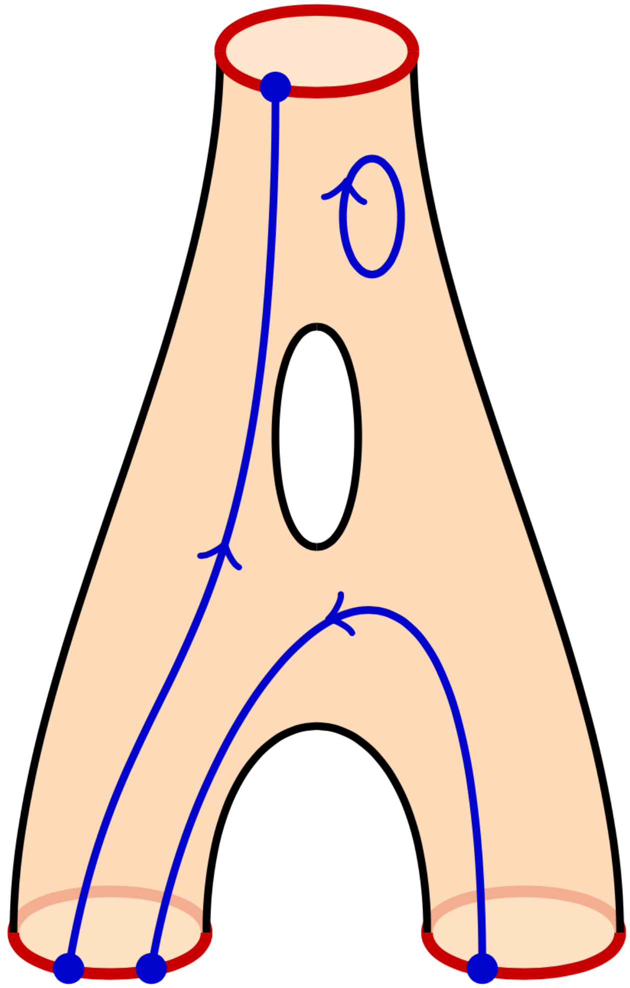

Bordism & Topological Field Theory

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The mathematical facts worthy of being studied
are those which, by their analogy with other facts,
are capable of leading us to the knowledge of a mathematical law
just as experimental facts lead us to the knowledge of a physical law.
They reveal the kinship between other facts, long known,
but wrongly believed to be strangers to one another.

–H. Poincaré

For the reader

These are the lecture notes of the course *"Bordism and Topological Field Theory"* held by Prof. Dr. Claudia Scheimbauer (scheimbauer@ma.tum.de) during the Wintersemester 2023/2024 at the *Technische Universität München*. The tutorial sessions are held by Anja Švraka (svr@ma.tum.de). The notes are typed by Luca Ipsale (luca.ipsale@campus.lmu.de), William Luciani (w.luciani@campus.lmu.de), Andrea Sittoni (sittoniandrea@gmail.com) and Üzeyir Saçıkay (uzeyirsacikay@gmail.com). Exercises are also implemented in the notes by Jacob Skarby (jacob.skarby@tum.de). If you find errors or if you have suggestions of any kind, please write us an e-mail.

⚠ These notes have not been proofread by Prof. Scheimbauer, use at your own risk.

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Notational Conventions

We use the symbol ’✈’ at the end of titles of sections, definitions or theorems that depart from the content of the lecture and deepen some aspect on a topic with an outlook towards research topics.

Notation. Throughout these notes we will abuse notation indicating a collection with some structure on it just by writing down the collection, e.g. denoting a monoid $(M, \cdot)$ by M, a metric space (X, d) by X etc...

Notation. n-manifold and n-bordism mean respectively n dimensional manifold and n dimensional bordism.

Notation. We often denote equivalence classes just with a representative thereof.

Notation. We denote the category of spaces/∞-groupoids both with $\mathcal{S}$, Grpd_∞ and with ∞-Grpd. This is motivated by the homotopy hypothesis 3.8.15.

Notation. When there is no ambiguity, we denote $(\infty, 1)$-categories as ∞-categories.

Prerequisites

The notes strive to be as self-contained as possible. We do assume however knowledge of linear algebra, basic notions from analysis, e.g. C^∞ differentiability, and basic notions of topology, e.g. paracompactness. Knowledge of algebraic topology is helpful but not necessary to understand the most important parts.

Acknowledgements

Apart from Prof. Scheimbauer’s lectures and the cited references, we often got inspiration from the nLab (`https://ncatlab.org/nlab/show/HomePage`) and from notes on algebraic topology by Prof. Land (available on `https://www.mathematik.uni-muenchen.de/~gritscha/TOP1-23.php`).

Contents

Chapter 1

Why should you care? An informal introduction

There are two ways one can approach topological field theories:

1. As a way to make (some[1]) quantum field theories more mathematically rigorous
2. As a way to refine bordism invariants, or, roughly, as a kind of homology for smooth manifolds of a certain dimension, instead of general topological spaces.

We now sketch how these two approaches work.

1.1 QFT

> Physics is very interesting: There are many, many interesting theorems. Unfortunately, there are no definitions.
>
> — David Kazhdan

Start with a (smooth) manifold M and, generally, some extra structure. For example:

- $M^4 = \mathbb{R}^4$ with a metric with signature $(+,+,+,-)$, so that we can distinguish a strictly spatial part and a temporal part ($\mathbb{R}^4 = \mathbb{R}^3_{space} \times \mathbb{R}^1_{time}$). This is called *Minkowski Spacetime* and is particularly important in QFT being the geometric foundation of Special Relativity,
- $M^3 = \mathbb{R}^3$ with the Euclidean metric,
- M^2 with a conformal structure,
- M^{11} as an "Elliptic Fibration", something which *locally* looks like $\left(S^1 \times S^1\right) \times something$. These things are useful in areas with high-sounding names such as "M-Theory" or, more generically, "String Theory".

[1]Sadly, TFTs cannot axiomatize many quantum field theories that are particularly useful in physics, e.g. the quantum field theory behind the standard model.

However, in general, these manifolds must be thought of with other structures, like connections, bundles...
Now we can "define" a Quantum Field Theory on M via a list of ingredients:

1. ***Fields***. The space of fields $\mathscr{F}$, associated usually to a bundle $E \xrightarrow{p} M$, is defined as sections[2] of E over M. In the case the bundle is trivial ($E \cong M \times X$), then $\mathscr{F} = \Gamma(M, E) = \mathrm{Maps}(M \to X)$. But what mathematical object are we dealing with? What do we mean by "space" here? Is it a set, a topological space, a category, a scheme, a stack...?

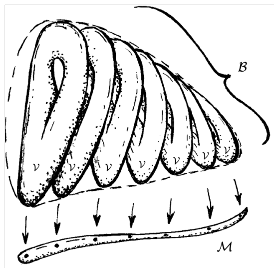


Fig. 15.2 A bundle $\mathcal{B}$, with base space $\mathcal{M}$ and fibre $\mathcal{V}$ may be thought of as constituted as an '$\mathcal{M}$'s worth of $\mathcal{V}$s'. The canonical projection from $\mathcal{B}$ down to $\mathcal{M}$ may be viewed as the collapsing of each fibre $\mathcal{V}$ down to a single point.

2. ***Partition Function***. We need a measure against which we can compute "correlation functions" of fields ψ_1, ψ_2 i.e. the "likelihood of ψ_1 given ψ_2". We thus define a Partition Function
$$Z(M) = \int_{\psi \in \mathscr{F}} e^{iS(\psi)} \mathscr{D}\psi,$$
with
$$S(\psi) = \int_M \mathscr{L}(\psi),$$
called the Action Functional. Generally $\mathscr{L}$ is a polynomial in the fields and it has derivatives. From the Partition Function one can obtain the correlation functions. Although physicists use this formula all the time, formally there is a problem: the measure $\mathscr{D}\psi$ is, in most cases, ill defined.

3. ***Quantization***. Often QFT arises from "quantizing" something classical. But what does this mean? And in what way does this thing behave when changing input?

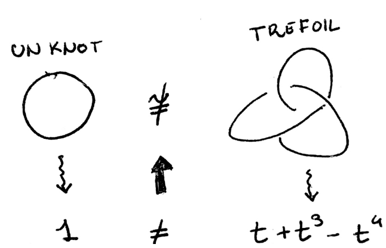
[2] A section of a bundle $E \xrightarrow{p} M$ is a map (in this context, smooth) $s : M \to E$ such that $p \circ s = id_M$.

Physicists use all sorts of techniques (Feynman Diagrams, renormalization...) to make sense of undefined measures and divergences of all kinds emerging from calculations, dealing with things like $\infty - \infty$ or ∞/∞ and obtaining finite and testable results.
This black box that physicists have (successfully) developed frustrates mathematicians because they do not understand why it works! Therefore, axiomatizations have been developed exploiting new tools from geometry, algebra and topology to develop a formal and rigorous framework[3].
The Partition Function Z behaves well when "smoothly" changing the metric and is (in most cases) independent of most extra data of the manifold, so $Z(M)$ depends only on the smooth manifold and as such is purely topological!

If someone is into mathematics for the money or the prestige[4], the field of topological field theories is the one to specialize in:

- René Thom, the mathematician who laid the foundation of cobordism theory[5] in his PhD thesis, received the Fields Medal for this.
- Shiing-Shen Chern and Jim Simons discovered geometric invariants of 3-dimensional Riemannian manifolds called (classical) Chern-Simons invariants. They are a generalization of the total geodesic curvature, which is in turn a generalization of the curvature of a plane curve that = 0 when the curve is a geodesic. Such invariants are the basic building blocks Witten used to define the earliest example we have of a TFT: 3d Chern-Simons theory[6]. Simons then went on to found an incredibily successful hedge fund and became a billionaire. Chern continued to do groundbreaking work in differential geometry and topology; so much that some years ago the International Congress of Mathematicians named a prize after him: the Chern Medal.
- Edward Witten received his Fields medal mainly because he found a link between 3d-TFTs, in particular Chern-Simons theory, and knot theory, in particular with the Jones polynomial. See [Wit89] for details.

The Jones polynomial is a topological invariant of a knot, meaning that you can assign to each knot a Jones Polynomial in such a way that if two Knots have different polynomials, then they must be different.



What the heck do TFTs and knots have in common?! There is actually a deep connection between the two. When writing his paper, Witten drew several pictures like the following:

[3]More specifically to the path integral, see [CR18, 2.1] for an introduction on how topological field theories can be seen as a way to axiomatize some properties of the path integral as a tool to compute correlation functions.
[4]If not already sufficiently clear, we explicitly state that this is a joke.
[5]Which is at the root of TFTs: TFTs can be seen as a refinement of his work.
[6]One can find more on this in [Fre08]

Quantum Field Theory and the Jones Polynomial

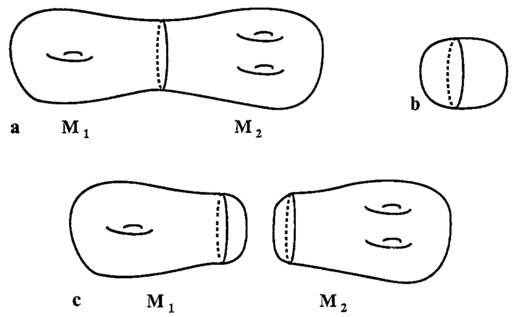


Fig. 5a–c. In **a** is sketched a three manifold M which is the connected sum of two pieces M_1 and M_2, joined along a sphere S^2. Similarly, a three sphere S^3 can be cut along its equator, as in **b**. Cutting both M and S^3 as indicated in **a** and **b**, the pieces can be rearranged into the *disconnected* sum of M_1 and M_2, as in **c**

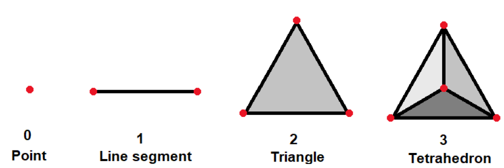
Two mathematicians, Segal and Atiyah, recognized a hidden symmetric monoidal functor and pinned down Witten's intuitions rigorously, thereby axiomatizing TQFTs (and CFTs).

1.2 Topology

Topology is the science of fundamental patterns
and structural relationships of event constellations.
-Fuller

The ideas behind cobordisms were developed already by Poincaré together with homology (and their group structure was discovered and investigated by Emmy Noether) but a proper definition was established by Pontryangin and Thom in the 20$^{\text{th}}$ century.

The strategy in general in (algebraic) topology is to probe a topological space by mapping into it, a way of viewing things very reminiscent of the Yoneda Lemma. In the case of (singular) homology, we map simplices into the space of interest S. These maps are indexed by the dimension of the simplex as in the following figure. We want to construct an



algebraic structure around these maps $\sigma^{(n)} : \Delta^n \to S$ and so we choose a set of coefficients, say $\mathbb{R}$, and construct the set of "formal (finite) sums" of n-simplices as

$$\sum a_i \sigma_i^{(n)},$$

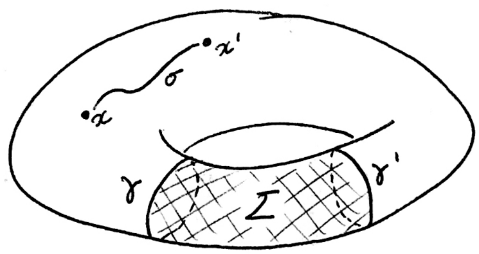
with $a_i \in \mathbb{R}$ and call this set $C_n(S)$. This is now an $\mathbb{R}$-Vector Space generated by the simplices. On the set of simplices we can introduce a map ∂, often called the differential or boundary, that takes each simplex to the alternating sum of the $n-1$ simplices that make up its boundary:

$$\partial\sigma^{(n)} := \sum_{i=0}^{n} (-1)^i \sigma^{(n)}|_{i^{\text{th}}-boundary},$$

and then extend it linearly to $C_n(S)$. Calling $\partial_n := \partial|_{C_n(S)}$, we have $\partial_n : C_n(S) \to C_{n-1}(S)$ and it can also be shown that $\partial_n \circ \partial_{n+1} = 0$, a property which is often simply written as $\partial^2 = 0$. This defines a chain complex structure. Given this property of ∂, note that $\mathrm{im}(\partial_{n+1}) \subset \ker(\partial_n)$ and therefore, we can define the n^{th} homology of S as

$$H_n(S) = \left\{c^{(n)} : \ \partial_n c^{(n)} = 0\right\} / \left\{f^{(n)} : \ f^{(n)} = \partial_{n+1}(f'^{(n+1)})\right\} = \frac{\ker(\partial_n)}{\mathrm{im}(\partial_{n+1})},$$

and this is a vector space (by construction).



Note that $x - x' = \partial\sigma$ so x and x' define the same element in H_0, similarly $\gamma - \gamma' = \partial\Sigma$ so that γ and γ' define the same element in H_1.

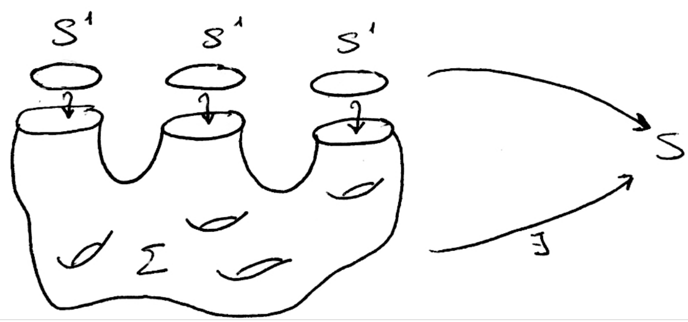
The elements of $H_1(S)$ and $H_2(S)$ can be characterized in different ways:

1. The elements of $H_1(S)$ can be represented by a collection of oriented loops mapped in S.
$$\amalg\, S^1 \longrightarrow S$$
2. The elements of $H_2(S)$ can be represented by a collection of maps from closed oriented surfaces (e.g. genus $g-$surfaces) in S.
$$\amalg\, \Sigma \longrightarrow S$$
3. Let $c \in H_1(S)$, then $\partial c = 0$. Now by 1. this is some map $c : \amalg\, S^1 \to S$ which is $0 \in H_1(S)$ if and only if it extends to a map $\tilde{c}$ of oriented surfaces in S, i.e. if there is the map on the bottom right such that the following diagram commutes.
$$\begin{tikzcd}
\amalg\, S^1 \arrow[d, hook] \arrow[dr, "c"] & \\
\amalg\, \Sigma \arrow[r, dashed, "\tilde{c}"'] & S
\end{tikzcd}$$

A drawing might be helpful:

Can we then construct something like homology but characterized more like this last property? This is exactly the idea of *(co)bordism*:

1. Let M^n be a $n-$dimensional smooth compact manifold generally with boundary. Now, instead of maps from simplices we consider maps
$$M^n \xrightarrow{f} S.$$



2. Instead of the boundary map of simplices we consider something like:

$$\partial\big(M^n \xrightarrow{f} S\big) = \big(\partial M^n \xrightarrow{f|_{\partial M^n}} S\big),$$

   where, if $\partial M^n = \emptyset$ then define the map to be 0.

3. Let $H_n^{\text{bord}}(S)$ denote this "homology theory" of degree n on the space S. Let M^n be closed, then $M^n \xrightarrow{f} S$ is zero in $H_n^{\text{bord}}(S)$ if and only if the map f extends to a $(n+1)-$dimensional smooth compact manifold W such that $\partial W = M$.

Using these ideas one obtains something very similar to singular homology even though different, indeed this bordism theory constitutes a generalized homology theory[7]!

In particular, consider the one point space $S = \{*\} = pt$. What are elements in $H_n^{\text{bord}}(pt)$? Consider a closed manifold M^n. This defines a class in $H_n^{\text{bord}}(pt)$ since $\partial M^n = \emptyset$. Now, this is $0 \in H_n^{\text{bord}}(pt)$ if we can find a compact $n+1$ manifold W^{n+1} and maps into pt such that the following diagram commutes

$$\begin{tikzcd}
M^n \arrow[dd, hook] \arrow[dr, dashed] & \\
& pt \\
W^{n+1} \arrow[ur, dashed] &
\end{tikzcd}$$

however the maps into pt are trivial and carry no extra information. Now, "surely" not all closed manifolds are a boundary of compact manifold! And so we can be sure that, in general, $H_n^{\text{bord}}(pt) \neq 0$ for $n > 0$.

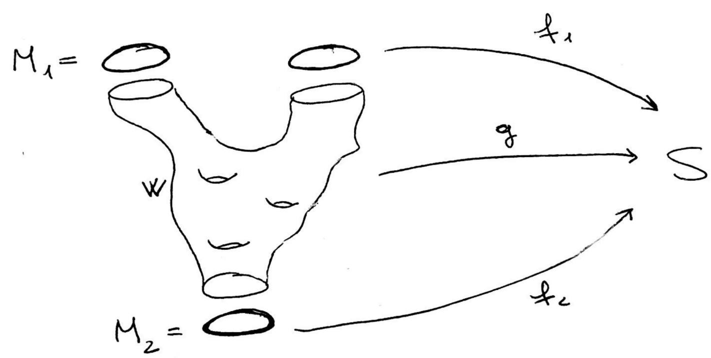
But when do two n manifolds represent the same class in $H_n^{\text{bord}}(S)$? In the case of $S = pt$, they should be the boundary of the same manifold. In general, we require also that the maps into S extend to the $n+1$ manifold, i.e. let M_1 and M_2 be closed n manifolds, with maps $M_1 \xrightarrow{f_1} S$ and $M_2 \xrightarrow{f_2} S$, then they represent the same element in $H_n^{\text{bord}}(S)$ if and only if there exists a compact $n+1$ manifold with boundary W with a map $W \xrightarrow{g} S$ with the

[7] There exist a list of axioms called *Eilenberg–Steenrod axioms* that is used to define what a homology theory is since it may come in different flavours. A generalized homology theory is a theory that has every property required but one, generally (as in the case of Bordism as seen below) that one is the *dimension axiom*.

property that $\partial W = M_1 \amalg M_2$ and such that it makes the following diagram commute

$$\begin{tikzcd}
M_1 \arrow[dr, hook] \arrow[drr, bend left, "f_1"] & & \\
& W \arrow[r, "g"] & S \\
M_2 \arrow[ur, hook] \arrow[urr, bend right, "f_2"'] & &
\end{tikzcd}$$

Pictorially We then say that M_1 and M_2 are *cobordant* and W is called a *bordism* from M_1



to M_2.

Restricting to $S = pt$ allows us to define the *Cobordism Group*

$$\Omega_n = H_n^{\text{bord}}(pt) = \Big\{\text{Closed } n \text{ manifold}\Big\} \Big/ \Big\{(n+1) \text{ dimensional bordisms}\Big\}.$$

But are all these notions are actually useful? Well, classifying manifolds up to diffeomorphism is hard: dimension 0, 1 and 2 can be done without too much trouble but already in dimension 3 things get complicated (think of the Poincaré Conjecture!)... so this classifications can be done (more easily) up to bordism!

Part I

Classical cobordism theory

Chapter 2

Manifolds and bordisms

2.1 Some definitions

Before venturing into a definition of cobordism, let us recall some useful definitions. This section mainly relies on [Lee12] and [Hir76].

Definition 2.1.1 (*Topological Manifold*)**.** A topological manifold of dimension n is a paracompact Hausdorff topological space X such that every point $x \in X$ has an open neighborhood U which is homeomorphic to an open set[1] in $\mathbb{R}^n$. The latter property means that for each $x \in X$ there exist:

- an open subset $U \subseteq X$ containing x, i.e. an open neighbourhood of x,
- a corresponding open subset $\tilde{U} \in \mathbb{R}^n$,
- a homeomorphism $\phi_U : U \to \tilde{U} = \phi(U)$ called coordinate chart, or just chart (see Fig. 2.1).

If (U,ϕ), (V,ψ) are two coordinate charts of a topological manifold X and $U \cap V \neq \emptyset$, then $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a transition function from ϕ to ψ (see Fig. 2.2).

From the introduction it's clear that we will also deal with manifolds with boundary, so we recall also this definition. In order to do this we first introduce the following notation.

[1] Equivalently to $\mathbb{R}^n$.

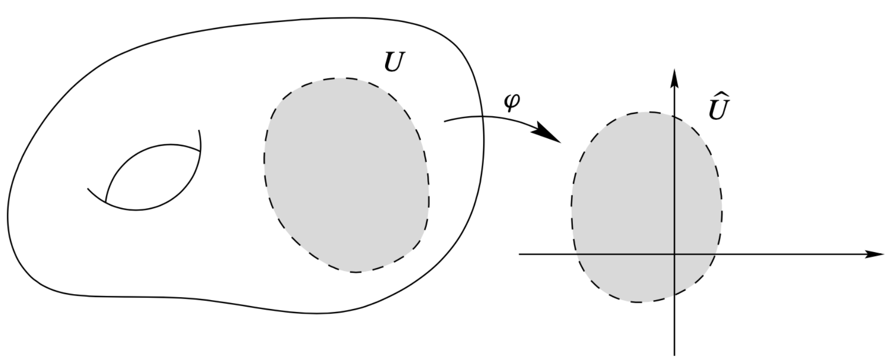


Figure 2.1: The visualization of a coordinate chart map from Lee's textbook on smooth manifolds [Lee12]

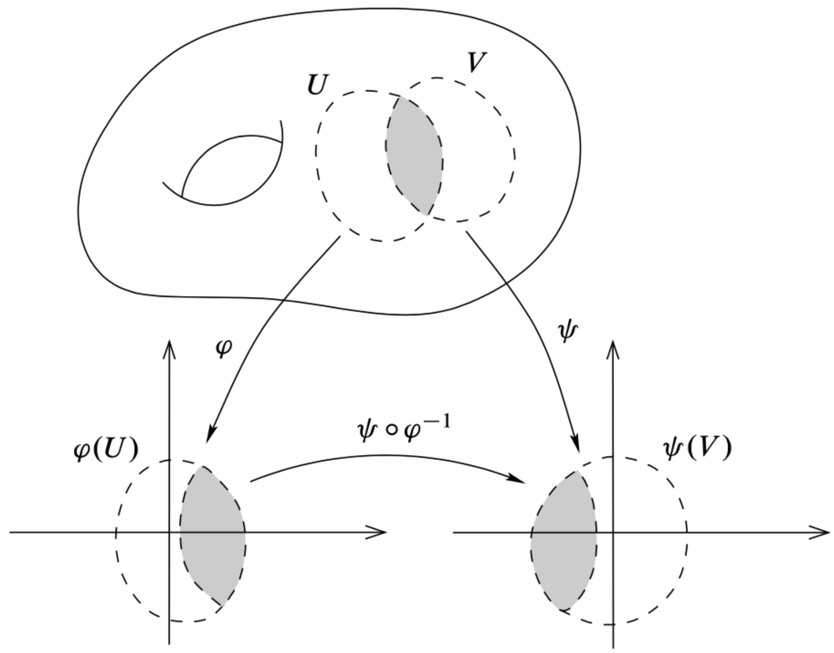


Figure 2.2: The picture of a transition function from [Lee12]

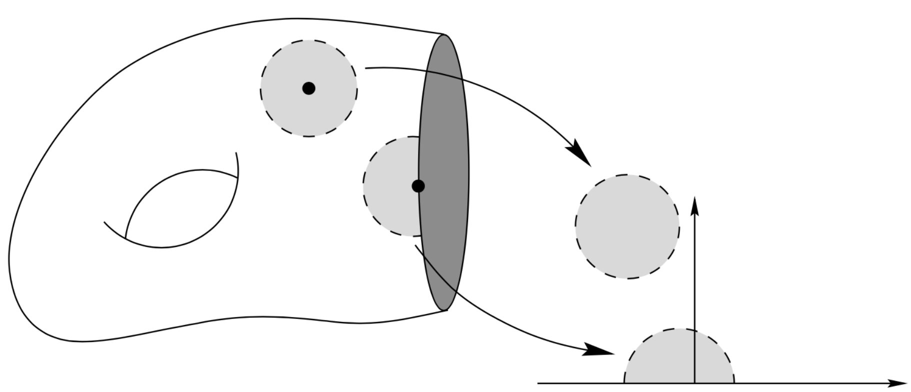


Figure 2.3: The picture of a 2-dimensional manifold with boundary from [Lee12]

Notation (Half-space)**.** By $\mathbb{H}^n$ we denote the n-dimensional upper (closed) half-space,

$$\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n | x_1 \leq 0\}$$

Definition 2.1.2 (Manifold with Boundary)**.** A manifold with boundary of dimension n is a paracompact Hausdorff topological space X such that every point $p \in X$ has an open neighborhood U_p which is homeomorphic to an open set V in $\mathbb{H}^n$, i.e. the closed[2] half space, via the homeomorphism ϕ of the coordinate chart (U_p, ϕ).

Definition 2.1.3 (Boundary of a Manifold)**.** If for $p \in X$ and some chart ϕ it is the case that $x_1(\phi(p)) = 0$ (meaning $\phi_p(x) \in \{(0, x_2, \ldots, x_n)\} \subseteq \mathbb{H}^n$), then it does in every chart. We then say that p is in the *boundary* of X,

$$\partial X := \{p \in X : x_1(p) = 0\}$$

[2]in topological sense and not in the manifold sense we later define.

otherwise, p is in the *interior* of X, which we denote with Int X.
Equivalently, $p \in \partial X$ means that p has neighborhood V that is the domain of a coordinate chart $\psi : V \to \mathbb{H}^n$ such that $\psi(V) \cap \partial\mathbb{H}^n \neq \emptyset$ and sending p to $\partial\mathbb{H}^n$.
Specularly, we could have defined the interior of X to be the set of points $q \in X$ that have a neighourhood U that is the domain of a chart $\psi : U \to \mathbb{R}^n$.

Lemma 2.1.4. *If X is an n dimensional manifold with boundary, then ∂X is a $n-1$ dimensional manifold.*

Proof. Let $p \in \partial X$ be an arbitrary point on the boundary of an n manifold with boundary. Hence, there is an open neighbourhood U_p of p which is homeomorphic to an open set V in $\mathbb{H}^n$, $\phi(U_p) \cong V$. Since $p \in \partial X$ we know that $x_1(\phi(p)) = 0$ and $\phi(p) \in V \cap \mathbb{H}^n$. $\phi : U_p \to V$ can be restricted to a homemorphism $\phi \mid_{\phi^{-1}(V \cap \mathbb{H}^n)}: \phi^{-1}(V \cap \mathbb{H}^n) \xrightarrow{\cong} V \cap \mathbb{H}^n$. Note that $\phi^{-1}(V \cap \mathbb{H}^n) = U_p \cap \partial X$ because p is sent to the boundary $\partial\mathbb{H}^n$ for every chart. Note also that $\partial\mathbb{H}^n = \{(0, x_2, \ldots, x_n)\}$ and $\partial\mathbb{H}^n \cong \mathbb{R}^{n-1}$. Since $V \cap \mathbb{H}^n \in \partial\mathbb{H}^n$ and $V \cap \mathbb{H}^n \cong U_p \cap \partial X$, $U_p \cap \partial X$ is homeomorphic to an open set in $\mathbb{R}^{n-1}$. □

Definition 2.1.5 (Closed Manifold)**.** A manifold is *closed* if it is compact and without boundary, i.e. $\partial X = \emptyset$. Conversely, an *open* manifold is also a manifold without boundary but with no closed components, i.e. it has only non-compact components.

In some contexts, it can be interesting to do calculus on a manifold. In order to make sense of this, one needs to add some extra structure to the topology of the manifold which enables investigating if a map between manifolds is smooth.

Definition 2.1.6 (Smooth Function)**.** Given $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ a function $f : X \to Y$ is smooth[3] if each of its component functions has continuous partial derivatives of all order.

The right notion of isomorphism when talking about smooth functions is that of diffeomorphism:

Definition 2.1.7 (Diffeomorphism)**.** A smooth map $f : X \to Y$ is a diffeomorphism when it is bijective and has a smooth inverse map.

Definition 2.1.8 (Smooth Manifold Without Boundary)**.** A smooth manifold X is a topological manifold, together with a collection of charts $\phi_i : U_i \to \mathbb{R}^n$, called smooth structure, (U_i, ϕ_i) such that:

1. $X = \bigcup_{i \in I} U_i$
2. the pairwise transition functions are smooth in the usual sense of $\mathbb{R}^n$
3. it is maximal with respect to 1. and 2.

Some call a collection of charts (U_i, ϕ_i) such that $\{U_i\}_i \in I$ is a cover of the whole manifold X an atlas. A smooth atlas is an atlas where the transition functions and their inverses are smooth. A smooth structure is thus a maximal smooth atlas. We call charts forming the smooth structure smooth charts.

Example 2.1.9. A lot of the spaces we think of are smooth manifolds, such as:

[3] Also called infinitely differentiable or C^∞.

1. the circle, a dimension 1 manifold
2. a genus g surface, a dimension 2 manifold
3. $\emptyset$, a smooth manifold of any dimension

Notation. In these lecture notes, manifolds are *always* smooth, unless explicitly specified.

Remark. The definition of smoothness we just provided does not work for manifolds with boundary since to say that the transition functions are smooth we used the notion of smoothness of $\mathbb{R}^n$ and not of $\mathbb{H}^n$, the actual codomain of manifolds *with* boundary. Hence, we characterize now smoothness for $\mathbb{H}^n$.

Recall that a map $f : M \to \mathbb{R}^m$, where $M \subseteq \mathbb{R}^n$ is not necessarily open, is smooth if in a neighbourhood of each $x \in M$ it can be extended to a smooth function defined on an open subset in $\mathbb{R}^n$.

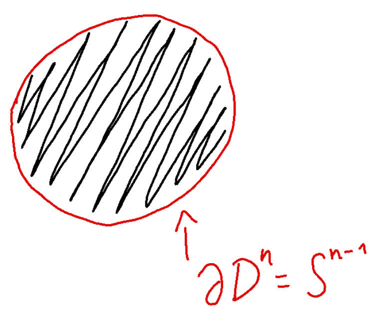
Definition 2.1.10 (Smoothness in the Half Space)**.** If X is an open subset of $\mathbb{H}^n$, we define $f : X \to \mathbb{R}^m$ to be smooth for each $x \in X$ if X can be extended to an open subset $\tilde{X} \in \mathbb{R}^n$, and f to a smooth map $\tilde{f} : \tilde{X} \to \mathbb{R}^m$ such that $\tilde{f}|_{\tilde{X}\cap\mathbb{H}^n} = f$.

Definition 2.1.11 (Smooth Manifold with Boundary)**.** A topological manifold with boundary is smooth if it has a maximal smooth atlas where the transition functions and their inverses are smooth according to 2.1.10.

Corollary 2.1.12. *If X is a smooth n dimensional manifold with boundary, then ∂X is a smooth $n-1$ dimensional manifold. It follows from the proof for topological manifolds.*

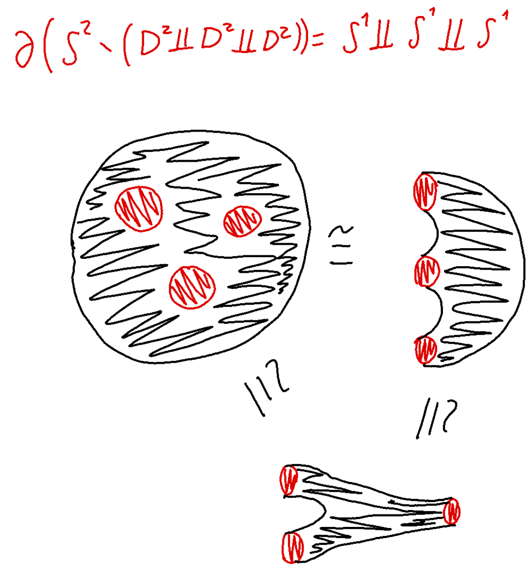
Example 2.1.13.

1. $D^n = \{x \in \mathbb{R}^n \mid \ ||x|| \leq 1\}$ the n dimensional disk, which has a sphere as boundary: $\partial D^n = S^{n-1}$

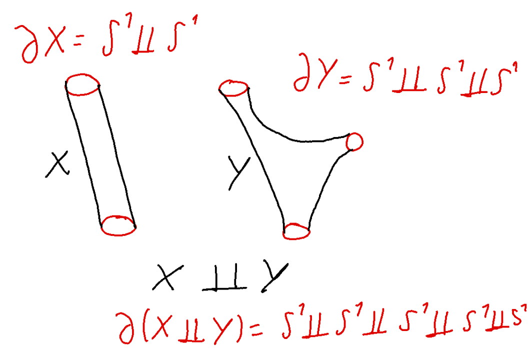


2. Consider 2 dimensional *open* disks, i.e. $D^2 = \{x \in \mathbb{R}^2 \mid \ ||x|| < 1\}$. We can remove such shapes from the sphere and get the manifold $S^2 \setminus D^2_1 \amalg D^2_2 \amalg D^2_3$ which we can visualize

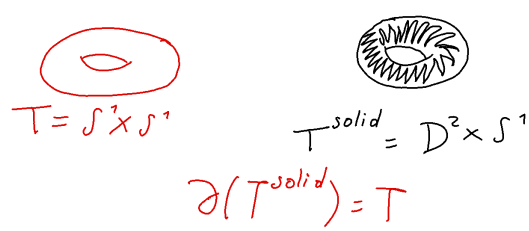
in different ways.



3. Let X, Y be n manifolds with boundary, then $X \amalg Y$ is an n manifold with boundary.



4. $T = S^1 \times S^1$ is a 2 manifold with $\partial T = \emptyset$ and $T^{\text{solid}} = D^2 \times S^1$ is a 3-manifold with $\partial T^{\text{solid}} = T$.



Definition 2.1.14 (Smooth Map to $\mathbb{R}^n$)**.** Let X be a smooth manifold. $f : X \to \mathbb{R}^n$ is smooth if for each $x \in X$ there is a smooth chart (U, ϕ) such that $f = f \circ \phi^{-1}$

This definition can be generalized to maps between smooth manifolds.

Definition 2.1.15 (Smooth Map between Manifolds)**.** Let X and Y be smooth manifolds of dimensions m and n and $f : X \to Y$ a function between them. f is smooth if for every $x \in X$ there is a smooth chart (U, ϕ) of X where $x \in U$ and a smooth chart (V, ψ) of Y where

$f(x) \in V$ and $f(U) \subseteq V$ such that $\tilde{f} = \psi \circ f \circ \phi^{-1}$ is smooth. $\tilde{f}$ can be represented as in the following diagram:

$$\begin{tikzcd} X \supseteq U \arrow[r, "f|_U"] \arrow[d, "\phi_x"', "\cong"] & V \subseteq Y \arrow[d, "\cong"', "\psi_x"] \\ \mathbb{H}^m \supseteq \tilde{U} \arrow[r, "\tilde{f}"'] & \tilde{V} \subseteq \mathbb{H}^n \end{tikzcd}$$

This also allows to define the natural notion of isomorphism in the category of smooth manifolds:

Definition 2.1.16 (Diffeomorphism of Manifolds)**.** A diffeomorphism is a smooth map with a smooth inverse, i.e. an isomorphism in the category of smooth manifolds.

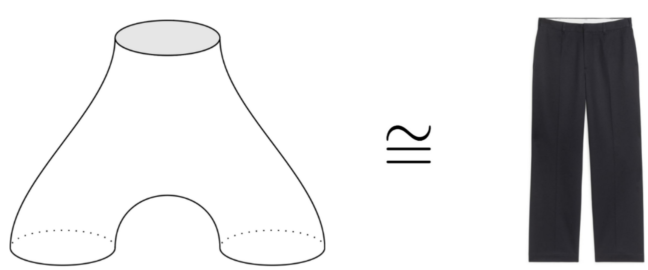
Exercise 2.1.17. Come up with the natural notion of smooth map between manifolds *with* boundary.

2.2 What is a Bordism?

Definition 2.2.1 (1^{st} variant of the definition of bordism: via *strict equalities*)**.** Let Y_0, Y_1 be closed n manifolds. A bordism (X, p) from Y_0 to Y_1 consists of a compact $n+1$ manifold with boundary together with a map $p : \partial X \to \{0, 1\}$ such that $Y_0 = p^{-1}(0)$ and $Y_1 = p^{-1}(1)$. Thus, $\partial X = Y_0 \amalg Y_1$.
We then say that Y_0 and Y_1 are *cobordant*.

We call Y_0 the *incoming* boundary and Y_1 the *outgoing* boundary. We sometimes write $\partial_{in} X = Y_0$ and $\partial_{out} X = Y_1$.

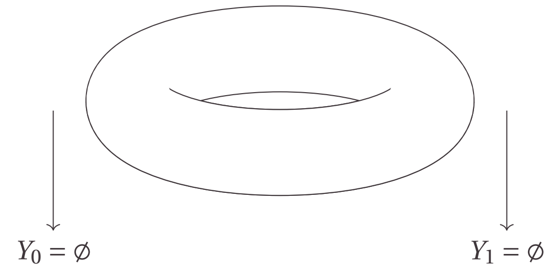
Example 2.2.2. "pair of pants"



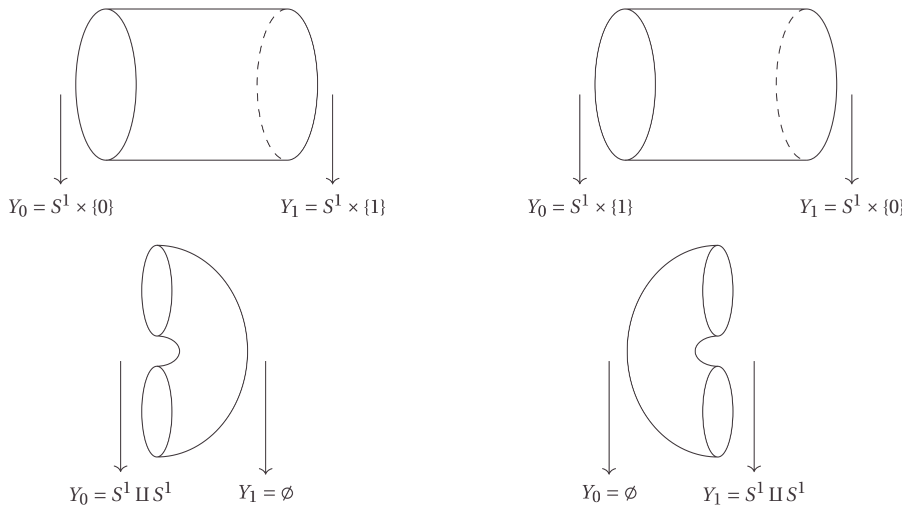
Bordism from S^1 to $S^1 \amalg S^1$

Example 2.2.3.

- The torus is a bordism from $\emptyset$ to $\emptyset$. ($\partial T = \emptyset = Y_0 \amalg Y_1 \implies Y_0 = Y_1 = \emptyset$)



- $X = S^1 \times [0,1]$, now $\partial X = S_1 \amalg S_1$ and we can view it as a bordism in 4 ways:



The latter two are sometimes calles 'macaroni'.

This shows that different bordisms can arise from the same underlying manifold. We will have a way of differentiating them when we will introduce tangential structures on a manifold, which will enable us to explain in which direction a manifold is oriented.

- Given two n manifolds M, N, their *connected sum* is

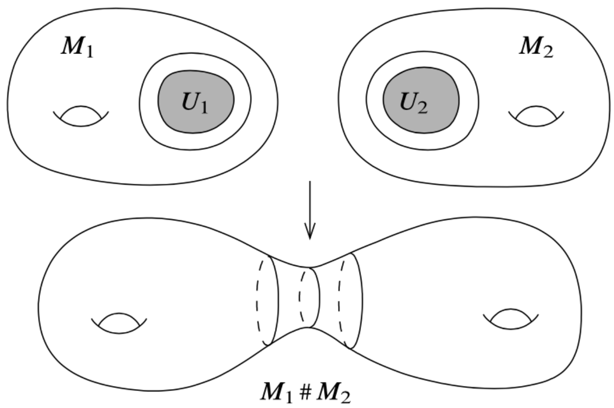
$$M\#N = M \setminus (D^n)^\circ \coprod_{S^{n-1}} N \setminus (D^n)^\circ$$

where $^\circ$ is for taking the interior and $\coprod_{S^{n-1}}$ is glueing along the new boundaries in N and M.

Proposition 2.2.4. *$M\#N$ is an n manifold.*

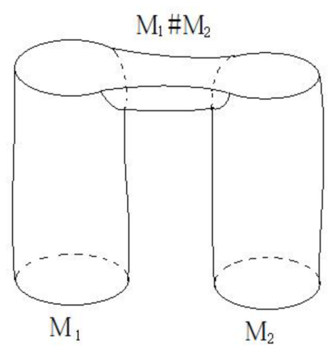
Lemma 2.2.5. *There is a bordism between $M \coprod N$ and $M\#N$.*

Proof. A bordism between $M_1 \coprod M_2$ and $M_1\#M_2$ may be constructed in the following manner (proof taken from the Manifold Atlas Project). Consider a cylinder $M_1 \times I$, from which we remove an ϵ-neighbourhood $U_\epsilon(v_1 \times 1)$ of the point $v_1 \times 1$. Similarly, remove the neighbourhood $U_\epsilon(v_2 \times 1)$ from $M_2 \times I$ (each of these two neighbourhoods can be identified with the half of a standard open $(n+1)$-ball). Now connect the two remainders of cylinders by a "half pipe" $(S_n \leq 0) \times I$ in such a way that the half-sphere $S_n \leq 0$ is identified with the half-sphere on the boundary of $U_\epsilon(v_1 \times 1)$, and $(S_n \leq 0) \times I$ is identified with the



The picture of a connected sum from Wikipedia

half-sphere on the boundary of $U_\epsilon(v_2 \times 1)$. Smoothening the angles we obtain a manifold with boundary $(M_1 \coprod M_2) \coprod (M_1 \# M_2)$.



The illustration by the Manifold Atlas Project of the bordism constructed in the proof above

□

Theorem 2.2.6. *Being cobordant is an equivalence relation on closed n-manifolds.*

Proof. We need to show that the relation satisfies the properties of reflexivity, symmetry and transitivity:

- Reflexive: $Y_0 \sim Y_0$ since we can take $X = Y_0 \times [0,1]$ as in the previous example. We then have $\partial X = Y_0 \times \{0,1\} \xrightarrow{p} \{0,1\}$.

- Symmetric: assume $Y_0 \sim Y_1$. Thus, we have (X, p) from Y_0 to Y_1. We can use the same manifold and compose p with the $swap$ of 0 and 1 in order to get

$$\tilde{p} = swap \circ p : \partial X \xrightarrow{p} \{0,1\} \xrightarrow{swap} \{1,0\}$$

  Consequently, $(X, swap \circ p)$ is a bordism from Y_1 to Y_0.

- Transitive: we have $Y_0 \sim Y_1$ via (X_1, p_1) and $Y_1 \sim Y_2$ via (X_2, p_2). Then we can use as a manifold $X = X_1 \cup_{Y_1} X_2$ where $\cup_{Y_1}$ indicates the gluing along Y_1. This is a manifold but there's some work involved in showing it's a *smooth* manifold and it will be done later on (2.4), via an equivalent definition of bordism. We then think of the boundary in the following way $\partial X = \partial_{in} X_1 \amalg \partial_{out} X_2$.

□

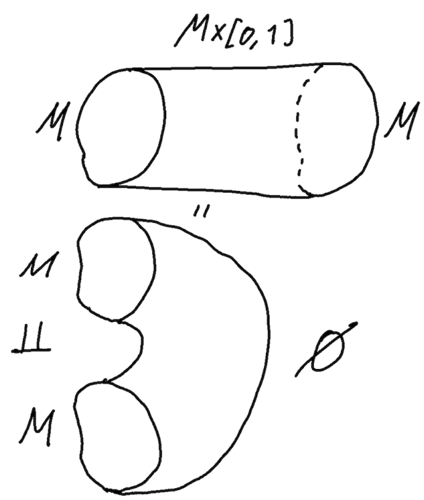
2.3 Cobordism groups

Definition 2.3.1. The set underlying the n-th cobordism group is

$$\Omega_n = \frac{\{\text{closed } n \text{ manifolds}\}}{\{n+1 \text{ cobordisms}\}}$$

$(\Omega_n, \amalg)$ is an abelian group with operation given by the disjoint union $\amalg$.

1. the disjoint union is associative and commutative
2. the identity element[4] is the cobordism class of the empty set[5] $[\emptyset]$, i.e. the set of n-manifolds which are bordant with the empty set
3. every element has an inverse: $[M] \amalg [M] = [\emptyset]$ since $M \amalg M = (\partial(M \times [0,1]))$ and such a cylinder can be bent into a macaroni, which is bordant to the empty set:



Remark. There is an insidious set-theoretic issue when we define the n-th cobordism group Ω_n (2.3.1): is the collection of all closed n manifolds a set? And what about the collection of all $n+1$ bordisms? It could be the case that it is something bigger than a usual set, thus not a set and problematic since we would not know how to treat them[6]. For example, we know that the collection of all sets is strictly greater than any set[7] and thus not

[4]We will sometimes write $0 = [\emptyset]$ or more often $\emptyset = [\emptyset]$

[5]Reminder: $\emptyset$ is an n-manifold for every n.

[6]We are assuming to be working in ZFC. There are other set theories in which one can define collections greater than sets, e.g. Von Neumann–Bernays–Gödel set theory.

[7]Because of famous set-theoretic paradoxes like Cantor's paradox.

a set. Similarly, also the collection of topological spaces is not a set because we can regard any set as a topological space via the discrete topology[8]. One could wonder if the collection of all manifolds of a certain dimension n is likewise not a set. This is fortunately for us not the case. The following theorem allows us to happily treat {closed n manifolds} and {$n+1$ cobordisms} as sets by replacing abstract manifolds and cobordism by manifolds and cobordisms embedded in $\mathbb{R}^\infty$.

Theorem 2.3.2 (Whitney Embedding theorem[9])**.** *Any n manifold can be embedded in $\mathbb{R}^\infty (= \bigcup_{n\in\mathbb{N}} \mathbb{R}^n)$. The space of such embeddings is contractible.*[10]

Remark. Actually $(\Omega_n, \amalg)$ is a finitely generated abelian group, but this is a hard theorem. In particular, it is a finite product of cyclic groups of order 2 (from 2.)

Definition 2.3.3 (Bordism invariant)**.** A bordism invariant is a homomorphism of abelian groups

$$(\Omega_n, \amalg, \emptyset) \to (A, \cdot, e)$$

the abelian group A can be $\mathbb{Z}$, $\mathbb{R}$ or $\mathbb{C}$ for instance.

Remark. Many important manifold invariants are also bordism invariants.

Example 2.3.4.

- χ mod 2 (the Euler characteristic)
- signature
- characteristic classes such as Pontrjagin, Stiefel-Whitney or Chern classes

2.3.1 The 0-th Cobordism Group, Ω_0

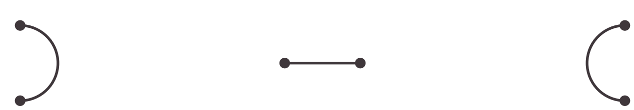
Consider

$$\Omega_0 = \{\text{finite disjoint unions of points}\}/\{\text{1-dimensional cobordisms}\}.$$

To compute this, consider the following classification of 1 manifolds:

Proposition 2.3.5. *Any* 1 *dimensional compact manifold with boundary is diffeomorphic to a finite disjoint union of closed intervals* $[0,1]$ *and circles* S^1.

