Induction term in spherical coordinates

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1 The problem with the traditional induction terms

We consider the ideal (resistance-free) magnetohydrodynamic (MHD) induction equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) := \boldsymbol{I}, \tag{1}$$

$$= \boldsymbol{B} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{B} - (\nabla \cdot \boldsymbol{u})\boldsymbol{B}, \tag{2}$$

where $\boldsymbol{B}$ and $\boldsymbol{u}$ are the vector magnetic and velocity fields (respectively) and $\boldsymbol{I}$ is the vector induction. The three terms on the right-hand-side of Equation (2) are often interpreted as "shear," "advection," and "compression," respectively. However, this interpretation is problematic in general for two reasons:

1. The so-called shear and compression terms contain sub-terms that cancel; in particular, only velocity motions *transverse* to magnetic-field lines can shear or compress.

2. Solid-body rotation (which is a non-shearing motion that simply rotates the whole field configuration) shows up in the so-called shear and advection terms in a strange way.

Finally, even after these issues have been addressed, resolving the final terms into a particular curvilinear system (e.g., spherical coordinates) seems to present a major headache. Our goal here is to explain fully how these problems emerge and propose a tentative solution for the case of spherical coordinates.

2 Transverse shear and compression

To see how Problem 1 arises, we decompose the velocity field into components parallel and perpendicular to the local direction of $\boldsymbol{B}$:

$$\boldsymbol{u} := u_{\parallel} \hat{\boldsymbol{e}}_{\parallel} + \boldsymbol{u}_{\perp} \tag{3}$$

Obviously $\boldsymbol{B} = B x_{\parallel}$, where $B = |\boldsymbol{B}|$. We denote the Cartesian distance along $\boldsymbol{B}$ by $x_{\parallel}$. We also decompose $\boldsymbol{u}$ into its parallel and perpendicular components:

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_{\parallel}}{\partial x_{\parallel}} + \nabla_{\perp} \cdot \boldsymbol{u}_{\perp} \tag{4}$$

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We then calculate

$$\boldsymbol{B} \cdot \nabla \boldsymbol{u} - (\nabla \cdot \boldsymbol{u})\boldsymbol{B} = B\frac{\partial}{\partial x_\parallel}(u_\parallel \hat{\boldsymbol{e}}_\parallel + \boldsymbol{u}_\perp) - \left(\frac{\partial u_\parallel}{\partial x_\parallel} + \nabla_\perp \cdot \boldsymbol{u}_\perp\right) B\hat{\boldsymbol{e}}_\parallel$$

$$= B\frac{\cancel{\partial u_\parallel}}{\cancel{\partial x_\parallel}}\hat{\boldsymbol{e}}_\parallel + B\frac{\partial \boldsymbol{u}_\perp}{\partial x_\parallel} - \frac{\cancel{\partial u_\parallel}}{\cancel{\partial x_\parallel}}B\hat{\boldsymbol{e}}_\parallel - (\nabla_\perp \cdot \boldsymbol{u}_\perp)B\hat{\boldsymbol{e}}_\parallel$$

$$= \boldsymbol{B} \cdot \nabla \boldsymbol{u}_\perp - (\nabla_\perp \cdot \boldsymbol{u}_\perp)\boldsymbol{B}. \tag{5}$$

Thus, only motions transverse to the local field line (i.e., $\boldsymbol{u}_\perp$) can shear or compress $\boldsymbol{B}$.

3 Rigid rotation

To see how Problem 2 arises, we consider a velocity field due to rigid rotation at constant angular velocity Ω about the z-axis in a cylindrical coordinate system:

$$\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{e}}_z = \text{constant} \tag{6a}$$

$$\boldsymbol{u} = \boldsymbol{\Omega} \times \boldsymbol{r} = \Omega\lambda\hat{\boldsymbol{e}}_\phi \tag{6b}$$

Here, λ is the cylindrical radius, ϕ the longitude, and z the axial coordinate. In general, $\hat{\boldsymbol{e}}_{(\dots)}$ denotes a unit vector in the direction of its subscript. We calculate:

$$\nabla \cdot \boldsymbol{u} = \nabla \cdot (\boldsymbol{\Omega} \times \boldsymbol{r})$$

$$= \boldsymbol{\Omega} \cdot \nabla \times \boldsymbol{r} - \boldsymbol{r} \cdot \nabla \times \boldsymbol{\Omega}$$

$$= 0 \quad \text{no compression for rigid rotation (obviously)}. \tag{7}$$

Then:

$$\boldsymbol{B} \cdot \nabla \boldsymbol{u} = (\boldsymbol{B} \cdot \nabla)(\boldsymbol{\Omega} \times \boldsymbol{r})$$

$$= \boldsymbol{\Omega} \times [(\boldsymbol{B} \cdot \nabla)(\boldsymbol{r})] \tag{8}$$

$$= \boldsymbol{\Omega} \times \boldsymbol{B} \quad \text{``shear'' for rigid rotation}. \tag{9}$$

