

Hydrostatic, ideal gas reference states in spherical coordinates

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1 Basic assumptions

We wish to derive the thermodynamic state for an ideal, hydrostatic gas in spherical coordinates, assuming a known specific entropy stratification $s(r)$. In mathematical terms, this means that

$$\frac{dp}{dr} = -\rho g(r), \quad (1)$$

$$p(r) = \rho(r)\mathcal{R}T(r), \quad (2)$$

and

$$s'(r) := \frac{ds}{dr} \quad (3)$$

is a known function of radius. Here $p(r)$ is the pressure, $\rho(r)$ the density, $T(r)$ the temperature, $g(r)$ the gravitational acceleration (unspecified for now except that it is spherically symmetric), and \mathcal{R} the gas constant. We further assume an ideal gas with three translational degrees of freedom (d.o.f.) throughout, so that

$$c_v = \frac{3}{2}\mathcal{R}, \quad (4a)$$

$$c_p = c_v + \mathcal{R} = \frac{5}{2}\mathcal{R}, \quad (4b)$$

$$\gamma := \frac{c_p}{c_v} = \frac{5}{3}, \quad (4c)$$

$$\text{and} \quad \mathcal{R} = (\gamma - 1)c_v, \quad (4d)$$

which in turn implies that

$$c_v = \frac{\mathcal{R}}{\gamma - 1} \quad (5a)$$

$$\text{and} \quad c_p = \frac{\gamma\mathcal{R}}{\gamma - 1}. \quad (5b)$$

For quasistatic processes, the First Law of Thermodynamics states that

$$Tds = de + pdv = c_v dT - \frac{p}{\rho^2} d\rho, \quad (6)$$

where e is the specific energy, $v = 1/\rho$ is the specific volume, and the second equality follows from the fact that $e = c_v T$ for an ideal gas (we assume the variation of c_v with T —at least in the infinitesimal sense—is negligible). Noting that $p/c_v \rho T = \rho \mathcal{R} T / c_v \rho T = \mathcal{R}/c_v = \gamma - 1$, we can divide (6) by $e = c_v T$ to yield

$$\frac{ds}{c_v} = \frac{dT}{T} - (\gamma - 1) \frac{d\rho}{\rho} = d \ln T - (\gamma - 1) d \ln \rho. \quad (7)$$

As long as there are consistently only three d.o.f. throughout the domain of interest (i.e., T is not so high that rotational and vibrational d.o.f. are excited, for example), then c_v , c_p , and γ are constants and relation (7) can be applied across radius to yield

$$\frac{d \ln T}{dr} - (\gamma - 1) \frac{d \ln \rho}{dr} = \frac{s'}{c_v}. \quad (8)$$

Taking the radial derivative of the logarithm of the ideal gas law (2) also gives

$$\frac{d \ln p}{dr} = \frac{d \ln \rho}{dr} + \frac{d \ln T}{dr}. \quad (9)$$

To isolate the temperature derivative, we plug (9) into (8) use the hydrostatic equation (1) to give

$$\begin{aligned} \frac{d \ln T}{dr} - (\gamma - 1) \frac{d \ln \rho}{dr} &= \frac{d \ln T}{dr} - (\gamma - 1) \left(\frac{d \ln p}{dr} - \frac{d \ln T}{dr} \right) \\ &= \gamma \frac{d \ln T}{dr} - (\gamma - 1) \frac{(-\rho g)}{p} \\ &= \gamma \frac{d \ln T}{dr} + \frac{\gamma - 1}{\mathcal{R}} \frac{g}{T} \end{aligned}$$

and thus

$$\begin{aligned} \frac{\gamma}{T} \frac{dT}{dr} + \frac{\gamma - 1}{\mathcal{R}} \frac{g}{T} &= \frac{s'}{c_v}, \\ \text{or} \quad \frac{dT}{dr} + \frac{\gamma - 1}{\gamma \mathcal{R}} g &= \frac{T s'}{c_p}. \end{aligned}$$

Using (5b), our equation for temperature in the standard form for ordinary differential equations is

$$\boxed{\frac{dT}{dr} - \left[\frac{s'(r)}{c_p} \right] T(r) = -\frac{g(r)}{c_p}}. \quad (10)$$

