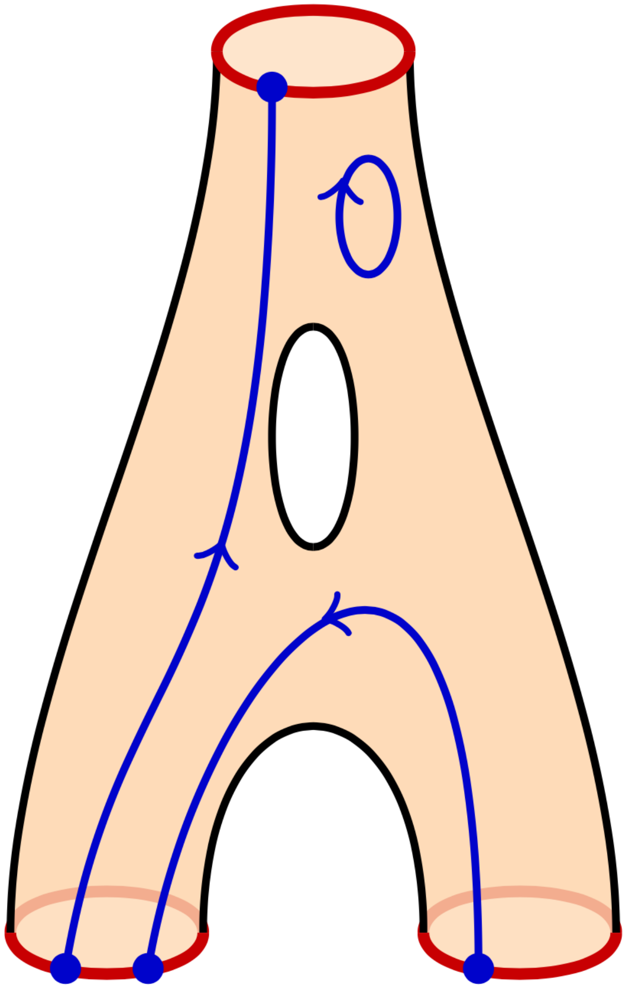


Bordism & Topological Field Theory

Claudia Scheimbauer

Wintersemester 2023

Tutor:
Anja Švraka

Notes created by:
Luca Ipsale
William Luciani
Andrea Sittoni
Üzeyir Saçıkay
Jacob Skarby

The mathematical facts worthy of being studied
are those which, by their analogy with other facts,
are capable of leading us to the knowledge of a mathematical law
just as experimental facts lead us to the knowledge of a physical law.
They reveal the kinship between other facts, long known,
but wrongly believed to be strangers to one another.

—H. Poincaré

For the reader

These are the lecture notes of the course *"Bordism and Topological Field Theory"* held by Prof. Dr. Claudia Scheimbauer (scheimbauer@ma.tum.de) during the Wintersemester 2023/2024 at the *Technische Universität München*. The tutorial sessions are held by Anja Švraka (svr@ma.tum.de). The notes are typed by Luca Ipsale (luca.ipsale@campus.lmu.de), William Luciani (w.luciani@campus.lmu.de), Andrea Sittoni (sittoniandrea@gmail.com) and Üzeyir Saçıkay (uzeyirsacikay@gmail.com). Exercises are also implemented in the notes by Jacob Skarby (jacob.skarby@tum.de). If you find errors or have suggestions of any kind, please write us an e-mail.

There is another version with some outlooks towards related topics. This is however the more faithful representation of what we did in class. Some references might be broken (when you find **??**), if it is something you need or want to understand, check out the other version.

⚠ These notes have not been proofread by Prof. Scheimbauer, use at your own risk.

Last updated on February 27, 2024

Notational Conventions

Notation. Throughout these notes we will abuse notation indicating a collection with some structure on it just by writing down the collection, e.g. denoting a monoid $(M, \cdot)$ by M, a metric space (X, d) by X etc...

Notation. By 'n-manifold' and 'n-bordism', we mean respectively a manifold of dimension n and bordism a bordism of dimension n.

Notation. We often denote equivalence classes just with a representative thereof.

Prerequisites

The notes strive to be as self-contained as possible. We do assume however knowledge of linear algebra, basic notions from analysis, e.g. smoothness, and basic notions of topology, e.g. paracompactness. Knowledge of algebraic topology is helpful but not necessary to understand the most important parts.

Acknowledgements

Apart from Prof. Scheimbauer's lectures and the cited references, we often got inspiration from the nLab (`https://ncatlab.org/nlab/show/HomePage`) and from notes on algebraic topology by Prof. Land (available on `https://www.mathematik.uni-muenchen.de/~gritscha/TOP1-23.php`).

Contents

Chapter 1

Why should you care? An informal introduction

There are two ways one can see topological field theories:

1. As a way to make (some[1]) quantum field theories more mathematically rigorous
2. As a way to refine bordism invariants

We now sketch how these two approaches work.

1.1 QFT

> Physics is very interesting: There are many, many interesting theorems. Unfortunately, there are no definitions.
>
> David Kazhdan

Start with a (smooth) manifold M and, generally, some extra structure. For example:

- $M^4 = \mathbb{R}^4$ with a metric with signature $(+,+,+,-)$, so that we can distinguish a strictly spatial part and a temporal part ($\mathbb{R}^4 = \mathbb{R}^3_{space} \times \mathbb{R}^1_{time}$). This is called *Minkowski Spacetime* and is particularly important in QFT being the geometric foundation of Special Relativity,
- $M^3 = \mathbb{R}^3$ with the Euclidean metric,
- M^2 with a conformal structure,
- M^{11} as an "Elliptic Fibration", something which *locally* looks like $\left(S^1 \times S^1\right) \times something$. These things are useful in areas with high-sounding names such as "M-Theory" or, more generically, "String Theory".

However, in general, these manifolds must be thought of with other structures, like connections, bundles...
Now we can "define" a Quantum Field Theory on M via a list of ingredients:

[1] Sadly, TFTs cannot axiomatize many quantum field theories that are particularly useful in physics, e.g. the quantum field theory behind the standard model.

1. ***Fields***. The <u>space</u> of fields $\mathscr{F}$, associated usually to a bundle $E \xrightarrow{p} M$, is defined as sections[2] of E over M. In the case the bundle is trivial ($E \cong M \times X$), then $\mathscr{F} = \Gamma(M, E) = \mathrm{Maps}(M \to X)$. But what mathematical object are we dealing with? What do we mean by "space" here? Is it a set, a topological space, a category, a scheme, a stack...?

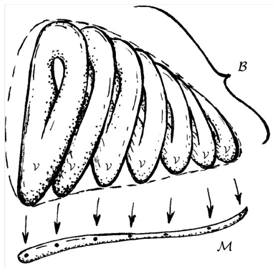


Fig. 15.2 A bundle $\mathcal{B}$, with base space $\mathcal{M}$ and fibre $\mathcal{V}$ may be thought of as constituted as an '$\mathcal{M}$'s worth of $\mathcal{V}$s'. The canonical projection from $\mathcal{B}$ down to $\mathcal{M}$ may be viewed as the collapsing of each fibre $\mathcal{V}$ down to a single point.

2. ***Partition Function***. We need a measure against which we can compute "correlation functions" of fields ψ_1, ψ_2 i.e. the "likelihood of ψ_1 given ψ_2". We thus define a Partition Function
$$Z(M) = \int_{\psi \in \mathscr{F}} e^{iS(\psi)} \mathscr{D}\psi,$$
with
$$S(\psi) = \int_M \mathscr{L}(\psi),$$
called the Action Functional. Generally $\mathscr{L}$ is a polynomial in the fields and it has derivatives. From the Partition Function one can obtain the correlation functions. Although physicists use this formula all the time, formally there is a problem: the measure $\mathscr{D}\psi$ is, in most cases, ill defined.

3. ***Quantization***. Often QFT arises from "quantizing" something classical. But what does this mean? And in what way does this thing behave when changing input?

Physicists use all sorts of techniques (Feynman Diagrams, renormalization...) to make sense of undefined measures and divergences of all kinds emerging from calculations, dealing with things like $\infty - \infty$ or ∞/∞ and obtaining finite and testable results.

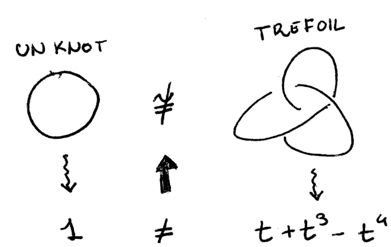
[2] A section of a bundle $E \xrightarrow{p} M$ is a map (in this context, smooth) $s : M \to E$ such that $p \circ s = id_M$.

This black box that physicists have (successfully) developed frustrates mathematicians because they do not understand why it works! Therefore, axiomatizations have been developed exploiting new tools from geometry, algebra and topology to develop a formal and rigorous framework[3].
The Partition Function Z behaves well when "smoothly" changing the metric and is (in most cases) independent of most extra data of the manifold, so $Z(M)$ depends only on the smooth manifold and as such is purely topological!

If someone is into mathematics for the money or the prestige[4], the field of topological field theories is the one to specialize in:

- René Thom, the mathematician who laid the foundation of cobordism theory[5] in his PhD thesis, received the Fields Medal for this.
- Shiing-Shen Chern and Jim Simons discovered geometric invariants of 3-dimensional Riemannian manifolds called (classical) Chern-Simons invariants. They are a generalization of the total geodesic curvature, which is in turn a generalization of the curvature of a plane curve that = 0 when the curve is a geodesic. Such invariants are the basic building blocks Witten used to define the earliest example we have of a TFT: 3d Chern-Simons theory[6]. Simons then went on to found an incredibily successful hedge fund and became a billionaire. Chern continued to do groundbreaking work in differential geometry and topology; so much that some years ago the International Congress of Mathematicians named a prize after him: the Chern Medal.
- Edward Witten received his Fields medal mainly because he found a link between 3d-TFTs, in particular Chern-Simons theory, and knot theory, in particular with the Jones polynomial. See [Wit89] for details.

The Jones polynomial is a topological invariant of a knot, meaning that you can assign to each knot a Jones Polynomial in such a way that if two Knots have different polynomials, then they must be different.



What the heck do TFTs and knots have in common?! There is actually a deep connection between the two. When writing his paper, Witten drew several pictures like the following:

[3] More specifically to the path integral, see [CR18, 2.1] for an introduction on how topological field theories can be seen as a way to axiomatize some properties of the path integral as a tool to compute correlation functions.

[4] If not already sufficiently clear, we explicitly state that this is a joke.

[5] Which is at the root of TFTs: TFTs can be seen as a refinement of his work.

[6] One can find more on this in [Fre08]

Quantum Field Theory and the Jones Polynomial

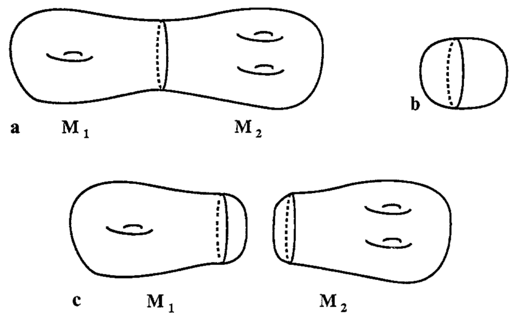


Fig. 5a–c. In **a** is sketched a three manifold M which is the connected sum of two pieces M_1 and M_2, joined along a sphere S^2. Similarly, a three sphere S^3 can be cut along its equator, as in **b**. Cutting both M and S^3 as indicated in **a** and **b**, the pieces can be rearranged into the *disconnected* sum of M_1 and M_2, as in **c**

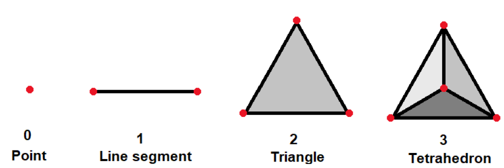
Two mathematicians, Segal and Atiyah, recognized a hidden symmetric monoidal functor and pinned down Witten's intuitions rigorously, thereby axiomatizing TQFTs (and CFTs).

1.2 Topology

> Topology is the science of fundamental patterns and structural relationships of event constellations.
>
> Fuller

The ideas behind cobordisms were developed already by Poincaré together with homology (and their group structure was discovered and investigated by Emmy Noether) but a proper definition was established by Pontryangin and Thom in the 20^{th} century.

The strategy in general in (algebraic) topology is to probe a topological space by mapping into it, a way of viewing things very reminiscent of the Yoneda Lemma. In the case of (singular) homology, we map simplices into the space of interest S. These maps are indexed by the dimension of the simplex as in the following figure. We want to construct an



algebraic structure around these maps $\sigma^{(n)} : \Delta^n \to S$ and so we choose a set of coefficients, say $\mathbb{R}$, and construct the set of "formal (finite) sums" of n-simplices as

$$\sum a_i \sigma_i^{(n)},$$

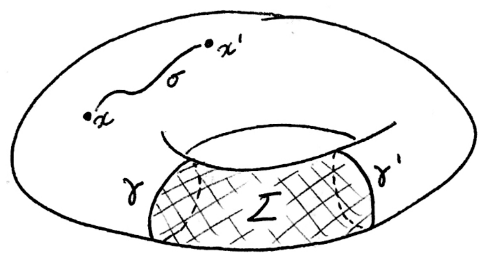
with $a_i \in \mathbb{R}$ and call this set $C_n(S)$. This is now an $\mathbb{R}$-Vector Space generated by the simplices. On the set of simplices we can introduce a map ∂, often called the differential or boundary, that takes each simplex to the alternating sum of the $n-1$ simplices that make up its boundary:

$$\partial\sigma^{(n)} := \sum_{i=0}^{n} (-1)^i \sigma^{(n)}|_{i^{\text{th}}-boundary},$$

and then extend it linearly to $C_n(S)$. Calling $\partial_n := \partial|_{C_n(S)}$, we have $\partial_n : C_n(S) \to C_{n-1}(S)$ and it can also be shown that $\partial_n \circ \partial_{n+1} = 0$, a property which is often simply written as $\partial^2 = 0$. This defines a chain complex structure. Given this property of ∂, note that $\mathrm{im}(\partial_{n+1}) \subset \ker(\partial_n)$ and therefore, we can define the n^{th} homology of S as

$$H_n(S) = \left\{c^{(n)} :\ \partial_n c^{(n)} = 0\right\} / \left\{f^{(n)} :\ f^{(n)} = \partial_{n+1}(f'^{(n+1)})\right\} = \frac{\ker(\partial_n)}{\mathrm{im}(\partial_{n+1})},$$

and this is a vector space (by construction).



Note that $x - x' = \partial\sigma$ so x and x' define the same element in H_0, similarly $\gamma - \gamma' = \partial\Sigma$ so that γ and γ' define the same element in H_1.

The elements of $H_1(S)$ and $H_2(S)$ can be characterized in different ways:

1. The elements of $H_1(S)$ can be represented by a collection of oriented loops mapped in S.
$$\amalg\, S^1 \longrightarrow S$$

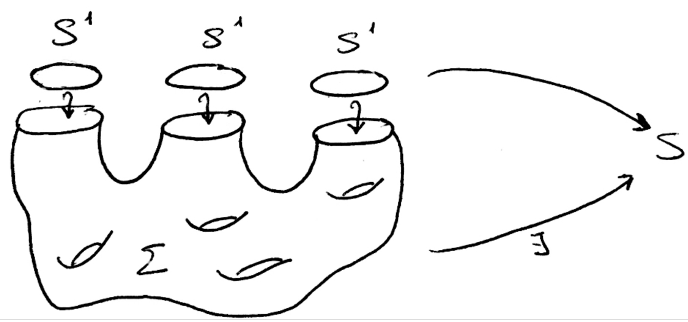
2. The elements of $H_2(S)$ can be represented by a collection of maps from closed oriented surfaces (e.g. genus $g-$surfaces) in S.
$$\amalg\, \Sigma \longrightarrow S$$

3. Let $c \in H_1(S)$, then $\partial c = 0$. Now by 1. this is some map $c : \amalg\, S^1 \to S$ which is $0 \in H_1(S)$ if and only if it extends to a map $\tilde{c}$ of oriented surfaces in S, i.e. if there is the map on the bottom right such that the following diagram commutes.
$$\begin{tikzcd} \amalg\, S^1 \arrow[d, hook] \arrow[dr, "c"] & \\ \amalg\, \Sigma \arrow[r, dashed, "\tilde{c}"'] & S \end{tikzcd}$$

A drawing might be helpful:

Can we then construct something like homology but characterized more like this last property? This is exactly the idea of *(co)bordism*:

1. Let M^n be a $n-$dimensional smooth compact manifold generally with boundary. Now, instead of maps from simplices we consider maps
$$M^n \xrightarrow{f} S.$$



2. Instead of the boundary map of simplices we consider something like:

$$\partial\big(M^n \xrightarrow{f} S\big) = \big(\partial M^n \xrightarrow{f|_{\partial M^n}} S\big),$$

   where, if $\partial M^n = \emptyset$ then define the map to be 0.

3. Let $H_n^{\text{bord}}(S)$ denote this "homology theory" of degree n on the space S. Let M^n be closed, then $M^n \xrightarrow{f} S$ is zero in $H_n^{\text{bord}}(S)$ if and only if the map f extends to a $(n+1)-$dimensional smooth compact manifold W such that $\partial W = M$.

Using these ideas one obtains something very similar to singular homology even though different, indeed this bordism theory constitutes a generalized homology theory[7]!

In particular, consider the one point space $S = \{*\} = pt$. What are elements in $H_n^{\text{bord}}(pt)$? Consider a closed manifold M^n. This defines a class in $H_n^{\text{bord}}(pt)$ since $\partial M^n = \emptyset$. Now, this is $0 \in H_n^{\text{bord}}(pt)$ if we can find a compact $n+1$ manifold W^{n+1} and maps into pt such that the following diagram commutes

$$\begin{tikzcd}
M^n \arrow[d, hook] \arrow[dr, dashed] & \\
 & pt \\
W^{n+1} \arrow[ur, dashed] &
\end{tikzcd}$$

however the maps into pt are trivial and carry no extra information. Now, "surely" not all closed manifolds are a boundary of compact manifold! And so we can be sure that, in general, $H_n^{\text{bord}}(pt) \neq 0$ for $n > 0$.

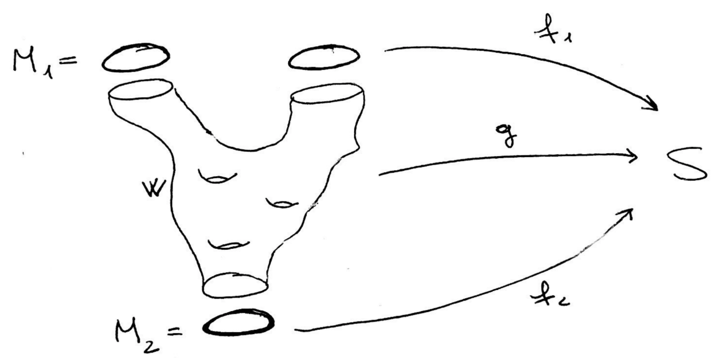
But when do two n-manifolds represent the same class in $H_n^{\text{bord}}(S)$? In the case of $S = pt$, they should be the boundary of the same manifold. In general, we require also that the maps into S extend to the $n+1$ manifold, i.e. let M_1 and M_2 be closed n manifolds, with maps $M_1 \xrightarrow{f_1} S$ and $M_2 \xrightarrow{f_2} S$, then they represent the same element in $H_n^{\text{bord}}(S)$ if and only if there exists a compact $n+1$ manifold with boundary W with a map $W \xrightarrow{g} S$ with the

[7]There exist a list of axioms called *Eilenberg–Steenrod axioms* that is used to define what a homology theory is since it may come in different flavours. A generalized homology theory is a theory that has every property required but one, generally (as in the case of Bordism as seen below) that one is the *dimension axiom*.

property that $\partial W = M_1 \amalg M_2$ and such that it makes the following diagram commute

$$\begin{tikzcd}
M_1 \arrow[dr, hook] \arrow[drr, bend left, "f_1"] & & \\
& W \arrow[r, "g"] & S \\
M_2 \arrow[ur, hook] \arrow[urr, bend right, "f_2"'] & &
\end{tikzcd}$$

Pictorially We then say that M_1 and M_2 are *cobordant* and W is called a *bordism* from M_1



to M_2.

Restricting to $S = pt$ allows us to define the *Cobordism Group*

$$\Omega_n = H_n^{\text{bord}}(pt) = \Big\{\text{Closed } n \text{ manifold}\Big\} \Big/ \Big\{(n+1) \text{ dimensional bordisms}\Big\}.$$

But are all these notions are actually useful? Well, classifying manifolds up to diffeomorphism is hard: dimension $0, 1$ and 2 can be done without too much trouble but already in dimension 3 things get complicated (think of the Poincaré Conjecture!)... so this classifications can be done (more easily) up to bordism!

Part I

Classical cobordism theory

Chapter 2

Manifolds and bordisms

2.1 What are manifolds?

Before venturing into a definition of cobordism, let us recall some useful definitions. This section mainly relies on [Lee12] and [Hir76].

Definition 2.1.1 (*Topological Manifold*)**.** A topological manifold of dimension n is a paracompact Hausdorff topological space X such that every point $x \in X$ has an open neighborhood U which is homeomorphic to an open set[1] in $\mathbb{R}^n$. The latter property means that for each $x \in X$ there exist:

- an open subset $U \subseteq X$ containing x, i.e. an open neighbourhood of x,
- a corresponding open subset $\tilde{U} \in \mathbb{R}^n$,
- a homeomorphism $\phi_U : U \to \tilde{U} = \phi(U)$ called coordinate chart, or just chart (see Fig. 2.1).

If (U,ϕ), (V,ψ) are two coordinate charts of a topological manifold X and $U \cap V \neq \emptyset$, then $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a transition function from ϕ to ψ (see Fig. 2.2).

From the introduction it's clear that we will also deal with manifolds with boundary, so we recall also this definition. In order to do this we first introduce the following notation.

[1] Equivalently to $\mathbb{R}^n$.

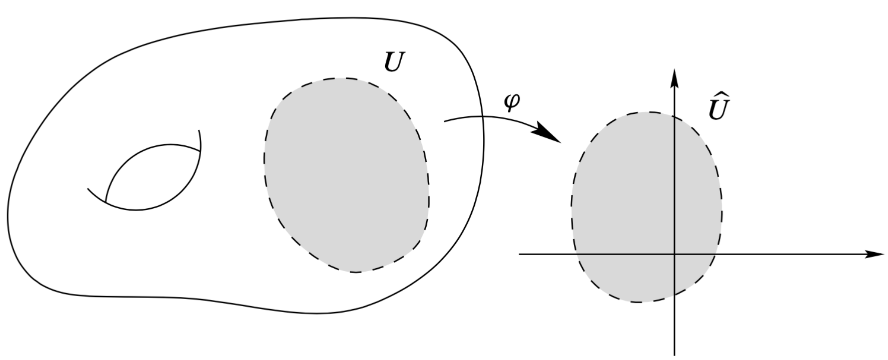


Figure 2.1: The visualization of a coordinate chart map from Lee's textbook on smooth manifolds [Lee12]

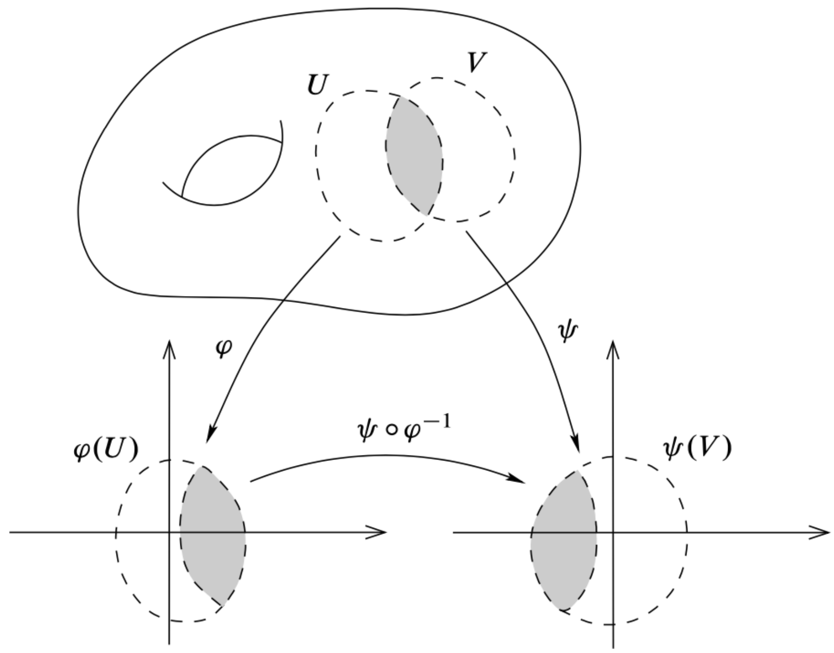


Figure 2.2: The picture of a transition function from [Lee12]

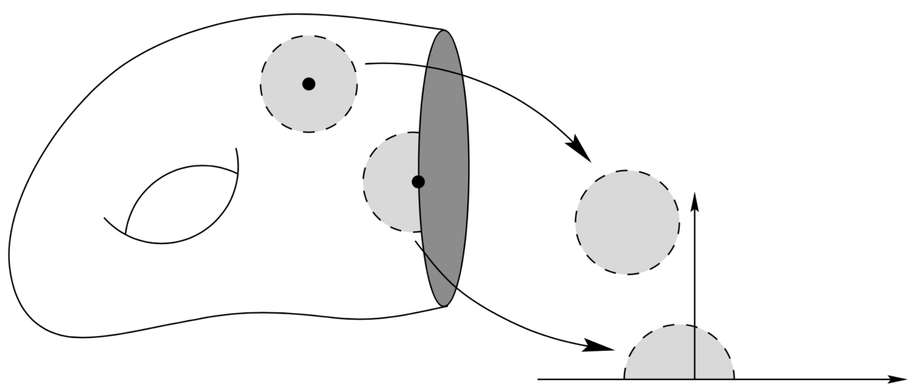


Figure 2.3: The picture of a 2-dimensional manifold with boundary from [Lee12]

Notation (Half-space)**.** By $\mathbb{H}^n$ we denote the n-dimensional upper (closed) half-space,

$$\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n | x_1 \leq 0\}$$

Definition 2.1.2 (Manifold with Boundary)**.** A manifold with boundary of dimension n is a paracompact Hausdorff topological space X such that every point $p \in X$ has an open neighborhood U_p which is homeomorphic to an open set V in $\mathbb{H}^n$, i.e. the closed[2] half space, via the homeomorphism ϕ of the coordinate chart (U_p, ϕ).

Definition 2.1.3 (Boundary of a Manifold)**.** If for $p \in X$ and some chart ϕ it is the case that $x_1(\phi(p)) = 0$ (meaning $\phi(p) \in \{(0, x_2, \ldots, x_n)\} \subseteq \mathbb{H}^n$), then it does in every chart. We then say that p is in the *boundary* of X,

$$\partial X := \{p \in X : x_1(p) = 0\}$$

[2] in topological sense and not in the manifold sense we later define.

otherwise, p is in the *interior* of X, which we denote with Int X.
An equivalent definition is: $p \in \partial X$ if p has neighborhood V that is the domain of a coordinate chart $\psi : V \to \mathbb{H}^n$ such that $\psi(V) \cap \partial\mathbb{H}^n \neq \emptyset$ and sending p to $\partial\mathbb{H}^n$.
Specularly, we could have defined the interior of X to be the set of points $q \in X$ that have a neighourhood U that is the domain of a chart $\psi : U \to \mathbb{R}^n$.

Lemma 2.1.4. *If X is an n dimensional manifold with boundary, then ∂X is a $n-1$ dimensional manifold.*

Proof. Let $p \in \partial X$ be an arbitrary point on the boundary of an n-manifold with boundary. Hence, there is an open neighbourhood U_p of p which is homeomorphic to an open set V in $\mathbb{H}^n$, $\phi(U_p) \cong V$. Since $p \in \partial X$ we know that $x_1(\phi(p)) = 0$ and $\phi(p) \in V \cap \mathbb{H}^n$. $\phi : U_p \to V$ can be restricted to a homemorphism $\phi\ |_{\phi^{-1}(V\cap\mathbb{H}^n)}: \phi^{-1}(V \cap \mathbb{H}^n) \xrightarrow{\cong} V \cap \mathbb{H}^n$. Note that $\phi^{-1}(V \cap \mathbb{H}^n) = U_p \cap \partial X$ because p is sent to the boundary $\partial\mathbb{H}^n$ for every chart. Note also that $\partial\mathbb{H}^n = \{(0, x_2, \dots, x_n)\}$ and $\partial\mathbb{H}^n \cong \mathbb{R}^{n-1}$. Since $V \cap \mathbb{H}^n \in \partial\mathbb{H}^n$ and $V \cap \mathbb{H}^n \cong U_p \cap \partial X$, $U_p \cap \partial X$ is homeomorphic to an open set in $\mathbb{R}^{n-1}$. □

Definition 2.1.5 (Closed Manifold)**.** A manifold is *closed* if it is compact and without boundary, i.e. $\partial X = \emptyset$. Conversely, an *open* manifold is also a manifold without boundary but with no closed components, i.e. it has only non-compact components.

In some contexts, it can be interesting to do calculus on a manifold. In order to make sense of this, one needs to add some extra structure to the topology of the manifold which enables investigating if a map between manifolds is smooth.

Definition 2.1.6 (Smooth Function)**.** Given $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ a function $f : X \to Y$ is smooth[3] if each of its component functions has continuous partial derivatives of all order.

The right notion of isomorphism when talking about smooth functions is that of diffeomorphism:

Definition 2.1.7 (Diffeomorphism)**.** A smooth map $f : X \to Y$ is a diffeomorphism when it is bijective and has a smooth inverse map.

Definition 2.1.8 (Smooth Manifold Without Boundary)**.** A smooth manifold X is a topological manifold, together with a collection of charts $\phi_i : U_i \to \mathbb{R}^n$, called smooth structure, (U_i, ϕ_i) such that:

1. $X = \bigcup_{i\in I} U_i$
2. the pairwise transition functions are smooth in the usual sense of $\mathbb{R}^n$
3. it is maximal with respect to 1. and 2.

Some call a collection of charts (U_i, ϕ_i) such that $\{U_i\}_i \in I$ is a cover of the whole manifold X an atlas. A smooth atlas is an atlas where the transition functions and their inverses are smooth. A smooth structure is thus a maximal smooth atlas. We call charts forming the smooth structure smooth charts.

Example 2.1.9. A lot of the spaces we think of are smooth manifolds, such as:

[3] Also called infinitely differentiable or C^∞.

1. the circle, a dimension 1 manifold
2. a genus g surface, a dimension 2 manifold
3. $\emptyset$, a smooth manifold of any dimension

Notation. In these lecture notes, manifolds are *always* smooth, unless explicitly specified.

Remark. The definition of smoothness we just provided does not work for manifolds with boundary since to say that the transition functions are smooth we used the notion of smoothness of $\mathbb{R}^n$ and not of $\mathbb{H}^n$, the actual codomain of manifolds *with* boundary. Hence, we characterize now smoothness for $\mathbb{H}^n$.

Recall that a map $f : M \to \mathbb{R}^m$, where $M \subseteq \mathbb{R}^n$ is not necessarily open, is smooth if in a neighbourhood of each $x \in M$ it can be extended to a smooth function defined on an open subset in $\mathbb{R}^n$.

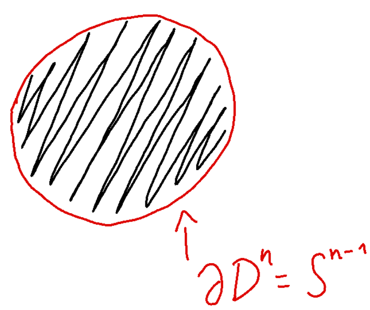
Definition 2.1.10 (Smoothness in the Half Space)**.** If X is an open subset of $\mathbb{H}^n$, we define $f : X \to \mathbb{R}^m$ to be smooth for each $x \in X$ if X can be extended to an open subset $\tilde{X} \in \mathbb{R}^n$, and f to a smooth map $\tilde{f} : \tilde{X} \to \mathbb{R}^m$ such that $\tilde{f}|_{\tilde{X}\cap\mathbb{H}^n} = f$.

Definition 2.1.11 (Smooth Manifold with Boundary)**.** A topological manifold with boundary is smooth if it has a maximal smooth atlas where the transition functions and their inverses are smooth according to 2.1.10.

Corollary 2.1.12. *If X is a smooth n dimensional manifold with boundary, then ∂X is a smooth $n-1$ dimensional manifold. It follows from the proof for topological manifolds.*

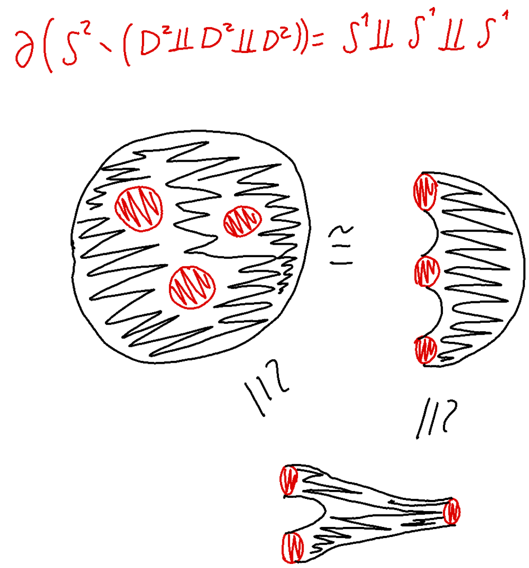
Example 2.1.13.

1. $D^n = \{x \in \mathbb{R}^n \mid \ ||x|| \leq 1\}$ the n dimensional disk, which has a sphere as boundary: $\partial D^n = S^{n-1}$

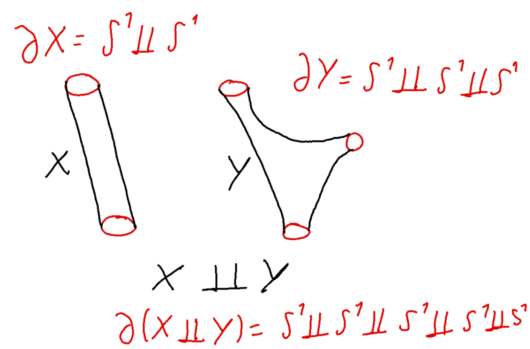


2. Consider 2 dimensional *open* disks, i.e. $D^2 = \{x \in \mathbb{R}^2 \mid \ ||x|| < 1\}$. We can remove such shapes from the sphere and get the manifold $S^2 \setminus D_1^2 \amalg D_2^2 \amalg D_3^2$ which we can visualize

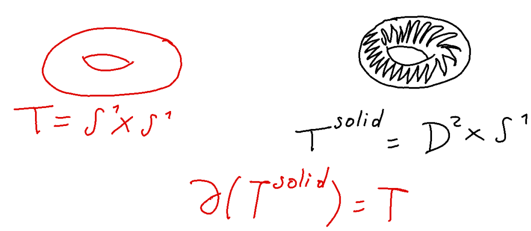
in different ways.



3. Let X, Y be n manifolds with boundary, then $X \amalg Y$ is an n manifold with boundary.



4. $T = S^1 \times S^1$ is a 2 manifold with $\partial T = \emptyset$ and $T^{\text{solid}} = D^2 \times S^1$ is a 3-manifold with $\partial T^{\text{solid}} = T$.



Definition 2.1.14 (Smooth Map to $\mathbb{R}^n$)**.** Let X be a smooth manifold. $f : X \to \mathbb{R}^n$ is smooth if for each $x \in X$ there is a smooth chart (U, ϕ) such that $f = f \circ \phi^{-1}$

This definition can be generalized to maps between smooth manifolds.