Example 2.3.6. An example of a 1-dimensional cobordism is $W = [0,1]$. Here are the 3 different way of seeing it as a bordism: from $Y_0 = \{0,1\}$ to $Y_1 = \emptyset$, from $Y_0 = \{0\}$ to $Y_1 = \{1\}$ and from $Y_0 = \emptyset$ to $Y_1 = \{0,1\}$.

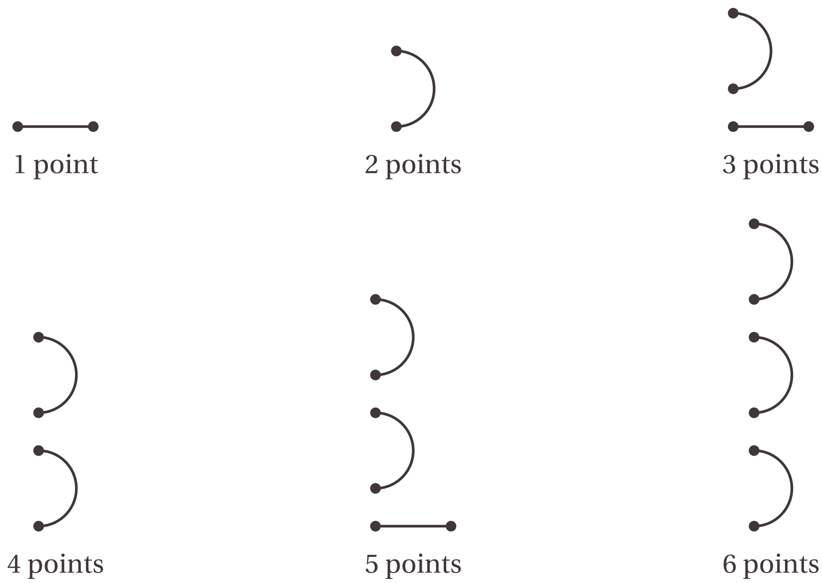


[8] The topology where the set of open sets is the powerset of the set in question, or in other words where any subset is open.

[9] This result is not important only in this case, but notably for the interest of this class it was used to construct an apt category of bordisms 4 in order to sketch a partial proof of the most important conjecture in the field of TFTs: the cobordism hypothesis; see for details on this [Lur09] and [CS19].

[10] Note that there are refinements of the latter result.

Example 2.3.7. Here is a list of various finite disjoint unions of points with different cardinalities. Can you see a pattern emerging?



Keeping this pattern in mind, consider collection of k points, with k finite (i.e. the only closed 0 manifolds):

- If k is even, we can find a bordism to $\varnothing$
- If k is odd, we can find a bordism to $\{*\}$

We then have

$$2k \text{ points} \sim \varnothing, \qquad 2k+1 \text{ points} \sim 1 \text{ point}$$

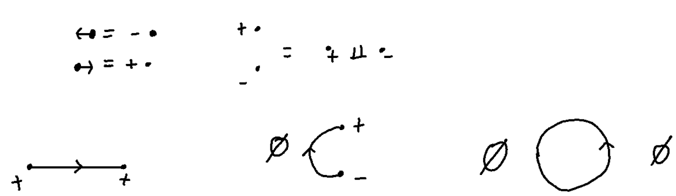
Therefore, the 0-th cobordism group is given by

$$\Omega_0 = \{\varnothing, \{*\}\} \cong \mathbb{Z}_2. \tag{2.1}$$

A possible variation is: add a "decoration", called *orientation* (will be made more precise later), so we now have:

$$\Omega_0^{or} = \big\{\text{finite sets of points } S \text{ with a map } S \to \{+,-\}\big\} / \{\text{oriented cobordism}\}$$

An oriented cobordism in 1 dimension is given by $[0,1]$ or S^1 which comes with an orientation. Here are some examples



Exercise 2.3.8. What is Ω_0^{or}? Try to think about it!

2.3.2 The 1st cobordism group, Ω_1

Now we have

$$\Omega_1 = \{\text{closed 1-dimensional manifolds}\}/\{\text{2-dimensional cobordisms}\}.$$

In studying this group, the following result, which is a restriction of 2.3.5:

Theorem 2.3.9. *Any closed* 1*-dimensional manifold is a finite disjoint union of circles*

However, a circle is the boundary of a 2-disk, which gives a cobordism from S^1 to $\emptyset$, we then have $S^1 \sim \emptyset$. Hence, also finite disjoint unions of S^1 are cobordant to the empty set. Therefore, the 1st cobordism group is trivial:

$$\Omega_1 = 0 \tag{2.2}$$

2.3.3 The 2nd cobordism group, Ω_2

In order to find the 2nd cobordism group, we first need a classification of 2 manifolds.

Proposition 2.3.10 (Classification of 2-dimensional manifolds)**.** *Every connected closed* 2 *manifold is diffeomorphic to*

1. S^2*, orientable*

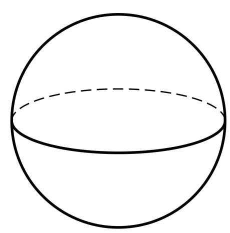


Figure 2.9: 2-sphere, S^2

2. $\Sigma_g = \underbrace{T\#\ldots\#T}_{g-times}$*, orientable*

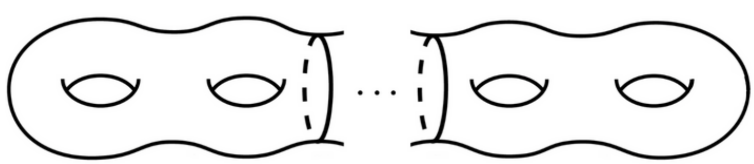


Figure 2.10: Surface of genus g, Σ_g

3. $P_k = \underbrace{\mathbb{RP}^2\#\ldots\#\mathbb{RP}^2}_{k-times}$*, non-orientable*

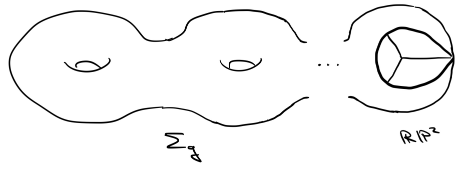
We can now use this knowledge to compute

$$\Omega_2^{\text{or}} = \{\text{closed oriented 2 manifolds}\}/\{\text{3-dimensional oriented cobordisms}\}$$

by observing that S^2 and Σ_g are both cobordant to $\emptyset$ since they can simply be "filled" to a 3 dimensional manifold with boundary, we get

$$\Omega_2^{or} = 0. \tag{2.3}$$

Exercise 2.3.11. What about the non-oriented part Ω_2?



(a) $P_{2g+1} \cong \Sigma_g \# \mathbb{RP}^2$

(b) $P_{2g+2} \cong \Sigma_g \# K$

2.4 Different definitions of bordisms

Reminder (Definition 2.2.1 of Bordism)**.** Let Y_0, Y_1 be closed n manifolds. A bordism (X, p) from Y_0 to Y_1 consists of a compact $n+1$ manifold with boundary together with a map $p : \partial X \to \{0,1\}$ such that $Y_0 = p^{-1}(0)$ and $Y_1 = p^{-1}(1)$ (hence $\partial X = Y_0 \amalg Y_1$). We say that Y_0 and Y_1 are *cobordant*.

There was a complaint: why don't we use diffeomorphisms instead of the highlighted equalities? This question arises because even with simple examples the equality seems too strict:

Reminder (from 2.2.6)**.** Being cobordant is an equivalence relation.

In particular, for any closed n manifold Y we have $Y \sim Y$. The bordism is given by the cylinder $Y \times [0,1]$, where $p : Y \times \{0,1\} \to \{0,1\}$. Then $Y \times \{0\} = p^{-1}(0)$, but $Y \neq Y \times \{0\}$! It seems like our previous definition does not actually give rise to an equivalence relation. Instead, if we substitute the strict equalities with diffeomorphisms (isomorphisms in the category of smooth manifolds) things seem to work out flawlessly:

Definition 2.4.1 (2^{nd} variant of the definition of bordism: via diffeomorphisms)**.** Let Y_0, Y_1 be closed n manifolds. A bordism from Y_0 to Y_1 is a quadruple (X, p, ϕ_0, ϕ_1) consisting of a compact $n+1$ manifold with boundary ∂X together with maps

$$p : \partial X \to \{0,1\}$$
$$\phi_0 : Y_0 \xrightarrow{\cong} p^{-1}(0)$$
$$\phi_1 : Y_1 \xrightarrow{\cong} p^{-1}(1)$$

(we therefore have $\partial X \cong Y_0 \amalg Y_1$).

Example 2.4.2. Let $\psi : M \xrightarrow{\cong} N$ be a diffeomorphism. Then $(M \times [0,1], M \times \{0,1\} \xrightarrow{p} \{0,1\}, \phi_0, \phi_1)$, with maps

$$\phi_0 : M \xrightarrow{id \times \{0\}} p^{-1}(0) = M \times \{0\}$$
$$\phi_1 : N \xrightarrow{\psi^{-1} \times \{1\}} p^{-1}(1) = M \times \{1\}$$

gives a bordism from M to N. This is called the mapping cylinder of ψ.
$\Rightarrow$ under the new definition, <u>any</u> two diffeomorphic manifolds are also cobordant.

Why does this not change the definition of bordism? Because any diffeomorphism also gives a bordism in the old sense: Given $M \xrightarrow{\cong} N$, take the gluing $(M \times I) \underset{M \times \{1\}}{\coprod_{\psi}} N$ so that $Y_1 = N$ and $Y_0 = M \times \{0\}" = "M$ (abusing notation).

Now we prove the transitivity of the relation 'being cobordant'. We do that by showing that given X from Y_0 to Y_1 and X' from Y_1 to Y_2, we can glue them along the boundary in common, Y_1, to obtain a bordism from Y_0 to Y_2. In order to achieve this, we need to introduce the following notion:

Definition 2.4.3 (Collar of a boundary)**.** Let X be a manifold with boundary. A collar of the boundary is an open set $U \subseteq X$ containing ∂X together with a diffeomorphism $(-\epsilon, 0] \times \partial M \to U$ for some $\epsilon > 0$.

Theorem 2.4.4. *The boundary of a manifold with boundary always has a collar.*

Idea of the proof. We start with a manifold with boundary, locally at $x \in \partial X$ we have

$$(-\epsilon_x, 0] \times V_x \xrightarrow{\cong} U_x \subseteq \mathbb{H}^n$$

where V_x is an open neighbourhood of x in ∂X. Now we use a standard trick in differential geometry/topology (see [Hir76]): globally patch these diffemorphisms using a partition of unity. It works if X is compact. □

Definition 2.4.5 (3^{rd} variant of the definition of bordism: via collars)**.** Let Y_0 and Y_1 be closed n manifolds. A bordism from Y_0 to Y_1 is a quadruple (M, p, ϕ_0, ϕ_1) with M and p as before. ϕ_0, ϕ_1 are given by

$$\phi_0 : [0, \epsilon) \times Y_0 \to U \supseteq p^{-1}(0)$$
$$\phi_1 : (-\epsilon, 0] \times Y_1 \to V \supseteq p^{-1}(1)$$

where U and V together form a collar of the boundary of M.

Moreover,

$$([0,\epsilon) \times Y_0) \amalg ((-\epsilon, 0] \times Y_1) \xrightarrow{\phi_0 \amalg \phi_1} \left(U \amalg V\right) \supseteq \left(p^{-1}(0) \amalg p^{-1}(1)\right) = \partial M$$

This variant of the definition is equivalent to the other two but we would unfortunately need a lot of differential topology in order to prove this.

However, with the third definition we can finally prove the following claim which then is what we were missing to prove transitivity of being cobordant.

Claim. $X \underset{Y_1}{\cup} X'$ *admits a smooth structure.*

Transitivity of cobordant. We need a maximal atlas with smooth transition functions. Clearly, in the interior points of X and X' this exists since we can use directly the charts of X and X'. The problem is the double collar around Y_1. So we need to construct charts around points on Y_1. Take $U_1^- \underset{Y_1}{\cup} U_1^+$ in an open neigbourhood. Then via $U_1^- \underset{Y_1}{\cup} U_1^+ \xrightarrow{\phi_1 \cup \phi_0'} (-\epsilon, \epsilon) \times Y_0 = (-\epsilon, 0] \cup [0, \epsilon) \times Y_0$ which contains $(-\epsilon, \epsilon) \times V_x$ which maps to $\mathbb{R}^n$ via ψ_x and thereby generating a maximal atlas.

Let X be a bordism between Y_0 to Y_1 and X' between Y_1 and Y_0.

$$[0, \epsilon) \times Y_0 \xrightarrow{\phi_0} U$$

$$(-\epsilon, 0] \times Y_1 \xrightarrow{\phi_1} U_1^-$$

$$[0, \epsilon) \times Y_1 \xrightarrow{\cong} (-\epsilon, 0] \times Y_1 \xrightarrow{\phi_0'} U_1^+$$

$$(-\epsilon, 0] \times Y_2 \xrightarrow{\phi_1'} U_2$$

□

2.5 Tangential Structures

2.5.1 Orientations and the tangent bundle

Definition 2.5.1 (Tangent bundle)**.** The tangent bundle TM of a bundle M is the vector bundle over M of rank n with

1. fibers given by the tangent spaces T_pM at each point $p \in M$:

$$(TM)_p = T_pM$$

2. (as a set) $TM = \coprod_{p\in M} T_pM$ with natural projection:

$$TM = \coprod_{p\in M} T_pM \xrightarrow{p} M \tag{2.4}$$

$$(p, v) \mapsto p \tag{2.5}$$

It is possible to define a topology on TM such that it is a smooth $2n$ manifold: (U,ϕ) chart of M, we have a chart of TM, $(p^{-1}(U),\tilde{\phi})$ by looking at preimages of U under the projection map and we construct the local trivialization with the usual construction. This is summarized by the fact that the following diagram commutes:

$$\begin{tikzcd}
p^{-1}(U) = TU = \coprod_{p\in U} T_pM \arrow[r, "\tilde{\phi}=(\phi,d\phi)"'] \arrow[d, "p"] & T\tilde{U} = \tilde{U} \times \mathbb{R}^n \arrow[d, "pr_{\tilde{U}}"] \\
U \arrow[r, "\cong", "\phi"'] & \tilde{U} \subseteq \mathbb{R}^n
\end{tikzcd}$$

This is actually an atlas with smooth structure. Transition functions $g_{ij} = \tilde{\phi}_j \circ \tilde{\phi}_i^{-1}|_{\tilde{\phi}_i(U_i \cap U_j)}$ are actually the jacobian and we have $g_{ij} \in GL_n(\mathbb{R})$.

Definition 2.5.2 (Orientability and admissibility of G structure)**.** If TM has local trivializations such that the transition functions between overlapping trivializations are in $GL_n^+(\mathbb{R}) \subseteq GL_n(\mathbb{R})$ then we say that M is orientable.
In other words "The structure group of TM can be reduced to $GL_n^+(\mathbb{R}) \subset GL_n(\mathbb{R})$".
Analogously, given some $G \subset GL_n(\mathbb{R})$, we can say M admits a G structure if the local trivialisations are such that $\phi_{i,j} \in G$, i.e. "the structure group can be reduced to G". In this sense, being orientable is the same as admitting a $GL_n^+(\mathbb{R})$ structure.

Example 2.5.3.

- $\mathbb{R}^n$: we have $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$, meaning the tangent bundle is a trivial bundle.
- S^1: we again have the trivial bundle $TS^1 \cong S^1 \times \mathbb{R}$
- S^2: now $TS^2 \ncong S^2 \times \mathbb{R}^2$, it is not a trivial bundle. This follows from the hairy ball theorem.

All these are orientable. Instead examples of non orientable manifolds are the Klein bottle, the Möbius strip and $\mathbb{RP}^2$.

Definition 2.5.4 (Orientation and G structure)**.** An orientation (equivalently, a $GL_n^+(\mathbb{R})$ structure) of M is an equivalence class of trivialisations $\{(\tilde{U},\tilde{\phi})\}$ of TM such that $\det g_{ij} > 0$ (i.e. $g_{ij} \in GL^+(n,\mathbb{R})$).
Analogously, given some $G \subset GL_n(\mathbb{R})$, a G structure on TM is an equivalence class of trivialisations $\{(\tilde{U},\tilde{\phi})\}$ such that $g_{ij} \in G$.

We can now note the following lemma which is related to the fact that on a manifold one can always construct a Riemannian metric.

Lemma 2.5.5. *A $GL_n^+(\mathbb{R})$ structure is the same as an $SO(n)$ structure.*

How many orientations can M have? 0 if it's not orientable, 2 if it's connected and orientable, in general $2^{\#\text{ connected components}}$ if it's orientable.

Definition 2.5.6 (Opposite orientation)**.** M oriented with charts $\{(U_i,\phi_i)\}$. The opposite orientation on M is given by: $\{(U_i,\bar{\phi}_i)\}$ in which $\bar{\phi}_i$ is given by the following composition

$$U_i \xrightarrow{\phi_i} \mathbb{R}^n \to \mathbb{R}^n \tag{2.6}$$

We will indicate M with opposite orientation by $\overline{M}$.

Example 2.5.7. S^1

Definition 2.5.8 (Orientation preserving and reversing diffeomorphism)**.** A diffeomorphism $\psi : M \to N$ between oriented manifolds is orientation preserving (reversing) if the orientation on N induced by the one on M is the (opposite) orientation on N.

Example 2.5.9. Consider maps $S^1 \to S^1$ (implicitly using the same orientation on each S^1). The identity is an orientation preserving map, while the map $x \mapsto \bar{x}$ is orientation reversing (considering $S^1 \subset \mathbb{C}$).
We could instead consider maps $S^1 \to \overline{S^1}$. In this case the identity is orientation reversing.

Given a manifold M with boundary, its tangent bundle (how is it defined?) contains a copy of $T(\partial M)$. This is because we can "glue another collar" $[0,\epsilon]$ onto the boundary so that the points that were on the boundary are now interior points. This allows us to define T_xM for $x \in \partial M$, the tangent bundle for boundary points. Instead, considering ∂M as its own manifold we have $T_x\partial M$ and we have that $T_x\partial M \subset T_xM$ and it has codimension 1. We can then define the quotient

$$\nu_x = T_xM/T_x\partial M$$

and we can write the following short exact sequence of bundles on ∂M.

$$0 \to T(\partial M) \to i^*TM \to \nu \to 0 \tag{2.7}$$

in which $i : \partial M \to M$ is the inclusion and $i^* : TM$ (bundle over M) $\to TM$ (bundle over ∂M) is its pullback. If we fix a local chart with coordinates $x_1,\dots,x_n$ we have a basis $\frac{\partial}{\partial x_1}$ for ν.
This has the following consequences:

1. an orientation on M induces one on ∂M
2. ν has two "orientations" ($\cong$ normal directions)

3. an orientation on ∂M induces one on $\partial M \times (0,1)$ (if I choose an orientation on $(0,1)$) but in general we can't extend this construction to all of M.

   More generally, note that if M and N are oriented, then $M \times N$ is oriented. [Fre13]

Special case: 0 dimensional manifold. $M = \{x_1, \dots, x_k\}$, but $T_x M = 0$ so our definition fails! So in this case the definition does not capture what we would like, in particular the consequences we found before are either tautological or boring. We can however use 3. to construct a new definition.

Definition 2.5.10. An orientation on a 0 dimensional manifold is a map $M \to \{\pm 1\}$ ($\iff$ orientation of $M \times (0,1)$ if I fixed a chosen orientation of (0,1)).

Definition 2.5.11 (Oriented bordism)**.** Let Y_0 and Y_1 be oriented closed n manifolds. An oriented bordism from Y_0 to Y_1 is an oriented $n+1$ manifold X together with $p : \partial X \to \{0,1\}$ and $\psi : \partial X \to \overline{Y}_0 \amalg Y_1$ an orientation preserving diffeomorphism. (Note: now p is actually too much data ..., since it basically tells us which of the Y_i we're reversing, so we can get rid of it.)

Again this is an equivalence relation which we call "being oriented cobordant", which allows us to give the following definition.

Definition 2.5.12. The oriented bordism group is:

$$\Omega_n^{or} = \{\text{closed oriented } n \text{ manifolds}\} / \{\text{oriented } n+1 \text{ cobordisms}\} \tag{2.8}$$

Remark. Note that $\emptyset$ is an oriented n manifold with a unique orientation.

2.5.2 Framings

Recall the definition of a G structure given above. As stated previously, if $G = GL_n^+(\mathbb{R})$ a G structure is an orientation. Another choice is simply to take $G = \{e\}$, this gives a *framing*, which amounts to a smooth choice of basis for every point. We can talk about *frameable* or *parallelizable* manifolds, meaning they admit a framing. As a consequence, the manifold has a trivial tangent bundle we have $TM \cong M \times \mathbb{R}^n$ and a choice of framing corresponds to a choice of isomorphism.

Example 2.5.13. Some examples of frameable manifolds are the following:

- For $\mathbb{R}^n$, we have $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$
- S^1
- $S^1 \times S^1$
- Any Lie group H has $TH \cong H \times \mathbb{R}^{\dim}$
- What about S^2? Not frameable because of the hairy ball theorem (we would need two vector fields that are a basis at each point of S^2 but because of the theorem we don't even have one).

In fact, there aren't many closed connected 2 manifolds with framing:

Fact. *The only closed connected 2 manifolds with framing is the torus.*

This follows from the Poincaré–Hopf index theorem[11].

Lemma 2.5.14. *Any framing on an n manifold M induces an orientation on M.*

From our definition this is immediate because $\{e\} \subseteq GL_n^+$ (however there are other definitions of orientation for which it's not immediate).

2.6 Classification of 2 manifolds with boundary

The aim of this chapter is to prove the aforementioned classification of compact 2-dimensional manifolds with the use of Morse theory (see 2.3.10). We remind the reader: let M be a 2 manifold with genus g, then

- if M is oriented, M is classified by its genus g. In particular M it is diffeomorphic to $\Sigma_g := T\#\ldots\#T$, the connected sum of g-tori, where in particular $\Sigma_0 = S^2$.
- Instead if M is non orientable[12], M is diffeomorphic to g copies of the real projective plane, i.e. $\mathbb{R}P^2\#\ldots\#\mathbb{R}P^2$.

One can find a thorough covering of this result in [Hir76].
In addition, we will also find a classification of compact 2 manifolds with boundary (which we will simply call *surfaces*). To do it we will need Morse theory. What is the idea? Given a surface M (it works in greater generality but now we're only interested in surfaces), find a map $f : M \to \mathbb{R}$ which is "nice" is a way we'll specify later on. Then by analizing properties of f we can recover the topology of the surface (i.e. the surface up to diffeomorphism).
This section is based on [Hir76] and [Kos13].

2.6.1 Introduction to Morse Theory

Take as an example the projection map from the torus, see 2.12. Here we have 4 special points: the top and bottom, and then the two points in the middle. What happens in these points is that the shape of the preimage changes:

- the preimage of the top (and bottom) is a point
- the preimage of points between the top and middle point is a circle
- the preimage of the middle points is a wedge of circles
- the preimage of points between the middle points is a disjoint union of circles.

Note that at the special points the preimage is generally not a smooth manifold. These "special" points are critical points of f:

Definition 2.6.1 (Critical point and critical value)**.** Let M be a closed n manifold. A critical point of $f : M \to \mathbb{R}$ is a point $p \in M$ such that $(df)_p : T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$ is *not* surjective. A critical value is the image of a critical point.

[11] This differential topology theorem relates the index of a vector field to the Euler characteristic of the manifold. In particular, it implies that a nonvanishing vector field can only exist if the Euler characteristic of the manifold is zero.

[12] see `https://en.wikipedia.org/wiki/Genus_(mathematics)` for what is meant by genus of a non orientable surface

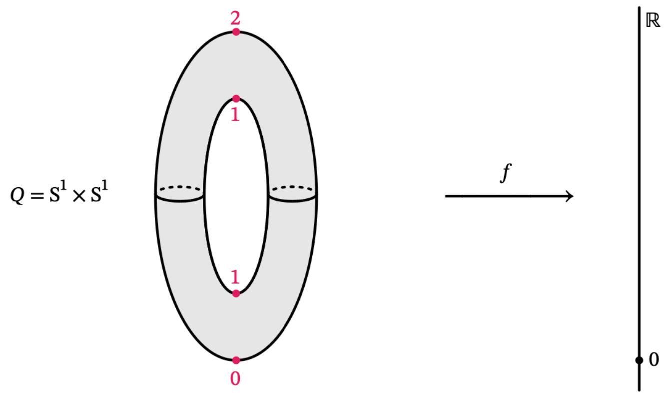


Figure 2.12: Image of the Morse function of a torus from [Tan22]

The main goal then is to recover (up to diffeomorphism) M from knowing the critical points and the behaviour in a neighborhood thereof.
Firstly we can define what we mean by a "nice" function:

Definition 2.6.2. A Morse function is a smooth map $f : M \to \mathbb{R}$ such that all critical points are nondegenerate, that is $\mathrm{Hess}(f)_p := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)_{i,j}$ is non-singular.

Remark. This is independent of the choice of chart.

Example 2.6.3. Let's start with a nonexample: the parabolic cylinder. With coordinates x_1, x_2, the Morse function drawn can be written as $f(x_1, x_2) = -x_1^2$. Then at every critical point the Hessian is given by

$$\mathrm{Hess}(f)_p = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.9}$$

which is clearly singular.
Another nonexample is given by the function $f(x) = x^3$. Consider the map that goes from the graph of f to its y coordinate. Again the Hessian is singular at the critical point.

We're then interested in the local picture at a critical point.

Definition 2.6.4 (Index of a critical point)**.** If $p \in M$ is a nondegenerate critical point of $f : M \to \mathbb{R}$, the index of f at p is

$$\mathrm{ind}_f(p) = \text{index of } \mathrm{Hess}(f)_p = \#\text{ negative eigenvalues of} \mathrm{Hess}(f)_p. \tag{2.10}$$

Example 2.6.5. Consider the following basic examples:

- $f(x_1, x_2) = -x_1^2 - x_2^2$, this has critical point $p = (0,0)$ and there the Hessian is given by $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$. We therefore see that p is nondegenerate and that $\mathrm{ind}_f(p) = 2$.
- $f(x_1, x_2) = -x_1^2 + x_2^2$, very similar, but $\mathrm{ind}_f(p) = 1$,
- $f(x_1, x_2) = +x_1^2 - x_2^2$, again $\mathrm{ind}_f(p) = 1$,
- $f(x_1, x_2) = +x_1^2 + x_2^2$, now $\mathrm{ind}_f(p) = 0$.

The examples above are very important, because locally Morse functions are always of one of those forms.

Lemma 2.6.6 (Morse Lemma I). *Let $p \in M^n$ be a nondegenerate critical point of $f : M^n \to \mathbb{R}$ of index k. Then there is a chart (U, ϕ) around p such that the map $\tilde{f}$ defined as*

$$\begin{tikzcd} U \arrow[r, "\phi"] \arrow[d, "f"] & \mathbb{R}^n \arrow[dl, "\tilde{f}"] \\ \mathbb{R} & \end{tikzcd} \tag{2.11}$$

is given by:

$$\tilde{f}(x_1, \dots, x_n) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{j=k+1}^{n} x_j^2 \tag{2.12}$$

Idea of proof. Given a nondenerate critical point of index k we only know that $\mathrm{Hess}(f)_p$ has k negative eigenvalues, is nonsingular and is diagonalizable. That allows me to change the basis so that the function takes that form, essentially doing a Taylor expansion. In particular then the chart can be chosen such that the function is *exactly* that expression. □

Given a Morse function $f : M \to \mathbb{R}$, $p \in M$, then we have the following immediate consequences of the lemma:

- if p is critical (non degenerate because of the *Morse* function), because of the lemma we have a chart as above. Then the level sets $f^{-1}(f(p))$ look like a neighborhood of 0 in a quadric (locally).
- if p is not critical (it's regular), by the implicit function theorem there are coordinates such that $f(x_1, \dots, x_n) = x_1$. So the level sets $f^{-1}(f(p))$ look like hyperplanes in $\mathbb{R}^n$ (locally).

This explains what we noticed initially for the example of the torus, that the level set changes when passing a critical point. In particular it always changes in one of the ways shown in 2.6.5. This observation will be more complete after Morse Lemma II and Theorem 2.6.13 below.

The following theorem justifies the use of Morse theory to study manifolds.

Theorem 2.6.7. *For any manifold Morse functions exist.*

It's not trivial but it can be proven with tools from differential topology.

Example of something we can prove with Morse functions:

Theorem 2.6.8. *The Euler characteristic can be calculated by knowning the critical points and the corresponding indices of a Morse function:*

$$\chi(M) = \sum (-1)^k c_k \tag{2.13}$$

where c_k is the number of critical points of index k.

This is quite clear for the sphere and the genus g surfaces with the usual Morse functions, but it's interesting that for *any* Morse function we have this result.

The following theorem also seems very plausible from the drawings:

Lemma 2.6.9 (Morse Lemma II)**.** *Let $f : M \to \mathbb{R}$, $a < b \in \mathbb{R}$ such that f does not have a critical value in $[a, b]$. Then*

$$M_a = f^{-1}((-\infty, a]) \hookrightarrow M_b = f^{-1}((-\infty, b]) \tag{2.14}$$

is a deformation retract. In particular, it induces a diffeomorphism $f^{-1}(a) \cong f^{-1}(b)$.

Concretely this tells us that the level set *only* changes when crossing a critical point.

Remark. The proof uses flow along vector field $\frac{\text{grad} f}{|\text{grad} f|}$...

We will restrict to the following class of Morse functions:

Definition 2.6.10 (Admissibile Morse function)**.** A Morse function $f : M \to [a, b]$ is admissibile if $\partial M = f^{-1}(a) \cup f^{-1}(b)$ and a, b are regular values.

The fact that they're regular values is important, since it gives us that a neighborhood of $f^{-1}(a)$ is diffeomorphic to a cylinder $f^{-1}(a) \times [0, \epsilon]$, i.e. a collar (and the same is true for a neighborhood of $f^{-1}(b)$). So an admissible Morse function naturally equips M with the structure of a cobordism from $f^{-1}(a)$ to $f^{-1}(b)$.
Another useful theorem, analogous to 2.6.7 is the following:

Theorem 2.6.11. *Admissible Morse functions exist.*

We would now like to know *how* the level set changes before and after the critical point. In order to do this we first introduce the concept of handle attachment.

Definition 2.6.12 (Handle attachment)**.** Let M^n be an n manifold and let $H^j :- D^j \times D^{n-j}$ for $j = 0, 1, \ldots, n$. In addition let $f : \partial D^j \times D^{n-j} \hookrightarrow \partial M^n$ be an embedding and note that $\partial D^j \times D^{n-j}$ also embeds into H^j. We can then *attach a j-handle* to M^n by gluing along f to obtain the manifold $M^n \amalg_f H^j$.

This concept will be studied in one of the exercises. The following claim makes the definition meaningful:

Claim. *$M^n \amalg_f H^j$ has a smooth structure.*

Since we're interested in 2 manifolds, let's make explicit what it means to attach a j handle to a 2 manifold:

- $j = 0$, $H^0 = D^0 \times D^2 \cong D^2$ and we attach along the empty set (since $\partial D^0 = \emptyset$), i.e. we take the disjoint union with D^2.
- $j = 1$, $H^1 = D^1 \times D^1$ and we attach along $\partial D^1 \times D^1$, i.e. along two segments
- $j = 2$, $H^2 = D^2 \times D^0$ and we attach along $\partial D^2 \times D^0 \cong S^1$, i.e. along a circle

We can now see how the level sets change at a critical point.

Theorem 2.6.13 ([Hir76] Theorem 3.2, p.157)**.** *Let M^n be compact and $f : M^n \to [a, b]$ an admissible Morse function. Suppose that f has a unique critical point z of index j.*
There is an embedding $\iota : D^j \hookrightarrow M^n$ with image $e^j := \operatorname{im} \iota$ (called "belt disk"), satisfying:

- *$z \in e^j$;*

- $e^j \subset M^n \setminus f^{-1}(b)$;
- $f^{-1}(a) \cap e^j = \partial e^j = \iota(\partial D^j)$, *this is also called "belt sphere";*
- *M deformation retracts onto* $f^{-1}(a) \cup e^j$.

The embedding can then be extended to $e^j \times D^{n-j} =: H^j$. *We can now choose an* a' *with* $a < a' < f(z)$ *and we then have*

$$M^n \cong f^{-1}([a, a']) \cup_{\bar{\iota}} H^j \tag{2.15}$$

(for $n = 2$ *this is a unique diffeomorphism up to isotopy*[13]*).*

This theorem explains what happens to the level sets when we meet a critical point of index j: we attach a j handle! To use the terms above: we extend the belt disk to a j handle and attach it to the boundary by gluing along the belt sphere (which is also extended). Of course, all the drawings presented are for 2 manifolds and we will only apply these results to such manifolds, but this theorem works more in general!

Example 2.6.14.

- index 0: $M = S^2$ we then have $e^0 = \{z\}$, $\iota : D^0 = pt \hookrightarrow M$ and $\bar{\iota} : D^0 \times D^2 \hookrightarrow M$.
- Now what happens when attaching a 1-handle im$\bar{\iota}$? Chose: $D^{n-k} \hookrightarrow \partial(S^1 \times [0,1] \amalg S^1 \times [0,1])$
- Start with $S^1 \times [0,1]$. Want to attach a 1-handle $D^1 \times D^1$. But now we can attach in two ways. With a simple band or with a twisted one.
- Let's start again with a cylinder and attach a 2-handle $D^2 \times D^0$, so I kind of close one of the two side of the cylinder. We see that the 1-handle and the 2-handle kind of cancel each other out!
- Analogously, starting with a disk and attaching a 0-handle and then a 1-handle, these also cancel each other out!

Proposition 2.6.15 (VI 7.1 in [Kos13], "Handle slides")**.** *If* $\tilde{M} := (M \coprod_f H^j) \coprod_g H^i$ *and* $i \le j$, *then* $\tilde{M}$ *can be obtained by first attaching* H^i *and then* H^j.

Note that the word *can* in the proposition is important: if we were to simply commute the two attachments $(M \coprod_g H^i) \coprod_f H^j$ we would get something that in general doesn't make sense, since the map g goes into $M \coprod_f H^j$, not just M, and H^i may no longer be attached at the boundary, leading to a space that is not even a manifold. The proposition instead says that there exist maps $\tilde{g}$ and $\tilde{f}$, into M and $M \coprod_{\tilde{g}} H^i$ respectively, such that $(M \coprod_f H^j) \coprod_g H^i \cong (M \coprod_{\tilde{g}} H^i) \coprod_{\tilde{f}} H^j$.

Example 2.6.16.

- $j = 1, i = 0$
- $j = 2, i = 1$ same picture but upside down!

[13] See 3.10.17 for the definition of isotopy.

- $j = i = 1$ do it as an exercise.

What happens if $i > j$? In general it's not so simple, but if $i = j+1$ the following result holds.

Proposition 2.6.17 (VI 7.4 in [Kos13], "Handle cancellation"). *Let $\tilde{M} := (M \coprod_f H^j) \coprod_g H^{j+1}$, where the attaching sphere of H^{j+1} intersects the belt sphere of H^j "transversely" in one point. Then $\tilde{M} \cong M$.*

Example 2.6.18. $j = 0$

Theorem 2.6.19 ([Hir76] 8.3.4, p. 187). *Let M be a surface admitting a Morse function that has exactly two critical points. Then*

$$M \cong S^2 \tag{2.16}$$

Remark. In dimension 1,2 and 3 all the homeomorphic manifolds are also diffeomorphic.

Proof. Because of the remark it's enough to prove the homeomorphism, rather than the diffeomorphism.
Assume $p_+, p_- \in M$ are critical points. Since M is compact, also $f(M)$ is compact and therefore has a maximum and a minimum, which exactly correspond to the two critical points. Now assume p_+ is the maximum, then $\operatorname{ind} p_+ = 2$. Now, because of Morse Lemma I we have $\exists U_+$ a neighborhood of p_+ with coordinates x_1, x_2 such that

$$f|_{U_+} = -x_1^2 - x_2^2 + f(p_+) \tag{2.17}$$

Then we have $\exists b < f(p_+)$ such that $D_+ = f^{-1}([b, +\infty)) \cong D^2$.
Similarly, $\exists U_-$ a neighborhood of p_- with coordinates x_1, x_2 such that

$$f|_{U_-} = x_1^2 + x_2^2 + f(p_-) \tag{2.18}$$

and now we have $\exists a > f(p_-)$ such that $D_- = f^{-1}((-\infty, a]) \cong D^2$.
Let B_+, B_- be disjoint caps around the poles of S^2 and denote $C := S^2 \operatorname{int}(B_+ \amalg B_-) \cong S^1 \times [0,1]$. Now, notice $\partial D_+ \cong S^1 \cong \partial D_-$ and we have a diffeomorphism $h_0 : D_+ \to B_+$ as they are both diffeomorphic to D^2. In addition $h_0|_{\partial D_+} \partial D_+ \to \partial B_+$. Notice that between $[a, b]$ there are no critical values, that is $f^{-1}([a,b])$ has no crtiical points Now, using Morse Lemma II we get:

- $f^{-1}((-\infty, b]) =: M_b \hookrightarrow M_a := f^{-1}((-\infty, a])$, $f^{-1}(a) \cong f^{-1}(b)$
- $f^{-1}([a,b]) \cong f^{-1}(a) \times [0,1] \cong S^1 \times [0,1]$

Now we can extend $h_0|_{\partial D_+}$ to $h_1 : \partial D_+ \times [0,1] \to \partial B_+ \times [0,1]$. Now we can glue h_0 and h_1

$$h : D_+ \cup (\partial D_+ \times [0,1]) \to B_+ \cup (\partial B_+ \times [0,1]) \tag{2.19}$$

Claim. *If $g : S^1 \to S^1$ is a homeomorphism, then it can be extended to a homeomorphism $\tilde{g} : D^2 \to D^2$.*

$\tilde{g}$ can simply be defined as follows

$$\tilde{g}(x) = \begin{cases} ||x|| g\left(\frac{x}{||x||}\right), & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases} \tag{2.20}$$

From the claim h can be extended to a homeomorphism $M \to S^2$. $\square$

Remark.

1. For any closed n manifold that has two critical points, we have $M \cong S^n$ (*homeomorphic*, not *diffeomorphic*).

2. (From Milnor) There are manifolds M homeomorphic to a sphere but not diffeomorphic (e.g. S^7), we then talk about *exotic spheres*.

Theorem 2.6.20 (Classification of surfaces)**.** *Let M^2 be an oriented, connected, closed surface. Then it can be obtained as in the following drawing:*

Proof.
Step 1: Choose an admissible Morse function f.
Step 2a: If necessary, "perturb" f to get distinct critical values.
Step 2b: Chop into pieces with exactly one critical point and apply Morse Lemma I. We then get that $f^{-1}((-\infty, a_i])$ is obtained from $f^{-1}((-\infty, a_{i-1}])$ by attaching a handle.
Goal: normal form (for closed connected surfaces)
Step 3: Claim: We can obtain M by first attaching 0-handles, then 1-handles, ..., in ascending order.
To prove this there are two strategies:

1. Use proposition about handle cancellation (first attaching i handle and then $i+1$ handle these cancel) and handle slides: with these you can write down an algorithm to get the normal form from any handle decomposition.

2. One can change the Morse function. Claim: We can choose f such that if $\operatorname{ind} p_1 < \operatorname{ind} p_2 \implies f(p_1) < f(p_2)$. This can be proven by changing the Morse function locally around a critical point (see [Hir76] for more details).

Step 4: If $\partial M = \emptyset$, then either $M = \emptyset$ or I have at least two critical points.
Case 1: We have exactly 2 critical points which gives us $M \cong S^2$ because of Lemma.
Case 2: If we have > 2 critical points, look at minimum, then there exist a neighborhood D_- of the minimum such that $D_- \cong D^2$, i.e. a 0 handle attached to $\emptyset$ (because of Morse Lemma I f locally looks like a cup).
Starting from the minimum, attach one 0-handle. If then we attach another 0-handle, then we can use the handle slides and cancellations to get rid of all 0 handles but one.
Next we can attach one handles and we have two options, only one of which gives an orientable manifold.
At "end", same argument as for why only one 0-handle read backwards shows that we only have one 2-handle (for example change f to $-f$).
This ends the proof. $\square$

Question: What about the unoriented case? (*Hint: enough to have one attachment of Möbius strip*)
We now consider the case with boundary:

Theorem 2.6.21 (Classification of surfaces with boundary)**.** *Let M^2 be an oriented, connected surface. Then it can be obtained as in the following drawing:*

Proof. Now $\partial M \neq \emptyset$, then $\partial M = S^1 \amalg \cdots \amalg S^1$. If f is an admissible Morse function f onto $[a, b]$ with $f^{-1}(a) \cong (S^1)^{\amalg m}$ and $f^{-1}(b) \cong (S^1)^{\amalg n}$. Now one can prove that we need neither 0-handles (if $m \neq 0$) nor 2-handles (if $n \neq 0$). $\square$

2.7 More on cobordism groups

We now go back to cobordism groups to give some important results.

We had found

$$\Omega_0 \cong \mathbb{Z}_2 \tag{2.21}$$
$$\Omega_1 \cong 0 \tag{2.22}$$
$$\Omega_2 \cong \mathbb{Z}_2 \tag{2.23}$$

while for the oriented case we have

$$\Omega_0^{or} \cong \mathbb{Z} \tag{2.24}$$
$$\Omega_1^{or} \cong 0 \tag{2.25}$$
$$\Omega_2^{or} \cong 0 \tag{2.26}$$

Definition 2.7.1 (Commutative $\mathbb{Z}$ -Graded Ring[14])**.** A commutative $\mathbb{Z}$ -graded ring is a ring R if there is a family of subgroups $\{R_n\} n \in \mathbb{Z}$ such that

- the underlying abelian group can be decomposed as $R = \bigoplus_{n\in\mathbb{Z}} R_n$
- $R_n \cdot R_k \subset R_{n+k}$ for all $n, k \in \mathbb{Z}$)

A non-zero element $x \in R_n$ is called a homogeneous element of R of degree n.

Proposition 2.7.2. $(\Omega_\bullet = \bigoplus_{n\geq 0} \Omega_n, \amalg, \times)$ *is a commutative* $\mathbb{Z}$ *-graded ring.*

Proof. Firstly we need to check that the product respects the degree: this is true because $M^m \times N^n = (M \times N)^{m+n}$, so $[M \times N] \in \Omega_{m+n}$.
Then the products descends to equivalence classes: if $Y_0 \simeq Y_1$ are cobordant via cobordism (X, p) $(p : \partial X \to 0, 1)$, M any manifold, then $\partial X \times M = \partial X \times M \xrightarrow{p\circ pr_X} \{0, 1\}$. Therefore $(X \times M, p \circ pr_X)$ is a cobordism from $Y_0 \times M$ to $Y_1 \times M$. □

Theorem 2.7.3 (Thom)**.** *There is an isomorphism of* $\mathbb{Z}$ *-graded commutative rings*

$$\Omega_\bullet \cong \mathbb{Z}_2[x_2, x_4, x_5, x_6, x_8, \dots],\ \textit{with } \deg x_k = k, k \neq 2^i - 1 \tag{2.27}$$

and the generators of the even degrees are given by $x_{2k} = [\mathbb{RP}^{2k}]$.

Remark. Note that $(\deg x_2)^2 = \deg 4$ so maybe we could imagine that $\mathbb{RP}^2 \times \mathbb{RP}^2 \sim \mathbb{RP}^4$ however that's not true since $x_2^2 \neq x_4$.

There are versions for any tangential structure, such as orientation or stable framing.

Theorem 2.7.4. *There is an isomorphism of* $\mathbb{Z}$ *graded commutative rings:*

$$\Omega_\bullet^{or} \otimes \mathbb{Q} \cong \mathbb{Q}[y_4, y_8, y_{12}, \dots]$$
$$y_{4k} \mapsto [\mathbb{CP}^{2k}]$$

We write it in this way because in this case there is nontrivial torsion. We could also write

$$\Omega_\bullet^{or} / torsion \cong \mathbb{Z}[z_4, z_8, z_{12}, \dots]$$

where the generators are given by Milnor hypersurfaces.

[14]Note that there is a difference between commutative graded ring and graded commutative ring! A commutative graded ring is a commutative ring that is graded (our notion), a graded commutative ring is a different notion that depends on the degree of homogeneous elements.

Example 2.7.5. In particular, the groups in various degrees are given by the following list:

$$\Omega_0^{\text{or}} = \mathbb{Z}$$
$$\Omega_1^{\text{or}} = 0$$
$$\Omega_2^{\text{or}} = 0$$
$$\Omega_3^{\text{or}} = 0$$
$$\Omega_4^{\text{or}} = \mathbb{Z}$$
$$\Omega_5^{\text{or}} = \mathbb{Z}_2$$
$$\Omega_6^{\text{or}} = 0$$
$$\Omega_7^{\text{or}} = 0$$
$$\Omega_8^{\text{or}} = \mathbb{Z} \oplus \mathbb{Z}$$
$$\Omega_{n\geq 9}^{\text{or}} \neq 0$$

For the last result see [MS05, p. 203].

2.7.1 Cobordism groups and the sphere spectrum ✈

Very interestingly, the (stable) framed version is isomorphic to the stable homotopy groups of the sphere and it is *not* computed to all degrees. This is one way to phrase a theorem named after Thom. It is fascinating because the stable homotopy groups of the sphere are central objects in stable homotopy theory. They are equivalently the homotopy groups of the sphere spectrum.

Now characterize spectra the bare minimum in order to talk about the sphere spectrum. Later, after some detours in the magical world of ∞-categories, we will give a deeper sketch of what they are 4.1.1.

Definition 2.7.6 (Suspension)**.** Let X be a pointed topological space. The suspension ΣX of X is the smash product of X with S^1, i.e.

$$\Sigma X = X \wedge S^1 = \frac{X \times S^1}{(\{*\} \times S^1) \amalg X \times \{1\}}$$

Example 2.7.7. $\Sigma(S^1) = S^2$ more generally $\Sigma(S^n) = S^{n+1}$

Definition 2.7.8 (Sequential spectrum[15])**.** A sequential spectrum is a sequence of pointed spaces $\{X_k\}_{k\in\mathbb{Z}}$ with maps preserving basepoints $\Sigma X_n \xrightarrow{\sigma_n} X_{n+1}$ called structure maps.

Definition 2.7.9 (Suspension Spectrum)**.** The suspension spectrum $\Sigma^\infty X$ has Σ_n as the n-th space in the sequence and structure maps $\Sigma\Sigma_n \cong \Sigma_{n+1}$.

Example 2.7.10 (Sphere Spectrum)**.** The sphere spectrum $\mathbb{S}$ is the suspension spectrum of the point, S^0. In fact, $\Sigma S^0 = S^1$, $\Sigma^2 S^0 = \Sigma\Sigma S^0 = \Sigma S^1 = S^2$ and in general $\Sigma^n S^0 = S^n$

[15] Sometimes it is called just spectrum. We call it *pre*spectrum in order to distinguish it from other notions with more structure, such as the suspension spectrum

Notation (Stable Homotopy Groups of the Sphere)**.** The stable homotopy groups of the sphere are the homotopy groups of the sphere spectrum. Alternatively, one can define them as the homotopy groups of the sphere $\pi_{n+i}(S^n)$ such that $n > i+1$. This latter characterization explains why they are called 'stable': due to Freudenthal's suspension theorem, such homotopy groups are independent of n.

Remark. Note that the stable homotopy groups of the sphere can be made into a commutative $\mathbb{Z}$ -graded ring via direct sums. We denote it with $\pi_\bullet(\mathbb{S})$

$$\pi_\bullet(\mathbb{S}) = \bigoplus_{n\geq 0} \pi_n(\mathbb{S})$$

Theorem 2.7.11 (Thom's theorem)**.** *One way of phrasing Thom's theorem is*

$$\Omega_\bullet^{\mathrm{fr}} \cong \pi_\bullet(\mathbb{S})$$

This is also called Pontrjagin-Thom isomorphism.

Example 2.7.12. We list some examples of such commutative rings.

$$\Omega_0^{fr} \cong \mathbb{Z}$$

$$\Omega_1^{fr} \cong \mathbb{Z}_2$$
$$\Omega_2^{fr} \cong \mathbb{Z}_2$$
$$\Omega_3^{fr} \cong \mathbb{Z}_{24}$$
$$\Omega_4^{fr} \cong 0$$
$$\Omega_5^{fr} \cong 0$$
$$\Omega_6^{fr} \cong \mathbb{Z}_2$$
$$\Omega_7^{fr} \cong \mathbb{Z}_{240},$$
$$\Omega_1^{fr}1 \cong \mathbb{Z}_{504}$$
$$\Omega_1^{fr}5 \cong \mathbb{Z}_{480} \oplus \mathbb{Z}_2$$

Part II

Topological field theories

Chapter 3

A summary of category theory

3.1 A modern perspective on cobordisms

A bordism invariant was characterized as a homomorphism from the cobordism group to some other abelian group. To extract the cobordism group from bordism we took the following steps:

1. observed that being cobordant is an equivalence relation,
2. considered the equivalence classes of closed n manifolds up to $n+1$ cobordisms, obtaining the *sets* Ω_n,
3. took such equivalence classes together with disjoint unions, resulting in an abelian group[1].