The preceding equation is “nice” and can be multiplied by the integrating factor e^{-s/c_p} to yield

$$\begin{aligned} \frac{d}{dr}(e^{-s/c_p}T) &= -\frac{ge^{-s/c_p}}{c_p}, \\ \text{or} \quad e^{-s(r)/c_p}T(r) &= -\frac{1}{c_p} \int_{r_0}^r g(\tilde{r})e^{-s(\tilde{r})/c_p}d\tilde{r} + A, \\ \text{or} \quad T(r) &= -\frac{e^{s(r)/c_p}}{c_p} \int_{r_0}^r g(\tilde{r})e^{-s(\tilde{r})/c_p}d\tilde{r} + Ae^{s(r)/c_p}. \end{aligned}$$

Here we have defined r_0 as an arbitrary radius from which to start the integration (in practice it will often be convenient to set $r_0 = r_i$, the radius at the inner boundary of the shell). The integration constant A depends on the temperature at the chosen radius, $T_0 := T(r_0)$. We can see that $T_0 = 0 + Ae^{s_0/c_p}$, or $A = T_0e^{-s_0/c_p}$, where s_0 is the entropy at r_0 . Thus, the full expression for $T(r)$ is

$$\boxed{T(r) = -\frac{e^{s(r)/c_p}}{c_p} \int_{r_0}^r g(\tilde{r})e^{-s(\tilde{r})/c_p}d\tilde{r} + T_0e^{[s(r)-s_0]/c_p}. \quad (11)}$$

Note that in equation (11), adding a constant σ to the entropy ($s \rightarrow s + \sigma$) has no effect on the value of T . *The absolute value of the entropy nowhere appears in the final state; only the relative stratification of entropy is important.*

Once $T(r)$ is obtained, we can obtain $p(r)$ through the hydrostatic assumption:

$$\frac{dp}{dr} = -\rho g = -\frac{p}{\mathcal{R}T}g, \quad \text{or} \quad \frac{d \ln p}{dr} = -\frac{1}{\mathcal{R}} \frac{g}{T}.$$

Integrating, we find

$$\ln p - \ln p_0 = \ln(p/p_0) = -\frac{1}{\mathcal{R}} \int_{r_0}^r \frac{g(\tilde{r})}{T(\tilde{r})}d\tilde{r},$$

or

$$p(r) = p_0 \exp \left[-\frac{1}{\mathcal{R}} \int_{r_0}^r \frac{g(\tilde{r})}{T(\tilde{r})}d\tilde{r} \right],$$

where p_0 is the pressure at r_0 . Using (10), we can further simplify:

$$\begin{aligned} \int_{r_0}^r \frac{g(\tilde{r})}{T(\tilde{r})}d\tilde{r} &= \int_{r_0}^r \left[s'(\tilde{r}) - c_p \frac{d \ln T}{d\tilde{r}} \right]d\tilde{r} = s(r) - s_0 - c_p \ln(T/T_0) \Rightarrow \\ p(r) &= p_0 \exp \left[-\frac{s(r) - s_0}{\mathcal{R}} + \frac{c_p}{\mathcal{R}} \ln \left(\frac{T}{T_0} \right) \right], \end{aligned}$$

or

$$\boxed{p(r) = p_0 \exp \left[-\frac{s(r) - s_0}{\mathcal{R}} \right] \left[\frac{T(r)}{T_0} \right]^{\gamma/(\gamma-1)}. \quad (12)}$$

Here p_0 is the pressure at r_0 , and we have used (5b) to write $c_p/\mathcal{R} = \gamma/(\gamma - 1)$.

Finally, $\rho(r)$ is now determined from the ideal gas law (2):

$$\boxed{\rho(r) = \rho_0 \exp \left[- \frac{s(r) - s_0}{\mathcal{R}} \right] \left[\frac{T(r)}{T_0} \right]^{1/(\gamma-1)},} \quad (13)$$

where $\rho_0 = p_0/\mathcal{R}T_0$ is the density at r_0 .

The full state of the atmosphere is purely determined by the entropy stratification $s'(r)$, the gravitational acceleration $g(r)$, and the two constants T_0 and p_0 . (We assume we already know the specific heats and gas constant).