Definition 2.1.15 (Smooth Map between Manifolds)**.** Let X and Y be smooth manifolds of dimensions m and n and $f : X \to Y$ a function between them. f is smooth if for every $x \in X$ there is a smooth chart (U, ϕ) of X where $x \in U$ and a smooth chart (V, ψ) of Y where

$f(x) \in V$ and $f(U) \subseteq V$ such that $\tilde{f} = \psi \circ f \circ \phi^{-1}$ is smooth. $\tilde{f}$ can be represented as in the following diagram:

$$\begin{tikzcd}
X \supseteq U \arrow[r, "f|_U"] \arrow[d, "\phi_x"', "\cong"] & V \subseteq Y \arrow[d, "\cong"', "\psi_x"] \\
\mathbb{H}^m \supseteq \tilde{U} \arrow[r, "\tilde{f}"'] & \tilde{V} \subseteq \mathbb{H}^n
\end{tikzcd}$$

This also allows to define the natural notion of isomorphism in the category of smooth manifolds:

Definition 2.1.16 (Diffeomorphism of Manifolds)**.** A diffeomorphism is a smooth map with a smooth inverse, i.e. an isomorphism in the category of smooth manifolds.

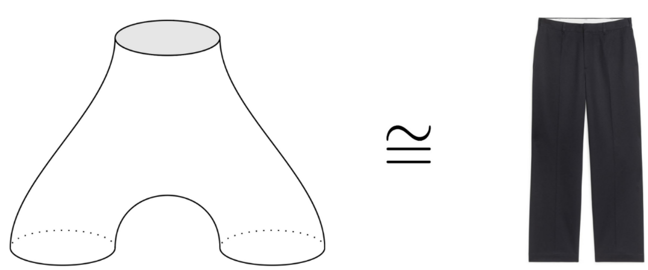
Exercise 2.1.17. Come up with the natural notion of smooth map between manifolds *with* boundary.

2.2 What is a bordism?

Definition 2.2.1 (1^{st} variant of the definition of bordism: via *strict equalities*)**.** Let Y_0, Y_1 be closed n manifolds. A bordism (X, p) from Y_0 to Y_1 consists of a compact $(n+1)$-manifold with boundary together with a map $p : \partial X \to \{0, 1\}$ such that $Y_0 = p^{-1}(0)$ and $Y_1 = p^{-1}(1)$. Thus, $\partial X = Y_0 \amalg Y_1$.
We then say that Y_0 and Y_1 are *cobordant*.

We call Y_0 the *incoming* boundary and Y_1 the *outgoing* boundary. We sometimes write $\partial_{in} X = Y_0$ and $\partial_{out} X = Y_1$.

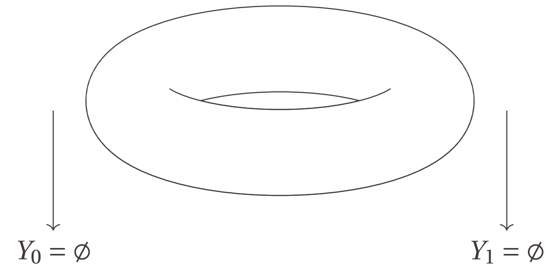
Example 2.2.2. "pair of pants"



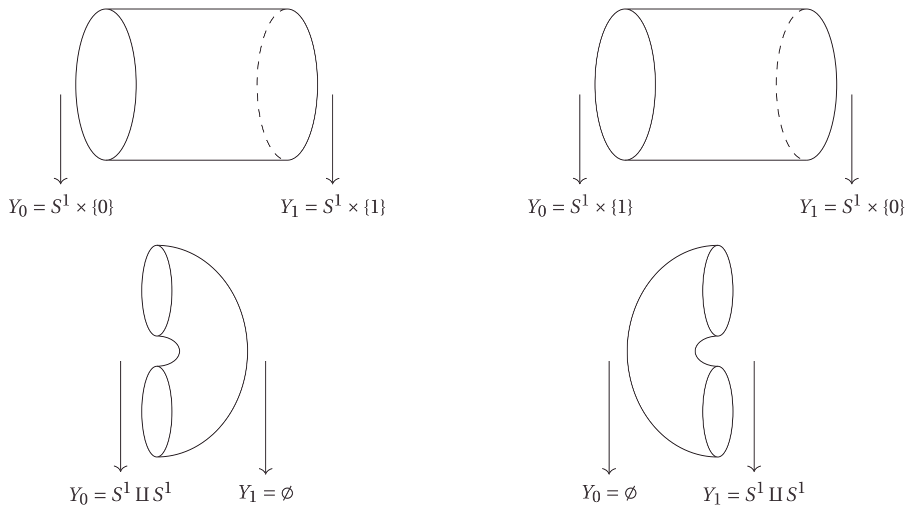
Bordism from S^1 to $S^1 \amalg S^1$

Example 2.2.3.

- The torus is a bordism from $\emptyset$ to $\emptyset$. ($\partial T = \emptyset = Y_0 \amalg Y_1 \implies Y_0 = Y_1 = \emptyset$)



- $X = S^1 \times [0,1]$, now $\partial X = S_1 \amalg S_1$ and we can view it as a bordism in 4 ways:



The latter two are sometimes calles 'macaroni'.

This shows that different bordisms can arise from the same underlying manifold. We will have a way of differentiating them when we will introduce tangential structures on a manifold, which will enable us to explain in which direction a manifold is oriented.

- Given two n-manifolds M, N, their *connected sum* is

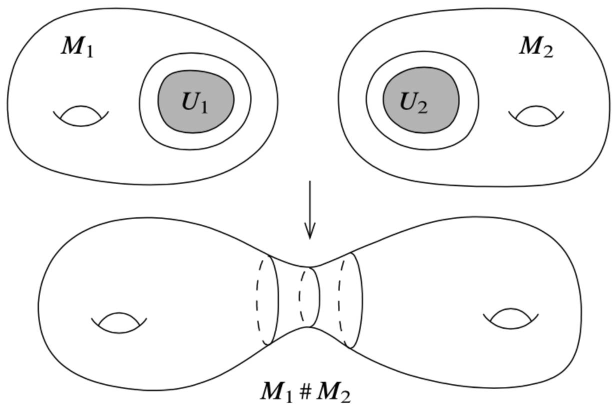
$$M\#N = M \setminus (D^n)^\circ \coprod_{S^{n-1}} N \setminus (D^n)^\circ$$

where $^\circ$ is for taking the interior and $\coprod_{S^{n-1}}$ is glueing along the new boundaries in N and M.

Proposition 2.2.4. *$M\#N$ is an n-manifold.*

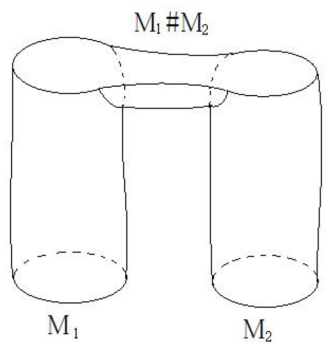
Lemma 2.2.5. *There is a bordism between $M \coprod N$ and $M\#N$.*

Proof. A bordism between $M_1 \coprod M_2$ and $M_1\#M_2$ may be constructed in the following manner (proof taken from the Manifold Atlas Project). Consider a cylinder $M_1 \times I$, from which we remove an ϵ-neighbourhood $U_\epsilon(v_1 \times 1)$ of the point $v_1 \times 1$. Similarly, remove the neighbourhood $U_\epsilon(v_2 \times 1)$ from $M_2 \times I$ (each of these two neighbourhoods can be identified with the half of a standard open $(n+1)$-ball). Now connect the two remainders of cylinders by a "half pipe" $(S_n \le 0) \times I$ in such a way that the half-sphere $S_n \le 0$ is identified with the half-sphere on the boundary of $U_\epsilon(v_1 \times 1)$, and $(S_n \le 0) \times I$ is identified with the



The picture of a connected sum from Wikipedia

half-sphere on the boundary of $U_\epsilon(v_2 \times 1)$. Smoothening the angles we obtain a manifold with boundary $(M_1 \coprod M_2) \coprod (M_1 \# M_2)$.



The illustration by the Manifold Atlas Project of the bordism constructed in the proof above

□

Theorem 2.2.6. *Being cobordant is an equivalence relation on closed n-manifolds.*

Proof. We need to show that the relation satisfies the properties of reflexivity, symmetry and transitivity:

- Reflexive: $Y_0 \sim Y_0$ since we can take $X = Y_0 \times [0,1]$ as in the previous example. We then have $\partial X = Y_0 \times \{0,1\} \xrightarrow{p} \{0,1\}$.
- Symmetric: assume $Y_0 \sim Y_1$. Thus, we have (X, p) from Y_0 to Y_1. We can use the same manifold and compose p with the $swap$ of 0 and 1 in order to get

$$\tilde{p} = swap \circ p : \partial X \xrightarrow{p} \{0,1\} \xrightarrow{swap} \{1,0\}$$

Consequently, $(X, swap \circ p)$ is a bordism from Y_1 to Y_0.

- Transitive: we have $Y_0 \sim Y_1$ via (X_1, p_1) and $Y_1 \sim Y_2$ via (X_2, p_2). Then we can use as a manifold $X = X_1 \cup_{Y_1} X_2$ where $\cup_{Y_1}$ indicates the gluing along Y_1. This is a manifold but there's some work involved in showing it's a *smooth* manifold and it will be done later on (2.3), via an equivalent definition of bordism. We then think of the boundary in the following way $\partial X = \partial_{in} X_1 \amalg \partial_{out} X_2$.

$\square$

2.3 Different definitions of bordisms

Reminder (Definition 2.2.1 of Bordism)**.** Let Y_0, Y_1 be closed n-manifolds. A bordism (X, p) from Y_0 to Y_1 consists of a compact $(n+1)$-manifold with boundary together with a map $p : \partial X \to \{0,1\}$ such that $Y_0 = p^{-1}(0)$ and $Y_1 = p^{-1}(1)$ (hence $\partial X = Y_0 \amalg Y_1$). We say that Y_0 and Y_1 are *cobordant*.

There was a complaint: why don't we use diffeomorphisms instead of the highlighted equalities? This question arises because even with simple examples the equality seems too strict:

Reminder (from 2.2.6)**.** Being cobordant is an equivalence relation.

In particular, for any closed n-manifold Y we have $Y \sim Y$. The bordism is given by the cylinder $Y \times [0,1]$, where $p : Y \times \{0,1\} \to \{0,1\}$. Then $Y \times \{0\} = p^{-1}(0)$, but $Y \neq Y \times \{0\}$! It seems like our previous definition does not actually give rise to an equivalence relation. Instead, if we substitute the strict equalities with diffeomorphisms (isomorphisms in the category of smooth manifolds) things seem to work out flawlessly:

Definition 2.3.1 (2^{nd} variant of the definition of bordism: via diffeomorphisms)**.** Let Y_0, Y_1 be closed n-manifolds. A bordism from Y_0 to Y_1 is a quadruple (X, p, ϕ_0, ϕ_1) consisting of a compact $(n+1)$-manifold with boundary ∂X together with maps

$$p : \partial X \to \{0,1\}$$
$$\phi_0 : Y_0 \xrightarrow{\cong} p^{-1}(0)$$
$$\phi_1 : Y_1 \xrightarrow{\cong} p^{-1}(1)$$

(we therefore have $\partial X \cong Y_0 \amalg Y_1$).

Example 2.3.2. Let $\psi : M \xrightarrow{\cong} N$ be a diffeomorphism. Then $(M \times [0,1], M \times \{0,1\} \xrightarrow{p} \{0,1\}, \phi_0, \phi_1)$, with maps

$$\phi_0 : M \xrightarrow{id \times \{0\}} p^{-1}(0) = M \times \{0\}$$

$$\phi_1 : N \xrightarrow{\psi^{-1} \times \{1\}} p^{-1}(1) = M \times \{1\}$$

gives a bordism from M to N. This is called the mapping cylinder of ψ.
$\Rightarrow$ under the new definition, <u>any</u> two diffeomorphic manifolds are also cobordant.

Why does this not change the definition of bordism? Because any diffeomorphism also gives a bordism in the old sense: Given $M \xrightarrow{\cong} N$, take the gluing $(M \times I) \coprod_{\substack{\psi \\ M \times \{1\}}} N$ so that $Y_1 = N$ and $Y_0 = M \times \{0\}" = "M$ (abusing notation).

Now we prove the transitivity of the relation 'being cobordant'. We do that by showing that given X from Y_0 to Y_1 and X' from Y_1 to Y_2, we can glue them along the boundary in common, Y_1, to obtain a bordism from Y_0 to Y_2. In order to achieve this, we need to introduce the following notion:

Definition 2.3.3 (Collar of a boundary)**.** Let X be a manifold with boundary. A collar of the boundary is an open set $U \subseteq X$ containing ∂X together with a diffeomorphism $(-\epsilon, 0] \times \partial M \to U$ for some $\epsilon > 0$.

Theorem 2.3.4. *The boundary of a manifold with boundary always has a collar.*

Idea of the proof. We start with a manifold with boundary, locally at $x \in \partial X$ we have

$$(-\epsilon_x, 0] \times V_x \xrightarrow{\cong} U_x \subseteq \mathbb{H}^n$$

where V_x is an open neighbourhood of x in ∂X. Now we use a standard trick in differential geometry/topology (see [Hir76]): globally patch these diffemorphisms using a partition of unity. It works if X is compact. □

Definition 2.3.5 (3^{rd} variant of the definition of bordism: via collars)**.** Let Y_0 and Y_1 be closed n-manifolds. A bordism from Y_0 to Y_1 is a quadruple (M, p, ϕ_0, ϕ_1) with M and p as before. ϕ_0, ϕ_1 are given by

$$\phi_0 : [0, \epsilon) \times Y_0 \to U \supseteq p^{-1}(0)$$
$$\phi_1 : (-\epsilon, 0] \times Y_1 \to V \supseteq p^{-1}(1)$$

where U and V together form a collar of the boundary of M.

Moreover,

$$([0, \epsilon) \times Y_0) \amalg ((-\epsilon, 0] \times Y_1) \xrightarrow{\phi_0 \amalg \phi_1} \left(U \amalg V\right) \supseteq \left(p^{-1}(0) \amalg p^{-1}(1)\right) = \partial M$$

This variant of the definition is equivalent to the other two but we would unfortunately need a lot of differential topology in order to prove this.

However, with the third definition we can finally prove the following claim which then is what we were missing to prove transitivity of being cobordant.

Claim. $X \underset{Y_1}{\cup} X'$ *admits a smooth structure.*

Transitivity of cobordant. We need a maximal atlas with smooth transition functions. Clearly, in the interior points of X and X' this exists since we can use directly the charts of X and X'. The problem is the double collar around Y_1. So we need to construct charts around points on Y_1. Take $U_1^- \underset{Y_1}{\cup} U_1^+$ in an open neigbourhood. Then via $U_1^- \underset{Y_1}{\cup} U_1^+ \xrightarrow{\phi_1 \cup \phi_0'} (-\epsilon, \epsilon) \times Y_0 = (-\epsilon, 0] \cup [0, \epsilon) \times Y_0$ which contains $(-\epsilon, \epsilon) \times V_x$ which maps to $\mathbb{R}^n$ via ψ_x and thereby generating a maximal atlas.

Let X be a bordism between Y_0 to Y_1 and X' between Y_1 and Y_0.

$$[0, \epsilon) \times Y_0 \xrightarrow{\phi_0} U$$
$$(-\epsilon, 0] \times Y_1 \xrightarrow{\phi_1} U_1^-$$
$$[0, \epsilon) \times Y_1 \xrightarrow{\cong} (-\epsilon, 0] \times Y_1 \xrightarrow{\phi_0'} U_1^+$$
$$(-\epsilon, 0] \times Y_2 \xrightarrow{\phi_1'} U_2$$

□

2.4 Tangential Structures

2.4.1 Orientations and the tangent bundle

Definition 2.4.1 (Tangent bundle)**.** The tangent bundle TM of a bundle M is the vector bundle over M of rank n with

1. fibers given by the tangent spaces T_pM at each point $p \in M$:

$$(TM)_p = T_pM$$

2. (as a set) $TM = \coprod_{p\in M} T_pM$ with natural projection:

$$TM = \coprod_{p\in M} T_pM \xrightarrow{p} M \tag{2.1}$$

$$(p, v) \mapsto p \tag{2.2}$$

It is possible to define a topology on TM such that it is a smooth $2n$ manifold: (U,ϕ) chart of M, we have a chart of TM, $(p^{-1}(U),\tilde{\phi})$ by looking at preimages of U under the projection map and we construct the local trivialization with the usual construction. This is summarized by the fact that the following diagram commutes:

$$\begin{tikzcd}
p^{-1}(U) = TU = \coprod_{p\in U} T_pM \arrow[r, "\tilde{\phi}=(\phi,d\phi)"'] \arrow[d, "p"] & T\tilde{U} = \tilde{U} \times \mathbb{R}^n \arrow[d, "pr_{\tilde{U}}"] \\
U \arrow[r, "\cong", "\phi"'] & \tilde{U} \subseteq \mathbb{R}^n
\end{tikzcd}$$

This is actually an atlas with smooth structure. Transition functions $g_{ij} = \tilde{\phi}_j \circ \tilde{\phi}_i^{-1}|_{\tilde{\phi}_i(U_i \cap U_j)}$ are actually the jacobian and we have $g_{ij} \in GL_n(\mathbb{R})$.

Definition 2.4.2 (Orientability and admissibility of G structure)**.** If TM has local trivializations such that the transition functions between overlapping trivializations are in $GL_n^+(\mathbb{R}) \subseteq GL_n(\mathbb{R})$ then we say that M is orientable.
In other words “”The structure group of TM can be reduced to $GL_n^+(\mathbb{R}) \subset GL_n(\mathbb{R})$”.
Analogously, given some $G \subset GL_n(\mathbb{R})$, we can say M admits a G structure if the local trivialisations are such that $\phi_{i,j} \in G$, i.e. "the structure group can be reduced to G". In this sense, being orientable is the same as admitting a $GL_n^+(\mathbb{R})$ structure.

Example 2.4.3.

- $\mathbb{R}^n$: we have $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$, meaning the tangent bundle is a trivial bundle.
- S^1: we again have the trivial bundle $TS^1 \cong S^1 \times \mathbb{R}$
- S^2: now $TS^2 \ncong S^2 \times \mathbb{R}^2$, it is not a trivial bundle. This follows from the hairy ball theorem.

All these are orientable. Instead examples of non orientable manifolds are the Klein bottle, the Möbius strip and $\mathbb{RP}^2$.

Definition 2.4.4 (Orientation and G structure)**.** An orientation (equivalently, a $GL_n^+(\mathbb{R})$ structure) of M is an equivalence class of trivialisations $\{(\tilde{U},\tilde{\phi})\}$ of TM such that $\det g_{ij} > 0$ (i.e. $g_{ij} \in GL^+(n,\mathbb{R})$).
Analogously, given some $G \subset GL_n(\mathbb{R})$, a G structure on TM is an equivalence class of trivialisations $\{(\tilde{U},\tilde{\phi})\}$ such that $g_{ij} \in G$.

We can now note the following lemma which is related to the fact that on a manifold one can always construct a Riemannian metric.

Lemma 2.4.5. *A $GL_n^+(\mathbb{R})$ structure is the same as an $SO(n)$ structure.*

How many orientations can M have? 0 if it's not orientable, 2 if it's connected and orientable, in general $2^{\#\text{ connected components}}$ if it's orientable.

Definition 2.4.6 (Opposite orientation)**.** M oriented with charts $\{(U_i,\phi_i)\}$. The opposite orientation on M is given by: $\{(U_i,\bar{\phi}_i)\}$ in which $\bar{\phi}_i$ is given by the following composition

$$U_i \xrightarrow{\phi_i} \mathbb{R}^n \xrightarrow{x_1 \mapsto -x_1} \mathbb{R}^n \tag{2.3}$$

We will indicate M with opposite orientation by $\overline{M}$.

Example 2.4.7. S^1

Note that a diffeomorphism between two orientable manifolds can induce orientations: if we have $f: M \to N$ and choose an orientation on M i.e. a proper choice of trivialization (U_α,ϕ_α), then on N we can define a trivialization (V_α,ψ_α) with $V_\alpha := f(U_\alpha)$ and $\psi_\alpha := \phi_\alpha \circ f^{-1}$. First note that, being f a diffeomorphism, everything in perfectly well defined. Then note: $\psi_\alpha \circ \psi_\beta^{-1} = (\phi_\alpha \circ f^{-1}) \circ (f \circ \phi_\beta^{-1}) = \phi_\alpha \circ \phi_\beta^{-1}$ and thus the determinand of the transition functions on N is the same as the ones of M!

Definition 2.4.8 (Orientation preserving and reversing diffeomorphism)**.** A diffeomorphism $f: M \to N$ between oriented manifolds is orientation preserving (reversing) if the orientation on N induced by the one on M is the (opposite) orientation on N $\iff$ $\det(df_p) > 0$ (or < 0) for all $p \in M$.

Example 2.4.9. Consider maps $S^1 \to S^1$ (implicitly using the same orientation on each S^1). The identity is an orientation preserving map, while the map $x \mapsto \bar{x}$ is orientation reversing (considering $S^1 \subset \mathbb{C}$).
We could instead consider maps $S^1 \to \overline{S^1}$. In this case the identity is orientation reversing.

Consider a oriented manidold M with boundary. By a previous theorem we know that ∂M is a manifold itself and thus we can talk of it's tangent bundle $T\partial M$. At the same time, we can "glue another collar" $[0,\epsilon)$ onto the boundary so that the points of the boundary are now in the interior of the new extended manifold thus allowing us to define the tangent space T_pM for $p \in \partial M$ (this construction is independent from the extension). Now, we can consider $T\partial M$ as a subspace of $TM|_{\partial M}$ via the inclusion $i: \partial M \to M$ which induces $i_*: T(\partial M) \to TM$. We can then define the quotient

$$\nu_p = T_pM / i_* T_p\partial M \cong \mathbb{R}$$

and we can write the following short exact sequence of bundles on ∂M.

$$0 \to i_* T(\partial M) \to TM \to \nu \to 0 \tag{2.4}$$

This is saying that the tangent bundle over M on a boudary point is the tangent bundle of the boundary plus an extra "normal" direction ν.
We can fix a local chart on M with coordinates $x_1, \dots, x_n$ so that we have a positive oriented basis of TM with $\frac{\partial}{\partial x_1}$ basis for ν and so that it's "pointing outword" (i.e. under pushforward of the chart $(\phi_\alpha)_*(\partial/\partial x_1)$ has positive first component, x_1), so that removing it from the basis of TM we have an induced orientation on the boundary.
From this, we can infer the following consequences:

1. ν has "two orientations" ($\cong$ normal directions, $\pi_0(\mathbb{R} \setminus 0)$)

2. an orientation on M induces one on ∂M

3. an orientation on ∂M induces one on $\partial M \times (0,1)$ (if I choose an orientation on $(0,1)$) but in general we can't extend this construction to all of M.

   More generally, note that if M and N are oriented, then $M \times N$ is oriented. [Fre13]

Special case: 0 dimensional manifold. $M = \{x_1, \dots, x_k\}$, but $T_x M = 0$ so our definition fails! So in this case the definition does not capture what we would like, in particular the consequences we found before are either tautological or boring. We can however use 3. to construct a new definition.

Definition 2.4.10. An orientation on a 0 dimensional manifold is a map $M \to \{\pm 1\}$ ($\iff$ orientation of $M \times (0,1)$ if I fixed a chosen orientation of (0,1)).

Definition 2.4.11 (Oriented bordism)**.** Let Y_0 and Y_1 be oriented closed n-manifolds. An oriented bordism from Y_0 to Y_1 is an oriented $(n+1)$-manifold X together with $p : \partial X \to \{0,1\}$ and $\psi : \partial X \to \overline{Y}_0 \amalg Y_1$ an orientation preserving diffeomorphism.

Note: the choice of ψ already clarifies what is the incoming boundary, i.e. $\overline{Y}_0$, and the outgoing one, i.e. Y_1, so that p is actually too much data and so it can be dropped.
This is an equivalence relation which we call "being oriented cobordant", which allows us to give the following definition.

Definition 2.4.12. The oriented bordism group is:

$$\Omega_n^{or} = \{\text{closed oriented } n \text{ manifolds}\}/\{\text{oriented } n+1 \text{ cobordisms}\} \tag{2.5}$$

Remark. Note that $\emptyset$ is an oriented n manifold with a unique orientation.

2.4.2 Framings

Recall the definition of a G structure given above. As stated previously, if $G = GL_n^+(\mathbb{R})$ a G structure is an orientation. Another choice is simply to take $G = \{e\}$, this gives a *framing*, which amounts to a smooth choice of basis for every point. We can talk about *frameable* or *parallelizable* manifolds, meaning they admit a framing. As a consequence, the manifold has a trivial tangent bundle we have $TM \cong M \times \mathbb{R}^n$ and a choice of framing corresponds to a choice of isomorphism.

Example 2.4.13. Some examples of frameable manifolds are the following:

- For $\mathbb{R}^n$, we have $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$

- S^1
- $S^1 \times S^1$
- Any Lie group H has $TH \cong H \times \mathbb{R}^{\dim}$
- What about S^2? Not frameable because of the hairy ball theorem (we would need two vector fields that are a basis at each point of S^2 but because of the theorem we don't even have one).

In fact, there aren't many closed connected 2 manifolds with framing:

Fact. *The only closed connected 2 manifolds with framing is the torus.*

This follows from the Poincaré–Hopf index theorem[4].

Lemma 2.4.14. *Any framing on an n manifold M induces an orientation on M.*

From our definition this is immediate because $\{e\} \subseteq GL_n^+$ (however there are other definitions of orientation for which it's not immediate).

2.5 Classifications of 1- and 2-manifolds

In this section we classify 1- and 2-manifolds up to diffeomorphism. This will make our life much easier when we will later classify such manifolds up to cobordism, via cobordism invariants.

Note that any manifold, of any dimension, is diffeomorphic to the disjoint union of its connected components. Hence, two manifolds are diffeomorphic if and only if there is a 1-to-1 correpsondence between their connected components such that the correpsonding components are diffeomorphic. For this reason, when classifying n-manifolds it is sufficient to classify connected ones.

In the next two subsections we aim at doing this: classifying connected 1- and 2-manifolds, so that we classify all 1- and 2-manifolds.

2.5.1 Classification of 1-manifolds

We took the proof from [MW97, Appendix] and `https://math.mit.edu/classes/18.966/2014SP/965/class.pdf`, which is essentially the proof in Milnor's book but rephrased without explicitly using the concept of 'arc-length'.

Theorem 2.5.1 (Classification of (connected) 1-manifolds)**.** *Any connected 1-manifold is diffeomorphic to:*

- $[0,1]$
- $(0,1]$
- $(0,1)$

[4]This differential topology theorem relates the index of a vector field to the Euler characteristic of the manifold. In particular, it implies that a nonvanishing vector field can only exist if the Euler characteristic of the manifold is zero.

- S^1

In order to prove this, we make use of parametrizations.

Definition 2.5.2 (Parametrization)**.** Let M be an n-manifold, $U_x \subseteq M$ a neighbourhood of x and $V \subseteq \mathbb{R}^n$. Any diffeomorphism $g : V \to U_x$ is a parametrization of U_x.
The inverse map $U_x \to V$ is called a system of coordinates on U_x.

Now, we can state the following lemma, which is the key to our proof of the classification of 1-manifolds

Lemma 2.5.3. *Let M be a 1-manifold, and let $f, g : (0,1) \to M$ be two parametrizations. Then $f(0,1) \cap g(0,1)$ has at most two components. If it has one component, then there is a parametrization $h : (0,1) \to M$ so that $h(0,1) = f(0,1) \cap g(0,1)$. If it has two components, then $f(0,1) \cup g(0,1)$*

Proof. Let $\Gamma = \{(s,t) \mid f(s) = g(t)\} \subseteq (0,1) \times (0,1)$. Γ is closed, since it is the preimage of the continuous function $f \times g : (0,1) \times (0,1) \to M \times M$, which is in sometimes called diagonal map and denoted Δ. Let s_0 be a point contained in the topological boundary of $f^{-1}(g(0,1)) \subseteq (0,1) \subseteq [0,1]$. Since $f^{-1}(g(0,1))$ is open, $s_0 \notin f^{-1}(g(0,1))$. Let s_k be a sequence in $f^{-1}(g(0,1))$ that converges to s_0. Then

□

2.5.2 Classification of 2-manifolds with boundary

The aim of this chapter is to prove the aforementioned classification of compact 2-dimensional manifolds with the use of Morse theory (see 3.2.1). We remind the reader: let M be a compact 2-manifold with genus g, then

- if M is oriented, M is classified by its genus g. In particular M it is diffeomorphic to $\Sigma_g := T \# \ldots \# T$, the connected sum of g tori, and where we define $\Sigma_0 := S^2$.
- Instead if M is non orientable[5], M is diffeomorphic to g copies of the real projective plane, i.e. $\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2$.

One can find a thorough covering of this result in [Hir76, Chapter 3, page 200].
In addition, we will also find a classification of compact 2-manifolds with boundary (which we will simply call *surfaces*). To do it we will need Morse theory. What is the idea? Given a surface M (it works in greater generality but now we're only interested in surfaces), find a map $f : M \to \mathbb{R}$ which is "nice". Then by analizing properties of f we can recover the topology of the surface (i.e. the surface up to diffeomorphism).
This section is based on [Hir76] and [Kos13].

2.5.3 Introduction to Morse Theory

Morse theory is a way to extract informations on a manifold M from smartly chosen functions

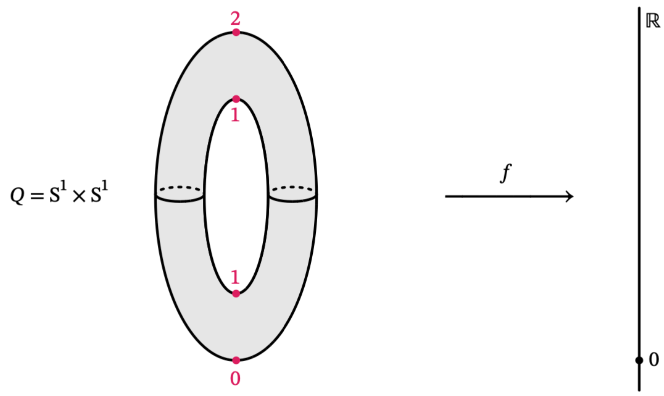
$$f : M \to \mathbb{R}$$

[5]see `https://en.wikipedia.org/wiki/Genus_(mathematics)` for what is meant by genus of a non orientable surface

Such smart functions are called Morse functions. It is very powerful, for instance, it plays a major role in the sketch of the proof of the cobordism hypothesis, see [Lur09], and in symplectic topology, see [Tan22]. Thus, it is very useful to have an idea of how it works!

However, we will only use it to classify 2-dimensional manifolds. Now, we introduce the fundamental tool of Morse theory: the Morse function.

Take as an example the projection map from the torus (stolen from [Tan22]):



Here we have 4 special points: the top and bottom, and then the two points in the middle. What happens in these points is that the shape of the preimage changes:

- the preimage of the top (and bottom) is a point
- the preimage of points between the top and middle point is a circle
- the preimage of the middle points is a wedge of circles
- the preimage of points between the middle points is a disjoint union of circles.

Note that at the special points the preimage is generally not a smooth manifold. These "special" points are critical points of f:

Definition 2.5.4 (Critical point and critical value)**.** Let M be a closed n-manifold and $f : M \to \mathbb{R}$ a smooth map. A critical point of f is a point $p \in M$ such that $(df)_p : T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$ is *not* surjective. A critical value is the image of a critical point.

So a critical point is a point of M with tangent bundle “vertical” with respect to $\mathbb{R}$ (thinking of f as some kind of projection).

The main goal then is to recover (up to diffeomorphism) M from knowing the critical points and the behaviour in a neighborhood thereof.
Firstly we can define what we mean by a “nice” function:

Definition 2.5.5. A Morse function is a smooth map $f : M \to \mathbb{R}$ such that all critical points are nondegenerate, that is if p is a critical point of f, then $\mathrm{Hess}(f)_p := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)_{i,j}$ is non-singular (i.e. $\det(\mathrm{Hess}(f)_p) \neq 0$).

Remark. This is independent of the choice of chart.

Example 2.5.6. Let's start with a nonexample: the parabolic cylinder. With coordinates x_1, x_2, the Morse function drawn can be written as $f(x_1, x_2) = -x_1^2$. Then at every critical point the Hessian is given by

$$\mathrm{Hess}(f)_p = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.6}$$

which is clearly singular.
Another nonexample is given by the function $f(x) = x^3$. Consider the map that goes from the graph of f to its y coordinate. Again the Hessian is singular at the critical point.

We're then interested in the local picture at a critical point.

Definition 2.5.7 (Index of a critical point)**.** If $p \in M$ is a nondegenerate critical point of $f : M \to \mathbb{R}$, the index of f at p is

$$\mathrm{ind}_f(p) = \text{index of } \mathrm{Hess}(f)_p = \#\text{ negative eigenvalues of} \mathrm{Hess}(f)_p. \tag{2.7}$$

Example 2.5.8. Consider the following basic examples:

- $f(x_1, x_2) = -x_1^2 - x_2^2$, this has critical point $p = (0,0)$ and there the Hessian is given by $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$. We therefore see that p is nondegenerate and that $\mathrm{ind}_f(p) = 2$.
- $f(x_1, x_2) = -x_1^2 + x_2^2$, very similar, but $\mathrm{ind}_f(p) = 1$,
- $f(x_1, x_2) = +x_1^2 - x_2^2$, again $\mathrm{ind}_f(p) = 1$,
- $f(x_1, x_2) = +x_1^2 + x_2^2$, now $\mathrm{ind}_f(p) = 0$.

The examples above are very important, because locally Morse functions are always of one of those forms.

Lemma 2.5.9 (Morse Lemma I)**.** *Let $p \in M^n$ be a nondegenerate critical point of $f : M^n \to \mathbb{R}$ of index k. Then there is a chart (U, ϕ) around p such that the map $\tilde{f}$ defined as*

$$\begin{tikzcd} U \arrow[r, "\phi"] \arrow[d, "f"] & \mathbb{R}^n \arrow[dl, "\tilde{f}"] \\ \mathbb{R} & \end{tikzcd} \tag{2.8}$$

is given by:

$$\tilde{f}(x_1, \ldots, x_n) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{j=k+1}^{n} x_j^2 \tag{2.9}$$

Idea of proof. Given a nondenerate critical point of index k we only know that $\mathrm{Hess}(f)_p$ has k negative eigenvalues, is nonsingular and is diagonalizable. That allows me to change the basis so that the function takes that form, essentially doing a Taylor expansion. In particular then the chart can be chosen such that the function is *exactly* that expression. □

Given a Morse function $f : M \to \mathbb{R}$, $p \in M$, then we have the following immediate consequences of the lemma:

- if p is critical (and nondegenerate since is a *Morse* function), because of the lemma we have a chart as above. Then the level sets $f^{-1}(f(p))$ look like a neighborhood of 0 in a quadric (locally).
- if p is regular (that is, non-singular), by the implicit function theorem there are coordinates such that $f(x_1,\ldots,x_n) = x_1$. So the level sets $f^{-1}(f(p))$ look like a submanifold in $\mathbb{R}^n$ (locally).

This explains what we noticed initially for the example of the torus, that the level set changes when passing a critical point. In particular it always changes in one of the ways shown in 2.5.8. This observation will be more complete after Morse Lemma II and Theorem 2.5.16 below.

The following theorem justifies the use of Morse theory to study manifolds.

Theorem 2.5.10. *For any manifold Morse functions exist.*

It's not trivial but it can be proven with tools from differential topology.

Example of something we can prove with Morse functions:

Theorem 2.5.11. *The Euler characteristic can be calculated by knowning the critical points and the corresponding indices of a Morse function:*

$$\chi(M) = \sum(-1)^k c_k \tag{2.10}$$

where c_k is the number of critical points of index k.

This is quite clear for the sphere and the genus g surfaces with the usual Morse functions, but it's interesting that for *any* Morse function we have this result.