This strategy was the key for classifying manifolds up to cobordism. However, the cobordism groups merely record that *there is* a bordism between two manifolds (since two manifolds are equivalent just if there is a bordism, independently of what kind of bordism it is), thereby forgetting other properties of the bordism itself. We swich now perspective and analyze a more sophisticated structure remembering how two manifolds are cobordant, e.g. indicating the manifold that bounds them and the direction of the bordism: the symmetric monoidal category $\mathrm{Bord}_{n,n-1}$ where objects are $(n-1)$ manifolds and morphisms are n-cobordisms. This is an instance of a process called categorification[2]: adding categorical structure to things, e.g.[3] passing from set-theoretic notions like set or function categorical ones like category or functor. The invariants will become in turn functors from $\mathrm{Bord}_{n,n-1}$ to categories of algebraic nature like Vect_k, the category of vector spaces on a field k. Such categorified cobordism invariants are exactly topological field theories (TFTs).

The following table summarizes the comparison between additional structures in the two perspectives:

[1] And eventually in a $\mathbb{Z}$ graded commutative ring, but this will not be as important for us from now on.

[2] Sometimes, e.g. in the nLab, also called vertical categorification.

[3] Note that although this way of categorifying is the most prominent one, it is strictly speaking not the only way to categorify, one could also go from category theory to higher category theory.

Ω_n	$\mathrm{Bord}_{n,n-1}$
set	category
monoid	monoidal category
commutative monoid	symmetric monoidal category
abelian group	Picard groupoid

Analogous to the comparison between the set-theoretic and category-theoretic perspective, we could also have a linear-algebraic perspective. Since an associative algebra on a vector space is the parallel construction to a monoid with set, and a commutative algebra corresponds to a commutative monoid. The following table adds this perspective.

Ω_n	Vect_k
set	vector space
monoid	associative algebra
commutative monoid	commutative algebra

We make both these comparisons more rigorous by later (3.9.2) showing that

1. a monoid is a monoid object in the category of sets.
2. an associative algebra is a monoid object in the category of k-vector spaces $Vect_k$
3. a (*strict*) monoidal category is a monoid object in the category of small categories Cat

The same holds for the commutative case, commutative monoids, commutative algebras and (*strict*[4]) symmetric monoidal categories are all examples of commutative monoid objects (see 3.10.5).

3.2 Category theory language

Definition 3.2.1. A locally small[5] category $\mathcal{C}$ consists of the following data:

- A class $\mathrm{ob}(\mathcal{C})$ whose elements are called the objects of $\mathcal{C}$,
- For any $X, Y \in \mathrm{ob}(\mathcal{C})$ a set $Hom_{\mathcal{C}}(X, Y)$ whose elements are called morphisms from X to Y,
- For any objects $X, Y, Z \in \mathrm{ob}(\mathcal{C})$ a map[6]

[4]We will also sketch a way how to get general symmetric monoidal categories, i.e. not necessarily strict, in an analogous way. See 3.10.1.

[5]A category is locally small if every hom $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is not bigger than a set. A locally small category is small if the collection of objects is also a set. A large category is a category which is not small. A category is essentially small if it is locally small and the collection of isomorphism classes (collections of isomorphic objects, i.e. objects with a morphism between them which has a left- and right-inverse) is a set. Questions of size of collections play an important role in category theory. For example, one cannot naively take the set of all sets as the collection of objects of the category of sets because of famous set-theoretic paradoxes like Cantor's, Burali-Forti's or Russell's. See [Shu08] for an account on possible set-theoretic foundations for category theory.

[6]Note that we can simply define composition as a map between sets because we are working with a locally small category.

$$\mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{\circ} \mathrm{Hom}_{\mathcal{C}}(X,Z)$$
$$(g,f) \mapsto g \circ f$$

which is called composition of morpisms,

- For every object $X \in \mathrm{ob}(\mathcal{C})$ an element $id_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ called the identity of X.

Such data must fulfill the following axioms:

- $\forall f \in \mathrm{Hom}_{\mathcal{C}}(X,Y), f \circ id_X = f = id_Y \circ f$ (unitality)
- $\forall f \in \mathrm{Hom}_{\mathcal{C}}(X,Y), g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z), h \in \mathrm{Hom}_{\mathcal{C}}(W,Z)$:

$$(h \circ g) \circ f = h \circ (g \circ f) \qquad \text{(associativity)}$$

Example 3.2.2. A few examples of categories are the following:

category	objects	morphisms
Set	class of all sets	functions between sets
Mon	class of all monoids	monoid homomorphisms
Grp	class of all groups	group homomorphisms
AbGrp	class of all abelian groups	group homomorphisms
Ring	class of all rings	ring homomorphisms
Vect_k	class of all k vector spaces	linear maps
Alg_k	class of all algebras over k	algebra homomorphisms
Top	class of all topological spaces	continuous functions
FinSet	class of all finite sets	functions between sets
SmoothMfld	set of all smooth manifolds	smooth functions

1. Let $(P, \leq)$ be a set with a transitive and reflexive relation $\leq$ (a preordered set). Define a category $\mathbf{P}$ with:

$$\mathrm{ob}(\mathbf{P}) = P$$
$$\mathrm{Hom}_{\mathbf{P}}(X,Y) = \begin{cases} \{*\} & \text{if } x \leq y \\ \emptyset & \text{else} \end{cases}$$

2. Given categories $\mathcal{C}, \mathcal{D}$ we can define a category $\mathcal{C} \times \mathcal{D}$ (the product category) by:

$$\mathrm{ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{ob}(\mathcal{C}) \times \mathrm{ob}(\mathcal{D})$$
$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((X,Y),(X',Y')) = \mathrm{Hom}_{\mathcal{C}}(X,X') \times \mathrm{Hom}_{\mathcal{D}}(Y,Y')$$

3. Given a category $\mathcal{C}$, define a category $\mathcal{C}^{op}$ by:

$$\mathrm{ob}(\mathcal{C}^{op}) = \mathrm{ob}(\mathcal{C})$$
$$\mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$$
$$g \circ_{\mathcal{C}^{op}} f = f \circ_{\mathcal{C}} g$$

this is called the opposite category of $\mathcal{C}$.

Definition 3.2.3. An isomorphism $f \in \mathrm{Hom}(X,Y)$ is a morphism such that $\exists g \in \mathrm{Hom}(Y,X)$ with $g \circ f = id_X, f \circ g = id_Y$.

Definition 3.2.4. A groupoid is a category where each morphism is an isomorphism.

Example 3.2.5. Let $\mathcal{C}$ be a category with $\mathrm{ob}(\mathcal{C}) = \{*\}$.
Then, $(\mathrm{Hom}_{\mathcal{C}}(*,*),\circ)$ is a monoid since composition is associative and unital with neutral element given by the identity morphism id_*. Conversely, every monoid $(M,\cdot)$ defines a category $\mathbf{B}M$ with

$$\mathrm{ob}(\mathbf{B}M) = \{*\}, \mathrm{Hom}_{\mathbf{B}M}(*,*) = M, m \circ_{\mathbf{B}M} m' = m \cdot m', id_* = 1_M$$

$\mathbf{B}M$ is called the delooping of the monoid $(M,\cdot)$ [7]. The same holds for groups and one-object groupoids: every group $(G,\cdot)$ defines a one-object groupoid $\mathbf{B}G$ and vice versa

More generally, monoids of the form $(\mathrm{Hom}_{\mathcal{C}}(X,X),\circ)$ are called endomorphism monoids and an interesting example thereof is endomorphism monoids in the category $\mathrm{TopVect}_k$ of topological vector spaces and continuous linear operators. Such endomorphism monoids $(\mathrm{Hom}_{\mathrm{TopVect}_k}(X,X),\circ)$ and submonoids thereof are called operator algebras. They are important in functional analysis and in quantum theory.

Definition 3.2.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ consists of the following data:

- an assignment

$$F : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D}) \tag{3.1}$$
$$X \mapsto F(X) \tag{3.2}$$

- for every two objects $X, Y \in \mathrm{ob}(\mathcal{C})$ a map

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)) \tag{3.3}$$
$$f \mapsto F(f) \tag{3.4}$$

such that

- $F(id_X) = id_{F(X)}$
- $F(g \circ f) = F(g) \circ F(f)$

Example 3.2.7. Some examples of functors:

1. There are forgetful functors
   - Ring $\to$ Grp $\to$ Set
     $(R,+,\cdot) \mapsto (R,+) \mapsto R$
   - Ring $\to$ Mon
     $(R,+,\cdot) \mapsto (R,\cdot)$
   - $\mathrm{Vect}_k \to$ AbGrp $\to$ Set

[7] We will see a generalization of such deloopings for certain categories, monoidal categories (see 3.7) where any monoidal category $\mathcal{C}$, will be a one-object *bi*category $\mathbf{B}\mathcal{C}$ called the delooping $\mathcal{C}$ (see 3.10.28).

- $\text{Alg}_k \to \text{Vect}_k$ where the multiplicative structure on algebras is forgotten

2. An action of a group $(G,\cdot)$ on a set X is a functor $A : \mathbf{B}G \xrightarrow{\rho} \text{Set}$ where $A(*) = X$ and every $g \in \text{Hom}_{\mathbf{B}G}(*,*)$ is mapped to an automorphism[8] on X, $\text{Hom}_{\mathbf{B}G}(*,*) \to \text{Hom}_{\text{Set}}(X,X)$. By the same reasoning, a linear representation of a group $(G,\cdot)$ is a functor $\mathbf{B}G \to \text{Vect}_k$.

3. Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, their composite $G \circ F : \mathcal{C} \to \mathcal{E}$ is also a functor: $G \circ F(X) = G(F(X)), G \circ F(f) = G(F(f))$.

4. $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ with $id_{\mathcal{C}}(X) = X, id_{\mathcal{C}}(f) = f$ is also a functor.

Remark. Since the composition of functors is associative[9] and unital there is a category of all (small[10]) categories Cat whose objects are (small) categories and whose morphisms are functors. For the same reason we also have a category Gpd of (small) groupoids.

Example 3.2.8. ✈ Some more examples related to topological spaces.

1. Let X be a topological space. Observe the following equivalence relation: $x \sim x'$ if and only if there is a continuous map $f : [0,1] \to X$, also known as "path", such that $f(x) = 0$ and $f(x') = 1$. Denote with $\pi_0(X)$ the set of equivalence classes of path-connected points, also known as the set of path-connected components, in X. This gives a functor[11] $\pi_0 : \text{Top} \to \text{Set}$

2. Let X be a topological space. Its fundamental groupoid $\pi_{\leq 1}$ is the groupoid having:

   - the points of X as objects, $\text{ob}(\pi_{\leq 1}(X)) = X$;
   - $\text{Hom}_{\pi_{\leq 1}(X)}(x,y)$ is given by equivalence classes of continuous paths from x to y that are homotopic[12] relative to their endpoints. We can spell out what this means in the following way

$$\text{Hom}_{\pi_{\leq 1}(X)}(x,y) = \frac{\{\gamma \in \text{Hom}_{\text{Top}}([0,1],X) | \gamma(0) = x, \gamma(1) = y\}}{\text{homotopy relative to } \partial[0,1]}$$

   - For $x,y,z \in X$, composition of $[\gamma] \in \text{Hom}_{\pi_{\leq 1}(X)}(x,y)$ and $[\eta] \in \text{Hom}_{\pi_{\leq 1}(X)}(y,z)$ is given by the concatenation of paths with appropriate reparametrization,

$$[\eta] \circ [\gamma] = [\gamma * \eta]$$

[8]It is an automorphism and not a simple endomorphism because of a very important property of functors: they preserve isomorphisms: given a functor $F : \mathcal{C} \to \mathcal{D}$, if $f \in \text{Hom}_{\mathcal{C}}(X,Y)$ is an isomorphism, then $F(f) \in \text{Hom}_{\mathcal{D}}(F(X),F(Y))$ is an isomorphism as well because $F(f^{-1}) \circ F(f) = F(f^{-1} \circ f) = F(id_X) = id_{F(X)}$ and simmetrically, $F(f) \circ F(f^{-1}) = id_{F(Y)}$. For example, this can be used in the converse direction to show that two topological spaces are not homeomorphic by sending them (with a functor) to their non-isomorphic fundamental group(oid)s.

[9]Try to convince yourself that it is so!

[10]Let $\mathcal{C},\mathcal{D}$ be categories, the collection of all functors $F : \mathcal{C} \to \mathcal{D}$ is generally a class. Hence the category of all categories without any restriction would not be a locally small category and thereby not a category according to our definition 3.2.1. However, if $\mathcal{C}$ is small, then the collection of all functors $F : \mathcal{C} \to \mathcal{D}$ is a set. Therefore, the category of all small categories is indeed a category according to our definition of category.

[11]There is also a functor $\pi : \text{Top} \to \text{Set}$ sending topological spaces to the sets of their connected components.

[12]We later define a homotopy, see Example 3 below.

Units are constant paths, i.e. $c : [0,1] \to X$ such that $\forall t \in [0,1], c(t) = x$. Additionally, this is indeed a groupoid since inverses are given by the same paths run in the opposite direction. We have a functor

$$\pi_{\leq 1} : \mathrm{Top} \to \mathrm{Gpd}$$

3. Let X be a topological space and $x \in X$ an arbitrary basepoint. The assignement of a fundamental group of X at x, $\pi_1(X,x)$[13] is a functor $\pi_1 : \mathrm{Top} \to \mathrm{Grp}$

3.3 Natural transformations

Definition 3.3.1. Given two functors $F, G : \mathcal{C} \to \mathcal{D}$ a natural transformation from F to G, $\alpha : F \Rightarrow G$ is a collection of morphisms indexed by objects in $\mathcal{C}$, $\alpha_X : F(X) \to G(X)$ such that $\forall f : X \to Y$ in $\mathcal{C}$ the following diagram commutes

$$\begin{tikzcd} F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\alpha_Y"] & G(Y) \end{tikzcd}$$

If for every $X \in \mathrm{ob}(\mathcal{C})$, α_X is an isomorphism, then α is a natural isomorphism.

Remark. Note that because of functoriality, if a diagram commutes in $\mathcal{C}$, then its image under a functor $F : \mathcal{C} \to \mathcal{D}$ also commutes in $\mathcal{D}$. Let for example $g \circ f = h \circ l$, then $F(g) \circ F(f) = F(g \circ f) = F(h \circ l) = F(h) \circ F(l)$

Notation. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We denote with $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the functor category where objects are functors $\mathcal{C} \to \mathcal{D}$ and morphisms are natural transformations between such functors. We encounter soon an example of functor category, the category of G-linear representations, see Example 1 below.

Remark. Note that given a category $\mathcal{C}$ and a *groupoid* $\mathcal{D}$, then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is a groupoid since every component of any natural transformation is invertible because they are morphisms in $\mathcal{D}$ and therefore any natural transformation is a natural isomorphism.

Example 3.3.2.

1. As we have seen in Example 2 in 3.2.7, a linear representation of a group G is a functor $\mathbf{B}G \to \mathrm{Vect}_k$. A morphism between $G-$representations V and W, $f : V \to W$ is a $k-$linear map which is equivariant, i.e. $\forall g \in G, \forall v \in V$ we have $f(gv) = gf(v)$. Since linear representations are functors, one might wonder if a morphism between functors $V, W : \mathbf{B}G \to \mathrm{Vect}_k$, i.e. a natural transformation, is an equivariant map. Let $f : V \Rightarrow W$ be a natural transformation. Then, for any $g \in \mathrm{Hom}_{\mathbf{B}G}(*, *)$ the following diagram commutes

$$\begin{tikzcd} V(*) \arrow[r, "f(*)"] \arrow[d, "V(g)=g\cdot"'] & W(*) \arrow[d, "W(g)=g\cdot"] \\ V(*) \arrow[r, "f(*)"] & W(*) \end{tikzcd}$$

[13] *Reminder*: $\pi_1(X,x) = \pi_0(\Omega_x(X))$, where $\Omega_x(X)$ is the based loop space of X at x

and hence the map is equivariant, it does not matter if we first act on the vector space and subsequently apply the map or viceversa.

Since one can compose unitally and associatively natural transformations[14], if we take the collection of all functors $\mathbf{B}G \to \mathrm{Vect}_k$ and the natural transformations between them we get the category of linear representations of the group G. Such categories where the objects are functors are called functor categories.

2. The determinant can also be seen as a natural transformation. Let Mat be the functor Ring → Mon taking a commutative ring R to the monoid $\mathrm{Mat}(R)$ of matrices with coefficients in the ring R. Another such functor is the forgetful functor which forgets addition in the ring and forgets that the product is commutative $U : \mathrm{Ring} \to \mathrm{Mon}$. The determinant is then the following map of monoids:

$$\begin{aligned} \mathrm{Mat}(R) &\xrightarrow{\det} R \\ M &\longmapsto \det M \end{aligned} \tag{3.5}$$

The product rule for the determinant makes the map into a monoid homomorphism. The naturality diagram for rings R, S for a map $f : R \to S$ is then the following:

$$\begin{tikzcd} \mathrm{Mat}(R) \arrow[r, "\det"] \arrow[d, "\mathrm{Mat}(f)"'] & R \arrow[d, "f"] \\ \mathrm{Mat}(S) \arrow[r, "\det"] & S \end{tikzcd} \tag{3.6}$$

In words, this means that to calculate the determinant with coefficients in S we can proceed in two equivalent ways:

- change coefficients from R to S and then calculate the determinant,
- calculate the determinant using the matrix with R coefficients and then map into S.

Instead of taking *all* matrices we could take the general linear group $GL(-) : \mathrm{Ring} \to \mathrm{Grp}$. In that case det is a natural transformation between $GL(-)$ and the functor $(-)^\times : \mathrm{Ring} \to \mathrm{Grp}$ taking the units in the ring.

3. Let $X, Y \in \mathrm{Top}$ and $f, g \in \mathrm{Hom}_{\mathrm{Top}}(X, Y)$. A homotopy from f to g is a continuous map

$$h : [0,1] \times X \to Y$$

such that for every $x \in X$, $h(0,x) = f(x)$ and $h(1,x) = g(x)$. Two maps are homotopic if there is a homotopy between them. Although the homotopy seems intuitively like a map between maps, like the natural transformation is, this seems still very far from a natural transformation. However, there is an equivalent formulation of natural transformation, given below, which shows that homotopies and natural transformations are related. We will later show a way to make this comparison more rigorous, see 3.8.14.

[14] By composining their components.

Definition 3.3.3. ✈ (Homotopy Analogue of Natural Transformation). Let Δ^1 be the category with two objects $0, 1$ and one nonidentity morphism $u : 0 \to 1$. Given two functors $F, G : \mathcal{C} \to \mathcal{D}$ we can see a natural transformation $\tau : F \Rightarrow G$ as a functor $N : \mathcal{C} \times \Delta^1 \to \mathcal{D}$ where $N|_{\mathcal{C}\times\{0\}} = F$ and $N|_{\mathcal{C}\times\{1\}} = G$ and thus for every $X \in \mathcal{C}$, $N(X, 0) = F(X)$ and $N(X, 1) = G(X)$.

This is equivalent to the previous definition for the following reason: let $X \xrightarrow{f} Y$ be an arbitrary arrow in $\mathcal{C}$ and consider this commutative[15] diagram in $\mathcal{C} \times \Delta^1$

$$\begin{tikzcd}
(X,0) \arrow[r, "(id_X,u)"] \arrow[d, "(f,id_0)"'] & (X,1) \arrow[d, "(f,id_1)"] \\
(Y,0) \arrow[r, "(id_Y,u)"'] & (Y,1)
\end{tikzcd}$$

The image of the latter diagram under N is the following commutative diagram in $\mathcal{D}$

$$\begin{tikzcd}
F(X) \arrow[r, "N(id_X,u)"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "N(id_Y,u)"'] & G(Y)
\end{tikzcd}$$

Hence the components of τ are just the image under N of the pairs of maps (id, u), i.e. $\forall X \in \mathcal{C}, \tau_X = N(id_X, u)$

3.4 Equivalence of categories ✈

We did not cover this topic in the lecture, but the concept of equivalence of categories is important for results such as 5.1.2.

Definition 3.4.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. F is named

1. faithful if $\forall X, Y \in \mathcal{C}$ the map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is injective,
2. full if $\forall X, Y \in \mathcal{C}$ the map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is surjective,
3. fully faithful, if F is full and faithful,
4. conservative, if for $f : X \to Y$ in $\mathcal{C}$ such that if $F(f)$ is an isomorphism, then f is an isomorphism,
5. an isomorphism if $\exists G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F = id_{\mathcal{C}}$ and $F \circ G = id_{\mathcal{D}}$,
6. an equivalence if $\exists G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $G \circ F \overset{\epsilon}{\cong} id_{\mathcal{C}}$ and $F \circ G \overset{\eta}{\cong} id_{\mathcal{D}}$,
7. essentially surjective if $\forall Z \in \mathcal{D}, \exists Z' \in \mathcal{D}, X \in \mathcal{C}$ and an isomorphsim $F(X) \overset{f}{\cong} Z'$.

Definition 3.4.2 (Equivalence and isomorphism of Categories)**.** Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent (isomorphic) if and only if there is an equivalence (isomorphism) between them.

[15]It commutes because of unitality in both categories: $f \circ id_X = id_Y \circ f$ and $u \circ id_0 = id_1 \circ u$.

From the definition above we can see that equivalence of categories is a *weaker* notion than that of being isomorphic. However finding naturally isomorphic categories is very rare, so the notion of equivalence is often more useful. Here we also use the term *weak* which in category theory often refers to the substitution of an equality by an appropriate natural isomorphism.

Notation. We denote that two categories $\mathcal{C}, \mathcal{D}$ are equivalent by $\mathcal{C} \simeq \mathcal{D}$. Whereas, we write down $\mathcal{C} \cong \mathcal{D}$ if they are naturally isomorphic, i.e. there is a natural isomorphism between them.

Theorem 3.4.3 (Fundamental Theorem of Category Theory[16])**.** *Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if and only if it there is a fully faithful and essentially surjective functor $F : \mathcal{C} \to \mathcal{D}$.*

Proof. $\implies$ Suppose that there is an equivalence between $\mathcal{C} \xrightarrow{F} \mathcal{D}$. Since it is an equivalence, there must be a functor $\mathcal{D} \xrightarrow{G} \mathcal{C}$ such that their compositions are naturally isomorphic to the identities on $\mathcal{C}$ and $\mathcal{D}$ thanks respectively to natural isomorphisms ϵ and η. Thus, for any $Y \in \mathcal{D}$ there is an object $G(Y) \in \mathcal{C}$ and an isomorphism $\epsilon_Y : (F \circ G)(Y) \to Y$ making F essentially surjective. To prove the faithfulness of F suppose that there is a pair of parallel arrows $f, g : X \to X'$ in $\mathcal{C}$ that are mapped by F to the same arrow in $\mathcal{D}$, i.e. $F(f) = F(g)$, and hence having the following commuting diagram in $\mathcal{C}$ because of the naturality of ϵ

$$\begin{tikzcd}
X \arrow[r, "\epsilon_X"] \arrow[d, "f"'] & G(F(X)) \arrow[d, dashed, "G(F(f))=G(F(g))"] & X \arrow[l, "\epsilon_X"'] \arrow[d, "g"] \\
X' \arrow[r, "\epsilon_{X'}"'] & G(F((X'))) & X' \arrow[l, "\epsilon_{X'}"]
\end{tikzcd}$$

Note that $G \circ F \overset{\epsilon}{\cong} id_{\mathcal{D}}$, and so $\epsilon_X = id_X$, $\epsilon_{X'} \cong id_{X'}$. We can conclude that $f = g$: $id_{X'} \circ f = G(F(g)) \circ id_X$ and $id_{X'} \circ g = G(F(g)) \circ id_X$ because of the commutativity of the latter diagram, thanks to unitality $f = G(F(g)) = g$. To prove that F is full take an arbitrary $g : F(X) \to F(X')$ in $\mathcal{D}$ and consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\epsilon_X", "\cong"'] \arrow[d, dashed, "g^*"'] & G(F(X)) \arrow[d, shift right=2, "G(F(g*))"'] \arrow[d, shift left=2, "G(g)"] \\
X' \arrow[r, "\cong", "\epsilon_{X'}"'] & G(F(X'))
\end{tikzcd}$$

Since we can compose $\epsilon_{X'}^{-1} \circ G(g) \circ \epsilon_X$ there is an arrow $X \to X'$ and it is unique since the diagram commutes by naturality of ϵ, we denote it by g^*. Also because of the commutativity of the latter diagram it must be the case that $G(F(g^*)) = G(g)$ since $G(F(g^*)) = \epsilon_{X'}^{-1} \circ G(g) \circ \epsilon_X = G(g)$. In the first part of the proof we proved the faithfulness of F but by a specular argument, we could have proven the faithfulness of G. So, $g = F(g^*)$ and thus F is full.
$\impliedby$ For every $Y \in \mathcal{D}$ there is an isomorphic $F(X)$ because F is essentially surjective and for every $X \in \mathcal{C}$ there is an isomorphic $G(Y)$ because G is essentially surjective, thus for every $Y \in \mathcal{D}$ we can *choose* an object $G(Y) \in \mathcal{C}$ and an isomorphism between them. We denote

[16] We call it fundamental theorem of category theory following [Rez21] to emphasize how important it is. It is central to category theory because it is often very interesting to prove that two categories are equivalent and virtually always one uses the "fully faithful and essentially surjective criterion". This is however an unorthodox denomination because usually such theorem just lacks a name.

the isomorphism by $\epsilon_Y : Y \cong F(G(Y))$. In order for ϵ_Y to be a natural transformation there must be a unique arrow such that for any $f : Y \to Y'$ in $\mathcal{D}$ the following diagram commutes

$$\begin{tikzcd} F(G(Y)) \arrow[r, dashed, "F(G(f))"] \arrow[d, "\eta_Y"'] & F(G(Y')) \arrow[d, "\eta_{Y'}"] \\ Y \arrow[r, "f"'] & Y' \end{tikzcd}$$

Then, we get an arrow $\eta_{Y'}^{-1} \circ f \circ \eta_Y : F(G(Y)) \to F(G(Y'))$ making the the diagram commute, such an arrow exists because F is full and is unique because F is faithful. We denote it with $F(G(f))$. Now we prove that G is actually a functor. Both $F(G(id_Y))$ and $F(id_{G(Y)})$ make the following diagram commute

$$\begin{tikzcd} F(G(Y)) \arrow[r, shift left, "F(id_{G(Y)})"] \arrow[r, shift right, "F(G(id_Y))"'] \arrow[d, "\eta_Y"'] & F(G(Y)) \arrow[d, "\eta_Y"] \\ Y \arrow[r, "id_Y"'] & Y \end{tikzcd}$$

Since the diagram commutes $F(G(id_Y)) = \eta_Y^{-1} \circ id_Y \circ \eta_Y = F(id_{G(Y)})$. Moreover, given $f : Y \to Y'$ and $f' : Y' \to Y'$ consider the following commutative diagram

$$\begin{tikzcd} F(G(Y)) \arrow[r, shift left, "F(G(f' \circ f))"] \arrow[r, shift right, "F(G(f') \circ G(f))"'] \arrow[d, "\eta_Y"'] & F(G(Y'')) \arrow[d, "\eta_{Y''}"] \\ Y \arrow[r, "f' \circ f"'] & Y'' \end{tikzcd}$$

By essentially the same argument we just provided for the functoriality of F on the identities we get $F(G(f') \circ G(f)) = \eta_{Y''}^{-1} \circ (f' \circ f) \circ \eta_Y = F(G(f' \circ f))$.

Now we just need to prove that there is a natural isomorphism $\epsilon : id_{\mathcal{C}} \cong G \circ F$. Since F is full and faithful we can find the components of ϵ by looking at their image under F. We denote $\eta_F^{-1}(X)$ by $F(\epsilon_X)$ and take into consideration a morphism $f : X \to X'$ from $\mathcal{C}$ and the following commuting outer rectangle

$$\begin{tikzcd} F(X) \arrow[r, "F(\epsilon_X)"] \arrow[d, "F(f)"'] & F(G(F(X))) \arrow[r, "\eta_{F(X)}", "\cong"'] \arrow[d, "F(G(F(f)))"] & F(X) \arrow[d, "F(f)"] \\ F(X') \arrow[r, "F(\epsilon_{X'})"'] & F(G(F(X'))) \arrow[r, "\cong", "\eta_{F(X')}"'] & F(X') \end{tikzcd}$$

The right hand square commutes because of the naturality of η. Since the right hand square commutes $F(G(F(f))) = \eta_{F(X')}^{-1} \circ F(f) \circ \eta_{F(X)}$. Since the outer square commmutes $F(\epsilon_{X'}) \circ F(f) = \eta_{F(X')}^{-1} \circ F(f) \circ \eta_{F(X)} \circ F(\epsilon_X)$. Thus, the left hand square commutes because $F(\epsilon_{X'}) \circ F(f) = F(G(F(f))) \circ F(\epsilon_X)$ and thereby ϵ is natural because $\epsilon_{X'} \circ f = G(F(f)) \circ \epsilon_X$ thanks to the faithfulness of F. $\square$

3.5 Adjunction

Definition 3.5.1 (Adjunction)**.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. We say that F and G form an adjunction if there are natural transformations

$$\eta : id_{\mathcal{C}} \Rightarrow G \circ F$$

$$\epsilon : G \circ F \Rightarrow id_{\mathcal{D}}$$

that make the following two diagrams commute, $\forall X \in \mathcal{C}$ for the first one and $\forall Y \in \mathcal{D}$ for the second one

$$\begin{tikzcd}
F(X) = F(id_{\mathcal{C}}(X)) \arrow[rr, "id_{F(X)}"] \arrow[dr, "F(\eta_X)"'] & & id_{\mathcal{D}}(F(X)) = F(X) \\
& F(G(F(X))) \arrow[ur, "\epsilon_{F(X)}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
G(Y) = id_{\mathcal{C}}(G(Y)) \arrow[rr, "id_{G(Y)}"] \arrow[dr, "\eta_{G(Y)}"'] & & G(id_{\mathcal{D}}(Y)) = G(Y) \\
& G(F(G(Y))) \arrow[ur, "G(\epsilon_Y)"'] &
\end{tikzcd}$$

Equivalently, we could have asked that the following two diagrams commute, the first in $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ and the second in $\mathrm{Fun}(\mathcal{D},\mathcal{C})$.

$$\begin{tikzcd}
F = F \circ id_{\mathcal{C}} \arrow[rr, Rightarrow, "id_F"] \arrow[dr, Rightarrow, "F(\eta)"'] & & id_{\mathcal{D}} \circ F = F \\
& F \circ G \circ F \arrow[ur, Rightarrow, "\epsilon_F"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
G = id_{\mathcal{C}} \circ G \arrow[rr, Rightarrow, "id_G"] \arrow[dr, Rightarrow, "\eta_G"'] & & G \circ id_{\mathcal{D}} = G \\
& G \circ F \circ G \arrow[ur, Rightarrow, "G(\epsilon)"'] &
\end{tikzcd}$$

We say that F is left adjoint to G denoted by $F \dashv G$, and reciprocally G is right adjoint to F denoted by $G \vdash F$.

See 4.1.17 for an equivalent formulation[17]

Example 3.5.2. A trivial example is given by any functor that is an isomorphism. In that case the natural transformations are actually identities.
An equivalence of categories instead is in general *not* an adjunction, however it can always be made into one by appropriately changing the natural isomorphisms.
The more typical examples are instead related to forgetful functors.

3.6 Higher categories

Remark. We previously remarked that there is a category of all categories with functors as morphisms (3.2). That was however not the end of the story since we also defined a

[17]Although in the ∞-categorical context, it is easily tranlatable by forgetting about the ∞s and substituting 'set' for '∞-groupoid'.

morphism between functors, the natural transformation. Cat is in fact not just a "normal" category, also called a 1-category, but a strict 2-category, a category that also has morphisms between morphisms between objects and morphisms between objects compose up to strict equality[18]. Morphisms between objects are called 1-morphisms and are functors in the case of Cat. Morphisms between morphisms between objects are called 2-morphisms and are natural transformations in the case of Cat.

Notation. An (n, k)-category is a category in which all m-morphisms with $n \geq m > k$ are invertible and all j-morphisms with $j \leq k$ are not necessarily invertible.

Following this convention an ordinary category is a strict $(1,1)$-category, a strict 2-category is a strict (2,2)-category, a groupoid is a strict (1,0) groupoid, and more generally an n-groupoid is an $(n,0)$-category. In order to make this more rigorous we need to loosely characterize enriched categories.

Definition 3.6.1 (Loose definition of enriched category)**.** Given a category $\mathcal{D}$ a category $\mathcal{C}$ enriched over $\mathcal{D}$ is a category such that for every $X, Y \in \mathcal{C}$

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathcal{D}$$

Remark. Note that this is just a loose characterization of what an enriched category is! For instance, one might wonder, how could we define the composition map, since for ordinary categories we defined it as a map of sets out of apt *products* of Hom-sets. There is a way to sensibly talk about products of objects in many several other categories. We spell out the rigorous definition later on, see 3.8.1. We gave this loose characterization just to be able to characterize higher strict categories.

Example 3.6.2.

- A locally small category is a category enriched over Set
- Cat is enriched over Cat, since instead of Hom-sets one has functor categories, which are themselves categories and hence in Cat. More generally categories with Hom-categories/functor categories instead of Hom-sets are called strict 2-categories, or (2,2)-categories following the convention we just stated.
- Grpd is enriched over groupoid, since (as we previously remarked, see 3.3) all natural transformations between functors with a groupoid in the codomain are natural isomorphisms and therefore all functor categories (i.e. the Hom-objects of Grpd) are groupoids and hence in Grpd. One calls categories enriched over Grpd strict (2,1)-categories by the notational convention we just spelled out.

Having loosely characterized enriched categores We can provide a definition of strict n-categories. Following the notational convention we previously spelled out in a previous example, they amount to (n, n)-categories.

Definition 3.6.3. We define strict n-categories inductively. A 0-category is a set. A strict n-category is a category $\mathcal{C}$ enriched in a (small[19]) strict $(n-1)$-category. This means that for any two $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is not a set as usual, but a small strict $(n-1)$-category.

[18] We remark soon after (see 3.6) in which sense Cat is a *strict* 2-category. See also for comparison the definition of Bicategory, a weak 2-category, 3.10.29.

[19] Because of size issues, as we remarked before, we work with locally small categories and the size of the Hom of any two objects cannot be greater than a set.

Example 3.6.4.

- In the case that $n = 1$ we end up with our usual definition of a (locally small) category (see 3.2.1).
- Cat is a 2-category, in fact for any two $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$, $\mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}, \mathcal{D})$ is a $1-$category, more specifically a functor category, i.e. a category where objects are functors and morphisms are natural transformations. Specifically:
$$\mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}, \mathcal{D}) = Fun(\mathcal{C}, \mathcal{D})$$

Remark. We call such categories, *strict* n-categories because $(n-1)$-morphisms are strictly associative and unital, associativity and unitality hold with strict identities, on the nose. For example in Cat it holds that $\forall F : \mathcal{C} \to \mathcal{D}, id_{\mathcal{D}} \circ F = F = F \circ id_{\mathcal{C}}$ and $\forall F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{E}, H : \mathcal{E} \to \mathcal{B}$
$$H \circ (G \circ F) = (H \circ G) \circ F.$$
We will later encounter a notion of weak 2-category (see 3.10.29).

3.7 Monoidal Categories

Recall that a monoid is a group where not necessarily every element is invertible: a set M with a distinguished object $e \in M$ called unit, also known as neutral element, and a map of sets $m : M \times M \to M$ such that m is:

- associative
$$\forall a, b, c \in M, \quad m(m(a, b), c) = m(a, m(b, c))$$
usually written
$$\forall a, b, c \in M, \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$$
- and unital with respect to e:
$$m(e, -) = \mathrm{id}(-), \quad m(-, e) = \mathrm{id}(-)$$
where $\forall x \in M, id(x) = x$. The latter equation is usually written
$$\forall x \in M, \quad e \cdot x = x = x \cdot e$$

A monoidal category generalizes this structure with respect to objects *and* morphisms of a category.

Definition 3.7.1. Let $\mathcal{C}$ be a category. A monoidal structure on $\mathcal{C}$ is

- (O) an object $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}$, the *unit*
- (M) bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, the *tensor product*
- (A) a natural isomorphism $\alpha : - \otimes (- \otimes -) \Rightarrow (- \otimes -) \otimes -$ that witnesses associativity:
$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, bend left, "-\otimes(-\otimes-)"] \arrow[r, bend right, "(-\otimes-)\otimes-"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{C} \end{tikzcd}$$
the *associator*

- (U) natural isomorphisms $\lambda : \mathbb{1}_{\mathcal{C}} \otimes (-) \Rightarrow \mathrm{id}_{\mathcal{C}} = (-)$ and $\rho : - \otimes \mathbb{1}_{\mathcal{C}} \Rightarrow \mathrm{id}\,\mathcal{C} = (-)$ witnessing unitality:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathbb{1}\otimes -"] \arrow[r, bend right, "id_{\mathcal{C}}=(-)"'] \arrow[r, phantom, "\lambda\Downarrow"] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "-\otimes\mathbb{1}"] \arrow[r, bend right, "id_{\mathcal{C}}=(-)"'] \arrow[r, phantom, "\rho\Downarrow"] & \mathcal{C} \end{tikzcd}$$

respectively the *left and right unitors*

such that

- $\forall X, Y$ the following diagram commutes

$$\begin{tikzcd} (X\otimes\mathbb{1})\otimes Y \arrow[rr, "\alpha_{X,\mathbb{1},\otimes Y}"] \arrow[dr, "\rho_X\otimes id_Y"'] & & X\otimes(\mathbb{1}\otimes Y) \arrow[dl, "id_X\otimes\lambda_Y"] \\ & X\otimes Y & \end{tikzcd}$$

  This diagram is called the triangle identity. It explains how the associator and the two unitors interact.

- $\forall W, X, Y, Z$ the following diagram commutes

$$\begin{tikzcd} & ((W\otimes X)\otimes Y)\otimes Z \arrow[dl, "\alpha_{W,X,Y}\otimes id_Z"'] \arrow[dr, "\alpha_{W\otimes X,Y,Z}"] & \\ (W\otimes(X\otimes Y))\otimes Z \arrow[d, "\alpha_{W,X\otimes Y,Z}"'] & & (W\otimes X)\otimes(Y\otimes Z) \arrow[d, "\alpha_{W,X,Y\otimes Z}"] \\ W\otimes((X\otimes Y)\otimes Z) \arrow[rr, "id_W\otimes\alpha_{X,Y,Z}"'] & & W\otimes(X\otimes(Y\otimes Z)) \end{tikzcd}$$

  The latter diagram is called Mac Lane's pentagon. Thanks to 3.10.4 it pins down the associativity of more than 3 objects.

Notation. As we usually abuse notation and denote monoids $(M,\cdot)$ just with M, although a monoid is a set *equipped with* a binary operation; we will denote monoidal categories $(\mathcal{C},\otimes,\mathbb{1}_{\mathcal{C}})$ just with $\mathcal{C}$, although a monoidal category is a category *equipped with* a monoidal structure.

Definition 3.7.2 (Strict Monoidal Category)**.** A strict monoidal category is a monoidal category where objects and morphisms are associative and unital strictly, not up to specific natural isomorphisms, i.e. the associator and the left/right unitors are the identity. This means that for every $f : A, B, C \in \mathcal{C}$ and every $f : A \to B, g : B \to C, h : C \to D \in \mathcal{C}$ where $\mathcal{C}$ is a strict monoidal category:

- $A\otimes(B\otimes C) = (A\otimes B)\otimes C$
- $A\otimes\mathbb{1}_{\mathcal{C}} = A = \mathbb{1}_{\mathcal{C}}\otimes A$
- $f\otimes(g\otimes h) = (f\otimes g)\otimes h$
- $f\otimes id_{\mathbb{1}} = f = id_{\mathbb{1}}\otimes f$

Example 3.7.3 (Examples of *strict* monoidal categories)**.**

- Given an arbitrary category $\mathcal{C}$, the set[20] of its endofunctors $\mathrm{End}(\mathcal{C},\mathcal{C}) = \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C},\mathcal{C})$ are the objects of a monoidal category with morphisms given by natural transformations between them. The tensor product is given by composition and it is associative since composition of functors is associative. The monoidal unit is $id_{\mathcal{C}}$ and is indeed left and right unital. Note however that this is an example of *strict* monoidal category since functors compose strictly and not up to some associator and unitors.

- A monoid $(M,\cdot,e)$, seen as a category with objects $m \in M$ and only the identity morphisms, is a discrete strict monoidal category. The monoidal structure is simply given by multiplication of the elements of the monoid, which is clearly (strictly) associative and unital. A discrete category is a category where there are only identity morphisms.

Example 3.7.4 (Examples of monoidal categories)**.** Examples of monoidal categories are

- $(\mathrm{AbGrp},\otimes)$
- $(\mathrm{AbGrp},\oplus)$
- $(\mathrm{Vect}_k,\otimes)$
- $(\mathrm{Vect}_k,\oplus)$
- $(\mathrm{Set},\coprod)$
- $(\mathrm{Set},\times)$
- $(\mathrm{Top},\times)$
- $(\mathrm{Cat},\times)$

3.7.1 Cartesian Categories: cartesian product

In particular, $(\mathrm{AbGrp},\oplus),(\mathrm{Vect}_k,\oplus),(\mathrm{Set},\times),(\mathrm{Top},\times),(Cat,\times)$ are all examples of special monoidal categories: cartesian monoidal categories. A cartesian monoidal category is a monoidal category where the monoidal structure is given by a suitable notion of cartesian product in such category. We give a definition via a *universal property* which is therefore valid for every category (although *existance* must then be checked in the categories of interest).

Definition 3.7.5 (Cartesian Product)**.** Given $A,B \in \mathcal{C}$, their cartesian product, if it exists, is the tuple $(A\times B, pr_1, pr_2)$, where $A\times B \in \mathrm{ob}(\mathcal{C})$ and $pr_1 : A\times B \to A$ $pr_2 : A\times B \to B$ such that for every other object Y with morphisms $f : Y \to A$ and $g : Y \to B$ there is a unique morphism $u : Y \to A\times B$. This definition is summarised by the fact that the following commuting diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "f"'] \arrow[d, dashed, "\exists! u"] \arrow[dr, "g"] & \\
A & A\times B \arrow[l, "pr_1"] \arrow[r, "pr_2"'] & B
\end{tikzcd}$$

[20] A set and not a larger collection since we defined categories to be locally small.

If we regard $\mathcal{C}$ as a specific category such construction gives indeed rise to the expected notion of product in such a category: the cartesian product in AbGrp and Vect_k is indeed the direct sum, the cartesian product of sets with its projections in Set, the product category in Cat and the product of spaces with the product topology in Top.

The following uniqueness property is typical of objects defined via a universal property:

Lemma 3.7.6. *Every product is unique up to isomorphism.*

Proof. Suppose $(A \times B, pr_1, pr_2)$ and (Y, f, g) are both products of $A, B \in \mathcal{C}$. Then there are unique morphisms $u : A \times B \to Q$ and $u' : Q \to A \times B$ such that the following diagram commutes

$$\begin{tikzcd}
& Y \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dashed, "\exists! u"] & \\
A & A \times B \arrow[l, "pr_1"] \arrow[r, "pr_2"'] \arrow[d, dashed, "\exists! u'"] & B \\
& Y \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Since everything commutes $f = f \circ u' \circ u$ and $g = g \circ u' \circ u$. Therefore, $u' \circ u = id_Y$ because $f = f \circ id_Y$ and $g = g \circ id_Y$ and id_Y is the unique morphism such that the latter two equations hold. $\square$

Lemma 3.7.7. *The cartesian product is commutative up to isomorphism.*

Proof. If $(A \times B, pr_1, pr_2)$ and $(B \times A, pr_1, pr_2)$ are both products of $A, B \in \mathcal{C}$ then, they are isomorphic because of 3.7.6. $\square$

Lemma 3.7.8. *The cartesian product is associative up to isomorphism.*

Proof. Same reasoning as in the proof of the commutativity of the cartesian product. $\square$

In a cartesian monoidal category $\mathcal{C}$ the unit is a terminal object, i.e. an object $* \in \mathcal{C}$ such that every object $X \in \mathcal{C}$ has a unique morphism to it: $\exists! \phi_X : X \to *$. This is the case because given an arbitrary object $A \in \mathcal{C}$ and a product with the terminal one $(A \times *, pr_1, pr_2)$ the units $\rho : A \times * \to A$ and $\lambda : * \times A \to A$ are the same projection $pr_1 : A \times * \to A$ since the cartesian product is commutative. Such projection is indeed an isomorphism because (A, id_A, ϕ_A) is also a product of $*, A \in \mathcal{C}$: any object $Y \in \mathcal{C}$ with morphisms $f : Y \to A$ and $\phi_Y \to *$ has indeed a unique morphism into A such that the diagram of the product commutes, namely f itself. Thus, since both $(A \times *, pr_1, pr_2)$ and (A, id_A, ϕ_A) are products of the same two elements, they must be isomorphic because of 3.7.6.
Hence, we see how any category with finite cartesian products is a monoidal category. Cartesian monoidal categories are in a sense special because they have some maps which are not possessed by every monoidal category, notably the diagonal map $\Delta_X : X \to X \times X$ and $\phi_X : X \to \mathbb{1}_{\mathcal{C}}$ giving every object a comonoid structure. Moreover, they are all symmetric (defined later, 3.10.2) because the cartesian product is symmetric.

3.7.2 Noncartesian monoidal categories ✈

Notation. We denote the category of abelian groups AbGrp also with Ab.

In the previous paragraph, we have talked about cartesian monoidal categories and left out some examples of *noncartesian* monoidal categories like (AbGrp, $\otimes$) and (Vect$_k$, $\otimes$). We now encounter some examples of noncartesian monoidal categories, which will be central in this class. They will be all similar to (AbGrp, $\otimes$), so we will first explain this category and then draw parallels with the others. To introduce the tensor product between abelian groups we first have to define what is a bilinear map:

Definition 3.7.9 (Bilinear Map)**.** For $A, B, C \in \text{Ab}$, a bilinear map, aka group bihomomorphism, is a function

$$f : A \times B \to C$$

which is a group homomorphism, aka linear map, in both arguments.

An explanation of what happens to the elements of A, B might be clearer. Note that a cartesian product $A \times B$ in Ab is an abelian group where the operation is

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2).$$

So, we can spell out what it means to be a group bihomomorphism in the following way

Definition 3.7.10 (Bilinear Map via Elements)**.** A bilinear map is a function $f : A \times B \to C$ such that for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$ it holds that

$$f(a_1 + a_2, b_1) = f(a_1, b_1) + f(a_2, b_1)$$

and

$$f(a_1, b_1 + b_2) = f(a_1, b_1) + f(a_1, b_2)$$

Having defined what a bilinear map is, we can arrive at the definition of tensor product in Ab

Definition 3.7.11 (Tensor Product of Abelian Groups)**.** The tensor product of two abelian groups A and B is given by $(A \otimes B, \phi)$ where $A \otimes B$ is an abelian group and $\phi : A \times B \to A \otimes B$ is a bilinear map. $(A \otimes B, \phi)$ has the universal property that any bilinear map out of $A \times B$ factors uniquely through $A \otimes B$, i.e. $\forall C, h : A \times B \to C$ there is a unique linear map $\bar{h} : A \otimes B \to C$ such that the following diagram commutes

$$\begin{tikzcd}
A \times B \arrow[r, "h"] \arrow[d, "\phi"'] & C \\
A \otimes B \arrow[ur, dashed, "!\bar{h}"'] &
\end{tikzcd}$$

It can be equivalently defined via elements.

Definition 3.7.12 (Tensor Product of Abelian Groups via Elements)**.** For $A, B \in \text{Ab}$ let $a, a' \in A$ and $b, b' \in B$ and consider the free abelian group on $A \times B$. Then, consider the free abelian group generated by elements of the form:

$$(a + a', b) - (a, b) - (a', b)$$

$$(a, b + b') - (a, b) - (a, b').$$

We denote such group by R. $A \otimes B$ is then the following quotient: $A \otimes B = \frac{A \times B}{R}$.

3.8 Enriched categories ✈

Ab is very important as an enriching category. Before, we gave a loose characterization of what an enriched category is by just stating that the Homs are objects of whatever category we are enriching over. However, we glossed over an essential requirement: the category we are enriching on must be monoidal in order to define composition. We now give a fully satisfactory definition of an enriched category:

Definition 3.8.1 (Enriched Category)**.** Given a *monoidal* category $\mathcal{D}$, a $\mathcal{D}$-enriched category $\mathcal{C}$, aka $\mathcal{D}$-category or category enriched over $\mathcal{D}$, is a category $\mathcal{C}$ such that

1. $\forall X, Y \in \mathcal{C}, \mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathcal{D}$. eW call $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ a Hom-object
2. Composition is a morphism in $\mathcal{D}$:
$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y,Z) \otimes \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$$
3. For any $X \in \mathcal{C}$ there is
$$id_X : \mathbb{1}_{\mathcal{D}} \to \mathrm{Hom}_{\mathcal{C}}(X,X)$$
4. There is a Mac-Lane pentagon for Hom-objects in order to specify how $\circ$ is associative and two triangle diagrams in order to pin down how $\circ$ is left and right unital with respect to composable identity morphisms. One can check the spelled out diagram at definition 1.2.34 of [Lan21]

Example 3.8.2. We recall some examples spelled out before and provide a new one coming from topology (the last one):

1. A locally small category is a category enriched over Set. Set is a category enriched over itself.
2. A 2-category is a category enriched over Cat. Cat is a category enriched over itself, since Cat is a 2-category because Homs are functor *categories*.
3. A strict n-category is a category enriched over a strict $(n-1)$-category.
4. A strict $(2,1)$-category is a category in which 2-morphisms are invertible and 1-morphisms not necessarily. It is a Grpd-enriched category. An example of (2,1)-category is Grpd itself, since all $\mathrm{Hom}_{\mathrm{Grpd}}$ are indeed groupoids (see 3.6.2 for an explanation).
5. A topologically enriched category is a category enriched over Top. The category of compactly generated Hausdorff spaces CGHaus, i.e. a category containing all CW-complexes, is a topologically enriched category, since one can take the mapping space $Map(X,Y)$ with the compact-open topology for any $X, Y \in$ CGHaus. Such categories can be used to model $(\infty,1)$-categories see 3.8.15 for a sketch of how this might work.