Finally:

$$-\boldsymbol{u} \cdot \nabla \boldsymbol{B} = -\Omega\lambda\hat{\boldsymbol{e}}_\phi \cdot \nabla \boldsymbol{B}$$

$$= -\Omega\frac{\partial}{\partial \phi}(B_\lambda\hat{\boldsymbol{e}}_\lambda + B_\phi\hat{\boldsymbol{e}}_\phi + B_z\hat{\boldsymbol{e}}_z)$$

$$= -\Omega\sum_\alpha \left(\frac{\partial B_\alpha}{\partial \phi}\right)\hat{\boldsymbol{e}}_\alpha - \Omega\sum_\alpha B_\alpha\frac{\partial \hat{\boldsymbol{e}}_\alpha}{\partial \phi},$$

where the index α runs over the three cylindrical coordinates. Note that in the cylindrical coordinate system (or indeed any coordinate system with an axis of rotational symmetry), $\partial \hat{\boldsymbol{e}}_\alpha/\partial \phi = \hat{\boldsymbol{e}}_z \times \hat{\boldsymbol{e}}_\alpha$ for each α. Thus,

$$-\Omega\sum_\alpha B_\alpha\frac{\partial \hat{\boldsymbol{e}}_\alpha}{\partial \phi} = -\Omega\sum_\alpha B_\alpha\hat{\boldsymbol{e}}_z \times \hat{\boldsymbol{e}}_\alpha$$

$$= -\Omega\hat{\boldsymbol{e}}_z \times \sum_\alpha B_\alpha\hat{\boldsymbol{e}}_\alpha$$

$$= -\boldsymbol{\Omega} \times \boldsymbol{B}$$

and so

$$-\boldsymbol{u} \cdot \nabla \boldsymbol{B} = -\Omega \sum_{\alpha} \left(\frac{\partial B_{\alpha}}{\partial \phi} \right) \hat{\boldsymbol{e}}_{\alpha} - \boldsymbol{\Omega} \times \boldsymbol{B} \qquad \text{``advection'' for rigid rotation.} \qquad (10)$$

Mathematically, in any coordinate system with a z-axis of rotational symmetry, the action of rigid rotation is as follows:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \left[(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{B} \right]$$

$$= \sum_{\alpha} \left(-\Omega \frac{\partial B_{\alpha}}{\partial \phi} \right) \hat{\boldsymbol{e}}_{\alpha} \qquad (11a)$$

$$\text{or} \qquad \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) B_{\alpha} = 0 \qquad \text{for each } \alpha. \qquad (11b)$$

If you think about it, this makes sense: All the rigid rotation does is rotate the whole field configuration around the z-axis at the rate Ω. If you decide to also rotate at Ω (so your personal Eulerian time derivative is $\partial/\partial t + \Omega \partial/\partial \phi$), then each component of the magnetic-field configuration should remain the same in your frame. Note that rotation does *not* advect the vector magnetic field (like the term $-\boldsymbol{u} \cdot \nabla \boldsymbol{B}$ viewed on its own would suggest), but rather advects the field *components* (as if they were scalars) in any coordinate system with an axis of rotational symmetry.

4   Solution for spherical coordinates

Resolving these issues fully for the spherical coordinate system seems complicated and I am not fully sure how to do it! In particular (for "full" resolution) we should, separately at each point (r,θ,ϕ):

1. Form a local Cartesian coordinate system, say (x_1, x_2, x_3), whose origin lies at the point (r, θ, ϕ). At the origin, $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ will coincide with $(\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta, \hat{\boldsymbol{e}}_\phi)$. But slightly away from the origin, $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ will stay fixed while $(\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta, \hat{\boldsymbol{e}}_\phi)$ curve away.

2. Calculate the velocity-gradient tensor: $\partial u_1/\partial x_1, \partial u_2/\partial x_1, \partial u_3/\partial x_1$, etc. While calculating derivatives, be careful to differentiate along the Cartesian coordinates, *not* the curvilinear ones or along the actual $\boldsymbol{B}$-line). Express the final tensor components in spherical coordinates.

