly driven flows must about $\epsilon^{-1/2}$ times *larger* than the sound crossing time of the region. Sound waves, which imply rapid temporal variations on the order of the sound crossing time, are thus eliminated from the anelastic equations, making them ideal for numerical integration, where large time steps are required to capture significant evolution of the system.

In the original asymptotic expansion of Ogura & Phillips (1962), the internal energy equation was replaced by a heat (or entropy) equation for the evolution of potential temperature, *before* nondimensionalizing the equations. The approach of considering an entropy equation instead of the internal energy equation seems to be repeated in all modern implementations of the anelastic approximation that we are aware of (e.g., Gilman & Glatzmaier 1981; Lipps & Hemler 1982; Glatzmaier 1984; Lantz 1992; Braginsky & Roberts 1995; Lantz & Fan 1999; Clune et al. 1999; Rogers & Glatzmaier 2005; Brown et al. 2012; Vasil et al. 2013; Wilczyński et al. 2022). The resulting energy equation is also used in all anelastic numerical codes that we are aware of, e.g., the ASH code (Brun et al., 2004), the MagIC code (Gastine & Wicht, 2012), the Rayleigh code (Featherstone & Hindman, 2016; Featherstone et al., 2023), the EULAG code (Smolarkiewicz & Prusa, 2004), and the Dedalus code (Burns et al., 2020; Brown et al., 2020).

While nondimensionalizing the entropy equation instead of the internal energy equation may at first appear to be an arbitrary (and harmless) choice, we show in the

present work that it has led to an asymptotically inconsistent set of equations when strong background entropy gradients exist. Gough (1969), by contrast, took a different approach than Ogura & Phillips (1962) and performed a formal asymptotic expansion in ϵ after nondimensionalizing the internal energy equation and assuming zero mean mass flux for the fully compressible flow. We show that this equation set, which we write in a slighlty different form and dub the "Energy-conserving Generalized Gough" (AnEGG) anelastic equations, conserves energy for arbitrary fluid motions and for all hydrostatic background states (whether stably or unstably stratified). The reason that nondimensionalizing the internal energy equation directly is so important is that it naturally leads to the retention of a term that allows for the proper conversion of internal energy into kinetic energy via the buoyancy work. The absence of this term is the reason that current codes do not conserve energy when (e.g.) the background is strongly stably stratified.

In the first part of this paper (which leads to writing down the AnEGG approximation; Sections 2–6), we essentially perform an accounting exercise for energy. We show how the conservation of energy has been violated by the anelastic equations currently in use and isolate the missing term in the internal energy equation that balances the anelastic energy budget.[1] In the second part of this paper , we argue that this new term epitomizes two related phenomena that are fundamental to anelasticity. First, for the anelastic approximation to be energetically consistent, it must be assumed that there is zero mean mass flux in the unapproximated fully compressible motion at all times. Second, because there is zero mean mass flux, there is never any build-up of mass at any vertical level in the compressible system. One consequence of this is that there can be *no exchange between potential and kinetic energy energy in either the fully compressible or anelastic system*. In the anelastic approximation, this means that the potential energy disappears entirely from the energy budget. Instead, the internal energy of a fluid parcel takes on two roles: it acts as an entropy (accounting for the fluid parcel's absorption of heat) and it acts *sort of* as a potential energy (allowing the parcel to convert its entropy into kinetic energy via falling—i.e., via buoyancy work).

In Section 2, we describe conservation of total energy in the fully compressible equations. In Section 3, we describe Gough (1969)'s original equations (slightly modified here to allow slow time-variation in the mean horizontally averaged background state) and show that these conserve total energy for arbitrary equations of state and all magnitudes of the background entropy gradient. In Section 4, we write Gough (1969)'s in terms of the perturbed pressure and entropy, which has the effect of relating the new terms to the background entropy gradient. In Section 5, we demonstrate that (perhaps intuitively obviously) the anelastic equations derived by expanding about a time-varying mean state are formally equivalent to those derived by expanding about a fixed reference state, provided the total evolution of the horizontal mean away from the reference state is sufficiently small. In Section (6), we write down the final AnEGG equation set, showing once again that total energy is conserved under all circumstances.

In Section 7, we discuss the dual role of internal energy in the anelastic system and the related absence of potential energy. In Section 8, we discuss the anelastic second law of thermodynamics and show that in general it is violated because mass is not conserved. Finally, we do not claim that the mere achievement of general energy

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[1]It should be remarked that this term is not a fudge factor, but properly *should* be in the internal energy equation after the formal asymptotic derivation of the approximation.

conservation makes it wise to apply the AnEGG equation set in all problems including stable stratification! After all, Gough (1969)'s original scale analysis was performed specifically with the problem of convection in mind. Nonetheless, in the third part of the paper, we argue (Section 9) that the AnEGG approximation may remain consistent even in regions of strong stable stratification (e.g., for convective overshoot problems), provided that the thermodynamic perturbations remain small during overshoot. That should be true if the buoyancy frequency is much less than the acoustic cutoff frequency. We also argue (Section 10) that the nondimensionalization associated with wave motion is fundamentally different than that associated with nonlinear motion, as in convection or overshoot. That is the fundamental reason that the anelastic approximation (even AnEGG) does not conserve energy for internal gravity waves (e.g., Brown et al. 2012; Vasil et al. 2013) and seriously fails to accurately describe the gravity-wave disperion relation and eigenfunctions (e.g., Hindman & Julien 2024). Accurately treating both the convection and linear waves using the same sound-proof equation set thus remains an outstanding problem.

2 Energy conservation under the fully compressible fluid equations

We begin by writing down the unapproximated fully compressible equations of motion for a nonrotating, nonmagnetic fluid. These are Gough (1969)'s Equations (2.1)–(2.5), consisting of the continuity equation,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{u}) \tag{1}$$

the momentum equation,

$$\frac{\partial}{\partial t}(\rho \boldsymbol{u}) = -\nabla \cdot (\rho \boldsymbol{u} \boldsymbol{u}) - \nabla P + \rho \boldsymbol{g} + \nabla \cdot \overleftrightarrow{D}, \tag{2a}$$

$$\text{where} \quad D_{ij} = \mu \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}(\nabla \cdot \boldsymbol{u})\delta_{ij} \right], \tag{2b}$$

the internal energy equation,

$$\frac{\partial}{\partial t}(\rho U) + \nabla \cdot (\rho U \boldsymbol{u}) + P \nabla \cdot \boldsymbol{u} = D_{ij}\frac{\partial u_i}{\partial x_j} + Q - \nabla \cdot \boldsymbol{F}, \tag{3}$$

and a general equation of state,

$$U = U(P, T). \tag{4}$$

Here, t is the time, the x_i are Cartesian spatial coordinates, ρ is the density, P the pressure, T the temperature, U the internal energy per unit mass, $\mu = \mu(P,T)$ the dynamic viscosity, $\boldsymbol{g} \coloneqq -\nabla \Phi$ the gravitational acceleration field, Φ the gravitational potential, Q a combined internal heating and cooling source, and $\boldsymbol{F}$ the combined conductive and radiative heat flux. The subscripts i and j (taking on the values 1, 2, and 3) denote vector or tensor components in any of the three Cartesian coordinate directions. The gravity $\boldsymbol{g}$ is assumed to point in the vertical direction $\hat{\boldsymbol{k}}$ (either the

upward Cartesian direction for a plane-parallel fluid layer or the radial direction for spherical shell). Additionally, $\boldsymbol{g}$ is assumed to depend only on the vertical coordinate q (either the upward Cartesian coordinate x_3 or the radial coordinate r) and to be time-independent (meaning that, among other consequences, self-gravity is ignored). The symbol "$\leftrightarrow$" in the viscous stress tensor $\overleftrightarrow{D}$ denotes a second-order tensor, as does the dyadic notation $\boldsymbol{uu}$. We use the Einstein summation convention and δ_{ij} denotes the Kronecker delta. These equations are not written in the exact form of Gough (1969) and use slightly different notation, but are mathematically equivalent.

An equation for the evolution of kinetic energy can be formed from $\boldsymbol{u}$ dotted into Equation (2),

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) = -\nabla \cdot \left(\frac{1}{2}\rho u^2 \boldsymbol{u}\right) + \boldsymbol{u} \cdot \nabla P - \rho \boldsymbol{u} \cdot \nabla \Phi + u_i \frac{\partial D_{ij}}{\partial x_j}. \tag{5}$$

Equation (1) multiplied by Φ yields an equation for the evolution of potential energy,

$$\frac{\partial}{\partial t}(\rho \Phi) = -\Phi \nabla \cdot (\rho \boldsymbol{u}). \tag{6}$$

Adding Equations (3), (5), and (6) yields an equation for the evolution of total energy,

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}u^2 + U + \Phi\right)\right] = -\nabla \cdot \left\{\left[\rho\left(\frac{1}{2}u^2 + U + \Phi\right) + P\right]\boldsymbol{u} - \boldsymbol{u} \cdot \overleftrightarrow{D} + \boldsymbol{F}\right\} + Q. \tag{7}$$

Integrating Equation (7) over the volume V of the fluid layer, using the divergence theorem, and assuming that the sum of the surface-integrated fluxes balances the volume-integrated heating and cooling yields[2]

$$E_{\text{tot}} := \int_V \left(\underbrace{\frac{1}{2}\rho u^2}_{:=W_{\text{KE}}} + \underbrace{\rho U}_{:=W_{\text{int}}} + \underbrace{\rho \Phi}_{:=W_{\text{pot}}}\right) dV \equiv \text{constant}. \tag{8}$$

We additionally define

$$W_{\text{tot}} := W_{\text{KE}} + W_{\text{int}} + W_{\text{pot}} = \rho\left(\frac{1}{2}u^2 + U + \Phi\right) \tag{9a}$$

$$\text{and} \quad W_{\text{NKE}} := W_{\text{tot}} - W_{\text{KE}} = W_{\text{int}} + W_{\text{pot}} = \rho(U + \Phi) \tag{9b}$$

as the total energy density and the combined internal and potential energy density (i.e., the "nonkinetic" energy), respectively.

Equation (8) expresses the conservation of total energy that is implicit to the fully compressible fluid equations (1)–(4). The volume-integrated total energy is conserved, as long as the total surface fluxes balance the total internal heat sources. Locally by contrast, fluid parcels can exchange total energy density with one another, and these exchanges are further partitioned into conversions between kinetic, internal, and potential energy density.

One of the main goals of this work is to elucidate how Equation (8) is modified under the anelastic approximation for a general equation of state and nonadiabatic

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[2]Throughout this work, we use the symbol "$:=$" for definitions (i.e., for "is defined to be equal to") and "$\equiv$" for homogeneous equivalence (i.e., for "is equal to [some constant value] everywhere and for all time").

stratification. It will be found that *only* exchange between internal and kinetic energy is possible for an energetically constistent anelastic approximation. *Conversion to and from gravitational potential energy, by contrast, is fundamentally impossible.* This seems to be an underappreciated result, which is easy to overlook because of a pathological coincidence: for a perfect, adiabatically stratified gas, the internal energy *looks* exactly like a gravitational potential energy.[3] For nonadiabatic stratification, or for an imperfect gas, the coincidence no longer holds, and the more general law derived here for conservation of total energy, which includes only internal and kinetic energy, must be used.

The other new and related result is that total energy *should* be conserved under the anelastic approximation, for arbitrary fluid motions and equations of state and for all stratifications, even for strongly subadiabatic ones. The anelastic equations of Gough (1969) satisfy such a general conservation law, while modern implementations of the anelastic approximation do not. Therefore, a major part of this paper is devoted to translating Gough (1969)'s equations into more conventional notation and elucidating the origin and form of the term (which should be present in the internal energy equation for asymptotic consistency) that is missing from modern anelastic implementations.

Finally, we note that Gough (1969) assumed zero net vertical transport of mass for the "true" fully compressible fluid whose motion the anelastic equations are intended to approximate. This may at first seem like an arbitrary choice to make the small-parameter expansion simpler. It is concluded in this work, however, that the "zero mean mass flux" assumption is in fact fundamental to ensure the energetic consistency of the anelastic approximation.

3   The anelastic approximation of Gough (1969)

To set the stage, we define some new thermodynamic variables, namely, the specific enthalpy,

$$h := U + \frac{P}{\rho} \tag{10}$$

and the specific entropy,

$$S = S(P, T). \tag{11}$$

Equation (11) is also an equation of state, containing the equivalent information of Equation (4).