The state specified in equations (11)–(13) is completely general, relying only on the assumptions of a hydrostatic, ideal gas. To move forward, we must specify $s'(r)$ and $g(r)$. For the convection zone of a star (and much of the radiative zone below), it is safe to assume that the gravitational acceleration comes purely from a central spherically distributed mass, i.e.,

$$g(r) = \frac{GM}{r^2}, \quad (14)$$

where G is the Universal Gravitational Constant and M is the stellar mass.

2 Specific atmospheres

Assuming equations (11)–(14), we now derive atmospheres for various cases of the entropy stratification $s'(r)$.

2.1 Adiabatic atmosphere

Considering the mathematical form of (11), the simplest atmosphere to derive is one that is adiabatic (constant-entropy):

$$s = s_0 \equiv \text{const.} \quad \text{and} \quad s' \equiv 0. \quad (15)$$

In this case all the exponentials in (11) cancel to 1, and after choosing $r_0 = r_i$ (with $T_i := T(r_i)$, $p_i := p(r_i)$, etc.), we find

$$T(r) = -\frac{GM}{c_p} \int_{r_i}^r \frac{d\tilde{r}}{\tilde{r}^2} + T_i = T_i + \frac{GM}{c_p} \frac{1}{\tilde{r}} \Big|_{r_i}^r = T_i + \frac{GM}{c_p} \left(\frac{1}{r} - \frac{1}{r_i} \right),$$

which can be rearranged to give

$$\boxed{T(r) = T_i \left[a_0 \left(\frac{r_i}{r} \right) + (1 - a_0) \right]} \quad (\text{adiabatic atmosphere}), \quad (16)$$

where

$$a_0 := \frac{GM}{c_p T_i r_i}. \quad (17)$$

Then (12) \implies

$$p(r) = p_i \left[a_0 \left(\frac{r_i}{r} \right) + (1 - a_0) \right]^{\gamma/(\gamma-1)} \quad (\text{adiabatic atmosphere}). \quad (18)$$

Finally, plugging (16) and (18) into the ideal gas law $\rho = p/\mathcal{R}T$ yields

$$\rho(r) = \rho_i \left[a_0 \left(\frac{r_i}{r} \right) + (1 - a_0) \right]^{1/(\gamma-1)} \quad (\text{adiabatic atmosphere}), \quad (19)$$

where $\rho_i = p_i/\mathcal{R}T_i$ is the density at the inner boundary.

2.2 Polytropes

The starting assumption for a “polytrope” is that the temperature gradient is divergenceless—i.e., there is a constant flux due to radiative diffusion throughout the fluid layer. In spherical coordinates this condition becomes

$$\frac{\kappa_r}{r^2} \frac{d}{dr} \left(r^2 \frac{dT}{dr} \right) = 0 \iff r^2 \frac{dT}{dr} = \text{const.} \iff \frac{dT}{dr} = \frac{\text{const.}}{r^2},$$

or

$$T(r) = \frac{A}{r} + B. \quad (20)$$

Here κ_r is the radiative diffusivity; the polytrope assumes the diffusivity *cannot* vary with radius—an assumption that obviously breaks down in a real star with, say, a Kramer’s opacity where $\kappa_r \sim \rho^\alpha T^\beta$ (α and β are dimensionless constants). The integration constants A and B in (20) are arbitrary, but one is eliminated from the condition $T(r_i) = T_i$, leaving

$$T(r) = T_i \left[a \left(\frac{r_i}{r} \right) + (1 - a) \right] \quad (\text{polytropic atmosphere}), \quad (21)$$

where the new constant $a := A/T_i r_i$ is dimensionless. Note the similarity of the argument $[a(r_i/r) + (1 - a)]$ in (21) (which was derived by assuming a divergenceless temperature gradient) to the argument in (16) (which was derived by assuming the gas was adiabatic).