3. Rotate "into $\boldsymbol{B}$" to form a new primed coordinate system (such that $\hat{\boldsymbol{e}}_1'$ points along $\boldsymbol{B}$), calculate the $\partial u_j'/\partial x_i'$, and thus form $\partial u_\parallel/\partial x_\parallel$ and $\partial \boldsymbol{u}_\perp/\partial x_\parallel$ (note that $\nabla_\perp \cdot \boldsymbol{u}_\perp = \nabla \cdot \boldsymbol{u} - \partial u_\parallel/\partial x_\parallel$). Note that the direction cosines $\hat{\boldsymbol{e}}_1'$ makes with $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ are simply $B_r/|\boldsymbol{B}|, B_\theta/|\boldsymbol{B}|, B_\phi/|\boldsymbol{B}|$, respectively. Note that both unit vectors and vector components transform like $x_1' = \sum_{j=1}^3 R_{1j} x_j$, where R_{1j} is the jth direction cosine.

4. Subtract the part of $\boldsymbol{u}_\perp$ corresponding to solid-body rotation and put it in the form of Equation (11a).

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5. Exult, because I haven't been able to do this!

Instead of addressing the full problem as just described, I have brute-forced my way into a quasi-solution, canceling obvious terms and expressing what seem to be "transverse shear with no solid-body rotation," "transverse compression," and advection.

We write:

$$I_r = \cancel{B_r \frac{\partial u_r}{\partial r}} + \frac{B_\theta}{r} \frac{\partial u_r}{\partial \theta} + \frac{B_\phi}{r \sin\theta} \frac{\partial u_r}{\partial \phi} - \cancel{\frac{B_\theta u_\theta + B_\phi u_\phi}{r}}$$
$$- \boldsymbol{u} \cdot \nabla B_r + \cancel{\frac{u_\theta B_\theta + u_\phi B_\phi}{r}}$$
$$- \left[\cancel{\frac{\partial u_r}{\partial r}} + \frac{2u_r}{r} + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + \cot\theta u_\theta \right) + \frac{1}{r \sin\theta} \frac{\partial u_\phi}{\partial \phi} \right] B_r$$

or

$$I_r = \left(\frac{B_\theta}{r} \frac{\partial}{\partial \theta} + \frac{B_\phi}{r \sin\theta} \frac{\partial}{\partial \phi} \right) u_r \qquad\qquad r\text{: transverse shear} \qquad (12a)$$
$$- \boldsymbol{u} \cdot \nabla B_r \qquad\qquad\qquad\qquad\qquad\qquad r\text{: advection} \qquad (12b)$$
$$- \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial u_\phi}{\partial \phi} + \frac{2u_r}{r} + \frac{\cot\theta u_\theta}{r} \right) B_r \qquad r\text{: transverse compression} \qquad (12c)$$

Then, for I_θ:

$$I_\theta = B_r \frac{\partial u_\theta}{\partial r} + \cancel{\frac{B_\theta}{r} \frac{\partial u_\theta}{\partial \theta}} + \frac{B_\phi}{r \sin\theta} \frac{\partial u_\theta}{\partial \phi} + \frac{B_\theta u_r}{r} - \cancel{\frac{\cot\theta B_\phi u_\phi}{r}}$$
$$- \boldsymbol{u} \cdot \nabla B_\theta - \frac{u_\theta B_r}{r} + \cancel{\frac{\cot\theta u_\phi B_\phi}{r}}$$
$$- \left[\frac{\partial u_r}{\partial r} + \frac{2u_r}{r} + \frac{1}{r}\left(\cancel{\frac{\partial u_\theta}{\partial \theta}} + \cot\theta u_\theta \right) + \frac{1}{r \sin\theta} \frac{\partial u_\phi}{\partial \phi} \right] B_\theta$$

or

$$I_\theta = r \left(B_r \frac{\partial}{\partial r} + \frac{B_\phi}{r \sin\theta} \frac{\partial}{\partial \phi} \right) \left(\frac{u_\theta}{r} \right) \qquad\qquad \theta\text{: transverse shear} \qquad (13a)$$
$$- \boldsymbol{u} \cdot \nabla B_\theta \qquad\qquad\qquad\qquad\qquad\qquad \theta\text{: advection} \qquad (13b)$$
$$- \left(\frac{\partial u_r}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{\cot\theta u_\theta}{r} \right) B_\theta \qquad \theta\text{: transverse compression} \qquad (13c)$$