There is an interesting example of strict (2,1)-category from topology and it will let us rigorously show in which sense a homotopy is a natural transformations and why one might want to talk about ∞-categories. Take a look at the following subsection for more on this 3.8.1.

Having characterized satisfactorily enriched categories we can now define what is an Ab-enriched category.

Definition 3.8.3 (Ab-Category)**.** An Ab-category, aka ringoid, Ab-enriched, $\mathcal{C}$ is a category $\mathcal{C}$ enriched over Ab. This is equivalent to saying that

1. $\forall X, Y \in \mathcal{C}. \mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathrm{Ab}$
2. Composition is a morphism in Ab:
$$+ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \otimes_{\mathrm{Ab}} \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$
such that it is associative and unital with respect to composable identity morphisms; in short Hom-objects of $\mathcal{C}$ posses an abelian structure with + as a binary operation
3. The composition of the underlying Hom-<u>sets</u> is bilinear with respect to +, meaning that $\circ$ of Hom-sets distributes over + of Hom-abelian groups i.e.
$$f \circ (g + h) = (f \circ g) + (f \circ h)$$
$$(f + g) \circ h = (f \circ h) + (f \circ h)$$

Remark. Note that in Ab-enriched categories there is an object 0 such that it for any $X \in \mathcal{C}$ there is a unique morphism $0 \to X$ and a unique morphism $X \to 0$. This means that 0 is an initial and final object, such objects are usually called zero objects.

Example 3.8.4.

- A ring is a one object ringoid. More explicitly, a ring $(R, \cdot, +) = (\mathrm{Hom}_{\mathbf{B}R}(*, *), \circ, +)$
- Ab is an Ab-category.

Definition 3.8.5 (Additive category)**.** A category $\mathcal{C}$ is additive if it is an Ab-enriched category such that it has finite coproducts.

Note that in an Ab-enriched category, finite coproducts coincide with finite products, any finite product is a finite coproduct and viceversa.

Definition 3.8.6 (Abelian category)**.** A category $\mathcal{C}$ is abelian if it is an additive category such that

- every morphism has kernel and cokernel
- every monomorphism is a kernel and every epimorphism is a cokernel, or equivalently, the evident map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism

Remark. The category of abelian groups is very important for homological algebra, a field with important applications to many fields, e.g. algebraic geometry (in particular intersection theory) and algebraic topology. The standard reference is [Wei94], check out part I of [MG24] for a modern introduction to the subject geared towards higher categorical generalizations. Such higher categorical generalizations are of major interest to us in the context of TFTs, we will later sketch for example how they can be used to calculate topological phases of matter (4.1.1).

Another very important example for us of noncartesian[21] monoidal category is $(\mathrm{Vect}_k, \otimes)$.

Definition 3.8.7 (Tensor Product between Vector Spaces)**.** The tensor product is usually defined from bases (see wikipedia entry on tensor product) or via quotient spaces[22] along these lines: let $V, W \in \mathrm{Vect}_K$. Consider L, a vector space that has $V \times W$ as basis. Let R be the subspace whose elements are linear combinations of one of the following forms

$$(v + v', w) - (v, w) - (v', w)$$

$$(v, w + w') - (v, w) - (v, w')$$

$$(kv, w) - k(v, w)$$

$$(v, kw) - k(v, w)$$

where $v, v' \in V,\ w, w' \in W$ and $k \in K$. The tensor product is then the quotient space $L/R = V \otimes W$, where the image (v, w) in this quotient is denoted by $v \otimes w$.
The cartesian product construction helps us find an equivalent definition of the tensor product: it is an endofunctor $\otimes : \mathrm{Vect}_K \times \mathrm{Vect}_K \to \mathrm{Vect}_K$ where $V \otimes W$ has the following universal property:

1. there is a unique bilinear map out of the cartesian product of the underlying sets $\phi : V \times W \to V \otimes W$

2. any other bilinear map out of the cartesian product of the underlying sets $h : V \times W \to X$ factors through ϕ:

$$\begin{tikzcd} V \times W \arrow[r, "h"] \arrow[d, "\phi"'] & X \\ V \otimes W \arrow[ur, "\bar{h}"'] & \end{tikzcd}$$

As with Ab, Vect_k can also be used as enriching category.

Definition 3.8.8. A linear category $\mathcal{C}$, or algebroid, is a category enriched over Vect_k.

Example 3.8.9.

- A k-algebra is a one object algebroid.

- Vect_k is a linear category, since also $\mathrm{Hom}_{\mathrm{Vect}_k}(X, Y)$ is a vector space

More generally the category of modules is also a very important example of noncartesian monoidal category.

Definition 3.8.10 (Module)**.**

Example 3.8.11. The category of bimodules is an example of symmetric monoidal category

[21] Interestingly, the fact that Vect_k is not cartesian and in particular $\mathrm{Hilb}_{\mathbb{C}}$ (the category of complex Hilbert spaces) is the main reason behind the no-cloning theorem from quantum information theory, more on this in [Bae04]. This characterizing non-cartesian aspect is nicely summarized by Freed in [Fre12], "We will see that a characteristic property of quantum systems is that disjoint unions map to tensor products. The passage from classical to quantum is (poetically) a passage from addition to multiplication, a kind of exponentiation."

[22] We use the definition via quotient spaces because it is easier to see that it fulfills the equivalent one via universal properties and the cartesian product.

Definition 3.8.12 (Bimodule)**.**

Another example of non-cartesian monoidal category, which is not algebraic in nature but topological and will be useful for an outlook in the next section is the category of disks.

Example 3.8.13. The category of oriented 1-dimensional open disks Disk_1^{or} is also an important example of symmetric monoidal category for us in the context of TFTs. Recall that $D^1 \cong (0,1) \cong \mathbb{R}$.

- Its objects are finite disjoint unions of 1-dimensional open disks

  $$\emptyset, \qquad \mathbb{R}, \qquad \mathbb{R} \amalg \mathbb{R}, \qquad \dots, \qquad \mathbb{R}^{\amalg k}$$

  with $k \in \mathbb{N}$ arbitrary.

- $\mathrm{Hom}_{\mathrm{Disk}_{1,0}^{or}}(X, Y)$ consists of all smooth embeddings $j : X \hookrightarrow Y$ respecting orientations, i.e. there is an orientation preserving diffeomorphism from X onto its image, .

- The tensor product is given by the disjoint union: given X a disjoint union of n oriented 1-disks and Y a disjoint union of m oriented 1-disks, $X \amalg Y$ is a disjoint union of $n+m$ oriented disks.

- The unit is the empty set since given an arbitrary disjoint union of disks X, $\emptyset \amalg X \cong X \cong X \amalg \emptyset$.

- There is a diffeomorphism $X \amalg Y \cong Y \amalg X$ making $\mathrm{Disk}_{1,0}^{or}$ a *symmetric* monoidal category.

We will use it to give an equivalent definition of k-algebra (see 3.10.1), but in general they can be used to define factorization algebras, a very important tool in the field of TFTs, see [Tan20].

3.8.1 Grothendieck's Homotopy Hypothesis and ∞-categories ✈

Example 3.8.14. We previously spelled out an analogue formulation of the natural transformation (see 3). This shows that the notion of homotopy and natural transformation are related. A way to more rigorously pin down that a homotopy is some kind of natural transformation is via the the homotopy 2-category of topological spaces. Instead of taking paths up to homotopies between paths, as in the more famous hTop, we take homotopies between paths up to homotopies between homotopies between paths. This results in a strict (2,1)-category $\mathrm{h}_2\mathrm{Top}$ where objects are topological spaces, 1-morphisms are continuous maps and 2-morphisms are homotopies between continuous maps up to higher homotopies, i.e. homotopies between homotopies between continuous maps. A homotopy between a homotopy between continuous maps is defined in the following manner: given continuous functions $f, g : X \to Y$ and homotopies between them $k, h : X \times [0,1] \to Y$, a homotopy L between these two homotopies k and h is a continuous function $L : X \times [0,1] \times [0,1] \to Y$ such that

- $\forall t \in [0,1] \forall x \in X. H(0,t,x) = h(t,x)$

- $\forall t \in [0,1] \forall x \in X. H(1,t,x) = k(t,x)$

- $\forall t' \in [0,1] \forall x \in X. H(t',0,x) = f(x)$
- $\forall t' \in [0,1] \forall x \in X. H(t',1,x) = g(x)$

If one takes topological spaces as objects, continuous maps between them as 1-morphisms, and homotopies between continuous maps *up to* homotopies between homotopies one gets the homotopy 2-category of topological spaces $\mathrm{h_2Top}$ and is an example of a strict $(2,1)$-category. If one restricts themselves to spaces admitting the structure of CW-complexes, i.e. something like weakly Hausdorff compactly generated spaces, and then take the homotopy 1-types thereof, we get a subcategory of $\mathrm{h_2Top}$ called 2-1Type. Via the Eilenberg-MacLane space of a groupoid one can establish an equivalence of categories

$$2\text{-}1Type \simeq Grpd$$

This is Grothendieck's infamous homotopy hypothesis in dimension 1. It was formulated in *Pursuing Stacks* in full generality for ∞-groupoids. We now sketch how this works in full glory.

Notation. By ∞-category we mean $(\infty,1)$-category.

Example 3.8.15. Regarding the last example, one could ask why stop at homotopies between homotopies and not go all the way to homotopies between homotopies between homotopies, homotopies between ... for ∞ many times? This is a great question and also the way to provide a universal properties for constructions similar to the already seen cartesian product called homotopy co/limits (for more on this see chapters 1 and 2 of [Har19]). In the end, one would get an $(\infty,1)$-category, a category in which the only morphisms that are not necessarily (weakly[23]) invertible are morphisms between objects, all other morphisms between morphisms are invertible; in other words, a category in which all $n > 1$-morphisms are invertible and 1-morphisms not necessarily.

Moreover, one gets the fundamental ∞-groupoid of a space via a similar reasoning: instead of considering paths up to homotopies between paths, one can go all the way up and consider homotopies between homotopies between... without forgetting any information. All paths and all homotopies are clearly invertible so for all $k \in \mathbb{N}_0$ all k-morphisms are invertible and that is why it is called ∞-*groupoid*. A synonym for ∞-groupoid is $(\infty,0)$-category. Note that the category of all ∞-groupoids is not an ∞-groupoids, because functors between ∞-groupoids are not necessarily invertible, parallelly to functors between 1-groupoids. The previously mentioned homotopy hypothesis states that the $(\infty,1)$-category of (nice) topological spaces we just sketched is equivalent to the $(\infty,1)$-category of ∞-groupoids. The intuition behinds this is that this should hold thanks to the construction of fundamental ∞-groupoids of spaces, whereas, in the other direction, from ∞-groupoids one can construct spaces via something called the geometric realization. This equivalence is the reason why in the literature the category of ∞-groupoids is sometimes called the category of Spaces. If the equivalence holds or not of course depends on how we define 'space' and ∞-groupoid, but it holds for many reasonable definitions, for instance a category of convenient topological spaces like the one of compactly generated weakly Hausdorff spaces.

[23]To be precise, higher morphisms in such a category are weakly invertible and not strictly. Weakly invertible means that it is not the case the sense that $f \circ g = id$, but $f \circ g \simeq id$. This is not at all negative. That things are not strict but up to homotopy/equivalence is exactly what one desires in this context.

Similarly to (2,1)-categories, which are enriched over ordinary groupoids, the sketchy idea behind an $(\infty,1)$-category is being enriched over ∞-groupoids.

More concretely, one way to model this kind of categories with infinitely many invertible morphisms is with topologically enriched categories (which we previously described 3.8.2), since mapping spaces are Hom-objects, 2-morphisms are paths in such mapping spaces and higher morphisms are given by the homotopies between homotopies.

A further way is via some specific simplicially enriched categories, i.e. categories enriched with the cartesian monoidal category of simplicial sets sSet (sometimes denoted Set_Δ). A synonym of simplicially enriched category is simplicial category.

Definition 3.8.16 (The Simplex Category)**.** We call simplex category, the category with all linearly ordered sets $[n]$ as objects and weakly monotonic maps as morphisms. We denote such category with Δ.

Definition 3.8.17 (Simplicial Set)**.** A simplicial set is a Set-valued presheaf on Δ, i.e. a functor $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$. Simplicial sets form a functor category where morphisms are natural transformations denoted sSet. It is bicomplete, see the proof in [Lan21, 1.1.24]

However, we do not want simplicial sets in general, but simplicial sets that are Kan complexes[24]. This is because these special simplicial sets give a geometric model of ∞-groupoids and hence by enriching a category via such simplicial sets we get the desired $(\infty,1)$-category, similarly to getting (2,1)-categories by enriching with ordinary groupoids.

Quasi-categories are another construction to model ∞-categories that has been proven very useful and apt to create useful constructions. It also relies heavily on simplicial sets and there is strong connection to simplicially enriched categories (see [Lan21]).

These three different ways to model ∞-categories are equivalent in some apt sense (see [Jac09] and [Ber06]).

Remark. Another important example of $(\infty,1)$-category is the category of spectra, a category similar[25] to the category of abelian groups which we will encounter later (see 4.1.1).

3.9 Objects Internal to a Monoidal Category

Definition 3.9.1 (Monoid Object)**.** A monoid object in a monoidal category $\mathcal{C}$ is an object $M \in \mathcal{C}$ equipped with two distinguished morphisms, $\mu : M \otimes M \to M$ and $\eta : \mathbb{1}_{\mathcal{C}} \to M$ called respectively multiplication and unit such that the following diagram commutes, so that the binary operation is associative

$$\begin{tikzcd}
(M\otimes M)\otimes M \arrow[r, "\alpha_{M,M,M}"] \arrow[d, "\mu\otimes id_M"'] & M\otimes(M\otimes M) \arrow[r, "id_M\otimes\mu"] & M\otimes M \arrow[d, "\mu"] \\
M\otimes M \arrow[rr, "\mu"'] & & M
\end{tikzcd}$$

and the following also commutes, so that the binary operation is left and right unital

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{C}}\otimes M \arrow[r, "\eta\otimes id_M"] \arrow[dr, "\lambda_M"'] & M\otimes M \arrow[d, dashed, "\mu"] & M\otimes\mathbb{1}_{\mathcal{C}} \arrow[l, "id_M\otimes\eta"'] \arrow[dl, "\rho_M"] \\
& M &
\end{tikzcd}$$

[24]See https://ncatlab.org/nlab/show/Kan+complex.

[25]Abelian Groups:Algebra~Spectra:Higher Algebra, for more on this check out [MG24].

Making multiplication associative and unital.

Example 3.9.2. A monoid object is a

- Monoid when in (Set, $\times$)
- Algebra when in ($\text{Vect}_k, \otimes$)
- Algebra when in ($BMod, \otimes$)
- *Strict* monoidal category when in ($Cat, \times$)
- Topological monoids in (Top, $\times$)

This example clarifies the parallel between monoids, algebras and monoidal categories we made at the start of this section. However, one could be bothered by the fact that we do not actually get a general monoidal category (i.e. not necessarily strict). To do this we will briefly talk about $\mathbb{E}_1$-algebras in section 3.10.1.

Definition 3.9.3 (Morphism of Monoids)**.** Let $M, M' \in \mathcal{C}$ be monoid objects (M, μ, η) and (M', μ', η'). A morphism of monoids is a morphism $f : M \to M'$ such that both the following two diagrams commute

$$\begin{tikzcd} M \otimes M \arrow[r, "f \otimes f"] \arrow[d, "\mu"'] & M' \otimes M' \arrow[d, "\mu'"] \\ M \arrow[r, "f"'] & M' \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1}_{\mathcal{C}} \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & M \arrow[d, "f"] \\ & M' \end{tikzcd}$$

Definition 3.9.4 (Comonoid Object)**.** Let $M \in \mathcal{C}$, M together with a comultiplication Δ and a counit ϵ is a comonoid object iff (M, μ, η) a monoid object in $\mathcal{C}^{op}$ (see 3) such that $\mu = \Delta^{op}$ and $\eta = \epsilon^{op}$. Straightforwardly, the definition we gave for monoid object, but with all the arrows inverted.

Definition 3.9.5 (Bimonoid Object)**.** A bimonoid object in a monoidal category $\mathcal{C}$ is simultaneously a monoid and a comonoid object in $\mathcal{C}$ in a compatible way. It has a unit $\eta : \mathbb{1}_{\mathcal{C}} \to M$ and a counit $\epsilon : \mathcal{C} \to \mathbb{1}_{\mathcal{C}}$, a multiplication $\mu : M \otimes M \to M$ and and a comultiplication $\Delta : M \to M \otimes M$ such that comultiplication and multiplication are morphisms of monoids. In the case of comultiplication, this means that given a monoid object $(M \otimes M, \mu', \eta')$ with $\mu' : (M \otimes M) \otimes (M \otimes M) \to M \otimes M$ and $\eta' : M \otimes M \to \mathbb{1}_{\mathcal{C}}$ both the following diagrams commute

$$\begin{tikzcd} M \otimes M \arrow[r, "\Delta \otimes \Delta"] \arrow[d, "\mu"'] & (M \otimes M) \otimes (M \otimes M) \arrow[d, "\mu'"] \\ M \arrow[r, "\Delta"'] & M \otimes M \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1}_{\mathcal{C}} \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & M \arrow[d, "\Delta"] \\ & M \otimes M \end{tikzcd}$$

We leave the case of the counit for the reader.

Definition 3.9.6 (Frobenius Algebra)**.** A Frobenius algebra in an arbitrary monoidal category $\mathcal{C}$ is simultaneously a monoid and a comonoid object in $\mathcal{C}$ with a compatibility condition different from the one above:

$$(\mathbb{1}_{\mathcal{C}} \otimes \mu) \circ (\Delta \otimes \mathbb{1}_{\mathcal{C}}) = \Delta \circ \mu = (\mu \otimes \mathbb{1}_{\mathcal{C}}) \circ (\mathbb{1}_{\mathcal{C}} \otimes \Delta)$$

called the Frobenius relation.

Definition 3.9.7 (Group Object)**.** A group object in a cartesian monoidal category $\mathcal{C}$ is a monoid object M that also has an inverse map $(-)^{-1} : M \to M$ such that the following diagram commutes, meaning that the inverse behave as expected

$$\begin{tikzcd} M \arrow[r, "(-)^{-1}\times id_M"] \arrow[d, "id_M\times(-)^{-1}"'] \arrow[dr, "id_M"] & M \times M \arrow[d, "\mu"] \\ M \times M \arrow[r, "\mu"'] & M \end{tikzcd}$$

Example 3.9.8.

- A topological group is a group object in Top
- A Lie group is a group object in SmoothMfld
- A group is a group object in Set

3.10 Symmetric Monoidal Categories

In order to get the categorical parallel of a commutative monoid, we need to define a *symmetric* monoidal category. We want to achieve something similar to $a \cdot b = b \cdot a$ in a monoid. However, we will not characterize this behavior with strict identities, but with a natural isomorphism[26]. Given the functor

$$swap : \mathcal{C} \to \mathcal{C}$$

$$swap : (X, Y) \mapsto (Y, X)$$
$$swap : (f, g) \mapsto (g, f)$$

we could try to achieve our objective with a natural isomorphism $\beta : \otimes \to \otimes \circ swap$ that can be visualized in the category of all categories Cat with the following diagram

$$\begin{tikzcd} \mathcal{C}\times\mathcal{C} \arrow[r, bend left, "\otimes"] \arrow[r, bend right, "\otimes\circ swap"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{C} \end{tikzcd}$$

However, this does not characterize an actually symmetric structure, but rather a *braiding*, meaning that inverting two times the order of two tensor multiplied elements, i.e. objects or morphisms, does not necessarily equal the original tensor product. What this means will become clearer with the example of the braid group and the definition of symmetric monoidal category.

Definition 3.10.1. A braiding on a monoidal category is a natural transformation β with components $\beta_{X,Y} : X \otimes Y \Rightarrow Y \otimes X$

$$\begin{tikzcd} \mathcal{C}\times\mathcal{C} \arrow[r, bend left, "-_i\otimes-_j"] \arrow[r, bend right, "-_j\otimes-_i"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{C} \end{tikzcd}$$

[26] Just as we did for associativity and unitality.

In addition, we need to impose some compatibility conditions with the associator:

$$\begin{tikzcd}
& (X \otimes Y) \otimes Z \arrow[dr, "\beta_{X,Y} \otimes id_Z"] & \\
X \otimes (Y \otimes Z) \arrow[ur, "\alpha^{-1}_{X,Y,Z}"] \arrow[d, "\beta_{X,Y\otimes Z}"'] & & (Y \otimes X) \otimes Z \arrow[d, "\alpha_{Y,X,Z}"] \\
(Y \otimes Z) \otimes X \arrow[dr, "\alpha_{Y,Z,X}"'] & & Y \otimes (X \otimes Z) \\
& Y \otimes (Z \otimes X) \arrow[ur, "id_Y \otimes \beta_{Z,X}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& X \otimes (Y \otimes Z) \arrow[dr, "id_X \otimes \beta_{Y,Z}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "\alpha_{X,Y,Z}"] \arrow[d, "\beta_{X\otimes Y,Z}"'] & & X \otimes (Z \otimes Y) \arrow[d, "\alpha^{-1}_{Y,X,Z}"] \\
Z \otimes (X \otimes Y) \arrow[dr, "\alpha^{-1}_{Z,X,Y}"'] & & (X \otimes Z) \otimes Y \\
& (Z \otimes X) \otimes Y \arrow[ur, "\beta_{Z,X} \otimes id_Y"'] &
\end{tikzcd}$$

The latter two diagrams are known as the hexagon diagrams. We call such categories braided monoidal.

A braided monoidal category is *not* the category corresponding to an abelian monoid. A special type of braided monoidal category is: the symmetric monoidal category.

Definition 3.10.2. A symmetric monoidal category is a braided monoidal category $(C, \otimes, 1, \alpha, \lambda, \rho, \beta)$ such that $\beta^2 = id$ (i.e. $\beta_{y,x} \circ \beta_{x,y} = id$).

Example 3.10.3.

- All the previous examples of monoidal categories! (Vect, $\oplus$), (AbGrp, $\otimes$), ...
- Importantly for us the category of bimodules $BMod$.
- The category of algebras over a module.

The following theorem assures us of the associativity of higher products that we had mentioned and also generalizes this result to braided and symmetric categories.

Theorem 3.10.4 (MacLane's coherence theorem)**.** *In any monoidal category, any formal diagram, i.e. a diagram made up just of associators, unitors (and braidings, in the case of braided and symmetric monoidal categories) commutes.*

We do not provide the proof here but refer to chapter 7 of [Lan71]. The analogy in a monoid is that it does not make a difference however I put my parentheses:

$$(a_1 a_2)((a_3 a_4) a_5) = ((a_1(a_2 a_3)) a_4) a_5$$

but instead of an equality we have objects equivalent up to isomorphisms given by associators, unitors, etc... note that we could have more than one way to compose unitors, e.g. in the diagram we previously imposed as conditions, but these will all form a commutative diagram and hence be equivalent.

Earlier on we defined a monoid object (3.9.2) in a monoidal category, now in a symmetric monoidal category we can also define a commutative monoid object.

Definition 3.10.5 (Commutative Monoid Object)**.** A commutative monoid object, is a monoid object in a symmetric monoidal category for which additionally the following diagram commutes

$$\begin{tikzcd} M \otimes M \arrow[rr, "\beta_{M,M}"] \arrow[dr, "\mu"'] & & M \otimes M \arrow[dl, "\mu"] \\ & M & \end{tikzcd}$$

Example 3.10.6. A commutative monoid object is a

- commutative monoid when in $(\mathrm{Set}, \times)$,
- commutative algebra when in $(\mathrm{Vect}_k, \otimes)$,
- *strict* symmetric monodial category when in $(Cat, \times)$,
- commutative topological monoids in $(\mathrm{Top}, \times)$.

Definition 3.10.7 (Cocommutative Comonoid Object)**.** Let (M, Δ, ϵ) be a comonoid object (3.9.4) in a symmetric monoidal category $\mathcal{C}$. It is cocommutative iff $(M, \mu, \eta, \beta_{M,M})$ a commutative monoid object in $\mathcal{C}^{op}$ (see 3) such that $\mu = \Delta^{op}$, $\eta = \epsilon^{op}$ and $\beta^{op}_{M,M} = \beta_{M,M}$. Straightforwardly, the definition we gave for commutative monoid object, but with all the arrows inverted.

Definition 3.10.8 (Commutative Bimonoid Object)**.** A commutative bimonoid object in a symmetric monoidal category $\mathcal{C}$ is a bimonoid object (3.9.5) such that the underlying monoid object is commutative. Note that the underlying comonoid object is cocommutative if and only if the underlying monoid object is commutative.

Definition 3.10.9 (Commutative Frobenius Algebra)**.** A commutative Frobenius algebra in a symmetric monoidal category $\mathcal{C}$ is a Frobenius algebra whose monoid structures is commutative (this also implies that the comonoid structure is cocommutative).

We need a notion of homomorphism between symmetric monoidal categories, a suitable definition of functor.

Definition 3.10.10 (Symmetric monoidal functor)**.** Let $\mathcal{B}, \mathcal{C}$ be symmetric monoidal categories. A symmetric monoidal functor is a functor $F : \mathcal{B} \to \mathcal{C}$ compatible with all of the structure:

- an isomorphism taking the monoidal unit in $\mathcal{C}$ to the monoidal unit in $\mathcal{B}$: $\phi : \mathbb{1}_{\mathcal{C}} \xrightarrow{1_F} F(\mathbb{1}_{\mathcal{B}})$
- a natural isomorphism respecting the tensor product:

$$\begin{tikzcd} \mathcal{B} \times \mathcal{B} \arrow[r, bend left=50, "F(-\otimes-)", ""{name=U, below}] \arrow[r, bend right=50, "F(-)\otimes F(-)"', ""{name=D, above}] & \mathcal{C} \arrow[Rightarrow, from=D, to=U, "\psi"] \end{tikzcd}$$

such that it interacts reasonably with the associator by making the following diagram commute for every $X, Y, Z \in \mathcal{C}$

$$\begin{tikzcd}
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[r, "\psi_{X,Y} \otimes id_{F(Z)}"] \arrow[d, "\alpha_{F(X),F(Y),F(Z)}"'] & F(X \otimes Y) \otimes F(Z) \arrow[r, "\psi_{X \otimes Y, Z}"] & F((X \otimes Y) \otimes Z) \arrow[d, "F(\alpha_{X,Y,Z})"] \\
F(X) \otimes (F(Y) \otimes F(Z)) \arrow[r, "id_{F(X)} \otimes \psi_{Y,Z}"'] & F(X) \otimes F(Y \otimes Z) \arrow[r, "\psi_{X, Y \otimes Z}"'] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

it interacts well with the unit[27] by making the following diagram also commute for every $X \in \mathcal{C}$

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{C}} \otimes F(X) \arrow[r, "\lambda_{F(X)}"] \arrow[d, "\phi \otimes id_{F(X)}"'] & F(X) \\
F(\mathbb{1}_{\mathcal{B}}) \otimes F(X) \arrow[r, "\psi_{\mathbb{1},X}"'] & F(\mathbb{1}_{\mathcal{B}} \otimes X) \arrow[u, "F(\lambda_X)"']
\end{tikzcd}$$

finally we just need a commutative diagram for all $X, Y \in \mathcal{C}$ specifying how it interacts with the braiding

$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "\beta_{F(X),F(Y)}"] \arrow[d, "\psi_{X,Y}"'] & F(Y) \otimes F(X) \arrow[d, "\psi_{Y,X}"] \\
F(X \otimes Y) \arrow[r, "F(\beta_{X,Y})"'] & F(Y \otimes X)
\end{tikzcd}$$

This notion will be central in this course since a TFT is just a symmetric monoidal functor with a special domain: the cobordism category.

Example 3.10.11.

1. The path-connected component functor (1) $\pi_0 : (\text{Top}, \times) \to (\text{Set}, \times)$ is a symmetric monoidal functor.

2. ...

We also need a suitable notion of natural transformation between symmetric monoidal functors.

Definition 3.10.12 ((Symmetric) Monoidal Natural Transformation)**.** Given monoidal functors (F, ψ, ϕ) and (G, ξ, γ) from monoidal categories $\mathcal{C}$ to $\mathcal{D}$, a natural transformation $\eta : F \Rightarrow G$ is monoidal if and only if the two following diagrams commute

$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "\eta_X \otimes \eta_Y"] \arrow[d, "\psi_{X,Y}"'] & G(X) \otimes G(Y) \arrow[d, "\xi_{X,Y}"] \\
F(X \otimes Y) \arrow[r, "\eta_{X \otimes Y}"'] & G(X \otimes Y)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{D}} \arrow[d, "\phi"'] \arrow[dr, "\gamma"] & \\
F(\mathbb{1}_{\mathcal{C}}) \arrow[r, "\eta_{\mathbb{1}_{\mathcal{C}}}"'] & G(\mathbb{1}_{\mathcal{C}})
\end{tikzcd}$$

[27] We just need the diagram for one unit since in a symmetric monoidal category $\lambda = \rho$. We chose arbitrarily to give the diagram for the left unit

No other conditions are needed to be specified for a symmetric monoidal natural transformation, a monoidal natural transformation between symmetric monoidal functors.

Example 3.10.13.

- Let $\mathbb{Z}[-] : \text{Set} \to \text{AbGrp}$ be the free functor[28] generating free groups from sets. Define $\mathbb{Z}[- \times -] : \text{Set} \to \text{AbGrp}$, the free functor generating free groups out of cartesian products in Set, and $\mathbb{Z}[-] \otimes \mathbb{Z}[-]$, the free functor first generating free groups and then tensor-multiply them with the tensor product in AbGrp. There is a monoidal natural isomorphism α between them, $\forall A, B \in \text{Set}, \mathbb{Z}[A \times B] \cong \mathbb{Z}[A] \otimes \mathbb{Z}[B]$. The notation $\mathbb{Z}[-]$ comes from the fact that abelian groups are exactly modules on the integers, the categories of abelian groups and of modules on the integers are not just equivalent categories, but isomorphic, something very rare.
- Let $V \in \text{Vect}_k$. Then $V \to V^{**}$ are the components of...
- The determinant...
- There is a monoidal natural isomorphism between id_{Grp} and $\text{Grp} \xrightarrow{op} \text{Grp}, G \mapsto G^{op}$, the functor sending every group $(G, *)$ to its opposite group, i.e. $(G^{op}, *^{op})$, where $a *^{op} b = b * a$.

Remark. One way in which (symmetric) monoidal natural transfomations are important is that they let us define a suitable notion of equivalence between (symmetric) monoidal categories. This will be fundamental in a forthcoming section (5).

Definition 3.10.14 (Monoidal Equivalence)**.** There is a (symmetric) monoidal equivalence between (symmetric) monoidal categories $\mathcal{C}$ and $\mathcal{D}$ if and only if there are (symmetric) monoidal functors $F, G : \mathcal{D} \to \mathcal{C}$ and (symmetric) monoidal natural isomorphisms $G \circ F \overset{\epsilon}{\cong} id_{\mathcal{C}}$ and $F \circ G \overset{\eta}{\cong} id_{\mathcal{D}}$.

Remark. There is a strict 2-category SymmMonCat similar to Cat (see section 3.6), where 1-morphisms are symmetric monoidal functors and 2-morphisms are symmetric monoidal natural transformations.

Notation. Note that then $\text{Hom}_{\text{SymmMonCat}}(\mathcal{C}, \mathcal{D})$ is a special functor category. A functor category where objects are symmetric monoidal functors. An alternative way to denote $\text{Hom}_{\text{SymmMonCat}}(\mathcal{C}, \mathcal{D})$ is $\text{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$.

We have previously seen (2) that a linear representation of a group G is just a functor $\mathbf{B}G \to \text{Vect}_k$. More generally, some, e.g. [Koc03, p. 34], give a definition of linear representation for symmetric monoidal categories:

Definition 3.10.15 (Linear Representation)**.** Let $\mathcal{C}$ be a symmetric monoidal category. A linear representation of such category is a symmetric monoidal functor

$$\mathcal{C} \to \text{Vect}_k$$

Such functors are the objects of the category of representation of $\mathcal{C}$ whereas symmetric monoidal natural transformations are the morphisms,

$$\text{Rep}(\mathcal{C}) = \text{Fun}^{\otimes}(\mathcal{C}, \text{Vect}_k) \tag{3.7}$$

[28]A free functor is a left adjoint to the forgetful functor.

3.10.1 Outlook towards $\mathbb{E}_n$-Algebras ✈

This outlook is based on [Tan22], [MG24] and [Max21]. It has two motivations:

1. understand how we can define (symmetric/braided) monoidal categories internally to Cat
2. provide the necessary machinery to sensibly talk about spectra and $\mathbb{E}_n$ spaces in 4.1.1

A monoid object in Cat[29] is a strict monoidal category. Take a look at associativity for instance: a monoid object $\mathcal{C} \in$ Cat makes the following diagram commute strictly

$$\begin{tikzcd}
(\mathcal{C} \times \mathcal{C}) \times \mathcal{C} \arrow[r, "\alpha_{\mathcal{C},\mathcal{C},\mathcal{C}}"] \arrow[d, "\mu \times id_M"'] & \mathcal{C} \times (\mathcal{C} \times \mathcal{C}) \arrow[r, "id_{\mathcal{C}} \times \mu"] & \mathcal{C} \times \mathcal{C} \arrow[d, "\mu"] \\
\mathcal{C} \times \mathcal{C} \arrow[rr, "\mu"'] & & \mathcal{C}
\end{tikzcd}$$

This means that if we take $X, Y, Z \in \mathcal{C}$, then $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$, but the tensor product in general monoidal categories is associative and unital via some specific natural isomorphisms, the associator and the two unitors! So, this definition of monoid object does not work well with our initial comparison between monoids, algebras and monoidal categories. The issue is that the definition of monoid object is tailored for 1-categories, and thus cannot regard morphisms higher than 1 and makes everything commute strictly. There is however a way to have the **morally right** notion of monoid object, meaning a monoid object that yields monoidal categories in the case of cats and more generally monoids that cohere via higher morphisms (including the ones higher than 2, given that the category in question has them). The idea is to exploit the homotopical information of the category of n-disks[30], i.e. the isotopies between embeddings from disjoint unions of $\mathbb{R}^n$, and import it to the monoidal category we want our monoid object to be defined in via a symmetric monoidal functor. This notion is called $\mathbb{E}_1$-algebra for monoid objects, and $\mathbb{E}_\infty$-algebra for commutative ones. We can summarize the idea with the following mottos

$\mathbb{E}_1$*-algebras are homtopy coherently associative and unital monoid objects*

$\mathbb{E}_\infty$*-algebras are homotopy coherently commutative, associative and unital monoid objects*

And generally:

$\mathbb{E}_n$*-algebras are homotopy-coherent monoid objects*
that are commutative via homotopies up to the n-th order

Now we try to sketch what being homotopy coherent means by means of an example: homotopy coherent associativity. Let $\mathscr{A}$ be an $\mathbb{E}_1$-algebra in an arbitrary symmetric monoidal ∞-category $\mathcal{C}$, let $W, X, Y, Z \in \mathscr{A}$ and let us denote the multiplication of such algebra as $\cdot : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$. Then,

- there are two ways one can bracket three elements, for any $X, Y, Z \in \mathscr{A}$ there is a 2-associator[31] $\alpha : X \cdot (Y \cdot Z) \simeq (X \cdot Y) \cdot Z$ instead of the usual strict equality $X \cdot (Y \cdot Z) \simeq (X \cdot Y) \cdot Z$

[29] Recall that the monoidal structure on Cat is given by the product of categories, i.e. a cartesian product in Cat.

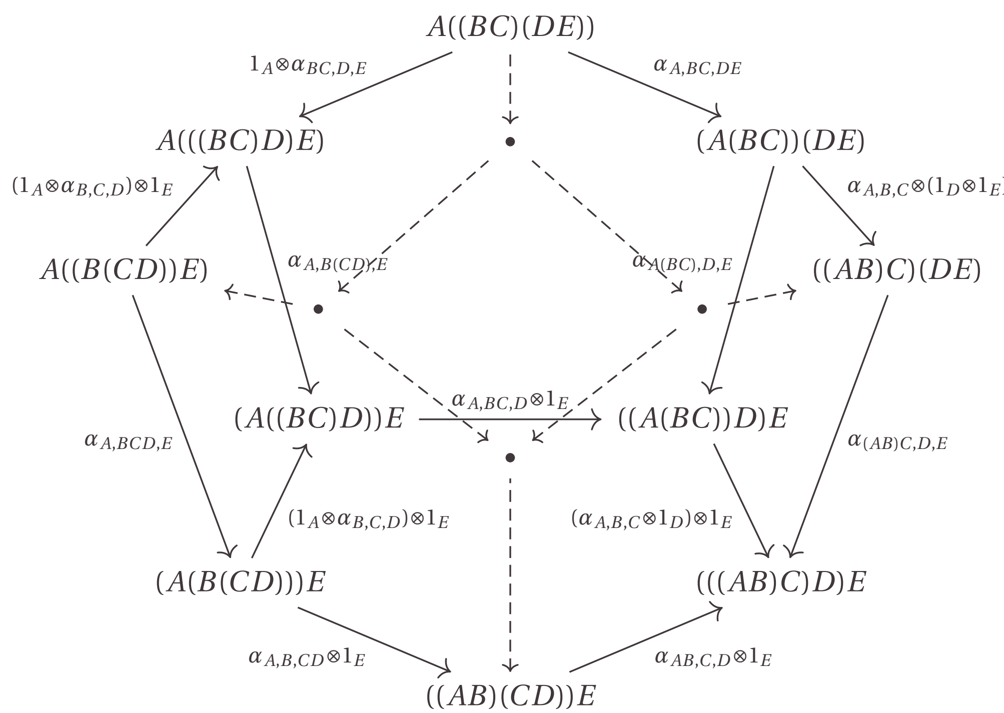
[30] Using disks is not the only way to do it, but it is arguably the simplest. Historically, it is done via things called operads.

[31] We say 'n-associators' to make explicit that they are n-morphisms.

- there are five ways one can bracket 4 elements, for any $W, X, Y, Z \in \mathscr{A}$ there is a 3-associator between two compositions of 2-associators making the following pentagon cohere

$$\begin{tikzcd}
& (X\cdot Y)\cdot(Z\cdot W) \arrow[dr] & \\
((X\cdot Y)\cdot Z)\cdot W \arrow[ur] \arrow[d] & \Uparrow & X\cdot(Y\cdot(Z\cdot W)) \\
(X\cdot(Y\cdot Z))\cdot W \arrow[rr] & & X\cdot((Y\cdot Z)\cdot W) \arrow[u]
\end{tikzcd}$$

- there are 14 ways to bracket 5 elements, for any $A, B, C, D, E \in \mathscr{A}$ there is a 4-associator between the two possible compositions of 3-associators. The next diagram is stolen from Luke Trujillo's undergraduate thesis ([Luk20]). It depicts from one point of view, the 2-associators and the objects of the coherence diagram for 3-associators that commutes via the 4-associator. Here the 3-associators should be arrows on the faces of this polyhedron[32] and the 4-associator should be a 4-morphism inside the polyhedron going from one possible composition of 3-associators to the other.



- the story then goes on and on to ∞-associators

Example 3.10.16. Like a one object 1-category can be seen as a monoid, a one object ∞-category can be seen as a homotopy coherent monoid $(\mathscr{A}, cdot, e)$, where

- 1-Mor $= \mathscr{A}$
- $\cdot = \circ$ where $\circ$ refers to composition of 1-morphisms

[32]Such polyhedra are called associahedra.

- $e = id_*$
- the higher homotopies making everything homotopy coherently associative and unital coincide

We recall now what an isotopy is since this will be the information in the source category of disks that we will stamp onto the category where the desired monoid object lives.

Definition 3.10.17 (Isotopy)**.** An isotopy is a stricter form of homotopy. Let $f, g : X \to Y$ be embeddings. Then $H : X \times [0,1] \to Y$ is an isotopy if

- $H(0,-) = f$
- $H(1,-) = g$

thereby making an isotopy a homotopy and

- $\forall t \in [0,1], H(t,-) \hookrightarrow Y$ is an embedding.

Moreover, if f, g are *smooth* embeddings and $\forall t \in [0,1], H(t,-) \hookrightarrow Y$ is also a smooth embedding, then H is a *smooth* isotopy.

Definition 3.10.18 ($\mathbb{E}_1$-Algebra)**.** Let $\mathrm{Disk}^{or}_{1,0}$ be the category of 1-disks (see 3.8.13). Then a $\mathbb{E}_1$-algebra is a symmetric monoidal functor

$$\mathrm{Disk}^{or}_1 \to \mathcal{C}$$

such that if $j, i \in \mathrm{Hom}_{\mathrm{Disk}^{or}_1}(X, Y)$ are smoothly isotopic embeddings, then $F(j) = F(i)$, i.e. they are sent to the same morphism in $\mathcal{C}$. More generally.

Notation. A synonym for $\mathbb{E}_1$-algebra is A_∞-algebra.

Theorem 3.10.19. *$\mathbb{E}_1$-algebras with* Vect_k *as a target category correspond to k-algebras. Look into [Tan20] for a sketch of the proof.*

Theorem 3.10.20. *$\mathbb{E}_1$-algebras with* Cat *as a target category are monoidal categories.*

Proof. □

We summarize the correspondence with the following table

	Set	Vect_k	Cat
$\mathbb{E}_1$-algebra	monoid	k-algebra	monoidal category

Remark. Remember that a monoid object in Cat is a *strict* monoidal category. Via $\mathbb{E}_1$-algebras is instead the way to precisely pin down the correspondence we laid down at the start of the section between monoidal categories, k-algebras and monoids, without collapsing monoidal categories to *strict* monoidal category, as we did with the monoid object.

We now start to define the necessary tools we need to define monoid objects that are (more) commutative.

Definition 3.10.21 (Category of n-disks)**.** Let $\mathrm{Disk}_n^{\mathrm{fr}}$ be the ∞-category with finite disjoint unions of framed n-dimensional disks as objects, i.e. $(\mathbb{R}^n)^{\amalg i}$ for $i \geq 0$, and spaces of framed smooth embeddings (with the compact-open topology) as Hom-objects, e.g. $\mathrm{Hom}_{\mathrm{Disk}_n^{\mathrm{fr}}}(\mathbb{R}^n \amalg \mathbb{R}^n, \mathbb{R}^n)$.

Definition 3.10.22 ($\mathbb{E}_n$ algebra)**.** Fix a symmetric monoidal ∞-category $\mathcal{C}$. For any $n \geq 1$, an $\mathbb{E}_n$-algebra in $\mathcal{C}$ is a symmetric monoidal functor

$$\mathrm{Disk}_n^{\mathrm{fr}} \xrightarrow{A} \mathcal{C}$$

This is actually an algebra because it induces a parametrized family of i-ary multiplications

$$\mathrm{Hom}_{\mathrm{Disk}_n^{\mathrm{fr}}}((\mathbb{R}^n)^{\amalg i}, \mathbb{R}^n) \xrightarrow{A} \mathrm{Hom}_{\mathcal{C}}(A(\mathbb{R}^n)^{\otimes i}, A(\mathbb{R}^n))$$

Which do not compose strictly! But up to the higher isotopies of the mappig space in $\mathrm{Disk}_n^{\mathrm{fr}}$!

In short, such $\mathbb{E}_n$-algebras allow us to define objects internal to categories up to coherent homotopies.

$\mathbb{E}_n$-algebras in a symmetric monoidal ∞-category $\mathcal{C}$ form an ∞-category

$$\mathrm{Alg}_{\mathbb{E}_n} := \mathrm{Fun}^{\otimes}(\mathrm{Disk}_n^{\mathrm{fr}}, \mathcal{C})$$

Similarly to how monoid objects internal to a category $\mathcal{C}$ form a category $\mathrm{Mon}(\mathcal{C})$, e.g. $\mathrm{Mon}(\mathrm{Set}) = \mathrm{Mon}$ (the usual category of monoids).

	Set	Vect_k	Cat
$\mathbb{E}_1$-algebra	monoid	k-algebra	monoidal category
$\mathbb{E}_2$-algebra	commutative monoid	commutative k-algebra	braided monoidal category
$\mathbb{E}_3$-algebra	commutative monoid	commutative k-algebra	symmetric monoidal category
$\mathbb{E}_4$-algebra	commutative monoid	commutative k-algebra	symmetric monoidal category
...	...	...	...
$\mathbb{E}_\infty$-algebra	commutative monoid	commutative k-algebra	symmetric monoidal category

Set and Vect_k cannot detect the difference between $\mathbb{E}_n$-algebras for $n \geq 2$ while Cat can go one step further. This is because Cat is a 2-category, and so has the necessary homotopies in order to express more fine grained notions of commutativity like braided and symmetric.

However, some categories have more homotopical information and hence are even more refined. This is one of the reasons they are the categories in which higher algebra takes place.

	∞-Grpd	∞-Cat	Sp
$\mathbb{E}_1$-algebra	$\mathbb{E}_1$-space	monoidal ∞-category	$\mathbb{E}_1$-ring spectrum
$\mathbb{E}_\infty$-algebra	$\mathbb{E}_\infty$-space	symmetric monoidal ∞-category	$\mathbb{E}_\infty$-ring spectrum

Notation. Sp denotes the category of spectra, see 4.1.1 for a definition.

Notation. Under the homotopy hypothesis ∞-Grpd corresponds to a nice category of spaces, hence it is sometimes denoted as $\mathbb{S}$ for $\mathbb{S}$paces. Other synonyms nowadays are:

- ∞-Set, because it is the category one enriches over to get ∞-categories, similarly to ordinary sets and locally small categories, and it plays a parallel role in higher algebra to set in ordinary algebra (see the table in 4.1.1)
- An, standing for anima, terminology coming from condensed mathematics

Notation. Synonyms for $\mathbb{E}_1$-space are A_∞-space and monoidal ∞-groupoid. A synonym for $\mathbb{E}_\infty$-space is symmetric monoidal ∞-groupoid.

Example 3.10.23. ΩX is an $\mathbb{E}_1$-space: composition of loops induces an operation which is not strictly unital or associative, but only via homotopies. See 4.1.1 for a definition of loop spaces.

Theorem 3.10.24 (Dunn additivity theorem)**.** *The Dunn additivity theorem is a higher categorical refinement of the Eckmann-Hilton argument. The Eckmann-Hilton argument states that if there are two monoid multiplications on the same set, then they coincide and the operation is commutative. However, in categories with morphisms higher than 1, we do not just have commutativity or non-commutativity of monoid objects but an infinite sequence of shades of commutativity. The Dunn additivity theorem states that for* $0 < n, m \leq \infty$

$$\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Alg}_{\mathbb{E}_m}(\mathcal{C})) \simeq \mathrm{Alg}_{\mathbb{E}_{m+n}}(\mathcal{C})$$

See [Jac17, 5.1.2.2] for a proof.

Example 3.10.25. By the Dunn additivity argument, n-iterated loop spaces are $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S})$. In particular, loop spaces which are iterated infinitely many times are $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{S})$. Such ∞-iterated loop spaces are called infinite loop spaces, see 4.1.21 for a definition.

Later we will state a result that shows that grouplike $\mathbb{E}_n$-spaces and n-iterated loop spaces coincide: not only every n-iterated loop space is an $\mathbb{E}_n$-space, but also viceversa, every $\mathbb{E}_n$-space is an n-iterated loop space. See 4.1.26.

As one can expect, there is a way to define homotopy coherent group objects in a cartesian monoidal ∞-category, as there is a way to define group objects in a cartesian monoidal 1-category. Nonetheless, we concentrate solely on such homotopy coherent group objects in the category of spaces because it is what we will later need in 4.1.1 to prove that invertible field theories are maps between certain spectra. See [Jac17, 5.2.6.6][33] for a definition of homotopy coherent group objects in full generality, i.e. in any cartesian monoidal ∞-category.

Definition 3.10.26 (Grouplike $\mathbb{E}_n$-space)**.** An $\mathbb{E}_n$-algebra in the category of spaces $A \in \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S})$, i.e. an $\mathbb{E}_n$-space, is grouplike, if it lands in the category of groups $\mathrm{Grp} \subset \mathrm{Mon}$ when sent via the composition of

- the functor $\pi_0 : \mathcal{C} \to \mathrm{Set}$, sending objects in groupoids to their isomorphism classes[34]
- and the one forgetting all the commutative structure, but just remembering the homotopy-coherent associativity[35]

[33]Note that there is a typo, instead of 'Let $\mathcal{C}$ be an ∞-category which admits finite products.' there should be 'Let $\mathcal{X}$ be an ∞-category which admits finite products.'.

[34]The notation is no coincidence: under the homotopy hypothesis this functor in the category of spaces and the one sending points to their path-connected components are the same.