We define several first derivatives of thermodynamic quantities derivable from the generalized equation of state (4) or (11): the specific heat at constant volume,

$$C_{\mathrm{v}} = C_{\mathrm{v}}(P, T) := T \left(\frac{\partial S}{\partial T} \right)_{\rho}, \tag{12}$$

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[3]We use the nomenclature that an "ideal gas" refers to a fluid for which the internal energy and specific enthalpy depend only on the temperature (i.e., $U = U(T)$ and $h = h(T)$). It can be shown from Equations (17) these two conditions yield the ideal gas law, $P = \mathcal{R} \rho T$, where $\mathcal{R} := C_{\mathrm{p}} - C_{\mathrm{v}}$ is a constant and the individual specific heats depend only on temperature. A "perfect gas" is a gas that is not only ideal, but has specific heats that are individually constants, independent of temperature.

the specific heat at constant pressure,

$$C_{\mathrm{p}} = C_{\mathrm{p}}(P,T) := T\left(\frac{\partial S}{\partial T}\right)_P, \tag{13}$$

the squared adiabatic sound speed,

$$c_{\mathrm{s}}^2 = c_{\mathrm{s}}^2(P,T) := \left(\frac{\partial P}{\partial \rho}\right)_S, \tag{14}$$

the (isobaric) thermal expansion coefficient,

$$\delta = \delta(P,T) := -\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_P, \tag{15}$$

and the (isochoric) thermal pressure coefficient,

$$\eta = \eta(P,T) := \left(\frac{\partial \ln P}{\partial \ln T}\right)_\rho. \tag{16}$$

The first law of thermodynamics[4] then takes the following forms:

$$T dS = dU - \frac{P}{\rho^2} d\rho \tag{17a}$$

$$= dh - \frac{dP}{\rho} \tag{17b}$$

$$= C_{\mathrm{v}} dT - \frac{P\eta}{\rho^2} d\rho \tag{17c}$$

$$= C_{\mathrm{p}} dT - \frac{\delta}{\rho} dP \tag{17d}$$

$$= \frac{C_{\mathrm{p}} T}{\rho\delta}\left(\frac{dP}{c_{\mathrm{s}}^2} - d\rho\right). \tag{17e}$$

3.1 Gough (1969)'s anelastic scale analysis for convection

We will not repeat the full asymptotic expansion in ϵ performed by Gough (1969) of Equations (1)–(4) here. Instead, we reiterate the salient assumptions in the case where the layer depth is thicker than the typical pressure scale height.[5] The main assumption is that the thermodynamic perturbations from the horizontally averaged background state of the fully compressible system are small, i.e.,

$$f = \overline{f}(q,t) + f_1(x_i,t) \quad \text{with} \quad f_1/\overline{f} = O(\epsilon) \ll 1, \tag{18}$$

where f denotes either ρ, P, T, U, h, C_{p}, μ, c_{s}^2, δ, η, $\boldsymbol{F}$, or Q.[6] Here, the overbars denote horizontal averages (taken at a particular instant in time so that they depend

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[4]We assume local thermodynamic equilibrium always holds everywhere in the fluid.

[5]Gough (1969) also considers thin layers, in which the anelastic equations become the Boussinesq equations.

[6]We use the notation "$O(\epsilon)$" to mean "about equal to ϵ." We also use the notation "$+O(\epsilon^n)$" (where n is a nonnegative integer) on the right-hand sides of equations to mean "plus higher-order terms, where the next-higher order is n higher than the explicitly listed terms."

on both q and t) and the "1" subscripts denote the perturbations about this average, which are time-dependent and three-dimensional functions of space. Note that it is *not* correct to write "$S = \overline{S}(q,t) + S_1(x_i,t)$ with $S_1/\overline{S} = O(\epsilon)$." The fully compressible equations of motion contain only differences in entropy and so no meaningful absolute value of $\overline{S}$ can be defined. Instead, we must write

$$S = \overline{S}(q,t) + S_1(x_i,t) \qquad \text{with} \qquad S_1/\overline{C_{\mathrm{p}}} = O(\epsilon) \ll 1 \tag{19}$$

or some equivalent expression.

The second central assumption is that the coordinate system can be chosen such that there is no mass flux across any horizontal surface, i.e.,

$$\overline{\rho u_i} \equiv 0. \tag{20}$$

This equation is presumed to hold for the fully compressible system that the anelastic approximation seeks to approximate. It is then *enforced* to hold in the anelastic system up to first order in ϵ. Physically, Equation (20) ensures that there is no net build-up of mass at any vertical level at any time, such as during pulsations of the atmosphere (and it means that the inertial coordinate frame in which the equations of motion are written must coincide with the center of mass of the system). In a spherical system (e.g., a spherical shell), the horizontal average would be a spherically symmetric average and the coordinates would point along the spatially varying curvilinear coordinate directions.

In writing $\overline{f} = \overline{f}(q,t)$ in Equation (18), we make a third central assumption not made by Gough (1969), namely, that the mean state of the compressible system may be time-dependent.[7] In order to still be consistent with Gough (1969)'s asymptotic expansion, we must assume that the time-variation of the mean state is sufficiently slow, i.e., comparable in magnitude to the time variation of the fluctuations:

$$\left| \frac{\partial \overline{f}}{\partial t} \right| = O\left(\left| \frac{\partial f_1}{\partial t} \right| \right). \tag{21}$$

If the mean state varies in time due to restratification by the convection, then Equation (21) is natural and fits in with the spirit of slow-moving, small thermal perturbations.

The rest of the assumptions of Gough (1969) concern scale analysis, which is appropriate for convection in the presence of mean density stratification $\partial \overline{\rho}/\partial q \neq 0$. The characteristic length scale of variation of the fluid is assumed to be a typical value H_a for the pressure scale height. The flow is assumed to be buoyantly driven by the $O(\epsilon)$ thermal perturbations, i.e.,

$$|\boldsymbol{u}| = O(\sqrt{\epsilon g_a H_a}) = O(\sqrt{\epsilon} c_{\mathrm{s}a}), \tag{22}$$

where the "a" subscripts denote typical atmospheric background-state values. Thus, the squared Mach number of the flow is assumed to be $O(\epsilon)$. The characteristic time scale of variation of the fluid is assumed to be advective, i.e.,

$$\left| \frac{\partial}{\partial t} \right| = O\left(\sqrt{\frac{\epsilon g_a}{H_a}} \right) = O\left(\sqrt{\epsilon} \frac{1}{H_a/c_{\mathrm{s}a}} \right). \tag{23}$$

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[7]Gough (1969) also considers time-dependent atmospheres, but this is in the different context of pulsations, where the atmosphere itself is moving. In the present work, we consider "Eulerian-only" time-dependence of the mean state.

Thus, the characteristic time scale of variation for the convection is $O(\epsilon^{-1/2})$ times longer than the time it takes a sound wave to cross a pressure scale height. From Equation (21), the characteristic time scale of variation for the mean state is $O(\epsilon^{-3/2})$ times longer than the sound crossing time.

Finally, the vertical convective heat flux, which maximally could transport an energy flux of order $\overline{\rho}\,\overline{T}w\Delta\overline{S}$, where

$$w := \hat{\boldsymbol{k}} \cdot \boldsymbol{u} \tag{24}$$

is the vertical velocity and $\Delta\overline{S}$ is the total drop in background entropy across the convecting layer, is assumed to be limited primarily by the thermal diffusion $\boldsymbol{F}$. This should be true if the conductive heating $-\nabla \cdot \boldsymbol{F}$ in Equation (3) is at at least as large as the viscous and internal heatings. In the case of negligible heatings (high Rayleigh number), one expects

$$\frac{|\Delta\overline{S}|}{C_{\mathrm{p}a}} = O(\epsilon), \tag{25}$$

i.e., the convecting layer should be nearly adiabatically stratified for vigorous convection.

One consequence of Equation (20) is that the horizontally averaged velocity $\overline{\boldsymbol{u}}$ is $O(\epsilon)$ *smaller* than the perturbed velocity $\boldsymbol{u}_1$. To see this, we write

$$0 = \overline{(\overline{\rho} + \rho_1)(\overline{\boldsymbol{u}} + \boldsymbol{u}_1)} = \overline{\rho}\,\overline{\boldsymbol{u}} + \overline{\rho_1\boldsymbol{u}_1},$$

$$\text{or} \quad |\overline{\boldsymbol{u}}| = \left|-\frac{\overline{\rho_1\boldsymbol{u}_1}}{\overline{\rho}}\right| = O(\epsilon|\boldsymbol{u}_1|). \tag{26}$$

For the total mass flux (or equivalently, the momentum density), $\boldsymbol{m} := \rho\boldsymbol{u} = (\overline{\rho} + \rho_1)(\overline{\boldsymbol{u}} + \boldsymbol{u}_1)$ to have vanishing mean at all orders, we thus write

$$\boldsymbol{m} = \boldsymbol{m} - \overline{\boldsymbol{m}} = \overline{\rho}\boldsymbol{u}_1 + \rho_1\boldsymbol{u}_1 - \overline{\rho_1\boldsymbol{u}_1} + O(\epsilon^2). \tag{27}$$

Hence, at $O(\epsilon)$, only the perturbation velocity $\boldsymbol{u}_1$ appears in the equations, so we subsequently use $\boldsymbol{u}$ as shorthand for $\boldsymbol{u}_1$ and drop the subscript "1." If we need to reference the total velocity, we explicitly write $\boldsymbol{u}_{\mathrm{tot}}$.

Under this convention,

$$\overline{\boldsymbol{u}} \equiv 0 \tag{28}$$

by definition and Equation (27) becomes

$$\boldsymbol{m} = \overline{\rho}\boldsymbol{u} + \rho_1\boldsymbol{u} - \overline{\rho_1\boldsymbol{u}} + O(\epsilon^2). \tag{29}$$

Each of the two terms $\rho_1\boldsymbol{u}$ and $-\overline{\rho_1\boldsymbol{u}}$ are $O(\epsilon)$. In most cases, we can thus write $\boldsymbol{m} \approx \overline{\rho}\boldsymbol{u}$ to translate from Gough (1969) to the current notation (in which we use the fluctuating velocity $\boldsymbol{u}$ as the primary field variable), except when multiplying by potentially zeroth-order quantities. Physically, using Equation (29) to translate from Gough (1969)'s equations to ours also ensures that, to $O(\epsilon)$, there is no mean mass flux in the final anelastic equations.

One other change in notation concerns the background stratification. Gough (1969) uses the superadiabatic mean background temperature gradient

$$\beta := -\frac{1}{\overline{C_{\mathrm{p}}}}\hat{\boldsymbol{k}} \cdot \left[\nabla\overline{h} - \frac{1}{\overline{\rho}}\nabla\overline{P}\right]$$

$$= -\frac{\overline{T}}{\overline{C_{\mathrm{p}}}}\hat{\boldsymbol{k}} \cdot \nabla\overline{S} + O(\epsilon^2), \tag{30}$$

whereas we will use $\nabla\overline{S}$. Compare to Gough (1969)'s Equation (3.7). We translate using Equation (35c) below.

3.2   Gough (1969)'s anelastic equations

Once all of the above scaling assumptions have been made, Equations (1), (2), (3), and (4) are nondimensionalized, each dependent variable is expanded in powers of ϵ, terms up to zeroth-order in the continuity equation and first-order in the other equations are retained, and redimensionalization then yields the anelastic equations. Specifically, we discuss Gough (1969)'s Equations (4.3)–(4.7) and (4.15)–(4.22). We translate these equations using the change of variables outlined in Equations (29) and (30).

Under Gough (1969)'s anelastic approximation (with the additional assumption (21)), the continuity equation (1) becomes

$$\nabla \cdot (\overline{\rho}\boldsymbol{u}) \equiv 0, \tag{31}$$

the momentum equation (2) becomes

$$\frac{\partial}{\partial t}(\overline{\rho}\boldsymbol{u}) = -\nabla \cdot (\overline{\rho}\boldsymbol{u}\boldsymbol{u}) - \nabla P_1 + \rho_1\boldsymbol{g} + \nabla \cdot \overleftrightarrow{D} + [-\nabla\overline{P} + \overline{\rho}\boldsymbol{g}], \tag{32a}$$

$$\text{where now} \quad D_{ij} = \overline{\mu}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}(\nabla \cdot \boldsymbol{u})\delta_{ij}\right), \tag{32b}$$

the energy equation (3) becomes

$$\overline{\rho}\overline{C_{\mathrm{p}}}\left(\frac{\partial T_1}{\partial t} + \frac{\partial \overline{T}}{\partial t}\right) - \overline{\delta}\left(\frac{\partial P_1}{\partial t} + \frac{\partial \overline{P}}{\partial t}\right) = -\overline{\rho}\boldsymbol{u} \cdot \left(\nabla h_1 - \frac{1}{\overline{\rho}}\nabla P_1\right) - \overline{\rho}\overline{T}\boldsymbol{u} \cdot \nabla\overline{S}$$

$$D_{ij}\frac{\partial u_i}{\partial x_j} + Q_1 - \nabla \cdot \boldsymbol{F}_1 - \rho_1\boldsymbol{u} \cdot \boldsymbol{g} - \overline{T}(\rho_1\boldsymbol{u} - \overline{\rho_1\boldsymbol{u}}) \cdot \nabla\overline{S}$$

$$+ [\overline{Q} - \nabla \cdot \overline{\boldsymbol{F}}], \tag{33}$$

and the equation of state (4) or (11) becomes linearized via the first law of thermody-

10

namics (17), yielding

$$\overline{T}S_1 = \overline{C_\mathrm{p}}T_1 - \frac{\overline{\delta}}{\overline{\rho}}P_1 \tag{34a}$$

$$= U_1 - \frac{\overline{P}}{\overline{\rho}^2}\rho_1 \tag{34b}$$

$$= h_1 - \frac{P_1}{\overline{\rho}} \tag{34c}$$

$$= \overline{C_\mathrm{v}}T_1 - \frac{\overline{P\eta}}{\overline{\rho}^2}\rho_1 \tag{34d}$$

$$= \frac{\overline{C_\mathrm{p}}\,\overline{T}}{\overline{\delta}\overline{\rho}}\left[\frac{P_1}{c_\mathrm{s}^2} - \rho_1\right]. \tag{34e}$$

We have used Equation (29) to yield the term $-\overline{T}(\rho_1\boldsymbol{u} - \overline{\rho_1\boldsymbol{u}}) \cdot \nabla\overline{S}$ in Equation (33).