We again use (12) to find $p(r)$, yielding

$$p(r) = p_i \left[a \left(\frac{r_i}{r} \right) + (1 - a) \right]^{n+1} \quad (\text{polytropic atmosphere}), \quad (22)$$

where we have defined the *polytropic index* n to be

$$n := \frac{1}{a} \frac{GM}{\mathcal{R}T_i r_i} - 1 \iff a = \frac{GM}{(n+1)\mathcal{R}T_i r_i}. \quad (23)$$

From the ideal gas law, the density is then

$$\rho(r) = \rho_i \left[a \left(\frac{r_i}{r} \right) + (1 - a) \right]^n \quad (\text{polytropic atmosphere}). \quad (24)$$

We can also easily compute the logarithmic derivatives of the reference variables by defining

$$\zeta(r) = a \left(\frac{r_i}{r} \right) + (1 - a), \quad \frac{d\zeta}{dr} = -\frac{ar_i}{r^2} \quad (25)$$

and computing

$$\frac{d \ln T}{dr} = -\frac{ar_i}{r^2 \zeta}, \quad (26)$$

$$\frac{d \ln p}{dr} = -(n + 1) \frac{ar_i}{r^2 \zeta}, \quad (\text{polytropic atmosphere}) \quad (27)$$

$$\frac{d \ln \rho}{dr} = -n \frac{ar_i}{r^2 \zeta}, \quad (28)$$

$$\frac{d^2 \ln \rho}{dr^2} = nar_i \left[\frac{2}{r^2 \zeta} - \frac{ar_i}{r^4 \zeta^2} \right], \quad \text{etc...} \quad (29)$$

The equations (16)–(19) for the adiabatic atmosphere are a special case of equations (21)–(24) for the polytropic atmosphere, provided we define the polytropic constants for an adiabatic atmosphere as

$$n_0 := \frac{1}{\gamma - 1} \quad \text{or} \quad a_0 := \frac{GM}{c_p T_i r_i}. \quad (30)$$

Note that for the polytrope, in addition to the two constants p_i and T_i , there is another constant n (or equivalently a), which we now show is equivalent to specifying the entropy gradient. Plugging equations (21)–(30) into the First Law of Thermodynamics (8) yields

$$s'(r) = \left(\frac{n}{n_0} - 1 \right) \left[\frac{c_v}{r + (1 - a)r^2/(ar_i)} \right] \quad (\text{polytropic atmosphere}). \quad (31)$$

From the preceding equation, it is obvious that $n = n_0$ corresponds to an adiabatic atmosphere (as we have already shown), while

$$n > n_0 \iff s' > 0 \quad (\text{atmosphere is stable to convection}) \quad (32)$$

$$\text{and} \quad n < n_0 \iff s' < 0 \quad (\text{atmosphere is unstable to convection}). \quad (33)$$

If the actual entropy is desired instead of the entropy gradient, (31) may be integrated via partial fractions to yield

$$s(r) = c_v \left(\frac{n}{n_0} - 1 \right) \left\{ \ln \left(\frac{r}{r_i} \right) - \ln \left[a + (1 - a) \left(\frac{r}{r_i} \right) \right] \right\} \quad (\text{polytropic atmosphere}), \quad (34)$$

where we have chosen $s_i = 0$ for simplicity.

Assuming the entropy stratification (34), one could of course derive the polytropic equations (21)–(24) via the general equations (11)–(13). But this would be a rather contrived form of the entropy profile a priori, and its only benefit would be to enforce a divergenceless temperature gradient, so it makes the most sense to start from the divergenceless temperature gradient (constant-radiative-flux) equation (20).

2.2.1 Polytrope in terms of overall density stratification

In the Rayleigh code and Jones et al. (2011), the polytrope is initiated using the inner density ρ_i and the number of density scale heights N_ρ , as opposed to the two inner-boundary values p_i and T_i . We can recast equations (21)–(24) in terms of ρ_i and N_ρ by defining the density at the inner and outer boundaries as ρ_i and ρ_o , noting the definition $e^{N_\rho} := \rho_i/\rho_o$, and computing

$$\begin{aligned} e^{N_\rho/n} = (\rho_i/\rho_o)^{1/n} &= \frac{1}{a(r_i/r_o) + (1-a)} = \frac{1}{1 - (1-\beta)a} \iff e^{N_\rho/n} - (1-\beta)e^{N_\rho/n}a = 1 \\ &\iff (1-\beta)e^{N_\rho/n}a = e^{N_\rho/n} - 1 \iff a = \frac{e^{N_\rho/n} - 1}{(1-\beta)e^{N_\rho/n}}, \end{aligned}$$

where we have defined the aspect ratio of the shell,

$$\beta := \frac{r_i}{r_o} < 1. \quad (35)$$

Noting the definition of a (23), we see that

$$\frac{GM}{(n+1)\mathcal{R}T_i r_i} = \frac{e^{N_\rho/n} - 1}{(1-\beta)e^{N_\rho/n}} \quad \text{or} \quad T_i = \frac{(1-\beta)e^{N_\rho/n}}{e^{N_\rho/n} - 1} \frac{GM}{(n+1)\mathcal{R}r_i}. \quad (36)$$

Thus, instead of specifying T_i and p_i , we can specify ρ_i and N_ρ ; T_i is then determined through equation (36), and $p_i = \rho_i \mathcal{R}T_i$.