[35]Similarly to a forgetful functor from the category of commutative monoids to the category of monoids.

such composition of morphisms can be visualized with the following diagram

$$\begin{tikzcd} \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S}) \arrow[rr] \arrow[dr, "\mathrm{Alg}_{\mathbb{E}_n}(\pi_0)"'] & & \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Set}) \simeq \mathrm{Mon} \\ & \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Set}) \arrow[ur, "\mathrm{forget}"'] & \end{tikzcd}$$

One may denote the subcategory of grouplike $\mathbb{E}_n$-spaces with $\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_n}(\mathcal{S}) \subset \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S})$

Example 3.10.27. ΩX is not just an $\mathbb{E}_1$-space, but also a grouplike $\mathbb{E}_1$-space since the inversion of based loops is homotopy equivalent to the costant map on the point. Moreover, by the Dunn additivity theorem we just stated infinite loop spaces are grouplike $\mathbb{E}_\infty$-spaces

Remark. Grouplike $\mathbb{E}_\infty$-spaces in the category of spaces are Picard ∞-groupoids! The intuition is that the $\mathbb{E}_\infty$ provides the symmetric monoidal structure and being grouplike provides the invertibility objects.

Notation. Sometimes grouplike $\mathbb{E}_\infty$-spaces are called abelian ∞-groups because:

- ∞-groupoids are also known as ∞-sets
- group objects in Set are groups
- group objects (in the morally right sense, i.e. grouplike $\mathbb{E}_1$-algebras) in $\infty-$Set, aka ∞-Grpd, are monoidal groupoids with invertible objects, which are usually called ∞-groups
- abelian groups are groups where the operation is commutative and grouplike $\mathbb{E}_\infty$-spaces are ∞-groups where the operation is homotopy coherently commutative

Such objects are interesting for us because invertible field theories are equivalently maps of such grouplike $\mathbb{E}_\infty$-spaces! See 4.1.1.

3.10.2 Interlude on Bicategories ✈

Definition 3.10.28 (Delooping of a Monoidal Category)**.** Let $\mathcal{C}$ be a monoidal category. The delooping category $\mathbf{B}\mathcal{C}$ is a one object 2-category with:

- $\mathrm{ob}(\mathbf{B}\mathcal{C}) = *$
- $1\text{-}mor(\mathbf{B}\mathcal{C}) = \mathrm{ob}(\mathcal{C})$
- $2\text{-}mor(\mathbf{B}\mathcal{C}) = mor(\mathcal{C})$

Composition of 2-morphisms is given by the usual composition of morphisms between objects in $\mathcal{C}$. The composition of 1-morphisms is given by the tensor product. This makes sense since there is a monoidal unit $\mathbb{1}_{\mathcal{C}}$ that works as the identity arrow for any object, since it is left and right unital for any object and the tensor product is associative. However, in monoidal categories associativity and unitality hold up to isomorphism and this is witnessed by the associator and left/right unitors!

Remark. Notice that the delooping $\mathbf{B}\mathcal{C}$ is generally not a 2-category in the same sense that Cat and SymmMonCat are. 1-morphisms compose strictly in Cat, "on the nose", i.e. $F \circ (G \circ H) = (F \circ G) \circ H$ and $F \circ id_{\mathcal{C}} = F = id_{\mathcal{D}} \circ F$; whereas 1-morphisms in $\mathbf{B}\mathcal{C}$ compose up to natural isomorphism, more precisely up to the associator and left/right unitors.

Definition 3.10.29 (Bicategory)**.** A bicategory $\mathcal{B}$ consists of

- A collection of objects $\mathrm{ob}(\mathcal{B})$, whose elements X, Y, Z are also called 0-cells
- For each objects X, Y, a 1-category $\mathrm{Hom}_{\mathcal{B}}(X, Y)$, whose objects $f, g, \ldots$ are called 1-cells or 1-morphisms and whose morphisms $\alpha, \beta, \ldots$ are called 2-cells or 2-morphisms; composition of (2-)morphisms in this category is also called vertical composition
- For any three objects X, Y, Z there are composition functors, also called horizontal composition
$$\circ_{X,Y,Z} : \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$$
$$(f, g) \mapsto f \circ g$$
$$(\alpha, \beta) \mapsto \alpha \circ \beta$$
in the context of bicategories we denote composition of functors with $\bigcirc$ instead of the usual $\circ$ in order to not get confused between a composition functor and a composition of functors
- For every object X there is an identity functor from the one-object category[36] $\mathbb{1}_{\mathrm{Cat}}$
$$identity_X : \mathbb{1}_{\mathrm{Cat}} \to \mathrm{Hom}(X, X)$$
$$* \mapsto id_X$$
we name the image of the single object the identity 1-morphism, i.e. $identity_X(*) = id_X$
- There are natural isomorphisms a, r, l expressing associativity and unitality of the composition functor:
  - associativity, as a natural isomorphism in Cat
  $$a : \circ_{W,X,Y} \bigcirc (\circ_{X,Y,Z} \times id_{\mathrm{Hom}(W,X)}) \overset{\cong}{\Rightarrow} (id_{\mathrm{Hom}(Y,Z)} \times id_{W,X,Y}) \bigcirc \circ_{W,Y,Z}$$
  given $h : W \to X$, $f : X \to Y$ and $g : Y \to Z$, a is a 2-isomorphism in $\mathcal{B}$
  $$a_{g,f,h} : g \circ (f \circ h) \overset{\cong}{\Rightarrow} (g \circ f) \circ h$$
  - right unitality, as a natural isomorphism in Cat[37]
  $$r : \circ_{X,Y,Y} \bigcirc (id_{\mathrm{Hom}(X,Y)} \times identity_Y) \overset{\cong}{\Rightarrow} \rho^{\mathrm{Cat}}_{\mathrm{Hom}(X,Y)}$$
  and as a 2-isomorphism in $\mathcal{B}$
  $$r_f : f \circ id_Y \overset{\cong}{\Rightarrow} f$$

[36]This is the terminal object of Cat and thereby monoidal unit of the cartesian monoidal category $(\mathrm{Cat}, \times, \mathbb{1}_{\mathrm{Cat}})$ with the cartesian product as the tensor product. With $\mathrm{ob}(\mathbb{1}_{\mathrm{Cat}}) = \{*\}$ and $\mathrm{Hom}_{\mathrm{Cat}}(*, *) = id_*$.

[37]Note that $\rho^{\mathrm{Cat}}_{\mathrm{Hom}(X,Y)}$ is the component of the right unitor of the monoidal category $(\mathrm{Cat}, \times, \mathbb{1}_{\mathrm{Cat}})$ at $\mathrm{Hom}(X, Y)$.

– left unitality, as a natural isomorphism in Cat[38]

$$l : \circ_{X,X,Y} \bigcirc (identity_X \times id_{\mathrm{Hom}(X,Y)}) \overset{\cong}{\Rightarrow} \lambda^{\mathrm{Cat}}_{\mathrm{Hom}(X,Y)}$$

and as a 2-isomorphism in $\mathcal{B}$

$$l_f : id_X \circ f \overset{\cong}{\Rightarrow} f$$

Such natural isomorphisms can be more clearly visualized with the following diagrams[39]

$$\begin{tikzcd}
\mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \times \mathrm{Hom}(W,X) \arrow[r, "id_{\mathrm{Hom}(Y,Z)} \times id_{W,X,Y}"] \arrow[d, dashed, "\circ_{X,Y,Z} \times id_{\mathrm{Hom}(W,X)}"'] & \mathrm{Hom}(Y,Z) \times \mathrm{Hom}(W,Y) \arrow[d, "\circ_{W,Y,Z}"] \\
\mathrm{Hom}(X,Z) \times \mathrm{Hom}(W,X) \arrow[r, "\circ_{W,X,Z}"'] \arrow[ur, phantom, "\Rightarrow a_{W,X,Y,Z}"] & \mathrm{Hom}(W,Z)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}(X,Y) \times \mathbb{1}_{\mathrm{Cat}} \arrow[d, "id_{\mathrm{Hom}(X,Y)} \times identity_Y"'] \arrow[dr, "\cong"] & \\
\mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Y) \arrow[r, "\circ_{X,Y,Y}"'] \arrow[ur, phantom, "\Rightarrow r_{X,Y}"] & \mathrm{Hom}(X,Y)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1}_{\mathrm{Cat}} \times \mathrm{Hom}(X,Y) \arrow[d, "identity_X \times id_{\mathrm{Hom}(X,Y)}"'] \arrow[dr, "\cong"] & \\
\mathrm{Hom}(X,X) \times \mathrm{Hom}(X,Y) \arrow[r, "\circ_{X,X,Y}"'] \arrow[ur, phantom, "\Rightarrow l_{X,Y}"] & \mathrm{Hom}(X,Y)
\end{tikzcd}$$

These data must satisfy the following axioms: an apt version of the Mac Lane's pentagon for the associator and the triangle identity for the unitors must commute so that the associator and left/right unitors behave reasonably (see 3.7 for such diagrams for monoidal categories and see [Sch23] for the spelled out diagrams for bicategories).

Note that in $\mathbb{1}_{\mathrm{Cat}}$ there is also an identity morphism of $*$, id_*, that gets mapped to a 2-morphism id_{id_X}

$$identity_X(id_*) = id_{id_X} : id_X \to id_X$$

The fact that id_* is unital in $*$ ensures that d_{id_X} is unital with respect to vertical composition thanks to the functoriality of $identity_X$.

Bicategories are also known as weak 2-categories. Note that every strict 2-category is a bicategory where 1-morphisms compose strictly instead of up to associatiors and unitors; similarly to monoidal categories and strict monoidal categories

Example 3.10.30. A monoidal category $\mathcal{C}$ is a one-object bicategory $\mathbf{B}\mathcal{C}$ where $\mathcal{C} = \mathrm{Hom}(\mathbf{B}\mathcal{C})$. A strict monoidal category is a one-object strict 2-category.

[38]Note that $\lambda^{\mathrm{Cat}}_{\mathrm{Hom}(X,Y)}$ is the component of the left unitor of the monoidal category $(\mathrm{Cat}, \times, \mathbb{1}_{\mathrm{Cat}})$ at $\mathrm{Hom}(X,Y)$.

[39]Note that these are homotopy coherent diagrams, meaning that they **do not** commute strictly necessarily, but only up to the indicated 2-morphisms!

We now list some examples of 2-categories, to convey a more concrete intuition of the difference between bicategories and strict 2-categories. More generally, try to think how this difference between weak and strict categories carries on in higher dimensions.

Example 3.10.31.

1. Any strict 2-category is a bicategory but not viceversa.
2. Cat and SymmMonCat are both *strict* since functors compose strictly
3. The fundamental 2-groupoid of a topological space X is a *strict* 2-category where objects are the points in X, 1-morphisms are paths and homotopy classes of homotopies between paths are 2-morphisms.
4. Every 1-category is a strict 2-category where the only 2-morphisms are identities on 1-morphisms
5. There is a bicategory Alg_k^2 where objects are algebras over a vector space, 1-morphisms are bimodules (see 3.8.12) and 2-morphisms are maps between bimodules, also known as intertwiners

Chapter 4

The cobordism category

We now get to the most important example of symmetric monoidal category in this course: the symmetric monoidal category of cobordisms[1]. This is what will allow us to define a topological field theory.

Notation. Note that, if not explicitly stated we will mean *smooth* manifold when we write manifold.

Definition 4.0.1 (Bordism Category)**.** The objects are closed $(n-1)$-dimensional manifolds. A morphism from M to N is a *diffeomorphism class of* bordisms from M (*source*) to N (*target*), that is, a compact manifold with boundary Σ together with embeddings

$$\theta_0 : [0,1) \times M \hookrightarrow \Sigma$$

$$\theta_1 : (-1,0] \times N \hookrightarrow \Sigma$$

with a partition $p : \partial\Sigma \to 0,1$ such that

$$\theta_0(0, M) = (\partial\Sigma)_0 := p^{-1}(0)$$

$$\theta_1(0, N) = (\partial\Sigma)_1 := p^{-1}(1)$$

Composition is given by gluing[2] bordisms together along a common boundary for Σ from M to $N1$ and Σ' from N to P we have $\Sigma' \circ \Sigma = \Sigma \cup_N \Sigma'$. The gluing of manifolds is associative. In order for this to actually be a category there must be unital identity morphisms. For an object M it is given by $M \times [0,1]$. However, one could also take $M \times [0,2]$ which is diffeomorphic to $M \times [0,1]$ but *not* strictly identical, on the nose. That is why we took a *diffeomorphism class* of bordisms. In particular, the identity morphisms for any closed manifold $X \in \mathrm{Bord}_{\text{n,n-1}}$, $X \xrightarrow{id_X} X$ is given by $[X \times [0,1]]_{\cong}$ where $\cong$ denotes isomorphisms between smooth manifolds, i.e. diffeomorphisms.

Notation. $\mathrm{nCob} = \mathrm{Bord}_{\text{n,n-1}}$, meaning that objects are closed $n-1$-dimensional manifolds and morphisms are diffeomorphism classes of bordisms. Moreover, we indicate that the closed manifolds are oriented and the bordisms orientation preserving with $\mathrm{Bord}^{\text{or}}_{\text{n,n-1}}$ or equivalently $\mathrm{nCob}^{\text{or}}$.

Let us explicitly state what is meant by diffeomorphism of bordisms:

[1] See chapter 14 of [Fre13].

[2] We define bordisms with the definition with collars because it is easier to see how to glue them.

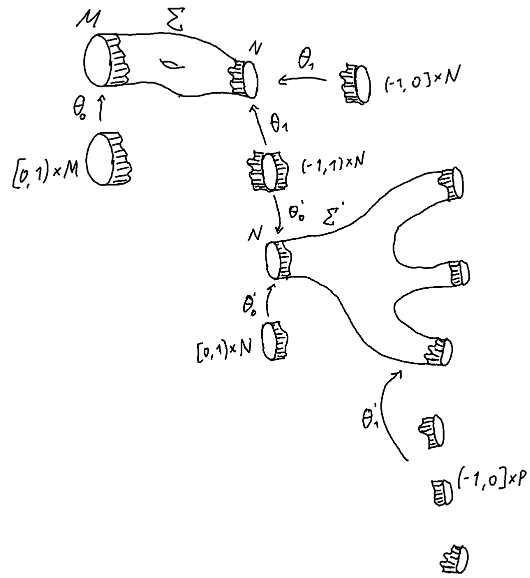


Figure 4.1: This is an illustration of gluing of two bordisms along their common boundary.

Definition 4.0.2. A diffeomorphism of bordisms from M to N, $(\Sigma, p, \theta_0, \theta_1) \to (\Sigma', p', \theta_0', \theta_1')$ is a diffeomorphism $\phi : \Sigma \to \Sigma'$ of manifolds with boundary

$$\begin{tikzcd}
& M \arrow[dl, hook', "\theta(0,-)"'] \arrow[dr, hook, "\theta'(0,-)"] & \\
\Sigma \arrow[rr, "\cong", "\phi"'] & & \Sigma' \\
& N \arrow[ul, hook, "\theta(-,1)"] \arrow[ur, hook', "\theta'(-,1)"'] &
\end{tikzcd}$$

The commutativity of the latter diagram also implies that also extra data commute accordingly, e.g. with the partitions:

$$\begin{tikzcd}
\partial\Sigma \arrow[rr, "\phi|_{\partial\Sigma}"] \arrow[dr, "p"'] & & \partial\Sigma' \arrow[dl, "p'"] \\
& \{0,1\} &
\end{tikzcd}$$

Example 4.0.3. We have that

$$\mathrm{Hom}_{\mathrm{nCob}}(\emptyset, \emptyset) = \{\text{closed } n \text{ manifolds}\}/\{\text{diffeomorphism}\}.$$

For $n = 2$, $\Sigma = S^2$ is a composition of

while a genus g surface is the composite:

which is very reminiscent of what we did in Morse theory. There we were actually also learning how the composition of bordisms works, this is called handle decomposition.

We just saw that $\mathrm{Bord}_{\mathrm{n,n\text{-}1}}$ is a category. It has importantly also a symmetric monoidal structure:

- (M) The tensor product is given by the disjoint union of manifolds, $\otimes = \amalg$, both on objects ($M \otimes N := M \amalg N$) and on morphisms ($\Sigma \otimes \Sigma' := \Sigma \amalg \Sigma'$). Disjoint union is functorial since for $\Sigma : M \to N$, $\Sigma' : M' \to N'$, $\Lambda : N \to P$, $\Lambda' : N' \to P'$ in $\mathrm{Bord}_{\mathrm{n,n\text{-}1}}$ it holds that:

- $(\Sigma \cup_N \Lambda) \amalg (\Sigma' \cup_{N'} \Lambda') = (\Sigma \amalg \Sigma') \cup_{N \amalg N'} (\Lambda \amalg \Lambda')$
- $(M \xrightarrow{M\times[0,1]} M) \amalg (M' \xrightarrow{M'\times[0,1]} M') = M \amalg M' \xrightarrow{(M\times[0,1])\amalg(M'\times[0,1])} M \amalg M'$

- (O) $\mathbb{1}_{\mathrm{Bord_{n,n-1}}} = \emptyset$ (note that $\emptyset \amalg M \neq M$, so $\mathrm{Bord_{n,n-1}}$ is not a *strict* symmetric monoidal category[3]. However, there is an isomorphism in the category since they're clearly diffeomorphic, and any diffeomorphic manifolds are cobordant. So, $\emptyset \amalg M \cong M$ in $\mathrm{Bord_{n,n-1}}$, as we expect the monoidal unit to behave.

- (A) $M_1 \amalg (M_2 \amalg M_3) \xrightarrow{\cong} (M_1 \amalg M_2) \amalg M_3$

- (B) $\beta_{M,N} : M \amalg N \xrightarrow{\cong} N \amalg M$ and $\beta^2 = id$

Remark. Associativity and commutativity also hold up to diffeomorphism and not strictly if one defines the disjoint union as a coproduct[4]: every coproduct is unique up to isomorphism[5], the appropriate notion of isomorphism in the category of smooth manifolds is diffeomorphism and all diffeomorphic manifolds are cobordant.

4.1 Definition of topological field theories

Recall that a group homomorphism $\Omega_n \to X$ with X also an abelian group is a bordism invariant, which is generally quite useful and interesting. Now that we have refined our tools and have a cobordism *category* instead of the cobordism *group*, the bordism invariants become symmetric monoidal functors into another symmetric monoidal category since symmetric monoidal functors are the appropriate notion of homomorphism of symmetric monoidal categories.

In particular, that's how we get a topological field theory:

Definition 4.1.1. A topological field theory is a symmetric monoidal functor[6] with source nCob into a symmetric monoidal category $\mathcal{C}$

$$\mathcal{Z} : \mathrm{nCob} \to \mathcal{C}$$

Exercise 4.1.2.

- Compare this definition to other definitions of quantum field theory.

- Skim through Atiyah's original paper [MF88], how does this definition compare to his set of axioms?

After skimming through [MF88] one might wonder why we defined the target category of a topological field to be just a general symmetric monoidal category and not specifically $(\mathrm{Vect}_k, \otimes, k)$, as Atiyah did, or at least something similar, in the end we are studying topological quantum field theories! There are probably several reasons why someone would want to do so, for example for representation theoretic ones[7]. One of the crucial ones in

[3] Note that objects were *not* taken up to diffeomorphism, only morphisms.

[4] A coproduct is the dual notion of (cartesian) product (see 3.7.5), i.e. a coproduct in $\mathcal{C}$ is a product in $\mathcal{C}^{op}$.

[5] One can prove this in a specular way in which we proved the uniqueness of the cartesian product, see 3.7.6.

[6] See 3.10.10 for the definition of symmetric monoidal functor.

[7] e.g. [Str22], later we will also show that one must pick categories different to Vect_k in order to get interesting 3dTFTs and connect them with the Jones polynomial from knot theory.

particular is that one of the guiding conjectures of this field, the cobordism hypothesis, is formulated and provable without mentioning specifically Vect_k as a target category. See 5.6 for more on this conjecture[8]

Remark. A TFT is a functor in a similar way that a linear representation of a group[9] is (see 2). We can construct the category of TFTs in an analagous way to the one we used to define the category of linear representations of a group (see 1): by constructing a functor category where the objects are TFTs (hence particular symmetric monoidal functors) and morphisms apt natural transformations between them[10]. The category of n-dimensional TFTs will then be $\mathrm{Hom}_{\mathrm{SymmMonCat}}(\mathrm{nCob}, \mathcal{C})$ (see 3.10 for the definition of SymmMonCat, the category of all symmetric monoidal categories). Similarly as with groups, one can have linear and non-linear representations, although linear representations appear more frequenly. Typical examples of linear target categories are $(\mathrm{AbGrp}, \otimes)$, $(\mathrm{Mod}_R, \otimes)$, $(\mathrm{Vect}_k, \otimes)$.

Notation (Category of TFTs)**.** The category of TFTs is the functor category

$$\mathrm{Hom}_{\mathrm{SymmMonCat}}(\mathrm{nCob}, \mathcal{C}) = \mathrm{Fun}^{\otimes}(\mathrm{nCob}, \mathcal{C}) = \mathrm{TFT}_n(\mathcal{C}) = \mathrm{nTFT}(\mathcal{C}).$$

This functor category is symmetric monoidal by defining the tensor product between TFTs pointwise: for $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathrm{TFT}_n(\mathcal{C})$ and $M \in \mathrm{nCob}$

$$(\mathcal{Z}_1 \otimes \mathcal{Z}_2)(M) := \mathcal{Z}_1(M) \otimes \mathcal{Z}_2(M)$$

TFTs were originally defined with Vect as the target category, $\mathcal{Z} : (\mathrm{nCob}, \amalg) \to (\mathrm{Vect}, \otimes))$, see [MF88]. In this case we can use the notation in equation 3.7 and write $\mathrm{TFT}_n(\mathrm{Vect}_k) = \mathrm{Rep}(\mathrm{Bord}_{n,n-1}$. We can infer a few things about such TFTs:

1. Σ closed n manifold, we can then view it as a bordism $\emptyset \xrightarrow{\Sigma} \emptyset$ and then we can apply Z and we get $Z(\emptyset) \xrightarrow{Z(\Sigma)} Z(\emptyset)$ but $Z(\emptyset) = k$, since that's the unit in Vect. So $\mathcal{Z}(\Sigma)$ is a linear map from k to k, which is entirely determined by where it sends one element. We then get that $\mathcal{Z}(\Sigma)(\mathbb{1})$ is a diffeomorphism invariant.

   In fact, the original hope was to use TFTs to find new diffeomorphism invariants.

2. There is a "trivial" TFT which on objects acts as $\mathcal{Z}(M) = k$ and on morphisms as $\mathcal{Z}(\Sigma) = id_k$ and we write $Z = 1$.

[8]As a spoiler: the moral of the cobordism hypothesis is that "The history of the Baez-Dolan conjecture *[i.e. the cobordism hypothesis]* goes most directly through quantum field theory and its adaptation to low-dimensional topology. Yet in retrospect it is a theorem about the structure of manifolds in all dimensions..." [Fre12] and hence TFTs are of interest with respect to any suitable target category, not only the ones of interest from the viewpoint of physics.

[9]In fact, some regard it a kind of representation, check out for instance around minute 18:00 of Catharina Stroppel: The beauty of braids - from knot invariants to higher categories; or the following quote from a poster for a conference on TQFTs and their connections to representation theory and mathematical physics "...a TQFT is a symmetric monoidal functor from the cobordism category to some symmetric monoidal category. It can thus be seen as a representation of a fundamental geometric category on a target category and thereby organizes interesting algebraic structures, e.g. representations of mapping class groups, in terms of cobordism categories. ..."; or this quote from Daniel Freed "An extended topological field theory is a representation of the bordism category..." [Fre12].

[10]See 3.10.12 for the definition of a symmetric monoidal natural transformation.

3. An *invertible* TFT is a TFT in which $Z(M) \cong k$ and $Z(\Sigma)$ is an isomorphism. Which is very similar but in the trivial TFT we have chosen not only $Z(M) \cong k$ but specifically $Z(M) = k$. This is related to "anomalies" in physics[11].

Example 4.1.3. Euler characteristic ($\to$ Euler theory)
Fix $\lambda \in \mathbb{C}^*$

$$Z_\lambda(\Sigma) := \lambda^{\chi(\Sigma)}, \quad Z(M) = \mathbb{C} \tag{4.1}$$

Here functoriality follows from additivity

$$\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y) \tag{4.2}$$

Definition 4.1.4. Let $(\mathcal{C}, \otimes, \mathbb{1}_\mathcal{C})$ be a monoidal category. An object $X \in \mathcal{C}$ is invertible if there is a $Y \in \mathcal{C}$ such that $X \otimes Y \cong \mathbb{1}_\mathcal{C}$.

Example 4.1.5. In $(\mathrm{Vect}_k, \otimes)$, X is invertible iff there is a vector space Y such that $X \otimes Y \cong k$.

Notation. $\mathcal{C}^\times \subset \mathcal{C}$ subset of invertible objects and isomorphisms. We will later see that this is the underlying Picard groupoid (see 4.1.47)

Notation. For some symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1}_\mathcal{C})$ we sometimes explicitly denote its tensor product by $\mathcal{C}^\otimes$. For example, $\mathrm{Set}^\times$, $\mathrm{Set}^\amalg$, $\mathrm{Vect}^\oplus$, $\mathrm{Vect}^\otimes$.

We can now rigorously define what we previously called an invertible TFT:

Definition 4.1.6. An invertible TFT is an invertible object in $\mathrm{Fun}^\otimes(\mathrm{Bord}_{\text{n,n-1}}, \mathcal{C})$. This is equivalent to the fact that $\mathcal{Z}$ factors through $\mathcal{C}^\times$:

$$\begin{tikzcd} {\mathrm{Bord}_{\text{n,n-1}}}^\amalg \arrow[rr, "\mathcal{Z}"] \arrow[dr, dashed, "\tilde{\mathcal{Z}}"'] & & \mathcal{C}^\otimes \\ & \mathcal{C}^\times \arrow[ur, hook] & \end{tikzcd}$$

Definition 4.1.7 (Groupoid Completion)**.** Let $\mathcal{C}$ be a category. A groupoid completion $(|\mathcal{C}|, i)$ is a groupoid $|\mathcal{C}|$ with a functor $i : \mathcal{C} \to |\mathcal{C}|$ such that if $\mathcal{D}$ is a groupoid and $f : \mathcal{C} \to \mathcal{D}$ a functor, then there is a unique map $\tilde{f} : |\mathcal{C}| \to \mathcal{D}$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "i"] \arrow[dr, "f"'] & & {|\mathcal{C}|} \arrow[dl, dashed, "\tilde{f}"] \\ & \mathcal{D} & \end{tikzcd}$$

From the universal property of the groupoid completion, there is a further factorization of an invertible field theory:

$$\begin{tikzcd} \mathrm{Bord}^\amalg \arrow[r, "\mathcal{Z}"] \arrow[d, "i"'] & \mathcal{C}^\otimes \\ {|\mathrm{Bord}^\amalg|} \arrow[r, "\tilde{\mathcal{Z}}"'] & \mathcal{C}^\times \arrow[u, hook] \end{tikzcd}$$

Remark. Note that the groupoid completion of the bordism category $|{\mathrm{Bord}_{\text{n,n-1}}}^\amalg|$ is a Picard groupoid, see 4.1.47.

Remark. A nTFT is invertible if the TFT is itself an invertible (see 4.1.4) object in the category of nTFTs.

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[11]To precisely see how, one needs to work with *extended* TFTs, a higher categorical refinement of what we are working with now. See [Mü20].

4.1.1 Invertible field theories and stable homotopy theory ✈

These theories can be studied with stable homotopy theory. Stable homotopy homotopy theory is the branch of homotopy theory that studies phenomena and structures that are stable, i.e. that can occur in any dimension, or in any sufficiently large dimension independently of the exact dimension. The tool used to reach higher dimensions is very often suspension[12], this is why sometimes stable homotopy theory is characterized as the phenomena that are stable under suspension.

The paradigmatic example of stable phenomena are the stable homotopy groups of the sphere are the homotopy groups of the sphere $\pi_{n+i}(S^n)$ such that $n > i+1$. They are stable because thanks to Freudenthal's suspension theorem implies that the suspension functor[13] $S^n \xrightarrow{\Sigma} S^{n+1}$ induces an isomorphism on certain homotopy groups of the sphere: the stable ones, i.e. where $n > i+1$

$$\pi_{n+i}(S^n) \cong \pi_{n+i+1}(S^{n+1})$$

The investigation of stable phenomena is achieved via spectra, i.e. sequence of pointed spaces with maps relating them one another. For instance the stable homotopy group of the sphere are exactly the homotopy groups of the sphere spectrum. This begs the question: what are spectra?

What are spectra? ✈

This subsection is based off [Jac17], [ADA78], [Rok19], [MG24], [Den20] and [Max21].

We already have seen some spectra and definitions of some kinds of spectra in 2.7.1, e.g. the sphere spectrum and the suspension spectrum. In that section, we defined suspension as a quotient. However, one can define a suspension in full generality, i.e. for any pointed ∞-category, via certain homotopy colimits. Moreover, via such homotopy co/limits one can define other fundamental notions for spectra, so in what follows, we try to convey an idea for how they work.

Homotopy co/limits are objects making certain diagrams commute via homotopies and with certain universal properties. Such diagrams commute in a very similar sense to how $\mathbb{E}_n$-algebras are associative via homotopies of every order up to ∞. Every co/limit in an ∞-category works as a homotopy co/limit[14], see [Lan21, Section 4] for a thorough introduction to how that works.

We now provide a rough sketch of what are homotopy co/limits. To do this, we first need to define what are co/limits in ordinary categories.

Definition 4.1.8 (Terminal, initial, zero objects)**.** A terminal object in a category $\mathcal{C}$ is an object $* \in \mathcal{C}$ such that for every $X \in \mathcal{C}$ there is a unique morphism $X \to *$

An initial object in a category $\mathcal{C}$ is an object $\emptyset \in \mathcal{C}$ such that for every $X \in \mathcal{C}$ there is a unique morphism $\emptyset \to X$

A zero object is an object that is both initial and final. A category is called pointed if it has a zero object.

---

[12] We define it in the next subsection.

[13] Do not fear if you do not know what it is yet, we define it in the next subsection.

[14] Being able to define homotopy co/limits via usual universal properties is indeed one of the motivations for ∞-categories, as said in 3.8.15.

Definition 4.1.9 (Diagram)**.** Until now, we spoke of diagrams informally, there is however a precise definition. Given a small[15] category $\mathcal{J}$, a diagram of shape $\mathcal{J}$ in $\mathcal{C}$ is a functor $D : \mathcal{J} \to \mathcal{C}$

Definition 4.1.10 (Co/limits)**.** Given a category $\mathcal{J}$ we can construct

- the cone[16] of $\mathcal{J}$: $\mathcal{J}^{\lhd}$ by adding an initial object to $\mathcal{J}$
- the cocone[17] of $\mathcal{J}$: $\mathcal{J}^{\rhd}$ by adding a terminal object to $\mathcal{J}$

Given a diagram D of shape $\mathcal{J}$ in $\mathcal{C}$ and $X \in \mathcal{C}$ we have

- a cone over D that is a natural transformation $\gamma : X \Rightarrow D$, where X denotes the functor sending every object in $\mathcal{J}$ to X. So, because of naturality of γ for $M \xrightarrow{f} N$ in $\mathcal{J}$ there is the following commutative diagram:

$$\begin{tikzcd} & X(N) = X = X(M) \arrow[dl, "\gamma_N"'] \arrow[dr, "\gamma_M"] & \\ D(N) \arrow[rr, "D(f)"'] & & D(M) \end{tikzcd}$$

  We have then also the category of cones over D. An equivalent description of a cone over D is a functor $\mathcal{J}^{\lhd} \to \mathcal{C}$ hence the category of cones is the functor category $\mathrm{Fun}_D(\mathcal{J}^{\lhd}, \mathcal{C})$.

- a cocone over D that is a natural transformation $\eta : D \Rightarrow X$ and because of naturality of γ for $M \xrightarrow{f} N$ in $\mathcal{J}$ there is the following commutative diagram:

$$\begin{tikzcd} D(N) \arrow[rr, "D(f)"] \arrow[dr, "\eta_N"'] & & D(M) \arrow[dl, "\eta_M"] \\ & X(N) = X = X(M) & \end{tikzcd}$$

  We have then also the category of cocones over D. An equivalent description of a cocone over D is a functor $\mathcal{J}^{\rhd} \to \mathcal{C}$ hence the category of cocones is the functor category $\mathrm{Fun}_D(\mathcal{J}^{\rhd}, \mathcal{C})$

Then,

- the limit of the diagram D is a terminal object in the category of cones over $\mathcal{J}$, i.e. $\mathrm{Fun}_D(\mathcal{J}^{\lhd}, \mathcal{C})$
- the colimit of the diagram D is an initial object in the category of cocones over $\mathcal{J}$, i.e. $\mathrm{Fun}_D(\mathcal{J}^{\rhd}, \mathcal{C})$

In order to get a definition of homotopy co/limit or $(\infty, 1)$-co/limit, just substitute your favourite model of ∞-category with 'category' in this definition.

[15] This means that the collection of objects and morphisms are both sets
[16] Aka left cone
[17] Aka right cone

Example 4.1.11 (Cartesian product)**.** We previously defined the cartesian product of two objects by saying that some diagram commutes and the product has a certain universal property (3.7.5). Such cartesian product is a limit over a diagram shaped as a discrete category with two objects. Let us name this discrete category with two different objects $\mathcal{J} = \{0, 1\}$, discrete means that there are only the identity morphisms id_0 and id_1. Given $\mathcal{C}$ an arbitrary category, observe a diagram

$$D : \mathcal{J} \to \mathcal{C}$$

$$0 \mapsto A$$

$$1 \mapsto B$$

Then, a cartesian product $A \times B$ is the limit over D

$$\begin{tikzcd} & Y \arrow[dl, "f"'] \arrow[d, dashed, "\exists! u"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "pr_1"] \arrow[r, "pr_2"'] & B \end{tikzcd}$$

The two projections exist because $A \times B$ is a cone: a cone is an initial object in the shape category $\mathcal{J}$ and D is a functor, so it imports the unique morphisms from the initial object to where 0 and 1 are mapped, i.e. to A and B.
u exists and is unique because $A \times B$ is the terminal object in the category of cones.
The two smaller triangles commute because of the naturality of the morphism u: recall that the category of cones is a functor category hence morphisms between cones are natural transformations and note that pr_1 and f for instance are images of morphisms in $\mathcal{J}^{\triangleleft}$

We already sketched what might mean that a diagram is homotopy coherently commutative in the case of an $\mathbb{E}_1$-algebra (3.10.1). For the sake of clarity we sketch it here again, in a more general setting. Recall that all morphisms higher than 1 in an ∞-category are weakly invertible.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "h\ \ l\ \ \cdots\ \ m\ \ k"] & C \arrow[d, "r"] \\ B \arrow[r, "q"'] & D \end{tikzcd}$$

This diagram does not commute strictly, but via homotopies. This means that $h : q \circ f \simeq r \circ g : k$, instead of what we are used to: $q \circ f = r \circ g$. Not only this however, it is the case that $k \circ h \simeq id_{q \circ f}$ and $h \circ k \simeq id_{r \circ g}$, instead of the usual strict invertibility, and thus via the 3-morphisms l and m, the 2-morphisms k and h are homotopic, i.e. $l : k \simeq h : m$. Then there will be some explicit 4-morphisms witnessing that $m \simeq l$, and so on to ∞.

Example 4.1.12 (Homotopy pullback/$(\infty, 1)$-pullback)**.** Let $\mathcal{J}$ be an ∞-category with objects and 1-morphisms $0 \to 1 \leftarrow 2$ and $D : \mathcal{J} \to \mathcal{C}$ be a diagram. To be rigorous, we would need to define what is an ∞-functor. However, since this necessitates getting our hands dirty with a model, we prefer to remain loose: an ∞-functor is an assignment that respects composition and identity morphisms[18] not only on 1-morphisms, but for also all higher invertible morphisms. Then, an ∞-pullback of is the homotopy limit of D.

[18] i.e. is functorial.

We now unpack what this means. A cone of $\mathcal{J}$, where $\varnothing$ is the new initial object looks like this:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] \arrow[dr] & 2 \arrow[d] \arrow[dl, phantom, "\Leftrightarrow", near start] \\ 0 \arrow[r] \arrow[ur, phantom, "\Leftrightarrow", near start] & 1 \end{tikzcd}$$

Where the two small triangles commute via higher morphisms denoted with $\Leftrightarrow$.

Let $D(1) = X$, $D(2) = Y$ and $D(0) = Z$. Then, the ∞-pullback of D is the terminal object in the ∞-functor ∞-category $\mathrm{Fun}_D(\mathcal{J},\mathcal{C})$. Denote the image of $\varnothing$ of such pullback with $Y \times_X Z$. Then there is the following homotopy coherent diagram

$$\begin{tikzcd} Y \times_X Z \arrow[r] \arrow[d] \arrow[dr] & X \arrow[d] \arrow[dl, phantom, "\Leftrightarrow", near start] \\ Y \arrow[r] \arrow[ur, phantom, "\Leftrightarrow", near start] & Z \end{tikzcd}$$

Because the ∞-pullback is a functor and thus preserves the homotopy-coherent commutativity of diagrams in the shape $\mathcal{J}$, by preserving compositionality for all n-morphisms. Since it is the terminal object in the category of cones over the diagram D for every other cone mapping $\varnothing$ to an object $Q \in \mathcal{C}$, then there is a unique morphism $Q \to Y \times_X Z$ making the following diagram commute homotopy coherently

$$\begin{tikzcd} Q \arrow[dr] \arrow[drr] \arrow[ddr] & & \\ & Y \times_X Z \arrow[r] \arrow[d] \arrow[dr] & X \arrow[d] \arrow[dl, phantom, "\Leftrightarrow", near start] \\ & Y \arrow[r] \arrow[ur, phantom, "\Leftrightarrow", near start] & Z \end{tikzcd}$$

The two small triangles formed by morphisms that have Q as source and by the projections of $Y \times_X Z$ (the ones on the upper left) commute because of the naturality of morphisms in $\mathrm{Fun}_D(\mathcal{J}^{\triangleleft},\mathcal{C})$

Notation. From now on, when we are talking about commutative diagrams in an ∞-category, we do not write the double arrows $\Leftrightarrow$ symbolizing the higher morphisms that make the diagram commute.

There is the dual notion of homotopy pullback: the homotopy pushout, which we now use to define suspensions.

Definition 4.1.13 (Suspension)**.** Let $\mathcal{C}$ be a pointed[19] ∞-category with the zero object denoted by 0 and let $X \in \mathcal{C}$. Then the suspension ΣX of X is the pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Definition 4.1.14 (Suspension functor)**.** Given a pointed ∞-category $\mathcal{C}$ the assignment $X \mapsto \Sigma X$ gives rise to a functor $\Sigma : \mathcal{C} \to \mathcal{C}$

Dually, one can define the loop space of an object

[19]A category with a zero object: an object that is both initial and final.

Definition 4.1.15 (Loop space)**.** Let $\mathcal{C}$ be a pointed ∞-category and let $X \in \mathcal{C}$. Then the loop space ΩX is the pullback

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd}$$

Definition 4.1.16 (Loop space functor)**.** Given a pointed ∞-category $\mathcal{C}$ the assignment $X \mapsto \Omega X$ gives rise to a functor $\Omega : \mathcal{C} \to \mathcal{C}$

Remark. As already mentioned in 3.10.1, a loop space is an example of $\mathbb{E}_1$-space and n-iterated loop spaces are $\mathbb{E}_n$-spaces by the Dunn additivity theorem.

Definition 4.1.17 (Alternative definition of adjunction)**.** We defined adjoint functors via the natural tranformations unit and counit (3.5.1). There is however an equivalent formulation: two ∞-functors $\mathcal{C} \underset{L}{\overset{R}{\leftrightarrows}} \mathcal{D}$ form an adjunction $L \dashv R$, i.e. where L is the left adjoint and R is the right adjoint, if between the two Hom-∞-functors

$$\mathrm{Hom}_{\mathcal{C}}(L(-),-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Grpd}_\infty$$

and

$$\mathrm{Hom}_{\mathcal{D}}(-,R(-)) : \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Grpd}_\infty$$

there is a natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(L(-),-) \simeq \mathrm{Hom}_{\mathcal{D}}(-,R(-))$

This definition is easily derivable from the one with units we gave previously, given the unit $\epsilon : id_{\mathcal{D}} \Rightarrow R \circ L$ and $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, the following holds because of one of the triangle identities:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(L(Y),X) \arrow[rr, "\simeq"] \arrow[dr] & & \mathrm{Hom}_{\mathcal{D}}(Y,R(X)) \\ & \mathrm{Hom}_{\mathcal{D}}(R(L(Y)),R(X)) \arrow[ur] & \end{tikzcd}$$

Remark (Suspension-loop adjunction). There is an adjunction $\Sigma \dashv \Omega$ and thus for any pointed ∞-category $\mathcal{C}$ it holds that $\mathrm{Hom}_{\mathcal{C}}(\Sigma-,-) \simeq \mathrm{Hom}_{\mathcal{C}}(-,\Omega-)$. If it is not sufficiently clear from the fact that $\Omega X = 0 \times_X 0$ and $\Sigma X = 0 \amalg_X 0$, do not worry. In the case that it interests us the most it can be seen from another perspective. Recall that for $X \in \mathrm{Top}_*$ $\Sigma X = X \wedge S^1$ and $\Omega_x = \mathrm{Map}_*(S^1, X)$ and that Top_* is cartesian monoidal closed, i.e. for any $A,B,C \in \mathrm{Top}_*$ it holds that $\mathrm{Hom}_{\mathrm{Top}_*}(A \wedge B, C) \cong \mathrm{Hom}_{\mathrm{Top}_*}(A, \mathrm{Map}_*(B,C))$ and hence $\mathrm{Hom}_{\mathrm{Top}_*}(- \wedge S^1, -) \cong \mathrm{Hom}_{\mathrm{Top}_*}(-, \mathrm{Map}_*(S^1,-))$

Definition 4.1.18 ((Ω-)Spectrum[20])**.** A spectrum is a collection of pointed spaces $\{E_n\}_{n \in \mathbb{Z}}$ with equivalences $\delta_n : E_n \simeq \Omega E_{n+1}$

Remark. Note that via the adjunction $\Sigma \dashv \Omega$ we can get a sequential spectrum from a spectrum: given $X_i, X_{i+1} \in \mathcal{S}_*$ it holds that $\mathrm{Hom}_{\mathcal{S}_*}(X_i, \Omega X_{i+1}) \simeq \mathrm{Hom}_{\mathcal{S}_*}(\Sigma X_i, X_{i+1})$ and thus we can get the pointed maps we need in order to define a sequential spectrum. This means that every spectrum is a sequential spectrum, i.e. the category of spectra is a subcategory of the category of sequential spectra. This is counterintuitive and the reason why sequential spectra are sometimes called prespectra, signalling that they are something more general, less structured.

[20] It is denoted Ω-spectrum in the older literature, e.g. books or articles by May or Adams like [ADA78].

Definition 4.1.19. Inhalt...

Definition 4.1.20 (∞-category of spectra)**.** The ∞-category of spectra is the category where objects are spectra.

This is equivalent to saying that the category of spectra is the sequential homotopy limit

$$\lim(\ldots \leftarrow \mathcal{S}_* \leftarrow \mathcal{S}_* \leftarrow \mathcal{S}_* \leftarrow \ldots)$$

where we can see the diagram we are taking the limit over as a functor $D : \mathbb{Z}^{\mathrm{op}} \to \infty\text{-Cat}$ where $\mathbb{Z}$ is considered as a category via its ordering ($\leq$ corresponds to a morphism $\to$) and for each $i \in \mathbb{Z}$ $D(i) = \mathcal{S}_*$. Via this definition of the category of spectra as such sequential homotopy limit, we avoid defining explicitly what are the morphisms between spectra.

The initial object of such category is the sphere spectrum $\mathbb{S}$.

Definition 4.1.21 (Infinite loop space)**.** An infinite loop space is a space X_0 with an infinite sequence of pointed spaces $X_0, X_1, X_2, \ldots$ with weak equivalences for $n \geq 0$

$$X_n \simeq \Omega X_{n+1}$$

Infinite loop spaces seem very similar to what we defined as spectra. However, there is a crucial difference: there is no space to start from in a spectrum, since there are deloopings for all spaces, i.e. $X_{-1} \simeq \Omega X_0$, $X_{-2} \simeq \Omega X_{-1}$ and so on, while in an infinite loop space there are not.

Definition 4.1.22 (Underlying (infinite loop) space functor)**.** Given a spectrum, the assignment of a spectrum $\{X_i\}_{i\in\mathbb{Z}}$ to its underlying infinite loop space, aka 0-th space, is a functor.

$$\Omega^\infty : \mathrm{Sp} \to \mathcal{S}_*$$

$$\{X_i\}_{i\in\mathbb{Z}} \mapsto X_0$$

Such assignment has a left adjoint.

Definition 4.1.23 (Suspension spectrum functor)**.**

$$\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$$

$$X \mapsto \{\Sigma^i X\}_i$$

sending pointed spaces to suspension spectra, i.e. sequences of spaces $\{\Sigma^i X\}_i$ with pointed maps $\Sigma(\Sigma^i X) = \Sigma^{i+1} X$. Such suspension spectra are indeed spectra las we defined then (aka Ω-spectra) thanks to the adjunction $\Sigma \dashv \Omega$.

An example of such spectra is the sphere spectrum we already encountered in 2.7.1.

Definition 4.1.24 (Classifying spectrum)**.** Let M be an $\mathbb{E}_\infty$-space, i.e. an $\mathbb{E}_\infty$-algebra in $\mathcal{S}_*$ (see 3.10.1). Then we define the classifying spectrum of M

$$\mathbf{B}^\infty(M) := \{\mathbf{B}^n\}$$

Definition 4.1.25 (Classifying spectrum functor)**.**

$$\mathbf{B}^\infty$$

Picard ∞-groupoids do not only correspond to infinite loop spaces but also to connective spectra.

Remark (Important observation: Infinite loop spaces are grouplike $\mathbb{E}_\infty$-spaces=Picard groupoids). We already argued in 3.10.1 that such infinite loop spaces are grouplike $\mathbb{E}_\infty$-spaces because of the Dunn additivity theorem 3.10.24. Moreover, also in 3.10.1 we explained how grouplike $\mathbb{E}_\infty$-spaces coincide with Picard ∞-groupoids:

- the $\mathbb{E}_\infty$ makes them homotopy coherent commutative monoid objects in Grpd_∞, i.e. symmetric monoidal ∞-groupoids
- since they are grouplike, they have inverses because if we first apply π_0, then forget about the commutative structure, and land in the category of groups, that means that the inverses must have been there from the start

The following result proves the other direction: all grouplike $\mathbb{E}_\infty$-spaces (and thus Picard ∞-groupoids) are infinite loop spaces.

Theorem 4.1.26 (Recognition theorem for n-iterated loop spaces, Boardman-Vogt, May)**.** *For any $n < 0$, n-iterated loop spaces coincide with grouplike $\mathbb{E}_n$-spaces. More rigorously: the n-iterated loop space functor $\Omega^n : \mathcal{S}_* \to \mathcal{S}_*$ restricts to an equivalence of ∞-categories between n-connective pointed spaces and grouplike $\mathbb{E}_n$-spaces*

$$\Omega^n : \mathcal{S}_*^{\geq n} \simeq \mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_n}(\mathcal{S})$$

Notation. An n-connective, or alternatively n-connected, space is a space with trivial homotopy groups for all $k \leq n$. Hence 0-connective spaces are the usual path-connected spaces.

See [Jac17, 5.2.6.10] for a rigorous proof. Note anyway that it follows from an easier fact to prove (see [Den20, Theorem 3.9]):

Theorem 4.1.27 (Recognition theorem for loop spaces)**.** *Loop spaces coincide with grouplike $\mathbb{E}_1$-spaces. More rigorously: the loop space functor $\Omega : \mathcal{S}_* \to \mathcal{S}_*$ restricts to an equivalence of ∞-categories between 0-connective pointed spaces and grouplike $\mathbb{E}_1$-spaces*

$$\mathcal{S}_*^{\geq 0} \overset{\Omega}{\simeq} \mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S})$$

Combining this with the Dunn additivity theorem (3.10.24) allows to prove the recognition theorem for n-iterated loop spaces by induction. The base case: $n = 1$ is the recognition theorem for loop spaces and the inductive step holds via the Dunn additivity theorem.

As said in the preamble of this section, stable homotopy groups of a space $A \in \mathcal{S}$, denoted $\pi_i^s(A)$, are exactly the homotopy groups of the spectrum of A, $\Sigma^\infty A$. Thus, $\pi_i^s(A) \cong \pi_i(\Sigma^\infty A)$. There is however a natural question: what are the homotopy groups of a spectrum? While the degree homotopy groups for usual topological spaces are defined to be positive, homotopy groups of spectra can be defined and be non-trivial also for negative degrees.

Definition 4.1.28 (Homotopy groups of spectra)**.** Given a sequential spectrum $\{E_i\}_{i\in\mathbb{Z}}$ with pointed maps $\sigma_i : \Sigma E_i \to E_{i+1}$, for $n, k \in \mathbb{Z}$, the nth homotopy group of $\{E_i\}_{i\in\mathbb{Z}}$ is

$$\pi_n(\{E_i\}_{i\in\mathbb{Z}}) := \varinjlim(\ldots \to \pi_{n+k}(E_k) \xrightarrow{\Sigma} \pi_{n+k+1}(\Sigma E_k) \xrightarrow{\pi_{n+k+1}(\sigma_k)} \pi_{n+k+1}(E_{k+1}) \to \ldots)$$

where all the π_* inside the parentheses indicate the homotopy groups of the various pointed spaces $E_k \in \{E_i\}_{i\in\mathbb{Z}}$. Since $k \in \mathbb{Z}$, there are indeed also homotopy groups with negative degrees.