Note that the continuity equation (31) is unchanged from Gough (1969)'s Equation (4.16), despite $\partial\overline{\rho}/\partial t$ being nonzero. This is because $\partial\overline{\rho}/\partial t$ shows up only at $O(\epsilon)$ by Equation (21) and only zeroth-order terms in the continuity equation are retained. Similarly, the momentum equation (32) takes the same form regardless of whether the mean state is time-dependent or not, since only first-order terms are retained and thus we can replace $\partial\boldsymbol{m}/\partial t$ with $\partial(\overline{\rho}\boldsymbol{u})/\partial t$ by virtue of Equations (21) and (29). The left-hand side of the internal energy equation (33) and the right-hand side of the mean internal energy equation (37) below, where we must keep the time-variation of the mean state, is the only place our equations differ materially from those of Gough (1969), specifically, his Equations (4.17) and (4.7), respectively.

The differentials in Equation (17) can be converted into gradients (e.g., $T\nabla S = \nabla h - \nabla P/\rho$) and the horizontally averaged form of these relations yields

$$\overline{T}\nabla\overline{S} = \overline{C_\mathrm{p}}\nabla\overline{T} - \frac{\overline{\delta}}{\overline{\rho}}\nabla\overline{P} + O(\epsilon^2) \tag{35a}$$

$$= \nabla\overline{U} - \frac{\overline{P}}{\overline{\rho}^2}\nabla\overline{\rho} + O(\epsilon^2) \tag{35b}$$

$$= \nabla\overline{h} - \frac{\nabla\overline{P}}{\overline{\rho}} + O(\epsilon^2) \tag{35c}$$

$$= \overline{C_\mathrm{v}}\nabla\overline{T} - \frac{\overline{P\eta}}{\overline{\rho}^2}\nabla\overline{\rho} + O(\epsilon^2) \tag{35d}$$

$$= \frac{\overline{C_\mathrm{p}}\,\overline{T}}{\overline{\delta}\overline{\rho}}\left[\frac{\nabla\overline{P}}{\overline{c_\mathrm{s}^2}} - \nabla\overline{\rho}\right] + O(\epsilon^2). \tag{35e}$$

The horizontal averages of Equations (32) and (33) satisfy

$$-\nabla\overline{P} + \overline{\rho}\boldsymbol{g} = \nabla(\overline{\rho w^2}) \tag{36}$$

and

$$\overline{Q} - \nabla\cdot\overline{\boldsymbol{F}} = \overline{\rho\boldsymbol{u}\cdot\left(\nabla h_1 - \frac{1}{\overline{\rho}}\nabla P_1\right)} + \boldsymbol{g}\cdot\overline{\rho_1\boldsymbol{u}} - \overline{D_{ij}\frac{\partial u_i}{\partial x_j}} + \overline{\rho}\,\overline{C_\mathrm{p}}\frac{\partial\overline{T}}{\partial t} - \overline{\delta}\frac{\partial\overline{P}}{\partial t} \tag{37}$$

11

Equations (31)–(37) are mathematically equivalent to Gough (1969)'s Equations (4.3)–(4.7) and (4.15)–(4.22), with the minor exception of the mean-state time derivatives in Equations (33) and (37). Together (with a set of appropriate initial and boundary conditions), they form a complete system that can be solved for the evolution of the fluctuating thermal variables and velocity, and then (using the correlations between fluctuations) for the evolution of the mean thermal profiles.

Dotting $\boldsymbol{u}$ into Equation (32a) and using Equation (36) yields Gough (1969)'s anelastic kinetic energy equation,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{\rho}u^2\right) = -\nabla\cdot\left(\frac{1}{2}\overline{\rho}u^2\boldsymbol{u}\right) - \boldsymbol{u}\cdot\nabla P_1 + \rho_1\boldsymbol{u}\cdot\boldsymbol{g} + u_i\frac{\partial D_{ij}}{\partial x_j} + \boldsymbol{u}\cdot\nabla(\overline{\rho w^2}). \qquad (38)$$

Converting the differentials in Equation (17d) into Eulerian differentials in time, the left-hand side of Equation (33) can be written in terms of the total entropy $S_1 + \overline{S}$,

$$\overline{\rho C_{\mathrm{p}}}\left(\frac{\partial T_1}{\partial t} + \frac{\partial \overline{T}}{\partial t}\right) - \overline{\delta}\left(\frac{\partial P_1}{\partial t} + \frac{\partial \overline{P}}{\partial t}\right) = \overline{\rho T}\frac{\partial S_1}{\partial t} + \overline{\rho T}\frac{\partial \overline{S}}{\partial t}. \qquad (39)$$

Using Equations (31), (34c), and (39) and then adding Equations (33) and (38) yields Gough (1969)'s total energy equation,

$$\frac{\partial}{\partial t}\left\{\overline{\rho}\left[\frac{1}{2}u^2 + \overline{T}(S_1 + \overline{S})\right]\right\} = -\nabla\cdot\left\{\left[\overline{\rho}\left(\frac{1}{2}u^2 + \overline{T}S_1\right) + P_1\right]\boldsymbol{u} - \boldsymbol{u}\cdot\overset{\leftrightarrow}{D} + \overline{\boldsymbol{F}} + \boldsymbol{F}_1\right\} + \overline{Q} + Q_1$$
$$+ \left[-\overline{\rho T}\boldsymbol{u}\cdot\nabla\overline{S} + \boldsymbol{u}\cdot\nabla(\overline{\rho w^2}) - \overline{T}(\rho_1\boldsymbol{u} - \overline{\rho_1\boldsymbol{u}})\cdot\nabla\overline{S}\right]. \qquad (40)$$

Because of Equation (28), or more specifically, the condition

$$\overline{w} \equiv 0, \qquad (41)$$

each of the rightmost terms in brackets in Equation (40) vanishes after volume integration over V (i.e., none of the terms can transport any net energy). Note that Equation (41) is also a consequence of integrating Equation (31) over volumes bounded by horizontal surfaces (and also assuming w vanishes on the boundaries). Thus, the vanishing of the horizontal components of $\overline{\boldsymbol{u}}$ is not strictly necessary for the bracketed terms to conserve energy.

Integrating Equation (40) over V, using the divergence theorem, and assuming that the sum of the surface-integrated fluxes balances the volume-integrated heating yields

$$E_{\mathrm{tot}}^{\mathrm{an}} := \int_V\left[\underbrace{\frac{1}{2}\overline{\rho}u^2}_{:=W_{\mathrm{KE}}^{\mathrm{an}}} + \underbrace{\overline{\rho T}(S_1 + \overline{S})}_{:=W_{\mathrm{int}}^{\mathrm{an}}}\right]dV \equiv \text{constant}. \qquad (42)$$

We have used the superscript "an" to denote energies and densities under Gough (1969)'s anelastic approximation and defined the kinetic and internal energy densities $W_{\mathrm{KE}}^{\mathrm{an}}$ and $W_{\mathrm{int}}^{\mathrm{an}}$ (respectively). We additionally define

$$W_{\mathrm{tot}}^{\mathrm{an}} := W_{\mathrm{KE}}^{\mathrm{an}} + W_{\mathrm{int}}^{\mathrm{an}} = \overline{\rho}\left[\frac{1}{2}u^2 + \overline{T}(S_1 + \overline{S})\right] \qquad (43a)$$

$$\text{and} \quad W_{\mathrm{NKE}}^{\mathrm{an}} := W_{\mathrm{tot}}^{\mathrm{an}} - W_{\mathrm{KE}}^{\mathrm{an}} = W_{\mathrm{int}}^{\mathrm{an}} = \overline{\rho T}(S_1 + \overline{S}) \qquad (43b)$$

as the total anelastic energy density and the internal anelastic energy density (which makes up the total "nonkinetic" energy), respectively. Note that an arbitrary constant can be added to $\overline{S}$ without affecting the dynamics.

The conservation law (42) holds for arbitrary fluid motions that obey Equations (31), (32), (33), and (34) and for arbitrary magnitudes of $|\nabla \overline{S}|$. Note that the partition of energy density is fundamentally different than in the fully compressible conservation law (8). Most notably, the potential energy density is absent under the anelastic approximation. Furthermore, $W_{\text{int}}^{\text{an}}$ takes a fundamentally different form than might be expected from Equation (34b). We return to this central point in Section 7.

4 Gough (1969)'s anelastic equations in terms of pressure and entropy

The anelastic equations are often written using the pressure and entropy (P_1, S_1, and $\overline{S}$) in place of the quantities T_1, $\overline{T}$, P_1, $\overline{P}$, ρ_1, and h_1 that appear in Equations (32) and (36). This is more than just a matter of convention, since transforming the variables to pressure and entropy isolates terms in the equations that depend on the mean entropy gradient $\nabla \overline{S}$. Historically, these terms were neglected on the grounds that $\nabla \overline{S}$ should be small for a fully and vigorously convecting fluid. To accommodate stable stratification, however (while retaining the exact total energy conservation law of Equation (42)), these terms must be kept.

In each of Equations (36) and (37), each term on the right-hand side is $O(\epsilon)$ compared to each term on the left-hand side. In particular, we can make the mean hydrostatic approximation,

$$\nabla \overline{P} \approx \overline{\rho} \boldsymbol{g}, \tag{44}$$

when multiplying by terms that are already of first order in ϵ.

We can change variables using Equations (34) and (35) and the approximation (44). In the momentum equation, we find

$$-\nabla P_1 + \rho_1 \boldsymbol{g} = -\overline{\rho} \nabla \left(\frac{P_1}{\overline{\rho}} \right) - \overline{\delta \rho} \left(\frac{S_1}{C_{\text{p}}} \right) \boldsymbol{g} + \frac{\overline{\delta} P_1}{\overline{C_{\text{p}}}} \nabla \overline{S}. \tag{45}$$

In the internal energy equation, we find

$$-\overline{\rho} \boldsymbol{u} \cdot \left(\nabla h_1 - \frac{1}{\overline{\rho}} \nabla P_1 \right) = -\overline{\rho} \overline{T} \boldsymbol{u} \cdot \nabla S_1 - \overline{\rho} S_1 \boldsymbol{u} \cdot \nabla \overline{T} + \left(\frac{P_1}{\overline{\rho}} \right) \boldsymbol{u} \cdot \nabla \overline{\rho}. \tag{46}$$

We then compute, with some effort,

$$-\overline{\rho} S_1 \boldsymbol{u} \cdot \nabla \overline{T} + \left(\frac{P_1}{\overline{\rho}} \right) \boldsymbol{u} \cdot \nabla S_1 - \rho_1 \boldsymbol{u} \cdot \boldsymbol{g} = -\overline{\rho} T_1 \boldsymbol{u} \cdot \nabla \overline{S} + O(\epsilon^2). \tag{47}$$

Substituting Equation (45) into Equation (32), and Equations (39), (46), and (47) into Equation (33), we find

$$\frac{\partial}{\partial t} (\overline{\rho} \boldsymbol{u}) = - \nabla \cdot (\overline{\rho} \boldsymbol{u} \boldsymbol{u}) - \overline{\rho} \nabla \left(\frac{P_1}{\overline{\rho}} \right) - \underbrace{\overline{\delta \rho} \left(\frac{S_1}{C_{\text{p}}} \right) \boldsymbol{g} + \frac{\overline{\delta} P_1}{\overline{C_{\text{p}}}} \nabla \overline{S}}_{:=\boldsymbol{f}_{\text{NLBR}}} + \nabla \cdot \overset{\leftrightarrow}{D}$$

$$+ [-\nabla \overline{P} + \overline{\rho} \boldsymbol{g}], \tag{48}$$

13

and

$$\overline{\rho}\overline{T}\left(\frac{\partial S_1}{\partial t} + \frac{\partial \overline{S}}{\partial t}\right) = -\overline{\rho}\overline{T}\boldsymbol{u}\cdot\nabla S_1 - \overline{\rho}\overline{T}\boldsymbol{u}\cdot\nabla\overline{S}\underbrace{-\overline{\rho}T_1\boldsymbol{u}\cdot\nabla\overline{S}}_{:=Q_{\text{NLBR}}}+D_{ij}\frac{\partial u_i}{\partial x_j} + Q_1 - \nabla\cdot\boldsymbol{F}_1$$

$$+ [\overline{Q} - \nabla\cdot\overline{\boldsymbol{F}} - \overline{T}(\rho_1\boldsymbol{u} - \overline{\rho_1\boldsymbol{u}})\cdot\nabla\overline{S}], \tag{49a}$$

$$\text{where} \qquad \rho_1 = \frac{P_1}{\overline{c_s^2}} - \frac{\overline{\delta}\overline{\rho}S_1}{\overline{C_{\text{p}}}} \tag{49b}$$

$$\text{and} \qquad T_1 = \frac{\overline{T}S_1}{\overline{C_{\text{p}}}} + \frac{\overline{\delta}P_1}{\overline{\rho}\overline{C_{\text{p}}}}. \tag{49c}$$

Finally, we make no changes to the mean momentum equation (36) (since it does not contain any perturbed thermal variables), but eliminate h_1 in the mean internal energy equation (37) by using Equations (46) and (47):

$$\overline{Q} - \nabla\cdot\overline{\boldsymbol{F}} = \overline{\rho}\overline{T}\,\overline{\boldsymbol{u}\cdot\nabla S_1} - \overline{D_{ij}\frac{\partial u_i}{\partial x_j}} + \overline{\rho}(\overline{T_1\boldsymbol{u}})\cdot\nabla\overline{S} + \overline{\rho}\overline{T}\frac{\partial\overline{S}}{\partial t}. \tag{50}$$

The essential terms required for energy conservation when $\nabla\overline{S} \neq 0$ are thus the "non-LBR" force density $\boldsymbol{f}_{\text{NLBR}}$, which we have defined in Equation (48), and the "non-LBR" heating Q_{NLBR}, which we have defined in Equation (49). Both these terms vanish for a nearly adiabatic background state (where $\nabla\overline{S} = O(\epsilon)$), which is expected for a fully (and sufficiently vigorously) convecting fluid layer. Neglecting $\boldsymbol{f}_{\text{NLBR}}$ was first done independently by Lantz (1992) and Braginsky & Roberts (1995) and is therefore referred to as the "Lantz-Braginsky-Roberts" (LBR) approximation. The term Q_{NLBR} in the internal energy equation, which was implicitly contained in the equations of Gough (1969), seems to be absent in the other forms of the anelastic equations currently in use. Its neglect seems to have not been explicitly considered.