The following theorem also seems very plausible from the drawings:

Lemma 2.5.12 (Morse Lemma II)**.** *Let $f : M \to \mathbb{R}$ be a Morse function, $a < b \in \mathbb{R}$ such that f does not have a critical value in $[a,b]$. Then*

$$M_a = f^{-1}((-\infty,a]) \hookrightarrow M_b = f^{-1}((-\infty,b]) \tag{2.11}$$

is a (smooth) deformation retract. In particular, it induces a diffeomorphism $f^{-1}(a) \cong f^{-1}(b)$.

Concretely this tells us that the level set *only* changes when crossing a critical point.

Remark. The proof uses flow along vector field $\frac{\operatorname{grad} f}{|\operatorname{grad} f|}$...

We will restrict to the following class of Morse functions:

Definition 2.5.13 (Admissibile Morse function)**.** A Morse function $f : M \to [a,b]$ is admissibile if $\partial M = f^{-1}(a) \cup f^{-1}(b)$ and a,b are regular values.

The fact that they're regular values is important, since it gives us that a neighborhood of $f^{-1}(a)$ is diffeomorphic to a cylinder $f^{-1}(a) \times [0,\epsilon]$, i.e. a collar (and the same is true for a neighborhood of $f^{-1}(b)$). So an admissible Morse function naturally equips M with the structure of a cobordism from $f^{-1}(a)$ to $f^{-1}(b)$.
Another useful theorem, analogous to 2.5.10 is the following:

Theorem 2.5.14. *Admissible Morse functions exist.*

We would now like to know *how* the level set changes before and after the critical point. In order to do this we first introduce the concept of handle attachment.

Definition 2.5.15 (Handle attachment)**.** Let M^n be an n-manifold and let $H^j := D^j \times D^{n-j}$ for $j = 0, 1, \dots, n$. In addition let $f : \partial D^j \times D^{n-j} \hookrightarrow \partial M^n$ be an embedding and note that $\partial D^j \times D^{n-j}$ also embeds into H^j. We can then *attach a j-handle* to M^n by gluing along f to obtain the manifold $M^n \amalg_f H^j$.

This concept will be studied in one of the exercises. The following claim makes the definition meaningful:

Claim. *$M^n \amalg_f H^j$ has a smooth structure.*

Since we're interested in 2 manifolds, let's make explicit what it means to attach a j handle to a 2 manifold:

- $j = 0$, $H^0 = D^0 \times D^2 \cong D^2$ and we attach along the empty set (since $\partial D^0 = \emptyset$), i.e. we take the disjoint union with D^2.
- $j = 1$, $H^1 = D^1 \times D^1$ and we attach along $\partial D^1 \times D^1$, i.e. along two segments
- $j = 2$, $H^2 = D^2 \times D^0$ and we attach along $\partial D^2 \times D^0 \cong S^1$, i.e. along a circle

We can now see how the level sets change at a critical point.

Theorem 2.5.16 ([Hir76] Theorem 3.2, p.157)**.** *Let M^n be compact and $f : M^n \to [a, b]$ an admissible Morse function. Suppose that f has a unique critical point z of index j.*
There is an embedding $\iota : D^j \hookrightarrow M^n$ with image $e^j := \operatorname{im}\iota$ (called "belt disk"), satisfying:

- *$z \in e^j$;*
- *$e^j \subset M^n \setminus f^{-1}(b)$;*
- *$f^{-1}(a) \cap e^j = \partial e^j = \iota(\partial D^j)$, this is also called "belt sphere";*
- *M deformation retracts onto $f^{-1}(a) \cup e^j$.*

The embedding can then be extended to $e^j \times D^{n-j} =: H^j$. We can now choose an a' with $a < a' < f(z)$ and we then have

$$M^n \cong f^{-1}([a, a']) \cup_{\bar{\iota}} H^j \tag{2.12}$$

(for $n = 2$ this is a unique diffeomorphism up to isotopy[6]).

This theorem explains what happens to the level sets when we meet a critical point of index j: we attach a j handle! To use the terms above: we extend the belt disk to a j handle and attach it to the boundary by gluing along the belt sphere (which is also extended). Of course, all the drawings presented are for 2 manifolds and we will only apply these results to such manifolds, but this theorem works more in general!

Example 2.5.17.

- index 0: $M = S^2$ we then have $e^0 = \{z\}$, $\iota : D^0 = pt \hookrightarrow M$ and $\bar{\iota} : D^0 \times D^2 \hookrightarrow M$.

[6]See **??** for the definition of isotopy.

- Now what happens when attaching a 1-handle im$\bar{\iota}$? Chose: $D^{n-k} \hookrightarrow \partial(S^1 \times [0,1] \amalg S^1 \times [0,1])$
- Start with $S^1 \times [0,1]$. Want to attach a 1-handle $D^1 \times D^1$. But now we can attach in two ways. With a simple band or with a twisted one.
- Let's start again with a cylinder and attach a 2-handle $D^2 \times D^0$, so I kind of close one of the two side of the cylinder. We see that the 1-handle and the 2-handle kind of cancel each other out!
- Analogously, starting with a disk and attaching a 0-handle and then a 1-handle, these also cancel each other out!

Proposition 2.5.18 (VI 7.1 in [Kos13], "Handle slides"). *If $\tilde{M} := (M \coprod_f H^j) \coprod_g H^i$ and $i \le j$, then $\tilde{M}$ can be obtained by first attaching H^i and then H^j.*

Note that the word *can* in the proposition is important: if we were to simply commute the two attachments $(M \coprod_g H^i) \coprod_f H^j$ we would get something that in general doesn't make sense, since the map g goes into $M \coprod_f H^j$, not just M, and H^i may no longer be attached at the boundary, leading to a space that is not even a manifold. The proposition instead says that there exist maps $\tilde{g}$ and $\tilde{f}$, into M and $M \coprod_{\tilde{g}} H^i$ respectively, such that $(M \coprod_f H^j) \coprod_g H^i \cong (M \coprod_{\tilde{g}} H^i) \coprod_{\tilde{f}} H^j$.

Example 2.5.19.

- $j = 1, i = 0$
- $j = 2, i = 1$ same picture but upside down!
- $j = i = 1$ do it as an exercise.

What happens if $i > j$? In general it's not so simple, but if $i = j + 1$ the following result holds.

Proposition 2.5.20 (VI 7.4 in [Kos13], "Handle cancellation"). *Let $\tilde{M} := (M \coprod_f H^j) \coprod_g H^{j+1}$, where the attaching sphere of H^{j+1} intersects the belt sphere of H^j "transversely" in one point. Then $\tilde{M} \cong M$.*

Example 2.5.21. $j = 0$

Theorem 2.5.22 ([Hir76] 8.3.4, p. 187). *Let M be a surface admitting a Morse function that has exactly two critical points. Then*

$$M \cong S^2 \tag{2.13}$$

Remark. In dimension 1,2 and 3 all the homeomorphic manifolds are also diffeomorphic.

Proof. Because of the remark it's enough to prove the homeomorphism, rather than the diffeomorphism.
Assume $p_+, p_- \in M$ are critical points. Since M is compact, also $f(M)$ is compact and therefore has a maximum and a minimum, which exactly correspond to the two critical points. Now assume p_+ is the maximum, then ind $p_+ = 2$. Now, because of Morse Lemma I we have $\exists U_+$ a neighborhood of p_+ with coordinates x_1, x_2 such that

$$f|_{U_+} = -x_1^2 - x_2^2 + f(p_+) \tag{2.14}$$

Then we have $\exists b < f(p_+)$ such that $D_+ = f^{-1}([b, +\infty)) \cong D^2$.
Similarly, $\exists U_-$ a neighborhood of p_- with coordinates x_1, x_2 such that

$$f|_{U_-} = x_1^2 + x_2^2 + f(p_-) \tag{2.15}$$

and now we have $\exists a > f(p_-)$ such that $D_- = f^{-1}((-\infty, a]) \cong D^2$.
Let B_+, B_- be disjoint caps around the poles of S^2 and denote $C := S^2 \operatorname{int}(B_+ \amalg B_-) \cong S^1 \times [0,1]$. Now, notice $\partial D_+ \cong S^1 \cong \partial D_-$ and we have a diffeomorphism $h_0 : D_+ \to B_+$ as they are both diffeomorphic to D^2. In addition $h_0|_{\partial D_+} \partial D_+ \to \partial B_+$. Notice that between $[a,b]$ there are no critical values, that is $f^{-1}([a,b])$ has no crtiical points Now, using Morse Lemma II we get:

- $f^{-1}((-\infty, b]) =: M_b \hookrightarrow M_a := f^{-1}((-\infty, a])$, $f^{-1}(a) \cong f^{-1}(b)$
- $f^{-1}([a,b]) \cong f^{-1}(a) \times [0,1] \cong S^1 \times [0,1]$

Now we can extend $h_0|_{\partial D_+}$ to $h_1 : \partial D_+ \times [0,1] \to \partial B_+ \times [0,1]$. Now we can glue h_0 and h_1

$$h : D_+ \cup (\partial D_+ \times [0,1]) \to B_+ \cup (\partial B_+ \times [0,1]) \tag{2.16}$$

Claim. *If $g : S^1 \to S^1$ is a homeomorphism, then it can be extended to a homeomorphism $\tilde{g} : D^2 \to D^2$.*

$\tilde{g}$ can simply be defined as follows

$$\tilde{g}(x) = \begin{cases} ||x|| g\left(\frac{x}{||x||}\right), & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases} \tag{2.17}$$

From the claim h can be extended to a homeomorphism $M \to S^2$. □

Remark.

1. For any closed n manifold that has two critical points, we have $M \cong S^n$ (*homeomorphic*, not *diffeomorphic*).

2. (From Milnor) There are manifolds M homeomorphic to a sphere but not diffeomorphic (e.g. S^7), we then talk about *exotic spheres*.

Theorem 2.5.23 (Classification of surfaces)**.** *Let M^2 be an oriented, connected, closed surface. Then it can be obtained as in the following drawing:*

Proof.
Step 1: Choose an admissible Morse function f.
Step 2a: If necessary, "perturb" f to get distinct critical values.
Step 2b: Chop into pieces with exactly one critical point and apply Morse Lemma I. We then get that $f^{-1}((-\infty, a_i])$ is obtained from $f^{-1}((-\infty, a_{i-1}])$ by attaching a handle.
Goal: normal form (for closed connected surfaces)
Step 3: Claim: We can obtain M by first attaching 0-handles, then 1-handles, ..., in ascending order.
To prove this there are two strategies:

1. Use proposition about handle cancellation (first attaching i handle and then $i+1$ handle these cancel) and handle slides: with these you can write down an algorithm to get the normal form from any handle decomposition.

2. One can change the Morse function. Claim: We can choose f such that if $\operatorname{ind} p_1 < \operatorname{ind} p_2 \implies f(p_1) < f(p_2)$. This can be proven by changing the Morse function locally around a critical point (see [Hir76] for more details).

Step 4: If $\partial M = \emptyset$, then either $M = \emptyset$ or I have at least two critical points.
Case 1: We have exactly 2 critical points which gives us $M \cong S^2$ because of Lemma.
Case 2: If we have > 2 critical points, look at minimum, then there exist a neighborhood D_- of the minimum such that $D_- \cong D^2$, i.e. a 0 handle attached to $\emptyset$ (because of Morse Lemma I f locally looks like a cup).
Starting from the minimum, attach one 0-handle. If then we attach another 0-handle, then we can use the handle slides and cancellations to get rid of all 0 handles but one.
Next we can attach one handles and we have two options, only one of which gives an orientable manifold.
At "end", same argument as for why only one 0-handle read backwards shows that we only have one 2-handle (for example change f to $-f$).
This ends the proof. □

Question: What about the unoriented case? (*Hint: enough to have one attachment of Möbius strip*)

We now consider the case with boundary:

Theorem 2.5.24 (Classification of surfaces with boundary)**.** *Let M^2 be an oriented, connected surface. Then it can be obtained as in the following drawing:*

Proof. Now $\partial M \neq \emptyset$, then $\partial M = S^1 \amalg \cdots \amalg S^1$. If f is an admissible Morse function f onto $[a,b]$ with $f^{-1}(a) \cong (S^1)^{\amalg m}$ and $f^{-1}(b) \cong (S^1)^{\amalg n}$. Now one can prove that we need neither 0-handles (if $m \neq 0$) nor 2-handles (if $n \neq 0$). □

Chapter 3

Cobordism groups

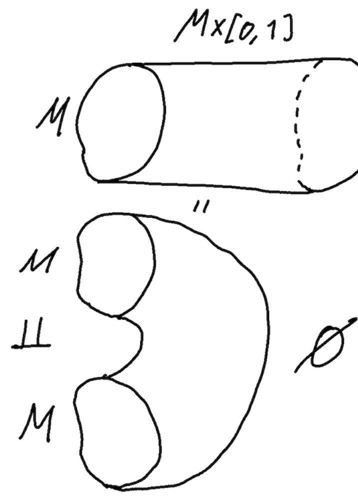
We now define the most fundamental object in cobordism theory: the cobordism group. It is an algebraic structure characterizig if two manifolds are cobordant and how this relates to the disjoint union of manifolds. From this one can establish cobordism invariants, i.e. homomorphisms from a certain cobordism group to another abelian group. To do this, we will exploit the classifications of 1- and 2-manifolds up to diffeomorphism we proved in the previous chapters. To be diffeomorphic is a much stricter and finer relation then being cobordant, that is why it is much harder to classify manifolds up to diffeomorphism, but it also will make our life much easier when establishing cobordism invariants.

Definition 3.0.1. The set underlying the n-th cobordism group is

$$\Omega_n = \frac{\{\text{closed } n \text{ manifolds}\}}{\{n+1 \text{ cobordisms}\}}$$

$(\Omega_n, \amalg)$ is an abelian group with operation given by the disjoint union $\amalg$.

1. the disjoint union is associative and commutative
2. the identity element[1] is the cobordism class of the empty set[2] $[\emptyset]$, i.e. the set of n-manifolds which are bordant with the empty set
3. every element has an inverse: $[M] \amalg [M] = [\emptyset]$ since $M \amalg M = (\partial(M \times [0,1]))$ and such a cylinder can be bent into a macaroni, which is bordant to the empty set:



[1]We will sometimes write $0 = [\emptyset]$ or more often $\emptyset = [\emptyset]$

[2]Reminder: $\emptyset$ is an n-manifold for every n.

Remark. There is an insidious set-theoretic issue when we define the n-th cobordism group Ω_n (3.0.1): is the collection of all closed n-manifolds a set? And what about the collection of all $(n+1)$-bordisms? It could be the case that it is something bigger than a usual set, thus not a set and problematic since we would not know how to treat them[3]. For example, we know that the collection of all sets is strictly greater than any set[4] and thus not a set. Similarly, also the collection of topological spaces is not a set because we can regard any set as a topological space via the discrete topology[5]. One could wonder if the collection of all manifolds of a certain dimension n is likewise not a set. This is fortunately for us not the case. The following theorem allows us to happily treat {closed n manifolds} and $\{n+1$ cobordisms} as sets by replacing abstract manifolds and cobordism by manifolds and cobordisms embedded in $\mathbb{R}^\infty$.

Theorem 3.0.2 (Whitney Embedding theorem[6])**.** *Any n manifold can be embedded in* $\mathbb{R}^\infty(=\bigcup_{n\in\mathbb{N}}\mathbb{R}^n)$. *The space of such embeddings is contractible.*[7]

Remark. Actually $(\Omega_n, \amalg)$ is a finitely generated abelian group, but this is a hard theorem. In particular, it is a finite product of cyclic groups of order 2 (from 2.)

Definition 3.0.3 (Bordism invariant)**.** A bordism invariant is a homomorphism of abelian groups

$$(\Omega_n, \amalg, \emptyset) \to (A, \cdot, e)$$

the abelian group A can be $\mathbb{Z}$, $\mathbb{R}$ or $\mathbb{C}$ for instance.

Remark. Many important manifold invariants are also bordism invariants.

Example 3.0.4.

- χ mod 2 (the Euler characteristic)
- signature
- characteristic classes such as Pontrjagin, Stiefel-Whitney or Chern classes

3.1 The 0-th Cobordism Group, Ω_0

Consider

$$\Omega_0 = \{\text{finite disjoint unions of points}\}/\{\text{1-dimensional cobordisms}\}.$$

To compute this, consider the following classification of 1 manifolds:

Proposition 3.1.1. *Any* 1 *dimensional compact manifold with boundary is diffeomorphic to a finite disjoint union of closed intervals* $[0,1]$ *and circles* S^1.

[3]We are assuming to be working in ZFC. There are other set theories in which one can define collections greater than sets, e.g. Von Neumann–Bernays–Gödel set theory.

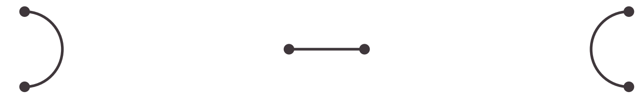
[4]Because of famous set-theoretic paradoxes like Cantor's paradox.

[5]The topology where the set of open sets is the powerset of the set in question, or in other words where any subset is open.

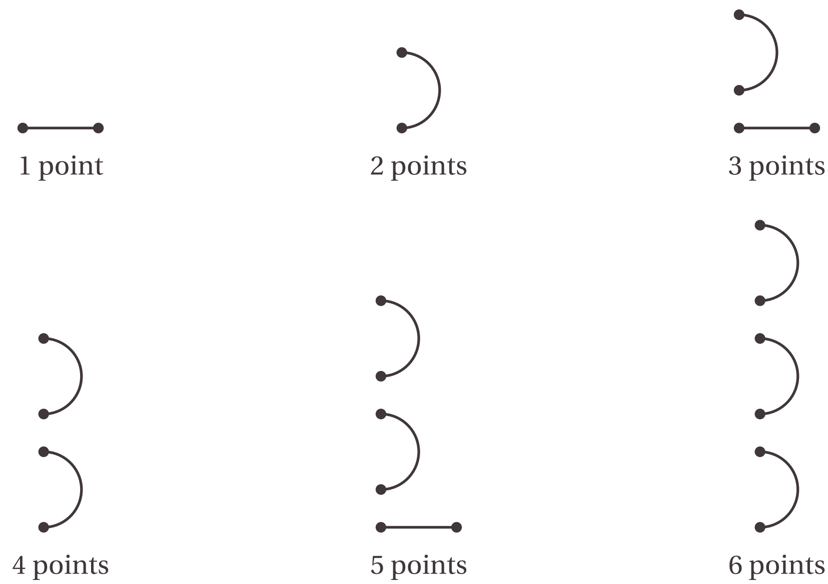
[6]This result is not important only in this case, but notably for the interest of this class it was used to construct an apt category of bordisms 5 in order to sketch a partial proof of the most important conjecture in the field of TFTs: the cobordism hypothesis; see for details on this [Lur09] and [CS19].

[7]Note that there are refinements of the latter result.

Example 3.1.2. An example of a 1-dimensional cobordism is $W = [0,1]$. Here are the 3 different way of seeing it as a bordism: from $Y_0 = \{0,1\}$ to $Y_1 = \emptyset$, from $Y_0 = \{0\}$ to $Y_1 = \{1\}$ and from $Y_0 = \emptyset$ to $Y_1 = \{0,1\}$.



Example 3.1.3. Here is a list of various finite disjoint unions of points with different cardinalities. Can you see a pattern emerging?



Keeping this pattern in mind, consider collection of k points, with k finite (i.e. the only closed 0 manifolds):

- If k is even, we can find a bordism to $\emptyset$
- If k is odd, we can find a bordism to $\{*\}$

We then have

$$2k \text{ points} \sim \emptyset, \qquad 2k+1 \text{ points} \sim 1 \text{ point}$$

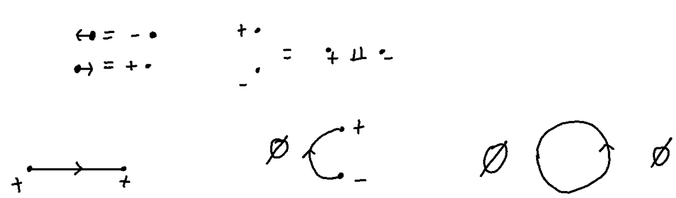
Therefore, the 0-th cobordism group is given by

$$\Omega_0 = \{\emptyset, \{*\}\} \cong \mathbb{Z}_2. \tag{3.1}$$

A possible variation is: add a "decoration", called *orientation* (will be made more precise later), so we now have:

$$\Omega_0^{or} = \big\{\text{finite sets of points } S \text{ with a map } S \to \{+,-\}\big\} / \{\text{oriented cobordism}\}$$

An oriented cobordism in 1 dimension is given by $[0,1]$ or S^1 which comes with an orientation. Here are some examples



Exercise 3.1.4. What is Ω_0^{or}? Try to think about it!

3.1.1 The 1st cobordism group, Ω_1

Now we have

$$\Omega_1 = \{\text{closed 1-dimensional manifolds}\}/\{\text{2-dimensional cobordisms}\}.$$

In studying this group, the following result, which is a restriction of 3.1.1:

Theorem 3.1.5. *Any closed* 1*-dimensional manifold is a finite disjoint union of circles*

However, a circle is the boundary of a 2-disk, which gives a cobordism from S^1 to $\emptyset$, we then have $S^1 \sim \emptyset$. Hence, also finite disjoint unions of S^1 are cobordant to the empty set. Therefore, the 1st cobordism group is trivial:

$$\Omega_1 = 0 \tag{3.2}$$

3.2 The 2nd cobordism group, Ω_2

In order to find the 2nd cobordism group, we first need a classification of 2 manifolds.

Proposition 3.2.1 (Classification of 2-dimensional manifolds)**.** *Every connected closed* 2 *manifold is diffeomorphic to*

1. S^2*, orientable*

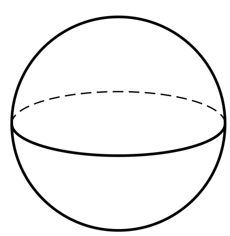


Figure 3.2: 2-sphere, S^2

2. $\Sigma_g = \underbrace{T\#\ldots\#T}_{g-times}$*, orientable*

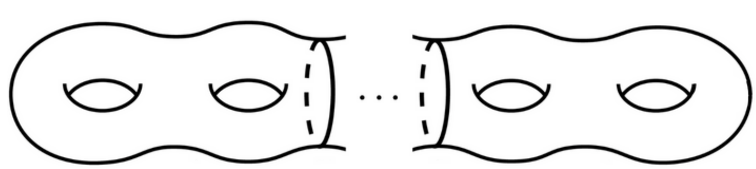
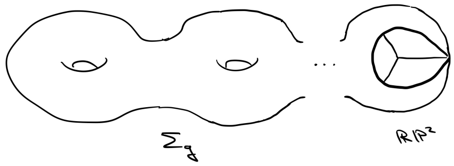


Figure 3.3: Surface of genus g, Σ_g

3. $P_k = \underbrace{\mathbb{RP}^2\#\ldots\#\mathbb{RP}^2}_{k-times}$*, non-orientable*



(a) $P_{2g+1} \cong \Sigma_g \# \mathbb{R}\mathbb{P}^2$

(b) $P_{2g+2} \cong \Sigma_g \# K$

We can now use this knowledge to compute

$$\Omega_2^{\text{or}} = \{\text{closed oriented 2 manifolds}\}/\{\text{3-dimensional oriented cobordisms}\}$$

by observing that S^2 and Σ_g are both cobordant to $\emptyset$ since they can simply be "filled" to a 3 dimensional manifold with boundary, we get

$$\Omega_2^{or} = 0. \tag{3.3}$$

Exercise 3.2.2. What about the non-oriented part Ω_2?

3.3 More on cobordism groups

We now go back to cobordism groups to give some important results.

We had found

$$\Omega_0 \cong \mathbb{Z}_2 \tag{3.4}$$
$$\Omega_1 \cong 0 \tag{3.5}$$
$$\Omega_2 \cong \mathbb{Z}_2 \tag{3.6}$$

while for the oriented case we have

$$\Omega_0^{or} \cong \mathbb{Z} \tag{3.7}$$
$$\Omega_1^{or} \cong 0 \tag{3.8}$$
$$\Omega_2^{or} \cong 0 \tag{3.9}$$

Definition 3.3.1 (Commutative $\mathbb{Z}$ -Graded Ring[8])**.** A commutative $\mathbb{Z}$ -graded ring is a ring R if there is a family of subgroups $\{R_n\} n \in \mathbb{Z}$ such that

- the underlying abelian group can be decomposed as $R = \bigoplus_{n\in\mathbb{Z}} R_n$
- $R_n \cdot R_k \subset R_{n+k}$ for all $n, k \in \mathbb{Z}$)

A non-zero element $x \in R_n$ is called a homogeneous element of R of degree n.

Proposition 3.3.2. $(\Omega_\bullet = \bigoplus_{n\geq 0} \Omega_n, \amalg, \times)$ *is a commutative* $\mathbb{Z}$ *-graded ring.*

Proof. Firstly we need to check that the product respects the degree: this is true because $M^m \times N^n = (M \times N)^{m+n}$, so $[M \times N] \in \Omega_{m+n}$.
Then the products descends to equivalence classes: if $Y_0 \simeq Y_1$ are cobordant via cobordism (X, p) $(p : \partial X \to 0, 1)$, M any manifold, then $\partial X \times M = \partial X \times M \xrightarrow{p \circ pr_X} \{0, 1\}$. Therefore $(X \times M, p \circ pr_X)$ is a cobordism from $Y_0 \times M$ to $Y_1 \times M$. □

[8] Note that there is a difference between commutative graded ring and graded commutative ring! A commutative graded ring is a commutative ring that is graded (our notion), a graded commutative ring is a different notion that depends on the degree of homogeneous elements.

Theorem 3.3.3 (Thom)**.** *There is an isomorphism of $\mathbb{Z}$ -graded commutative rings*

$$\Omega_\bullet \cong \mathbb{Z}_2[x_2, x_4, x_5, x_6, x_8, \dots], \ \textit{with } \deg x_k = k, k \neq 2^i - 1 \tag{3.10}$$

and the generators of the even degrees are given by $x_{2k} = [\mathbb{RP}^{2k}]$.

Remark. Note that $(\deg x_2)^2 = \deg 4$ so maybe we could imagine that $\mathbb{RP}^2 \times \mathbb{RP}^2 \sim \mathbb{RP}^4$ however that's not true since $x_2^2 \neq x_4$.

There are versions for any tangential structure, such as orientation or stable framing.

Theorem 3.3.4. *There is an isomorphism of $\mathbb{Z}$ graded commutative rings:*

$$\Omega_\bullet^{or} \otimes \mathbb{Q} \cong \mathbb{Q}[y_4, y_8, y_{12}, \dots]$$
$$y_{4k} \mapsto [\mathbb{CP}^{2k}]$$

We write it in this way because in this case there is nontrivial torsion. We could also write

$$\Omega_\bullet^{or} / \textit{torsion} \cong \mathbb{Z}[z_4, z_8, z_{12}, \dots]$$

where the generators are given by Milnor hypersurfaces.

Example 3.3.5. In particular, the groups in various degrees are given by the following list:

$$\Omega_0^{\text{or}} = \mathbb{Z}$$
$$\Omega_1^{\text{or}} = 0$$
$$\Omega_2^{\text{or}} = 0$$
$$\Omega_3^{\text{or}} = 0$$
$$\Omega_4^{\text{or}} = \mathbb{Z}$$
$$\Omega_5^{\text{or}} = \mathbb{Z}_2$$
$$\Omega_6^{\text{or}} = 0$$
$$\Omega_7^{\text{or}} = 0$$
$$\Omega_8^{\text{or}} = \mathbb{Z} \oplus \mathbb{Z}$$
$$\Omega_{n \geq 9}^{\text{or}} \neq 0$$

For the last result see [MS05, p. 203].

3.3.1 Cobordism groups and the sphere spectrum

Very interestingly, the (stable) framed version is isomorphic to the stable homotopy groups of the sphere and it is *not* computed to all degrees. This is one way to phrase a theorem named after Thom. It is fascinating because the stable homotopy groups of the sphere are central objects in stable homotopy theory. They are equivalently the homotopy groups of the sphere spectrum.

Now characterize spectra the bare minimum in order to talk about the sphere spectrum. Later, after some detours in the magical world of ∞-categories, we will give a more comprehensive sketch of what they are **??**.

Definition 3.3.6 (Suspension)**.** Let X be a pointed topological space. The suspension ΣX of X is the smash product of X with S^1, i.e.

$$\Sigma X = X \wedge S^1 = \frac{X \times S^1}{(\{*\} \times S^1) \amalg X \times \{1\}}$$

Example 3.3.7. $\Sigma(S^1) = S^2$ more generally $\Sigma(S^n) = S^{n+1}$

Definition 3.3.8 (Prespectrum[9])**.** A prespectrum is a sequence of pointed spaces $\{X_k\}_{k\in\mathbb{Z}}$ with maps preserving basepoints $\Sigma X_n \xrightarrow{\sigma_n} X_{n+1}$ called structure maps.

Definition 3.3.9 (Suspension Spectrum)**.** The suspension spectrum $\Sigma^\infty X$ has Σ_n as the n-th space in the sequence and structure maps $\Sigma\Sigma_n \cong \Sigma_{n+1}$.

Example 3.3.10 (Sphere Spectrum)**.** The sphere spectrum $\mathbb{S}$ is the suspension spectrum of the point, S^0. In fact, $\Sigma S^0 = S^1$, $\Sigma^2 S^0 = \Sigma\Sigma S^0 = \Sigma S^1 = S^2$ and in general $\Sigma^n S^0 = S^n$

Notation (Stable Homotopy Groups of the Sphere)**.** The stable homotopy groups of the sphere are the homotopy groups of the sphere spectrum. Alternatively, one can define them as the homotopy groups of the sphere $\pi_{n+i}(S^n)$ such that $n > i+1$. This latter characterization explains why they are called 'stable': due to Freudenthal's suspension theorem, such homotopy groups are independent of n.

Remark. Note that the stable homotopy groups of the sphere can be made into a commutative $\mathbb{Z}$ -graded ring via direct sums. We denote it with $\pi_\bullet(\mathbb{S})$

$$\pi_\bullet(\mathbb{S}) = \bigoplus_{n\geq 0} \pi_n(\mathbb{S})$$

Theorem 3.3.11 (Thom's theorem)**.** *One way of phrasing Thom's theorem is*

$$\Omega_\bullet^{\mathrm{fr}} \cong \pi_\bullet(\mathbb{S})$$

This is also called Pontrjagin-Thom isomorphism.

Example 3.3.12. We list some examples of such commutative rings.

$$\Omega_0^{fr} \cong \mathbb{Z}$$
$$\Omega_1^{fr} \cong \mathbb{Z}_2$$
$$\Omega_2^{fr} \cong \mathbb{Z}_2$$
$$\Omega_3^{fr} \cong \mathbb{Z}_{24}$$
$$\Omega_4^{fr} \cong 0$$
$$\Omega_5^{fr} \cong 0$$
$$\Omega_6^{fr} \cong \mathbb{Z}_2$$
$$\Omega_7^{fr} \cong \mathbb{Z}_{240},$$
$$\Omega_1^{fr}1 \cong \mathbb{Z}_{504}$$
$$\Omega_1^{fr}5 \cong \mathbb{Z}_{480} \oplus \mathbb{Z}_2$$

[9] In older literature it is called just spectrum, or sometimes it is called sequential spectrum. We call it *pre*spectrum in order to distinguish it from other notions with more structure, such as the suspension spectrum

Part II

Topological field theories

Preamble: a modern perspective on cobordisms

A bordism invariant was characterized as a homomorphism from the cobordism group to some other abelian group. To extract the cobordism group from bordism we took the following steps:

1. observed that being cobordant is an equivalence relation,
2. considered the equivalence classes of closed n manifolds up to $n+1$ cobordisms, obtaining the sets Ω_n,
3. took such equivalence classes together with disjoint unions, resulting in an abelian group[10].

This strategy was the key for classifying manifolds up to cobordism. However, the cobordism groups merely record that *there is* a bordism between two manifolds (since two manifolds are equivalent just if there is a bordism, independently of what kind of bordism it is), thereby forgetting other properties of the bordism itself. We swich now perspective and analyze a more sophisticated structure remembering <u>how</u> two manifolds are cobordant, e.g. indicating the manifold that bounds them and the direction of the bordism: the symmetric monoidal category $\mathrm{Bord}_{n,n-1}$ where objects are $(n-1)$ manifolds and morphisms are n-cobordisms. This is an instance of a process called categorification[11]: adding categorical structure to things, e.g.[12] passing from set-theoretic notions like set or function categorical ones like category or functor. The invariants will become in turn functors from $\mathrm{Bord}_{n,n-1}$ to categories of algebraic nature like Vect_k, the category of vector spaces on a field k. Such categorified cobordism invariants are exactly topological field theories (TFTs).

The following table summarizes the comparison between additional structures in the two perspectives:

Ω_n	$\mathrm{Bord}_{n,n-1}$
set	category
monoid	monoidal category
commutative monoid	symmetric monoidal category
abelian group	Picard groupoid

[10] And eventually in a $\mathbb{Z}$ graded commutative ring, but this will not be as important for us from now on.

[11] Sometimes, e.g. in the nLab, also called vertical categorification.

[12] Note that although this way of categorifying is the most prominent one, it is strictly speaking not the only way to categorify, one could also go from category theory to higher category theory.

Analogous to the comparison between the set-theoretic and category-theoretic perspective, we could also have a linear-algebraic perspective. Since an associative algebra on a vector space is the parallel construction to a monoid with set, and a commutative algebra corresponds to a commutative monoid. The following table adds this perspective.

Ω_n	Vect_k
set	vector space
monoid	associative algebra
commutative monoid	commutative algebra

We make both these comparisons more rigorous by later (4.4.2) showing that

1. a monoid is a monoid object in the category of sets.
2. an associative algebra is a monoid object in the category of k-vector spaces $Vect_k$
3. a (*strict*) monoidal category is a monoid object in the category of small categories Cat

The same holds for the commutative case, commutative monoids, commutative algebras and (*strict*[13]) symmetric monoidal categories are all examples of commutative monoid objects (see 4.5.5).

[13]We will also sketch a way how to get general symmetric monoidal categories, i.e. not necessarily strict, in an analogous way. See **??**.

Chapter 4

A summary of category theory

4.1 Category theory: basic definitions

Definition 4.1.1. A locally small[1] category $\mathcal{C}$ consists of the following data:

- A class $\mathrm{ob}(\mathcal{C})$ whose elements are called the objects of $\mathcal{C}$,
- For any $X, Y \in \mathrm{ob}(\mathcal{C})$ a set $Hom_{\mathcal{C}}(X, Y)$ whose elements are called morphisms from X to Y,
- For any objects $X, Y, Z \in \mathrm{ob}(\mathcal{C})$ a map[2]

$$\mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{\circ} \mathrm{Hom}_{\mathcal{C}}(X,Z)$$
$$(g, f) \mapsto g \circ f$$

  which is called composition of morpisms,
- For every object $X \in \mathrm{ob}(\mathcal{C})$ an element $id_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ called the identity of X.

Such data must fulfill the following axioms:

- $\forall f \in \mathrm{Hom}_{\mathcal{C}}(X,Y), f \circ id_X = f = id_Y \circ f$ (unitality)
- $\forall f \in \mathrm{Hom}_{\mathcal{C}}(X,Y), g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z), h \in \mathrm{Hom}_{\mathcal{C}}(W,Z)$:

$$(h \circ g) \circ f = h \circ (g \circ f) \qquad \text{(associativity)}$$

Example 4.1.2. A few examples of categories are the following:

[1] A category is locally small if every hom $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is not bigger than a set. A locally small category is small if the collection of objects is also a set. A large category is a category which is not small. A category is essentially small if it is locally small and the collection of isomorphism classes (collections of isomorphic objects, i.e. objects with a morphism between them which has a left- and right-inverse) is a set. Questions of size of collections play an important role in category theory. For example, one cannot naively take the set of all sets as the collection of objects of the category of sets because of famous set-theoretic paradoxes like Cantor's, Burali-Forti's or Russell's. See [Shu08] for an account on possible set-theoretic foundations for category theory.