If a sequential spectrum $\{X_i\}_{i\in\mathbb{Z}}$ is also a spectrum, then from the definition we just provided it follows that for

$$\pi\{X_i\}_{i\in\mathbb{Z}} \begin{cases} \pi_{n+k}(X_k) & k+n \geq 0 \\ \pi_n(X_0) & n \geq 0 \\ \pi_0 X_{|n|} & n < 0 \end{cases}$$

Definition 4.1.29 (Connective spectra)**.** Connective spectra are spectra that only have positive non-trivial homotopy groups: let $\{X_i\}_{i\in\mathbb{Z}}$ be a connective spectra, then for all $n < 0$ it holds that $\pi_n(\{X_i\}_{i\in\mathbb{Z}}) = 0$.

Connective spectra form a subcategory of spectra denoten $Sp_{\geq 0}$

Theorem 4.1.30 (Recognition theorem for connective spectra)**.** *There is an equivalence:*

$$\mathbf{B}^\infty : \mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathcal{S}) \leftrightarrows \mathrm{Sp}_{\geq 0} : \Omega^\infty$$

where $\mathrm{Sp}_{\geq 0}$ *denotes the category of connective spectra.*

See [HW, Corollary II.24] or [Den20, 3.29] for a proof.

Remark. Combining the recognition theorem for connective spectra and the recognition theorem for infinite loop spaces we get that infinite loop spaces coincide with connective spectra. Another way to see this is to note that

- for a spectra $X \in \mathrm{Sp}$ it holds that $\pi_i(X) \cong \pi_i(\Omega^\infty X)$, where π_i on the left denotes a homotopy group of spectra while π_i on the right refers to the usual homotopy groups of spaces.

In sum, spectra are a higher categorical analog of abelian groups (see 3.8.3), they have been in fact called "the abelian groups of the 21st century" [MG24] because they provide very powerful techniques for many areas of mathematics where there is currently much progress, e.g. in algebraic K-theory. One can do algebra with spectra as one does ordinary algebra with abelian groups and rings, this subject is usually called 'higher algebra'[21]. This is not just a vague analogy, but something that can be motivated formally, for instance, abelian groups are discrete spectra:

$$\pi_0 : \mathrm{Sp} \to \mathrm{Ab}$$

where in particular $\pi_0(\mathbb{S}) = \mathbb{Z}$

We make the comparison clearer via the following table[22] :

[21] A funny synonym for higher algebra is brave new algebra, referring to Huxley's dystopian novel *Brave New World*. It was coined by Waldhausen, one of the pioneers of this discipline, hinting at the fact that it seems to have myriads of intriguing opportunities but at the same could lead into a world of pure abstractions detached from mathematics with actual content. Examples of successful applications of such machinery to concrete problems, e.g. the counterexamples to Ravenel's telescope conjecture [BHLS23], convince the author of this section that this nightmare was avoided, although it must be said that they are clearly biased.

[22] his table was inspired by a table in Thomas Nikolaus' talk: `https://www.youtube.com/watch?v=SJNKs1PxT9g`.

algebra	higher algebra
set	∞-groupoid
abelian group	spectrum
Ab	Sp
$\mathbb{Z}$	$\mathbb{S}$
ring	$\mathbb{E}_1$-ring spectrum
commutative ring	$\mathbb{E}_\infty$-ring spectrum
Ab-enriched category	Sp-enriched category
abelian category	stable category

See 3.8.3 and 3.8.6 for the definitions of Ab-enriched category and abelian category. See 3.10.1 for the definitions of $\mathbb{E}_1$-ring spectrum and of $\mathbb{E}_\infty$-ring spectrum. A stable category is a pointed ∞-category where $\Sigma \dashv \Omega$ do not just form an adjunction, but are equivalent. There is an equivalent definition of stable category that more closely resembles the definition of abelian category, i.e. an additive ∞-category with further conditions, see [Jac17, Section 1].

Invertible field theories are maps of spectra ✈

In order to allow the application of stable homotopy theory, one must fully extend topological field theories (see 5.6), i.e. with (∞, n)-categories of bordisms as a source, because only with the homotopy coherent additional structure of such categories one gets that $|\mathrm{Bord}|$ and $\mathcal{C}^\times$ are particular kinds of spectra.

In the case of extended TFTs, $|\mathrm{Bord}|$ and $\mathcal{C}^\times$ both are ∞-groupoids since

1. we get $|\mathrm{Bord}|$ by adjoining all inverses for all morphisms and thus all morphisms become invertible

2. $\mathcal{C}^\times$ is obtained by forgetting about non-invertible morphisms (and objects) of $\mathcal{C}$ and thus we are left only with invertible morphisms

However, we do not have just ∞-groupoids but also a symmetric monoidal structure with duals on both $|\mathrm{Bord}|$ and $\mathcal{C}^\times$. In short, they are Picard ∞-groupoid. Thanks to 4.1.1, we know that they are thereby grouplike $\mathbb{E}_n$-spaces and infinite loop spaces. Via the recognition principle for infinite loop spaces **??**, we know that they are connective spectra.

This is why $\tilde{\mathcal{Z}}$ is not just a map of spaces but a map of spectra, and this enables the application of stable homotopy theory, making life easier. We can conclude this interlude on stable homotopy theory and invertible TFTs with the following motto

Invertible TFTs are maps of connective spectra!

This is not only of theoretical interest but it also allows the application of topological field theories to condensed matter theory! Classical phases of matter are governed by Landau theory of phase transition. However, the phases of a quantum material do not behave in the same manner. Fortunately, there are ways to describe them, or at least specific types of quantum matter (with gapped Hamiltonians for instance)[23]. The topological field theories

[23] Kitaev for instance proposed in [KLF09] that the topological insulators and superconductors of a certain type of condensed matter are classifiable via K-theory[24] and this was further refined via twisted equivariant K-theory by Dan Freed and Greg Moore in [FM13].

describing the (topological) phases of matter of some condensed matter systems (with short range entanglement) are *invertible* field theories. The connection to stable homotopy theory allows to compute topological invariants of spectra and thereby of phases of matter. This was done by Freed and Hopkins in [FH21] and in [Fre14].

4.1.2 Dualizability in the context of topological field theories

Given $\mathcal{Z} : \mathrm{Bord}_{\mathrm{n,n\text{-}1}} \to \mathrm{Vect}$ and $M \in \mathrm{Bord}_{\mathrm{n,n\text{-}1}}$, one can prove that $\mathcal{Z}(M)$ is a finite dimensional vector space. Specifically, we have that $\dim \mathcal{Z}(M) = \mathcal{Z}(M \times S^1)(1)$, where $M \times S^1$ is an n dimensional cobordism from $\emptyset$ to $\emptyset$.

Proof. Decompose $M \times S^1$ into two semicircles $M \times [0,1]$. Then it gets mapped by $\mathcal{Z}$ to Vect_k in the following manner

$$\begin{tikzcd}
\emptyset \arrow[rr, "M\times[0,1]"] \arrow[dd, bend right, mapsto, "\mathcal{Z}"'] & & M \amalg M \arrow[rr, "M\times[0,1]"] & & \emptyset \arrow[dd, bend left, mapsto, "\mathcal{Z}"] \\
& M\times \arrow[d, mapsto, "\mathcal{Z}"] & & M\times \arrow[d, mapsto, "\mathcal{Z}"'] & \\
k \cong \mathcal{Z}(\emptyset) \arrow[rr, "\mathcal{Z}(M\times[0,1])"'] & & \mathcal{Z}(M)\otimes\mathcal{Z}(M) \arrow[rr, "\mathcal{Z}(M\times[0,1])"'] & & \mathcal{Z}(\emptyset) \cong k
\end{tikzcd}$$

We claim that the maps will roughly be $k \to V^\vee \otimes V \xrightarrow{evaluate} k$, where the first map sends $1 \mapsto \sum_{i=1}^n f_i \otimes e_i$ where e_i is a basis of V and f_i is the dual basis. □

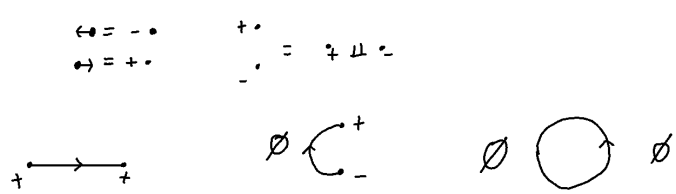
Definition 4.1.31. Let $\mathrm{Bord}^{\mathrm{or}}_{\mathrm{n,n\text{-}1}}$ be the category with:

- objects: closed $(n-1)$ dimensional manifolds
- morphisms: $\mathrm{Hom}_{\mathrm{Bord}^{\mathrm{or}}_{\mathrm{n,n\text{-}1}}}(M,N)$ = orientation preserving diffeomorphism classes of oriented bordisms. i.e. for a bordism $(\Sigma, p, \theta_0, \theta_1)$ Σ has an orientation and $\partial\Sigma \cong \overline{M} \amalg N$ is an orientation preserving diffeomorphism.

Notation. We denote with $\bullet_+$ the positively oriented point, i.e. the closed 0-dimensional manifold from where there is an outgoing 1-dimensional bordism.

We denote with $\bullet_-$ the negatively, i.e. the closed 0-dimensional manifold which is the incoming boundary of a 1-dimensional bordism.

Example 4.1.32 (Some objects and morphisms from $\mathrm{Bord}^{\mathrm{or}}_{1,0}$)**.**



Example 4.1.33 (Some objects and morphisms from $\mathrm{Bord}_{2,1}^{\mathrm{or}}$).

Note that the orientation on the outgoing boundary is opposite to the induced orientation on the incoming one.

We can also define analogous categories for other tangential structures, e.g. a framing.

Example 4.1.34. In 1 and 2 dimensions...

Definition 4.1.35. Let $\mathcal{C}$ be a monoidal category. A left dual of an object $X \in \mathcal{C}$ is an object Y together with $ev_X : Y \otimes X \to \mathbb{1}_{\mathcal{C}}$ and $coev_X : \mathbb{1}_{\mathcal{C}} \to X \otimes Y$ such that

$$\begin{tikzcd} \mathbb{1}_{\mathcal{C}} \otimes X \cong X \arrow[rr, "id_X"] \arrow[dr, "coev_X \otimes id_X"'] & & X \otimes \mathbb{1}_{\mathcal{C}} \cong X \\ & X \otimes Y \otimes X \arrow[ur, "id_X \otimes ev_X"'] & \end{tikzcd} \tag{4.3}$$

and

$$\begin{tikzcd} Y \otimes \mathbb{1}_{\mathcal{C}} \cong Y \arrow[rr, "id_Y"] \arrow[dr, "id_Y \otimes coev_Y"'] & & \mathbb{1}_{\mathcal{C}} \otimes Y \cong Y \\ & Y \otimes X \otimes Y \arrow[ur, "ev_Y \otimes id_Y"'] & \end{tikzcd} \tag{4.4}$$

The fact that these two diagrams commute is called snake relations. If the diagrams commute, then X is the right dual of Y and Y is the left dual of X.

Remark. If $\mathcal{C}$ is braided, and in particular for us symmetric, any right dual is a left dual and viceversa.

Notation. We denote the dual of an object X with $X^\vee$. Sometimes it is also denoted with X^*.

Corollary 4.1.36. *Let $F : \mathcal{C} \to \mathcal{D}$ be a monoidal functor. Thanks to monoidal functoriality, the monoidal unit $\mathbb{1}_{\mathcal{C}}$ is sent to $\mathbb{1}_{\mathcal{D}}$ (up to isomorphism), and hence one gets the evaluation map and coevaluation map: $ev_{F(X)} : F(Y) \otimes F(X) \to \mathbb{1}_{\mathcal{D}} \cong F(\mathbb{1}_{\mathcal{C}})$ and $coev_{F(X)} : F(\mathbb{1}_{\mathcal{C}}) \cong \mathbb{1}_{\mathcal{D}} \to F(X) \otimes F(Y)$. Moreover, the image of a commuting diagram under a functor still commutes (see 3.3). So, the dual objects are sent to dual objects by monoidal functors and $F(X^\vee) = F(X)^\vee$.*

Example 4.1.37.

1. Any finite dimensional vector space V has a dual, namely $V^\vee$:

$$ev_V : V^\vee \otimes V \to k \tag{4.5}$$
$$coev_V : k \to V \otimes V^\vee \tag{4.6}$$

2. As an exercise try Set, ×

3. $\mathcal{C} = \mathrm{Bord}^{\mathrm{or}}_{\mathrm{n,n\text{-}1}}$. The point $\bullet_+$ is dualizable with dual $\bullet_-$.

   This construction actually works for any object in $\mathrm{Bord}^{\mathrm{or}}_{\mathrm{n,n\text{-}1}}$ (and $\mathrm{Bord}_{\mathrm{n,n\text{-}1}}$).

Definition 4.1.38 (Dual Morphisms)**.** Let X, Y be dualizable objects in a symmetric monoidal category $\mathcal{C}$ and $f : X \to Y$ be a morphism. The dual morphism is given by

$$f^\vee : Y^\vee \xrightarrow{id_{Y^\vee}\otimes coev_X} Y^\vee \otimes X \otimes X^\vee \xrightarrow{id_Y^\vee \otimes f \otimes id_{X^\vee}} Y^\vee \otimes Y \otimes X^\vee \xrightarrow{ev_Y \otimes id_{X^\vee}} X^\vee$$

Definition 4.1.39 (Picard Groupoid)**.** A Picard Groupoid is a symmetric monoidal category where every object is invertible with respect to $\otimes$ (i.e. $\forall A \in \mathcal{C} \exists A^{-1}. A \otimes A^{-1} \cong \mathbb{1}_{\mathcal{C}}$) and every morphism is an isomorphism, and hence it is a groupoid.

Lemma 4.1.40. *In every bordism category* $\mathrm{Bord}^{(\mathrm{or})}_{n,n-1}$ *every object is dualizable.*

Lemma 4.1.41. *Given symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$, symmetric monoidal functors $F, G : \mathcal{C} \to \mathcal{D}$ and a symmetric monoidal natural transfromation $\alpha : F \Rightarrow G$, if $X \in \mathcal{C}$ is dualizable, then $\alpha_X : F(X) \to G(X)$ is invertible.*

Proof. We claim that the inverse is given by $\alpha_{(X^\vee)^\vee}$. Following 4.1.38, $\alpha_{(X^\vee)^\vee} : G(X^\vee)^\vee \to F(X^\vee)^\vee$. Note that $F(X^\vee) = F(X)^\vee$ and thus $F(X^\vee)^\vee = F(X)^{\vee\vee} = F(X)$ and so $\alpha_{(X^\vee)^\vee} : G(X) \to F(X)$. Remember that $ev_X : X^\vee \otimes X \to \mathbb{1}_{\mathcal{C}}$ and $coev_X : \mathbb{1}_{\mathcal{C}} \to X \otimes X^\vee$. We prove that the following diagram commutes

$$\begin{tikzcd}
& G(X) \cong G(X)\otimes \mathbb{1}_{\mathcal{D}} \arrow[dl, "id_{G(X)}\otimes G(coev_X)"'] \arrow[dr, "id_{G(X)}\otimes F(coev_X)"] & \\
G(X)\otimes G(X^\vee)\otimes G(X) \arrow[dd, "G(ev_X)\otimes id_{G(X)}"'] & & G(X)\otimes F(X^\vee)\otimes F(X) \arrow[ll, "id_{G(X)}\otimes\alpha_{X^\vee}\otimes\alpha_X"'] \arrow[d, "id_{G(X)}\otimes\alpha_{X^\vee}\otimes id_{F(X)}"] \\
& & G(X)\otimes G(X^\vee)\otimes F(X) \arrow[ull, "id_{G(X)}\otimes id_{G(X)}\otimes\alpha_X"] \arrow[d, "G(ev_X)\otimes id_{F(X)}"] \\
\mathbb{1}_{\mathcal{D}}\otimes G(X) \cong G(X) & & \mathbb{1}_{\mathcal{D}}\otimes F(X) \cong F(X) \arrow[ll, "\alpha_X"]
\end{tikzcd}$$

First of all, the triangle on the top commutes, i.e. $(id_{G(X)} \otimes \alpha_{X^\vee} \otimes \alpha_X) \circ (id_{G(X)} \otimes F(coev_X)) = id_{G(X)} \otimes G(coev_X)$, because of the naturality of α. The triangle underneath it also commutes, i.e. $id_{G(X)} \otimes \alpha_{X^\vee} \otimes \alpha_X = (id_{G(X)} \otimes id_{G(X)} \otimes \alpha_X) \circ (id_{G(X)} \otimes \alpha_{X^\vee} \otimes id_{id_{F(X)}})$ because of the unitality of $id_{G(X)}$ and $id_{F(X)}$. Lastly, the bottom trapezoid commutes thanks to the naturality of α and thus the whole big diagram commutes. Consider now first mapping leftwards and successively downwards, i.e. $(G(ev_X) \otimes id_{G(X)}) \circ id_{G(X)} \otimes G(coev_X)$, this equals to the identity by one of the two snake relations of dual objects. Take now the other route of the outer diagram, on the right; note that $(\alpha_X \vee)\vee = (G(ev_X) \otimes id_{F(X)}) \circ (id_{G(X)} \otimes \alpha_{X^\vee} \otimes id_{F(X)}) \circ id_{G(X)} \otimes F(coev_X)$ by definition (4.1.38) and hence the right route on the outer diagram corresponds to $\alpha_X \circ (\alpha_{X^\vee})^\vee$. Since the outer diagram commutes we obtain that $\alpha_X \circ (\alpha_{X^\vee})^\vee = id_{G(X)}$, i.e. $(\alpha_{X^\vee})^\vee$ is the right inverse of α_X. One constructs a symmetric diagram by substituting Fs with Gs to prove that it is also the left inverse. □

Corollary 4.1.42. *If every object in a symmetric monoidal category $\mathcal{C}$ is dualizable, then any symmetric monoidal natural transformation between symmetric monoidal functors that have $\mathcal{C}$ as the source category is invertible since a natural transfomation is invertible if and only if each of its components is invertible. Hence,* $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ *is a groupoid for any $\mathcal{D}$ when all objects in $\mathcal{C}$ are dualizable.*

Corollary 4.1.43. *The category of topological field theories* $\mathrm{TFT}^{or}_{n,n-1}(\mathcal{C})$ *is a groupoid.*

Lemma 4.1.44. *Any invertible object is dualizable.*

Proof. Let X be an invertible object. Then, there must be an X^{-1} such that $X \otimes X^{-1} \cong \mathbb{1}_{\mathcal{C}}$, the invertible object is the dual of X. The co/evaluation maps become isomorphisms the snake relations hold because if a commutative triangle of isomorphisms commutes in one direction, then it commutes also in the opposite one, i.e. for arbitrary isomorphisms f, g, h such that $f = h \circ g$, it holds that $f^{-1} = (h \circ g)^{-1} = g^{-1} \circ h^{-1}$. □

Lemma 4.1.45. *A dualizable object is invertible if and only if its co/evaluation maps are isomorphisms.*

Proof. $\Longrightarrow$ directly follows from 4.1.44 since any invertible object has as co/evaluation maps isomorphisms. Conversely, suppose that X is a dualizable object and the co/evaluation maps are isomorphisms. Then, there is a $Y \in \mathcal{C}$ such that $ev_X : Y \otimes X \cong \mathbb{1}$. □

Corollary 4.1.46. *If $\mathcal{C}$ is a monoidal groupoid, then dualizable objects are invertible.*

Proof. Since $\mathcal{C}$ is a groupoid, then also the co/evaluation maps are isomorphisms. From 4.1.45 it follows that the dualizable objects are invertible. □

Corollary 4.1.47. *A Picard groupoid (4.1.39) is equivalently a monoidal groupoid where every object is dualizable.*

Example 4.1.48. The underlying groupoid of $\mathrm{Bord}^{\mathrm{or}}_{\mathrm{n,n\text{-}1}}$ is a picard groupoid.

Notation. We denote with $\mathcal{C}^{\cong}$ the underlying groupoid of $\mathcal{C}$, i.e. the subcategory of $\mathcal{C}$ where we forget about non-invertible morphisms.

Notation. We denote with $\mathcal{C}^{\mathrm{fd}}$ (or sometiimes $\mathcal{C}^{\mathrm{dualizable}}$) the subcategory of $\mathcal{C}$ containing only dualizable objects. We chose 'fd' because it can stand for two things:

- 'finite dimensional' since dualizable objects in Vect are finite dimensional vector spaces and in general dualizability can be seen as a finiteness condition
- 'fully dualizable', a higher-categorical generalization of the notion of dualizability which we encounter in the next subsection

4.1.3 Dualizability in higher categories ✈

Equivalently, a left/right dual object X in a monoidal category $\mathcal{C}$ can also be characterized as a left/right adjoint when considered as a 1-morphism in the delooping (see 3.10.28) bicategory (see 3.10.29) $\mathbf{B}\mathcal{C}$ of $\mathcal{C}$. This makes sense because:

- the evaluation map ev_X corresponds to the counit ϵ of the adjunction

- the coevaluation map $coev_X$ to the unit η of the adjunction
- the two diagrams (4.3 and 4.4) to the triangle identities of the adjunction.

A natural question is: what is an adjoint 1-morphism? We defined only adjoint functors (see 3.5.1)! Fortunately the definition we gave for functors, i.e. 1-morphisms in Cat, is easily generalizable to 1-morphisms in an arbitrary bicategory, meaning that it is virtually the same: one just has to substitute '1-morphisms' for 'functors' and substitute '2-morphisms' for 'natural transformations' (when talking about the co/evaluation and the identity on the functors).

A sane individual, i.e. not a homo categoriens[25] , might ask why this is remotely useful. The answer is that one needs this intuition to talk about monoidal (∞, n)-categories with duals. This is interesting for us because, for instance, these objects are used in the cobordism hypothesis, stating that their objects classify fully extended TFTs, see 5.6.

Remark. The notion of delooping of a monoidal category as a one-object bicategory can be generalized to monoidal (∞, n)-categories. The delooping of a monoidal (∞, n)-category $\mathcal{C}$ is a one-object $(\infty, n+1)$-category $\mathbf{B}\mathcal{C}$ where the objects of $\mathcal{C}$ are the 1-morphisms of $\mathbf{B}\mathcal{C}$ and the tensor product of $\mathcal{C}$ is the composition $\circ$ of 1-morphisms.

Definition 4.1.49 ((∞, n)-monoidal categories with duals)**.** Let $\mathcal{C}$ be a monoidal (∞, n)-category. Then $\mathcal{C}$ has duals if and only if its delooping $\mathbf{B}\mathcal{C}$ has adjoints[26] for any k-morphism with $0 < k < n+1$. We say that any object in such a category is fully n-dualizable.

We have a vague idea of what are monoidal (∞, n)-categories and deloopings thereof. However, we did not sketch what it means that an $(\infty, n+1)$-category has adjoints, but it is essential to understand this definition since we characterized the duals of a monoidal (∞, n)-category in terms of the adjoints of its delooping. The idea is to trace this back to adjoints of 1-morphisms in bicategories.

Definition 4.1.50 (Homotopy 2-category of an (∞, n)-category)**.** A homotopy 2-category of an (∞, n)-category $\mathcal{C}$ with $n \geq 2$ is a bicategory $\mathrm{h}_2\mathcal{C}$ with

- the objects of $\mathcal{C}$ as objects
$$\mathrm{ob}(\mathcal{C}) = \mathrm{ob}(\mathrm{h}_2\,\mathcal{C})$$
- the 1-morphisms of $\mathcal{C}$ as 1-morphisms
$$1\text{-mor}(\mathcal{C}) = 1\text{-mor}(\mathrm{h}_2\,\mathcal{C})$$
- isomorphism classes of 2-morphisms in $\mathcal{C}$ as 2-morphisms, i.e. given 1-morphisms $f, g : X \to Y$ a 2-morphism in $\mathrm{h}_2\,\mathcal{C}$ is a 2-morphism in $\mathcal{C}$ up to 3-isomorphism
$$2\text{-mor}(\mathcal{C})/_{\cong} = 2\text{-mor}(\mathrm{h}_2\,\mathcal{C})$$

Definition 4.1.51 (Adjoints in (∞, n)-categories)**.** We define adjoints for k-morphisms by recursion. An (∞, n)-category $\mathcal{C}$ has adjoints for 1-morphisms if its homotopy 2-category $\mathrm{h}_2\,\mathcal{C}$ has adjoints for 1-morphisms, i.e. if all 1-morphisms of $\mathrm{h}_2\,\mathcal{C}$ are part of an adjunction. An (∞, n)-category $\mathcal{C}$ has adjoints for k-morphisms if, for every pair $X, Y \in \mathrm{ob}(\mathcal{C})$, the $(\infty, n-1)$-category $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has adjoints for $k-1$-morphisms.

[25]A subspecies of homo sapiens. Exemplars usually say stuff like 'I understand differential equations as sub-∞-groupoids of tangent and jet bundles' and have posters of Grothendieck or Lawvere in their bedroom.

[26]In the usual jargon, an (∞, n)-category has adjoints if it has adjoints for any k-morphism with $0 < k < n$. We spelled what it means anyway for the sake of clarity.

We say that an (∞, n)-category has adjoints if it has adjoints for any k-morphism with $0 < k < n$.

Having defined adjoints of morphisms of an (∞, n)-category, we now have all the necessary pieces of the puzzle constituiting the sketch of what it means that a monoidal (∞, n)-category has duals.

Remark. There is an equivalent definition of monoidal (∞, n)-category with duals, see [Lur09] and [GS18]. This intuition about adjoints is very useful also in this case.

Chapter 5

Classification of topological field theories

> **Existence**: Why is there something, rather than nothing?
> This does not seem very accessible by current methods. A more realistic goal may be
> **Classification**: Given that there's something, what could it be?
>
> Jack Morava in [Mor11]

In this chapter, we classify 1d-TFTs and 2d-TFTs, provide a sketch on how that might work for 3d-TFTs and show surprsising connections between knot theory and 3d-TFTs.

Roughly, our classifications theorems will be equivalence of categories of this kind

$$ev_{X\in\mathrm{Bord}_n} : \mathrm{Fun}^{\otimes}(\mathrm{Bord}_n, \mathcal{C}) \simeq (\mathcal{C}^{\mathrm{fd}})^{\cong}$$

meaning that evaluating a TFT on an object of Bord_n induces an equivalence of categories. To be clear, recall that:

- Bord_n is the category of n-bordisms
- $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n, \mathcal{C})$ is the category of n-dimensional topological field theories since they are symmetric monoidal functors from the category of n-bordisms to some symmetric monoidal category
- $(\mathcal{C}^{\mathrm{fd}})^{\cong}$ is the Picard groupoid underlying $\mathcal{C}$ since $\mathcal{C}^{\mathrm{fd}}$ is the subcategory of $\mathcal{C}$ contating only dualizable objects and $(\mathcal{C}^{\mathrm{fd}})^{\cong}$ is the maximal underlying groupoid, restricting dualizable objects in $\mathcal{C}^{\mathrm{fd}}$ to invertible ones

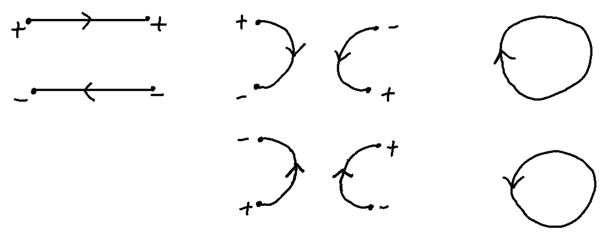
A natural question is: how is this a *classification*? This sort of statement certainly looks different from other classifications, for example the one of 2-dimensional manifolds we have seen in this course (2.3.10). Such an equivalence of categories classifies TFTs because every n-TFT is determined by where it sends an object in Bord_n and thus TFTs are classified by where they send such $n-1$-manifold, without specifying how the functor acts on morphism, i.e. the bordisms. Moreover, given a dualizable object of $\mathcal{C}$ one can construct a TFT since the equivalence goes also in the other direction, so one can say that TFTs are classified by the dualizable objects (up to isomorphism) of their target category.

Sketchily, our strategy to prove these results will be to find generators and relations for Bord_n and then check where they are sent to. It will follow that dualizable objects in $\mathcal{C}$ already provide all the information we need to know to determine what is the TFT in question: where the generators and relations of Bord_n are sent to.

5.1 Classification of 1d-TFTs

The main goal of this subsection is to prove that given an oriented 1d-TFT $\mathcal{Z} : \mathrm{Bord}_{1,0} \to \mathcal{C}$, $\mathcal{Z}(\bullet)$ has a dual and conversely, given an object $X \in \mathcal{C}$ in the target category of the TFT we can reconstruct a 1d-TFT.

Theorem 5.1.1 (Classification of 1-dimensional bordisms)**.** *Any connected oriented 1-dimensional manifold with boundaries is diffeomorphic to one of the following manifolds:*



For the proof we refer to the appendix of [MW97].

This means that any 1-dimensional bordism (i.e. including disconnected ones) is diffeomorphic to a disjoint union of such bordisms. Since in $\mathrm{Bord}_{1,0}$ bordisms are taken up to orientation preserving diffeomorphisms, this means that in $\mathrm{Bord}_{1,0}$ all morphisms *are* tensor products of the listed manifolds. Moreover, we know that the objects of $\mathrm{Bord}_{1,0}$ are just disjoint unions of $\bullet_+$ and $\bullet_-$. Hence, we now know the generators of $\mathrm{Bord}_{1,0}$, i.e. $\bullet_+$ and $\bullet_-$, and the relations, i.e. the possible connected 1-dimensional bordisms. Since we also know that $\mathcal{Z}$ is a *symmetric monoidal* functor, it means that we shall just check where such generators and relations are sent, see **??**.

Remark. By $(\mathcal{C}^{\mathrm{fd}})^{\cong}$ we denote the restriction of a symmetric monoidal category on its dualizable objects and its isomorphisms, said briefly: a restriction on the underlying Picard groupoid because in any symmetric monoidal groupoid an object is dualizable if and only if it is invertible (see 4.1.47).

Theorem 5.1.2. *Let $\mathcal{C}$ be a symmetric monoidal category. Then the map*[1]

$$\mathrm{TFT}^{or}_{1,0} = \mathrm{Fun}^{\otimes}(\mathrm{Bord}^{or}_{1,0}, \mathcal{C}) \xrightarrow{\mathcal{E}v_{\bullet_+}} (\mathcal{C}^{\mathrm{fd}})^{\cong} \tag{5.1}$$

$$\mathcal{Z} \mapsto \mathcal{Z}(\bullet_+) \tag{5.2}$$

[1]There is unfortunately a notational clash between what we called the evaluation map for dualizable objects, denoted by ev, and what we also call in this case evaluation map, denoted by $\mathcal{E}V$. In the latter case, we are referring to a similar convention we use for example in topology when we call a map of this genre $Map(*, X) \to X$ an evaluation map. The intuition is that we feed an element of the domain into a map, a functor in our case, and see what happens, "evaluate it".

is a symmetric monoidal equivalence of groupoids.

$$\begin{tikzcd}
\mathrm{TFT}^{or}_{1,0} \arrow[rr, "\mathscr{E}v_{\bullet_+}"] \arrow[dr, dashed] \arrow[dd, "\simeq"] & & \mathcal{C} \\
& \mathcal{C}^{\mathrm{fd}} \arrow[ur, hook, "j"'] & \\
(\mathcal{C}^{\mathrm{fd}})^{\cong} \arrow[ur, hook, "i"'] & &
\end{tikzcd}$$

The image of $\mathscr{E}v_{\bullet_+}$ is in $(\mathcal{C}^{\mathrm{fd}})^{\cong}$, since

1. $\mathrm{TFT}^{\mathrm{or}}_{1,0}$ is a groupoid, and functors preserve isomorphisms

2. Every object in $\mathrm{Bord}^{\mathrm{or}}_{1,0}$ is dualizable and, because of functoriality, the image of a dualizable object remains a dualizable object.

Proof. The following proof is the closest to what we did in class and is from [Fre13, Theorem 16.10].

We need to prove three things to demonstrate that $\mathscr{E}V_{\bullet_+}$ is an equivalence: that it is full, faithful and essentially surjective:

1. Trivially, note that since $\mathscr{E}V_{\bullet_+}$ lands in $(\mathcal{C}^{\mathrm{fd}})^{\cong}$ then it must be essentially surjective since any two objects are isomorphic.

2. It is faithful: let $\mathcal{Z},\mathcal{Z}' \in \mathrm{TFT}^{or}_{1,0}$ and $\alpha,\beta : \mathcal{Z} \Rightarrow \mathcal{Z}'$ be natural transformations between them and hence isomorphisms (see 4.1.43). Suppose that $\mathscr{E}V_{\bullet_+}(\alpha) = \mathscr{E}V(\beta)$, i.e. $\alpha^1_{\bullet_+} = \alpha^2_{\bullet_+}$ since $\mathscr{E}V_{\bullet_+}$ evaluates on the point. Recall that $\bullet_- = \bullet_+^\vee$. We have that for any natural isomorphism η it holds that $\eta\bullet_+^\vee = (\eta\bullet_+)^\vee$ because of functoriality and $\eta_{\bullet_-} = (\eta\bullet_+{}^\vee)^{-1}$ because of 4.1.41. From this it follows that $\alpha_{\bullet_-} = (\alpha^\vee_{\bullet_+})^{-1} = (\beta\bullet_+{}^\vee)^{-1} = \beta_{\bullet_-}$. With a specular proof we can prove that $\alpha_{\bullet_+} = \beta_{\bullet_+}$. Since 0-dimensional compact manifolds are just finite disjoint unions of $\bullet_+$ or $\bullet_-$ and α and β are symmetric monoidal natural transformations it follows that for any $X \in \mathrm{Bord}^{or}_{1,0}$, $\alpha_X = \beta_X$. This proves that $\mathscr{E}V$ is faithful because it implies that for any two natural transformations that are mapped to equal morphisms in $\mathcal{C}$ must be also equal in $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_{1,0},\mathcal{C})$

3. $\mathscr{E}V_{\bullet_+}$ is full. Given an arbitrary isomorphism $f : X \to X'$ in $(\mathcal{C}^{\mathrm{fd}})^{\cong}$ we now show that there is an isomorphism $\eta : \mathcal{Z} \Rightarrow \mathcal{Z}'$ in $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_{1,0},\mathcal{C})$ such that $\mathscr{E}V_{\bullet_+}(\eta) = \eta_{\bullet_+} = f$. So, we define $f = \eta$ and thus $(f^\vee)^{-1}$. Objects in $(\mathcal{C}^{\mathrm{fd}})^{\cong}$ are sequences of tensor products of X and $X^\vee$. Hence, using the trick we used before we can extend any natural transformation $\eta : \bullet_+ \to \bullet_+$ to a disjoint union of such natural transfomations, via the monoidal structure on $\mathrm{Bord}_{1,0}$, since any object $X \in \mathrm{Bord}_{1,0}$ is diffeomorphic to a finite disjoint union of $\bullet_+$ and $\bullet_-$, and we know that $(\alpha^\vee_{\bullet_+})^{-1}$. This is independent of the choice of the diffeomorphism between X and disjoint unions of $\bullet_+$ and $\bullet_-$ thanks to the coherence of the chosen map η_X. Moreover, the diffeomorphism is determined up to permutation also because of the coherence of η_X . It remains to show that η_X is an isomorphism. η_X can be a disjoint union of the 5 1-bordisms (5.1.1) and hence, thanks to symmetric monoidality of η, we can just check these

5. The two identities are trivially isomorphisms. Then we need to check that the following diagram commutes to check for the coevaluation

$$\begin{tikzcd}
\mathcal{Z}(\bullet_-)\otimes\mathcal{Z}(\bullet_+) \arrow[rr, "f\circ(f^\vee)^{-1}"] & & \mathcal{Z}'(\bullet_-)\otimes\mathcal{Z}'(\bullet_+) \\
& \mathbb{1}_{\mathcal{C}} \arrow[ul, "\mathcal{Z}(coev_X)"] \arrow[ur, "\mathcal{Z}'(coev_X)"'] &
\end{tikzcd}$$

The commutativity of this diagram follows from the fact that the following diagram commutes

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{C}} \arrow[rr, "\mathcal{Z}(coev_X)"] \arrow[d, "\mathcal{Z}'(coev_X)"'] & & \mathcal{Z}(\bullet_-)\otimes\mathcal{Z}(\bullet_+) \arrow[d, "id_{\mathcal{Z}(\bullet_-)}\otimes f"] \arrow[ddll, "id_{\mathcal{Z}(\bullet_-)}\otimes id_{\mathcal{Z}(\bullet_+)}\otimes\mathcal{Z}(coev_x)"'] \\
\mathcal{Z}'(\bullet_-)\otimes\mathcal{Z}'(\bullet_+) \arrow[d, "\mathcal{Z}(coev_x)\otimes id_{\mathcal{Z}'(\bullet_-)}\otimes id_{\mathcal{Z}'(\bullet_+)}"'] & & \mathcal{Z}(\bullet_-)\otimes\mathcal{Z}'(\bullet_+) \arrow[d, "id_{\mathcal{Z}(\bullet_-)}\otimes\mathcal{Z}'(coev_x)\otimes id_{\mathcal{Z}(\bullet_+)}"] \\
\mathcal{Z}(\bullet_-)\otimes\mathcal{Z}(\bullet_+)\otimes\mathcal{Z}'(\bullet_-)\otimes\mathcal{Z}'(\bullet_+) \arrow[rr, "id_{\mathcal{Z}(\bullet_-)}\otimes f\otimes id_{\mathcal{Z}'(\bullet_-)}\otimes id_{\mathcal{Z}'(\bullet_+)}"'] & & \mathcal{Z}(\bullet_-)\otimes\mathcal{Z}'(\bullet_+)\otimes\mathcal{Z}'(\bullet_-)\otimes\mathcal{Z}'(\bullet_+)
\end{tikzcd}$$

The argument for the evaluation of X is specular. From there also the map for the circle follows since the circle is just a composition of evaluation and a coevaluation with a braiding in between. Note that the commutativity of the circle means that $\mathcal{Z}(S^1) = \mathcal{Z}'(S^1)$. □

Remark. The author of this part of the notes was very confused for a long time by the proof just provided. They cannot understand Freed's line of argument when it comes to the proof of the fullness of $\mathcal{E}V_{\bullet_+}$, probably because of notational choices and some typos. Hence, we now give 2 alternative proofs, in order to make it as clear as possible.

Theorem 5.1.3 (Alternative formulation of the classification of 1TFTs)**.** *Let $X \in \mathcal{C}$ be a dualizable object. Specifying $\mathcal{Z}(\bullet_+) = X$ determines a 1d-TFT.*

Proof of the classification of 1-TFTs 2.0. An alternative proof relies proves the equivalence in the other direction and relies on the fact that since the functor is *symmetric* monoidal one can just check what the TFT does only for 5 of all the possible connected 1-dimensional bordisms (5.1.1), not considering what happens for the other 3 ones with the reversed orientations without loss of generality because $\mathrm{Bord}_{1,0}$ is symmetric monoidal. We took it from [Lur09, Example 1.1.9]

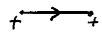
First we need to determine where $\bullet_+, \bullet_- \in \mathrm{Bord}_{1,0}$ are sent. Since they are duals of one another, they are dualizable objects and thus they are sent to $X, X^\vee \in \mathcal{C}^{\mathrm{fd}}$, e.g. finite dimensional vector spaces in the case of Vect_k. An arbitrary objects of $M \in \mathrm{Bord}_{1,0}$ is a finite disjoint union of $\bullet_+, \bullet_-$, i.e.

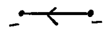
$$M = (\amalg_I \bullet_+) \amalg (\amalg_J \bullet_-)$$

where $I, J \subseteq \mathbb{N}$. Thanks to the symmetric monoidal functoriality of $\mathcal{Z}$ we have that

$$\mathcal{Z}(M) = (\bigotimes_I X) \otimes (\bigotimes_J X^\vee)$$

Now we need to understand where the 5 possible 1-bordisms are sent.

1. $\mathcal{Z}($  $) = id_X$

2. $\mathcal{Z}($  $) = id_{X^\vee}$

3. $\mathcal{Z}($  $) = ev_X$, hence $X \amalg X^\vee \xrightarrow{ev_X} \mathbb{1}_{\mathcal{C}}$ and more precisely for $v \in X \in \mathrm{Vect}_k$: $ev_X : (X, \lambda) \mapsto \lambda(v)$

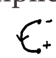
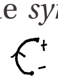
4. $\mathcal{Z}($  $) = coev_X$, hence $\mathbb{1}_{\mathcal{C}} \xrightarrow{coev_X} X \otimes X^\vee$ and more precisely for $\lambda \in k \in \mathrm{Vect}_k$, via the canonical isomorphism $X \otimes X^\vee \cong \mathrm{End}(X)$, $coev_X$ corresponds to taking the trace

5. $\mathcal{Z}($  $) = coev_X \circ ev_X$, hence $\mathbb{1}_{\mathcal{C}} \xrightarrow{coev_X \circ ev_X} \mathbb{1}_{\mathcal{C}}$, and, following the cases for Vect_k treated in the previous two points, to taking the trace of the identity matrix, i.e. the dimension of X

The moral of the story is that a TFT is uniquely determined, up to isomorphism, once one knows $X \in \mathcal{C}^{\mathrm{fd}}$ since then one knows

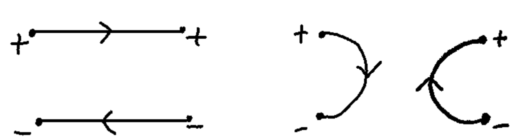
- what $X^\vee$ is, what finite sequences of tensored X and $X^\vee$ are, which will be sent to finite disjoint unions of $\bullet_+$ and $\bullet_-$ (aka all the objects of $\mathrm{Bord}_{1,0}$);
- what id_X and $id_{X^\vee}$ are
- what ev_X and $coev_X$ are and hence also $coev_X \circ ev_X$

□

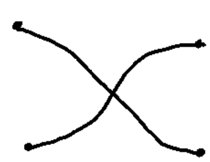
Remark. Lurie considers 5 possible bordisms and not all 8, thereby excluding some possible orientations. However, Lurie's argument is without loss of generality because it implicitly relies on the *symmetric* monoidality of $\mathrm{Bord}_{1,0}$ since, for instance, one can get  by composing  with a braiding.

Anyways, we hold that there is a clearer way to demonstrate this result in a similar vein, which relies on the following result.

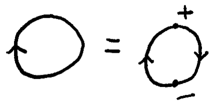
Lemma 5.1.4. *All 1-dimensional bordisms are generated by the generators:*

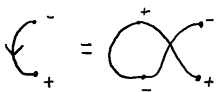


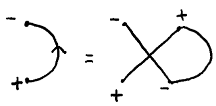
and the relation

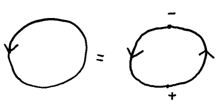


Proof. We need to show that there is a way of gluing our generators and the braiding relation to get the other 4 manifolds of 5.1.1.

1. 

2. 

3. 

4. 

□

Proof of the classification of 1-TFTs 2.1 . In this proof we rely on the lemma we just proved. We need to show where the objects, the generators and the relation are sent to. Fortunately, we showed already in the proof took from Lurie's paper (5.1) where the objects and our generators are sent to. It remains to show where the braiding relation  is sent to. Since $\mathcal{Z}$ is a symmetric monoidal functor, the braiding in $\mathrm{Bord}_{1,0}$ is sent to the braiding in the target category $\mathcal{C}$. □

The classification of 1-dimensional topological field theories is the simplest case of an important guiding hypothesis in the field of TFTs, the cobordism hypothesis, see 5.6. It is the only case in which so-called extended TFTs coincide with non-extended ones, i.e. the usual Atiyah-Segal definition as a simple symmetric monoidal functor we are using.

Recap: $\mathrm{TFT}^{or}_{1,0}(\mathcal{C}) \xrightarrow{\mathcal{E}v_{\bullet_+}} (\mathcal{C}^{\mathrm{fd}})^{\simeq} \hookrightarrow \mathcal{C}$. The map is the following: $\mathcal{Z} \mapsto \mathcal{Z}(\bullet_+)$.

Now we classified 1 dimensional TFTs, we would now like to do the same for 2 dimensional TFTs. We will find out that there is an equivalence between $\mathrm{TFT}^{or}_{2,1}(\mathcal{C})$ and commutative Frobenius algebras.

5.2 Classification of 2d-TFTs

As we have done for the 1-dimensional case, we now try to classify 2-dimensional TFTs. For a more detailed proof see Kock [Koc03] or the lecture notes from Schweigert [Sch23].

We have already defined the following notions in arbitrary monoidal categories (see 3.9.6, 3.9.2, 3.9.4, 3.10.9, 3.10.8). However, we now explicitly state how the abstract formal definitions are instantiated in Vect_k.

Definition 5.2.1 (Algebra over a Field)**.** An algebra[2] over a field k is a monoid object (3.9.2) in Vect_k. More generally a left/right $R-$algebra is a monoid object in the category of left/right $R-$modules. The latter generalization holds also for the next definitions of $k-$coalgebra, $k-$bialgebra and Frobenius $k-$algebra.

[2]Note that for us algebras are **always** associative and unital.

Definition 5.2.2 (Coalgebra over a Field)**.** A $k-$coalgebra is a comonoid object in Vect_k.

Definition 5.2.3 (Bialgebra over a Field)**.** A $k-$bialgebra is a bimonoid object (see 3.9.5) in Vect_k. More explicitly, it is simultaneously an $k-$algebra and a $k-$coalgebra, a monoid and a comonoid object in Vect_k. A $k-$bialgebra is commutative in Vect_k, if the underlying monoid is commutative, or viceversa, if the underlying comonoid is commutative.

Definition 5.2.4 (Frobenius $k-$Algebra)**.** A Frobenius $k-$algebra is a Frobenius algebra (see 3.9.6) in a Vect_k. It is commutative if it is also a commutative $k-$algebra (and therefore a cocommutative coalgebra).

These definitions are perfectly fine, if we disregard pedagogical considerations. However, they might still seem mysterious to someone not used to this abstract nonsense. For this reason, we now give three more concrete definitions of a Frobenius algebra. They are all equivalent.

Reminder. A pairing $K : W \otimes V \to k$ is non-degenerate if it is part of a data exhibiting W as the dual of V, i.e. $\exists \gamma : k \to V \otimes W$ such that ...

If V, W finite dimensional, this is equivalent to saying that

$$K^{\#} : V \to W^{\vee} := \mathrm{Hom}_k(W, k), \quad v \mapsto K(- \otimes v) \tag{5.3}$$

$$K_{\#} : W \to V^{\vee} := \mathrm{Hom}_k(V, k), \quad w \mapsto K(w \otimes -) \tag{5.4}$$

are isomorphisms.

Exercise 5.2.5. What's the analogous reformulation of "$X \in \mathcal{C}$ is dualizable"?

Definition 5.2.6. A k-algebra is a monoid object in Vect_k. More explicitly, it is a k-vector space A together with linear maps:

$$\mu : A \otimes A \to A \tag{5.5}$$

$$\eta : k \to A \tag{5.6}$$

which satisfy associativity and (right and left) unitality by making the following two diagrams commute

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[r, "\alpha_{A,A,A}"] \arrow[d, "\mu \otimes id_A"'] & A \otimes (A \otimes A) \arrow[r, "id_A \otimes \mu"] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[rr, "\mu"'] & & A
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{C}} \otimes A \arrow[r, "\eta \otimes id_A"] \arrow[dr, "\lambda_A"'] & A \otimes A \arrow[d, dashed, "\mu"] & M \otimes \mathbb{1}_{\mathcal{C}} \arrow[l, "id_A \otimes \eta"'] \arrow[dl, "\rho_A"] \\
& M &
\end{tikzcd}$$

A homomorphism $\phi : A \to B$ of k algebras is a linear map that preserves/commutes with (μ_A, η_A) and (μ_B, η_B), a monoid homomorphism.

Definition 5.2.7. A *co*algebra is a comonoid object in Vect_k. More explicitly, one simply takes

$$\Delta : A \to A \otimes A$$

$$\epsilon : A \to k$$

and reverses the arrows in the diagrams.

A homomorphisms of k-coalgebras accordingly preserve (Δ_A, ϵ_A) and (Δ_B, ϵ_B). It is a monoid homomorphism in Vect_k^{op}.

Example 5.2.8. Take the vector space $A = k[x]/x^2 = k \oplus kx$ and take the map $A \to A \otimes A$ which sends $x \mapsto 1 \otimes x + x \otimes 1$ and $1 \mapsto 1 \otimes 1$ and the map $\epsilon : A \to k$ which sends 1 to 1 and x to 0.

Definition 5.2.9 (First definition)**.** A (k-) Frobenius algebra is a (finite dimensional) k algebra (A, μ, ν) together with an associative (=invariant) non-degenerate pairing $k : A \otimes A \to k$, i.e.

$$K(ab, c) = K(a, bc) \tag{5.7}$$

The fact that the pairing is invariant actually tells us something about the algebra structure, not only the vector space structure.

Example 5.2.10. $A = \mathrm{Mat}_{n\times n}(k)$ with matrix multiplication, in which the pairing K is simply the composition of multiplication and taking the trace, i.e. $K = \mathrm{tr} \circ \mu$.