5 The partial equivalence between horizontally averaged atmospheres to fixed reference atmospheres

In the formalism of Gough (1969), the horizontally averaged atmosphere, denoted by the overbars, was originally assumed to be time-independent. We have relaxed that assumption here, but we must consider how the mean state depends on the flow via Equations (36) and (50). Many anelastic numerical codes (e.g., the `Rayleigh`, `EULAG`, and `MagIC` codes, which simulate the anelastic equations in spherical shells) treat the background state as a fixed-in-time, spherically symmetric, hydrostatic "reference" state and let the thermal perturbations about this reference state develop small but nonzero horizontal means. Some codes (e.g., the `ASH` code) alternatively solve for both the perturbations about the horizontal average and the mean state itself, choosing some initial mean state consistent with zero flow.[8] In the latter approach, horizontally

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[8] This initial state, satisfying Equations (35), (36), and (50) with $\boldsymbol{u} \equiv 0$, would resemble a one-dimensional stellar structure model, for example.

averaged terms are retained on the right-hand sides of the momentum and energy equations, similar to the bracketed terms in Equations (48) and (49a). As we now show, these two approaches are exactly equivalent to the order of the anelastic equations, provided we make a fourth central assumption: that the horizontal means of the thermal variables ultimately wander by no more than $O(\epsilon)$ away from their preordained reference-state values.

We formalize this new assumption by defining each horizontally averaged thermal profile as the sum of the reference-state profile (denoted by a tilde) and a horizontally symmetric deviation (denoted by a hat):

$$\overline{f} = \overline{f}(q,t) = \tilde{f}(q) + \hat{f}(q,t) \quad \text{with} \quad \hat{f}/\tilde{f} = O(\epsilon), \tag{51}$$

where again f denotes either ρ, P, T, U, h, C_{v}, C_{p}, μ, δ, η, c_{s}^2, $\boldsymbol{F}$, or Q. In physical terms, Equation (21) requires that the time variation of the mean state of the fully compressible system is slow (e.g., altered by the convection on an advective time scale), while Equation (51) additionally requires that the total variation of the mean state away from the reference state (integrated from $t = 0$) always remains small.

We denote the (temporally and horizontally dependent) deviations from the reference state by primes and note that by definition,

$$f' := f - \tilde{f} = \hat{f} + f_1. \tag{52}$$

The primed quantities, being the sum of two assumed-small quantities, thus remain small as long Equations (21) and (51) are both satisfied.

We assume that the reference state has been chosen to be hydrostatic,

$$\nabla \tilde{P} = \tilde{\rho} \boldsymbol{g}, \tag{53}$$

to be in thermal equilibrium,

$$\tilde{Q} = \nabla \cdot \tilde{\boldsymbol{F}}, \tag{54}$$

and to satisfy the first law of thermodynamics for gradients, exactly analogously to Equations (35),

$$\tilde{T}\nabla\tilde{S} = \tilde{C}_{\mathrm{p}}\nabla\tilde{T} - \frac{\tilde{\delta}}{\tilde{\rho}}\nabla\tilde{P} \tag{55a}$$

$$= \nabla\tilde{U} - \frac{\tilde{P}}{\tilde{\rho}^2}\nabla\tilde{\rho} \tag{55b}$$

$$= \nabla\tilde{h} - \frac{\nabla\tilde{P}}{\tilde{\rho}} \tag{55c}$$

$$= \tilde{C}_{\mathrm{v}}\nabla\tilde{T} - \frac{\tilde{P}\tilde{\eta}}{\tilde{\rho}^2}\nabla\tilde{\rho} \tag{55d}$$

$$= \frac{\tilde{C}_{\mathrm{p}}\tilde{T}}{\tilde{\delta}\tilde{\rho}}\left[\frac{\nabla\tilde{P}}{\tilde{c}_{\mathrm{s}}^2} - \nabla\tilde{\rho}\right]. \tag{55e}$$

We assume that there is no flow in the reference state, i.e.,[9]

$$\tilde{\boldsymbol{u}} \equiv 0 \tag{56}$$

and thus thus $\widehat{\boldsymbol{u}_{\text{tot}}} = \overline{\boldsymbol{u}_{\text{tot}}}$. From Equation (26), we find $\boldsymbol{u}'_{\text{tot}} = (\boldsymbol{u}_{\text{tot}})_1 + \widehat{\boldsymbol{u}_{\text{tot}}} = (\boldsymbol{u}_{\text{tot}})_1 + O(\epsilon) = \boldsymbol{u} + O(\epsilon)$. Thus, although $\boldsymbol{u}$ formally refers to $(\boldsymbol{u}_{\text{tot}})_1$, it can be replaced with $\boldsymbol{u}'_{\text{tot}}$ without affecting the validity of the anelastic approximation. For "symmetry," readers might therefore wish to interpret $\boldsymbol{u}$ as $\boldsymbol{u}'_{\text{tot}}$ in this section (while remembering $\overline{\boldsymbol{u}} \equiv 0$).

The linearized equation of state for the primed quantities is exactly analogous to Equations (34),

$$\tilde{T}S' = \tilde{C}_{\text{p}}T' - \frac{\tilde{\delta}}{\tilde{\rho}}P' \tag{57a}$$

$$= U' - \frac{\tilde{P}}{\tilde{\rho}^2}\rho' \tag{57b}$$

$$= h' - \frac{P'}{\tilde{\rho}} \tag{57c}$$

$$= \tilde{C}_{\text{v}}T' - \frac{\tilde{P}\tilde{\eta}}{\tilde{\rho}^2}\rho' \tag{57d}$$

$$= \frac{\tilde{C}_{\text{p}}\tilde{T}}{\tilde{\delta}\tilde{\rho}}\left[\frac{P'}{\tilde{c}_{\text{s}}^2} - \rho'\right]. \tag{57e}$$

as is the linearized equation of state for the hatted quantities. Note that in Equation (34), the overbars may be replace by tildes, making Equations (57) applicable to the "subscript 1" thermal perturbations as well.

To zeroth order in ϵ, Equation (31) becomes simply

$$\nabla \cdot (\tilde{\rho}\boldsymbol{u}) \equiv 0. \tag{58}$$

Because the right-hand side of Equations (32) is $O(\epsilon)$ compared to the left-hand side we write, using Equations (34) (for the hatted quantities) and (44),

$$[-\nabla\overline{P} + \overline{\rho}\boldsymbol{g}] = -\nabla\hat{P} + \hat{\rho}\boldsymbol{g}$$

$$= -\tilde{\rho}\nabla\left(\frac{\hat{P}}{\tilde{\rho}}\right) - \tilde{\delta}\tilde{\rho}\left(\frac{\hat{S}}{\tilde{C}_{\text{p}}}\right)\boldsymbol{g} + \frac{\tilde{\delta}\hat{P}}{\tilde{C}_{\text{p}}}\nabla\tilde{S}. \tag{59}$$

This is analogous to Equation (45).

Substituting Equation (59) into Equation (48) and noting that all terms in the new

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[9]To partially avoid likely unavoidable confusion, note that a tilde here does not refer to an operator. Therefore, the meaning of $\tilde{\boldsymbol{u}}$ is unambiguously "the total velocity associated with the reference state." The hats, primes, and "1 subscripts" are operators, as is writing $\boldsymbol{u}$ with no subscript. When using operators, we must explicitly refer to $\boldsymbol{u}_{\text{tot}}$ and $\boldsymbol{u} \coloneqq (\boldsymbol{u}_{\text{tot}})_1 = \boldsymbol{u}_1$. For example, $\boldsymbol{u}'_{\text{tot}} = \boldsymbol{u}_{\text{tot}} - \tilde{\boldsymbol{u}}$ and $\boldsymbol{u}' = \boldsymbol{u}_1 - \tilde{\boldsymbol{u}}$, so $\boldsymbol{u}'_{\text{tot}} \neq \boldsymbol{u}'$.

equation are of $O(\epsilon)$ (so that we can replace overbars with tildes), we find

$$\frac{\partial}{\partial t}(\tilde{\rho}\boldsymbol{u}) = -\nabla \cdot (\tilde{\rho}\boldsymbol{u}\boldsymbol{u}) - \tilde{\rho}\nabla\left(\frac{P'}{\tilde{\rho}}\right) - \tilde{\delta}\tilde{\rho}\left(\frac{S'}{\tilde{C}_{\mathrm{p}}}\right)\boldsymbol{g} + \frac{\tilde{\delta}P'}{\tilde{C}_{\mathrm{p}}}\nabla\tilde{S} + \nabla \cdot \overset{\leftrightarrow}{D},$$

(60a)

where now $\quad D_{ij} := \tilde{\mu}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}(\nabla \cdot \boldsymbol{u})\delta_{ij}\right],$ (60b)

Using Equation (28), Equation (29) can be additionally written as

$$\overline{\rho}\boldsymbol{u} + \rho_1\boldsymbol{u} - \overline{\rho_1\boldsymbol{u}} = \tilde{\rho}\boldsymbol{u} + \rho'\boldsymbol{u} - \overline{\rho'\boldsymbol{u}} + O(\epsilon^2).$$

(61)

We finally write

$$Q_{\mathrm{NLBR}} = -\tilde{\rho}T_1\boldsymbol{u} \cdot \nabla\tilde{S} + O(\epsilon^2)$$

(62)

and

$$-\overline{\rho}\overline{T}\boldsymbol{u} \cdot \nabla S_1 - \overline{\rho}\overline{T}\boldsymbol{u} \cdot \nabla\overline{S} = -\overline{\rho}\overline{T}\boldsymbol{u} \cdot \nabla S' - \overline{\rho}\overline{T}\boldsymbol{u} \cdot \nabla\tilde{S}$$
$$= -\tilde{\rho}\tilde{T}\boldsymbol{u} \cdot \nabla S' - \tilde{T}(\overline{\rho}\boldsymbol{u}) \cdot \nabla\tilde{S} - \tilde{\rho}\hat{T}\boldsymbol{u} \cdot \nabla\tilde{S} + O(\epsilon^2).$$ (63)

(These "mixed terms" make the substitutions easier). Substituting Equations (54) and (61)–(63) into Equation (49) yields

$$\tilde{\rho}\tilde{T}\frac{\partial S'}{\partial t} = -\tilde{\rho}\tilde{T}\boldsymbol{u} \cdot \nabla S' - \tilde{\rho}\tilde{T}\boldsymbol{u} \cdot \nabla\tilde{S} - \tilde{\rho}T'\boldsymbol{u} \cdot \nabla\tilde{S} + D_{ij}\frac{\partial u_i}{\partial x_j} + Q' - \nabla \cdot \boldsymbol{F}'$$
$$- \tilde{T}(\rho'\boldsymbol{u} - \overline{\rho'\boldsymbol{u}}) \cdot \nabla\tilde{S},$$ (64a)

where $\quad \rho' = \frac{P'}{\tilde{c}_{\mathrm{s}}^2} - \frac{\tilde{\delta}\tilde{\rho}S'}{\tilde{C}_{\mathrm{p}}},$ (64b)

$$T' = \frac{\tilde{T}S'}{\tilde{C}_{\mathrm{p}}} + \frac{\tilde{\delta}P'}{\tilde{\rho}\tilde{C}_{\mathrm{p}}},$$ (64c)

and D_{ij} is now given by Equation (60b).