2.2.2 Polytrope with respect to the center of the shell

Jones et al. (2011) defines the polytrope using the variables at the center of the shell (i.e., in our notation $r_0 = r_c := (r_i + r_o)/2$). In that case, equations (21)–(24) become

$$T(r) = T_c \left[a_c \left(\frac{r_c}{r} \right) + (1 - a_c) \right], \quad (37)$$

$$p(r) = p_c \left[a_c \left(\frac{r_c}{r} \right) + (1 - a_c) \right]^{n+1}, \quad (38)$$

$$\rho(r) = \rho_c \left[a_c \left(\frac{r_c}{r} \right) + (1 - a_c) \right]^n, \quad (39)$$

where

$$a_c := \frac{GM}{(n+1)\mathcal{R}T_c r_c}. \quad (40)$$

Using a similar computation as the one leading to (36), one can show that

$$a_c = \frac{2\beta(e^{N_\rho/n} - 1)}{(1 - \beta)(\beta e^{N_\rho/n} + 1)}. \quad (41)$$

Defining

$$d := r_o - r_i, \quad (42)$$

$$c_1 := \left(\frac{r_c}{d}\right) a_c = \frac{(1 + \beta)\beta(e^{N_\rho/n} - 1)}{(1 - \beta)^2(\beta e^{N_\rho/n} + 1)}, \quad (43)$$

$$c_0 := 1 - a_c = \frac{(1 + \beta)(1 - \beta e^{N_\rho/n})}{(1 - \beta)(\beta e^{N_\rho/n} + 1)}, \quad (44)$$

$$\text{and} \quad \tilde{\zeta}(r) := a_c \left(\frac{r_c}{r}\right) + (1 - a_c) = c_0 + c_1 \left(\frac{d}{r}\right), \quad (45)$$

equations (37)–(39) become

$$T(r) = T_c \tilde{\zeta}(r), \quad p(r) = p_c [\tilde{\zeta}(r)]^{n+1}, \quad \rho(r) = \rho_c [\tilde{\zeta}(r)]^n. \quad (46)$$

With some more algebra, we can also see that if we define

$$\tilde{\zeta}_o := \tilde{\zeta}(r_o) = \frac{1 + \beta}{\beta e^{N_\rho/n} + 1}, \quad (47)$$

then

$$c_0 = \frac{2\tilde{\zeta}_o - \beta - 1}{1 - \beta} \quad \text{and} \quad c_1 = \frac{(1 + \beta)(1 - \tilde{\zeta}_o)}{(1 - \beta)^2}, \quad (48)$$

yielding exactly the formulation for the polytrope in Jones et al. (2011).

Both the formulation with respect to the center of the shell (45)–(48) and the formulation with respect to the inner boundary (21)–(24) (also (36) if specifying ρ_i and N_ρ instead of p_i and T_i) are mathematically equivalent. However, given the substantial extra notation involved in defining the constants c_0 and c_1 , it seems preferable to use the latter formulation.

2.3 Two polytropes stitched together

In simulations involving a convection zone (CZ) on top of a non-convecting (stable) radiation zone (RZ), a common approximation is to use two polytropes: one with $n_1 \approx n_0$ in the CZ (domain 1) and one with $n_2 > n_0$ in the RZ (domain 2). We denote the inner and outer boundaries by r_i and r_o (as before) and the “middle boundary,” or interface between the convective layer and stable region, by r_m . If we take $r_0 = r_m$ and specify N_ρ , n_1 , and $\rho_m := \rho(r_m)$, the CZ polytrope is given by