[2] Note that we can simply define composition as a map between sets because we are working with a locally small category.

category	objects	morphisms
Set	class of all sets	functions between sets
Mon	class of all monoids	monoid homomorphisms
Grp	class of all groups	group homomorphisms
AbGrp	class of all abelian groups	group homomorphisms
Ring	class of all rings	ring homomorphisms
Vect_k	class of all k vector spaces	linear maps
Alg_k	class of all algebras over k	algebra homomorphisms
Top	class of all topological spaces	continuous functions
FinSet	class of all finite sets	functions between sets
SmoothMfld	set of all smooth manifolds	smooth functions

1. Let $(P, \leq)$ be a set with a transitive and reflexive relation $\leq$ (a preordered set). Define a category $\mathbf{P}$ with:
$$\text{ob}(\mathbf{P}) = P$$
$$\text{Hom}_{\mathbf{P}}(X, Y) = \begin{cases} \{*\} & \text{if } x \leq y \\ \emptyset & \text{else} \end{cases}$$

2. Given categories $\mathcal{C}, \mathcal{D}$ we can define a category $\mathcal{C} \times \mathcal{D}$ (the product category) by:
$$\text{ob}(\mathcal{C} \times \mathcal{D}) = \text{ob}(\mathcal{C}) \times \text{ob}(\mathcal{D})$$
$$\text{Hom}_{\mathcal{C}\times\mathcal{D}}((X, Y), (X', Y')) = \text{Hom}_{\mathcal{C}}(X, X') \times \text{Hom}_{\mathcal{D}}(Y, Y')$$

3. Given a category $\mathcal{C}$, define a category $\mathcal{C}^{op}$ by:
$$\text{ob}(\mathcal{C}^{op}) = \text{ob}(\mathcal{C})$$
$$\text{Hom}_{\mathcal{C}^{op}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$$
$$g \circ_{\mathcal{C}^{op}} f = f \circ_{\mathcal{C}} g$$
this is called the opposite category of $\mathcal{C}$.

Definition 4.1.3. An isomorphism $f \in \text{Hom}(X, Y)$ is a morphism such that $\exists g \in \text{Hom}(Y, X)$ with $g \circ f = id_X, f \circ g = id_Y$.

Definition 4.1.4. A groupoid is a category where each morphism is an isomorphism.

Example 4.1.5. Let $\mathcal{C}$ be a category with $\text{ob}(\mathcal{C}) = \{*\}$.
Then, $(\text{Hom}_{\mathcal{C}}(*, *), \circ)$ is a monoid since composition is associative and unital with neutral element given by the identity morphism id_*. Conversely, every monoid $(M, \cdot)$ defines a category $\mathbf{B}M$ with
$$\text{ob}(\mathbf{B}M) = \{*\},\ \ \text{Hom}_{\mathbf{B}M}(*, *) = M,\ \ m \circ_{\mathbf{B}M} m' = m \cdot m',\ \ id_* = 1_M$$
$\mathbf{B}M$ is called the delooping of the monoid $(M, \cdot)$ [3]. The same holds for groups and one-object groupoids: every group $(G, \cdot)$ defines a one-object groupoid $\mathbf{B}G$ and vice versa

More generally, monoids of the form $(\text{Hom}_{\mathcal{C}}(X, X), \circ)$ are called endomorphism monoids and an interesting example thereof is endomorphism monoids in the category TopVect_k of topological vector spaces and continuous linear operators. Such endomorphism monoids $(\text{Hom}_{\text{TopVect}_k}(X, X), \circ)$ and submonoids thereof are called operator algebras. They are important in functional analysis and in quantum theory.

[3] We will see a generalization of such deloopings for certain categories, monoidal categories (see 4.3) where any monoidal category $\mathcal{C}$, will be a one-object *bi*category $\mathbf{B}\mathcal{C}$ called the delooping $\mathcal{C}$ (see **??**).

Definition 4.1.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ consists of the following data:

- an assignment

$$F : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D}) \tag{4.1}$$
$$X \mapsto F(X) \tag{4.2}$$

- for every two objects $X, Y \in \mathrm{ob}(\mathcal{C})$ a map

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)) \tag{4.3}$$
$$f \mapsto F(f) \tag{4.4}$$

such that

- $F(id_X) = id_{F(X)}$
- $F(g \circ f) = F(g) \circ F(f)$

Example 4.1.7. Some examples of functors:

1. There are forgetful functors
   - Ring → Grp → Set
     $(R,+,\cdot) \mapsto (R,+) \mapsto R$
   - Ring → Mon
     $(R,+,\cdot) \mapsto (R,\cdot)$
   - $\mathrm{Vect}_k \to \mathrm{AbGrp} \to \mathrm{Set}$
   - $\mathrm{Alg}_k \to \mathrm{Vect}_k$ where the multiplicative structure on algebras is forgotten

2. An action of a group $(G,\cdot)$ on a set X is a functor $A : \mathbf{B}G \xrightarrow{\rho} \mathrm{Set}$ where $A(*) = X$ and every $g \in \mathrm{Hom}_{\mathbf{B}G}(*,*)$ is mapped to an automorphism[4] on X, $\mathrm{Hom}_{\mathbf{B}G}(*,*) \to \mathrm{Hom}_{\mathrm{Set}}(X,X)$. By the same reasoning, a linear representation of a group $(G,\cdot)$ is a functor $\mathbf{B}G \to \mathrm{Vect}_k$.

3. Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, their composite $G \circ F : \mathcal{C} \to \mathcal{E}$ is also a functor: $G \circ F(X) = G(F(X)), G \circ F(f) = G(F(f))$.

4. $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ with $id_{\mathcal{C}}(X) = X, id_{\mathcal{C}}(f) = f$ is also a functor.

Remark. Since the composition of functors is associative[5] and unital there is a category of all (small[6]) categories Cat whose objects are (small) categories and whose morphisms are functors. For the same reason we also have a category Gpd of (small) groupoids.

[4] It is an automorphism and not a simple endomorphism because of a very important property of functors: they preserve isomorphisms: given a functor $F : \mathcal{C} \to \mathcal{D}$, if $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an isomorphism, then $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ is an isomorphism as well because $F(f^{-1}) \circ F(f) = F(f^{-1} \circ f) = F(id_X) = id_{F(X)}$ and simmetrically, $F(f) \circ F(f^{-1}) = id_{F(Y)}$. For example, this can be used in the converse direction to show that two topological spaces are not homeomorphic by sending them (with a functor) to their non-isomorphic fundamental group(oid)s.

[5] Try to convince yourself that it is so!

[6] Let $\mathcal{C}, \mathcal{D}$ be categories, the collection of all functors $F : \mathcal{C} \to \mathcal{D}$ is generally a class. Hence the category of all categories without any restriction would not be a locally small category and thereby not a category according to our definition 4.1.1. However, if $\mathcal{C}$ is small, then the collection of all functors $F : \mathcal{C} \to \mathcal{D}$ is a set. Therefore, the category of all small categories is indeed a category according to our definition of category.

Example 4.1.8. Some more examples related to topological spaces.

1. Let X be a topological space. Observe the following equivalence relation: $x \sim x'$ if and only if there is a continuous map $f : [0,1] \to X$, also known as "path", such that $f(x) = 0$ and $f(x') = 1$. Denote with $\pi_0(X)$ the set of equivalence classes of path-connected points, also known as the set of path-connected components, in X. This gives a functor[7] $\pi_0 : \mathrm{Top} \to \mathrm{Set}$

2. Let X be a topological space. Its fundamental groupoid $\pi_{\leq 1}$ is the groupoid having:

   - the points of X as objects, $\mathrm{ob}(\pi_{\leq 1}(X)) = X$;
   - $\mathrm{Hom}_{\pi_{\leq 1}(X)}(x, y)$ is given by equivalence classes of continuous paths from x to y that are homotopic[8] relative to their endpoints. We can spell out what this means in the following way

   $$\mathrm{Hom}_{\pi_{\leq 1}(X)}(x, y) = \frac{\{\gamma \in \mathrm{Hom}_{\mathrm{Top}}([0,1], X) | \gamma(0) = x, \gamma(1) = y\}}{\text{homotopy relative to } \partial[0,1]}$$

   - For $x, y, z \in X$, composition of $[\gamma] \in \mathrm{Hom}_{\pi_{\leq 1}(X)}(x, y)$ and $[\eta] \in \mathrm{Hom}_{\pi_{\leq 1}(X)}(y, z)$ is given by the concatenation of paths with appropriate reparametrization,

   $$[\eta] \circ [\gamma] = [\gamma * \eta]$$

   Units are constant paths, i.e. $c : [0,1] \to X$ such that $\forall t \in [0,1], c(t) = x$. Additionally, this is indeed a groupoid since inverses are given by the same paths run in the opposite direction. We have a functor

   $$\pi_{\leq 1} : \mathrm{Top} \to \mathrm{Gpd}.$$

   In this language, the Seifert–Van Kampen theorem can be formulated as follows: $\pi_{\leq 1}$ preserves pushouts[9].

3. Let X be a topological space and $x \in X$ an arbitrary basepoint. The assignement of a fundamental group of X at x, $\pi_1(X, x)$[10] is a functor $\pi_1 : \mathrm{Top} \to \mathrm{Grp}$

4.2 Natural transformations

Definition 4.2.1. Given two functors $F, G : \mathcal{C} \to \mathcal{D}$ a natural transformation from F to G, $\alpha : F \Rightarrow G$ is a collection of morphisms indexed by objects in $\mathcal{C}$, $\alpha_x : F(X) \to G(X)$ such that $\forall f : X \to Y$ in $\mathcal{C}$ the following diagram commutes

$$\begin{tikzcd} F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\alpha_Y"] & G(Y) \end{tikzcd}$$

[7] There is also a functor $\pi : \mathrm{Top} \to \mathrm{Set}$ sending topological spaces to the sets of their connected components.

[8] We later define a homotopy, see Example 3 below.

[9] Pushouts are certain colimits, that is objects in a category defined, up to isomorphism, by a certain universal property, see **??** for a fast paced introduction to co/limits. An example of pushout is the gluing of two spaces along a common other, as we have already seen numerous times in this course. The resulting space is a pushout in Top

[10] *Reminder*: $\pi_1(X, x) = \pi_0(\Omega_x(X))$, where $\Omega_x(X)$ is the based loop space of X at x

If for every $X \in \mathrm{ob}(\mathcal{C})$, α_X is an isomorphism, then α is a natural isomorphism.

Remark. Note that because of functoriality, if a diagram commutes in $\mathcal{C}$, then its image under a functor $F : \mathcal{C} \to \mathcal{D}$ also commutes in $\mathcal{D}$. Let for example $g \circ f = h \circ l$, then $F(g) \circ F(f) = F(g \circ f) = F(h \circ l) = F(h) \circ F(l)$

Notation. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We denote with $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the functor category where objects are functors $\mathcal{C} \to \mathcal{D}$ and morphisms are natural transformations between such functors. We encounter soon an example of functor category, the category of G-linear representations, see Example 1 below.

Remark. Note that given a category $\mathcal{C}$ and a *groupoid* $\mathcal{D}$, then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is a groupoid since every component of any natural transformation is invertible because they are morphisms in $\mathcal{D}$ and therefore any natural transformation is a natural isomorphism.

Example 4.2.2.

1. As we have seen in Example 2 in 4.1.7, a linear representation of a group G is a functor $\mathbf{B}G \to \mathrm{Vect}_k$. A morphism between $G-$representations V and W, $f : V \to W$ is a $k-$linear map which is equivariant, i.e. $\forall g \in G, \forall v \in V$ we have $f(gv) = gf(v)$. Since linear representations are functors, one might wonder if a morphism between functors $V, W : \mathbf{B}G \to \mathrm{Vect}_k$, i.e. a natural transformation, is an equivariant map. Let $f : V \Rightarrow W$ be a natural transformation. Then, for any $g \in \mathrm{Hom}_{\mathbf{B}G}(*, *)$ the following diagram commutes

$$\begin{tikzcd} V(*) \arrow[r, "f(*)"] \arrow[d, "V(g)=g\cdot"'] & W(*) \arrow[d, "W(g)=g\cdot"] \\ V(*) \arrow[r, "f(*)"] & W(*) \end{tikzcd}$$

   and hence the map is equivariant, it does not matter if we first act on the vector space and subsequently apply the map or viceversa.

   Since one can compose unitally and associatively natural transformations[11], if we take the collection of all functors $\mathbf{B}G \to \mathrm{Vect}_k$ and the natural transformations between them we get the category of linear representations of the group G. Such categories where the objects are functors are called functor categories.

2. The determinant can also be seen as a natural transformation. Let Mat be the functor $\mathrm{Ring} \to \mathrm{Mon}$ taking a commutative ring R to the monoid $\mathrm{Mat}(R)$ of matrices with coefficients in the ring R. Another such functor is the forgetful functor which forgets addition in the ring and forgets that the product is commutative $U : \mathrm{Ring} \to \mathrm{Mon}$. The determinant is then the following map of monoids:

$$\begin{aligned} \mathrm{Mat}(R) &\xrightarrow{\det} R \\ M &\longmapsto \det M \end{aligned} \tag{4.5}$$

   The product rule for the determinant makes the map into a monoid homomorphism. The naturality diagram for rings R, S for a map $f : R \to S$ is then the following:

$$\begin{tikzcd} \mathrm{Mat}(R) \arrow[r, "\det"] \arrow[d, "\mathrm{Mat}(f)"'] & R \arrow[d, "f"] \\ \mathrm{Mat}(S) \arrow[r, "\det"] & S \end{tikzcd} \tag{4.6}$$

[11] By composing their components.

In words, this means that to calculate the determinant with coefficients in S we can proceed in two equivalent ways:

- change coefficients from R to S and then calculate the determinant,
- calculate the determinant using the matrix with R coefficients and then map into S.

Instead of taking *all* matrices we could take the general linear group $GL(-): \text{Ring} \to \text{Grp}$. In that case det is a natural transformation between $GL(-)$ and the functor $(-)^{\times}: \text{Ring} \to \text{Grp}$ taking the units in the ring.

3. Let $X, Y \in \text{Top}$ and $f, g \in \text{Hom}_{\text{Top}}(X, Y)$. A homotopy from f to g is a continuous map
$$h : [0,1] \times X \to Y$$
such that for every $x \in X, h(0,x) = f(x)$ and $h(1,x) = g(x)$. Two maps are homotopic if there is a homotopy between them. Although the homotopy seems intuitively like a map between maps, like the natural transformation is, this seems still very far from a natural transformation. However, there is an equivalent formulation of natural transformation, given below, which shows that homotopies and natural transformations are related. We will later show a way to make this comparison more rigorous, see **??**.

4.3 Monoidal categories

Recall that a monoid is a group where elements are not required to be invertible: a set M with a distinguished object $e \in M$ called unit, also known as neutral element, and a map of sets $m : M \times M \to M$ such that m is:

- associative
$$\forall a, b, c \in M, \quad m(m(a,b),c) = m(a, m(b,c))$$
usually written
$$\forall a, b, c \in M, \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$$
- and unital with respect to e:
$$m(e, -) = \text{id}(-), \quad m(-, e) = \text{id}(-)$$
where $\forall x \in M, id(x) = x$. The latter equation is usually written
$$\forall x \in M, \quad e \cdot x = x = x \cdot e$$

A monoidal category generalizes this structure with respect to objects *and* morphisms of a category.

Definition 4.3.1. Let $\mathcal{C}$ be a category. A monoidal structure on $\mathcal{C}$ is

- (O) an object $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}$, the *unit*
- (M) bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, the *tensor product*

- (A) a natural isomorphism $\alpha : - \otimes (- \otimes -) \Rightarrow (- \otimes -) \otimes -$ that witnesses associativity:

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, bend left, "-\otimes(-\otimes-)"] \arrow[r, bend right, "(-\otimes-)\otimes-"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{C} \end{tikzcd}$$

the *associator*

- (U) natural isomorphisms $\lambda : \mathbb{1}_{\mathcal{C}} \otimes (-) \Rightarrow \mathrm{id}_{\mathcal{C}} = (-)$ and $\rho : - \otimes \mathbb{1}_{\mathcal{C}} \Rightarrow \mathrm{id}_{\mathcal{C}} = (-)$ witnessing unitality:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathbb{1}\otimes-"] \arrow[r, bend right, "id_{\mathcal{C}}=(-)"'] \arrow[r, phantom, "\lambda\Downarrow"] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "-\otimes\mathbb{1}"] \arrow[r, bend right, "id_{\mathcal{C}}=(-)"'] \arrow[r, phantom, "\rho\Downarrow"] & \mathcal{C} \end{tikzcd}$$

respectively the *left and right unitors*

such that

- $\forall X, Y$ the following diagram commutes

$$\begin{tikzcd} (X \otimes \mathbb{1}) \otimes Y \arrow[rr, "\alpha_{X,\mathbb{1},\otimes Y}"] \arrow[dr, "\rho_X \otimes id_Y"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "id_X \otimes \lambda_Y"] \\ & X \otimes Y & \end{tikzcd}$$

This diagram is called the triangle identity. It explains how the associator and the two unitors interact.

- $\forall W, X, Y, Z$ the following diagram commutes

$$\begin{tikzcd} & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "\alpha_{W,X,Y} \otimes id_Z"'] \arrow[dr, "\alpha_{W \otimes X, Y, Z}"] & \\ (W \otimes (X \otimes Y)) \otimes Z \arrow[d, "\alpha_{W, X \otimes Y, Z}"'] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "\alpha_{W,X,Y \otimes Z}"] \\ W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "id_W \otimes \alpha_{X,Y,Z}"'] & & W \otimes (X \otimes (Y \otimes Z)) \end{tikzcd}$$

The latter diagram is called Mac Lane's pentagon. Thanks to 4.5.4 it pins down the associativity of more than 3 objects.

Notation. As we usually abuse notation and denote monoids $(M, \cdot)$ just with M, although a monoid is a set *equipped with* a binary operation; we will denote monoidal categories $(\mathcal{C}, \otimes, \mathbb{1}_{\mathcal{C}})$ just with $\mathcal{C}$, although a monoidal category is a category *equipped with* a monoidal structure.

Definition 4.3.2 (Strict Monoidal Category)**.** A strict monoidal category is a monoidal category where objects and morphisms are associative and unital strictly, not up to specific natural isomorphisms, i.e. the associator and the left/right unitors are the identity. This means that for every $f : A, B, C \in \mathcal{C}$ and every $f : A \to B, g : B \to C, h : C \to D \in \mathcal{C}$ where $\mathcal{C}$ is a strict monoidal category:

- $A \otimes (B \otimes C) = (A \otimes B) \otimes C$

- $A \otimes \mathbb{1}_{\mathcal{C}} = A = \mathbb{1}_{\mathcal{C}} \otimes A$
- $f \otimes (g \otimes h) = (f \otimes g) \otimes h$
- $f \otimes id_{\mathbb{1}} = f = id_{\mathbb{1}} \otimes f$

Example 4.3.3 (Examples of *strict* monoidal categories)**.**

- Given an arbitrary category $\mathcal{C}$, the set[12] of its endofunctors $\mathrm{End}(\mathcal{C},\mathcal{C}) = \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C},\mathcal{C})$ are the objects of a monoidal category with morphisms given by natural transformations between them. The tensor product is given by composition and it is associative since composition of functors is associative. The monoidal unit is $id_{\mathcal{C}}$ and is indeed left and right unital. Note however that this is an example of *strict* monoidal category since functors compose strictly and not up to some associator and unitors.
- A monoid $(M, \cdot, e)$, seen as a category with objects $m \in M$ and only the identity morphisms, is a discrete strict monoidal category. The monoidal structure is simply given by multiplication of the elements of the monoid, which is clearly (strictly) associative and unital. A discrete category is a category where there are only identity morphisms.

Example 4.3.4 (Examples of monoidal categories)**.** Examples of monoidal categories are

- $(\mathrm{AbGrp}, \otimes)$
- $(\mathrm{AbGrp}, \oplus)$
- $(\mathrm{Vect}_k, \otimes)$
- $(\mathrm{Vect}_k, \oplus)$
- $(\mathrm{Set}, \coprod)$
- $(\mathrm{Set}, \times)$
- $(\mathrm{Top}, \times)$
- $(\mathrm{Cat}, \times)$

4.4 Objects internal to a monoidal category

Definition 4.4.1 (Monoid Object)**.** A monoid object in a monoidal category $\mathcal{C}$ is an object $M \in \mathcal{C}$ equipped with two distinguished morphisms, $\mu : M \otimes M \to M$ and $\eta : \mathbb{1}_{\mathcal{C}} \to M$ called respectively multiplication and unit such that the following diagram commutes, so that the binary operation is associative

$$\begin{tikzcd}
(M \otimes M) \otimes M \arrow[r, "\alpha_{M,M,M}"] \arrow[d, "\mu \otimes id_M"'] & M \otimes (M \otimes M) \arrow[r, "id_M \otimes \mu"] & M \otimes M \arrow[d, "\mu"] \\
M \otimes M \arrow[rr, "\mu"'] & & M
\end{tikzcd}$$

[12] A set and not a larger collection since we defined categories to be locally small.

and the following also commutes, so that the binary operation is left and right unital

$$\begin{tikzcd} \mathbb{1}_{\mathcal{C}} \otimes M \arrow[r, "\eta\otimes id_M"] \arrow[dr, "\lambda_M"'] & M \otimes M \arrow[d, dashed, "\mu"] & M \otimes \mathbb{1}_{\mathcal{C}} \arrow[l, "id_M\otimes\eta"'] \arrow[dl, "\rho_M"] \\ & M & \end{tikzcd}$$

Making multiplication associative and unital.

Example 4.4.2. A monoid object is a

- Monoid when in (Set, $\times$)
- Algebra when in (Vect$_k$, $\otimes$)
- Algebra when in $(BMod, \otimes)$
- *Strict* monoidal category when in $(Cat, \times)$
- Topological monoids in (Top, $\times$)

This example clarifies the parallel between monoids, algebras and monoidal categories we made at the start of this section. However, one could be bothered by the fact that we do not actually get a general monoidal category (i.e. not necessarily strict). To do this we will briefly talk about $\mathbb{E}_1$-algebras in section **??**.

Definition 4.4.3 (Morphism of Monoids)**.** Let $M, M' \in \mathcal{C}$ be monoid objects (M, μ, η) and (M', μ', η'). A morphism of monoids is a morphism $f : M \to M'$ such that both the following two diagrams commute

$$\begin{tikzcd} M \otimes M \arrow[r, "f\otimes f"] \arrow[d, "\mu"'] & M' \otimes M' \arrow[d, "\mu'"] \\ M \arrow[r, "f"'] & M' \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1}_{\mathcal{C}} \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & M \arrow[d, "f"] \\ & M' \end{tikzcd}$$

Definition 4.4.4 (Comonoid Object)**.** Let $M \in \mathcal{C}$, M together with a comultiplication Δ and a counit ϵ is a comonoid object iff (M, μ, η) a monoid object in $\mathcal{C}^{op}$ (see 3) such that $\mu = \Delta^{op}$ and $\eta = \epsilon^{op}$. Straightforwardly, the definition we gave for monoid object, but with all the arrows inverted.

Definition 4.4.5 (Bimonoid Object)**.** A bimonoid object in a monoidal category $\mathcal{C}$ is simultaneously a monoid and a comonoid object in $\mathcal{C}$ in a compatible way. It has a unit $\eta : \mathbb{1}_{\mathcal{C}} \to M$ and a counit $\epsilon : \mathcal{C} \to \mathbb{1}_{\mathcal{C}}$, a multiplication $\mu : M \otimes M \to M$ and and a comultiplication $\Delta : M \to M \otimes M$ such that comultiplication and multiplication are morphisms of monoids. In the case of comultiplication, this means that given a monoid object $(M \otimes M, \mu', \eta')$ with $\mu' : (M \otimes M) \otimes (M \otimes M) \to M \otimes M$ and $\eta' : M \otimes M \to \mathbb{1}_{\mathcal{C}}$ both the following diagrams commute

$$\begin{tikzcd} M \otimes M \arrow[r, "\Delta\otimes\Delta"] \arrow[d, "\mu"'] & (M \otimes M) \otimes (M \otimes M) \arrow[d, "\mu'"] \\ M \arrow[r, "\Delta"'] & M \otimes M \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1}_{\mathcal{C}} \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & M \arrow[d, "\Delta"] \\ & M \otimes M \end{tikzcd}$$

We leave the case of the counit for the reader.

Definition 4.4.6 (Frobenius Algebra)**.** A Frobenius algebra in an arbitrary monoidal category $\mathcal{C}$ is simultaneously a monoid and a comonoid object in $\mathcal{C}$ with a compatibility condition different from the one above:

$$(id \otimes \mu) \circ (\Delta \otimes id) = \Delta \circ \mu = (\mu \otimes id) \circ (id \otimes \Delta)$$

called the Frobenius relation.

Definition 4.4.7 (Group Object)**.** A group object in a cartesian monoidal category $\mathcal{C}$ is a monoid object M that also has an inverse map $(-)^{-1} : M \to M$ such that the following diagram commutes, meaning that the inverse behave as expected

$$\begin{tikzcd}
M \arrow[r, "(-)^{-1} \times id_M"] \arrow[d, "id_M \times (-)^{-1}"'] \arrow[dr, "id_M"] & M \times M \arrow[d, "\mu"] \\
M \times M \arrow[r, "\mu"'] & M
\end{tikzcd}$$

Example 4.4.8.
- A topological group is a group object in Top
- A Lie group is a group object in SmoothMfld
- A group is a group object in Set

4.5 Symmetric monoidal categories

In order to get the categorical parallel of a commutative monoid, we need to define a *symmetric* monoidal category. We want to achieve something similar to $a \cdot b = b \cdot a$ in a monoid. However, we will not characterize this behavior with strict identities, but with a natural isomorphism[13]. Given the functor

$$swap : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$$

$$swap : (X, Y) \mapsto (Y, X)$$
$$swap : (f, g) \mapsto (g, f)$$

we could try to achieve our objective with a natural isomorphism $\beta : \otimes \to \otimes \circ swap$ that can be visualized in the category of all categories Cat with the following diagram

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \arrow[r, bend left, "\otimes"] \arrow[r, bend right, "\otimes \circ swap"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{C}
\end{tikzcd}$$

However, this does not characterize an actually symmetric structure, but rather a *braiding*, meaning that inverting two times the order of two tensor multiplied elements, i.e. objects or morphisms, does not necessarily equal the original tensor product. What this means will become clearer with the example of the braid group and the definition of symmetric monoidal category.

[13]Just as we did for associativity and unitality.

Definition 4.5.1. A braiding on a monoidal category is a natural transformation β with components $\beta_{X,Y} : X \otimes Y \Rightarrow Y \otimes X$

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, bend left, "-_i \otimes -_j"{name=U}] \arrow[r, bend right, "-_j \otimes -_i"'{name=D}] & \mathcal{C} \arrow[from=U, to=D, Rightarrow, "\beta"] \end{tikzcd}$$

In addition, we need to impose some compatibility conditions with the associator:

$$\begin{tikzcd} & (X \otimes Y) \otimes Z \arrow[dr, "\beta_{X,Y} \otimes id_Z"] & \\ X \otimes (Y \otimes Z) \arrow[ur, "\alpha^{-1}_{X,Y,Z}"] \arrow[d, "\beta_{X,Y \otimes Z}"'] & & (Y \otimes X) \otimes Z \arrow[d, "\alpha_{Y,X,Z}"] \\ (Y \otimes Z) \otimes X \arrow[dr, "\alpha_{Y,Z,X}"'] & & Y \otimes (X \otimes Z) \\ & Y \otimes (Z \otimes X) \arrow[ur, "id_Y \otimes \beta_{Z,X}"'] & \end{tikzcd}$$

$$\begin{tikzcd} & X \otimes (Y \otimes Z) \arrow[dr, "id_X \otimes \beta_{Y,Z}"] & \\ (X \otimes Y) \otimes Z \arrow[ur, "\alpha_{X,Y,Z}"] \arrow[d, "\beta_{X \otimes Y,Z}"'] & & X \otimes (Z \otimes Y) \arrow[d, "\alpha^{-1}_{Y,X,Z}"] \\ Z \otimes (X \otimes Y) \arrow[dr, "\alpha^{-1}_{Z,X,Y}"'] & & (X \otimes Z) \otimes Y \\ & (Z \otimes X) \otimes Y \arrow[ur, "\beta_{Z,X} \otimes id_Y"'] & \end{tikzcd}$$

The latter two diagrams are known as the hexagon diagrams. We call such categories braided monoidal.

A braided monoidal category is *not* the category corresponding to an abelian monoid. A special type of braided monoidal category is: the symmetric monoidal category.

Definition 4.5.2. A symmetric monoidal category is a braided monoidal category $(\mathcal{C}, \otimes, \mathbb{1}_{\mathcal{C}}, \alpha, \lambda, \rho, \beta)$ such that $\beta^2 = id$ (i.e. $\beta_{y,x} \circ \beta_{x,y} = id$).

Example 4.5.3.

- All the previous examples of monoidal categories! (Vect, $\oplus$), (AbGrp, $\otimes$), ...
- Importantly for us the category of bimodules $BMod$.
- The category of algebras over a module.

The following theorem assures us of the associativity of higher products that we had mentioned and also generalizes this result to braided and symmetric categories.

Theorem 4.5.4 (MacLane's coherence theorem)**.** *In any monoidal category, any formal diagram, i.e. a diagram made up just of associators, unitors (and braidings, in the case of braided and symmetric monoidal categories) commutes.*

We do not provide the proof here but refer to chapter 7 of [Lan71]. The analogy in a monoid is that it does not make a difference however I put my parentheses:

$$(a_1 a_2)((a_3 a_4) a_5) = ((a_1 (a_2 a_3)) a_4) a_5$$

but instead of an equality we have objects equivalent up to isomorphisms given by associators, unitors, etc... note that we could have more than one way to compose unitors, e.g. in the diagram we previously imposed as conditions, but these will all form a commutative diagram and hence be equivalent.

Earlier on we defined a monoid object (4.4.2) in a monoidal category, now in a symmetric monoidal category we can also define a commutative monoid object.

Definition 4.5.5 (Commutative Monoid Object)**.** A commutative monoid object, is a monoid object in a symmetric monoidal category for which additionally the following diagram commutes

$$\begin{tikzcd} M \otimes M \arrow[rr, "\beta_{M,M}"] \arrow[dr, "\mu"'] & & M \otimes M \arrow[dl, "\mu"] \\ & M & \end{tikzcd}$$

Example 4.5.6. A commutative monoid object is a

- commutative monoid when in $(\mathrm{Set}, \times)$,
- commutative algebra when in $(\mathrm{Vect}_k, \otimes)$,
- *strict* symmetric monodial category when in $(Cat, \times)$,
- commutative topological monoids in $(\mathrm{Top}, \times)$.

Definition 4.5.7 (Cocommutative Comonoid Object)**.** Let (M, Δ, ϵ) be a comonoid object (4.4.4) in a symmetric monoidal category $\mathcal{C}$. It is cocommutative iff $(M, \mu, \eta, \beta_{M,M})$ a commutative monoid object in $\mathcal{C}^{op}$ (see 3) such that $\mu = \Delta^{op}$, $\eta = \epsilon^{op}$ and $\beta^{op}_{M,M} = \beta_{M,M}$. Straightforwardly, the definition we gave for commutative monoid object, but with all the arrows inverted.

Definition 4.5.8 (Commutative Bimonoid Object)**.** A commutative bimonoid object in a symmetric monoidal category $\mathcal{C}$ is a bimonoid object (4.4.5) such that the underlying monoid object is commutative. Note that the underlying comonoid object is cocommutative if and only if the underlying monoid object is commutative.

Definition 4.5.9 (Commutative Frobenius Algebra)**.** A commutative Frobenius algebra in a symmetric monoidal category $\mathcal{C}$ is a Frobenius algebra whose monoid structures is commutative (this also implies that the comonoid structure is cocommutative).

We need a notion of homomorphism between symmetric monoidal categories, a suitable definition of functor.

Definition 4.5.10 (Symmetric monoidal functor)**.** Let $\mathcal{B}, \mathcal{C}$ be symmetric monoidal categories. A symmetric monoidal functor is a functor $F : \mathcal{B} \to \mathcal{C}$ compatible with all of the structure:

- an isomorphism taking the monoidal unit in $\mathcal{C}$ to the monoidal unit in $\mathcal{B}$: $\mathbb{1}_{\mathcal{C}} \xrightarrow{1_F} F(\mathbb{1}_{\mathcal{B}})$

- a natural isomorphism respecting the tensor product:

$$\begin{tikzcd}
\mathcal{B} \times \mathcal{B} \arrow[r, bend left, "F(-\otimes -)"] \arrow[r, bend right, "F(-)\otimes F(-)"'] \arrow[r, phantom, "\Uparrow \psi"] & \mathcal{C}
\end{tikzcd}$$

such that it interacts reasonably with the associator by making the following diagram commute for every $X, Y, Z \in \mathcal{C}$

$$\begin{tikzcd}
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[r, "\psi_{X,Y} \otimes id_{F(Z)}"] \arrow[d, "\alpha_{F(X),F(Y),F(Z)}"'] & F(X \otimes Y) \otimes F(Z) \arrow[r, "\psi_{X\otimes Y,Z}"] & F((X \otimes Y) \otimes Z) \arrow[d, "F(\alpha_{X,Y,Z})"] \\
F(X) \otimes (F(Y) \otimes F(Z)) \arrow[r, "id_{F(X)} \otimes \psi_{Y,Z}"'] & F(X) \otimes F(Y \otimes Z) \arrow[r, "\psi_{X,Y\otimes Z}"'] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

it interacts well with the unit[14] by making the following diagram also commute for every $X \in \mathcal{C}$

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{C}} \otimes F(X) \arrow[r, "\lambda_{F(X)}"] \arrow[d, "\phi \otimes id_{F(X)}"'] & F(X) \\
F(\mathbb{1}_{\mathcal{B}}) \otimes F(X) \arrow[r, "\psi_{\mathbb{1},X}"'] & F(\mathbb{1}_{\mathcal{B}} \otimes X) \arrow[u, "F(\lambda_X)"']
\end{tikzcd}$$

finally we just need a commutative diagram for all $X, Y \in \mathcal{C}$ specifying how it interacts with the braiding

$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "\beta_{F(X),F(Y)}"] \arrow[d, "\psi_{X,Y}"'] & F(Y) \otimes F(X) \arrow[d, "\psi_{Y,X}"] \\
F(X \otimes Y) \arrow[r, "F(\beta_{X,Y})"'] & F(Y \otimes X)
\end{tikzcd}$$

This notion will be central in this course since a TFT is just a symmetric monoidal functor with a special domain: the cobordism category.

Example 4.5.11.

1. The path-connected component functor (1) $\pi_0 : (\text{Top}, \times) \to (\text{Set}, \times)$ is a symmetric monoidal functor.

2. ...

We also need a suitable notion of natural transformation between symmetric monoidal functors.

Definition 4.5.12 ((Symmetric) Monoidal Natural Transformation)**.** Given monoidal functors (F, ψ, ϕ) and (G, ξ, γ) from monoidal categories $\mathcal{C}$ to $\mathcal{D}$, a natural transformation

[14] We just need the diagram for one unit since in a symmetric monoidal category $\lambda = \rho$. We chose arbitrarily to give the diagram for the left unit

$\eta : F \Rightarrow G$ is monoidal if and only if the two following diagrams commute

$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "\eta_X \otimes \eta_Y"] \arrow[d, "\psi_{X,Y}"'] & G(X) \otimes G(Y) \arrow[d, "\xi_{X,Y}"] \\
F(X \otimes Y) \arrow[r, "\eta_{X \otimes Y}"'] & G(X \otimes Y)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{D}} \arrow[d, "\phi"'] \arrow[dr, "\gamma"] & \\
F(\mathbb{1}_{\mathcal{C}}) \arrow[r, "\eta_{\mathbb{1}_{\mathcal{C}}}"'] & G(\mathbb{1}_{\mathcal{C}})
\end{tikzcd}$$

No other conditions are needed to be specified for a symmetric monoidal natural transformation, a monoidal natural transformation between symmetric monoidal functors.