Taking the horizontal means of Equations (60) and (64) and then using Equations (28), (53), (54), and (59) recovers the mean momentum and energy equations (36) and (50). The anelastic formulation with fixed reference states (Equations (58), (60) and (64)) is thus seen to be asymptotically equivalent to the formulation with horizontally averaged background states (Equations (31), (48), and (49), combined with Equations (36) and (50)), provided the restriction (51) always holds.

Intuitively, none of the results of this section should seem too surprising and may even seem obvious. However, formalizing the equivalence between mean-state expansions of the anelastic equations and reference-state expansions via (21) and (51) is important. The mean-state anelastic formulation consistently allows for time-varying mean thermal states of the fully compressible system, subject only to the slow-variation constraint (21). Furthermore, the mean-state equations can allow for secular change in the mean state (e.g., a growing or shrinking convection zone) self-consistently, as long as the evolution is slow. The reference-state formulation, by contrast, does not allow

17

for secular changes in the mean state, because the total change must remain small as per Equation (51).

One pernicious aspect of the reference-state formulation is that the equations have no way to "inform" the evolution if the reference state was chosen unwisely. For example, a fluid layer with a reference state violating hydrostatic balance at zeroth-order may behave as a "normal" low-Mach number fluid under Equations (31), (48), and (49) (a physically unreasonable result). Under the mean-state formalism by contrast, a fluid layer with initial mean state not in hydrostatic balance should explode or collapse via Equation (36) (a physically reasonable result). On the other hand, during the explosion or collapse, the fluid would likely develop very large thermal perturbations and the original assumption of anelasticity (either via the mean-state or reference-state formalism) would have been made unjustly.

To summarize: as long as the true fully compressible system is not expected to change its mean state by more than $O(\epsilon)$ during evolution, either the mean-state or reference-state anelastic equations may be used with asymptotically equivalent results. If secular variation of the real system *is* expected, then the mean-state formulation should be used. However, the expected variation must be slow enough to satisfy Equation (21).

6   The Anelastic Energy-conserving Generalized Gough (AnEGG) approximation

The final equations, representing what we call the Anelastic Energy-conserving Generalized Gough (AnEGG) approximation, written for time-independent reference states satisfying Equations (53)–(56), are repeated here for reference:

$$\nabla \cdot (\tilde{\rho}\boldsymbol{u}) \equiv 0, \tag{65a}$$

$$\frac{\partial}{\partial t}(\tilde{\rho}\boldsymbol{u}) = -\nabla \cdot (\tilde{\rho}\boldsymbol{u}\boldsymbol{u}) - \tilde{\rho}\nabla\left(\frac{P'}{\tilde{\rho}}\right) - \tilde{\delta}\tilde{\rho}\left(\frac{S'}{\tilde{C}_{\mathrm{p}}}\right)\boldsymbol{g} + \underbrace{\frac{\tilde{\delta}P'}{\tilde{C}_{\mathrm{p}}}\nabla\tilde{S}}_{:=\boldsymbol{f}_{\mathrm{NLBR}}} + \nabla \cdot \overleftrightarrow{D}, \tag{65b}$$

$$\text{and} \quad \tilde{\rho}\tilde{T}\frac{\partial S'}{\partial t} = -\tilde{\rho}\tilde{T}\boldsymbol{u}\cdot\nabla S' \underbrace{-\tilde{\rho}\tilde{T}\boldsymbol{u}\cdot\nabla\tilde{S}}_{:=Q_{\mathrm{adv}}} \underbrace{-\tilde{\rho}T'\boldsymbol{u}\cdot\nabla\tilde{S}}_{:=Q_{\mathrm{NLBR}}} \underbrace{-\tilde{T}(\rho'\boldsymbol{u} - \overline{\rho'\boldsymbol{u}})\cdot\nabla\tilde{S}}_{:=Q_{\mathrm{nomassflux}}}$$

$$+ D_{ij}\frac{\partial u_i}{\partial x_j} + Q' - \nabla\cdot\boldsymbol{F}', \tag{65c}$$

$$\text{where} \quad \rho' = \frac{P'}{\tilde{c}_{\mathrm{s}}^2} - \frac{\tilde{\delta}\tilde{\rho}S'}{\tilde{C}_{\mathrm{p}}} \tag{65d}$$

$$\text{and} \quad T' = \frac{\tilde{T}S'}{\tilde{C}_{\mathrm{p}}} + \frac{\tilde{\delta}P'}{\tilde{\rho}\tilde{C}_{\mathrm{p}}}. \tag{65e}$$

$\overleftrightarrow{D}$ is now defined by Equation (60b). We have redefined $\boldsymbol{f}_{\mathrm{NLBR}}$ and Q_{NLBR}. We also have defined the heating due to advection through the imposed reference-state stratification (Q_{adv}) and the heating due to the requirement (expressed by Equation (61)) that the anelastic equations contain no mean mass flux ($Q_{\mathrm{nomassflux}}$).

The AnEGG kinetic energy equation, derived from $\boldsymbol{u}$ dotted into Equation (65b), is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\tilde{\rho}u^2\right) = -\nabla \cdot \left(\frac{1}{2}\tilde{\rho}u^2\boldsymbol{u}\right) - \nabla \cdot (P'\boldsymbol{u}) - \tilde{\delta}\tilde{\rho}\left(\frac{S'}{\tilde{C}_{\mathrm{p}}}\right)\boldsymbol{u}\cdot\boldsymbol{g} + \frac{\tilde{\delta}P'}{\tilde{C}_{\mathrm{p}}}\boldsymbol{u}\cdot\nabla\tilde{S} + u_i\frac{\partial D_{ij}}{\partial x_j}. \tag{66}$$

Adding Equations (65c) and (66) yields the AnEGG total energy equation,

$$\frac{\partial}{\partial t}\left[\tilde{\rho}\left(\frac{1}{2}u^2 + \tilde{T}S'\right)\right] = -\nabla \cdot \left\{\left[\tilde{\rho}\left(\frac{1}{2}u^2 + \tilde{T}S'\right) + P'\right]\boldsymbol{u} - \boldsymbol{u}\cdot\overset{\leftrightarrow}{D} + \boldsymbol{F}'\right\} + Q'$$
$$- \tilde{\rho}\tilde{T}\boldsymbol{u}\cdot\nabla\tilde{S} - \tilde{T}(\rho'\boldsymbol{u} - \overline{\rho'\boldsymbol{u}})\cdot\nabla\tilde{S}. \tag{67}$$

Again using condition (28) (or equivalently, Equation (65a) and assuming $w \equiv 0$ on the boundaries), integrating Equation (67) over the volume V of the fluid layer, using the divergence theorem, and assuming that the sum of the surface-integrated fluxes balances the volume-integrated heating and cooling yields (analogously to Equation (42))

$$E_{\mathrm{tot}}^{\mathrm{EGG}} := \int_V \left[\underbrace{\frac{1}{2}\tilde{\rho}u^2}_{:=W_{\mathrm{KE}}^{\mathrm{EGG}}} + \underbrace{\tilde{\rho}\tilde{T}S'}_{:=W_{\mathrm{int}}^{\mathrm{EGG}}}\right]dV \equiv \text{constant}. \tag{68}$$

We have used the superscript "EGG" to denote energies and densities under the AnEGG approximation and defined the kinetic and internal energy densities $W_{\mathrm{KE}}^{\mathrm{EGG}}$ and $W_{\mathrm{int}}^{\mathrm{EGG}}$ (respectively). We additionally define

$$W_{\mathrm{tot}}^{\mathrm{EGG}} := W_{\mathrm{KE}}^{\mathrm{EGG}} + W_{\mathrm{int}}^{\mathrm{EGG}} = \tilde{\rho}\left(\frac{1}{2}u^2 + \tilde{T}S'\right) \tag{69a}$$

$$\text{and} \quad W_{\mathrm{NKE}}^{\mathrm{EGG}} := W_{\mathrm{tot}}^{\mathrm{EGG}} - W_{\mathrm{KE}}^{\mathrm{EGG}} = W_{\mathrm{int}}^{\mathrm{EGG}} = \tilde{\rho}\tilde{T}S'. \tag{69b}$$

This conservation law holds for arbitrary fluid motions obeying Equations (65) and for all magnitudes of $|\nabla\tilde{S}|$.

Note that in practice when simulating stiff systems (large $|\nabla\tilde{S}|$) numerically (e.g., Guerrero et al. 2016; Matilsky et al. 2022; Korre & Featherstone 2024; Matilsky et al. 2024), the term Q_{adv} may pointwise be quite large (though it can never alter the deviation of the mean stability from that of the reference state, $\partial\hat{S}/\partial q$). The degree to which energy is conserved numerically may thus be limited by the precision of the condition (41). In the streamfunction formulation of ASH and Rayleigh for example (e.g., Clune et al. 1999; Featherstone & Hindman 2016), Equation (41) holds to near machine precision.

To summarize this section, one set of conditions for anelastic codes implementing background stable layers to conserve energy are (1) including the non-LBR force density,

$$\boldsymbol{f}_{\mathrm{NLBR}} := \frac{\tilde{\delta}P'}{\tilde{C}_{\mathrm{p}}}\boldsymbol{u}\cdot\nabla\tilde{S} \tag{70}$$

in the momentum equation (i.e., not making the LBR approximation) and (2) including the non-LBR heating term,

$$Q_{\mathrm{NLBR}} := -\tilde{\rho}T'\boldsymbol{u}\cdot\nabla\tilde{S} \tag{71}$$

in the internal energy equation. Alternatively, the LBR approximation can still be made by setting $\boldsymbol{f}_{\mathrm{NLBR}} \equiv 0$ in Equation (65b); then total energy will be conserved if Q_{NLBR} in Equation (65c) is replaced by

$$Q_{\mathrm{LBR}} := -\left(\frac{\tilde{\rho}\tilde{T}}{\tilde{C}_{\mathrm{p}}}\right) S' \boldsymbol{u} \cdot \nabla \tilde{S}. \tag{72}$$

However, doing this would yield an asymptotically inconsistent equation set if $|\nabla \tilde{S}|$ is large.

Furthermore, the term $Q_{\mathrm{nomassflux}}$ in Equation (65c), although it does not affect the total energy conservation law (68), should be kept, both for asymptotic consistency and for consistency with the assumption of zero mean mass flux. Moreover, in rotating spherical shells, it is often the case that small deviations from "thermal wind balance" (e.g., Rempel 2005; Vallis 2017; Matilsky 2023) are the primary driver of the all-important meridional circulation (e.g., Choudhuri 2020). Thus, although $Q_{\mathrm{nomassflux}} \ll Q_{\mathrm{adv}}$ (as long as ρ' remains $\ll \tilde{\rho}$), it may still play a large role in the residual energy balance that determines the latitudinal thermal profiles of a rotating spherical system. Put more simply (for all geometries, not just spherical), if the anelastic $\boldsymbol{u}$ "wants" to transport net mass (i.e., if $\overline{\rho' \boldsymbol{u}} = \overline{\rho' \boldsymbol{u}'_{\mathrm{tot}}} + O(\epsilon^2) \neq 0$), the term $Q_{\mathrm{nomassflux}}$ ensures that there is no accompanying heat transport. This is consistent with the behavior of the fully compressible system, which should not transport any net mass or associated heat.

7 The meaning of total energy under the anelastic approximation

In this section, we discuss how the expression for total energy under the anelastic approximation $W_{\mathrm{tot}}^{\mathrm{EGG}}$ (or equivalently, $W_{\mathrm{tot}}^{\mathrm{an}}$) arises. The form of $W_{\mathrm{tot}}^{\mathrm{EGG}}$ places unique constraints on possible avenues for the transfer of energy under the anelastic approximation, and these fundamentally differ from the avenues available for the true compressible motion. It is found that all compressible effects, including the potential energy, are entirely eliminated from the energy budget.

One perhaps surprising corollary is that *no transfer between gravitational potential and kinetic energy is possible for anelastic convection.* Given the prevalence of the concept of "available potential energy" (APE), especially in atmospheric physics (e.g., Vallis 2017, p. 138), these may seem like fighting words, and so we tread with extra mathematical caution in this section. The true available energy (which fundamentally is due to changes in internal energy from heating or cooling, i.e., entropy changes) can resemble a potential energy because of a pathological coincidence that is unique to adiabatically stratified background states and constant $\tilde{C}_{\mathrm{p}}/\tilde{\delta}$. Since these conditions are satisfied for a fully and vigorously convecting perfect gas (which is thus nearly adiabatically stratified), in many cases the distinction between potential and heat energy is a philosophical one. However, in more general circumstances (such as convective overshoot into a stably stratified region), the distinction is important.

We note a priori that the zero mean mass flux condition (20) immediately eliminates possible conversion to and from potential energy for the fully compressible flow (since no net mass can be transferred vertically). Thus, it may not be surprising after all that

potential energy disappears from the anelastic energy budget. Perhaps one of the main new results of this work is that the assumption of zero mean mass flux is *fundamental* for the energetic consistency of the anelastic approximation.