Proof. Need to show nondegeneracy. Pick a basis of $\mathrm{Mat}_{n\times n}(k)$ given by $\{E_{ij}\}$ with 1 in the jth row and ith column. We then have the dual basis $\{E_{ji}\}$ and we have an isomorphism of A and $A^\vee$ given by $E_{ij} \mapsto E_{ji}$. Then we can compute: K is exactly the evaluation. □

Note that in the proof we used an isomorphism of A and $A^\vee$ to get the nondegenerate invariant pairing. This leads to the following alternative definition:

Definition 5.2.11 (Second definition)**.** A ($\Phi-$) Frobenius algebra is a (finite dimensional) algebra A with a left $A-$module isomorphism $\Phi : A \to A^\vee$.

The left $A-$module part again gives us the compatibility with the algebra structure.

Remark. For the definition to make sense A and $A^\vee$ should be left $A-$modules:

- $M = A$ is an A module via $A \otimes M \to M$ which sends $(a, m) \mapsto \mu(a, m)$.
- $M = A^\vee$ is an A module via $A \otimes M \to M = A^\vee = \mathrm{Hom}(A, k)$ which sends $(a, \phi) \mapsto \phi(\mu(a, -))$.

Then a left $A-$module map should satisfy $\Phi(a \cdot m) = a \cdot \Phi(m)$.

Example 5.2.12. Let's check that the map Φ in the previous example was actually a left $A-$module isomorphism. We would like

$$\Phi(A \cdot E_{ij}) = A \cdot \Phi(E_{ij}) \tag{5.8}$$

which is true when explicitly calculating both sides.

Definition 5.2.13 (Third definition)**.** A $(\Delta, \epsilon)-$ Frobenius algebra is a finite dimensional algebra (A, μ, ν) together with a coalgebra structure (A, Δ, ϵ) such that the *Frobenius relation*

$$(\mathbb{1}_{\mathcal{C}} \otimes \mu) \circ (\Delta \otimes \mathbb{1}_{\mathcal{C}}) = \Delta \circ \mu = (\mu \otimes \mathbb{1}_{\mathcal{C}}) \circ (\mathbb{1}_{\mathcal{C}} \otimes \Delta)$$

holds. This means that the following diagram commutes

$$\begin{tikzcd}
& A \otimes A \otimes A \arrow[dr, "\mu \otimes id_A"] & \\
A \otimes A \arrow[ur, "id_A \otimes \Delta"] \arrow[r, "\mu"] \arrow[dr, "\Delta \otimes id_A"'] & A \arrow[r, "\Delta"] & A \otimes A \\
& A \otimes A \otimes A \arrow[ur, "id_A \otimes \mu"'] &
\end{tikzcd}$$

Note that the the only way in which we exploited the fact that we are working with vector spaces is that we had the tensor product and the unit with respect to this product. In other words, we only used the fact that Vect is a symmetric monoidal category, so the definitions work in any symmetric monoidal category[3]. This is important to keep in mind because the target category of our 2d-TFTs might not be Vect_k.

Proposition 5.2.14. *The three definitions of Frobenius algebra are equivalent.*

Proof. (1) $\iff$ (2) is an exercise.

(3) $\implies$ (1): we set $K = \epsilon \circ \mu : A \otimes A \to k$ and $\gamma = \Delta \circ \eta$, however these are not duality data that show that K is a nondegenerate pairing.
We can then define $\Phi := (id_{A^\vee} \otimes K) \circ (coev_A \otimes id_A)$ which is an isomorphism with inverse $id_A \otimes ev_A) \circ (\gamma \otimes id_{A^\vee})$. Then using in addition this isomorphism one can prove that K is a nondegenerate pairing.

(1) $\implies$ (3) we can define $\Delta := \mu \circ (\gamma \otimes id_A)$ and $\epsilon := K \circ (id \otimes \mu)$. □

Reminder. $\mathrm{TFT}^{or}_{2,1}(\mathcal{C})$ is a groupoid (see 4.1.43).

Theorem 5.2.15 (Classification of 2 dimensional TFTs)**.** *The functor*

$$\mathrm{TFT}^{or}_{2,1}(\mathcal{C}) \to \mathrm{cFrob}_{\mathcal{C}}$$

$$\mathcal{Z} \mapsto \mathcal{Z}(S^1)$$

is an equivalence of groupoids[4].

The category $\mathrm{cFrob}_{\mathcal{C}}$ has as objects commutative Frobenius algebras on an arbitrary symmetric monoidal category $\mathcal{C}$, i.e. Frobenius algebras that are also commutative bimonoid objects (3.10.8), as morphisms it has Frobenius homomorphisms, i.e. morphisms of monoids that are also morphisms of comonoids and preserve all the structure of a Frobenius algebra. One could have also proven a statement with Vect_k as the target category and proven that $\mathrm{TFT}^{or}_{2,1}(\mathrm{Vect}_k) \simeq \mathrm{cFrob}_k$. Note that the category of commutative Frobenius algebras in an arbitrary symmetric monoidal category is indeed a groupoid:

Theorem 5.2.16. *The category of Frobenius algebras on an arbitrary symmetric monoidal category $\mathcal{C}$ is a groupoid.*

Proof. Let $(A, \epsilon, \eta, \Delta, \mu), (A', \epsilon', \eta', \Delta', \mu') \in \mathrm{cFrob}_{\mathcal{C}}$ and $\phi : A \to A'$ be a Frobenius homomorphism. Then ... □

Take a look at [Koc03] for a proof for Frobenius k-algebras, i.e. Frobenius algebras in Vect_k.

Proof. (1) We need to prove that $\mathcal{Z}(S^1) =: A$ is a commutative Frobenius algebra. We get the product from the pair of pants and the coproduct from the copants. The unit and counit maps are simply given by the cup and cocup. Showing that this is actually a Frobenius algebra now just amounts to drawing the commutative diagram for a Frobenius algebra in the category of bordisms, i.e. all maps are simply bordisms.

(2) We now want see that it's a functor. Let $\mathcal{Z}, \mathcal{Z}'$ be two TFTs and $\alpha : \mathcal{Z} \implies \mathcal{Z}'$ be a natural transformation.

[3] Take a detour in the subsection on monoidal categories (3.7), and more specifically take a look at 3.9.6, if you want to see a definition for general monoidal categories.

[4] We soon prove that also the category of Frobenius algebras is a groupoid.

Claim. *$f = \alpha(S) : A = \mathcal{Z}(S^1) \to \mathcal{Z}'(S^1) = B$ is a homomorphism of commutative Frobenius algebras.*

We just use naturality of α several times. □

For functoriality we did not check that compositions of morphisms (natural transformations) are sent to compositions (of morphisms in cFrob).

Now, we want to prove the converse direction: given a commutative Frobenius algebra A, construct an oriented 2d-TFT such that $\mathcal{Z}(S^1) = A$. We want to define a TFT $\mathcal{Z}$. On objects we simply set $\mathcal{Z}(S^1, or_+) \mapsto A$ and $\mathcal{Z}(S^1, or_-) \mapsto A^\vee$. This is enough because objects in $\mathrm{Bord}^{or}_{2,1}$ are closed 1 dimensional manifolds and therefore diffeomorphic to a disjoint union of S^1s. Therefore, if Y is a connected one dimensional manifold we have an orientation preserving diffeomorphism $Y \to S^1$, but in which sense is this unique?

Definition 5.2.17 (Diffeomorphism Group)**.** Let $X \in$ SmoothMfld. Diff(X) denotes the automorphism group of X, i.e. the group of diffeomorphisms $X \xrightarrow{\phi} X$.

Note that the diffeomorphism group can be considered a topological group (see 3.9.8) if we use an apt topology, for example one can consider Diff(X) a subspace of $(C^\infty(X,X), Whitney)$, where $Whitney$ denotes the Whitney C^∞ topology, and endow it with the subspace topology.

Remark. An isotopy (see 3.10.17 for the definition) is equivalently a path in Diff(X).

Fact. *Roations, SO(2) $\hookrightarrow$ $\mathrm{Diff}^{or}(S^1)$ and this is a retraction.*

Thus, up to "wiggling", the map $Y \to S^1$ is unique.

Upshot: for an oriented 1 manifold Y we define

$$\mathcal{Z}(Y) := A^{\#\pi_0(Y)} \tag{5.9}$$

and diffeos are sent to identities.

Now what do we do on morphisms? Just send everything to what we expect from the algebra and coalgebra structure! i.e. pants to multiplication, copants to comultiplication, cylinder to identity, cup to algebra unit, cocup to counit. Now, a morphism in the bordism category is a diffeomorphism class of oriented 2d bordisms and by the classification theorem we found that a 2d connected oriented manifold with boundary is diffeomorphic to a composition. A bordism then specifies where "in" and "outgoing" boundaries are. Since $\mathcal{Z}$ must be functorial, given a composite as in the drawing, we must define $\mathcal{Z}$ to be

$$\mathcal{Z}(\Delta)....$$

Question: is this well defined?

Going back to proof of classification, the local moves we had translate precisely to conditions of a Frobenius algebra. Subtlety: the following are not isomorphic as bordisms, while they are diffeomorphic as manifolds with boundary.

We then need to specify what that bordism is sent to:

$$\mathcal{Z}(..) : A \otimes A \xrightarrow{swap} A \otimes A \tag{5.10}$$

So the classification of (possibly disconnected) 2d bordisms is simply a manifold with boundary as given from the classification of manifolds with boundary, composed with a permutation bordism, i.e. a bordism such as the following.

Now, since $\mathcal{Z}$ is functorial we simply set

$$\mathcal{Z}(\ldots) \tag{5.11}$$

and for well definedness consider the following drawing:

but we see that this is true because the Frobenius algebra is commutative.

Lemma 5.2.18. *Every (possibly non connected) 2 cobordism is a composition of*

1. *a "permutation bordism". i.e. given a permutation $\sigma \in S_n$, then we get a bordism $S^{1^{1\amalg n}} \to (S^1)^{\amalg n}$ in which the two sides have the same orientation up to interchanging components. The bordism is simply $(S^1)^{\amalg n}$ in which on the right we use the permutation.*
2. *a disjoint union of connected 2 bordisms*
3. *another permutation bordism.*

Proposition 5.2.19. $\mathrm{Bord}_{2,1}^{or}$ *is the symmetric monoidal category with duals generated by (under composition and disjoint union):*

- *one object, S^1.*
- *morphisms the ones we've already mentioned: cup, pants, cocup, copants, swap (and cylinder but that's just the identity).*

with the following relations:

- *the cylinder is the identity, so composed with all the other bordisms it gives back the same boridsm*
- *sewing in disks*
- *(co)associativity*
- *(co)commutativity*
- *Frobenius relations*

In other words, $\mathrm{Bord}_{2,1}^{or}$ is free symmetric monoidal category with duals on one commutative Frobenius object S^1.

Usual depictions of orientations

Let's now do a recap of what we've done up to now. We proved

$$\mathrm{TFT}_{2,1}^{or}(\mathcal{C}) \xrightarrow[Ev]{\simeq} \mathrm{cFrob}(\mathcal{C})^{\simeq} \tag{5.12}$$

and the steps were the following:

- the functor is well defined: on objects we get $Ev(\mathcal{Z}) = \mathcal{Z}(S^1)$ which is a commutative Frobenius algebra. On morphisms we get an isomorphism of commutative Frobenius algebras.
- Ev is essentially surjective, actually we proved surjective.
- Ev is fully faithful

Remark. Last time we forgot about orientations...

5.3 Variants of TFTs

We now provide an outlook to some research programs that are tightly connected to TFTs.

- we could have used different tangential structures, such as
  - unoriented bordisms. In this case we would need an isomorphism $A \cong A^\vee$ which should be an isomorphism of commutative Frobenius algebras.
  - framing: not many framed closed 2 manifolds
  - spin structure
  - conformal structure and thereby we would have studied *conformal* field theories

- open-closed TFTs (Lauda-Pfeiffer, [LP08]): the idea is to enlarge the bordism category to include *compact* 1d manifolds (not necessarily closed) and we then also have additional morphisms.

  Now, what structure does the line segment have? It's still a Frobenius algebra! But it is not necessarily commutative.

  There is a more general result on the classification of such field theories by Kevin Costello [Cos06].

- extended TFTs ([Fre94],[BD95],[Lur09],[CS19]): one can extend TFTs downward, e.g. in 2-dimensional TFTs one might want to include the possibility of "composing" the line segment object with itself to get S^1. We would therefore modify the notion of symmetric monoidal category to be able to compose objects in $\mathrm{Bord}_{2,1,0}$. Thereby one gets a finer structure than a usual category, a.k.a. a 1-category: objects, morphisms between objects and morphism between morphisms. The last ones are usually called 2-morphisms and the morphisms between objects are then called 1-morphisms. Objects, 1-morphisms and 2-morphisms are in this case 0 manifolds, 1-dimensional bordisms and 2 dimensional bordisms with corners. We then no longer have the structure of a category, but rather that of a weak 2-category, or bicategory. See 3.6.3 for the definition of strict categories of any finite dimension, 3.10.29 for the definition of a bicategory, and 5.6 for a more detailed outlook on extended TFTs.

- cohomological TFTs ([Wit91]): aka families of $(n-1)$ manifolds and families of n bordisms. In practice, fix a space X, then objects are M closed $(n-1)$ manifolds together with $M \to X$ and morphisms are bordisms Σ together with $\Sigma \to X$.

  Often we take $X = BG$ the classifying space of a group G and having maps $M \to BG$ and $\Sigma \to BG$ gives us principle bundles on M and Σ. This is connected to the field of *gauge theory* and is sometimes called *cohomological TFT*.

✈ arithmetic field theories ([Maz73],[Kim16],[KW07], [Kap95],[Dav23]): although still very conjectural, some envision poetic connections between number theory and physics via TFTs. The original intuition behind this connection is a mysterious analogy of Barry Mazur stating that number fields form correspond to some kind of 3-dimensional manifolds and primes in such number fields behave like knots embedded in these 3-dimensional manifolds. How this connects to TFTs will become clearer after the next section, where we will have gone through the connection between the

Jones polynomial (a knot invariant) and 3d-Chern-Simons theory. For instance, the spectrum[5] of the commutative ring of the integers $\mathbb{Z}$ can be seen (roughly speaking[6]) as a 3-dimensional manifold[7] , and since the spectra prime fields can be canonically embedded in the spectra of the integers, they are analogous to knots since knots can be defined as embeddings of S^1 into some 3-dimensional manifold[8]. Mazur came up with this analogy in [Maz73], see this mathoverflow comment by Lieven Lebruyn and the nLab entry on Spec($\mathbb{Z}$) for accessible introductions to this specific correspondence. Some ([Kim16], Clark Barwick's SHIFTED project and this talk of his on his efforts towards geometric foundations for arithmetic field theories) are trying to define arithmetic quantum field theories in order to apply ideas of TFTs to number theory, for example to arithmetic curves[9], i.e. the spectra of rings of integers in algebraic number fields. While others ([Wit91],[Dav23]) more specifically, see a strong connection with the Langlands program. This view is summarized by the motto *the Langlands program is part of the study of 4-dimensional (arithmetic, topological) quantum field theory.* For a more accessible introduction to this research program, check out these notes [DBZ21] by Jackson Van Dyke of a course on this topic by David Ben-Zvi.

✈ algebraic quantum field theory ([HK64]): talking about endomorphism monoids, i.e. monoids of the form $(\mathrm{Hom}_{\mathcal{C}}(X,X),\circ)$ (see 3.2.5) we previously mentioned operator algebras, i.e. monoids and submonoids of the form $(\mathrm{Hom}_{\mathrm{TopVect}_k}(X,X),\circ)$. They are an object of interest in themselves but importantly for us algebraic quantum field theory makes extensive use of operator algebras. Algebraic quantum field theory is very roughly a way of translating the Heisenberg picture of quantum mechanics (sometimes called matrix mechanics) to quantum field theory by providing an attempt to axiomatize QFT in a mathematically rigorous way. Given some apt operators on an apt topological vector space, meaning that the topological vector space can represent a quantum state and the selected operators on it can represent observables, an AQFT provides a formal assignment of algebras of such operators to regions of spacetime according to some axioms, e.g. the Haag Kastler Axioms. In this way, it pins down how observables depend of spacetime while states are being fixed, as in the Heisenberg picture. Algebraic quantum field theory can be roughly considered to be a dual to topological quantum field theory since TFTs invert the direction by assigning geometric data to algebraic ones. In fact, topological quantum field theory can be seen as a way of translating the Schrödinger picture (also called wave mechanics) to quantum field theory: states are represented by objects in the target monoidal category, the functor from Bord captures the local time evolution, and thereby a TFT shows how states are propagated through time while the observables are held fixed, as in the Schrödinger picture (see more on this at this section of the nLab entry on TFTs). In quantum mechanics, the Heisenberg and the Schrödinger picture are two

[5] Not in the sense of a modern abelian group, which we previously described 4.1.1! This is the spectrum of a ring.

[6] To make this more precise Mazur talks about exotic stuff like étale cohomological dimension.

[7] This analogy has been very recently made precise by Peter Scholze via the framework of condensed mathematics. He talked about this in this talk `https://archive.mpim-bonn.mpg.de/id/eprint/4956/`.

[8] Often it is sufficient to pick $\mathbb{R}^3$ as the manifold we embed in and it is what we will do in the next section. However, taking arbitrary 3 manifolds does also make sense and in some cases, like this one, it is what one desires.

[9] Something from arithmetic geometry, see [Liu02]

faces of the same medal, i.e. matrix mechanics and wave mechanics are equivalent[10]; however, we are far from proving that AQFT and TQFT are also two faces of the same medal. One point of contact between the two is via factorization algebras (something related to E_n-algebras, see 3.10.1 for the definition of a E_1-algebra), see [CG23] for a summary on factorization algebras and how they relate to TQFT and AQFT.

In conclusion, algebraic quantum field theory is not a variant of topological field theory, but an alternative attempt to put quantum field theory on mathematically rigorous footings. One can discover more on algebraic quantum field theory in [HM06] and [FR19].

✈ supersymmetric field theory: supersymmetric field theories are field theories defined as a functor similarly to TFTs but instead of having usual manifolds in the source, they have objects called super manifolds. Stefan Stolz and Peter Teichner conjectured a deep relation between elliptic cohomology and such quantum field theories (see [TS04])

5.4 3d TFTs

From now on the main reference will be [KRT97].

For classifying TFTs in 1 and 2 dimensions the procedure we have been following is to find an algebraic category which is equivalent with such TFTs evaluated on some object. In particular, in order to classify 1d TFTs we established an equivalence with the evaluation functor of 1TFTs on a point and finite dimensional vector spaces and for 2d TFTs evaluated on the circle S^1 are equivalent to commutative Frobenius algebras in some symmetric monoidal category and to finite-dimensional Frobenius k-algebras when considering Vect_k as the target category. These equivalences of categories allow us to reconstruct 1d and 2d TFTs (up to some reasonable form of equivalence) from this purely algebraic data. One can ask themselves if a classification of this kind is possible also for higher dimension, such as 3d TFTs. First, 3d-TFTs can be constructed from algebraic data, although a bit more sophisticated than what we got in lower dimensions, we will find out that every *modular tensor category* gives a 3d TFT[11]. The algebraic data can be summarized diagramatically as follows

$$\mathrm{FinVect} \ni X \xrightarrow{\text{uniquely determines}} \mathcal{Z} \in \text{1-TFT}$$

$$\mathrm{CommFrob} \ni A \xrightarrow{\text{uniquely determines}} \mathcal{Z} \in \text{2-TFT}$$

$$\mathrm{ModTensor} \ni \mathcal{C} \xrightarrow{\text{uniquely determines}} \mathcal{Z} \in \text{3-TFT}$$

[10] Interestingly from a historical point of view, Schrödinger's (and Eckhart's) famous proof from 1926 that the two approaches are equivalent is faulty, but von Neumann fortunately provided a foolproof demonstration in 1932. See [Mul97] for more on this interesting trivia.

[11] So-called once-extended 3d TFTs, i.e. TFTs of the sort $\mathrm{Bord}_{3,2,1} \xrightarrow{\mathcal{Z}} \mathcal{C}$, can be classified by modular tensor categories. As we previously sketched, once-extended 2d TFTs have $\mathrm{Bord}_{2,1,0}$ as source and $\mathrm{Bord}_{2,1,0}$ is a symmetric monoidal bicategory (see 3.10.29 and 5.6). Also $\mathrm{Bord}_{3,2,1}$ is a symmetric monoidal bicategory with disjoint unions of S^1 as objects. In short, there is an equivalence between once-extended 3d-TFTs evaluated on the circle S^1 and modular tensor categories (see [Jor21] for more). This is a very difficult result, the details of which are not fully available yet, despite being expected/"known" for a decade

Here ModTensor is the category of *modular tensor categories*, which we will not fully define; rather, we give some ingredients below. As a rough sketch, it is a categorified commutative Frobenius algebra.

The main objective of this section is to sketch how this assignment could work and what is the relation between the 3d TFT assigned to a particular example of a modular tensor category (which is related to 3d Chern-Simons field theory), and the Jones polynomial, a knot invariant.

A modular tensor category is a tensor category with extra properties. We start with defining tensor categories.

Definition 5.4.1. A *linear category* is a Vect_k-enriched[12] category, i.e.

- $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a vector space $\forall X, Y$.
- composition is bilinear $\mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes \mathrm{Hom}_{\mathcal{C}}(Y, Z) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Z)$.

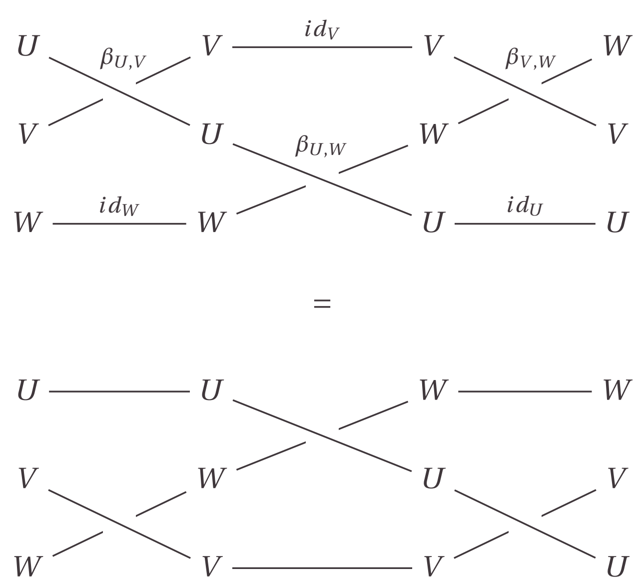
  A *tensor category* is a monoidal linear category, that is, a linear category which is monoidal, and the monoidal product on the Homs is linear.

5.4.1 The Yang-Baxter equation

Theorem 5.4.2. *Let $\mathcal{C}$ be a braided strict*[13] *monoidal category. Then, for $U, V, W \in \mathcal{C}$ we have*

$$(\beta_{V,W} \otimes id_U) \circ (id_V \otimes \beta_{U,W}) \circ (\beta_{U,V} \otimes id_W) = (id_W \otimes \beta_{U,V}) \circ (\beta_{U,W} \otimes id_V) \circ (id_U \otimes \beta_{V,W})$$

which we can visualize draw with the following diagram:



where a horizontal line is the identity and a crossing is the braiding β. In the drawing we simply "moved the middle string from above to below" and this is called a Reidemeister III move in knot theory.

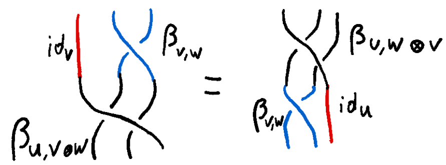
[12] See 3.8.1 for a rigorous definition of what enriched is and how that works for other categories.

[13] A braided strict monoidal category is a strict monoidal category, i.e. a monoidal category where associators and unitors are strict equalities instead of natural isomorphisms, with a braiding, i.e. a natural isomorphism $- \otimes - \cong - \otimes - \circ swap$. See the section on monoidal categories for more (3.7).

Proof. Recall that β is a *natural* isomorphism, so for $U \xrightarrow{id} U$, $V \otimes W \xrightarrow{\beta_{V,W}}$ we apply the naturality of $\beta_{U,-}$ and get the following diagram:

$$\begin{tikzcd} U \otimes (V \otimes W) \arrow[r, "\beta_{U,V\otimes W}"'] \arrow[d, "id_U \otimes \beta_{V,W}"'] & (V \otimes W) \otimes U \arrow[d, "\beta_{V,W} \otimes id_W"] \\ U \otimes (W \otimes V) \arrow[r, "\beta_{U,W\otimes V}"] & (W \otimes V) \otimes U \end{tikzcd}$$

We may visualize the commutativity also via string diagrams:



we get exactly two specular drawings! However, algebraically, the proof is not finished. We now apply

$$(id_Y \otimes \beta_{X,Z}) \circ (\beta_{X,Y} \otimes id_Z) = \beta_{X,Y\otimes Z} \tag{5.13}$$

which inserted for $\beta_{U,V\otimes W}$ and $\beta_{W\otimes V,U}$ gives the result. □

Now, what happens if we take $U = V = W$? Let $\beta := \beta_{V,V}$, we now get

$$(\beta \otimes id) \circ (id \otimes \beta) \circ (\beta \otimes id) = (id \otimes \beta) \circ (\beta \otimes id) \circ (id \otimes \beta) \tag{5.14}$$

which is reminiscent of the Yang-Baxter equation (YBE):

Definition 5.4.3 (Yang-Baxter Equation and R-matrix)**.** Let V be a vector space, $c \in \mathrm{Aut}(V \otimes V)$. The YBE for c is

$$(c \otimes id) \circ (id \otimes c) \circ (c \otimes id) = (id \otimes c) \circ (c \otimes id) \circ (id \otimes c) \tag{5.15}$$

A solution to the YBE is called R-matrix.

In coordinates, for v_i a basis of V, if

$$c(v_i \otimes v_j) = \sum_{k,l} c_{ij}^{kl} v_k \otimes v_l \tag{5.16}$$

then the YBE is given by

$$\sum_{p,q,y} c_{ij}^{pq} c_{qk}^{yn} c_{py}^{ln} = \sum_{y,q,r} c_{jk}^{qr} c_{iq}^{ly} c_{yr}^{mn} \tag{5.17}$$

Theorem 5.4.2 then tells us that, for *any* $V \in \mathrm{Vect}$, $\beta_{V,V}$ is an R-matrix.

Example 5.4.4. the automorphism on $V \in \mathrm{Vect}$: $V \otimes V \xrightarrow{swap} V \otimes V$ satisfies the YBE because

- it comes from a standard braiding β in Vect
- Coxeter relation in S_3:
$$(12)(23)(12) = (23)(12)(23)$$

V finite dimensional vector space with basis $e_1, \ldots, e_n$ and q an invertible scalar. Now define $c_q(e_i \otimes e_j) := q e_i \otimes e_j$ for $i = j$, $e_j \otimes e_i$ for $i < j$ and $e_j \otimes e_i + (q - q')e_i \otimes e_j$ for $i > j$. A computation then shows that this satisfies the YBE. Note that $c_1 = swap$ is a 1-parameter "deformation" of swap.

This comes from representation theory!

Definition 5.4.5. Let G be a group, a representation of G on V a vector space (or R module) is a group homomorphism $\rho : G \to \mathrm{Aut}(V)$, i.e.

$$\rho(gh) = \rho(g)\rho(h), \quad \forall g, h \in G \tag{5.18}$$

Now, a morphism of representations $\rho_i : G \to \mathrm{Aut}(V_i)$, $i = 1, 2$ is a linear map $\alpha : V_1 \to V_2$ such that the following diagram commutes:

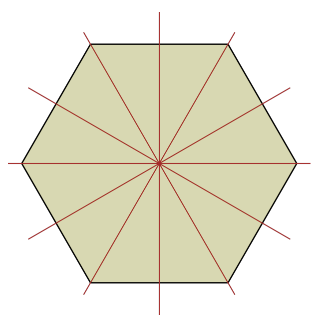
$$\begin{tikzcd} V_1 \arrow[r, "\alpha"] \arrow[d, "\rho_1(g)"'] & V_2 \arrow[d, "\rho_2(g)"] \\ V_1 \arrow[r, "\alpha"'] & V_2 \end{tikzcd} \tag{5.19}$$

This gives rise to the category of representations of G, Rep_G.

Remark. The diagram 5.19 looks suspiciously like a natural transformation, and it is! In particular it's exactly example 1 in 3.3.2. In other words, the category of representations is the functor category:

$$\mathrm{Rep}(G) = \mathrm{Fun}(\mathbf{B}G, \mathrm{Vect}) \tag{5.20}$$

Example 5.4.6. Symmetries of a polygon: $G = D_n$ dihedral group and $\rho : D_n \to \mathrm{Aut}(\mathbb{R}^2)$, where $D_n \circlearrowright \mathbb{R}^2$ via reflections and rotations. For example the reflections can be represented visually for the hexagon as follows



Solutions of YBE give representations of the braid group, which we explore in the next section.

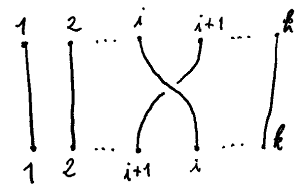
5.4.2 The braid group and the Braid category

Definition 5.4.7 (Braid groups)**.** Let $k \geq 3$. The braid group B_k with k strands has $k - 1$ generators $\sigma_1, \ldots, \sigma_{k-1}$ and 2 relations:

$$\begin{array}{lll} 1.\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if} & |i - j| > 1 \\ 2.\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{if} & 1 \leq i < k - 1 \end{array} \tag{5.21}$$

In addition B_2 is the free group on one generator σ, i.e. it is isomorphic to $\mathbb{Z}$. For even lower generators $B_1 = B_0 = e$.

The reason we call B_k the braid group with k strands is that we can picture the elements σ_i in the following way:



which resembles a braid.

Remark. There is a surjective homomorphism to the symmetric group

$$\begin{aligned} B_k &\to S_k \\ \sigma_k &\mapsto s_k \end{aligned} \tag{5.22}$$

which is clear since S_k has the same generators and relations with in addition $s_i^2 = e$.

Proposition 5.4.8. *Let $c \in \mathrm{Aut}(V \otimes V)$ be an R matrix (i.e. a solution of the YBE). Then, for any $k > 0$, there is a unique homomorphism $\rho_k^c : B_k \to \mathrm{Aut}(V^{\otimes k})$ (i.e. a representation of B_k on $\mathrm{Aut}(V^{\otimes k})$) such that*

$$\rho_k^c(\sigma_i) := c_i, \quad i = 1, \dots, k-1 \tag{5.23}$$

with the $c_i \in \mathrm{Aut}(V^{\otimes k})$ is defined as: take c in position $i, i+1$ and id otherwise, i.e.

$$c_i := \begin{cases} c \otimes id_{V^{\otimes(k-2)}}, & i = 1 \\ id_{V^{\otimes(i-1)}} \otimes c \otimes id_{V^{\otimes(k-i-1)}}, & 1 < i < k-1 \\ id_{V^{\otimes(k-2)}} \otimes c, & i = k-1 \end{cases} \tag{5.24}$$

(the second expression alone is enough is we forget about the identity when we have $id_{V^{\otimes 0}}$)

Remark. With this notation, YBE reads as

$$c_1 c_2 c_1 = c_2 c_1 c_2 \tag{5.25}$$

since the extra identities have no relevant effect. Equivalently then the YBE can also be written as

$$c_i c_{i+1} c_i = c_{i+1} c_i c_{i+1} \tag{5.26}$$

Proof. Define $\rho_k^c(\sigma_i) := c_i$ and check that the c_i satisfy the relations in Definition 5.4.7. For 1. we would like

$$c_j c_i = c_i c_j \tag{5.27}$$

for $i > j+1$. This follows from the fact that c_i and c_j don't "interact":

$$\begin{aligned} &(id_{V^{\otimes(j-1)}} \otimes c \otimes id_{V^{\otimes(k-j-1)}})(id_{V^{\otimes(i-1)}} \otimes c \otimes id_{V^{\otimes(k-i-1)}}) \\ =&(id_{V^{\otimes(j-1)}} \otimes c \otimes id_{V^{\otimes(i-j-2)}} \otimes id_{V^{\otimes 2}} \otimes id_{V^{\otimes(k-i-1)}})(id_{V^{\otimes(i-1)}} \otimes c \otimes id_{V^{\otimes(k-i-1)}}) \\ =&(id_{V^{\otimes(j-1)}} \otimes c \otimes id_{V^{\otimes(i-j-2)}} \otimes c \otimes id_{V^{\otimes(k-i-1)}}) \\ =&(id_{V^{\otimes(i-1)}} \otimes c \otimes id_{V^{\otimes(k-i-1)}})(id_{V^{\otimes(j-1)}} \otimes c \otimes id_{V^{\otimes(k-j-1)}}) \end{aligned}$$

2. simply follows from the remark above. $\square$

The braid group is also connected to the concept of configuration space.

Definition 5.4.9 (Unordered configuration space)**.** Let $\mathrm{uConf}_k(\mathbb{R}^2) \subset (\mathbb{R}^2)^k$ (the ordered configuration space of $\mathbb{R}^2$) be the subspace of k-tuples $(x_1, \ldots, x_k)$ such that $x_i \neq x_j$ for $i \neq j$. Then we have an action $S_k \circlearrowright \mathrm{uConf}_k(\mathbb{R}^2)$. The *unordered* configuration space is $\mathrm{Conf}_k(\mathbb{R}^2) := \mathrm{uConf}_k(\mathbb{R}^2)/S_k$.

We would like to talk about the fundamental group of $\mathrm{Conf}_k(\mathbb{R}^2)$, but for that we need to pick a basepoint. Let us indicate coordinates by working in $\mathbb{C}$ rather than in $\mathbb{R}^2$. Then let $p = [(1,2,\ldots,k)] \in \mathbb{C}^k$, that is to say that all points are placed on the x-axis in steps of 1 (and the square brackets are to indicate the class under the quotient by S_k). We then have a homomorphism $B \to \pi_1(\mathrm{Conf}_k(\mathbb{R}^2), p)$ which sends $\sigma_i \mapsto \hat{\sigma}_i = [f_i]$ where f^i is a loop at p, defined in $(\mathbb{R}^2)^k$ by:

$$f^i = (f_1^i, \ldots, f_k^i) : [0,1] \to (\mathbb{R}^2)^k \cong \mathbb{C}^k \tag{5.28}$$

$$f_j^i(s) = j, \quad \text{if } j \neq i, i+1 \tag{5.29}$$

$$f_i^i(s) = \frac{1}{2}(2i + 1 - e^{\pi i s}) \tag{5.30}$$

$$f_{i+1}^i(s) = \frac{1}{2}(2i + 1 - e^{\pi i s}) \tag{5.31}$$

this indeed induces a loop in $\mathrm{Conf}_k(\mathbb{R}^2)$.
For it to be a homomorphism we need to check that the images of σ_i satisfy the relations 5.21 of the B_k group.

1. $\hat{\sigma}_i \hat{\sigma}_j = \hat{\sigma}_j \hat{\sigma}_i$ for $|i - j| > 1$
2. $\hat{\sigma}_i \hat{\sigma}_{i+1} \hat{\sigma}_i = \hat{\sigma}_{i+1} \hat{\sigma}_i \hat{\sigma}_{i+1}$ for $1 \leq i < k - 1$

For 1. one finds the same result as in the proof of Proposition 5.4.8 above, i.e. for $|i - j| > 1$, $\hat{\sigma}_i$ and $\hat{\sigma}_j$ don't "interact" with one another. Instead 2. can be checked visually.
The following important result can be found in [ART50].

Theorem 5.4.10. *This homomorphism* $\phi : B_k \to \pi_1(\mathrm{Conf}_k(\mathbb{R}^2), p)$ *is an isomorphism.*

The drawings above may give us the idea to construct a category in which morphisms are given by the paths f which are morphisms between sets k points. This will be the Braid category. More in detail, given a representative $f = (f_1, \ldots, f_k) : [0,1] \to (\mathbb{R}^2)^k$ of an element in $\pi_1(\mathrm{Conf}_k(\mathbb{R}^2), p)$ we can define the following subset of $[0,1] \times \mathbb{R}^2$:

$$L_f = \bigcup_{j=1}^{k} \{(s, f_j(s)) : s \in [0,1]\} \tag{5.32}$$

which is pretty much what we were drawing above, the graph of the path considered. It is therefore simply the union of disjoint line segments. Note now that

1. $\partial L_f = \{0,1\} \times \{1,2,\ldots,k\}$ since the path starts and ends at $p = [(1,2,\ldots,k)]$.
2. $\forall s \in [0,1]$ we have that $L_f \cap (\{s\} \times \mathbb{R}^2) = k$ points.
3. L_f is a bordism from $\{1,\ldots,k\}$ to itself. Where in addition we can add an orientation by "flowing" from 0 to 1.

4. group structure gives the composition of bordisms.
5. always have just $\bullet_1, \dots, \bullet_k$ as source/target $\implies$ can take just this object.
6. bordism L_f comes with an embedding into $[0,1] \times \mathbb{R}^2$. This really depends on the representative! But the bordism itself only depends on $[f] \in \pi_1(\mathrm{Conf}_k(\mathbb{R}^2))$.

This allows us to define the following category:

Definition 5.4.11. The Braid category is given by:

- objects: natural numbers $0, 1, \dots, k, \dots$, which we think of as sets of k points,
- morphisms:

$$\mathrm{Hom}_{\mathrm{Braid}}(k,l) = \begin{cases} \emptyset & \text{if } k \neq l \\ \text{isotopy classes of "braids" from } k \text{ to } k & \text{if } k = l \end{cases} \tag{5.33}$$

By braids we mean a "permutation 1 bordism together with an embedding" $\bigcup_{\{1,\dots,k\}} [0,1] \hookrightarrow [0,1] \times \mathbb{R}^2$ such that 1. and 2. hold.

- composition is "stacking the pictures", i.e. composition of bordisms plus stacking embeddings (using $[0,1] \cup_{1=0} [0,1] \cong [0,1]$).
- braided monoidal structure given pictorially by stacking $\mathbb{R}^2$s next to each other, in the sense that $k \otimes l = k + l$ in which the additional l points are simply stacked next to the initial k points.

Exercise 5.4.12. One could now prove that Braid $\simeq \coprod_{k \geq 0} B_k$, where $\coprod_{k \geq 0} B_k$ was in one of the exercise sheets.

5.4.3 Expanding the Braid category: the Tangle category

Next step: allow more morphisms, namely all bordisms, together with suitable embeddings.

Definition 5.4.13. A tangle with k inputs and l outputs is a 1 dimensional bordism Σ from k to l points together with $\Sigma \hookrightarrow [0,1] \times \mathbb{R}^2$ smooth embedding, such that

$$\partial_{in}\Sigma = \{(0,1), (0,2), \dots, (0,k)\} \tag{5.34}$$
$$\partial_{out}\Sigma = \{(1,1), (1,2), \dots, (1,l)\} \tag{5.35}$$

We can now define the tangle category, of which Braid will be a (braided monoidal) subcategory.

Definition 5.4.14. The tangle category Tang_1 has:

- objects: natural numbers
- morphisms:

$$\mathrm{Hom}_{\mathrm{Tang}_1}(k,l) = \{\text{isotopy classes of tangles from } k \text{ to } l \text{ points}\} \tag{5.36}$$

- composition of underlying bordisms and stacking embeddings using $[0,1] \cup_{1=0} [0,1] \hookrightarrow [0,1]$
- braided monoidal structure as before

Note that we also have $\mathrm{Braid} \subset \mathrm{Tang}_1$.

Definition 5.4.15. Let $(\mathcal{C}, \otimes)$ be a monoidal category and $X \in \mathcal{C}$. We say that Y is a right dual of X if there is

$$ev_X : Y \otimes X \to 1 \tag{5.37}$$

$$coev_X : 1 \to X \otimes Y \tag{5.38}$$

such that the snake relations are satisfied. We then say that X is a left dual of Y. See 4.1.35 for a complete defintition.

Lemma 5.4.16. *In* Tang*, every object has a left and a right dual.*

Proof. Same pictures as for Bord_1. □

We can now define the concept of a framing on a tangle, which is different from what we previously meant as framing.

Definition 5.4.17. A framing on a tangle is a nonvanishing normal vector field on the tangle such that at the in-boundary and at the out-boundary it "points up".

This definition is made clearer through examples:

Example 5.4.18. ...

Considering framed tangles leads to the *framed* tangle category Tang_1^{fr}.
In fact, on the line segment as a bordism between two points we have $\mathbb{Z}$ many non isotopic framings corresponding to the *winding number* of normal vector fields. Similarly, S^1 also has $\mathbb{Z}$ many framings.
We noted last time that Tang_1 has left and right duals, one can check now that Tang_1^{fr} also does. We see that the framed tangle category is very similar but it has $\mathbb{Z}$ many morphisms between two points, instead of just one. This idea can be generalized with the concept of a ribbon category:

Definition 5.4.19. A ribbon category is a braided monoidal category $\mathcal{C}$ in which:

1. every object X has a right dual $X^\vee$, thus satisfying

   $$ev_X : X^\vee \otimes X \to 1$$

   $$coev_X : 1 \to X \otimes X^\vee$$

   which satisfies the snake relations/triangle identities (*right rigid category*),

2. there is a pivotal structure, that is a monoidal isomorphism

   $$w : id_{\mathcal{C}} \implies (-)^{\vee\vee} \tag{5.39}$$

   (these two properties are the defining ones for a *pivotal category*)

3. For any object X, we have a "twist"

$$\vartheta_X = (id \otimes ev_{X^\vee}) \circ (\beta_{X,X} \otimes id) \circ (id_X \otimes coev_X) \tag{5.40}$$

This satisfies

$$((\vartheta_X)^\vee = \vartheta_{X^\vee} \tag{5.41}$$

Here we're using an abuse of notation, by identifying $X^{\vee\vee}$ with X.

Example 5.4.20. In Tang_1^{fr}, the ϑ is exactly the picture with the blackboard framing. It remains to check property 3, which will be an exercise.

The definition of the dual of a map $f : X \to Y$ was given in 4.1.38 and is given by the following composition:

$$f^\vee : Y^\vee \xrightarrow{id_{Y^\vee} \otimes coev_X} Y^\vee \otimes X \otimes X^\vee \xrightarrow{id_Y^\vee \otimes f \otimes id_{X^\vee}} Y^\vee \otimes Y \otimes X^\vee \xrightarrow{ev_Y \otimes id_{X^\vee}} X^\vee$$

Lemma 5.4.21. *Let $\mathcal{C}$ be a braided pivotal category. Then every object has a left dual, the twist is invertible and natural in X and $\vartheta_{\mathbb{1}} = id_{\mathbb{1}}$ and*

$$\vartheta_{X\otimes Y} = \beta_{Y,X} \circ \beta_{X,Y} \circ \vartheta_X \otimes \vartheta_Y \tag{5.42}$$

Notation. We denote the left dual of X by $({}^\vee X, e\tilde{v}_X, co\tilde{e}v_x)$.

Example 5.4.22.

- $\mathrm{Vect}_k^{finitedim}$ is a ribbon category with the usual tensor product and braiding.
- $\mathrm{Mod}_R^{fpp} = R - \mathrm{Mod}^{fpp}$ is a ribbon category, where fpp stands for finitely presented and projective and R is a ring or k algebra.

However both these examples are *symmetric* monoidal

We would also like some more examples which are braided but *not* symmetric.
In order to do so, consider an algebra A, then $A - \mathrm{Mod}$ is a category. Now, which extra structure on A guarantees that $A - \mathrm{Mod}^{f.d.}$ is ribbon? Section 5.4.5 below is dedicated to answering this question. As a small preview we will be dealing with:

- representations of "quantum groups":
  - a "deformation" of $\mathrm{Rep}\, G = \mathrm{Mod}\, U\mathfrak{g}$,
  - equivalently, a deformation of $U\mathfrak{g}$ as a "Hopf algebra";
- representations of a "ribbon Hopf algebra" H which is a Hopf algebra with additional structure. We'll find $H - \mathrm{Mod}^{f.d.}$ to be a ribbon category. In particular we'll concentrate on $G = SL_2$ and we take $H = U_q(\mathfrak{sl}_2)$ where q is a root of unity.

Before that, we define a variation of the tangle category above and we'll have an interlude on knot theory.

Definition 5.4.23. Let $\mathcal{C}$ be a ribbon category. $\mathrm{Tang}_1^{fr}(\mathcal{C})$ is the following ribbon category, in which we "decorate with objects in $\mathcal{C}$ ".

- objects: finite sequences
$$(V_1,\epsilon_1),\dots,(V_k,\epsilon_k) \tag{5.43}$$
where $V_i \in \mathcal{C}$ and $\epsilon_i = \pm 1$. This corresponds to k points in $\mathrm{Tang}_1^{fr,or}$ in which ϵ gives us the orientation of each point.

- morphisms: has underlying (isotopy classes of) framed oriented tangles + each component is labelled by $V \in \mathcal{C}$ such that the source and target are labelled by $(V, \pm 1)$.

- Composition is stacking pictures but only if the labels match up.

The "correspondence" above is actually a forgetful functor $\mathrm{Tang}_1^{fr}(\mathcal{C}) \to \mathrm{Tang}_1^{fr,or}$, in which or means adding an orientation to the bordism.

Proposition 5.4.24. *If $\mathcal{C}$ is a ribbon category, then there is a unique braided monoidal functor $F : \mathrm{Tang}_1^{fr}(\mathcal{C}) \to \mathcal{C}$ such that*

- $F(V,+1) = V$ *and* $F(V,-1) = V^\vee$

- $\forall V, W \in \mathcal{C}$ *we have:*

Finally, what does all this have to do with 3d TFTs?

$$\begin{tikzcd}
\mathrm{Tang}_1^{\mathrm{fr}}(\mathcal{C}) \arrow[r] \arrow[d, leftrightsquigarrow, "\text{surgery}"'] & \mathcal{C} \arrow[r, "forget"] & \mathrm{Vect} \\
\mathrm{Bord}_3 \arrow[urr] & &
\end{tikzcd}$$

We will use:

- every 3 manifold arises from framed 1-tangles via *surgery*,

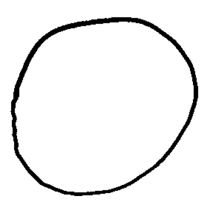
- two tangles giving the same 3 manifold can be related by *Kirby moves*,

- $F_{\mathcal{C}}$ is invariant under Kirby moves if $\mathcal{C}$ is a *modular* tensor category.

5.4.4 Interlude on knots, links and the Kones polynomial

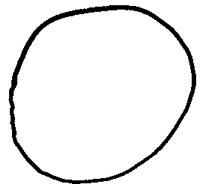
Example 5.4.25. Examples of knots are the following:

- the *unknot*



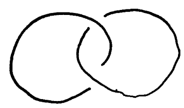
- the *trefoil*

- the *figure 8 knot*

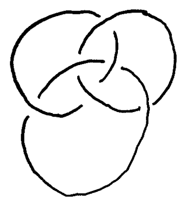


and of links:

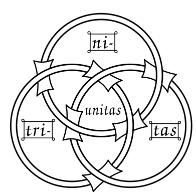
- the *Hopf link*



- the *Borromean rings*



Remark (Knots and links as symbols). The Borromean rings are called 'Borromean' because they appear on the coat of arms of a noble family from Northern Italy named 'Borromeo'. However, this is not the only instance of such link in history. They often appear elswhere with fascinating symbolism. For instance, they also symbolized the trinity



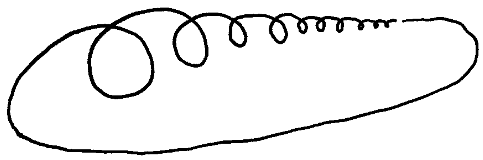
Interlocked triangles, which we now refer to as 'valknut', often appear in areas inhabited by Germanic peoples. Sometimes the valknut symbolizes the pagan God Odin. Some of these valknuts are topologically equivalent to the Borromean rings, such version is called the tricursal valknut. Here is an example of a tricursal valknut on an ornate slab of stone in Sweden called the Stora Hammars I.

Another version of the valknut called the unicursal valknut is not topologically equivalent to the Borromean link: it is a knot and ambient isotopic to the trefoil. It is the current symbol of the German Football Association.

Definition 5.4.26. A link is a finite collection of circles smoothly embedded in $\mathbb{R}^3$:

$$\coprod_{i=1}^{k} S^1 \hookrightarrow \mathbb{R}^3 \tag{5.44}$$

In which smoothness is to exclude some pathological situations. For instance the following one



This is pathological because it should be intuitively be equivalent to the unknot and to show this we would need infinitely many Reidemeister moves of type 1 (see 5.4.4) to deform it to the unknot. However, one the fundamental theorem of knot theory 5.4.31 manages to prove that two knots are equal if one can be deformed into the other with *finitely* many Reidemeister moves.

Definition 5.4.27. A knot is a link with one component ($k = 1$).

Definition 5.4.28. Two links $L_1, L_2 : \amalg_{i=1}^k S^1 \hookrightarrow \mathbb{R}^3$ are equivalent if there exists an ambient isotopy between them, i.e. $H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that

- $H(-,0) = id$
- $\forall\, t \in [0,1],\ H(-,t)$ is a diffeomorphism
- $H(-,1) \circ L_1 = L_2$

A goal of knot theory is to find an invariant of knots/links, i.e. an assignment

$$\{\text{knots/links}\} \to \mathbb{R}, \mathbb{Z}[A, A^{-1}], \ldots \tag{5.45}$$

such that if two links are equivalent we get the same number, polynomial, ...