Example 4.5.13.

- Let $\mathbb{Z}[-] : \text{Set} \to \text{AbGrp}$ be the free functor[15] generating free groups from sets. Define $\mathbb{Z}[- \times -] : \text{Set} \to \text{AbGrp}$, the free functor generating free groups out of cartesian products in Set, and $\mathbb{Z}[-] \otimes \mathbb{Z}[-]$, the free functor first generating free groups and then tensor-multiply them with the tensor product in AbGrp. There is a monoidal natural isomorphism α between them, $\forall A, B \in \text{Set}, \mathbb{Z}[A \times B] \cong \mathbb{Z}[A] \otimes \mathbb{Z}[B]$. The notation $\mathbb{Z}[-]$ comes from the fact that abelian groups are exactly modules on the integers, the categories of abelian groups and of modules on the integers are not just equivalent categories, but isomorphic, something very rare.
- Let $V \in \text{Vect}_k$. Then $V \to V^{**}$ are the components of...
- The determinant...
- There is a monoidal natural isomorphism between id_{Grp} and $\text{Grp} \xrightarrow{op} \text{Grp}, G \mapsto G^{op}$, the functor sending every group $(G, *)$ to its opposite group, i.e. $(G^{op}, *^{op})$, where $a *^{op} b = b * a$.

Remark. One way in which (symmetric) monoidal natural transfomations are important is that they let us define a suitable notion of equivalence between (symmetric) monoidal categories. This will be fundamental in a forthcoming section (6).

Definition 4.5.14 (Monoidal Equivalence)**.** There is a (symmetric) monoidal equivalence between (symmetric) monoidal categories $\mathcal{C}$ and $\mathcal{D}$ if and only if there are (symmetric) monoidal functors $F, G : \mathcal{D} \to \mathcal{C}$ and (symmetric) monoidal natural isomorphisms $G \circ F \overset{\epsilon}{\cong} id_{\mathcal{C}}$ and $F \circ G \overset{\eta}{\cong} id_{\mathcal{D}}$.

Remark. There is a strict 2-category SymmMonCat similar to Cat (see section **??**), where 1-morphisms are symmetric monoidal functors and 2-morphisms are symmetric monoidal natural transformations.

Notation. Note that then $\text{Hom}_{\text{SymmMonCat}}(\mathcal{C}, \mathcal{D})$ is a special functor category. A functor category where objects are symmetric monoidal functors. An alternative way to denote $\text{Hom}_{\text{SymmMonCat}}(\mathcal{C}, \mathcal{D})$ is $\text{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$.

[15] A free functor is a left adjoint to the forgetful functor.

We have previously seen (2) that a linear representation of a group G is just a functor $\mathbf{B}G \to \mathrm{Vect}_k$. More generally, some, e.g. [Koc03, p. 34], give a definition of linear representation for symmetric monoidal categories:

Definition 4.5.15 (Linear Representation)**.** Let $\mathcal{C}$ be a symmetric monoidal category. A linear representation of such category is a symmetric monoidal functor

$$\mathcal{C} \to \mathrm{Vect}_k$$

Such functors are the objects of the category of representation of $\mathcal{C}$ whereas symmetric monoidal natural transformations are the morphisms,

$$\mathrm{Rep}(\mathcal{C}) = \mathrm{Fun}^{\otimes}(\mathcal{C}, \mathrm{Vect}_k) \tag{4.7}$$

Chapter 5

The cobordism category

We now get to the most important example of symmetric monoidal category in this course: the symmetric monoidal category of cobordisms[1]. This is what will allow us to define a topological field theory.

Notation. Note that, if not explicitly stated we will mean *smooth* manifold when we write manifold.

Definition 5.0.1 (Bordism Category)**.** The objects are closed $(n-1)$-dimensional manifolds. A morphism from M to N is a *diffeomorphism class of* bordisms (to be soon defined) from M (*source*) to N (*target*), that is, a compact manifold with boundary Σ together with embeddings

$$\theta_0 : [0,1) \times M \hookrightarrow \Sigma$$

$$\theta_1 : (-1,0] \times N \hookrightarrow \Sigma$$

with a partition $p : \partial\Sigma \to \{0,1\}$ such that

$$\theta_0(0,M) = (\partial\Sigma)_0 := p^{-1}(0)$$

$$\theta_1(0,N) = (\partial\Sigma)_1 := p^{-1}(1)$$

Composition is given by gluing[2] bordisms together along a common boundary: if Σ is a morphism from M to N and Σ' from N to P we can define $\Sigma' \circ \Sigma := \Sigma \cup_N \Sigma'$ as (a rappresentative of) the composition morphism. The gluing of manifolds is associative.
In order for this to actually be a category there must be a identity morphisms for each object: take M to be a suitable manifold, we can then take $M \times [0,1]$ to be the identity morphism. However, one could also take $M \times [0,2]$ which is diffeomorphic to $M \times [0,1]$ but *not* strictly identical, on the nose.
That is why we took a *diffeomorphism class* of bordisms.

Notation. $\mathrm{nCob} = \mathrm{Bord}_{\mathrm{n,n\text{-}1}}$, meaning that objects are closed $(n-1)$-dimensional manifolds and morphisms are diffeomorphism classes of bordisms. Moreover, we indicate that the closed manifolds are oriented and the bordisms orientation preserving with $\mathrm{Bord}^{\mathrm{or}}_{\mathrm{n,n\text{-}1}}$ or equivalently $\mathrm{nCob}^{\mathrm{or}}$.

Let us explicitly state what is meant by diffeomorphism of bordisms:

[1] See chapter 14 of [Fre13].

[2] We define bordisms with the definition with collars because it is easier to see how to glue them.

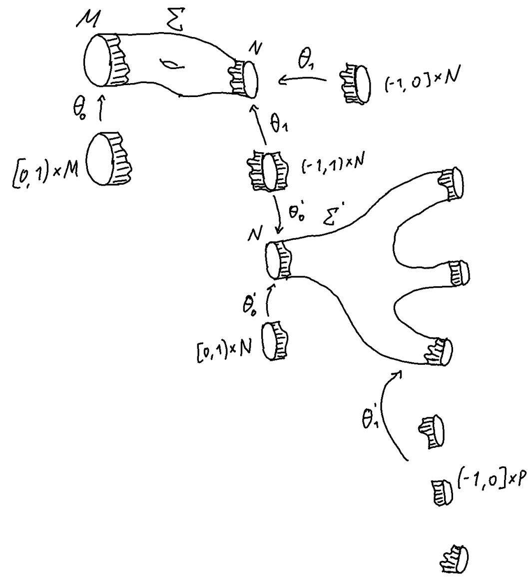


Figure 5.1: This is an illustration of gluing of two bordisms along their common boundary.

Definition 5.0.2. A diffeomorphism of bordisms from M to N, $(\Sigma, p, \theta_0, \theta_1) \to (\Sigma', p', \theta'_0, \theta'_1)$ is a diffeomorphism $\phi : \Sigma \to \Sigma'$ of manifolds with boundary

$$\begin{tikzcd} & M \arrow[dl, hook', "\theta(0,-)"'] \arrow[dr, hook, "\theta'(0,-)"] & \\ \Sigma \arrow[rr, "\cong", "\phi"'] & & \Sigma' \\ & N \arrow[ul, hook, "\theta(-,1)"] \arrow[ur, hook', "\theta'(-,1)"'] & \end{tikzcd}$$

The commutativity of the latter diagram also implies that also extra data commute accordingly, e.g. with the partitions:

$$\begin{tikzcd} \partial\Sigma \arrow[rr, "\phi|_{\partial\Sigma}"] \arrow[dr, "p"'] & & \partial\Sigma' \arrow[dl, "p'"] \\ & \{0,1\} & \end{tikzcd}$$

Example 5.0.3. We have that

$$\mathrm{Hom}_{\mathrm{nCob}}(\emptyset, \emptyset) = \{\text{closed } n\text{-manifolds}\}/\{\text{diffeomorphism}\}.$$

For $n = 2$, $\Sigma = S^2$ is a composition of

while a genus g surface is the composite:

which is very reminiscent of what we did in Morse theory. There we were actually also learning how the composition of bordisms works, this is called handle decomposition.

We just saw that $\mathrm{Bord}_{\mathrm{n,n\text{-}1}}$ is a category. It has importantly also a symmetric monoidal structure:

- (M) The tensor product is given by the disjoint union of manifolds, $\otimes = \amalg$, both on objects ($M \otimes N := M \amalg N$) and on morphisms ($\Sigma \otimes \Sigma' := \Sigma \amalg \Sigma'$). Disjoint union is functorial since for $\Sigma : M \to N$, $\Sigma' : M' \to N'$, $\Lambda : N \to P$, $\Lambda' : N' \to P'$ in $\mathrm{Bord}_{\mathrm{n,n\text{-}1}}$ it holds that:

  – $(\Sigma \cup_N \Lambda) \amalg (\Sigma' \cup_{N'} \Lambda') = (\Sigma \amalg \Sigma') \cup_{N \amalg N'} (\Lambda \amalg \Lambda')$
  – $(M \xrightarrow{M\times[0,1]} M) \amalg (M' \xrightarrow{M'\times[0,1]} M') = M \amalg M' \xrightarrow{(M\times[0,1])\amalg(M'\times[0,1])} M \amalg M'$

- (O) $\mathbb{1}_{\text{Bord}_{\text{n,n-1}}} = \emptyset$ (note that $\emptyset \amalg M \neq M$, so $\text{Bord}_{\text{n,n-1}}$ is not a *strict* symmetric monoidal category[3]. However, there is an isomorphism in the category since they're clearly diffeomorphic, and any diffeomorphic manifolds are cobordant. So, $\emptyset \amalg M \cong M$ in $\text{Bord}_{\text{n,n-1}}$, as we expect the monoidal unit to behave.

- (A) $M_1 \amalg (M_2 \amalg M_3) \xrightarrow{\cong} (M_1 \amalg M_2) \amalg M_3$

- (B) $\beta_{M,N} : M \amalg N \xrightarrow{\cong} N \amalg M$ and $\beta^2 = id$

Remark. Associativity and commutativity also hold up to diffeomorphism and not strictly if one defines the disjoint union as a coproduct[4]: every coproduct is unique up to isomorphism[5], the appropriate notion of isomorphism in the category of smooth manifolds is diffeomorphism and all diffeomorphic manifolds are cobordant.

5.1 Definition of topological field theories

Recall that a group homomorphism $\Omega_n \to X$ with X also an abelian group is a bordism invariant, which is generally quite useful and interesting. Now that we have refined our tools and have a cobordism *category* instead of the cobordism *group*, the bordism invariants become symmetric monoidal functors into another symmetric monoidal category since symmetric monoidal functors are the appropriate notion of homomorphism of symmetric monoidal categories.

In particular, that's how we get a topological field theory:

Definition 5.1.1. A topological field theory is a symmetric monoidal functor[6] with source nCob into a symmetric monoidal category $\mathcal{C}$

$$\mathcal{Z} : \text{nCob} \to \mathcal{C}$$

Exercise 5.1.2.

- Compare this definition to other definitions of quantum field theory.

- Skim through Atiyah's original paper [MF88], how does this definition compare to his set of axioms?

After skimming through [MF88] one might wonder why we defined the target category of a topological field to be just a general symmetric monoidal category and not specifically $(\text{Vect}_k, \otimes, k)$, as Atiyah did, or at least something similar, in the end we are studying topological quantum field theories! There are probably several reasons why someone would want to do so, for example for representation theoretic ones[7]. One of the crucial ones in

[3]Note that objects were *not* taken up to diffeomorphism, only morphisms.
[4]A coproduct is the dual notion of (cartesian) product (see **??**), i.e. a coproduct in $\mathcal{C}$ is a product in $\mathcal{C}^{op}$.
[5]One can prove this in a specular way in which we proved the uniqueness of the cartesian product, see **??**.
[6]See 4.5.10 for the definition of symmetric monoidal functor.
[7]e.g. [Str22], later we will also show that one must pick categories different to Vect_k in order to get interesting 3dTFTs and connect them with the Jones polynomial from knot theory.

particular is that one of the guiding conjectures of this field, the cobordism hypothesis, is formulated and provable without mentioning specifically Vect_k as a target category. See 6.6 for more on this conjecture[8]

Remark. A TFT is a functor in a similar way that a linear representation of a group[9] is (see 2). We can construct the category of TFTs in an analagous way to the one we used to define the category of linear representations of a group (see 1): by constructing a functor category where the objects are TFTs (hence particular symmetric monoidal functors) and morphisms apt natural transformations between them[10]. The category of n-dimensional TFTs will then be $\mathrm{Hom}_{\mathrm{SymmMonCat}}(\mathrm{nCob}, \mathcal{C})$ (see 4.5 for the definition of SymmMonCat, the category of all symmetric monoidal categories). Similarly as with groups, one can have linear and non-linear representations, although linear representations appear more frequenly. Typical examples of linear target categories are $(\mathrm{AbGrp}, \otimes)$, $(\mathrm{Mod}_R, \otimes)$, $(\mathrm{Vect}_k, \otimes)$.

Notation (Category of TFTs)**.** The category of TFTs is the functor category

$$\mathrm{Hom}_{\mathrm{SymmMonCat}}(\mathrm{nCob}, \mathcal{C}) = \mathrm{Fun}^{\otimes}(\mathrm{nCob}, \mathcal{C}) = \mathrm{TFT}_n(\mathcal{C}) = \mathrm{nTFT}(\mathcal{C}).$$

This functor category is symmetric monoidal by defining the tensor product between TFTs pointwise: for $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathrm{TFT}_n(\mathcal{C})$ and $M \in \mathrm{nCob}$

$$(\mathcal{Z}_1 \otimes \mathcal{Z}_2)(M) := \mathcal{Z}_1(M) \otimes \mathcal{Z}_2(M)$$

TFTs were originally defined with Vect as the target category, $\mathcal{Z} : (\mathrm{nCob}, \amalg) \to (\mathrm{Vect}, \otimes))$, see [MF88]. In this case we can use the notation in equation 4.7 and write $\mathrm{TFT}_n(\mathrm{Vect}_k) = \mathrm{Rep}(\mathrm{Bord}_{n,n-1})$. We can infer a few things about such TFTs:

1. Σ closed n-manifold, we can then view it as a bordism $\emptyset \xrightarrow{\Sigma} \emptyset$ and then we can apply Z and we get $Z(\emptyset) \xrightarrow{Z(\Sigma)} Z(\emptyset)$ but $Z(\emptyset) = k$, since that's the unit in Vect. So $\mathcal{Z}(\Sigma)$ is a linear map from k to k, which is entirely determined by where it sends one element. We then get that $\mathcal{Z}(\Sigma)(1)$ is a diffeomorphism invariant.

   In fact, the original hope was to use TFTs to find new diffeomorphism invariants.

2. There is a "trivial" TFT which on objects acts as $\mathcal{Z}(M) = k$ and on morphisms as $\mathcal{Z}(\Sigma) = id_k$ and we write $Z = 1$.

[8] As a spoiler: the moral of the cobordism hypothesis is that "The history of the Baez-Dolan conjecture *[i.e. the cobordism hypothesis]* goes most directly through quantum field theory and its adaptation to low-dimensional topology. Yet in retrospect it is a theorem about the structure of manifolds in all dimensions..." [Fre12] and hence TFTs are of interest with respect to any suitable target category, not only the ones of interest from the viewpoint of physics.

[9] In fact, some regard it a kind of representation, check out for instance around minute 18:00 of Catharina Stroppel: The beauty of braids - from knot invariants to higher categories; or the following quote from a poster for a conference on TQFTs and their connections to representation theory and mathematical physics "...a TQFT is a symmetric monoidal functor from the cobordism category to some symmetric monoidal category. It can thus be seen as a representation of a fundamental geometric category on a target category and thereby organizes interesting algebraic structures, e.g. representations of mapping class groups, in terms of cobordism categories. ..."; or this quote from Daniel Freed "An extended topological field theory is a representation of the bordism category..." [Fre12].

[10] See 4.5.12 for the definition of a symmetric monoidal natural transformation.

3. An *invertible* TFT is a TFT in which $Z(M) \cong k$ and $Z(\Sigma)$ is an isomorphism. Which is very similar but in the trivial TFT we have chosen not only $Z(M) \cong k$ but specifically $Z(M) = k$. This is related to "anomalies" in physics[11].

Example 5.1.3. Euler characteristic ($\to$ Euler theory)
Fix $\lambda \in \mathbb{C}^*$. If $M \xrightarrow{\Sigma} N$, then

$$Z_\lambda(\Sigma) := \lambda^{\chi(\Sigma)-\chi(M)}, \quad Z(M) = Z(N) = \mathbb{C} \tag{5.1}$$

Here functoriality follows from the following: if are open subsets of Σ and $\Sigma = int(X) \cup int(Y)$, then

$$\chi(\Sigma) = \chi(X) + \chi(Y) - \chi(X \cap Y) \tag{5.2}$$

Definition 5.1.4. Let $(\mathcal{C}, \otimes, \mathbb{1}_\mathcal{C})$ be a monoidal category. An object $X \in \mathcal{C}$ is invertible if there is a $Y \in \mathcal{C}$ such that $X \otimes Y \cong \mathbb{1}_\mathcal{C}$.

Example 5.1.5. In $(\mathrm{Vect}_k, \otimes)$, X is invertible iff there is a vector space Y such that $X \otimes Y \cong k$.

Notation. $\mathcal{C}^\times \subset \mathcal{C}$ subset of invertible objects and isomorphisms. We will later see that this is the underlying Picard groupoid (see 5.1.25)

Notation. For some symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1}_\mathcal{C})$ we sometimes abreviate that the category $\mathcal{C}$ is a monoidal category with tensor product $\otimes$ by writing $\mathcal{C}^\otimes$. For example, $\mathrm{Set}^\times$, $\mathrm{Set}^\amalg$, $\mathrm{Vect}^\oplus$, $\mathrm{Vect}^\otimes$.

We can now rigorously define what we previously called an invertible TFT:

Definition 5.1.6. An invertible TFT is an invertible object in $\mathrm{Fun}^\otimes(\mathrm{Bord}_{\text{n,n-1}}, \mathcal{C})$. This is equivalent to the fact that $\mathcal{Z}$ factors through $\mathcal{C}^\times$:

$$\begin{tikzcd} {\mathrm{Bord}_{\text{n,n-1}}}^\amalg \arrow[rr, "\mathcal{Z}"] \arrow[dr, dashed, "\tilde{\mathcal{Z}}"'] & & \mathcal{C}^\otimes \\ & \mathcal{C}^\times \arrow[ur, hook] & \end{tikzcd}$$

Indeed if $\mathcal{C}^\otimes = \mathrm{Vect}_k^\otimes$, invertible objects in $\mathrm{Vect}_k^\otimes$ are just k (up to isomorphism), so an invertible TFT Z must be such that for all $M \in \mathrm{Bord}_{\text{n,n-1}}$, $Z(M) \otimes Z'(M) \cong k$ and so $Z(M) \cong k \cong Z'(M)$ and so is clear why they must factor through $\mathcal{C}^\times$. This definition of invertible TFT is then equivalent to the one given above.

Definition 5.1.7 (Groupoid Completion)**.** Let $\mathcal{C}$ be a category. A groupoid completion $(|\mathcal{C}|, i)$ is a groupoid $|\mathcal{C}|$ with a functor $i : \mathcal{C} \to |\mathcal{C}|$ such that if $\mathcal{D}$ is a groupoid and $f : \mathcal{C} \to \mathcal{D}$ a functor, then there is a unique map $\tilde{f} : |\mathcal{C}| \to \mathcal{D}$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "i"] \arrow[dr, "f"'] & & {|\mathcal{C}|} \arrow[dl, dashed, "\tilde{f}"] \\ & \mathcal{D} & \end{tikzcd}$$

[11]To precisely see how, one needs to work with *extended* TFTs, a higher categorical refinement of what we are working with now. See [Mü20].

From the universal property of the groupoid completion, there is a further factorization of an invertible field theory:

$$\begin{tikzcd} \mathrm{Bord}^{\amalg} \arrow[r, "\mathcal{Z}"] \arrow[d, "i"'] & \mathcal{C}^{\otimes} \\ |\mathrm{Bord}^{\amalg}| \arrow[r, "\tilde{\mathcal{Z}}"'] & \mathcal{C}^{\times} \arrow[u] \end{tikzcd}$$

Remark. Note that the groupoid completion of the bordism category $|\mathrm{Bord}_{n,n\text{-}1}{}^{\amalg}|$ is a Picard groupoid, see 5.1.25.

5.1.1 Invertible field theories and stable homotopy theory

These theories can be studied with stable homotopy theory, in the sense that they are maps of spectra.

Stable homotopy homotopy theory is the branch of homotopy theory that studies phenomena and structures that are stable, i.e. that can occur in any dimension, or in any sufficiently large dimension independently of the exact dimension. The tool used to reach higher dimensions is very often suspension, this is why sometimes stable homotopy theory is characterized as the phenomena that are stable under suspension.

The investigation of stable phenomena is achieved via spectra, i.e. sequence of pointed spaces with maps relating them one another. For instance the stable homotopy group of the sphere are exactly the homotopy groups of the sphere spectrum

In what follows we sketch why invertible field theories are maps of spectra, this is however very handwavy, for a more detailed sketch, look in the appendix.

The key to realizing why invertible field theories are maps of spectra and thus computable via stable homotopy theory is the following theorem, due to May, which roughly says

Theorem 5.1.8 (Recognition theorem for connective spectra)**.** *Connective spectra and Picard ∞-groupoids coincide, they are the same thing.*

Since stable homotopy theory involves ∞-categorical machinery, one must fully extend topological field theories (see 6.6), i.e. with (∞, n)-categories of bordisms as a source. With the homotopy coherent additional structure of (∞, n)-categories one gets that $|\mathrm{Bord}|$ and $\mathcal{C}^{\times}$ are ∞-groupoids since

1. we get $|\mathrm{Bord}|$ by adjoining all inverses for all morphisms and thus all morphisms become invertible

2. $\mathcal{C}^{\times}$ is obtained by forgetting about non-invertible morphisms (and objects) of $\mathcal{C}$ and thus we are left only with invertible morphisms

However, we do not have just ∞-groupoids but also a symmetric monoidal structure with duals on both $|\mathrm{Bord}|$ and $\mathcal{C}^{\times}$. In short, they are Picard ∞-groupoid. Thanks to 5.1.8 we know that they are connective spectra.

This is why $\tilde{\mathcal{Z}}$ is not just a map of spaces but a map of spectra, and this enables the application of stable homotopy theory, making life easier. We can conclude this interlude on stable homotopy theory and invertible TFTs with the following motto

Invertible TFTs are maps of connective spectra!

The inquiring reader might rightly ask: why is this good news? The answer is that sadly we do not know much stuff about (∞, n)-categories, e.g. we do not know well how co/limits work[12], but in turn stable homotopy theory has been much more investigated and so we can understand them better than the usual extended TFT.

This is not only of pure mathematical interest but it also allows the application of topological field theories to condensed matter theory!

5.1.2 Dualizability in the context of topological field theories

Given $\mathcal{Z} : \mathrm{Bord}_{\text{n,n-1}} \to \mathrm{Vect}$ and $M \in \mathrm{Bord}_{\text{n,n-1}}$, one can prove that $\mathcal{Z}(M)$ is a finite dimensional vector space. Specifically, we have that $\dim \mathcal{Z}(M) = \mathcal{Z}(M \times S^1)(1)$, where $M \times S^1$ is an n dimensional cobordism from $\emptyset$ to $\emptyset$.

Proof. Decompose $M \times S^1$ into two semicircles $M \times [0,1]$. Then it gets mapped by $\mathcal{Z}$ to Vect_k in the following manner

$$\begin{tikzcd}
\emptyset \arrow[rr, "M\times[0,1]"] \arrow[dd, mapsto, bend right, "\mathcal{Z}"'] & & M \amalg M \arrow[rr, "M\times[0,1]"] & & \emptyset \arrow[dd, mapsto, bend left, "\mathcal{Z}"] \\
& M\times \arrow[d, mapsto, "\mathcal{Z}"] & & M\times \arrow[d, mapsto, "\mathcal{Z}"] & \\
k \cong \mathcal{Z}(\emptyset) \arrow[rr, "\mathcal{Z}(M\times[0,1])"'] & {} & \mathcal{Z}(M)\otimes\mathcal{Z}(M) \arrow[rr, "\mathcal{Z}(M\times[0,1])"'] & {} & \mathcal{Z}(\emptyset) \cong k
\end{tikzcd}$$

We claim that the maps will roughly be $k \to V^\vee \otimes V \xrightarrow{evaluate} k$, where the first map sends $1 \mapsto \sum_{i=1}^n f_i \otimes e_i$ where e_i is a basis of V and f_i is the dual basis. □

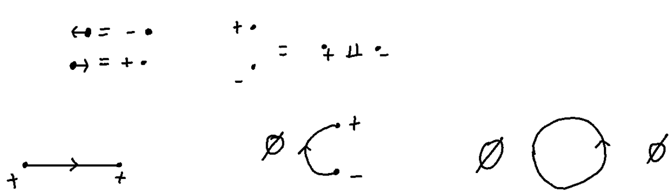
Definition 5.1.9. Let $\mathrm{Bord}^{\text{or}}_{\text{n,n-1}}$ be the category with:

- objects: closed and oriented $(n-1)$-manifolds
- morphisms: $\mathrm{Hom}_{\mathrm{Bord}^{\text{or}}_{\text{n,n-1}}}(M,N)$ = orientation preserving diffeomorphism classes of oriented bordisms. i.e. a bordism $(\Sigma, p, \theta_0, \theta_1)$ where Σ has an orientation and the diffeomorphism $\partial\Sigma \cong \overline{M} \amalg N$ is an orientation preserving one.

Notation. We denote with $\bullet_+$ the positively oriented point, i.e. the closed 0-dimensional manifold from where there is an outgoing 1-dimensional bordism.

We denote with $\bullet_-$ the negatively, i.e. the closed 0-dimensional manifold which is the incoming boundary of a 1-dimensional bordism.

Example 5.1.10 (Some objects and morphisms from $\mathrm{Bord}^{\text{or}}_{1,0}$).



[12]However, some people are fortunately working on this, see [MRR23].

Example 5.1.11 (Some objects and morphisms from $\mathrm{Bord}_{2,1}^{\mathrm{or}}$)**.**

Note that the orientation on the outgoing boundary is opposite to the induced orientation on the incoming one.

We can also define analogous categories for other tangential structures, e.g. a framing.

Example 5.1.12. In 1 and 2 dimensions...

Definition 5.1.13. Let $\mathcal{C}$ be a monoidal category. A left dual of an object $X \in \mathcal{C}$ is an object Y together with $ev_X : Y \otimes X \to \mathbb{1}_{\mathcal{C}}$ and $coev_X : \mathbb{1}_{\mathcal{C}} \to X \otimes Y$ such that

$$\begin{tikzcd} \mathbb{1}_{\mathcal{C}} \otimes X \cong X \arrow[rr, "id_X"] \arrow[dr, "coev_X \otimes id_X"'] & & X \otimes \mathbb{1}_{\mathcal{C}} \cong X \\ & X \otimes Y \otimes X \arrow[ur, "id_X \otimes ev_X"'] & \end{tikzcd} \tag{5.3}$$

and

$$\begin{tikzcd} Y \otimes \mathbb{1}_{\mathcal{C}} \cong Y \arrow[rr, "id_Y"] \arrow[dr, "id_Y \otimes coev_Y"'] & & \mathbb{1}_{\mathcal{C}} \otimes Y \cong Y \\ & Y \otimes X \otimes Y \arrow[ur, "ev_Y \otimes id_Y"'] & \end{tikzcd} \tag{5.4}$$

The fact that these two diagrams commute is called snake relations. If the diagrams commute, then X is the right dual of Y and Y is the left dual of X.

Remark. If $\mathcal{C}$ is braided, and in particular for us symmetric, any right dual is a left dual and viceversa.

Notation. We denote the dual of an object X with $X^{\vee}$. Sometimes it is also denoted with X^*.

Corollary 5.1.14. *Let $F : \mathcal{C} \to \mathcal{D}$ be a monoidal functor. Thanks to monoidal functoriality, the monoidal unit $\mathbb{1}_{\mathcal{C}}$ is sent to $\mathbb{1}_{\mathcal{D}}$ (up to isomorphism), and hence one gets the evaluation map and coevaluation map: $ev_{F(X)} : F(Y) \otimes F(X) \to \mathbb{1}_{\mathcal{D}} \cong F(\mathbb{1}_{\mathcal{C}})$ and $coev_{F(X)} : F(\mathbb{1}_{\mathcal{C}}) \cong \mathbb{1}_{\mathcal{D}} \to F(X) \otimes F(Y)$. Moreover, the image of a commuting diagram under a functor still commutes (see 4.2). So, the dual objects are sent to dual objects by monoidal functors and $F(X^{\vee}) = F(X)^{\vee}$.*

Example 5.1.15.

1. Any finite dimensional vector space V has a dual, namely $V^{\vee}$:

$$ev_V : V^{\vee} \otimes V \to k \tag{5.5}$$
$$coev_V : k \to V \otimes V^{\vee} \tag{5.6}$$

2. As an exercise try Set, $\times$

3. $\mathcal{C} = \mathrm{Bord}^{\mathrm{or}}_{\mathrm{n,n\text{-}1}}$. The point $\bullet_+$ is dualizable with dual $\bullet_-$.

   This construction actually works for any object in $\mathrm{Bord}^{\mathrm{or}}_{\mathrm{n,n\text{-}1}}$ (and $\mathrm{Bord}_{\mathrm{n,n\text{-}1}}$).

Definition 5.1.16 (Dual Morphisms)**.** Let X, Y be dualizable objects in a symmetric monoidal category $\mathcal{C}$ and $f : X \to Y$ be a morphism. The dual morphism is given by

$$f^\vee : Y^\vee \xrightarrow{id_{Y^\vee}\otimes coev_X} Y^\vee \otimes X \otimes X^\vee \xrightarrow{id_Y^\vee \otimes f \otimes id_{X^\vee}} Y^\vee \otimes Y \otimes X^\vee \xrightarrow{ev_Y \otimes id_{X^\vee}} X^\vee$$

Definition 5.1.17 (Picard Groupoid)**.** A Picard Groupoid is a symmetric monoidal category where every object is invertible with respect to $\otimes$ (i.e. $\forall A \in \mathcal{C}$, $\exists A^{-1}$ s.t $A \otimes A^{-1} \cong \mathbb{1}_{\mathcal{C}}$) and every morphism is an isomorphism, and hence it is a groupoid.

Lemma 5.1.18. *In every bordism category* $\mathrm{Bord}^{(\mathrm{or})}_{n,n-1}$ *every object is dualizable.*

Lemma 5.1.19. *Given symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$, symmetric monoidal functors $F, G : \mathcal{C} \to \mathcal{D}$ and a symmetric monoidal natural transfromation $\alpha : F \Rightarrow G$, if $X \in \mathcal{C}$ is dualizable, then $\alpha_X : F(X) \to G(X)$ is invertible.*

Proof. We claim that the inverse is given by $\alpha_{(X^\vee)^\vee}$. Following 5.1.16, $\alpha_{(X^\vee)^\vee} : G(X^\vee)^\vee \to F(X^\vee)^\vee$. Note that $F(X^\vee) = F(X)^\vee$ and thus $F(X^\vee)^\vee = F(X)^{\vee\vee} = F(X)$ and so $\alpha_{(X^\vee)^\vee} : G(X) \to F(X)$. Remember that $ev_X : X^\vee \otimes X \to \mathbb{1}_{\mathcal{C}}$ and $coev_X : \mathbb{1}_{\mathcal{C}} \to X \otimes X^\vee$. We prove that the following diagram commutes

$$\begin{tikzcd}
 & G(X) \cong G(X) \otimes \mathbb{1}_{\mathcal{D}} \arrow[dl, "id_{G(X)}\otimes G(coev_X)"'] \arrow[dr, "id_{G(X)}\otimes F(coev_X)"] & \\
G(X) \otimes G(X^\vee) \otimes G(X) \arrow[ddd, "G(ev_X)\otimes id_{G(X)}"'] & & G(X) \otimes F(X^\vee) \otimes F(X) \arrow[ll, "id_{G(X)}\otimes \alpha_{X^\vee} \otimes \alpha_X"'] \arrow[d, "id_{G(X)}\otimes \alpha_{X^\vee} \otimes id_{F(X)}"] \\
 & & G(X) \otimes G(X^\vee) \otimes F(X) \arrow[ull, "id_{G(X)}\otimes id_{G(X)}\otimes \alpha_X"] \arrow[dd, "G(ev_X)\otimes id_{F(X)}"] \\
 & & \\
\mathbb{1}_{\mathcal{D}} \otimes G(X) \cong G(X) & & \mathbb{1}_{\mathcal{D}} \otimes F(X) \cong F(X) \arrow[ll, "\alpha_X"]
\end{tikzcd}$$

First of all, the triangle on the top commutes, i.e. $(id_{G(X)} \otimes \alpha_{X^\vee} \otimes \alpha_X) \circ (id_{G(X)} \otimes F(coev_X)) = id_{G(X)} \otimes G(coev_X)$, because of the naturality of α. The triangle underneath it also commutes, i.e. $id_{G(X)} \otimes \alpha_{X^\vee} \otimes \alpha_X = (id_{G(X)} \otimes id_{G(X)} \otimes \alpha_X) \circ (id_{G(X)} \otimes \alpha_{X^\vee} \otimes id_{id_{F(X)}})$ because of the unitality of $id_{G(X)}$ and $id_{F(X)}$. Lastly, the bottom trapezoid commutes thanks to the naturality of α and thus the whole big diagram commutes. Consider now first mapping leftwards and successively downwards, i.e. $(G(ev_X) \otimes id_{G(X)}) \circ id_{G(X)} \otimes G(coev_X)$, this equals to the identity by one of the two snake relations of dual objects. Take now the other route of the outer diagram, on the right; note that $(\alpha_X \vee)\vee = (G(ev_X) \otimes id_{F(X)}) \circ (id_{G(X)} \otimes \alpha_{X^\vee} \otimes id_{F(X)}) \circ id_{G(X)} \otimes F(coev_X)$ by definition (5.1.16) and hence the right route on the outer diagram corresponds to $\alpha_X \circ (\alpha_{X^\vee})^\vee$. Since the outer diagram commutes we obtain that $\alpha_X \circ (\alpha_{X^\vee})^\vee = id_{G(X)}$, i.e. $(\alpha_{X^\vee})^\vee$ is the right inverse of α_X. One constructs a symmetric diagram by substituting Fs with Gs to prove that it is also the left inverse. $\square$

Corollary 5.1.20. *If every object in a symmetric monoidal category $\mathcal{C}$ is dualizable, then any symmetric monoidal natural transformation between symmetric monoidal functors that have $\mathcal{C}$ as the source category is invertible since a natural transfomation is invertible if and only if each of its components is invertible. Hence,* $\mathrm{Fun}^{\otimes}(\mathcal{C},\mathcal{D})$ *is a groupoid for any $\mathcal{D}$ when all objects in $\mathcal{C}$ are dualizable.*

Corollary 5.1.21. *The category of topological field theories* $\mathrm{TFT}^{or}_{n,n-1}(\mathcal{C})$ *is a groupoid.*

Lemma 5.1.22. *Any invertible object is dualizable.*

Proof. Let X be an invertible object. Then, there must be an X^{-1} such that $X \otimes X^{-1} \cong \mathbb{1}_{\mathcal{C}}$, the invertible object is the dual of X. The co/evaluation maps become isomorphisms the snake relations hold because if a commutative triangle of isomorphisms commutes in one direction, then it commutes also in the opposite one, i.e. for arbitrary isomorphisms f, g, h such that $f = h \circ g$, it holds that $f^{-1} = (h \circ g)^{-1} = g^{-1} \circ h^{-1}$. □

Lemma 5.1.23. *A dualizable object is invertible if and only if its co/evaluation maps are isomorphisms.*

Proof. $\Longrightarrow$ directly follows from 5.1.22 since any invertible object has as co/evaluation maps isomorphisms. Conversely, suppose that X is a dualizable object and the co/evaluation maps are isomorphisms. Then, there is a $Y \in \mathcal{C}$ such that $ev_X : Y \otimes X \cong \mathbb{1}$. □

Corollary 5.1.24. *If $\mathcal{C}$ is a monoidal groupoid, then dualizable objects are invertible.*

Proof. Since $\mathcal{C}$ is a groupoid, then also the co/evaluation maps are isomorphisms. From 5.1.23 it follows that the dualizable objects are invertible. □

Corollary 5.1.25. *A Picard groupoid (5.1.17) is equivalently a monoidal groupoid where every object is dualizable.*

Example 5.1.26. The underlying groupoid of $\mathrm{Bord}^{\mathrm{or}}_{\mathrm{n,n\text{-}1}}$ is a picard groupoid.