7.1 General expression for W'_{NKE}

From Equations (9b) and (57b), the perturbed[10] nonkinetic energy density for the fully compressible system is

$$
\begin{aligned}
W'_{\text{NKE}} &= \tilde{\rho}U' + \tilde{U}\rho' + \Phi\rho' \\
&= \underbrace{\tilde{\rho}\tilde{T}S'}_{:=W'_{\text{heat}}} + \underbrace{\left(\tilde{U} + \frac{\tilde{P}}{\tilde{\rho}} + \Phi\right)\rho'}_{:=W'_{\text{comp}}}.
\end{aligned}
\tag{73}
$$

Equation (57b) is really an expression of the first law of thermodynamics, which states that changes in the internal energy of fluid parcels come from the sum of irreversible heating processes and pressure work done by the environment during compression. Comparing to Equation (69b), it is clear that *only* changes in internal energy from heat sources (W'_{heat}, defined in Equation (73)) affect the nonkinetic energy density for the anelastic system (i.e., this is why $W^{\text{EGG}}_{\text{NKE}} = W^{\text{EGG}}_{\text{int}} = \tilde{\rho}\tilde{T}S'$). The changes in internal energy from density variations $\tilde{U}\rho'$, the pressure work $\tilde{P}(\rho'/\tilde{\rho})$, and the changes in potential energy $\Phi\rho'$ are all absent. Since all three of these latter effects are due to density variations (i.e., compression or expansion of the fluid), we identify the term W'_{comp}, defined in Equation (73).

Additionally, we write the perturbed form of Equation (9a) as

$$
W'_{\text{tot}} := \underbrace{\frac{1}{2}\tilde{\rho}u^2}_{:=W'_{\text{KE}}} + W'_{\text{NKE}} = W'_{\text{KE}} + W'_{\text{heat}} + W'_{\text{comp}},
\tag{74}
$$

where we have identified the kinetic energy due to small thermal perturbations about the reference state ($W'_{\text{KE}} = W^{\text{EGG}}_{\text{KE}}$). W'_{tot} represents the total energy density whose volume integral the true fully compressible fluid would conserve (at first order in ϵ) under the assumption (18) of small thermal perturbations. Thus, the fully compressible motion is formally able to convert kinetic energy to and from internal energy from heating W'_{heat} *and* to and from the compression effects accounted for by W'_{comp}. However, it *should not* convert (net) energy to and from W'_{comp} if the zero-mean-mass-flux assumption (20) is valid. By contrast, the anelastic motion is formally *unable* to convert kinetic energy to and from W'_{comp} under any circumstances and can only convert between kinetic energy and internal energy from heat sources. This is a direct consequence of (and is energetically consistent with) the assumption of zero mean mass flux.

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[10]As in, perturbed from the reference state. However, all these arguments go through analogously for the time-varying mean-state formulation of the anelastic approximation presented in Section 4.

7.2 Special form of W'_{NKE} for adiabatically stratified perfect gases

In contrast to Equation (73), Ogura & Phillips (1962) write

$$W'_{\mathrm{NKE}} := \underbrace{-\tilde{\rho}\Phi\left(\frac{S'}{C_{\mathrm{p}}}\right)}_{:=W'_{\mathrm{pot}}} + \underbrace{\tilde{\rho}C_{\mathrm{v}}\tilde{T}_0\left(\frac{P'}{\tilde{P}}\right)}_{:=W'_{\mathrm{elast}}}, \tag{75}$$

Here, $\tilde{C}_{\mathrm{p}} \equiv C_{\mathrm{p}}$ and $\tilde{C}_{\mathrm{v}} \equiv C_{\mathrm{v}}$ are constants, since Ogura & Phillips (1962) assumed a perfect gas. The gravitational potential Φ is measured from the location q_0 of the reference pressure for the potential temperature (in Ogura & Phillips 1962, this is the bottom of the fluid layer) and $\tilde{T}_0 := \tilde{T}(q_0)$.

The reference state is assumed to be adiabatic,

$$\frac{d\tilde{S}}{dq} \equiv 0, \tag{76}$$

and thus the background potential temperature is constant and equal to $\tilde{T}_0$.[11]

Eckart 1960 (p. 54) discussed the energy density associated with small perturbations about a background static atmosphere and identified a term with pressure perturbations, $P'^2/(2\tilde{\rho}\tilde{c}_{\mathrm{s}}^2)$, as the "elastic" energy density from acoustics (for the acoustics definition of the term, see Morse 1948, p. 237). Thus, in Equation (75), Ogura & Phillips (1962) identified two elements of the perturbed nonkinetic energy: the "potential" energy W'_{pot} and the "elastic" energy W'_{elast} due to pressure perturbations. Ogura & Phillips (1962) note that W'_{elast} disappears from the equation of total energy conservation (see Ogura & Phillips 1962's Equation (31)). This disappearance is apparently the origin of the term "anelastic," coined by Jule Charney.

However, in light of Equation (73), which must be the more general form of Equation (75), it is clear that the partition $W'_{\mathrm{NKE}} = W'_{\mathrm{pot}} + W'_{\mathrm{elast}}$ can only hold under very specific circumstances, namely one in which $\tilde{T}$ is proportional to Φ. It will be found in this section that the latter condition holds only for an adiabatically stratified fluid where also $\tilde{C}_{\mathrm{p}}/\tilde{\delta} \equiv$ constant. Furthermore, it turns out that the partition of Equation (75) *only* holds for the unique circumstance of a gas that is not only adiabatically stratified, but also perfect (we assume up front that the gas is hydrostatically stratified). This should not be a surprise (since Ogura & Phillips 1962 assumed a perfect gas to begin with), however, it is useful to go through the proof in order to understand that the appearance of W'_{pot} as a potential energy (which, it will be found, it fundamentally is not) is almost exclusively unique to adiabatically stratified perfect gases.

We use hydrostatic balance (53) and the first law of thermodynamics for the reference-state gradients (55) to yield

$$\frac{d\Phi}{dq} = -\frac{\tilde{C}_{\mathrm{p}}}{\tilde{\delta}}\frac{d\tilde{T}}{dq} + \frac{\tilde{T}}{\tilde{\delta}}\frac{d\tilde{S}}{dq}. \tag{77}$$

---

[11]See the original equation immediately following Ogura & Phillips (1962)'s Equation (31). To see the equivalence to our Equation (75), note that Ogura & Phillips (1962) use θ for the nondimensional order-unity perturbed potential temperature, gz' for the dimensional gravitational potential (equal to Φ here), and Θ for the constant dimensional potential temperature (equal to $\tilde{T}_0$ here) of the adiabatically stratified background. Also note the relation $S' = \epsilon C_{\mathrm{p}}\theta$.

If and only if

$$\frac{\tilde{C}_p}{\tilde{\delta}} \equiv \text{constant} := \frac{C_p}{\delta} \tag{78}$$

(the constancy of the individual coefficients comes later; we also write $\delta/C_p := (C_p/\delta)^{-1}$), Equation (77) can be integrated to yield

$$\Phi = -\frac{C_p}{\delta}(\tilde{T} - \tilde{T}_0) + \int_{q_0}^{q} \frac{\tilde{T}(q')}{\tilde{\delta}(q')} \frac{d\tilde{S}}{dq'} dq' \tag{79}$$

Then, if and only if the background is adiabatically stratified ($d\tilde{S}/dq \equiv 0$), there exists a simple proportionality between temperature and gravitational potential:[12]

$$\Phi = -\frac{C_p}{\delta}(\tilde{T} - \tilde{T}_0). \tag{80}$$

Using Equations (57e) and (80), Equation (73) becomes

$$W'_{\text{NKE}} = \tilde{\rho}\left[-\frac{\delta}{C_p}\Phi - \frac{\delta}{C_p}\left(\tilde{h} - \frac{C_p}{\delta}\tilde{T}\right)\right]S' + \left[\left(\tilde{h} - \frac{C_p}{\delta}\tilde{T}\right) + \frac{C_p}{\delta}\tilde{T}_0\right]\frac{P'}{\tilde{c}_s^2}. \tag{81}$$

This reduces to Equation (75) if and only if

$$\tilde{h} = \frac{C_p}{\delta}\tilde{T} = \tilde{h}(\tilde{T}). \tag{82}$$

Comparing Equations (55c) and (55a), Equation (82) implies $\tilde{\delta} \equiv \delta = 1$ and thus $\tilde{C}_p \equiv C_p = \text{constant}$ by Equation (78).

From Equations (55d), (55e), and (76), it can be shown that

$$\frac{C_p}{\tilde{c}_s^2} = \frac{\tilde{\rho}\tilde{C}_v}{\tilde{\eta}\tilde{P}}. \tag{83}$$

Substituting Equations (82) and (83) into Equation (81) then yields

$$W'_{\text{NKE}} = -\tilde{\rho}\Phi\left(\frac{S'}{C_p}\right) + \frac{\tilde{\rho}\tilde{C}_v\tilde{T}_0}{\tilde{\eta}}\left(\frac{P'}{\tilde{P}}\right), \tag{84}$$

which finally reduces to Equation (75) if and only if $\tilde{C}_v \equiv C_v = \text{constant}$ and $\tilde{\eta} \equiv 1$. It can be shown that the conditions $\tilde{\delta} \equiv \tilde{\eta} \equiv 1$ imply the ideal gas law $\tilde{P} = (C_p - C_v)\tilde{\rho}\tilde{T}$ (and by extension, $\tilde{U} = \tilde{U}(\tilde{T})$), and the condition $\tilde{C}_p = \text{constant}$ then implies that the gas is perfect.

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[12]We exclude the fringe case $\tilde{T}d\tilde{S} = (\tilde{C}_p + K\tilde{\delta})d\tilde{T}$, where K is a constant. This would yield $d\Phi = Kd\tilde{T}$ in Equation (77) even with nonzero $d\tilde{S}/dq$. However, from Equation (55a), this would be possible if and only if $\tilde{\delta} \equiv 0$ (as occurs, for example, in degenerate matter). In that case, hydrostatic balance would decouple from the rest of the thermodynamics.

7.3 The fundamental importance of the zero-mean-mass-flux assumption

Technically, the same conclusion reached by Ogura & Phillips (1962) (that the term W'_{pot} comes from residual potential energy) is suggested by Equation (84), even for the slightly more general case of adiabatically stratified but nonideal gases ($\tilde{C}_{\text{v}} \neq$ constant, $\tilde{\eta} \neq 1$, $\tilde{U} \neq \tilde{U}(\tilde{T})$). However, given that $\tilde{h} = \tilde{h}(\tilde{T})$, $\tilde{\delta} \equiv 1$, and $\tilde{C}_{\text{p}} \equiv$ constant are all necessary conditions for the term W'_{pot} to appear, it seems likely that adiabatically stratified perfect gases are the most common systems where the coincidence occurs.

The main point is that the term W'_{pot} in no way derives from a true gravitational potential energy and the pressure-fluctuation term W'_{elast} is not the term eliminated from anelastic energy conservation. The seeming partition of W'_{NKE} into potential and "elastic" components (either Equation (75) or Equation (84)) is a coincidence unique to adiabatic stratification and arises almost exclusively in perfect gases. The true partition of W'_{NKE} (Equation (73)) yields two components: one due due changes in internal energy due to to irreversible heating (W'_{heat}) and one due to compression effects (W'_{comp}). Under the anelastic approximation, W'_{comp} (which contains the potential energy) is entirely eliminated from the energy budget. In light of this discussion, a better name for the "anelastic" approximation might be the "incompressible" approximation, although such a rebranding would not be possible for obvious reasons.

For the true fully compressible fluid, Equation (20) applied to the continuity equation (1) yields the likely obvious result

$$\frac{\partial \overline{\rho}}{\partial t} \equiv 0. \tag{85}$$

In other words, there can be no net increase or decrease in mass at any particular vertical level (here the vanishing of the horizontal components of $\overline{\rho \boldsymbol{u}_{\text{tot}}}$ becomes necessary[13]). As such, no increase or decrease in energy due to compression effects (including potential energy) is possible. If at any q or t, Equation (85) does *not* hold for the compressible motion, then there will be net increase or decrease of W'_{comp}, which is not accounted for by the anelastic motion. *Therefore, the compressible motion is energetically consistent with the anelastic approximation if and only if the zero mean mass flux condition holds.*[14]

Furthermore, the anelastic approximation conserves total energy to begin with (for all magnitudes of $|\nabla \tilde{S}|$) if and only if at least the vertical component of the mass flux, $\overline{\rho w_{\text{tot}}}$, is assumed to vanish up to $O(\epsilon)$. To see this, first note that if we do not assume that any of the components of $\overline{\rho \boldsymbol{u}_{\text{tot}}}$ vanish a priori, then the anelastic condition takes the familiar form $\nabla \cdot (\tilde{\rho} \boldsymbol{u}_{\text{tot}}) \equiv 0$, where now the total velocity $\boldsymbol{u}_{\text{tot}}$ appears in the anelastic equations instead of $\boldsymbol{u} \coloneqq (\boldsymbol{u}_{\text{tot}})_1$ (e.g., Ogura & Phillips 1962). This is

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[13]Or the vanishing of the horizontal components of $\boldsymbol{u}$ on the horizontal boundaries, or semi-periodic boundaries, as in a spherical shell. The only requirement is that $\nabla \cdot (\overline{\rho \boldsymbol{u}_{\text{tot}}}) \equiv 0$.