Finally, for I_ϕ:

$$I_\phi = B_r \frac{\partial u_\phi}{\partial r} + \frac{B_\theta}{r} \frac{\partial u_\phi}{\partial \theta} + \cancel{\frac{B_\phi}{r \sin\theta} \frac{\partial u_\phi}{\partial \phi}} + \frac{B_\phi u_r}{r} + \cancel{\frac{\cot\theta B_\phi u_\theta}{r}}$$
$$- \boldsymbol{u} \cdot \nabla B_\phi - \frac{u_\phi B_r}{r} - \frac{\cot\theta u_\phi B_\theta}{r}$$
$$- \left[\frac{\partial u_r}{\partial r} + \frac{2u_r}{r} + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + \cancel{\cot\theta u_\theta} \right) + \frac{1}{\cancel{r \sin\theta}} \cancel{\frac{\partial u_\phi}{\partial \phi}} \right] B_\phi$$
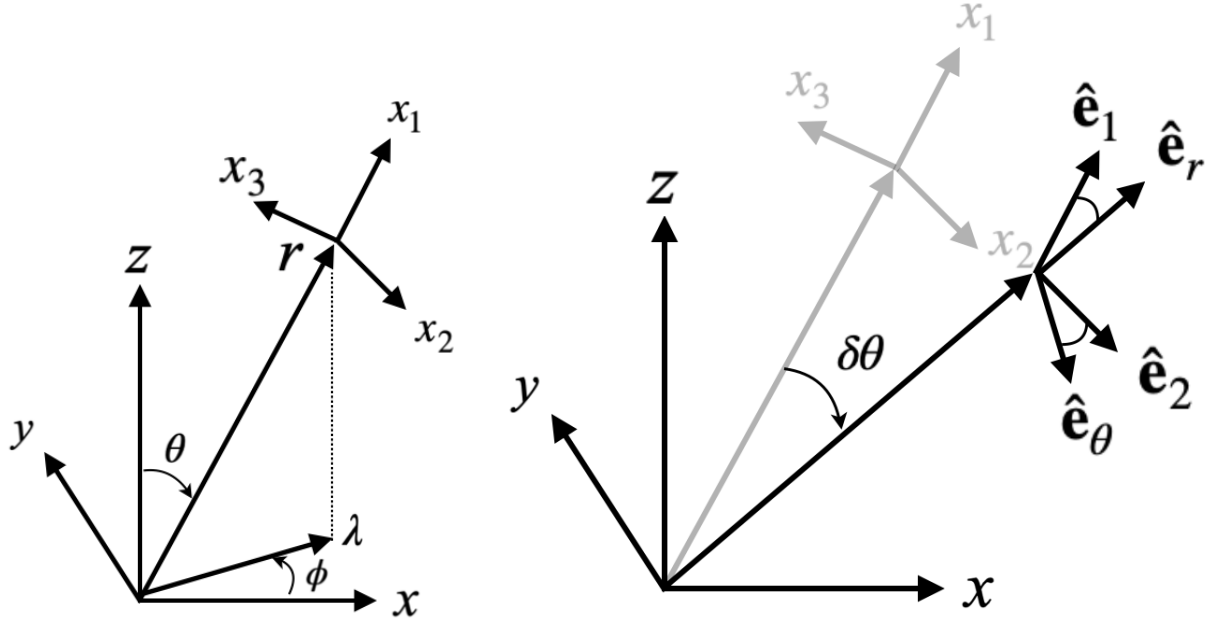
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Figure 1: *Left*: Schematic of coordinate systems used here, their relation to one another, and each coordinate's meaning. Note that $(x, y, z) = (\lambda \cos\phi, \lambda \sin\phi, z) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$. The local Cartesian system (x_1, x_2, x_3) has its origin at the point (r, θ, ϕ), where $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3) = (\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta, \hat{\boldsymbol{e}}_\phi)$. *Right*: Away from the origin, the unit vectors no longer coincide. We show the relationship between the spherical-coordinate unit vectors and the Cartesian unit vectors when $\theta \to \theta + \delta\theta$ (or equivalently, $x_2 = 0 \to \delta x_2$).

or

$$I_\phi = r \sin\theta \left(B_r \frac{\partial}{\partial r} + \frac{B_\theta}{r} \frac{\partial}{\partial \theta} \right) \left(\frac{u_\phi}{r \sin\theta} \right) \qquad \phi: \text{ transverse shear} \qquad (14\text{a})$$

$$- \; \boldsymbol{u} \cdot \nabla B_\phi \qquad\qquad\qquad\qquad\qquad \phi: \text{ advection} \qquad\qquad (14\text{b})$$

$$- \left(\frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) B_\phi \qquad\quad \phi: \text{ transverse compression} \qquad (14\text{c})$$

5   Transverse compression

Here we show that the "transverse compression" terms defined in Equations (12)–(14) make sense. The key point is that we must be careful (in calculating $\nabla_\perp \cdot \boldsymbol{u}_\perp$) when differentiating along θ or ϕ, because the local spherical unit vectors will all be changing. For example the divergence "transverse to $\hat{\boldsymbol{e}}_r$" is not simply $\nabla \cdot (u_\theta \hat{\boldsymbol{e}}_\theta + u_\phi \hat{\boldsymbol{e}}_\phi)$; there are also curvature terms.