Notation. We denote with $\mathcal{C}^{\cong}$ the underlying groupoid of $\mathcal{C}$, i.e. the subcategory of $\mathcal{C}$ where we forget about non-invertible morphisms.

Notation. We denote with $\mathcal{C}^{\mathrm{fd}}$ (or sometiimes $\mathcal{C}^{\mathrm{dualizable}}$) the subcategory of $\mathcal{C}$ containing only dualizable objects. We chose 'fd' because it can stand for two things:

- 'finite dimensional' since dualizable objects in Vect are finite dimensional vector spaces and in general dualizability can be seen as a finiteness condition
- 'fully dualizable', a higher-categorical generalization of the notion of dualizability

Chapter 6

Classification of topological field theories

> **Existence**: Why is there something, rather than nothing?
> This does not seem very accessible by current methods. A more realistic goal may be
> **Classification**: Given that there's something, what could it be?
>
> Jack Morava in [Mor11]

In this chapter, we classify 1d-TFTs and 2d-TFTs, provide a sketch on how that might work for 3d-TFTs and show surprsising connections between knot theory and 3d-TFTs.

Roughly, our classifications theorems will be equivalence of categories of this kind

$$ev_{X \in \mathrm{Bord}_n} : \mathrm{Fun}^{\otimes}(\mathrm{Bord}_n, \mathcal{C}) \simeq (\mathcal{C}^{\mathrm{fd}})^{\cong}$$

meaning that evaluating a TFT on an object of Bord_n induces an equivalence of categories. To be clear, recall that:

- Bord_n is the category of n-bordisms
- $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n, \mathcal{C})$ is the category of n-dimensional topological field theories since they are symmetric monoidal functors from the category of n-bordisms to some symmetric monoidal category
- $(\mathcal{C}^{\mathrm{fd}})^{\cong}$ is the Picard groupoid underlying $\mathcal{C}$ since $\mathcal{C}^{\mathrm{fd}}$ is the subcategory of $\mathcal{C}$ contating only dualizable objects and $(\mathcal{C}^{\mathrm{fd}})^{\cong}$ is the maximal underlying groupoid, restricting dualizable objects in $\mathcal{C}^{\mathrm{fd}}$ to invertible ones

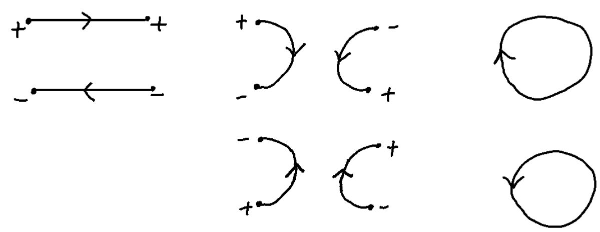
A natural question is: how is this a *classification*? This sort of statement certainly looks different from other classifications, for example the one of 2-dimensional manifolds we have seen in this course (3.2.1). Such an equivalence of categories classifies TFTs because every n-TFT is determined by where it sends an object in Bord_n and thus TFTs are classified by where they send such $n-1$-manifold, without specifying how the functor acts on morphism, i.e. the bordisms. Moreover, given a dualizable object of $\mathcal{C}$ one can construct a TFT since the equivalence goes also in the other direction, so one can say that TFTs are classified by the dualizable objects (up to isomorphism) of their target category.

Sketchily, our strategy to prove these results will be to find generators and relations for Bord_n and then check where they are sent to. It will follow that dualizable objects in $\mathcal{C}$ already provide all the information we need to know to determine what is the TFT in question: where the generators and relations of Bord_n are sent to.

6.1 Classification of 1d-TFTs

The main goal of this subsection is to prove that given an oriented 1d-TFT $\mathcal{Z} : \mathrm{Bord}_{1,0} \to \mathcal{C}$, $\mathcal{Z}(\bullet)$ has a dual and conversely, given an object $X \in \mathcal{C}$ in the target category of the TFT we can reconstruct a 1d-TFT.

Theorem 6.1.1 (Classification of 1-dimensional bordisms)**.** *Any connected oriented 1-dimensional manifold with boundaries is diffeomorphic to one of the following manifolds:*



For the proof we refer to the appendix of [MW97].

This means that any 1-dimensional bordism (i.e. including disconnected ones) is diffeomorphic to a disjoint union of such bordisms. Since in $\mathrm{Bord}_{1,0}$ bordisms are taken up to orientation preserving diffeomorphisms, this means that in $\mathrm{Bord}_{1,0}$ all morphisms *are* tensor products of the listed manifolds. Moreover, we know that the objects of $\mathrm{Bord}_{1,0}$ are just disjoint unions of $\bullet_+$ and $\bullet_-$. Hence, we now know the generators of $\mathrm{Bord}_{1,0}$, i.e. $\bullet_+$ and $\bullet_-$, and the relations, i.e. the possible connected 1-dimensional bordisms. Since we also know that $\mathcal{Z}$ is a *symmetric monoidal* functor, it means that we shall just check where such generators and relations are sent, see 6.1.

Remark. By $(\mathcal{C}^{\mathrm{fd}})^{\cong}$ we denote the restriction of a symmetric monoidal category on its dualizable objects and its isomorphisms, said briefly: a restriction on the underlying Picard groupoid because in any symmetric monoidal groupoid an object is dualizable if and only if it is invertible (see 5.1.25).

Theorem 6.1.2. *Let $\mathcal{C}$ be a symmetric monoidal category. Then the map*[1]

$$\mathrm{TFT}^{or}_{1,0} = \mathrm{Fun}^{\otimes}(\mathrm{Bord}^{or}_{1,0}, \mathcal{C}) \xrightarrow{\mathscr{E}v_{\bullet_+}} (\mathcal{C}^{\mathrm{fd}})^{\cong} \tag{6.1}$$

$$\mathcal{Z} \mapsto \mathcal{Z}(\bullet_+) \tag{6.2}$$

[1]There is unfortunately a notational clash between what we called the evaluation map for dualizable objects, denoted by ev, and what we also call in this case evaluation map, denoted by $\mathscr{E}V$. In the latter case, we are referring to a similar convention we use for example in topology when we call a map of this genre $Map(*, X) \to X$ an evaluation map. The intuition is that we feed an element of the domain into a map, a functor in our case, and see what happens, "evaluate it".

is a symmetric monoidal equivalence of groupoids.

$$\begin{tikzcd}
\mathrm{TFT}^{or}_{1,0} \arrow[rr, "\mathcal{E}v_{\bullet_+}"] \arrow[dd, "\simeq"] \arrow[dr, dashed] & & \mathcal{C} \\
& \mathcal{C}^{\mathrm{fd}} \arrow[ur, hook, "j"'] & \\
(\mathcal{C}^{\mathrm{fd}})^{\cong} \arrow[ur, hook, "i"'] & &
\end{tikzcd}$$

The image of $\mathcal{E}v_{\bullet_+}$ is in $(\mathcal{C}^{\mathrm{fd}})^{\cong}$, since

1. $\mathrm{TFT}^{\mathrm{or}}_{1,0}$ is a groupoid, and functors preserve isomorphisms
2. Every object in $\mathrm{Bord}^{\mathrm{or}}_{1,0}$ is dualizable and, because of functoriality, the image of a dualizable object remains a dualizable object.

This categorical equivalence means that

Theorem 6.1.3 (Alternative formulation of the classification of 1TFTs)**.** *Let $X \in \mathcal{C}$ be a dualizable object. Specifying $\mathcal{Z}(\bullet_+) = X$ determines a 1d-TFT.*

Firstly, we prove this alternative formulation, later we show that one can infer that the evaluation on a point is an equivalence. Note some important characteristics of symmetric monoidal functors in general.

Lemma 6.1.4. *Given a symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$ it holds that*

1. *identity morphisms are sent to identity morphisms because of functoriality, for any $X \in \mathcal{C}$ holds that $F(id_X) = id_{F(X)}$*
2. *dual objects are sent to dual objects because the functor preserves the tensor product and composition of morphisms; take for instance a look at the image of one of the triangle identities*

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{D}} \otimes F(X) \cong F(X) \arrow[rr, "id_{F(X)}"] \arrow[dr, "F(coev_X)\otimes id_{F(X)}"'] & & F(X) \otimes \mathbb{1}_{\mathcal{D}} \cong F(X) \\
& F(X) \otimes F(Y) \otimes F(X) \arrow[ur, "id_{F(X)}\otimes F(ev_X)"'] &
\end{tikzcd}$$

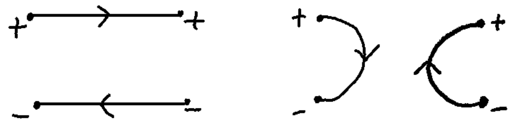
this must commute because of the functoriality of F: $F(id_X) = F((coev_X \otimes id_X) \circ (id_X \otimes ev_X))$ and thus $id_F(X) = F(coev_X \otimes id_X) \circ (id_{F(X)} \otimes F(ev_X))$; thereby the co/evaluation maps in $\mathcal{C}$, ev_X and $coev_X$, are sent to co/evaluation maps in $\mathcal{D}$, $ev_{F(X)}$ and $coev_{F(X)}$

3. *the components of the braiding of $\mathcal{C}$ are sent to components of the braiding in $\mathcal{D}$ since the following diagram composed of isomorphisms commutes for any $X, Y \in \mathcal{C}$*

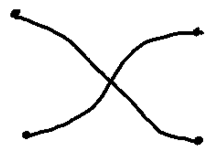
$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "\beta_{F(X),F(Y)}"] \arrow[d, "\psi_{X,Y}"'] & F(Y) \otimes F(X) \arrow[d, "\psi_{Y,X}"] \\
F(X \otimes Y) \arrow[r, "F(\beta_{X,Y})"'] & F(Y \otimes X)
\end{tikzcd}$$

Note that also the following lemma is true

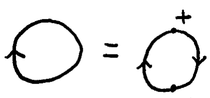
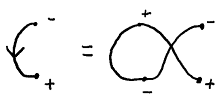
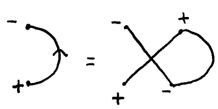
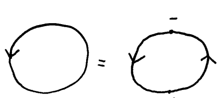
Lemma 6.1.5. *All 1-dimensional bordisms are generated by the generators:*



and the relation



Proof. We need to show that there is a way of gluing our generators and the braiding relation to get the other 4 manifolds of 6.1.1.

1. 
2. 
3. 
4. 

□

Proof of the classification of 1-TFTs. The moral of the latter lemma is that identity morphisms of $\bullet_+$ and $\bullet_-$, coevaluation and evaluation maps $coev_{\bullet_+}$ and $ev_{\bullet_+}$, and the braiding are enough to generate all the 1-dimensional bordisms. Thus, to determine what a TFT does, we need to only determine what it does on these morphisms, i.e. 1-dimensional bordisms. In the previous lemma on some properties of symmetric monoidal functors, we proved that identity morphisms are sent to identity morphisms, the co/evaluation maps are sent to co/evaluation maps and components of the braiding are sent to components of the braiding. Moreover, in that lemma we also proved that dual objects are sent to dual objects and $\bullet_+$ is a dual object, so for an arbitrary symmetric monoidal category $\mathcal{C}$, a 1-TFT sends it to a dual object. From a dual object X one can infer the co/evaluation maps and its dual. The braiding of a symmetric monoidal category is already given. So we can conclude that a one dimensional TFT is fully determined by where $\bullet_+$ is sent to. □

This proof mainly relies on abstract nonsense. So we now provide an alternative proof.

Proof. Also this proof relies on the fact that since the functor is *symmetric* monoidal one can just check what the TFT does only for 5 of all the possible connected 1-dimensional bordisms (6.1.1), not considering what happens for the other 3 ones with the reversed orientations without loss of generality because $\mathrm{Bord}_{1,0}$ is symmetric monoidal.

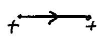
First we need to determine where $\bullet_+, \bullet_- \in \mathrm{Bord}_{1,0}$ are sent. Since they are duals of one another, they are dualizable objects and thus they are sent to $X, X^\vee \in \mathcal{C}^{\mathrm{fd}}$, e.g. finite dimensional vector spaces in the case of Vect_k. An arbitrary objects of $M \in \mathrm{Bord}_{1,0}$ is a finite disjoint union of $\bullet_+, \bullet_-$, i.e.

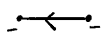
$$M = (\amalg_I \bullet_+) \amalg (\amalg_J \bullet_-)$$

where $I, J \subseteq \mathbb{N}$. Thanks to the symmetric monoidal functoriality of $\mathcal{Z}$ we have that

$$\mathcal{Z}(M) = (\bigotimes_I X) \otimes (\bigotimes_J X^\vee)$$

Now we need to understand where the 5 possible 1-bordisms are sent.

1. $\mathcal{Z}($  $) = id_X$

2. $\mathcal{Z}($  $) = id_{X^\vee}$

3. $\mathcal{Z}($  $) = ev_X$, hence $X \amalg X^\vee \xrightarrow{ev_X} \mathbb{1}_\mathcal{C}$ and more precisely for $v \in X \in \mathrm{Vect}_k$: $ev_X : (X, \lambda) \mapsto \lambda(v)$

4. $\mathcal{Z}($  $) = coev_X$, hence $\mathbb{1}_\mathcal{C} \xrightarrow{coev_X} X \otimes X^\vee$ and more precisely for $\lambda \in k \in \mathrm{Vect}_k$, via the canonical isomorphism $X \otimes X^\vee \cong \mathrm{End}(X)$, $coev_X$ corresponds to taking the trace

5. $\mathcal{Z}($  $) = coev_X \circ ev_X$, hence $\mathbb{1}_\mathcal{C} \xrightarrow{coev_X \circ ev_X} \mathbb{1}_\mathcal{C}$, and, following the cases for Vect_k treated in the previous two points, to taking the trace of the identity matrix, i.e. the dimension of X

The moral of the story is that a TFT is uniquely determined, up to isomorphism, once one knows $X \in \mathcal{C}^{\mathrm{fd}}$ since then one knows

- what $X^\vee$ is, what finite sequences of tensored X and $X^\vee$ are, which will be sent to finite disjoint unions of $\bullet_+$ and $\bullet_-$ (aka all the objects of $\mathrm{Bord}_{1,0}$);
- what id_X and $id_{X^\vee}$ are
- what ev_X and $coev_X$ are and hence also $coev_X \circ ev_X$

□

Remark. This proof considers 5 possible bordisms and not all 8, thereby excluding some possible orientations. However, Lurie's argument is without loss of generality because it implicitly relies on the *symmetric* monoidality of $\mathrm{Bord}_{1,0}$ since, for instance, one can get  by composing  with a braiding.

We stated that these two proofs by generators and relations prove the equivalence of categories in the classification of 1-TFTs. However, it might seem mysterious why this is the case. We now prove that the evaluation on a point is actually an equivalence of categories, a fully faithful essentially surjective functor by the fundamental theorem of category theory **??**.

Proof. This is from [Fre13, Theorem 16.10].

We need to prove three things to demonstrate that $\mathscr{E}V_{\bullet_+}$ is an equivalence: that it is full, faithful and essentially surjective:

1. Trivially, note that since $\mathscr{E}V_{\bullet_+}$ lands in $(\mathcal{C}^{\mathrm{fd}})^{\cong}$ for any $A \in \mathcal{C}^{\mathrm{fd}})^{\cong}$ one can define a TFT such that $\mathcal{Z}(\bullet_+) = A$

2. It is faithful: let $\mathcal{Z}, \mathcal{Z}' \in \mathrm{TFT}^{or}_{1,0}$ and $\alpha, \beta : \mathcal{Z} \Rightarrow \mathcal{Z}'$ be natural transformations between them and hence isomorphisms (see 5.1.21). Suppose that $\mathscr{E}V_{\bullet_+}(\alpha) = \mathscr{E}V(\beta)$, i.e. $\alpha^1_{\bullet_+} = \alpha^2_{\bullet_+}$ since $\mathscr{E}V_{\bullet_+}$ evaluates on the point. Recall that $\bullet_- = \bullet_+^{\vee}$. We have that for any natural isomorphism η it holds that $\eta\bullet_+^{\vee} = (\eta\bullet_+)^{\vee}$ because of functoriality and $\eta_{\bullet_-} = (\eta\bullet_+{}^{\vee})^{-1}$ because of 5.1.19. From this it follows that $\alpha_{\bullet_-} = (\alpha^{\vee}_{\bullet_+})^{-1} = (\beta\bullet_+{}^{\vee})^{-1} = \beta_{\bullet_-}$. With a specular proof we can prove that $\alpha_{\bullet_+} = \beta_{\bullet_+}$. Since 0-dimensional compact manifolds are just finite disjoint unions of $\bullet_+$ or $\bullet_-$ and α and β are symmetric monoidal natural transformations it follows that for any $X \in \mathrm{Bord}^{or}_{1,0}$, $\alpha_X = \beta_X$. This proves that $\mathscr{E}V$ is faithful because it implies that for any two natural transformations that are mapped to equal morphisms in $\mathcal{C}$ must be also equal in $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_{1,0}, \mathcal{C})$

3. $\mathscr{E}V_{\bullet_+}$ is full. Given an arbitrary isomorphism $f : X \to X'$ in $(\mathcal{C}^{\mathrm{fd}})^{\cong}$ we now show that there is an isomorphism $\eta : \mathcal{Z} \Rightarrow \mathcal{Z}'$ in $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_{1,0}, \mathcal{C})$ such that $\mathscr{E}V_{\bullet_+}(\eta) = \eta_{\bullet_+} = f$. So, we define $f = \eta$ and thus $(f^{\vee})^{-1}$. Objects in $(\mathcal{C}^{\mathrm{fd}})^{\cong}$ are sequences of tensor products of X and $X^{\vee}$. Hence, using the trick we used before we can extend any natural transformation $\eta : \bullet_+ \to \bullet_+$ to a disjoint union of such natural transfomations, via the monoidal structure on $\mathrm{Bord}_{1,0}$, since any object $X \in \mathrm{Bord}_{1,0}$ is diffeomorphic to a finite disjoint union of $\bullet_+$ and $\bullet_-$, and we know that $(\alpha^{\vee}_{\bullet_+})^{-1}$. This is independent of the choice of the diffeomorphism between X and disjoint unions of $\bullet_+$ and $\bullet_-$ thanks to the coherence of the chosen map η_X. Moreover, the diffeomorphism is determined up to permutation also because of the coherence of η_X . It remains to show that η_X is an isomorphism. η_X can be a disjoint union of the 5 1-bordisms (6.1.1) and hence, thanks to symmetric monoidality of η, we can just check these 5. The two identities are trivially isomorphisms. Then we need to check that the following diagram commutes to check for the coevaluation

$$\begin{tikzcd}
\mathcal{Z}(\bullet_-) \otimes \mathcal{Z}(\bullet_+) \arrow[rr, "f \circ (f^{\vee})^{-1}"] & & \mathcal{Z}'(\bullet_-) \otimes \mathcal{Z}'(\bullet_+) \\
& \mathbb{1}_{\mathcal{C}} \arrow[ul, "\mathcal{Z}(coev_X)"] \arrow[ur, "\mathcal{Z}'(coev_X)"'] &
\end{tikzcd}$$

The commutativity of this diagram follows from the fact that the following diagram commutes

$$
\begin{tikzcd}
\mathbb{1}_{\mathcal{C}} \arrow[r, "\mathcal{Z}(coev_X)"] \arrow[d, "\mathcal{Z}'(coev_X)"'] & \mathcal{Z}(\bullet_-) \otimes \mathcal{Z}(\bullet_+) \arrow[d, "id_{\mathcal{Z}(\bullet_-)}\otimes f"] \arrow[ddl, "id_{\mathcal{Z}(\bullet_-)}\otimes id_{\mathcal{Z}(\bullet_+)}\otimes \mathcal{Z}(coev_x)"] \\
\mathcal{Z}'(\bullet_-) \otimes \mathcal{Z}'(\bullet_+) \arrow[d, "\mathcal{Z}(coev_x)\otimes id_{\mathcal{Z}'(\bullet_-)}\otimes id_{\mathcal{Z}'(\bullet_+)}"'] & \mathcal{Z}(\bullet_-) \otimes \mathcal{Z}'(\bullet_+) \arrow[d, "id_{\mathcal{Z}(\bullet_-)}\otimes \mathcal{Z}'(coev_x)\otimes id_{\mathcal{Z}(\bullet_+)}"] \\
\mathcal{Z}(\bullet_-) \otimes \mathcal{Z}(\bullet_+) \otimes \mathcal{Z}'(\bullet_-) \otimes \mathcal{Z}'(\bullet_+) \arrow[r, "id_{\mathcal{Z}(\bullet_-)}\otimes f\otimes id_{\mathcal{Z}'(\bullet_-)}\otimes id_{\mathcal{Z}'(\bullet_+)}"'] & \mathcal{Z}(\bullet_-) \otimes \mathcal{Z}'(\bullet_+) \otimes \mathcal{Z}'(\bullet_-) \otimes \mathcal{Z}'(\bullet_+)
\end{tikzcd}
$$

The argument for the evaluation of X is specular. From there also the map for the circle follows since the circle is just a composition of evaluation and a coevaluation with a braiding in between. Note that the commutativity of the circle means that $\mathcal{Z}(S^1) = \mathcal{Z}'(S^1)$. $\square$

The classification of 1-dimensional topological field theories is the simplest case of an important guiding hypothesis in the field of TFTs, the cobordism hypothesis, see 6.6. It is the only case in which so-called extended TFTs coincide with non-extended ones, i.e. the usual Atiyah-Segal definition as a simple symmetric monoidal functor we are using.

Now we classified 1 dimensional TFTs, we would now like to do the same for 2 dimensional TFTs. We will find out that there is an equivalence between $\mathrm{TFT}^{or}_{2,1}(\mathcal{C})$ and commutative Frobenius algebra objects in $\mathcal{C}$.

6.2 Classification of 2d-TFTs

As we have done for the 1-dimensional case, we now try to classify 2-dimensional TFTs. For a more detailed proof see Kock [Koc03] or the lecture notes from Schweigert [Sch23].

We have already defined the following notions in arbitrary monoidal categories (see 4.4.6, 4.4.2, 4.4.4, 4.5.9, 4.5.8). However, we now explicitly state how the abstract formal definitions are instantiated in Vect_k.

Definition 6.2.1 (Algebra over a Field)**.** An algebra[2] over a field k is a monoid object (4.4.2) in Vect_k. More generally a left/right $R-$algebra is a monoid object in the category of left/right $R-$modules. The latter generalization holds also for the next definitions of $k-$coalgebra, $k-$bialgebra and Frobenius $k-$algebra.

Definition 6.2.2 (Coalgebra over a Field)**.** A $k-$coalgebra is a comonoid object in Vect_k.

Definition 6.2.3 (Bialgebra over a Field)**.** A $k-$bialgebra is a bimonoid object (see 4.4.5) in Vect_k. More explicitly, it is simultaneously an $k-$algebra and a $k-$coalgebra, a monoid and a comonoid object in Vect_k. A $k-$bialgebra is commutative in Vect_k, if the underlying monoid is commutative, or viceversa, if the underlying comonoid is commutative.

Definition 6.2.4 (Frobenius $k-$Algebra)**.** A Frobenius $k-$algebra is a Frobenius algebra (see 4.4.6) in a Vect_k. It is commutative if it is also a commutative $k-$algebra (and therefore a cocommutative coalgebra).

[2]Note that for us algebras are **always** associative and unital.

These definitions are perfectly fine, if we disregard pedagogical considerations. However, they might still seem mysterious to someone not used to this abstract nonsense. For this reason, we now give three more concrete definitions of a Frobenius algebra. They are all equivalent.

Reminder. A pairing $K : W \otimes V \to k$ is non-degenerate if it is part of a data exhibiting W as the dual of V, i.e. $\exists \gamma : k \to V \otimes W$ such that ...

If V, W finite dimensional, this is equivalent to saying that

$$K^{\#} : V \to W^{\vee} := \mathrm{Hom}_k(W, k), \quad v \mapsto K(- \otimes v) \tag{6.3}$$
$$K_{\#} : W \to V^{\vee} := \mathrm{Hom}_k(V, k), \quad w \mapsto K(w \otimes -) \tag{6.4}$$

are isomorphisms.

Exercise 6.2.5. What's the analogous reformulation of "$X \in \mathcal{C}$ is dualizable"?

Definition 6.2.6. A k-algebra is a monoid object in Vect_k. More explicitly, it is a k-vector space A together with linear maps:

$$\mu : A \otimes A \to A \tag{6.5}$$
$$\eta : k \to A \tag{6.6}$$

which satisfy associativity and (right and left) unitality by making the following two diagrams commute

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[r, "\alpha_{A,A,A}"] \arrow[d, "\mu \otimes id_A"'] & A \otimes (A \otimes A) \arrow[r, "id_A \otimes \mu"] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[rr, "\mu"'] & & A
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{C}} \otimes A \arrow[r, "\eta \otimes id_A"] \arrow[dr, "\lambda_A"'] & A \otimes A \arrow[d, dashed, "\mu"] & M \otimes \mathbb{1}_{\mathcal{C}} \arrow[l, "id_A \otimes \eta"'] \arrow[dl, "\rho_A"] \\
& M &
\end{tikzcd}$$

A homomorphism $\phi : A \to B$ of k algebras is a linear map that preserves/commutes with (μ_A, η_A) and (μ_B, η_B), a monoid homomorphism.

Definition 6.2.7. A *co*algebra is a comonoid object in Vect_k. More explicitly, one simply takes

$$\Delta : A \to A \otimes A$$

$$\epsilon : A \to k$$

and reverses the arrows in the diagrams.

A homomorphisms of k-coalgebras accordingly preserve (Δ_A, ϵ_A) and (Δ_B, ϵ_B). It is a monoid homomorphism in Vect_k^{op}.

Example 6.2.8. Take the vector space $A = k[x]/x^2 = k \oplus kx$ and take the map $A \to A \otimes A$ which sends $x \mapsto 1 \otimes x + x \otimes 1$ and $1 \mapsto 1 \otimes 1$ and the map $\epsilon : A \to k$ which sends 1 to 1 and x to 0.

Definition 6.2.9 (First definition)**.** A (k-) Frobenius algebra is a (finite dimensional) k algebra (A, μ, ν) together with an associative (=invariant) non-degenerate pairing $k : A \otimes A \to k$, i.e.

$$K(ab, c) = K(a, bc) \tag{6.7}$$

The fact that the pairing is invariant actually tells us something about the algebra structure, not only the vector space structure.

Example 6.2.10. $A = \mathrm{Mat}_{n\times n}(k)$ with matrix multiplication, in which the pairing K is simply the composition of multiplication and taking the trace, i.e. $K = \mathrm{tr} \circ \mu$.

Proof. Need to show nondegeneracy. Pick a basis of $\mathrm{Mat}_{n\times n}(k)$ given by $\{E_{ij}\}$ with 1 in the jth row and ith column. We then have the dual basis $\{E_{ji}\}$ and we have an isomorphism of A and $A^\vee$ given by $E_{ij} \mapsto E_{ji}$. Then we can compute: K is exactly the evaluation. □

Note that in the proof we used an isomorphism of A and $A^\vee$ to get the nondegenerate invariant pairing. This leads to the following alternative definition:

Definition 6.2.11 (Second definition)**.** A ($\Phi-$) Frobenius algebra is a (finite dimensional) algebra A with a left $A-$module isomorphism $\Phi : A \to A^\vee$.

The left $A-$module part again gives us the compatibility with the algebra structure.

Remark. For the definition to make sense A and $A^\vee$ should be left $A-$modules:

- $M = A$ is an A module via $A \otimes M \to M$ which sends $(a, m) \mapsto \mu(a, m)$.
- $M = A^\vee$ is an A module via $A \otimes M \to M = A^\vee = \mathrm{Hom}(A, k)$ which sends $(a, \phi) \mapsto \phi(\mu(a, -))$.

Then a left $A-$module map should satisfy $\Phi(a \cdot m) = a \cdot \Phi(m)$.

Example 6.2.12. Let's check that the map Φ in the previous example was actually a left $A-$module isomorphism. We would like

$$\Phi(A \cdot E_{ij}) = A \cdot \Phi(E_{ij}) \tag{6.8}$$

which is true when explicitly calculating both sides.

Definition 6.2.13 (Third definition)**.** A $(\Delta, \epsilon)-$ Frobenius algebra is a finite dimensional algebra (A, μ, ν) together with a coalgebra structure (A, Δ, ϵ) such that the *Frobenius relation*

$$(\mathbb{1}_{\mathcal{C}} \otimes \mu) \circ (\Delta \otimes \mathbb{1}_{\mathcal{C}}) = \Delta \circ \mu = (\mu \otimes \mathbb{1}_{\mathcal{C}}) \circ (\mathbb{1}_{\mathcal{C}} itimes \Delta)$$

holds. This means that the following diagram commutes

$$\begin{tikzcd}
& A \otimes A \otimes A \arrow[dr, "\mu\otimes id_A"] & \\
A \otimes A \arrow[ur, "id_A\otimes\Delta"] \arrow[r, "\mu"] \arrow[dr, "\Delta\otimes id_A"'] & A \arrow[r, "\Delta"] & A \otimes A \\
& A \otimes A \otimes A \arrow[ur, "id_A\otimes\mu"'] &
\end{tikzcd}$$

Note that the the only way in which we exploited the fact that we are working with vector spaces is that we had the tensor product and the unit with respect to this product. In other words, we only used the fact that Vect is a symmetric monoidal category, so the definitions work in any symmetric monoidal category[3]. This is important to keep in mind because the target category of our 2d-TFTs might not be Vect_k.

Proposition 6.2.14. *The three definitions of Frobenius algebra are equivalent.*

Proof. (1) $\iff$ (2) is an exercise.

(3) $\implies$ (1): we set $K = \epsilon \circ \mu : A \otimes A \to k$ and $\gamma = \Delta \circ \eta$, however these are not duality data that show that K is a nondegenerate pairing.
We can then define $\Phi := (id_{A^\vee} \otimes K) \circ (coev_A \otimes id_A)$ which is an isomorphism with inverse $id_A \otimes ev_A) \circ (\gamma \otimes id_{A^\vee})$. Then using in addition this isomorphism one can prove that K is a nondegenerate pairing.

(1) $\implies$ (3) we can define $\Delta := \mu \circ (\gamma \otimes id_A)$ and $\epsilon := K \circ (id \otimes \mu)$. □

Reminder. $\mathrm{TFT}^{or}_{2,1}(\mathcal{C})$ is a groupoid (see 5.1.21).

Theorem 6.2.15 (Classification of 2 dimensional TFTs)**.** *The functor*

$$\mathrm{TFT}^{or}_{2,1}(\mathcal{C}) \to \mathrm{cFrob}_{\mathcal{C}}$$

$$\mathcal{Z} \mapsto \mathcal{Z}(S^1)$$

is an equivalence of groupoids[4].

The category $\mathrm{cFrob}_{\mathcal{C}}$ has as objects commutative Frobenius algebras on an arbitrary symmetric monoidal category $\mathcal{C}$, i.e. Frobenius algebras that are also commutative bimonoid objects (4.5.8), as morphisms it has Frobenius homomorphisms, i.e. morphisms of monoids that are also morphisms of comonoids and preserve all the structure of a Frobenius algebra. One could have also proven a statement with Vect_k as the target category and proven that $\mathrm{TFT}^{or}_{2,1}(\mathrm{Vect}_k) \simeq \mathrm{cFrob}_k$. Note that the category of commutative Frobenius algebras in an arbitrary symmetric monoidal category is indeed a groupoid:

Theorem 6.2.16. *The category of Frobenius algebras on an arbitrary symmetric monoidal category $\mathcal{C}$ is a groupoid.*

Proof. Let $(A,\epsilon,\eta,\Delta,\mu),(A',\epsilon',\eta',\Delta',\mu') \in \mathrm{cFrob}_{\mathcal{C}}$ and $\phi : A \to A'$ be a Frobenius homomorphism. Then ... □

Take a look at [Koc03] for a proof for Frobenius k-algebras, i.e. Frobenius algebras in Vect_k.

Proof. (1) We need to prove that $\mathcal{Z}(S^1) =: A$ is a commutative Frobenius algebra. We get the product from the pair of pants and the coproduct from the copants. The unit and counit maps are simply given by the cup and cocup. Showing that this is actually a Frobenius algebra now just amounts to drawing the commutative diagram for a Frobenius algebra in the category of bordisms, i.e. all maps are simply bordisms.

(2) We now want see that it's a functor. Let $\mathcal{Z},\mathcal{Z}'$ be two TFTs and $\alpha : \mathcal{Z} \implies \mathcal{Z}'$ be a natural transformation.

[3] Take a detour in the subsection on monoidal categories (4.3), and more specifically take a look at 4.4.6, if you want to see a definition for general monoidal categories.

[4] We soon prove that also the category of Frobenius algebras is a groupoid.

Claim. *$f = \alpha(S) : A = \mathcal{Z}(S^1) \to \mathcal{Z}'(S^1) = B$ is a homomorphism of commutative Frobenius algebras.*

We just use naturality of α several times. □

For functoriality we did not check that compositions of morphisms (natural transformations) are sent to compositions (of morphisms in cFrob).

Now, we want to prove the converse direction: given a commutative Frobenius algebra A, construct an oriented 2d-TFT such that $\mathcal{Z}(S^1) = A$. We want to define a TFT $\mathcal{Z}$. On objects we simply set $\mathcal{Z}(S^1, or_+) \mapsto A$ and $\mathcal{Z}(S^1, or_-) \mapsto A^\vee$. This is enough because objects in $\text{Bord}^{or}_{2,1}$ are closed 1 dimensional manifolds and therefore diffeomorphic to a disjoint union of S^1s. Therefore, if Y is a connected one dimensional manifold we have an orientation preserving diffeomorphism $Y \to S^1$, but in which sense is this unique?

Definition 6.2.17 (Diffeomorphism Group)**.** Let $X \in$ SmoothMfld. Diff(X) denotes the automorphism group of X, i.e. the group of diffeomorphisms $X \xrightarrow{\phi} X$.

Note that the diffeomorphism group can be considered a topological group (see 4.4.8) if we use an apt topology, for example one can consider Diff(X) a subspace of $(C^\infty(X,X), Whitney)$, where $Whitney$ denotes the Whitney C^∞ topology, and endow it with the subspace topology.

Remark. An isotopy (see **??** for the definition) is equivalently a path in Diff(X).

Fact. *Roations, SO(2) $\hookrightarrow \text{Diff}^{or}(S^1)$ and this is a retraction.*

Thus, up to "wiggling", the map $Y \to S^1$ is unique.

Upshot: for an oriented 1 manifold Y we define

$$\mathcal{Z}(Y) := A^{\#\pi_0(Y)} \tag{6.9}$$

and diffeos are sent to identities.