[14]This statement needs to be tweaked slightly in, e.g., a rotating spherical system. In that case, there is clearly mean mass flux in the zonal direction. The true fundamental requirement for energetic consistency is that Equation (85) hold, which leaves slightly more general conditions for the mass flux than simply $\overline{\rho \boldsymbol{u}_{\text{tot}}} \equiv 0$ (i.e., all three vector components of the mass flux vanishing). If w_{tot} vanishes on the vertical boundaries at least, then the vertical component $\overline{\rho w_{\text{tot}}} \equiv 0$ must hold after integrating Equation (85) in q. Thus, under most circumstances, the total anelastic energy conservation equation (68) should be unaffected by nonvanishing (horizontal) mass-flux conditions.

vertically integrated to yield $\overline{w_{\mathrm{tot}}} \equiv 0$, as long as w_{tot} vanishes on the boundaries. We then expand the term representing Lagrangian evolution of heat energy in the fully compressible internal energy equation as

$$\rho T \frac{DS}{Dt} = \tilde{\rho}\tilde{T}\frac{\partial S'}{\partial t} + \tilde{\rho}\tilde{T}\boldsymbol{u}_{\mathrm{tot}} \cdot \nabla S' + \underbrace{\tilde{\rho}\tilde{T}\boldsymbol{u}_{\mathrm{tot}} \cdot \nabla \tilde{S}}_{:=-Q_{\mathrm{adv}}} + \underbrace{\rho'\tilde{T}\boldsymbol{u}_{\mathrm{tot}} \cdot \nabla \tilde{S}}_{:=-Q_{\mathrm{massflux}}} + \underbrace{T'\tilde{\rho}\boldsymbol{u}_{\mathrm{tot}} \cdot \nabla \tilde{S}}_{:=-Q_{\mathrm{NLBR}}} + O(\epsilon^2),$$

(86)

where we have defined the Lagrangian (or material) derivative $D/Dt := \partial/\partial t + \boldsymbol{u}_{\mathrm{tot}} \cdot \nabla$ and (slightly) redefined each heating term to accommodate $\boldsymbol{u} \to \boldsymbol{u}_{\mathrm{tot}}$, as well as the possibly nonvanishing mass flux (i.e., $Q_{\mathrm{nomassflux}} \to Q_{\mathrm{massflux}}$). Note that since $\tilde{\boldsymbol{u}} \equiv 0$ by Equation (56), $\boldsymbol{u}'_{\mathrm{tot}} = \boldsymbol{u}_{\mathrm{tot}}$ and we do not have to worry about expanding the velocity in ϵ.[15]

Since the continuity and momentum equations (65a) and (65b) are unmodified even if there may be mean mass flux (with the exception that $\boldsymbol{u} \to \boldsymbol{u}_{\mathrm{tot}}$), the replacement $Q_{\mathrm{nomassflux}} \to Q_{\mathrm{massflux}}$ is the only fundamental change to the equation set (and note that the volume integral of the new Q_{adv} still vanishes since $w_{\mathrm{tot}} \equiv 0$). Thus, total energy conservation remains possible if and only if the volume integral of Q_{massflux} vanishes. This is guaranteed if and only if we demand that $\overline{\rho'w_{\mathrm{tot}}} \equiv 0$, i.e., the vertical mass flux in both equation sets (fully compressible and anelastic) vanishes up to $O(\epsilon)$.

This whole discussion highlights that vanishing mean mass flux in the fully compressible system (or, from the most general Equation (85), no increase or decrease of mass at any vertical level at any time) is not just a choice used by Gough (1969) to make the expansion in ϵ neater. Instead, it is a fundamental assumption underpinning the anelastic approximation. It is necessary to guarantee not only energetic consistency between the anelastic and fully compressible motion, but also the ability for the anelastic motion to conserve total energy for all reference stratifications (values of $|\nabla \tilde{S}|$).

7.4 How current anelastic formulations became energetically nonconservative

From Equation (86), we begin to see why the AnEGG equations include the term Q_{NLBR} (which is crucial for energy conservation), but other anelastic equation sets do not (and hence do not conserve energy for nonadiabatic stratification). Gough (1969) nondimensionalized the internal energy equation (without using the entropy as a variable) and then linearized by expanding to first order in ϵ. The other approximations instead typically linearized the entropy (or potential temperature) equation (e.g., Batchelor 1953; Ogura & Phillips 1962; Lantz & Fan 1999). If this is done, the Lagrangian evolution of entropy under the fully compressible equations becomes

$$\frac{DS}{Dt} = \frac{\partial S'}{\partial t} + \boldsymbol{u}_{\mathrm{tot}} \cdot \nabla S' + \boldsymbol{u}_{\mathrm{tot}} \cdot \nabla \tilde{S} + O(\epsilon^2).$$

(87)

After multiplying by $\tilde{\rho}\tilde{T}$, the resulting anelastic internal energy equation then lacks the term Q_{massflux} (which is good for energy conservation, since vanishing fully compressible

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[15]Technically, we do, but Ogura & Phillips (1962) found that only the zeroth-order component of $\boldsymbol{u}_{\mathrm{tot}}$ appears in the equations anyway. The only difference now is that the velocity appearing in the anelastic equations may have a nonzero horizontal mean.

mass flux is typically not assumed a priori), but also lacks the crucial term Q_{NLBR} (which is bad for energy conservation). Interestingly, Gilman & Glatzmaier (1981) did expand the internal energy equation exactly in the form of Equation (86), but dropped both Q_{massflux} and Q_{NLBR} on the grounds that the background entropy gradient was assumed weak. Rogers & Glatzmaier (2005) technically did expand the internal energy equation for a perfect gas ($C_{\mathrm{v}}T$), but used the form $C_{\mathrm{v}}[DT/Dt + (\gamma - 1)T\nabla \cdot \boldsymbol{u}_{\mathrm{tot}}] =$ [sum of heating terms] before expanding $T = \tilde{T} + T' + O(\epsilon^2)$. Doing so implicitly assumes that $D\rho/Dt = -\rho\nabla \cdot \boldsymbol{u}_{\mathrm{tot}}$, or that the continuity equation is satisfied at all orders. Under the anelastic approximation of course, the continuity equation only holds at zeroth order (see Equation (65a)), resulting in inconsistent terms in Rogers & Glatzmaier (2005)'s internal energy equation that fail to conserve total energy.

By happenstance (or more likely a profound understanding of the true nature of anelasticity), Gough (1969) both assumed zero mass flux for the fully compressible flow and wrote the Lagrangian evolution of heat energy (i.e., $\rho T DS/Dt$) using different thermodynamic relations (specifically Equations (17d) and (17b)) in the Eulerian and advective components, respectively (see Gough (1969)'s Equation (2.9)). This had the fortunate consequence of properly subtracting the buoyancy work, $\rho'\boldsymbol{u}\cdot\boldsymbol{g}$, from the right-hand side of the internal energy equation. As we now show, this is the physical origin of the crucial term Q_{NLBR}, which at its essence, allows the proper conversion between heat internal energy and kinetic energy in the presence of background stratification.

Hewing more closely to the original notation of Gough (1969), Equation (65c) can be rewritten as (compare to Equations (33) and (47))

$$\frac{\partial}{\partial t}\underbrace{(\tilde{\rho}\tilde{T}S')}_{=W'_{\mathrm{heat}}} = \underbrace{-\nabla\cdot(\tilde{\rho}\tilde{T}S'\boldsymbol{u})}_{:=Q_{\mathrm{flux}}}\underbrace{-P'\nabla\cdot\boldsymbol{u}}_{:=Q_{\mathrm{comp}}}\underbrace{-\rho'\boldsymbol{u}\cdot\boldsymbol{g}}_{:=Q_{\mathrm{buoy}}}\underbrace{-\tilde{\rho}\tilde{T}\boldsymbol{u}\cdot\nabla\tilde{S}}_{=Q_{\mathrm{adv}}}\underbrace{-\tilde{T}(\rho'\boldsymbol{u} - \overline{\rho'\boldsymbol{u}})\cdot\nabla\tilde{S}}_{=Q_{\mathrm{nomassflux}}}$$
$$+ D_{ij}\frac{\partial u_i}{\partial x_j} + Q' - \nabla\cdot\boldsymbol{F}'. \tag{88}$$

Equation (88) shows once again that the internal energy from irreversible heating, $\tilde{\rho}\tilde{T}S'$, is the total nonkinetic energy under the anelastic approximation. Furthermore, this energy inside some finite volume can change via the integral of several heating densities, some of which we identify in Equation (88). The heating (or cooling) densities identified are due to: inward or outward advective flux of W'_{heat} through the surface of the volume (Q_{flux}), heating or cooling of the fluid by "compression" or "expansion" (Q_{comp}),[16] heating or cooling of the fluid via the buoyancy work (Q_{buoy}), and the familiar terms (which integrate to zero over any horizontal surface) Q_{adv} and $Q_{\mathrm{nomassflux}}$.

We write $Q_{\mathrm{buoy}} = g\rho'w$. In general, for buoyantly *accelerated* motion (as in convection, where $\partial\overline{S}/\partial q < 0$), cool overdense fluid should be falling ($\rho' > 0$ and $w < 0$) and warm underdense fluid should be rising ($\rho' < 0$ and $w > 0$). This is the essence of unstable stratification. Thus, for convection, we expect that $Q_{\mathrm{buoy}} < 0$ is actually a cooling term; the work done by buoyancy is positive and cools the fluid. By contrast, for buoyantly *decelerated* motion (as in convective overshoot, where $\partial\overline{S}/\partial q > 0$), cool

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[16]Note that there is no actual compression or expansion of the fluid in terms of changes in ρ', since the first-order continuity equation is not satisfied. Fluid parcels moving through background density stratification are "compressed" or "expanded" only in the sense that $D\ln\tilde{\rho}/Dt = \nabla\cdot\boldsymbol{u}$, which is another form of Equation (65a).

falling fluid should be underdense relative to the background state and warm rising fluid should be overdense. This is the essence of stable stratification. Thus, in a stable layer, we expect $Q_{\text{buoy}} > 0$ *is* a heating term; the work done by buoyancy is negative and heats the fluid.

Evidently, Q_{NLBR} is the residual of $Q_{\text{comp}} + Q_{\text{buoy}}$ that depends explicitly on the background stratification. Not including it when making the anelastic approximation does not allow the proper conversion between heat energy and kinetic energy when $d\tilde{S}/dq \neq 0$ and thus forces the system to be energetically nonconservative. It also seems likely that restratification of the fluid (as can occur during convective overshoot, for example) will not be accurately described unless the Q_{NLBR} is rightfully restored to the internal energy equation.

Finally, we note that the explicit presence of the buoyancy work in the heat equation means that W'_{heat} is not truly generated only by *irreversible* heating processes. Buoyancy work represents a 100% efficient conversion to and from kinetic energy and is therefore fully reversible (for example, what goes up in a convection zone must cool off and come down). This means that the anelastic approximation must violate the second law of thermodynamics in all circumstances, which we discuss briefly in the following section. Put another way, throwing away the first-order continuity equation means that the anelastic approximation has no way to conserve gravitational potential energy (recall the derivation of Equation (6)). Instead, the potential energy must be circumvented by forcing the buoyancy work to convert directly between internal and kinetic energy.

8 Anelastic nonconservation of mass and the second law of thermodynamics

Although there is no mean mass flux in both the fully compressible and anelastic equations (see Equations (20) and either (29) or (61)), total mass is *not* conserved at first order under the anelastic approximation. Instead of Equation (85), we have from (e.g.) Equation (64b),

$$\begin{aligned}
\frac{\partial \overline{\rho}}{\partial t} &= \frac{1}{\tilde{c}_{\text{s}}^2} \frac{\partial \overline{P}}{\partial t} - \frac{\tilde{\delta}\tilde{\rho}}{\tilde{C}_{\text{p}}} \frac{\partial \overline{S}}{\partial t} \\
&= \frac{1}{\tilde{c}_{\text{s}}^2} \frac{\partial \hat{P}}{\partial t} - \frac{\tilde{\delta}\tilde{\rho}}{\tilde{C}_{\text{p}}} \frac{\partial \hat{S}}{\partial t}.
\end{aligned} \tag{89}$$

$\partial \hat{P}/\partial t$ can be written in terms of $\partial \overline{w^2}/\partial t$ and $\partial \hat{S}/\partial t$ via Equations (36) and (59) and $\partial \hat{S}/\partial t$ comes directly from the mean of Equation (65c) (which is also Equation (50)). Thus, there is thus no guarantee that $\partial \overline{\rho}/\partial t \equiv 0$ and in general it will not. This should be expected, since one of the main purposes of the anelastic approximation is to "sound proof" the equations, which is accomplished by using only the zeroth-order part of the continuity equation. There is thus the somewhat odd result that the anelastic equations do not contain any mean mass flux, but *do* allow for net increases and decreases of mass at any particular vertical level.