$$T_1(r) = T_m \zeta_1(r), \quad p_1(r) = p_m [\zeta_1(r)]^{n_1+1}, \quad \rho(r) = \rho_m [\zeta_1(r)]^{n_1}, \quad (49)$$

$$\text{where} \quad \zeta_1(r) := a_1 \left(\frac{r_m}{r}\right) + (1 - a_1), \quad a_1 := \frac{e^{N_\rho/n_1} - 1}{(1 - \beta_1)e^{N_\rho/n_1}},$$

$$T_m := \frac{1 - \beta_1}{e^{N_\rho/n_1} - 1} \left[\frac{GM}{(n_1 + 1)\mathcal{R}r_m} \right], \quad p_m := \rho_m \mathcal{R}T_m, \quad \text{and} \quad \beta_1 := \frac{r_m}{r_o}. \quad (50)$$

The entropy profile is obtained from (31) and (34), yielding

$$s_1(r) = c_v \left(\frac{n_1}{n_0} - 1 \right) \left\{ \ln \left(\frac{r}{r_m} \right) - \ln \left[a_1 + (1 - a_1) \left(\frac{r}{r_m} \right) \right] \right\},$$

and

$$s'_1(r) = \left(\frac{n_1}{n_0} - 1 \right) \left[\frac{c_v}{r + (1 - a_1)r^2/(a_1 r_m)} \right]. \quad (51)$$

Note that if $n_1 = n_0$, $s'(r) = s(r) \equiv 0$.

The only free parameter to specify the RZ polytrope is n_2 , since for a continuous double-polytrope, the values of the thermodynamic variables at the top of the RZ must be ρ_m , T_m , p_m , and $s_m = 0$. We thus again set $r_o = r_m$ for domain 2 and obtain

$$T_2(r) = T_m \zeta_2(r), \quad p_2(r) = p_m [\zeta_2(r)]^{n_2+1}, \quad \rho(r) = \rho_m [\zeta_2(r)]^{n_2}, \quad (52)$$

where

$$\zeta_2(r) := a_2 \left(\frac{r_m}{r} \right) + (1 - a_2) \quad \text{and} \quad a_2 := \frac{GM}{(n_2 + 1)\mathcal{R}T_m r_m}. \quad (53)$$

The entropy profile for domain 2 is

$$s_2(r) = c_v \left(\frac{n_2}{n_0} - 1 \right) \left\{ \ln \left(\frac{r}{r_m} \right) - \ln \left[a_2 + (1 - a_2) \left(\frac{r}{r_m} \right) \right] \right\},$$

and

$$s'_2(r) = \left(\frac{n_2}{n_0} - 1 \right) \left[\frac{c_v}{r + (1 - a_2)r^2/(a_2 r_m)} \right]. \quad (54)$$

In order to smoothly match the two polytropes together, we define “smooth Heaviside functions” with width δr_m , δ being a dimensionless constant:

$$f_1(r) = \frac{1}{2} \left[1 + \tanh \left(\frac{r - r_m}{\delta r_m} \right) \right], \quad (55)$$

$$f_2(r) = \frac{1}{2} \left[1 - \tanh \left(\frac{r - r_m}{\delta r_m} \right) \right] = 1 - f_1(r), \quad (56)$$

In practice, usually $\delta \sim 0.01$ results in a transition that is sufficiently smooth and rapid.

We then define the thermodynamic profiles via

$$\rho(r) = f_1(r)\rho_1(r) + f_2(r)\rho_2(r), \quad (57a)$$

$$p(r) = f_1(r)p_1(r) + f_2(r)p_2(r), \quad (57b)$$

$$T(r) = f_1(r)T_1(r) + f_2(r)T_2(r), \quad (57c)$$

$$\text{and} \quad s(r) = f_1(r)s_1(r) + f_2(r)s_2(r), \quad (57d)$$

yielding asymptotic behavior (i.e., where $|r - r_i|/r_i \gg \delta$) that is a polytrope of index n_1 in domain 1 and a polytrope of index 2 in domain 2. One disadvantage of smoothing the two polytropes in this way is that the resulting atmosphere is not exactly in hydrostatic equilibrium nor exactly satisfies the ideal gas law. A better approach might be to specify $s(r)$ via (57d) and then numerically integrate equations (11) and (12).