Now what do we do on morphisms? Just send everything to what we expect from the algebra and coalgebra structure! i.e. pants to multiplication, copants to comultiplication, cylinder to identity, cup to algebra unit, cocup to counit. Now, a morphism in the bordism category is a diffeomorphism class of oriented 2d bordisms and by the classification theorem we found that a 2d connected oriented manifold with boundary is diffeomorphic to a composition. A bordism then specifies where "in" and "outgoing" boundaries are. Since $\mathcal{Z}$ must be functorial, given a composite as in the drawing, we must define $\mathcal{Z}$ to be

$$\mathcal{Z}(\Delta)....$$

Question: is this well defined?

Going back to proof of classification, the local moves we had translate precisely to conditions of a Frobenius algebra. Subtlety: the following are not isomorphic as bordisms, while they are diffeomorphic as manifolds with boundary.

We then need to specify what that bordism is sent to:

$$\mathcal{Z}(..) : A \otimes A \xrightarrow{swap} A \otimes A \tag{6.10}$$

So the classification of (possibly disconnected) 2d bordisms is simply a manifold with boundary as given from the classification of manifolds with boundary, composed with a permutation bordism, i.e. a bordism such as the following.

Now, since $\mathcal{Z}$ is functorial we simply set

$$\mathcal{Z}(\ldots) \tag{6.11}$$

and for well definedness consider the following drawing:

but we see that this is true because the Frobenius algebra is commutative.

Lemma 6.2.18. *Every (possibly non connected) 2 cobordism is a composition of*

1. *a "permutation bordism". i.e. given a permutation $\sigma \in S_n$, then we get a bordism $S^{1^{1 \amalg n}} \to (S^1)^{\amalg n}$ in which the two sides have the same orientation up to interchanging components. The bordism is simply $(S^1)^{\amalg n}$ in which on the right we use the permutation.*
2. *a disjoint union of connected 2 bordisms*
3. *another permutation bordism.*

Proposition 6.2.19. $\mathrm{Bord}_{2,1}^{or}$ *is the symmetric monoidal category with duals generated by (under composition and disjoint union):*

- *one object, S^1.*
- *morphisms the ones we've already mentioned: cup, pants, cocup, copants, swap (and cylinder but that's just the identity).*

with the following relations:

- *the cylinder is the identity, so composed with all the other bordisms it gives back the same boridsm*
- *sewing in disks*
- *(co)associativity*
- *(co)commutativity*
- *Frobenius relations*

In other words, $\mathrm{Bord}_{2,1}^{or}$ is free symmetric monoidal category with duals on one commutative Frobenius object S^1.

Usual depictions of orientations

Let's now do a recap of what we've done up to now. We proved

$$\mathrm{TFT}_{2,1}^{or}(\mathcal{C}) \xrightarrow[Ev]{\simeq} \mathrm{cFrob}(\mathcal{C})^{\simeq} \tag{6.12}$$

and the steps were the following:

- the functor is well defined: on objects we get $Ev(\mathcal{Z}) = \mathcal{Z}(S^1)$ which is a commutative Frobenius algebra. On morphisms we get an isomorphism of commutative Frobenius algebras.
- Ev is essentially surjective, actually we proved surjective.
- Ev is fully faithful

Remark. Last time we forgot about orientations...

6.3 Variants of TFTs

We now provide an outlook to some research programs that are tightly connected to TFTs.

- we could have used different tangential structures, such as
  - unoriented bordisms. In this case we would need an isomorphism $A \cong A^\vee$ which should be an isomorphism of commutative Frobenius algebras.
  - framing: not many framed closed 2 manifolds
  - spin structure
  - conformal structure and thereby we would have studied *conformal* field theories
- open-closed TFTs (Lauda-Pfeiffer, [LP08]): the idea is to enlarge the bordism category to include *compact* 1d manifolds (not necessarily closed) and we then also have additional morphisms.

  Now, what structure does the line segment have? It's still a Frobenius algebra! But it is not necessarily commutative.

  There is a more general result on the classification of such field theories by Kevin Costello [Cos06].
- extended TFTs ([Fre94],[BD95],[Lur09],[CS19]): one can extend TFTs downward, e.g. in 2-dimensional TFTs one might want to include the possibility of "composing" the line segment object with itself to get S^1. We would therefore modify the notion of symmetric monoidal category to be able to compose objects in $\mathrm{Bord}_{2,1,0}$. Thereby one gets a finer structure than a usual category, a.k.a. a 1-category: objects, morphisms between objects and morphism between morphisms. The last ones are usually called 2-morphisms and the morphisms between objects are then called 1-morphisms. Objects, 1-morphisms and 2-morphisms are in this case 0 manifolds, 1-dimensional bordisms and 2 dimensional bordisms with corners. We then no longer have the structure of a category, but rather that of a weak 2-category, or bicategory.
- cohomological TFTs ([Wit91]): aka families of $(n-1)$ manifolds and families of n bordisms. In practice, fix a space X, then objects are M closed $(n-1)$ manifolds together with $M \to X$ and morphisms are bordisms Σ together with $\Sigma \to X$.

  Often we take $X = BG$ the classifying space of a group G and having maps $M \to BG$ and $\Sigma \to BG$ gives us principle bundles on M and Σ. This is connected to the field of *gauge theory* and is sometimes called *cohomological TFT*.

6.4 3d TFTs

From now on the main reference will be [KRT97].

For classifying TFTs in 1 and 2 dimensions the procedure we have been following is to find an algebraic category which is equivalent with such TFTs evaluated on some object. In particular, in order to classify 1d TFTs we established an equivalence with the evaluation functor of 1TFTs on a point and finite dimensional vector spaces and for 2d TFTs evaluated on the circle S^1 are equivalent to commutative Frobenius algebras in some symmetric monoidal category and to finite-dimensional Frobenius k-algebras when considering Vect_k as the target category. These equivalences of categories allow us to reconstruct 1d and 2d TFTs (up to some reasonable form of equivalence) from this purely algebraic data. One can ask themselves if a classification of this kind is possible also for higher dimension, such as 3d TFTs. First, 3d-TFTs can be constructed from algebraic data, although a bit more sophisticated than what we got in lower dimensions, we will find out that every *modular tensor category* gives a 3d TFT[5]. The algebraic data can be summarized diagramatically as follows

$$\mathrm{FinVect} \ni X \xrightarrow{\text{uniquely determines}} \mathcal{Z} \in 1\text{-TFT}$$

$$\mathrm{CommFrob} \ni A \xrightarrow{\text{uniquely determines}} \mathcal{Z} \in 2\text{-TFT}$$

$$\mathrm{ModTensor} \ni \mathcal{C} \xrightarrow{\text{uniquely determines}} \mathcal{Z} \in 3\text{-TFT}$$

Here ModTensor is the category of *modular tensor categories*, which we will not fully define; rather, we give some ingredients below. As a rough sketch, it is a categorified commutative Frobenius algebra.

The main objective of this section is to sketch how this assignment could work and what is the relation between the 3d TFT assigned to a particular example of a modular tensor category (which is related to 3d Chern-Simons field theory), and the Jones polynomial, a knot invariant.

A modular tensor category is a tensor category with extra properties. We start with defining tensor categories.

Definition 6.4.1. A *linear category* is a Vect_k-enriched[6] category, i.e.

- $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a vector space $\forall X, Y$.
- composition is bilinear $\mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes \mathrm{Hom}_{\mathcal{C}}(Y, Z) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Z)$.

  A *tensor category* is a monoidal linear category, that is, a linear category which is monoidal, and the monoidal product on the Homs is linear.

6.4.1 The Yang-Baxter equation

Theorem 6.4.2. *Let $\mathcal{C}$ be a braided strict*[7] *monoidal category. Then, for $U, V, W \in \mathcal{C}$ we have*

$$(\beta_{V,W} \otimes id_U) \circ (id_V \otimes \beta_{U,W}) \circ (\beta_{U,V} \otimes id_W) = (id_W \otimes \beta_{U,V}) \circ (\beta_{U,W} \otimes id_V) \circ (id_U \otimes \beta_{V,W})$$

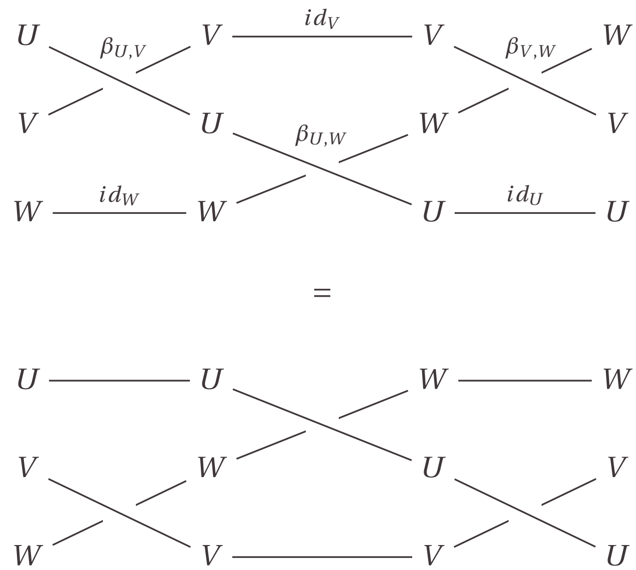
---

[5]So-called once-extended 3d TFTs, i.e. TFTs of the sort $\mathrm{Bord}_{3,2,1} \xrightarrow{\mathcal{Z}} \mathcal{C}$, can be classified by modular tensor categories. As we previously sketched, once-extended 2d TFTs have $\mathrm{Bord}_{2,1,0}$ as source and $\mathrm{Bord}_{2,1,0}$ is a symmetric monoidal bicategory (see **??** and 6.6). Also $\mathrm{Bord}_{3,2,1}$ is a symmetric monoidal bicategory with disjoint unions of S^1 as objects. In short, there is an equivalence between once-extended 3d-TFTs evaluated on the circle S^1 and modular tensor categories (see [Jor21] for more). This is a very difficult result, the details of which are not fully available yet, despite being expected/"known" for a decade

[6]See **??** for a rigorous definition of what enriched is and how that works for other categories.

[7]A braided strict monoidal category is a strict monoidal category, i.e. a monoidal category where associators and unitors are strict equalities instead of natural isomorphisms, with a braiding, i.e. a natural isomorphism $- \otimes - \cong - \otimes - \circ\, swap$. See the section on monoidal categories for more (4.3).

which we can visualize draw with the following diagram:

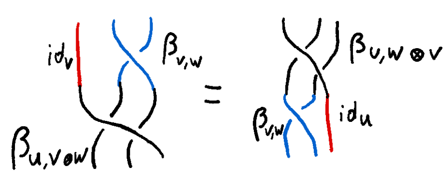


where a horizontal line is the identity and a crossing is the braiding β. In the drawing we simply "moved the middle string from above to below" and this is called a Reidemeister III move in knot theory.

Proof. Recall that β is a *natural* isomorphism, so for $U \xrightarrow{id} U$, $V \otimes W \xrightarrow{\beta_{V,W}}$ we apply the naturality of $\beta_{U,-}$ and get the following diagram:

$$\begin{tikzcd}
U \otimes (V \otimes W) \arrow[r, "\beta_{U,V\otimes W}"'] \arrow[d, "id_U \otimes \beta_{V,W}"'] & (V \otimes W) \otimes U \arrow[d, "\beta_{V,W} \otimes id_W"] \\
U \otimes (W \otimes V) \arrow[r, "\beta_{U,W\otimes V}"] & (W \otimes V) \otimes U
\end{tikzcd}$$

We may visualize the commutativity also via string diagrams:



we get exactly two specular drawings! However, algebraically, the proof is not finished. We now apply

$$(id_Y \otimes \beta_{X,Z}) \circ (\beta_{X,Y} \otimes id_Z) = \beta_{X,Y\otimes Z} \tag{6.13}$$

which inserted for $\beta_{U,V\otimes W}$ and $\beta_{W\otimes V,U}$ gives the result. □

Now, what happens if we take $U = V = W$? Let $\beta := \beta_{V,V}$, we now get

$$(\beta \otimes id) \circ (id \otimes \beta) \circ (\beta \otimes id) = (id \otimes \beta) \circ (\beta \otimes id) \circ (id \otimes \beta) \tag{6.14}$$

which is reminiscent of the Yang-Baxter equation (YBE):

Definition 6.4.3 (Yang-Baxter Equation and R-matrix)**.** Let V be a vector space, $c \in \mathrm{Aut}(V \otimes V)$. The YBE for c is

$$(c \otimes id) \circ (id \otimes c) \circ (c \otimes id) = (id \otimes c) \circ (c \otimes id) \circ (id \otimes c) \tag{6.15}$$

A solution to the YBE is called R-matrix.

In coordinates, for v_i a basis of V, if

$$c(v_i \otimes v_j) = \sum_{k,l} c_{ij}^{kl} v_k \otimes v_l \tag{6.16}$$

then the YBE is given by

$$\sum_{p,q,y} c_{ij}^{pq} c_{qk}^{yn} c_{py}^{ln} = \sum_{y,q,r} c_{jk}^{qr} c_{iq}^{ly} c_{yr}^{mn} \tag{6.17}$$

Theorem 6.4.2 then tells us that, for *any* $V \in \mathrm{Vect}$, $\beta_{V,V}$ is an R-matrix.

Example 6.4.4. the automorphism on $V \in \mathrm{Vect}$: $V \otimes V \xrightarrow{swap} V \otimes V$ satisfies the YBE because

- it comes from a standard braiding β in Vect
- Coxeter relation in S_3:

$$(12)(23)(12) = (23)(12)(23)$$

V finite dimensional vector space with basis $e_1, \ldots, e_n$ and q an invertible scalar. Now define $c_q(e_i \otimes e_j) := qe_i \otimes e_j$ for $i = j$, $e_j \otimes e_i$ for $i < j$ and $e_j \otimes e_i + (q - q')e_i \otimes e_j$ for $i > j$. A computation then shows that this satisfies the YBE. Note that $c_1 = swap$ is a 1-parameter "deformation" of swap.

This comes from representation theory!

Definition 6.4.5. Let G be a group, a representation of G on V a vector space (or R module) is a group homomorphism $\rho : G \to \mathrm{Aut}(V)$, i.e.

$$\rho(gh) = \rho(g)\rho(h), \quad \forall g, h \in G \tag{6.18}$$

Now, a morphism of representations $\rho_i : G \to \mathrm{Aut}(V_i), i = 1, 2$ is a linear map $\alpha : V_1 \to V_2$ such that the following diagram commutes:

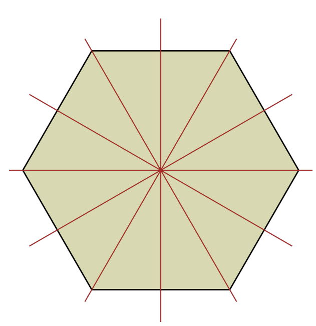
$$\begin{tikzcd}
V_1 \arrow[r, "\alpha"] \arrow[d, "\rho_1(g)"'] & V_2 \arrow[d, "\rho_2(g)"] \\
V_1 \arrow[r, "\alpha"'] & V_2
\end{tikzcd} \tag{6.19}$$

This gives rise to the category of representations of G, Rep_G.

Remark. The diagram 6.19 looks suspiciously like a natural transformation, and it is! In particular it's exactly example 1 in 4.2.2. In other words, the category of representations is the functor category:

$$\mathrm{Rep}(G) = \mathrm{Fun}(\mathbf{B}G, \mathrm{Vect}) \tag{6.20}$$

Example 6.4.6. Symmetries of a polygon: $G = D_n$ dihedral group and $\rho : D_n \to \mathrm{Aut}(\mathbb{R}^2)$, where $D_n \circlearrowright \mathbb{R}^2$ via reflections and rotations. For example the reflections can be represented visually for the hexagon as follows



Solutions of YBE give representations of the braid group, which we explore in the next section.

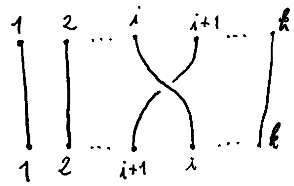
6.4.2 The braid group and the Braid category

Definition 6.4.7 (Braid groups)**.** Let $k \geq 3$. The braid group B_k with k strands has $k-1$ generators $\sigma_1, \ldots, \sigma_{k-1}$ and 2 relations:

$$\begin{array}{lll} 1. \, \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if} & |i-j| > 1 \\ 2. \, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{if} & 1 \leq i < k-1 \end{array} \tag{6.21}$$

In addition B_2 is the free group on one generator σ, i.e. it is isomorphic to $\mathbb{Z}$. For even lower generators $B_1 = B_0 = e$.

The reason we call B_k the braid group with k strands is that we can picture the elements σ_i in the following way:



which resembles a braid.

Remark. There is a surjective homomorphism to the symmetric group

$$\begin{aligned} B_k &\to S_k \\ \sigma_k &\mapsto s_k \end{aligned} \tag{6.22}$$

which is clear since S_k has the same generators and relations with in addition $s_i^2 = e$.

Proposition 6.4.8. *Let $c \in \mathrm{Aut}(V \otimes V)$ be an R matrix (i.e. a solution of the YBE). Then, for any $k > 0$, there is a unique homomorphism $\rho_k^c : B_k \to \mathrm{Aut}(V^{\otimes k})$ (i.e. a representation of B_k on $\mathrm{Aut}(V^{\otimes k})$) such that*

$$\rho_k^c(\sigma_i) := c_i, \quad i = 1, \ldots, k-1 \tag{6.23}$$

with the $c_i \in \mathrm{Aut}(V^{\otimes k})$ is defined as: take c in position $i, i+1$ and id otherwise, i.e.

$$c_i := \begin{cases} c \otimes id_{V^{\otimes(k-2)}}, & i = 1 \\ id_{V^{\otimes(i-1)}} \otimes c \otimes id_{V^{\otimes(k-i-1)}}, & 1 < i < k-1 \\ id_{V^{\otimes(k-2)}} \otimes c, & i = k-1 \end{cases} \tag{6.24}$$

(the second expression alone is enough is we forget about the identity when we have $id_{V^{\otimes 0}}$)

Remark. With this notation, YBE reads as

$$c_1 c_2 c_1 = c_2 c_1 c_2 \tag{6.25}$$

since the extra identities have no relevant effect. Equivalently then the YBE can also be written as

$$c_i c_{i+1} c_i = c_{i+1} c_i c_{i+1} \tag{6.26}$$

Proof. Define $\rho_k^c(\sigma_i) := c_i$ and check that the c_i satisfy the relations in Definition 6.4.7. For 1. we would like

$$c_j c_i = c_i c_j \tag{6.27}$$

for $i > j+1$. This follows from the fact that c_i and c_j don't "interact":

$$\begin{aligned}
&(id_{V^{\otimes(j-1)}} \otimes c \otimes id_{V^{\otimes(k-j-1)}})(id_{V^{\otimes(i-1)}} \otimes c \otimes id_{V^{\otimes(k-i-1)}}) \\
=&(id_{V^{\otimes(j-1)}} \otimes c \otimes id_{V^{\otimes(i-j-2)}} \otimes id_{V^{\otimes 2}} \otimes id_{V^{\otimes(k-i-1)}})(id_{V^{\otimes(i-1)}} \otimes c \otimes id_{V^{\otimes(k-i-1)}}) \\
=&(id_{V^{\otimes(j-1)}} \otimes c \otimes id_{V^{\otimes(i-j-2)}} \otimes c \otimes id_{V^{\otimes(k-i-1)}}) \\
=&(id_{V^{\otimes(i-1)}} \otimes c \otimes id_{V^{\otimes(k-i-1)}})(id_{V^{\otimes(j-1)}} \otimes c \otimes id_{V^{\otimes(k-j-1)}})
\end{aligned}$$

2. simply follows from the remark above. □

The braid group is also connected to the concept of configuration space.

Definition 6.4.9 (Unordered configuration space)**.** Let $\mathrm{uConf}_k(\mathbb{R}^2) \subset (\mathbb{R}^2)^k$ (the ordered configuration space of $\mathbb{R}^2$) be the subspace of k-tuples $(x_1,\dots,x_k)$ such that $x_i \neq x_j$ for $i \neq j$. Then we have an action $S_k \circlearrowright \mathrm{uConf}_k(\mathbb{R}^2)$. The *unordered* configuration space is $\mathrm{Conf}_k(\mathbb{R}^2) := \mathrm{uConf}_k(\mathbb{R}^2)/S_k$.

We would like to talk about the fundamental group of $\mathrm{Conf}_k(\mathbb{R}^2)$, but for that we need to pick a basepoint. Let us indicate coordinates by working in $\mathbb{C}$ rather than in $\mathbb{R}^2$. Then let $p = [(1,2,\dots,k)] \in \mathbb{C}^k$, that is to say that all points are placed on the x-axis in steps of 1 (and the square brackets are to indicate the class under the quotient by S_k). We then have a homomorphism $B \to \pi_1(\mathrm{Conf}_k(\mathbb{R}^2), p)$ which sends $\sigma_i \mapsto \hat{\sigma}_i = [f_i]$ where f^i is a loop at p, defined in $(\mathbb{R}^2)^k$ by:

$$f^i = (f_1^i,\dots,f_k^i) : [0,1] \to (\mathbb{R}^2)^k \cong \mathbb{C}^k \tag{6.28}$$

$$f_j^i(s) = j, \quad \text{if } j \neq i, i+1 \tag{6.29}$$

$$f_i^i(s) = \frac{1}{2}(2i+1-e^{\pi i s}) \tag{6.30}$$

$$f_{i+1}^i(s) = \frac{1}{2}(2i+1-e^{\pi i s}) \tag{6.31}$$

this indeed induces a loop in $\mathrm{Conf}_k(\mathbb{R}^2)$.
For it to be a homomorphism we need to check that the images of σ_i satisfy the relations 6.21 of the B_k group.

1. $\hat{\sigma}_i\hat{\sigma}_j = \hat{\sigma}_j\hat{\sigma}_i$ for $|i-j|>1$
2. $\hat{\sigma}_i\hat{\sigma}_{i+1}\hat{\sigma}_i = \hat{\sigma}_{i+1}\hat{\sigma}_i\hat{\sigma}_{i+1}$ for $1 \leq i < k-1$

For 1. one finds the same result as in the proof of Proposition 6.4.8 above, i.e. for $|i-j|>1$, $\hat{\sigma}_i$ and $\hat{\sigma}_j$ don't "interact" with one another. Instead 2. can be checked visually.
The following important result can be found in [ART50].

Theorem 6.4.10. *This homomorphism $\phi : B_k \to \pi_1(\mathrm{Conf}_k(\mathbb{R}^2), p)$ is an isomorphism.*

The drawings above may give us the idea to construct a category in which morphisms are given by the paths f which are morphisms between sets k points. This will be the Braid category. More in detail, given a representative $f = (f_1, \dots, f_k) : [0,1] \to (\mathbb{R}^2)^k$ of an element in $\pi_1(\mathrm{Conf}_k(\mathbb{R}^2), p)$ we can define the following subset of $[0,1] \times \mathbb{R}^2$:

$$L_f = \bigcup_{j=1}^{k} \{(s, f_j(s)) : s \in [0,1]\} \tag{6.32}$$

which is pretty much what we were drawing above, the graph of the path considered. It is therefore simply the union of disjoint line segments. Note now that

1. $\partial L_f = \{0,1\} \times \{1,2,\dots,k\}$ since the path starts and ends at $p = [(1,2,\dots,k)]$.
2. $\forall s \in [0,1]$ we have that $L_f \cap (\{s\} \times \mathbb{R}^2) = k$ points.
3. L_f is a bordism from $\{1,\dots,k\}$ to itself. Where in addition we can add an orientation by "flowing" from 0 to 1.
4. group structure gives the composition of bordisms.
5. always have just $\bullet_1, \dots, \bullet_k$ as source/target $\implies$ can take just this object.
6. bordism L_f comes with an embedding into $[0,1] \times \mathbb{R}^2$. This really depends on the representative! But the bordism itself only depends on $[f] \in \pi_1(\mathrm{Conf}_k(\mathbb{R}^2))$.

This allows us to define the following category:

Definition 6.4.11. The Braid category is given by:

- objects: natural numbers $0, 1, \dots, k, \dots$, which we think of as sets of k points,
- morphisms:

$$\mathrm{Hom}_{\mathrm{Braid}}(k,l) = \begin{cases} \varnothing & \text{if } k \neq l \\ \text{isotopy classes of "braids" from } k \text{ to } k & \text{if } k = l \end{cases} \tag{6.33}$$

  By braids we mean a "permutation 1 bordism together with an embedding" $\bigcup_{\{1,\dots,k\}} [0,1] \hookrightarrow [0,1] \times \mathbb{R}^2$ such that 1. and 2. hold.

- composition is "stacking the pictures", i.e. composition of bordisms plus stacking embeddings (using $[0,1] \cup_{1=0} [0,1] \cong [0,1]$).
- braided monoidal structure given pictorially by stacking $\mathbb{R}^2$s next to each other, in the sense that $k \otimes l = k + l$ in which the additional l points are simply stacked next to the initial k points.

Exercise 6.4.12. One could now prove that Braid $\simeq \coprod_{k \geq 0} B_k$, where $\coprod_{k \geq 0} B_k$ was in one of the exercise sheets.

6.4.3 Expanding the Braid category: the Tangle category

Next step: allow more morphisms, namely all bordisms, together with suitable embeddings.

Definition 6.4.13. A tangle with k inputs and l outputs is a 1 dimensional bordism Σ from k to l points together with $\Sigma \hookrightarrow [0,1] \times \mathbb{R}^2$ smooth embedding, such that

$$\partial_{in}\Sigma = \{(0,1),(0,2),\dots,(0,k)\} \tag{6.34}$$
$$\partial_{out}\Sigma = \{(1,1),(1,2),\dots,(1,l)\} \tag{6.35}$$

We can now define the tangle category, of which Braid will be a (braided monoidal) subcategory.

Definition 6.4.14. The tangle category Tang_1 has:

- objects: natural numbers
- morphisms:

$$\mathrm{Hom}_{\mathrm{Tang}_1}(k,l) = \{\text{isotopy classes of tangles from } k \text{ to } l \text{ points}\} \tag{6.36}$$

- composition of underlying bordisms and stacking embeddings using $[0,1] \cup_{1=0} [0,1] \hookrightarrow [0,1]$
- braided monoidal structure as before

Note that we also have $\mathrm{Braid} \subset \mathrm{Tang}_1$.

Definition 6.4.15. Let $(\mathcal{C},\otimes)$ be a monoidal category and $X \in \mathcal{C}$. We say that Y is a right dual of X if there is

$$ev_X : Y \otimes X \to 1 \tag{6.37}$$
$$coev_X : 1 \to X \otimes Y \tag{6.38}$$

such that the snake relations are satisfied. We then say that X is a left dual of Y. See 5.1.13 for a complete defintition.

Lemma 6.4.16. *In* Tang*, every object has a left and a right dual.*

Proof. Same pictures as for Bord_1. □

We can now define the concept of a framing on a tangle, which is different from what we previously meant as framing.

Definition 6.4.17. A framing on a tangle is a nonvanishing normal vector field on the tangle such that at the in-boundary and at the out-boundary it "points up".

This definition is made clearer through examples:

Example 6.4.18. ...

Considering framed tangles leads to the *framed* tangle category Tang_1^{fr}.
In fact, on the line segment as a bordism between two points we have $\mathbb{Z}$ many non isotopic framings corresponding to the *winding number* of normal vector fields. Similarly, S^1 also has $\mathbb{Z}$ many framings.
We noted last time that Tang_1 has left and right duals, one can check now that Tang_1^{fr} also does. We see that the framed tangle category is very similar but it has $\mathbb{Z}$ many morphisms between two points, instead of just one. This idea can be generalized with the concept of a ribbon category:

Definition 6.4.19. A ribbon category is a braided monoidal category $\mathcal{C}$ in which:

1. every object X has a right dual $X^\vee$, thus satisfying

$$ev_X : X^\vee \otimes X \to 1$$

$$coev_X : 1 \to X \otimes X^\vee$$

   which satisfies the snake relations/triangle identities (*right rigid category*),

2. there is a pivotal structure, that is a monoidal isomorphism

$$w : id_{\mathcal{C}} \implies (-)^{\vee\vee} \tag{6.39}$$

   (these two properties are the defining ones for a *pivotal category*)

3. For any object X, we have a "twist"

$$\vartheta_X = (id \otimes ev_{X^\vee}) \circ (\beta_{X,X} \otimes id) \circ (id_X \otimes coev_X) \tag{6.40}$$

   This satisfies

$$((\vartheta_X)^\vee = \vartheta_{X^\vee} \tag{6.41}$$

Here we're using an abuse of notation, by identifying $X^{\vee\vee}$ with X.

Example 6.4.20. In Tang_1^{fr}, the ϑ is exactly the picture with the blackboard framing. It remains to check property 3, which will be an exercise.

The definition of the dual of a map $f : X \to Y$ was given in 5.1.16 and is given by the following composition:

$$f^\vee : Y^\vee \xrightarrow{id_{Y^\vee} \otimes coev_X} Y^\vee \otimes X \otimes X^\vee \xrightarrow{id_Y^\vee \otimes f \otimes id_{X^\vee}} Y^\vee \otimes Y \otimes X^\vee \xrightarrow{ev_Y \otimes id_{X^\vee}} X^\vee$$

Lemma 6.4.21. *Let $\mathcal{C}$ be a braided pivotal category. Then every object has a left dual, the twist is invertible and natural in X and $\vartheta_{\mathbb{1}} = id_{\mathbb{1}}$ and*

$$\vartheta_{X\otimes Y} = \beta_{Y,X} \circ \beta_{X,Y} \circ \vartheta_X \otimes \vartheta_Y \tag{6.42}$$

Notation. We denote the left dual of X by $({}^\vee X, \tilde{ev}_X, co\tilde{e}v_x)$.

Example 6.4.22.

- $\mathrm{Vect}_k^{finitedim}$ is a ribbon category with the usual tensor product and braiding.

- $\mathrm{Mod}_R^{fpp} = R - \mathrm{Mod}^{fpp}$ is a ribbon category, where fpp stands for finitely presented and projective and R is a ring or k algebra.

However both these examples are *symmetric* monoidal

We would also like some more examples which are braided but *not* symmetric.
In order to do so, consider an algebra A, then $A - \mathrm{Mod}$ is a category. Now, which extra structure on A guarantees that $A - \mathrm{Mod}^{f.d.}$ is ribbon? Section 6.4.5 below is dedicated to answering this question. As a small preview we will be dealing with:

- representations of "quantum groups":
  - a "deformation" of $\mathrm{Rep}\, G = \mathrm{Mod}\, U\mathfrak{g}$,
  - equivalently, a deformation of $U\mathfrak{g}$ as a "Hopf algebra";
- representations of a "ribbon Hopf algebra" H which is a Hopf algebra with additional structure. We'll find $H - \mathrm{Mod}^{f.d.}$ to be a ribbon category. In particular we'll concentrate on $G = SL_2$ and we take $H = U_q(\mathfrak{sl}_2)$ where q is a root of unity.

Before that, we define a variation of the tangle category above and we'll have an interlude on knot theory.

Definition 6.4.23. Let $\mathcal{C}$ be a ribbon category. $\mathrm{Tang}_1^{fr}(\mathcal{C})$ is the following ribbon category, in which we "decorate with objects in $\mathcal{C}$ ".

- objects: finite sequences
$$(V_1, \epsilon_1), \ldots, (V_k, \epsilon_k) \tag{6.43}$$
where $V_i \in \mathcal{C}$ and $\epsilon_i = \pm 1$. This corresponds to k points in $\mathrm{Tang}_1^{fr,or}$ in which ϵ gives us the orientation of each point.
- morphisms: has underlying (isotopy classes of) framed oriented tangles + each component is labelled by $V \in \mathcal{C}$ such that the source and target are labelled by $(V, \pm 1)$.
- Composition is stacking pictures but only if the labels match up.

The "correspondence" above is actually a forgetful functor $\mathrm{Tang}_1^{fr}(\mathcal{C}) \to \mathrm{Tang}_1^{fr,or}$, in which or means adding an orientation to the bordism.

Proposition 6.4.24. *If $\mathcal{C}$ is a ribbon category, then there is a unique braided monoidal functor $F : \mathrm{Tang}_1^{fr}(\mathcal{C}) \to \mathcal{C}$ such that*

- *$F(V, +1) = V$ and $F(V, -1) = V^\vee$*
- *$\forall V, W \in \mathcal{C}$ we have:*

Finally, what does all this have to do with 3d TFTs?

$$\begin{tikzcd}
\mathrm{Tang}_1^{\mathrm{fr}}(\mathcal{C}) \arrow[r] & \mathcal{C} \arrow[r, "forget"] & \mathrm{Vect} \\
\mathrm{Bord}_3 \arrow[u, leftrightsquigarrow, "\text{surgery}"] \arrow[urr] & &
\end{tikzcd}$$

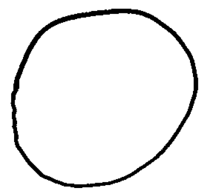
We will use:

- every 3 manifold arises from framed 1-tangles via *surgery*,
- two tangles giving the same 3 manifold can be related by *Kirby moves*,
- $F_{\mathcal{C}}$ is invariant under Kirby moves if $\mathcal{C}$ is a *modular* tensor category.

6.4.4 Interlude on knots, links and the Kones polynomial

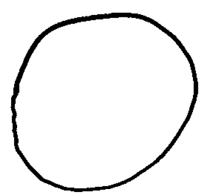
Example 6.4.25. Examples of knots are the following:

- the *unknot*



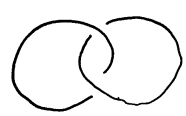
- the *trefoil*

- the *figure 8 knot*



and of links:

- the *Hopf link*

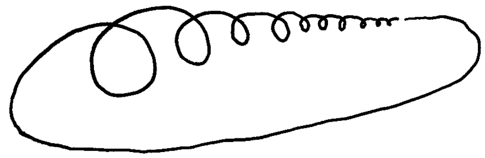


- the *Borromean rings*

Definition 6.4.26. A link is a finite collection of circles smoothly embedded in $\mathbb{R}^3$:

$$\coprod_{i=1}^{k} S^1 \hookrightarrow \mathbb{R}^3 \tag{6.44}$$

In which smoothness is to exclude some pathological situations. For instance the following one



This is pathological because it should be intuitively be equivalent to the unknot and to show this we would need infinitely many Reidemeister moves of type 1 (see 6.4.4) to deform it to the unknot. However, the fundamental theorem of knot theory 6.4.31 manages to prove that two knots are equal if one can be deformed into the other with *finitely* many Reidemeister moves.

Definition 6.4.27. A knot is a link with one component ($k = 1$).

Definition 6.4.28. Two links $L_1, L_2 : \amalg_{i=1}^{k} S^1 \hookrightarrow \mathbb{R}^3$ are equivalent if there exists an ambient isotopy between them, i.e. $H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that

- $H(-,0) = id$
- $\forall t \in [0,1],\ H(-,t)$ is a diffeomorphism
- $H(-,1) \circ L_1 = L_2$

A goal of knot theory is to find an invariant of knots/links, i.e. an assignment

$$\{\text{knots/links}\} \to \mathbb{R}, \mathbb{Z}[A, A^{-1}], \ldots \tag{6.45}$$

such that if two links are equivalent we get the same number, polynomial, ...

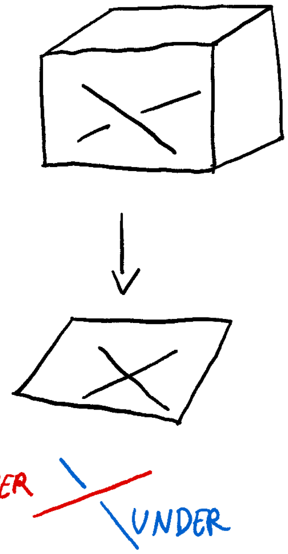
A link invariant is complete if it detects when 2 links are/are not equivalent.