Nonconservation of mass is also the reason the anelastic equations violate the second law of thermodynamics under all circumstances. In the absence of heat sources or sinks

27

$(Q \equiv 0, \boldsymbol{F} \equiv 0,$ and $\mu \equiv 0)$, the full compressible motion is adiabatic everywhere and thus conserves the system's total (extensive) entropy, $G_{\text{tot}} := \int_V \rho S dV$, at all orders. At first order, this means that for the fully compressible motion,

$$G'_{\text{tot}} := \int_V (\tilde{S}\rho' + \tilde{\rho}S')dV \equiv \text{constant}. \tag{90}$$

By contrast, for the anelastic motion, Equation (65c) (with $Q' \equiv 0$, $\boldsymbol{F}' \equiv 0$, and $\tilde{\mu} \equiv 0$) yields

$$\frac{dG'_{\text{tot}}}{dt} = \int_V \tilde{S}\frac{\partial\overline{\rho}}{\partial t}dV + \int_V \frac{Q_{\text{NLBR}}}{\tilde{T}}dV, \tag{91}$$

which $\neq 0$ (except at isolated instances) under any circumstances because $\partial\overline{\rho}/\partial t \neq 0$ and $\tilde{S} \neq 0$ (by the third law of thermodynamics, which states that $S > 0$ even when $T = 0$, i.e., absolute zero). Thus, there are no systems (even convection zones with $d\tilde{S}/dq \equiv 0$) for which the anelastic motion conserves total entropy.

The careful reader might at first complain about this sweeping statement for two obvious reasons. First, the fluid equations (1)–(4) are insensitive to the absolute value of S. In a convection zone, then (where Q_{NLBR} already vanishes identically), might we be free to set $\tilde{S} \equiv 0$ and thus conserve entropy? The answer is "no" and also a bit subtle. It is true that the fluid equations as written are insensitive to the absolute value of the *specific* entropy S. However, the second law of thermodynamics concerns the *total* (extensive) entropy G_{tot}, or for small thermal perturbations, G'_{tot}. The evolution of total entropy *does* depend on the absolute value of S for the simple reason that $\partial(\rho S)/\partial t = S\partial\rho/\partial t + \rho\partial S/\partial t$. Essentially then, Equation (91) states that because mass is not conserved at first order under the anelastic approximation, and because the total entropy can change via changes in mass, there is no way any anelastic approximation can satisfy the second law of thermodynamics.

The second complaint might be that *we* are (currently) the only ones advocating to introduce the terms Q_{NLBR} and $Q_{\text{nomassflux}}$. Anelastic simulations have for decades been perfectly fine not including these terms. Furthermore, Equation (91) clearly shows that by introducing Q_{NLBR} specifically, we are making the violation of the second law of thermodynamics worse.[17] If we examine the Lagrangian evolution of a fluid parcel's specific entropy with no heat sources or sinks, Equation (65c) yields

$$\frac{D}{Dt}(\tilde{S} + S') = \frac{Q_{\text{NLBR}} + Q_{\text{nomassflux}}}{\tilde{\rho}\tilde{T}}. \tag{92}$$

In other words, the AnEGG approximation has the seemingly undesirable result that fluid parcels do not conserve specific entropy during adiabatic motion. If we throw away Q_{NLBR} and $Q_{\text{nomassflux}}$, then perhaps energy would not be conserved, but at least specific entropy of fluid parcels would be conserved in the absence of heat sources and sinks. Our position is to keep Q_{NLBR} and $Q_{\text{nomassflux}}$ because specific entropy should *not* be conserved during anelastic adiabatic motion. Equation (91) clearly shows that the second law of thermodynamics is violated no matter what. On the righthand side of Equation (92), the missing term $-(\tilde{S}/\tilde{\rho})D\rho'/Dt$ is no less of an issue than the new term $(Q_{\text{NLBR}} + Q_{\text{nomassflux}})/(\tilde{\rho}\tilde{T})$.

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[17]The statements in this paragraph are, of course, only relevant for nonzero $d\tilde{S}/dq$, which is the only way to have nonzero Q_{NLBR} and $Q_{\text{nomassflux}}$.

We can summarize this section by stating that the anelastic asymptotic expansion is fundamentally a strange one, because all the equations are expanded to first order except the continuity equation, which is expanded only to zeroth order. This is bound to introduce some severe unphysical consequences. The root of all unphysicality is the nonconservation of mass at first order. As a result, the second law of thermodynamics seems unenforceable under all circumstances. By contrast, energy conservation when $d\tilde{S}/dq \neq 0$ is easily enforceable (and consistent with the true compressible motion) by assuming no mean mass flux and restoring the terms $\boldsymbol{f}_{\mathrm{NLBR}}$, Q_{NLBR}, and $Q_{\mathrm{nomassflux}}$ to the anelastic equations of motion.

9   Conditions for anelastic overshoot

Up to this point, we have carefully sidestepped the issue that Gough (1969) implicitly assumed small $|\nabla\tilde{S}|$.[18] Although we have shown that Gough (1969)'s equations conserve energy for all values of $|\nabla\tilde{S}|$, it still may be that Gough (1969)'s scale analysis presented in Section 3 breaks down, along with the subsequent nondimensionalization and asymptotic expansion leading to AnEGG. To systematically show that AnEGG can be used consistently for stably stratified layers would require a new scale analysis (considering overshoot and gravity-wave excitation instead of only convection) and subsequent formal asymptotic expansion of the fully compressible equations, which we do not attempt to do here. Instead, we argue that the anelastic approximation *may* remain valid in the limited scenario of convective overshoot, in the sense that the thermal perturbations should remain small when certain conditions are met.

The obvious issue with nonzero $d\tilde{S}/dq$ is that moving fluid may develop very large thermal perturbations via the term Q_{adv}, which violates the key assumption (18) of the anelastic approximation. For negative $d\tilde{S}/dq$ (i.e., in a convection zone accelerated by the reference state), there seems to be no way around this issue. If $|d\tilde{S}/dq| \sim C_{\mathrm{p}a}/H_a$, for example, a fluid parcel will be accelerated across the full length of H_a and develop a thermal perturbation $|S'/C_{\mathrm{p}a}| \sim H_a|d\tilde{S}/dq|/C_{\mathrm{p}a} = O(1)$. Instead, we must assume small $|d\tilde{S}/dq|$ for a convection zone only, making the whole discussion of energy conservation (and the new terms we introduce) irrelevant.

In a strongly stable layer, by contrast, an initially buoyant fluid parcel should be immediately decelerated and turned around before it has made it far enough to acquire a large thermal perturbation. The stronger the stable gradient is, the larger this restriction on total distance traveled will be.

In light of this idea, consider an overshooting convectively accelerated downflow plume with initial negative velocity w and initial thermal perturbation $|S'_{\mathrm{unstable}}|/C_{\mathrm{p}a} \sim \epsilon$. We ignore nonadiabatic processes (viscosity and heat conduction), as well as Q_{NLBR} and $Q_{\mathrm{nomassflux}}$, so that the plume conserves its total specific entropy $S' + \tilde{S}$ (for very strong stratification, Q_{adv} should be much larger than both Q_{NLBR} and $Q_{\mathrm{nomassflux}}$

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[18]This fact is "hidden" in Gough (1969)'s nondimensionalization of the background superadiabatic gradient, which appears in the unmarked equation immediately following his Equation (3.10). Here, Gough (1969) chooses the scale θ/d for β (recall Equation (30)), where d is the layer depth ($\sim H_a$ for our purposes) and θ is the integral of β across the convective region. Then, in Gough (1969)'s Equation (3.11), he implicitly assumes that $\theta/T_a \sim \epsilon$ for high Rayleigh number. Since $\theta/T_a \sim (1/C_{\mathrm{p}a})|\nabla\tilde{S}|$, this means $|\nabla\tilde{S}|$ is assumed to be small relative to $C_{\mathrm{p}a}/H_a$.

anyway). From the scale analysis of Section 3, we have

$$w^2 \sim g_a H_a \frac{|S'_{\text{unstable}}|}{C_{\text{pa}}} \sim g_a H_a \epsilon. \tag{93}$$

The plume then descends into a stably stratified layer, the top of which (also the base of the convection zone) is defined to be the lowermost location $q = q_0$ at which $\overline{S}$ equals its value at the top of the convection zone. Because we are considering strongly stable stratification (i.e., $|\nabla \overline{S}|$ is much greater in the stable layer than in the convection zone), we expect that q_0 is also quite close to the location of maximal $\overline{S}$. Therefore, we can expand $\tilde{S}$ about $q = q_0$ (and take $d\tilde{S}/dq = 0$ at $q = q_0$), provided the plume does not descend very far. How far is "very far" is made precise further below.[19] Because the plume conserves its total specific entropy, its entropy perturbation is given by

$$\frac{S'}{C_{\text{pa}}} \sim (q_0 - q)^2 |\nabla^2 \tilde{S}|_{q=q_0} \tag{94}$$

during descent.

The plume should ultimately reach a maximum depth d before it decelerates to rest ($w = 0$). The relative entropy perturbation at that depth should be roughly

$$\frac{|S'_{\text{stable}}|}{C_{\text{pa}}} \sim \frac{|\nabla^2 \tilde{S}|_{q=q_0}^2}{C_{\text{pa}}} d^2 = \frac{d^2}{H_{\nabla \overline{S}}^2}, \tag{95}$$

where we define $H_{\nabla \overline{S}} := (C_{\text{pa}}/|\nabla^2 \tilde{S}|)^{-1/2}$ as the scale height associated with the reference entropy *gradient* (not the background entropy). This is also the scale height associated with the squared buoyancy frequency $\tilde{N}^2$, since $\tilde{N}^2 = (g/\tilde{C}_{\text{p}})d\tilde{S}/dq$.

The total buoyancy work needed to decelerate the plume to $w = 0$ is thus roughly

$$g_a H_a \epsilon \sim w^2 \sim \frac{|S'_{\text{stable}}|}{C_{\text{pa}}} g_a d \sim \frac{g_a}{H_{\nabla \overline{S}}^2} d^3, \tag{96}$$

or

$$d \sim (H_{\nabla \overline{S}}^2 H_a)^{1/3} \epsilon^{1/3}. \tag{97}$$

Thus, $|S'_{\text{stable}}|$, at maximum, should be roughly

$$\frac{|S'_{\text{stable}}|}{C_{\text{pa}}} \sim \frac{H_a}{d} \epsilon \sim \left(\frac{H_a}{H_{\nabla \overline{S}}} \right)^{2/3} \epsilon^{2/3}, \tag{98}$$

which remains $\lesssim \epsilon$ provided that

$$\frac{H_a}{H_{\nabla \overline{S}}} \lesssim \epsilon^{1/2} \quad \text{or} \quad |\nabla^2 \tilde{S}|_{q=q_0} \lesssim \frac{C_{\text{pa}}}{H_a^2} \epsilon. \tag{99}$$

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[19] We are being a bit cavalier about expanding $\tilde{S}$ instead of $\overline{S}$. As long as the true maximum of $\overline{S}$ is sufficiently close to the base of the convection zone (more precisely, $\lesssim d$ away), then the distinction should be irrelevant for order of magnitude estimates. If the convection somehow restratifies the system so that everywhere $|\nabla \overline{S}| = |\nabla \tilde{S}| + O(\epsilon)$ (consistent with Equation (51)) but $|\nabla^2 \overline{S}| \neq |\nabla^2 \tilde{S}|$ at lowest order, then we would need to be more careful.

In other words, we expect the thermal perturbations to remain small in the stable layer, provided that the *second* derivative of $\tilde{S}$ is small enough at the base of the convection zone.

If Equation (99) holds, then Equation (97) yields $d \sim H_{\overline{\nabla S}}\epsilon^{1/2}$, which is consistent with the Taylor expansion (94). T

If instead, the stability transition is particularly abrupt (large $|\nabla^2 \tilde{S}|_{q=q_0}$) or ϵ is not particularly small, then the Taylor expansion of Equation (94) becomes insufficient because $d \gtrsim H_{\overline{\nabla S}}$. In other words, the plume descends all the way into regions of large $d\tilde{S}/dq$ and we would estimate

$$\frac{S'}{C_{\mathrm{p}a}} \sim (q_0 - q)\frac{|\nabla S_a|}{C_{\mathrm{p}a}} \sim (q_0 - q)\frac{N_a^2}{g_a} \tag{100}$$

during descent. Here, $|\nabla S_a|$ and N_a^2 are typical values of $|\nabla \tilde{S}|$ and $\tilde{N}^2$ in the bulk of the stable layer.

From arguments similar to the ones above, we then estimate

$$d \sim H_a \frac{\omega_{\mathrm{ac}}}{N_a}\epsilon^{1/2} \tag{101}$$

and

$$\frac{|S'_{\mathrm{stable}}|}{C_{\mathrm{p}a}} \sim \frac{H_a}{d}\epsilon \sim \frac{N_a}{\omega_{\mathrm{ac}}}\epsilon^{1/2}, \tag{102}$$

which remains $\lesssim \epsilon$ provided that

$$\frac{N_a^2}{\omega_{\mathrm{ac}}^2} \lesssim \epsilon. \tag{103}$$

Here, $\omega_{\mathrm{ac}} := c_{\mathrm{s}a}/H_a \sim \sqrt{g_a/H_a}$ is a typical value of the acoustic cutoff frequency and we have used hydrostatic balance to estimate $g_a \sim c_{\mathrm{s}a}^2/H_a$.

In other words, for abrupt stability transitions, we expect the thermal perturbations to remain small in the stable layer, provided the buoyancy frequency is sufficiently smaller than the acoustic cutoff frequency.

Which criteria is applicable to the overshoot (either Equations (99) or (103)) depends on the magnitudes of

10   Internal gravity waves

31

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