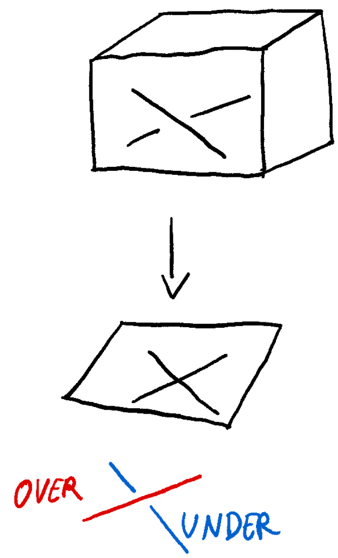
A link invariant is complete if it detects when 2 links are/are not equivalent.

Concretely, we represent links and nots by 2d drawings, these drawings are called *Link diagrams*. Let L be a link and $p : \mathbb{R}^3 \to \mathbb{R}^2$ a projection.

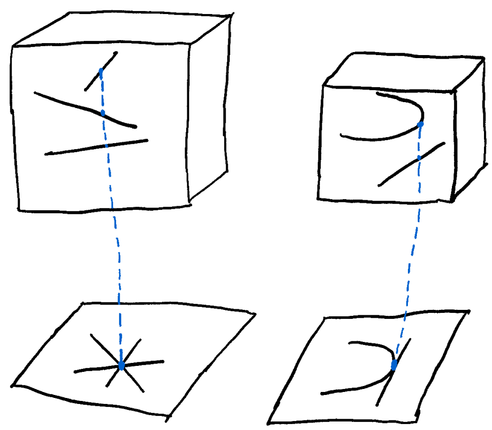
Definition 5.4.29. $p(L)$ is regular if

1. every point has at most 2 preimages in L
2. intersections are transversal

Definition 5.4.30. A link diagram is a regular $p(L)$ together with over/under information at each crossing. Visually:



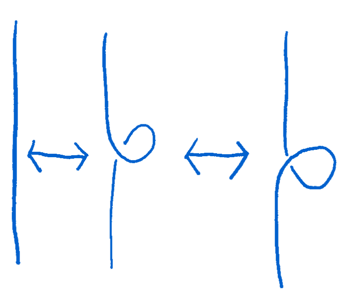
Remark. Given a link, we do *not* always get a link diagram $p(L)$, but we can always deform/perturb the embedding $L : \coprod_{i=1}^{k} S^1 \hookrightarrow \mathbb{R}^3$ by ambient isotopy to L' which gives a link diagram. For instance, the following two projections are irregular:



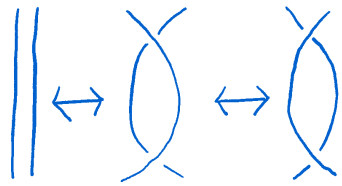
But such situation could be avoided by moving such links in a reasonable manner via ambient isotopies.

Our aim is to define a knot/link invariant by defining something on link diagrams. In order to do this we introduce Reidemeister moves, which are the following local changes in a link diagram:

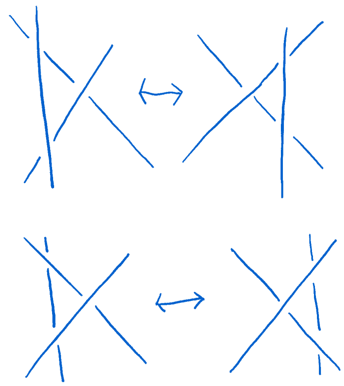
- (I or R1)



- (II or R2)



- (III or R3)



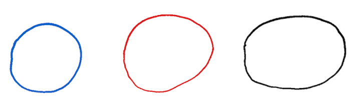
Theorem 5.4.31 (Reidemeister)**.** *Two links are equivalent if and only if their link diagrams are the same up to finitely many Reidemeister moves and isotopies in* $\mathbb{R}^2$.

This theorem is very important because it tells us that the link diagrams have all the information of the equivalence class of the link, so to determine (in)equivalence we can work with link diagrams.

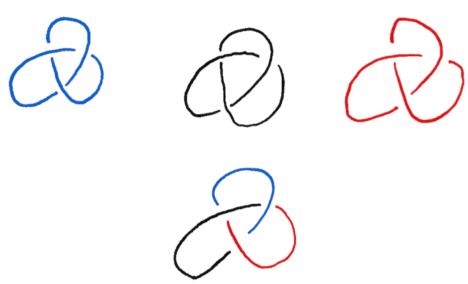
A first knot invariant is the "number of tricolorings": we color a link diagram with three colors according to the following rules

- each strand is one color
- at each crossing either all three strands are the same color, or they're all different.

Example 5.4.32. There are always three trivial colorings, the monochromatic ones. For the unknot, these are the only three tricolorings



while the trefoil has an additional one:



This shows that the trefoil knot is *not* equivalent to the unknot.

In order to prove that this is a knot invariant one should check that it is invariant under Reidemeister moves.

We now finally get to the Jones polynomial, found by Jones in 1984, and this presentation is from Kauffmann in 1987. The motivation for it comes from operator algebras and the braid group representations.

Definition 5.4.33. The Kauffmann bracket of a link diagram D is a Laurent polynomial $\langle D\rangle \in \mathbb{Z}[A, A^{-1}]$ defined by

- $\langle \bigcirc \rangle = 1$
- $\langle \times \rangle = A \cdot \langle)\,(\rangle + A^{-1} \cdot \langle \asymp \rangle$
- $\langle L \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle L\rangle$

Example 5.4.34.

$$\langle \infty\bigcirc \rangle = A \cdot \langle \bigcirc\,\bigcirc \rangle + A^{-1} \cdot \langle \bigcirc \rangle =$$
$$= A(-A^2 - A^{-2})\langle \bigcirc \rangle + A^{-1}\langle \bigcirc \rangle =$$
$$= -A^3 - A^{-1} + A^{-1} = -A^3$$

There is an issue:

$$\langle \infty \rangle = \langle \quad \rangle$$

and the last image is equivalent to the unknot, with an application of the first Reidemeister move, but

$$\langle \quad \rangle = A^3$$

and

$$\langle \bigcirc \rangle = 1$$

This shows that the Kauffmann bracket is *not* a knot invariant, in particular it is not invariant under R1. It *is* however invariant under R2 and R3, in fact the coefficients in the definition of the bracket are exactly the ones that preserve R2 and R3.

From now on: links are oriented, meaning S^1 comes with an orientation. So we can talk about oriented link invariants, for which we have oriented Reidemeister moves.

Now we would like to fix the problem with the Kauffmann bracket.

Definition 5.4.35. The sign of a crossing is

Definition 5.4.36. The writhe of an oriented link diagram D is the sum of the signs of the crossings:

$$w(D) = \sum_{c \text{ crossing in } D} \operatorname{sign}(c) \tag{5.46}$$

Lemma 5.4.37. *The writhe is also invariant under R2 and R3, and changes under R1 as ...*

Definition 5.4.38. For any link L, define $\chi(L) = (-A^3)^{-w(p(L))}\langle p(L)\rangle \in \mathbb{Z}[A^{-1}, A]$

Theorem 5.4.39. *This is a link invariant.*

Example 5.4.40. Hopf link vs two detached links

Definition 5.4.41. For any unoriented link L, the Jones polynomial of L is

$$V(L) = \chi(L)|_{A=t^{1/4}} \in \mathbb{Z}[t^{-1/2}, t^{1/2}] \tag{5.47}$$

Example 5.4.42. trefoil
$V(D) = t + t^3 - t^4$

Observe

- $V(unknot) = 1$
- skain relation: if L_+, L_-, L_0 are the same link except for at one crossing, according to the following picture

  then
  $$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0 \tag{5.48}$$

Explanation of problem of the Kauffmann bracket with R1:

Definition 5.4.43 (Alternative)**.** A framed link is (a link together with a nonvanishing normal vector field as before) the image of a smooth embedding of $\coprod_{i=1}^{k} S^{\times}[0,1] \hookrightarrow \mathbb{R}^3$.

Remark. This can be projected onto $\mathbb{R}^2$ such that annuli are "flat", this is the "blackboard framing" from previously.

Example 5.4.44. We can give two framings on the unknot which are not equivalent:

Theorem 5.4.45. *L, L' framed links, D, D' diagrams thereof using blackboard framings. Then $L \sim L'$ (isotopy equivalence with framing) if and only if D is related to D' via R2, R3 and the modification of R1 given by*

Theorem 5.4.46. *The Kauffmann bracket is a framed link invariant.*

Recall we had $\mathrm{Tang}_1^{or,fr}(\mathcal{C}) \to \mathcal{C}$ where $\mathcal{C}$ is a ribbon category. Now, fix a $V \in \mathcal{C}$, we get an injection $\mathrm{Link}^{or,fr} \hookrightarrow \mathrm{Tang}_1^{or,fr}(\mathcal{C})$ in which we decorate each strand (component) by V. The goal is then to find a specific ribbon category $\mathcal{C}$ such that we get back the Jones polynomial. We'll find $\mathcal{C} = U_q\mathfrak{sl}_2 - Mod^{f.d.}$.

5.4.5 Making $A - \mathrm{Mod}^{f.d.}$ into a ribbon category

We now want to add additional structures on an algebra A in such a way that $A - \mathrm{Mod}^{f.d.}$ becomes a ribbon category. We therefore need $A - \mathrm{Mod}^{f.d.}$ to have the following structures:

- monoidal structure,
- existence of (right) duals (right rigidity),
- braiding,
- pivotal structure,
- twist,

which we gradually construct in this order.

Monoidal structure

In order to do this, we will need additional structures on A, starting with the following.

Definition 5.4.47. A Hopf algebra in a braided strict monoidal category $\mathcal{C}$ is

- a bialgebra $(H, \mu, \eta, \Delta, \epsilon)$, in which we represent all these maps as the following drawings

  These maps should satisfy

$$(\mu \otimes \mu) \circ (id \otimes \beta \otimes id) \circ (\Delta \otimes \Delta) = \Delta \circ \mu \tag{5.49}$$

  and

- antipode map $S : H \to H$ depicted as

  satisfying

Example 5.4.48.

Take $H = k[G]$,

$$\begin{array}{lll} \mu(g,h) = gh & \eta(1) = 1 & \\ \Delta(g) = g \otimes g & \epsilon(g) = 1 & s(g) = g^{-1} \end{array} \tag{5.50}$$

for Lie algebra $\mathfrak{g}$, let $U\mathfrak{g}$ be its universal enveloping algebra, which we will define shortly. We then have

$$\mathrm{Rep}^{f.d.}\, G \simeq U\mathfrak{g} - \mathrm{Mod}^{f.d.} \tag{5.51}$$

we will then "deform" this in one of the exercises.

Recall that a Lie algebra is defined as follows.

Definition 5.4.49. A Lie algebra is a vector space with a bilinear map $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, sending $x \otimes y \mapsto [x, y]$, called the Lie bracket, satisfying:

- anticommutativity: $[x, x] = 0$ or $[x, y] = -[y, x]$
- Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

A homomorphism of Lie algebras $\mathfrak{g}, \mathfrak{h}$ is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ such that $\phi([x, y]_{\mathfrak{g}}) = [\phi(x), \phi(y)]_{\mathfrak{h}}$.

Example 5.4.50.

1. $\mathfrak{g} = \mathrm{End}(V) = \mathfrak{gl}_V$, with bracket given by the commutator $[f, g] = f \circ g - g \circ f$. There is also a sub Lie algebra $\mathfrak{sl}_V := \{f \in \mathfrak{gl}_V : \mathrm{tr}\, f = 0\}$.

   If for example we take $V = \mathbb{R}^2$ we have $\mathfrak{gl}_2 = \mathrm{Mat}_{2,2}$ and $\mathfrak{sl}_2 = \{A \in \mathrm{Mat}_{2,2} : \mathrm{tr}\, A = 0\}$.

2. Let A be an associative algebra, then if we take the underlying vector space, with bracket given by the commutator, we get a Lie algebra. In other words, there is a forgetful functor:

$$\mathrm{Alg} \to \mathrm{Lie} \tag{5.52}$$

   and its left adjoint is the concept of universal enveloping algebra which we will see more explicitly later on.

3.

4. smooth vector fields on a smooth manifold: $\mathfrak{X} = \mathrm{Der}(\mathbb{C}^\infty(M))$

Outlook: in physics we like to look at Lie groups, a smooth manifold with a group structure (i.e. a group object in the category of smooth manifolds). Since it's a manifold, we can look at its tangent space. The Lie algebra $\mathfrak{g}$ of G is defined as T_eG.

Now, fix $\mathfrak{g}$ a Lie algebra.

Definition 5.4.51. The tensor algebra of $\mathfrak{g}$ is the (graded) algebra given by:

$$T(\mathfrak{g}) := \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n} \tag{5.53}$$

with $\mathfrak{g}^{\otimes n} \otimes \mathfrak{g}^{\otimes k} \to \mathfrak{g}^{\otimes (n+k)}$.

There is a 2 sided ideal $I(\mathfrak{g}) \subset T(\mathfrak{g})$ generated by $\langle x \otimes y - y \otimes x - [x, y] \rangle$. We can then define the universal enveloping algebra of $\mathfrak{g}$:

$$U\mathfrak{g} := T(\mathfrak{g})/I(\mathfrak{g}) \tag{5.54}$$

The universal enveloping algebra has the universal property that given an algebra A and a Lie algebra map $\mathfrak{g} \to A$ there is a unique map $U\mathfrak{g} \to A$ making the following diagram commute

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[dr] & U\mathfrak{g} \arrow[d, dashed] \\ & A \end{tikzcd} \tag{5.55}$$

Claim. *$U\mathfrak{g}$ is a Hopf algebra, with coalgebra structure given by:*

- $\Delta : U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$
- $\epsilon : U\mathfrak{g} \to k$
- $S : U\mathfrak{g} \to U\mathfrak{g}$

In general, we'll now see that if A is a Hopf algebra, then $A-\mathrm{Mod}^{f.d.}$ is a monoidal category with duals.

Theorem 5.4.52. *Let (A, μ, η) be a unital associative algebra and $\Delta : A \to A \otimes A, \epsilon : A \to k$ linear maps. Then $(A, \mu, \eta, \Delta, \epsilon)$ is a bialgebra if and only if $(A-\mathrm{Mod}, \otimes, (k, \epsilon))$ is monoidal.*

- Let M, N be $A-$modules, then the vector space $M \otimes N$ is an $A-$module via

$$A \xrightarrow{\Delta} A \otimes A \longrightarrow \mathrm{End}(M) \otimes \mathrm{End}(N) \longrightarrow \mathrm{End}(M \otimes N) \tag{5.56}$$

- (k, ϵ) is an $A-$module via

$$A \xrightarrow{\epsilon} \mathrm{End}(k) \cong k \tag{5.57}$$

- associators and unitors are those from Vect.

Rigidity

Proposition 5.4.53. *Let H be a Hopf algebra and M an $H-$module, then $M^\vee := \mathrm{Hom}(M,k)$ has an H action via the transpose of S, i.e. given $\phi \in \mathrm{Hom}(M,k)$, $a \in H$ we get $a \cdot \phi \in \mathrm{Hom}(M,k)$ given by*

$$(a \cdot \phi)(m) := \phi(s(a) \cdot m) \tag{5.58}$$

This is a left dual if M is finite dimensional. Now, if S is invertible, ${}^\vee M = \mathrm{Hom}(M,k)$ with $(a \cdot \phi)(m) := \phi(s^{-1}(a) \cdot m)$ is a right dual if M is finite dimensional.

Corollary 5.4.54. *If H is a Hopf algebra with S invertible, then $H-\mathrm{Mod}^{f.d.}$ is rigid monoidal.*

Idea of proof of Proposition 5.4.53. Check that $ev_M, coev_M$ exhibiting dual in Vect are $H-$module maps. □

Braiding

Recall: we want a ribbon category and we had a Hopf algebra, so we had a monoidal category with duals.

We're now using [Sch23] as a reference.

Definition 5.4.55. A quasi-triangular structure on a bi/Hopf algebra is a "universal R-matrix", i.e. an invertible element $R \in A \otimes A$ such that $\forall x \in A$ with

$$\Delta^{coop} : A \xrightarrow{\Delta} A \otimes A \xrightarrow{\tau_{A,A}} A \otimes A \tag{5.59}$$

the following are satisfied

1. $\Delta^{coop}(x)R = R\Delta(x)$, and
2. In $A^{\otimes 3}$ we have
$$(\Delta \otimes id_A)(R) = R_{13}R_{23} \tag{5.60}$$
3. $$(id_A \otimes \Delta)(R) = R_{13}R_{12} \tag{5.61}$$
in which $R_{12} = R \otimes 1, R_{23} = 1 \otimes R, R_{13} = (\tau_{A,A} \otimes id_A)(1 \otimes R)$

Example 5.4.56. Any cocommutative bi/Hopf algebra has a canonical R matrix, $R = 1 \otimes 1$, since the first relation simply becomes $\Delta = \Delta^{coop}$, which is exactly the cocommutativity.

Proposition 5.4.57 (4.2.3. in [Sch23])**.** *Let A be a bialgebra. Then a braiding on $A-\mathrm{Mod}^{(f.d.)}$ uniquely determines a quasitriangular structure on A and viceversa.*

Proof strategy: Given a braiding β, define the universal R matrix to be

$$R := \tau_{A,A}(\beta_{A,A}(1_A \otimes 1_A)) \tag{5.62}$$

in which $\tau_{A,A}$ is the flip and $\beta_{A,A}$ is the braiding. Then one can check properties 1. 2. and 3. above.

Now, the following is also true:

Claim. *Given R as above, the braiding can be recovered as follows: $\beta_{U,V}(u\otimes v) = \tau_{U,V}(R(u\otimes v))$, for arbitrary $U, V \in A-\mathrm{Mod}$.*

So this motivates that if we are given a universal R matrix, we should define β as such, and one should check that it's actually a braiding. □

We're now using [Tin15] as a reference. Main example: $U_q\mathfrak{sl}_2$, Hopf algebra, not cocommutative but has a universal R matrix. Therefore $U_q\mathfrak{sl}_2-\mathrm{Mod}$ is braided monoidal. We get $U_q\mathfrak{sl}_2$ from the cocommutative Hopf algebra $U\mathfrak{sl}_2$ by "deforming" it.

$U_q\mathfrak{sl}_2$ is a $\mathbb{C}(q)$-algebra generators E, F, K, K^{-1} with relations:

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1 \\ KEK^{-1} &= q^2E \\ KFK^{-1} &= q^{-2}F \\ [E,F] &= \frac{K-K^{-1}}{q-q^{-1}} \end{aligned}$$

We have an important 2 dimensional module ("representation") for $U_q\mathfrak{sl}_2$: $V_1 \cong \mathbb{C}^2$ with basis v_1, v_{-1}. Now, how does $U_q\mathfrak{sl}_2$ act? In this basis we write:

$$\begin{aligned} E &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ F &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ K &= \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \end{aligned}$$

We also claimed there was a Hopf algebra structure, which we now make explicit. The comultiplication is given by

$$\begin{aligned} \Delta(E) &= 1\otimes E + E\otimes K \\ \Delta(F) &= K^{-1}\otimes F + F\otimes 1 \\ \Delta(K) &= K\otimes K \\ \Delta(K^{-1}) &= K^{-1}\otimes K^{-1} \end{aligned}$$

the counit map:

$$\begin{aligned} \epsilon(E) &= \epsilon(F) = 0 \\ \epsilon(K) &= \epsilon(K^{-1}) = 1 \end{aligned}$$

and we should check that these satisfy the required relations. In addition we need the antipode map:

$$\begin{aligned} S(E) &= -EK^{-1} \\ S(F) &= -KF \\ S(K) &= K^{-1} \\ S(K^{-1}) &= K \end{aligned}$$

We then get the monoidal category $U_q\mathfrak{sl}_2-\mathrm{Mod}$. If U,V are objects in $U_q\mathfrak{sl}_2-\mathrm{Mod}$, then so is $U\otimes V$ in which we have

$$E(u\otimes v)=E_U\otimes K_V+U\otimes E_V \tag{5.63}$$

It is also braided...

From S we get the dual of V_1: as vector space $V_1^\vee=\mathrm{Hom}(V_1,\mathbb{C})$, in which $\hat{v}_1,\hat{v}_{-1}$ is the dual basis. This has a $U_q\mathfrak{sl}_2-\mathrm{Mod}$ structure given by Equation 5.58.

Up to now we therefore get the following correspondence between the structures on A and those on $A-\mathrm{Mod}^{f.d.}$:

$$\begin{array}{ccc} \text{bialgebra} & \leftrightarrow & \text{monoidal} \\ \text{Hopf algebra} & \leftrightarrow & \text{right rigid} \\ \text{Hopf with } S \text{ invertible} & \leftrightarrow & \text{rigid} \\ \text{quasi-triangular} & \leftrightarrow & \text{braiding} \end{array}$$

Let's make explicit the dual representation to V_1, $V_1^\vee=\mathrm{Hom}(V,\mathbb{C})=\langle\hat{v}_1,\hat{v}_{-1}\rangle$, where $\hat{v}_1$ and $\hat{v}_{-1}$ are the dual basis.

Recall that we had the *decorated* tangle category. Let's take $\mathcal{C}=U_q\mathfrak{sl}_2-\mathrm{Mod}^{f.d.}$ and let $V=V_1$ the representation above. In particular we work with "$\mathrm{Tang}_1^{fr,or}(V_1)$"$\subset\mathrm{Tang}_1^{fr,or}(\mathcal{C})$ in which we decorate everything with just V_1.

Remark.
- $A=q^{-1/2}$
- slightly different normalization: $F(unknot)=-q-q^{-1}$.
- What about the third relation of the Kauffmann bracket? It follows from monoidality!

5.5 Outlook: How to get a 3D TFT?

5.5.1 From knots to 3 manifolds: surgery

Given: a framed link $L\subset S^3$ (so far $L\subset\mathbb{R}^3$ but we now embed that into S^3) with m components $L=L_1\cup\cdots\cup L_m$ with $L_i\cong S^1$. Now:

- Choose closed tubular neighborhood $U\subset S^3$ of link L such that $U=U_1\amalg\cdots\amalg U_m$:

$$\begin{array}{ccc} L_i & \cong & S^1\times\{0\} \\ \cap & & \\ U_i & \xrightarrow[\phi]{\cong} & S^1\times D^2 \end{array} \tag{5.64}$$

  framing of $L_i\cong$ constant normal vector field on $S^1\times\{0\}\hookrightarrow S^1\times D^2$. Could also have the nonconstant framing, that turns around once.

- $\partial(S^3\setminus\mathrm{int}(U_1\amalg\cdots\amalg U_m))=\partial(U_1\amalg\cdots\amalg U_n)\cong S^1\times S^1\amalg\cdots\amalg S^1\times S^1$, m tori. Now glue in solid tori $D^2\times S^1$ (note that the order here matters) using identity as gluing map to get a closed connected 3 manifold.

Exercise 5.5.1. Try unknot with trivial framing.

Objective: 3d tqft. Surgery.

Start with a link $L \subset S^3$ with $L = L_1 \amalg \cdots \amalg L_m$ with $L_i \cong S^1$. We take a tubular neighborhood of L, $U = U_1 \amalg \cdots \amalg U_m$ with $U_i \cong S^1 \times D^2$. Let $T := S^1 \times D^2$, the full torus, and let $\phi : U_i \xrightarrow{\cong} T$. Now we can define

$$M_{L,\phi} := (S^3 \setminus \mathrm{int}(\bigcup_{i=1}^{m} U_i)) \coprod_{\phi} T^m \tag{5.65}$$

surgery of S^3 along L via ϕ. However, the result depends on the choice of ϕ and we'll see that it corresponds to a choice of framing on the knot.

If L is framed and oriented, we can choose a ϕ as follows: let's restric to one component L_i. We then have $\partial U_i \cong S^1 \times S^1$ and the framing determines a "cycle" of $S^1 \times S^1$ as can be seen in the drawing. In particular if the framing winds around k times then we get a cycle that winds around once in the longitudinal direction and k times in the other direction. This determines a diffeomorphism on the torus up to to isotopy, called the "Dehn twist". We then compose with $swap : S^1 \times S^1 \to S^1 \times S^1$:

$$\phi_i : \begin{array}{ccc} T & \longrightarrow & T \\ \\ \alpha_i & \longmapsto & -\beta_i \\ \\ \beta_i & \longmapsto & \alpha_i \end{array} \tag{5.66}$$

In short, via ϕ_i the induced framing on $S^1 \times \{0\}$ is a constant normal framing. So framing on L determines/encodes/records the diffeomorphism ϕ.

Notation: L framed $M_L := M_{L,\phi}$ with ϕ as above.

Example 5.5.2.

- $L = \emptyset$, we then get $M_L = S^3$,
- $L = unknot$ with constant framing:

  $$\partial(S^3 \setminus \mathrm{int}\, T) \cong S^1 \times S^1 \tag{5.67}$$

  Now, as a warmup let's see how $T \coprod_{\phi'} T = S^2$:

  Now, $(S^3 \setminus \mathrm{int}\, T) \coprod_{\phi'} T$...

  We then have a well defined projection $M_L \to S^1$ and the fibers of this map are $S^2 = D^2 \coprod_{\partial D^2} D^2$. In particular, the bundle is trivial:

  Fact. $M_L \cong S^1 \times S^2$.

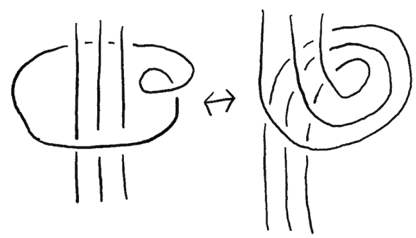
- $L = unknot$ with framing with one twist, we get $M_L = S^3$

The last example is important because it shows that multiple framed links can give the same 3 manifold. However we have the following result:

Theorem 5.5.3 (Lickorice-Wallace)**.** *Any connected closed oriented 3-manifold can be obtained as M_L for some framed link L in S^3 via surgery.*

We can therefore try to define a manifold invariant by defining it on framed links, but we have to make sure that links that give the same 3 manifold have the same invariant.

Theorem 5.5.4 (Kirby's theorem)**.** *$M_L \cong M_{L'}$ if and only if L' is obtained from L by a finite sequence of isotopies and Kirby moves which are as follows:*



We can now get a Reshetikin-Turaev invariant, see for details [KRT97].

Remark. If L_1 and L_2 are not linked, then $M_{L_1 \amalg L_2} = M_{L_1} \# M_{L_2}$ Moreover, $\tau(M_{L_1} \# M_{L_2}) = c\dot{\tau}(M_{L_1}) \cdot \tau(M_{L_2})$ where c is a constant depending on the modular tensor category

We can give a **rough sketch** of the following theorem

Theorem 5.5.5 (Turaev)**.** *For any modular tensor category we have an oriented 3d TFT which generalizes the Reshetikin-Turaev invariants.*

Modular tensor categories are hard to define. The following definition is from [Run].

Definition 5.5.6 (Modular Tensor Category)**.** A modular tensor category is a strict k-linear[14] abelian semisimple ribbon category such that the index set is finite, every simple object is absolutely simple, and for which the s-matrix $s = (s_{i,j})_{i,j \in I}$ with entries

$$s_{i,j} := s_{U_i,U_j} = \mathrm{tr}(c_{U_i,U_j} \circ c_{U_j,U_i})$$

is non-degenerate. An element $\mathscr{D} \in k$ is called rank of a modular tensor category $\mathcal{C}$ if

$$\mathscr{D}^2 = \Sigma_{i \in I}(dim(U_i))^2$$

Given a simple object $U_i \in \mathcal{C}$, also $U_i^\vee$ is simple.

See 3.8.6 for the definition of abelian category.

Remark. In the original definition by Turaev ([Tur16], [Tur]) some conditions were weaker. It was enriched over Mod_R, with R being a commutative ring instead of a field k, semisimplicity was replaced by the weaker dominance property and instead of it being abelian, it was just additive.

Example 5.5.7.

- Vect_k
- $U_q sl_2 - mod^{\mathrm{fd}}$ for q a root of unity

[14] I.e. Vect_k-enriched.

Definition 5.5.8 (Simple object)**.** An object $U \in \mathcal{C}$ where $\mathcal{C}$ is an abelian category is simple if it has no non-trivial subobjects, i.e. any injection $V \hookrightarrow U$ is either the 0 object or an isomorphism.

Definition 5.5.9 (Semisimple object)**.** An object $U \in \mathcal{C}$ where $\mathcal{C}$ is an abelian category is semisimple if it is isomorphic to a direct sum of simple objects.

Definition 5.5.10 (Semisimple category)**.** An abelian category is semisimple if it only has semisimple objects.

Definition 5.5.11 (Absolutely simple object)**.** An object $U \in \mathcal{C}$ where $\mathcal{C}$ is an abelian k-linear category is called absolutely simple if and only if $\mathrm{Hom}(V, V) = k\,\mathrm{id}_V$. If $\mathcal{C}$ is semisimple and k algebraically closed, then this is equivalent to a simple object.

5.6 Rough sketch of the cobordism hypothesis and extended topological field theories ✈

Recall that a TFT is

- a way to organize invariants of smooth manifolds and compute them
- a way to make (some) quantum field theories mathematically rigorous

From the point of view of mathematics one might ask if we classify TFTs as we did for dimension 1 and 2 for any dimension n. From the point of view of physics one might ask if a TFT describing a certain physical system can be determined by the behavior of the system around a point, i.e. if it is fully local. The answer to these questions is yes and is called the cobordism hypothesis.

Remark. The cobordism hypothesis informally states that any TFT, at any dimension, can be classified/constructed just by observing where the TFT in question sends the point[15], i.e. a bit more rigorously

Theorem 5.6.1 (Cobordism Hypothesis)*. Let $\mathcal{C}$ be a monoidal (∞, n)-category with duals*[16]*. Then the evaluation functor on a point, i.e. the functor sending $\mathcal{Z} \mapsto \mathcal{Z}(*)$, induces an equivalence*

$$ev_* : \mathrm{Fun}^{\otimes}(\mathrm{Bord}_n, \mathcal{C}) \to (\mathcal{C}^{\mathrm{fd}})^{\cong}$$

where $\mathcal{C}^{\mathrm{fd}}$ is the full[17] *monoidal (∞, n)-category with duals*[18]*, i.e. we forget about all non-dualizable objects, and $\mathcal{C}^{\cong}$ is the maximal underlying groupoid of $\mathcal{C}$, i.e. we forget about all morphisms that are not invertible.*

It was formulated by James Dolan and John Baez in [BD95] and now we have only partial proofs, [Lur09], [AF17] and [GP22]. The one by Ayala and Francis relies has a different strategy to the first sketch by Lurie and relies on an unproved conjecture regarding factorization homology, i.e. a homology theory for framed n-manifolds with coefficients in E_n-algebras (see 3.10.1).

[15] We will soon see a proof of the 1-dimensional case 5.1.

[16] We later give a definition of (∞, n)-category with duals, see 4.1.49.

[17] i.e. there are all morphisms between the objects of the subcategory, i.e. no morphism from the category is forgotten.

[18] We gave a definition of (∞, n)-category with duals, see 4.1.49.

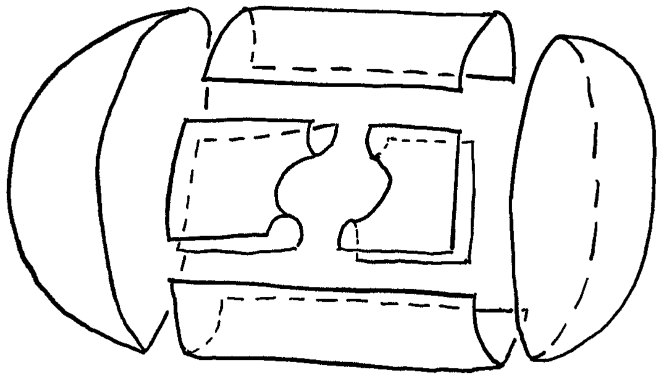


Figure 5.1: Visualization of a torus extended down to a point. The vertices of the corners of the 2-dimensional surfaces are points.

Remark (Remark on Extended TFTs). After defining the cobordism hypothesis in this manner, a natural question is: how can TFTs of any dimension greater than 1 be classified by evaluating them at a point? If we stick to our definition, there are no points in any category of bordisms of dimension greater than 1. Fortunately, one can *extend* the bordism category, and consequently TFTs, and also talk about lower dimensions. Take as an example $\mathrm{Bord}^{or}_{2,1}$. As we defined the bordism category, in this case the lowest dimension is 1, the objects are lines, not points. Nevertheless, one could treat the category of 2 dimensional bordisms as a 2-category, more specifically as a bicategory (see 3.10.29)[19], $\mathrm{Bord}_{2,1,0}$ where objects are disjoint unions of points, 1-morphisms are oriented cobordisms between points and 2-morphisms are cobordisms between the 1-morphisms; and then define 2d-TFTs as an appropriate notion of symmetric monoidal functor between symmetric monoidal bicategories[20]. However, to this we need to allow manifolds with corners, as the illustration of the downward extension of the torus shows (5.1).

Treating a TFT as a functor[21] from a symmetric monoidal[22] bicategory (instead of from a symmetric monoidal category) is sometimes named 'once-extended TFT', e.g. [Sch23]. There are two important results regarding once-extended TFTs:

1. One regards the classification of 2d-TFTs down to a point and is due to Christopher Schommer Pries who proved it in his PhD thesis

$$ev_* : \mathrm{Fun}^{\otimes}(\mathrm{Bord}^{\mathrm{fr}}_{2,1,0}, \mathcal{B}) \xrightarrow{\simeq} (\mathcal{B}^{2\text{-dualizable}})^{\cong}$$

2. The other, although still conjectural[23], classifies 3d-TFTs via the evaluation on the circle

$$ev_{S^1} : \mathrm{Fun}^{\otimes}(\mathrm{Bord}^{\mathrm{fr}}_{3,2,1}, \mathrm{ModTensor}) \xrightarrow{\simeq} (\mathrm{ModTensor}^{2\text{-dualizable}})^{\cong}$$

   hence $\mathcal{Z}(S^1) = \mathcal{C}$ where $\mathcal{C}$ is a modular tensor category, i.e. a special ribbon category which can be seen as a categorified commutative Frobenius algebra.

[19] Since 1-morphisms do not compose strictly, but up to an appropriate notion of isomorphism.

[20] A detailed treatment of the 2d case is found in [SP14].

[21] To be rigorous, one would need to precisify what is an appropriate notion of functor between bicategories.

[22] Although we did not define a symmetric monoidal bicategory, there is indeed a way to do this, see the appendix C of [SP14] for a definition and chapter 2 for an exposition.

[23] It was first stated by Kevin Walker and then some progress was made by Turaev, see [Tur16] for more on this.

However, as one can infer from the aforementioned cobordism hypothesis, one can also extended downward to dimension 0 any n-dimensional TFT and it was first proposed by Daniel Freed in [Fre94]. To be precise, one nowadays does not want to restrict themselves just to n-categories but wants to work with(∞, n)-categories, i.e. categories where morphisms of dimension strictly greater than n are invertible, because of technical reasons[24]. Sketchily, the (∞, n) category of bordisms will have points as objects, bordisms between points as 1-morphisms, bordisms between bordisms between points as 2-morphisms,..., bordisms between bordisms between bordisms...[25] as n-morphisms, diffeomorphisms between n-dimensional bordisms as $n+1$-morphisms, isotopies between diffeomorphisms as $n+2$-morphisms, isotopies between isotopies between diffeomorphsims as $n+3$-morphisms, ... and so on infinitely many times. This is an example of a (∞, n)-category, since isotopies of diffeomorphisms and diffeomorphisms are in fact invertible, whereas oriented bordisms not necessarily. One can find more on this higher category of bordisms in [CS19].

[24]For example, the argument sketched in [Lur09] is crucially a proof by induction on n and in order to understand the $n+1$-morphisms of Bord_{n+1} one must understand such $n+1$-morphisms in Bord_n which are absent if we treat Bord_n as an n-category and not (∞, n)-category. In short, that would just not be possible without such ∞-categorical machinery.

[25]There is exactly $n-1$ times 'between bordisms' after the first instance of 'bordisms', n-morphisms are morphisms between $(n-1)$-morphisms.

Part III

Hints and solutions to exercises

Sheet 1.

Exercise 5.6.2. ***Abelian structure of the cobordism group***
Show that the disjoint union induces an abelian group structure on the cobordism group Ω_n.

First we note that, under the disjoint union, each element of Ω_n is its own inverse, i.e. $a \sqcup a = \emptyset$. Using multiplicative notation (since we prefer not to use the standard additive notation before we know we are dealing with an abelian group) we have $ab = ab(ba)(ba) = a(bb)aba = (aa)ba = ba$ using associativity, hence proving that the group structure is abelian.

Exercise 5.6.3. ***Orientable Manifolds***

(a) Show that the circle S^1 is an orientable manifold.

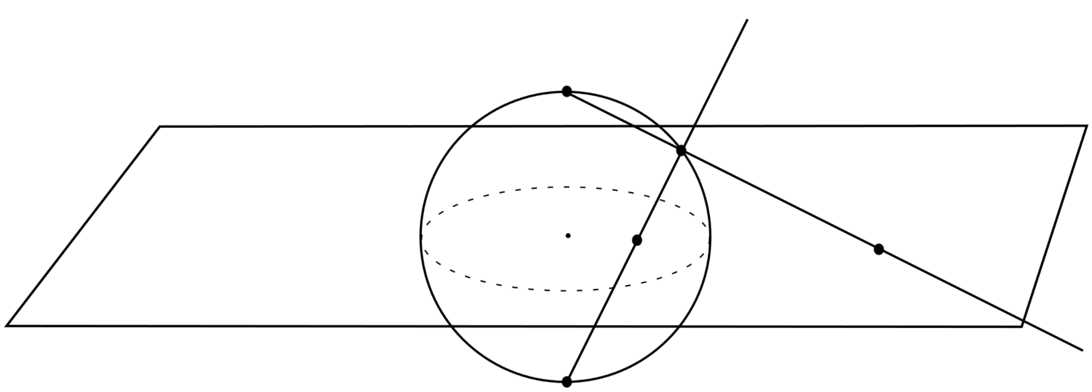
(b) Show that the sphere S^2 is an orientable manifold.

(c) Show that the total space of the tangent bundle of a smooth n-manifold is an orientable manifold.

A way of proving a manifold is orientable is finding a local trivialisation of the tangent bundle. To do this we first pick an open cover and then prove that the transition functions are orientation preserving (we understand what orientation means in Euclidean space).

(a) For S^1 we can use an open cover inspired by the universal cover. We choose $U_1 = (-\pi, \pi)$ and $U_2 = (0, 2\pi)$ along with the maps $\varphi_1 : x \mapsto e^{ix}$ and $\varphi_2 : x \mapsto e^{ix}$, respectively. Looking at the transition map $\varphi_1^- 1 \circ \varphi_2$ we get a map between $(0,\pi) \cup (\pi, 2\pi) \to (0,\pi) \cup (-\pi, 0)$ acting like the identity on the first constituent interval and like the identity plus 2π on the second one. By symmetry, we have a similar situation for the other transition function. Therefore, the differential acts by the identity on the tangent spaces, and so this has determinant $1 > 0$. Thus we conclude that S^1 is orientable, as required.

(b) **Attempt 1:** We try doing this using the standard stereographic projections from the north and south poles, respectively, as an atlas. Then, it is well known that the (both!) transition maps are simply circle inversions on $\mathbb{R}^2 \setminus \{0\}$. See the picture below for clarification (this is a simple exercise in standard Euclidean geometry).



In other words, the map is given by $(x, y) \mapsto (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. If we do the computations, we find that the Jacobian is

$$J = \frac{1}{(x^2+y^2)^4} \det \begin{pmatrix} y^2 - x^2 & -2yx \\ -2yx & x^2 - y^2 \end{pmatrix} = -\frac{1}{(x^2+y^2)^2} < 0$$

so this approach doesn't work...

Attempt 2: S^2 can be viewed as the level set of a smooth function $f : (x, y, z) \mapsto x^2 + y^2 + z^2$ with non-zero gradient and is therefore an orientable manifold.

Attempt 3: For those of you that would (for good reasons) argue that attempt 2 is somehow cheating because is doesn't use our definition, but rather a theorem we haven't presented, we will now present a proof using first principles. This will use a similar idea as in exercise (a). If we describe the sphere as the set of points $\{(\sin\phi\sin\theta, \sin\phi\cos\theta, \cos\phi) \in \mathbb{R}^3\}$. Let (U_1, φ_1) be the chart where...

(c) Show that the total space of the tangent bundle of a smooth n-manifold is an orientable manifold.

Exercise 5.6.4. ***Computation of*** Ω_0^{or}
Compute the oriented bordism group Ω_0^{or}.

We have that 0-dimensional oriented manifolds are classified by a finite collection of points, each with sign. By connecting points of different signs with oriented lines we get a cobordism to a finite collection of points all of the same sign. The sum of the signs in this sense is quite obviously invariant under cobordisms, and so we realise that $\Omega_0^{or} \cong \mathbb{Z}$.

Exercise 5.6.5. ***Computation of*** Ω_2^{or}

(a) Work through the argument in detail showing that Σ_g is cobordant to the empty set.

(b) Recall that the disjoint union is cobordant to the connected sum. Work through the details for an example different from what was shown in the lecture.

(c) Conclude that $\Omega_2^{\mathrm{or}} = 0$. (Here, you may omit details about the orientations of the 3-dimensional cobordisms.)

Let us now try to understand this cobordism group in dimension 2 a bit better.

(a) To understand that Σ_g is cobordant to the empty set we will use the "standard" embedding in $\mathbb{R}^3$. The manifold with boundary that is the interior of this surface in $\mathbb{R}^3$ forms a cobordism between Σ_g and $\emptyset$.

(b) Push along the normal bundle...

(c) This follows from classification of surfaces...

Exercise 5.6.6. ***Computation of*** Ω_2^{unor}***:***

(a) Show that the Klein bottle K is cobordant to the empty set.

(b) (Poincaré duality and Euler characteristic) Show that $\mathbb{RP}^2$ is non-zero in Ω_2^{unor}, i.e. that there is no compact 3-manifold X with boundary $\partial X = \mathbb{RP}^2$.
Hint: Consider the double $D = X \underset{\mathbb{RP}^2}{\cup} X$. What does Poincaré duality imply about the Euler characteristic of 3-dimensional closed manifolds?

(c) Conclude from the above that $\Omega_2^{\text{unor}} = \mathbb{Z}/2\mathbb{Z}$.

Sheet 2.

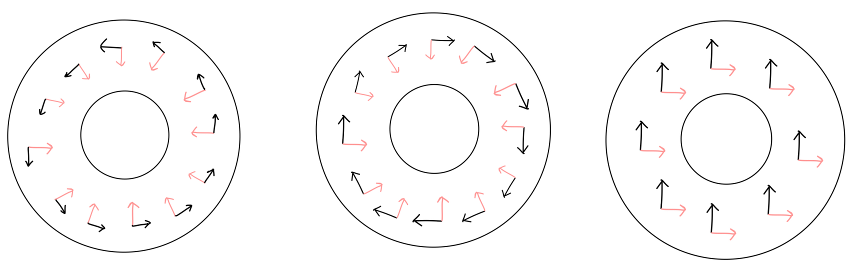
Exercise 5.6.7. ***Framings***
Justify your answers to the following questions:

(a) Can a Klein bottle be framed?

(b) Can S^2 be framed?

(c) An **isotopy between the framings** is a deformation given by a family of framings parameterized by the interval. Are any of the framed cylinders below isotopic?

(d) Which of the framings induce the same orientations?



(a) A Klein bottle cannot be framed because it is unoriented (existence of a framing is more restrictive).

(b) S^2 can also not be framed even though it is orientable. This is a consequence of the hairy ball theorem.

(c) An **isotopy between the framings** is a deformation given by a family of framings parameterized by the interval. Are any of the framed cylinders below isotopic? ...

(d) The two rightmost framings induce the same orientations.

Exercise 5.6.8. ***Attaching handles***
Definition. Let B^n denote the n-dimensional ball as a manifold with boundary and S^n the n-dimensional sphere.

Given a 2-dimensional manifold M, we *attach a j-handle* $H^j :- B^j \times B^{2-j}$, for $j \in \{0,1,2\}$ via and a smooth embedding $f : S^{j-1} \times D^{2-j} \hookrightarrow \partial M$ as follows:

$$M \cup_f H^j :- \left(M \sqcup (B^j \times B^{2-j})\right)/\sim$$

where for $(p,x) \in S^{j-1} \times B^{2-j} \subset B^j \times B^{2-j}$, we set $f(p,x) \sim (p,x)$.

(a) Convince yourself that there is a smooth structure on $M \cup_f H^j$.

(b) Which surface is obtained from attaching a 1-handle to a disk?

(c) Which surface is obtained from attaching two 1-handles to a disk, i.e. from attaching an additional 1-handle to the surface obtained in part (a)?

(d) Build the torus by successively attaching handles to a disk.

(a) "I can explain it to you, but I can't understand it for you."

(b) A cylinder (or a Moebius strip).

(c) A so-called pair of pants.

(d) To a disk we successively add a 1-handle, another 1-handle, and finally cap everything off with a 2-handle.

Exercise 5.6.9. ***Properties of the connected sum of manifolds***

(a) Given n-manifolds M, M', and M'', show that the connected sum satisfies the following properties.

(i) $M\#S^n \cong M$, *(neutral element)*

(ii) $M\#M' \cong M'\#M$, and *(commutativity)*

(iii) $(M\#M')\#M'' \cong M\#(M'\#M'')$. *(associativity)*

(b) If M and M' are smooth n-manifolds, construct a smooth structure on the connected sum $M\#M'$. Note that this is not unique but defines a well-defined diffeomorphism class. You may like to read more details using isotopies in Chapter 8, Section 2 in Hirsch, Differential Topology[26].

Exercise 5.6.10. ***Reading exercise***
Below is a list of several proofs of the classification theorem of 1-dimensional manifolds using different tools. Read through one (or several) of them, or find your own.

(i) `https://pnp.mathematik.uni-stuttgart.de/igt/eiserm/lehre/2014/Topologie/Gale%20-%201-manifolds.pdf`

(ii) Appendix of `https://www.maths.ed.ac.uk/~v1ranick/papers/milnortop.pdf`, starting at p.55.

[26] Can e.g. be accessed at `https://www.researchgate.net/publication/268035774_Differential_Topology`.

Sheet 3.

Exercise 5.6.11. ***Morse functions***

(a) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto x^3 - 3xy^2$. Find all the critical points and check if f is a Morse function. If it does not meet the criteria, perturb it in such a way that it becomes a Morse function.

(b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $(x, y) \mapsto x^2 y^2$. Find all the critical points and check if f is a Morse function. If it does not meet the criteria, perturb it in such a way that it becomes a Morse function.

(c) Show that if $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ are Morse functions, then $f + g : M \times N \to \mathbb{R}$ is also a Morse function, and the critical points are pairs of critical points of f and g. Visualize this for $M = N = S^1$ and $f\colon S^1 \subset \mathbb{R}^2 \to \mathbb{R}$ the projection onto the first coordinate.

(a) We have that $\operatorname{grad} f = (3x^2 - 3y^2, -6xy)$, and so the only critical point is the origin. The Hessian determinant is

$$\mathcal{H} = \det\begin{pmatrix} 6x & -6y \\ -6y & -6x \end{pmatrix}$$

and hence degenerate at the origin. This means f is not Morse. We can perturb it by adding $\varepsilon(x^2 + y^2)$ for a sufficiently small $\varepsilon > 0$ such that it becomes Morse. This has the origin as its only critical point and adds $2\varepsilon I_2$ to the Hessian matrix, thus making the determinant not vanish at the origin.

(b) By, again, computing the gradient $\operatorname{grad} f = (2xy^2, 2yx^2)$, we realise that the set of critical points of f consists precisely of the x- and y-axis. Therefore, f is obviously not Morse.

(c) Show that if $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ are Morse functions, then $f + g : M \times N \to \mathbb{R}$ is also a Morse function, and the critical points are pairs of critical points of f and g. Visualize this for $M = N = S^1$ and $f\colon S^1 \subset \mathbb{R}^2 \to \mathbb{R}$ the projection onto the first coordinate.

Exercise 5.6.12. ***Handle decomposition***
Definition 1. Let M be a compact 2-manifold. A **handle decomposition** of M is a finite sequence of manifolds

$$\emptyset = W_{-1} \subseteq W_0 \subseteq W_1 \subseteq W_2 = M$$

such that each W_i is obtained from W_{i-1} by attaching i-handles.

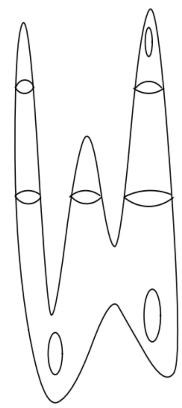
(a) Find two different handle decompositions of S^2.

(b) Find a handle decomposition of $\mathbb{RP}^2$.

(c) Find a handle decomposition of the Klein bottle.

(d) Explain why, for any non-empty closed connected surface, we can start a handle decomposition with a single 0-handle. *Hint*: The key argument was mentioned in lectures as "*handle cancellation*".

(e) Using the idea of handle cancellation, bring the surface below into *normal form*, i.e. such that read from bottom to top, the index of the critical points are non-decreasing.



Exercise 5.6.13. ***Reading exercise - Classification of closed 1-manifolds***
Prove the following theorem using Morse theory and/or read through the proof in `https://www.math.csi.cuny.edu/~abhijit/papers/classification.pdf` [Theorem 15].

Theorem 5.6.14. *Any closed 1-manifold is homeomorphic to S^1.*

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