2.4 Atmosphere with constant entropy gradient

A simple example of a fluid layer that is stable to convection is one with a constant “stiffness,” or entropy gradient. In this case, we can write

$$s(r) = kc_p \left(\frac{r}{r_i} - 1 \right) \quad \text{and} \quad s'(r) = \frac{kc_p}{r_i}, \quad (58)$$

where k is a dimensionless constant (> 0) representing the stiffness of the stable region. We can then use (11) to find $T(r)$:

$$T(r) = -\frac{GM}{c_p} \left\{ e^{k[(r/r_i)-1]} \right\} \underbrace{\int_{r_i}^r \frac{e^{-k[(r/r_i)-1]} d\tilde{r}}{\tilde{r}^2}}_{:=\mathcal{J}(r)} + T_i e^{k[(r/r_i)-1]}. \quad (59)$$

The integral $\mathcal{J}(r)$ does not have an analytic solution; however, it can be recast in terms of the exponential integral function

$$E_n(x) := \int_1^\infty \frac{e^{-xt}}{t^n} dt = x^{n-1} \int_x^\infty \frac{e^{-t}}{t^n} dt, \quad (60)$$

yielding

$$\begin{aligned} \mathcal{J}(r) &= e^k \int_{r_i}^r \frac{e^{-k\tilde{r}/r_i}}{\tilde{r}^2} d\tilde{r} = e^k \int_k^{kr/r_i} \frac{e^{-t}(r_i/k)}{(r_i/k)^2 t^2} dt \\ &= \frac{ke^k}{r_i} \int_k^{kr/r_i} \frac{e^{-t}}{t^2} dt = \frac{ke^k}{r_i} \left[\underbrace{\int_k^\infty \frac{e^{-t}}{t^2} dt}_{E_2(k)/k} - \underbrace{\int_{kr/r_i}^\infty \frac{e^{-t}}{t^2} dt}_{E_2(kr/r_i)/(kr/r_i)} \right] \\ &= \frac{e^k E_2(k)}{r_i} - \frac{e^k E_2(kr/r_i)}{r} = \frac{e^k}{r_i} \left[E_2(k) - \left(\frac{r_i}{r} \right) E_2\left(\frac{kr}{r_i} \right) \right] \end{aligned}$$

and

$$T(r) = -e^k \frac{GM}{c_p r_i} \left[E_2(k) - \left(\frac{r_i}{r} \right) E_2\left(\frac{kr}{r_i} \right) \right] e^{k[(r/r_i)-1]} + T_i e^{k[(r/r_i)-1]}$$

or

$$T(r) = T_i \left\{ \left[e^k a_0 \left(\frac{r_i}{r} \right) E_2\left(\frac{kr}{r_i} \right) + (1 - e^k E_2(k) a_0) \right] e^{k[(r/r_i)-1]} \right\}, \quad (61a)$$

$$\begin{aligned} \text{with} \quad a_0 &:= \frac{GM}{c_p T_i r_i} \text{ (again).} \\ &\text{(constant entropy gradient).} \end{aligned} \quad (61b)$$

Using (12) and (13) then yields

$$\begin{aligned}
& p(r) = p_i \exp \left[-\frac{\gamma}{\gamma-1} k \left(\frac{r}{r_i} \right) \right] \\
& \quad \times \left\{ \left[e^k a_0 \left(\frac{r_i}{r} \right) E_2 \left(\frac{kr}{r_i} \right) + (1 - e^k E_2(k) a_0) \right] e^{k[(r/r_i)-1]} \right\}^{\gamma/(\gamma-1)} \quad (62)
\end{aligned}$$

and

$$\begin{aligned}
& \rho(r) = \rho_i \exp \left[-\frac{\gamma}{\gamma-1} k \left(\frac{r}{r_i} \right) \right] \\
& \quad \times \left\{ \left[e^k a_0 \left(\frac{r_i}{r} \right) E_2 \left(\frac{kr}{r_i} \right) + (1 - e^k E_2(k) a_0) \right] e^{k[(r/r_i)-1]} \right\}^{1/(\gamma-1)} \quad (63)
\end{aligned}$$

(constant entropy gradient).

Clearly if $k = 0$, we recover equations (16)–(19) for an adiabatic atmosphere (note that $E_n(0) \equiv 1$ for all n).