Figure 1 shows schematically how this works. We set up a local Cartesian coordinate system (with origin at the point (r, θ, ϕ)) denoted by (x_1, x_2, x_3). As we differentiate $\boldsymbol{u}$, the spherical-coordinate unit vectors change, while the Cartesian ones stay fixed. In particular, if we move from $(r, \theta, \phi) \to (r + \delta r, \theta + \delta\theta, \phi + \delta\phi)$ (or equivalently in the Cartesian system,

$(0,0,0) \to (\delta x_1, \delta x_2, \delta x_3))$, we find the following (to first order in the δ's):

$$u_1 = u_r - (\delta\theta)u_\theta - (\sin\theta\delta\phi)u_\phi, \tag{15a}$$

$$u_2 = u_\theta + (\delta\theta)u_r - (\cos\theta\delta\phi)u_\phi, \tag{15b}$$

$$\text{and} \quad u_3 = u_\phi + (\sin\theta\delta\phi)u_r + (\cos\theta\delta\phi)u_\theta. \tag{15c}$$

And trivially:

$$\delta x_1 = \delta r \tag{16a}$$

$$\delta x_2 = r\delta\theta \tag{16b}$$

$$\delta x_3 = r\sin\theta\delta\phi \tag{16c}$$

We thus compute:

$$
\begin{aligned}
\frac{\partial u_1}{\partial x_1} &= \lim_{\delta x_1 \to 0} \frac{u_1(\delta x_1, 0, 0) - u_1(0,0,0)}{\delta x_1} \\
&= \lim_{\delta r \to 0} \frac{u_r(r + \delta r, \theta, \phi) - u_r(r, \theta, \phi)}{\delta r} \\
&= \frac{\partial u_r}{\partial r}
\end{aligned} \tag{17}
$$

That was the easy one! Now for the curvy ones:

$$
\begin{aligned}
\frac{\partial u_2}{\partial x_2} &= \lim_{\delta x_2 \to 0} \frac{u_2(0, \delta x_2, 0) - u_2(0,0,0)}{\delta x_2} \\
&= \lim_{\delta\theta \to 0} \frac{u_\theta(r, \theta + \delta\theta, \phi) + u_r(r, \theta + \delta\theta, \phi)\delta\theta - u_\theta(r, \theta, \phi)}{r\delta\theta} \\
&= \lim_{\delta\theta \to 0} \frac{u_\theta(r, \theta + \delta\theta, \phi) - u_\theta(r, \theta, \phi)}{r\delta\theta} + \lim_{\delta\theta \to 0} \frac{u_r(r, \theta + \delta\theta, \phi)}{r} \\
&= \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}
\end{aligned} \tag{18}
$$

And finally:

$$
\begin{aligned}
\frac{\partial u_3}{\partial x_3} &= \lim_{\delta x_3 \to 0} \frac{u_3(0, 0, \delta x_3) - u_3(0,0,0)}{\delta x_3} \\
&= \lim_{\delta\phi \to 0} \frac{u_\phi(r, \theta, \phi + \delta\phi) + (\sin\theta u_r + \cos\theta u_\theta)\delta\phi - u_\phi(r, \theta, \phi)}{r\sin\theta\delta\phi} \\
&= \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{\cot\theta u_\theta}{r}
\end{aligned} \tag{19}
$$

We thus find the various components of transverse divergence:

$$(\nabla \cdot \boldsymbol{u})_{\perp \text{ to } r} = \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r \sin \theta}\frac{\partial u_\phi}{\partial \phi} + \frac{2u_r}{r} + \frac{\cot \theta u_\theta}{r}, \qquad (20)$$

$$(\nabla \cdot \boldsymbol{u})_{\perp \text{ to } \theta} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}$$

$$= \frac{\partial u_r}{\partial r} + \frac{1}{r \sin \theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{\cot \theta u_\theta}{r} \text{and} \quad (\nabla \cdot \boldsymbol{u})_{\perp \text{ to } \phi} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}$$

$$= \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \qquad (21)$$