Concretely, we represent links and nots by 2d drawings, these drawings are called *Link diagrams*. Let L be a link and $p : \mathbb{R}^3 \to \mathbb{R}^2$ a projection.

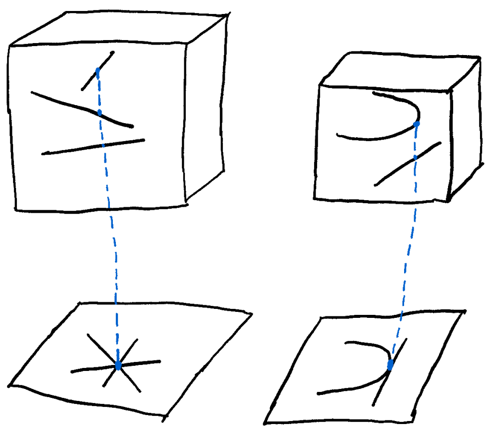
Definition 6.4.29. $p(L)$ is regular if

1. every point has at most 2 preimages in L
2. intersections are transversal

Definition 6.4.30. A link diagram is a regular $p(L)$ together with over/under information at each crossing. Visually:



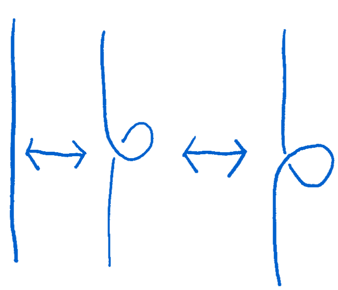
Remark. Given a link, we do *not* always get a link diagram $p(L)$, but we can always deform/perturb the embedding $L : \coprod_{i=1}^{k} S^1 \hookrightarrow \mathbb{R}^3$ by ambient isotopy to L' which gives a link diagram. For instance, the following two projections are irregular:



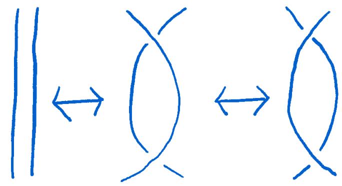
But such situation could be avoided by moving such links in a reasonable manner via ambient isotopies.

Our aim is to define a knot/link invariant by defining something on link diagrams. In order to do this we introduce Reidemeister moves, which are the following local changes in a link diagram:

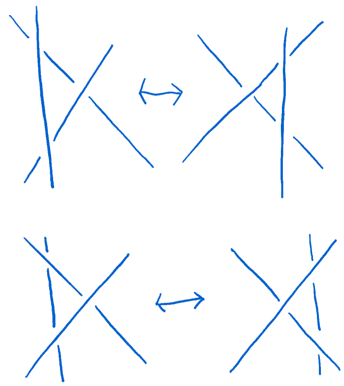
- (I or R1)



- (II or R2)



- (III or R3)



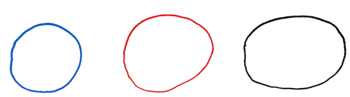
Theorem 6.4.31 (Reidemeister)**.** *Two links are equivalent if and only if their link diagrams are the same up to finitely many Reidemeister moves and isotopies in* $\mathbb{R}^2$.

This theorem is very important because it tells us that the link diagrams have all the information of the equivalence class of the link, so to determine (in)equivalence we can work with link diagrams.

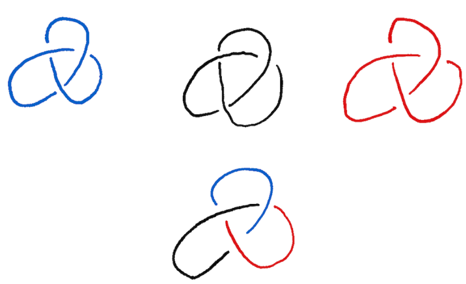
A first knot invariant is the "number of tricolorings": we color a link diagram with three colors according to the following rules

- each strand is one color
- at each crossing either all three strands are the same color, or they're all different.

Example 6.4.32. There are always three trivial colorings, the monochromatic ones. For the unknot, these are the only three tricolorings



while the trefoil has an additional one:

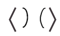


This shows that the trefoil knot is *not* equivalent to the unknot.

In order to prove that this is a knot invariant one should check that it is invariant under Reidemeister moves.

We now finally get to the Jones polynomial, found by Jones in 1984, and this presentation is from Kauffmann in 1987. The motivation for it comes from operator algebras and the braid group representations.

Definition 6.4.33. The Kauffmann bracket of a link diagram D is a Laurent polynomial $\langle D\rangle \in \mathbb{Z}[A, A^{-1}]$ defined by

- $\langle$  $\rangle = 1$
- $\langle$  $\rangle = A \cdot \langle$  $\rangle + A^{-1} \cdot \langle$  $\rangle$
- $\langle L \cup$  $\rangle = (-A^2 - A^{-2})\langle L\rangle$

Example 6.4.34.

$\langle$  $\rangle = A \cdot \langle$  $\rangle + A^{-1} \cdot \langle$  $\rangle =$

$= A(-A^2 - A^{-2})\langle$  $\rangle + A^{-1}\langle$  $\rangle =$

$$= -A^3 - A^{-1} + A^{-1} = -A^3$$

There is an issue:

$\langle$  $\rangle = \langle$  $\rangle$

and the last image is equivalent to the unknot, with an application of the first Reidemeister move, but

$\langle$  $\rangle = A^3$

and

$\langle$  $\rangle = 1$

This shows that the Kauffmann bracket is *not* a knot invariant, in particular it is not invariant under R1. It *is* however invariant under R2 and R3, in fact the coefficients in the definition of the bracket are exactly the ones that preserve R2 and R3.

From now on: links are oriented, meaning S^1 comes with an orientation. So we can talk about oriented link invariants, for which we have oriented Reidemeister moves.

Now we would like to fix the problem with the Kauffmann bracket.

Definition 6.4.35. The sign of a crossing is

Definition 6.4.36. The writhe of an oriented link diagram D is the sum of the signs of the crossings:

$$w(D) = \sum_{c \text{ crossing in } D} \mathrm{sign}(c) \tag{6.46}$$

Lemma 6.4.37. *The writhe is also invariant under R2 and R3, and changes under R1 as ...*

Definition 6.4.38. For any link L, define $\chi(L) = (-A^3)^{-w(p(L))}\langle p(L)\rangle \in \mathbb{Z}[A^{-1}, A]$

Theorem 6.4.39. *This is a link invariant.*

Example 6.4.40. Hopf link vs two detached links

Definition 6.4.41. For any unoriented link L, the Jones polynomial of L is

$$V(L) = \chi(L)|_{A=t^{1/4}} \in \mathbb{Z}[t^{-1/2}, t^{1/2}] \tag{6.47}$$

Example 6.4.42. trefoil
$V(D) = t + t^3 - t^4$

Observe

- $V(unknot) = 1$
- skain relation: if L_+, L_-, L_0 are the same link except for at one crossing, according to the following picture

  then
$$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0 \tag{6.48}$$

Explanation of problem of the Kauffmann bracket with R1:

Definition 6.4.43 (Alternative)**.** A framed link is (a link together with a nonvanishing normal vector field as before) the image of a smooth embedding of $\coprod_{i=1}^k S^\times [0,1] \hookrightarrow \mathbb{R}^3$.

Remark. This can be projected onto $\mathbb{R}^2$ such that annuli are "flat", this is the "blackboard framing" from previously.

Example 6.4.44. We can give two framings on the unknot which are not equivalent:

Theorem 6.4.45. *L, L' framed links, D, D' diagrams thereof using blackboard framings. Then $L \sim L'$ (isotopy equivalence with framing) if and only if D is related to D' via R2, R3 and the modification of R1 given by*

Theorem 6.4.46. *The Kauffmann bracket is a framed link invariant.*

Recall we had $\mathrm{Tang}_1^{or,fr}(\mathcal{C}) \to \mathcal{C}$ where $\mathcal{C}$ is a ribbon category. Now, fix a $V \in \mathcal{C}$, we get an injection $\mathrm{Link}^{or,fr} \hookrightarrow \mathrm{Tang}_1^{or,fr}(\mathcal{C})$ in which we decorate each strand (component) by V. The goal is then to find a specific ribbon category $\mathcal{C}$ such that we get back the Jones polynomial. We'll find $\mathcal{C} = U_q\mathfrak{sl}_2 - Mod^{f.d.}$.

6.4.5 Making $A - \mathrm{Mod}^{f.d.}$ into a ribbon category

We now want to add additional structures on an algebra A in such a way that $A - \mathrm{Mod}^{f.d.}$ becomes a ribbon category. We therefore need $A - \mathrm{Mod}^{f.d.}$ to have the following structures:

- monoidal structure,
- existence of (right) duals (right rigidity),
- braiding,
- pivotal structure,
- twist,

which we gradually construct in this order.

Monoidal structure

In order to do this, we will need additional structures on A, starting with the following.

Definition 6.4.47. A Hopf algebra in a braided strict monoidal category $\mathcal{C}$ is

- a bialgebra $(H, \mu, \eta, \Delta, \epsilon)$, in which we represent all these maps as the following drawings

  These maps should satisfy

$$(\mu \otimes \mu) \circ (id \otimes \beta \otimes id) \circ (\Delta \otimes \Delta) = \Delta \circ \mu \tag{6.49}$$

  and

- antipode map $S : H \to H$ depicted as

  satisfying

Example 6.4.48.

Take $H = k[G]$,

$$\begin{aligned} &\mu(g,h) = gh \qquad \eta(1) = 1 \\ &\Delta(g) = g \otimes g \qquad \epsilon(g) = 1 \qquad s(g) = g^{-1} \end{aligned} \tag{6.50}$$

for Lie algebra $\mathfrak{g}$, let $U\mathfrak{g}$ be its universal enveloping algebra, which we will define shortly. We then have

$$\mathrm{Rep}^{f.d.} G \simeq U\mathfrak{g} - \mathrm{Mod}^{f.d.} \tag{6.51}$$

we will then "deform" this in one of the exercises.

Recall that a Lie algebra is defined as follows.

Definition 6.4.49. A Lie algebra is a vector space with a bilinear map $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, sending $x \otimes y \mapsto [x, y]$, called the Lie bracket, satisfying:

- anticommutativity: $[x,x] = 0$ or $[x, y] = -[y, x]$
- Jacobi identity: $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

A homomorphism of Lie algebras $\mathfrak{g}, \mathfrak{h}$ is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ such that $\phi([x, y]_{\mathfrak{g}}) = [\phi(x), \phi(y)]_{\mathfrak{h}}$.

Example 6.4.50.

1. $\mathfrak{g} = \mathrm{End}(V) = \mathfrak{gl}_V$, with bracket given by the commutator $[f,g] = f \circ g - g \circ f$. There is also a sub Lie algebra $\mathfrak{sl}_V := \{f \in \mathfrak{gl}_V : \mathrm{tr}\, f = 0\}$.

   If for example we take $V = \mathbb{R}^2$ we have $\mathfrak{gl}_2 = \mathrm{Mat}_{2,2}$ and $\mathfrak{sl}_2 = \{A \in \mathrm{Mat}_{2,2} : \mathrm{tr}\, A = 0\}$.

2. Let A be an associative algebra, then if we take the underlying vector space, with bracket given by the commutator, we get a Lie algebra. In other words, there is a forgetful functor:

$$\mathrm{Alg} \to \mathrm{Lie} \tag{6.52}$$

   and its left adjoint is the concept of universal enveloping algebra which we will see more explicitly later on.

3.

4. smooth vector fields on a smooth manifold: $\mathfrak{X} = \mathrm{Der}(\mathbb{C}^\infty(M))$

Outlook: in physics we like to look at Lie groups, a smooth manifold with a group structure (i.e. a group object in the category of smooth manifolds). Since it's a manifold, we can look at its tangent space. The Lie algebra $\mathfrak{g}$ of G is defined as T_eG.

Now, fix $\mathfrak{g}$ a Lie algebra.

Definition 6.4.51. The tensor algebra of $\mathfrak{g}$ is the (graded) algebra given by:

$$T(\mathfrak{g}) := \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n} \tag{6.53}$$

with $\mathfrak{g}^{\otimes n} \otimes \mathfrak{g}^{\otimes k} \to \mathfrak{g}^{\otimes(n+k)}$.

There is a 2 sided ideal $I(\mathfrak{g}) \subset T(\mathfrak{g})$ generated by $\langle x \otimes y - y \otimes x - [x, y]\rangle$. We can then define the universal enveloping algebra of $\mathfrak{g}$:

$$U\mathfrak{g} := T(\mathfrak{g})/I(\mathfrak{g}) \tag{6.54}$$

The universal enveloping algebra has the universal property that given an algebra A and a Lie algebra map $\mathfrak{g} \to A$ there is a unique map $U\mathfrak{g} \to A$ making the following diagram commute

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[dr] & U\mathfrak{g} \arrow[d, dashed] \\ & A \end{tikzcd} \tag{6.55}$$

Claim. *$U\mathfrak{g}$ is a Hopf algebra, with coalgebra structure given by:*

- $\Delta : U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$
- $\epsilon : U\mathfrak{g} \to k$
- $S : U\mathfrak{g} \to U\mathfrak{g}$

In general, we'll now see that if A is a Hopf algebra, then $A - \mathrm{Mod}^{f.d.}$ is a monoidal category with duals.

Theorem 6.4.52. *Let (A, μ, η) be a unital associative algebra and $\Delta : A \to A \otimes A, \epsilon : A \to k$ linear maps. Then $(A, \mu, \eta, \Delta, \epsilon)$ is a bialgebra if and only if $(A - \mathrm{Mod}, \otimes, (k, \epsilon))$ is monoidal.*

- Let M, N be $A-$modules, then the vector space $M \otimes N$ is an $A-$module via

$$A \xrightarrow{\Delta} A \otimes A \longrightarrow \mathrm{End}(M) \otimes \mathrm{End}(N) \longrightarrow \mathrm{End}(M \otimes N) \tag{6.56}$$

- (k, ϵ) is an $A-$module via

$$A \xrightarrow{\epsilon} \mathrm{End}(k) \cong k \tag{6.57}$$

- associators and unitors are those from Vect.

Rigidity

Proposition 6.4.53. *Let H be a Hopf algebra and M an $H-$module, then $M^\vee := \mathrm{Hom}(M,k)$ has an H action via the transpose of S, i.e. given $\phi \in \mathrm{Hom}(M,k)$, $a \in H$ we get $a\cdot\phi \in \mathrm{Hom}(M,k)$ given by*

$$(a\cdot\phi)(m) := \phi(s(a)\cdot m) \tag{6.58}$$

This is a left dual if M is finite dimensional. Now, if S is invertible, ${}^\vee M = \mathrm{Hom}(M,k)$ with $(a\cdot\phi)(m) := \phi(s^{-1}(a)\cdot m)$ is a right dual if M is finite dimensional.

Corollary 6.4.54. *If H is a Hopf algebra with S invertible, then $H-\mathrm{Mod}^{f.d.}$ is rigid monoidal.*

Idea of proof of Proposition 6.4.53. Check that $ev_M, coev_M$ exhibiting dual in Vect are $H-$module maps. □

Braiding

Recall: we want a ribbon category and we had a Hopf algebra, so we had a monoidal category with duals.

We're now using [Sch23] as a reference.

Definition 6.4.55. A quasi-triangular structure on a bi/Hopf algebra is a "universal R-matrix", i.e. an invertible element $R \in A\otimes A$ such that $\forall x \in A$ with

$$\Delta^{coop} : A \xrightarrow{\Delta} A\otimes A \xrightarrow{\tau_{A,A}} A\otimes A \tag{6.59}$$

the following are satisfied

1. $\Delta^{coop}(x)R = R\Delta(x)$, and
2. In $A^{\otimes 3}$ we have
$$(\Delta\otimes id_A)(R) = R_{13}R_{23} \tag{6.60}$$
3. $$(id_A\otimes\Delta)(R) = R_{13}R_{12} \tag{6.61}$$
in which $R_{12} = R\otimes 1, R_{23} = 1\otimes R, R_{13} = (\tau_{A,A}\otimes id_A)(1\otimes R)$

Example 6.4.56. Any cocommutative bi/Hopf algebra has a canonical R matrix, $R = 1\otimes 1$, since the first relation simply becomes $\Delta = \Delta^{coop}$, which is exactly the cocommutativity.

Proposition 6.4.57 (4.2.3. in [Sch23])**.** *Let A be a bialgebra. Then a braiding on $A-\mathrm{Mod}^{(f.d.)}$ uniquely determines a quasitriangular structure on A and viceversa.*

Proof strategy: Given a braiding β, define the universal R matrix to be

$$R := \tau_{A,A}(\beta_{A,A}(1_A\otimes 1_A)) \tag{6.62}$$

in which $\tau_{A,A}$ is the flip and $\beta_{A,A}$ is the braiding. Then one can check properties 1. 2. and 3. above.

Now, the following is also true:

Claim. *Given R as above, the braiding can be recovered as follows: $\beta_{U,V}(u\otimes v) = \tau_{U,V}(R(u\otimes v))$, for arbitrary $U, V \in A-\mathrm{Mod}$.*

So this motivates that if we are given a universal R matrix, we should define β as such, and one should check that it's actually a braiding. □

We're now using [Tin15] as a reference. Main example: $U_q\mathfrak{sl}_2$, Hopf algebra, not cocommutative but has a universal R matrix. Therefore $U_q\mathfrak{sl}_2-\mathrm{Mod}$ is braided monoidal. We get $U_q\mathfrak{sl}_2$ from the cocommutative Hopf algebra $U\mathfrak{sl}_2$ by "deforming" it.

$U_q\mathfrak{sl}_2$ is a $\mathbb{C}(q)$-algebra generators E, F, K, K^{-1} with relations:

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1 \\ KEK^{-1} &= q^2 E \\ KFK^{-1} &= q^{-2} F \\ [E,F] &= \frac{K-K^{-1}}{q-q^{-1}} \end{aligned}$$

We have an important 2 dimensional module ("representation") for $U_q\mathfrak{sl}_2$: $V_1 \cong \mathbb{C}^2$ with basis v_1, v_{-1}. Now, how does $U_q\mathfrak{sl}_2$ act? In this basis we write:

$$\begin{aligned} E &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ F &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ K &= \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \end{aligned}$$

We also claimed there was a Hopf algebra structure, which we now make explicit. The comultiplication is given by

$$\begin{aligned} \Delta(E) &= 1\otimes E + E\otimes K \\ \Delta(F) &= K^{-1}\otimes F + F\otimes 1 \\ \Delta(K) &= K\otimes K \\ \Delta(K^{-1}) &= K^{-1}\otimes K^{-1} \end{aligned}$$

the counit map:

$$\begin{aligned} \epsilon(E) &= \epsilon(F) = 0 \\ \epsilon(K) &= \epsilon(K^{-1}) = 1 \end{aligned}$$

and we should check that these satisfy the required relations. In addition we need the antipode map:

$$\begin{aligned} S(E) &= -EK^{-1} \\ S(F) &= -KF \\ S(K) &= K^{-1} \\ S(K^{-1}) &= K \end{aligned}$$

We then get the monoidal category $U_q\mathfrak{sl}_2-\mathrm{Mod}$. If U,V are objects in $U_q\mathfrak{sl}_2-\mathrm{Mod}$, then so is $U\otimes V$ in which we have

$$E(u\otimes v)=E_U\otimes K_V+U\otimes E_V \tag{6.63}$$

It is also braided...

From S we get the dual of V_1: as vector space $V_1^\vee=\mathrm{Hom}(V_1,\mathbb{C})$, in which $\hat{v}_1,\hat{v}_{-1}$ is the dual basis. This has a $U_q\mathfrak{sl}_2-\mathrm{Mod}$ structure given by Equation 6.58.

Up to now we therefore get the following correspondence between the structures on A and those on $A-\mathrm{Mod}^{f.d.}$:

bialgebra	$\leftrightarrow$	monoidal
Hopf algebra	$\leftrightarrow$	right rigid
Hopf with S invertible	$\leftrightarrow$	rigid
quasi-triangular	$\leftrightarrow$	braiding

Let's make explicit the dual representation to V_1, $V_1^\vee=\mathrm{Hom}(V,\mathbb{C})=\langle\hat{v}_1,\hat{v}_{-1}\rangle$, where $\hat{v}_1$ and $\hat{v}_{-1}$ are the dual basis.

Recall that we had the *decorated* tangle category. Let's take $\mathcal{C}=U_q\mathfrak{sl}_2-\mathrm{Mod}^{f.d.}$ and let $V=V_1$ the representation above. In particular we work with "$\mathrm{Tang}_1^{fr,or}(V_1)$"$\subset\mathrm{Tang}_1^{fr,or}(\mathcal{C})$ in which we decorate everything with just V_1.

Remark.

- $A=q^{-1/2}$
- slightly different normalization: $F(unknot)=-q-q^{-1}$.
- What about the third relation of the Kauffmann bracket? It follows from monoidality!

6.5 Outlook: How to get a 3D TFT?

6.5.1 From knots to 3 manifolds: surgery

Given: a framed link $L\subset S^3$ (so far $L\subset\mathbb{R}^3$ but we now embed that into S^3) with m components $L=L_1\cup\cdots\cup L_m$ with $L_i\cong S^1$. Now:

- Choose closed tubular neighborhood $U\subset S^3$ of link L such that $U=U_1\amalg\cdots\amalg U_m$:

$$\begin{array}{ccc} L_i & \cong & S^1\times\{0\} \\ \cap & & \\ U_i & \xrightarrow[\phi]{\cong} & S^1\times D^2 \end{array} \tag{6.64}$$

  framing of $L_i\cong$ constant normal vector field on $S^1\times\{0\}\hookrightarrow S^1\times D^2$. Could also have the nonconstant framing, that turns around once.

- $\partial(S^3\setminus\mathrm{int}(U_1\amalg\cdots\amalg U_m))=\partial(U_1\amalg\cdots\amalg U_n)\cong S^1\times S^1\amalg\cdots\amalg S^1\times S^1$, m tori. Now glue in solid tori $D^2\times S^1$ (note that the order here matters) using identity as gluing map to get a closed connected 3 manifold.

Exercise 6.5.1. Try unknot with trivial framing.

Objective: 3d tqft. Surgery.

Start with a link $L \subset S^3$ with $L = L_1 \amalg \cdots \amalg L_m$ with $L_i \cong S^1$. We take a tubular neighborhood of L, $U = U_1 \amalg \cdots \amalg U_m$ with $U_i \cong S^1 \times D^2$. Let $T := S^1 \times D^2$, the full torus, and let $\phi : U_i \xrightarrow{\cong} T$. Now we can define

$$M_{L,\phi} := (S^3 \setminus \mathrm{int}(\bigcup_{i=1}^{m} U_i)) \coprod_{\phi} T^m \tag{6.65}$$

surgery of S^3 along L via ϕ. However, the result depends on the choice of ϕ and we'll see that it corresponds to a choice of framing on the knot.

If L is framed and oriented, we can choose a ϕ as follows: let's restric to one component L_i. We then have $\partial U_i \cong S^1 \times S^1$ and the framing determines a "cycle" of $S^1 \times S^1$ as can be seen in the drawing. In particular if the framing winds around k times then we get a cycle that winds around once in the longitudinal direction and k times in the other direction. This determines a diffeomorphism on the torus up to to isotopy, called the "Dehn twist". We then compose with $swap : S^1 \times S^1 \to S^1 \times S^1$:

$$\phi_i : \begin{array}{ccc} T & \longrightarrow & T \\ \\ \alpha_i & \longmapsto & -\beta_i \\ \\ \beta_i & \longmapsto & \alpha_i \end{array} \tag{6.66}$$

In short, via ϕ_i the induced framing on $S^1 \times \{0\}$ is a constant normal framing. So framing on L determines/encodes/records the diffeomorphism ϕ.

Notation: L framed $M_L := M_{L,\phi}$ with ϕ as above.

Example 6.5.2.

- $L = \emptyset$, we then get $M_L = S^3$,
- $L = unknot$ with constant framing:

  $$\partial(S^3 \setminus \mathrm{int}\, T) \cong S^1 \times S^1 \tag{6.67}$$

  Now, as a warmup let's see how $T \coprod_{\phi'} T = S^2$:

  Now, $(S^3 \setminus \mathrm{int}\, T) \coprod_{\phi'} T \ldots$

  We then have a well defined projection $M_L \to S^1$ and the fibers of this map are $S^2 = D^2 \coprod_{\partial D^2} D^2$. In particular, the bundle is trivial:

  Fact. $M_L \cong S^1 \times S^2$.

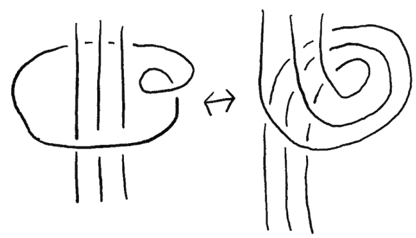
- $L = unknot$ with framing with one twist, we get $M_L = S^3$

The last example is important because it shows that multiple framed links can give the same 3 manifold. However we have the following result:

Theorem 6.5.3 (Lickorice-Wallace)**.** *Any connected closed oriented 3-manifold can be obtained as M_L for some framed link L in S^3 via surgery.*

We can therefore try to define a manifold invariant by defining it on framed links, but we have to make sure that links that give the same 3 manifold have the same invariant.

Theorem 6.5.4 (Kirby's theorem)**.** *$M_L \cong M_{L'}$ if and only if L' is obtained from L by a finite sequence of isotopies and Kirby moves which are as follows:*



We can now get a Reshetikin-Turaev invariant, see for details [KRT97].

Remark. If L_1 and L_2 are not linked, then $M_{L_1 \amalg L_2} = M_{L_1} \# M_{L_2}$ Moreover, $\tau(M_{L_1} \# M_{L_2}) = c\dot{\tau}(M_{L_1}) \cdot \tau(M_{L_2})$ where c is a constant depending on the modular tensor category

We can give a **rough sketch** of the following theorem

Theorem 6.5.5 (Turaev)**.** *For any modular tensor category we have an oriented 3d TFT which generalizes the Reshetikin-Turaev invariants.*

Modular tensor categories are hard to define. The following definition is from [Run].

Definition 6.5.6 (Modular Tensor Category)**.** A modular tensor category is a strict k-linear[8] abelian semisimple ribbon category such that the index set is finite, every simple object is absolutely simple, and for which the s-matrix $s = (s_{i,j})_{i,j \in I}$ with entries

$$s_{i,j} := s_{U_i,U_j} = \mathrm{tr}(c_{U_i,U_j} \circ c_{U_j,U_i})$$

is non-degenerate. An element $\mathscr{D} \in k$ is called rank of a modular tensor category $\mathcal{C}$ if

$$\mathscr{D}^2 = \Sigma_{i \in I}(dim(U_i))^2$$

Given a simple object $U_i \in \mathcal{C}$, also $U_i^\vee$ is simple.

See **??** for the definition of abelian category.

Remark. In the original definition by Turaev ([Tur16], [Tur]) some conditions were weaker. It was enriched over Mod_R, with R being a commutative ring instead of a field k, semisimplicity was replaced by the weaker dominance property and instead of it being abelian, it was just additive.

Example 6.5.7.

- Vect_k
- $U_q sl_2 - mod^{\mathrm{fd}}$ for q a root of unity

[8] I.e. Vect_k-enriched.

Definition 6.5.8 (Simple object)**.** An object $U \in \mathcal{C}$ where $\mathcal{C}$ is an abelian category is simple if it has no non-trivial subobjects, i.e. any injection $V \hookrightarrow U$ is either the 0 object or an isomorphism.

Definition 6.5.9 (Semisimple object)**.** An object $U \in \mathcal{C}$ where $\mathcal{C}$ is an abelian category is semisimple if it is isomorphic to a direct sum of simple objects.

Definition 6.5.10 (Semisimple category)**.** An abelian category is semisimple if it only has semisimple objects.

Definition 6.5.11 (Absolutely simple object)**.** An object $U \in \mathcal{C}$ where $\mathcal{C}$ is an abelian k-linear category is called absolutely simple if and only if $\mathrm{Hom}(V, V) = k\,\mathrm{id}_V$. If $\mathcal{C}$ is semisimple and k algebraically closed, then this is equivalent to a simple object.

6.6 Rough sketch of the cobordism hypothesis and extended topological field theories

Recall that a TFT is

- a way to organize invariants of smooth manifolds and compute them
- a way to make (some) quantum field theories mathematically rigorous

From the point of view of mathematics one might ask if we classify TFTs as we did for dimension 1 and 2 for any dimension n. From the point of view of physics one might ask if a TFT describing a certain physical system can be determined by the behavior of the system around a point, i.e. if it is fully local. The answer to these questions is yes and is called the cobordism hypothesis.

Remark (What is the cobordism hypothesis?). The cobordism hypothesis informally states that any TFT, at any dimension, can be classified/constructed just by observing where the TFT in question sends the point[9], i.e. a bit more rigorously

Theorem 6.6.1 (Cobordism hypothesis). *Let $\mathcal{C}$ be a monoidal (∞, n)-category with duals*[10]*. Then the evaluation functor on a point, i.e. the functor sending $\mathcal{Z} \mapsto \mathcal{Z}(*)$, induces an equivalence*

$$ev_* : \mathrm{Fun}^{\otimes}(\mathrm{Bord}_n, \mathcal{C}) \to (\mathcal{C}^{\mathrm{fd}})^{\cong}$$

where $\mathcal{C}^{\mathrm{fd}}$ is the full[11] *monoidal (∞, n)-category with duals*[12]*, i.e. we forget about all non-dualizable objects, and $\mathcal{C}^{\cong}$ is the maximal underlying groupoid of $\mathcal{C}$, i.e. we forget about all morphisms that are not invertible.*

It was formulated by James Dolan and John Baez in [BD95] and now we have only partial proofs, [Lur09], [AF17] and [GP22]. The one by Ayala and Francis relies has a different strategy to the first sketch by Lurie and relies on an unproved conjecture regarding factorization homology, see **??** for a definition of factorization homology.

[9] We will soon see a proof of the 1-dimensional case 6.1.

[10] We later give a definition of (∞, n)-category with duals, see **??**.

[11] i.e. there are all morphisms between the objects of the subcategory, i.e. no morphism from the category is forgotten.

[12] We gave a definition of (∞, n)-category with duals, see **??**.

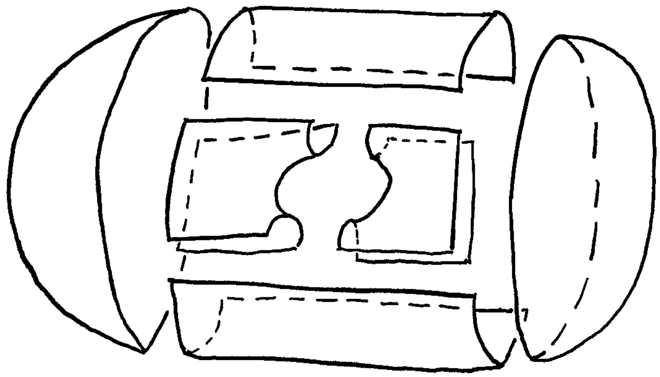


Figure 6.1: Visualization of a torus extended down to a point. The vertices of the corners of the 2-dimensional surfaces are points.

Remark (What are extended TFTs?). After defining the cobordism hypothesis in this manner, a natural question is: how can TFTs of any dimension greater than 1 be classified by evaluating them at a point? If we stick to our definition, there are no points in any category of bordisms of dimension greater than 1. Fortunately, one can *extend* the bordism category, and consequently TFTs, and also talk about lower dimensions. Take as an example $\mathrm{Bord}^{or}_{2,1}$. As we defined the bordism category, in this case the lowest dimension is 1, the objects are lines, not points. Nevertheless, one could treat the category of 2 dimensional bordisms as a 2-category, more specifically as a bicategory (see **??**)[13], $\mathrm{Bord}_{2,1,0}$ where objects are disjoint unions of points, 1-morphisms are oriented cobordisms between points and 2-morphisms are cobordisms between the 1-morphisms; and then define 2d-TFTs as an appropriate notion of symmetric monoidal functor between symmetric monoidal bicategories[14] (see **??** for a definition). However, to this we need to allow manifolds with corners, as the illustration of the downward extension of the torus shows (6.1).

Treating a TFT as a symmetric monoidal weak 2-functor (see **??** for a definition) from a symmetric monoidal bicategory $\mathrm{Bord}_{n,n-1,n-2}$ (see **??** for a definition), instead of from a symmetric monoidal category is named 'once-extended TFT', e.g. see [Sch23]. There are two important results regarding once-extended TFTs:

1. One regards the classification of 2d-TFTs down to a point and is due to Christopher Schommer Pries who proved it in his PhD thesis
$$ev_* : \mathrm{Fun}^{\otimes}(\mathrm{Bord}^{\mathrm{fr}}_{2,1,0}, \mathcal{B}) \xrightarrow{\simeq} (\mathcal{B}^{2\text{-dualizable}})^{\cong}$$
2. The other, although still conjectural[15], classifies 3d-TFTs via the evaluation on the circle
$$ev_{S^1} : \mathrm{Fun}^{\otimes}(\mathrm{Bord}^{\mathrm{fr}}_{3,2,1}, \mathrm{ModTensor}) \xrightarrow{\simeq} (\mathrm{ModTensor}^{2\text{-dualizable}})^{\cong}$$
hence $\mathcal{Z}(S^1) = \mathcal{C}$ where $\mathcal{C}$ is a modular tensor category, i.e. a special ribbon category which can be seen as a categorified commutative Frobenius algebra.

However, as one can infer from the aforementioned cobordism hypothesis, one can also extend downward to dimension 0 any n-dimensional TFT and it was first proposed

[13] Since 1-morphisms do not compose strictly, but up to an appropriate notion of isomorphism.

[14] A detailed treatment of the 2d case is found in [SP14].

[15] It was first stated by Kevin Walker and then some progress was made by Turaev, see [Tur16] for more on this.

by Daniel Freed in [Fre94]. To be precise, one nowadays does not want to only extend downwards but also upwards, i.e. to not work wirh n-categories, i.e. (n,n)-categories, but wants to work with(∞,n)-categories, i.e. categories where morphisms of dimension strictly greater than n are invertible, because of technical reasons[16]. Sketchily, the (∞,n) category of bordisms will have points as objects, bordisms between points as 1-morphisms, bordisms between bordisms between points as 2-morphisms,..., bordisms between bordisms between bordisms...[17] as n-morphisms, diffeomorphisms between n-dimensional bordisms as $n+1$-morphisms, isotopies between diffeomorphisms as $n+2$-morphisms, isotopies between isotopies between diffeomorphsims as $n+3$-morphisms, ... and so on infinitely many times. This is an example of a (∞,n)-category, since isotopies of diffeomorphisms and diffeomorphisms are in fact invertible, whereas oriented bordisms not necessarily. One can find more on this higher category of bordisms in [CS19].

[16]For example, the argument sketched in [Lur09] is crucially a proof by induction on n and in order to understand the $n+1$-morphisms of Bord_{n+1} one must understand such $n+1$-morphisms in Bord_n which are absent if we treat Bord_n as an n-category and not (∞,n)-category. In short, that would just not be possible without such ∞-categorical machinery.

[17]There is exactly $n-1$ times 'between bordisms' after the first instance of 'bordisms', n-morphisms are morphisms between $(n-1)$-morphisms.

Part III

Hints and solutions to exercises

Sheet 1.

Exercise 6.6.2. ***Abelian structure of the cobordism group***
Show that the disjoint union induces an abelian group structure on the cobordism group Ω_n.

First we note that, under the disjoint union, each element of Ω_n is its own inverse, i.e. $a \sqcup a = \emptyset$. Using multiplicative notation (since we prefer not to use the standard additive notation before we know we are dealing with an abelian group) we have $ab = ab(ba)(ba) = a(bb)aba = (aa)ba = ba$ using associativity, hence proving that the group structure is abelian.

Exercise 6.6.3. ***Orientable Manifolds***

(a) Show that the circle S^1 is an orientable manifold.

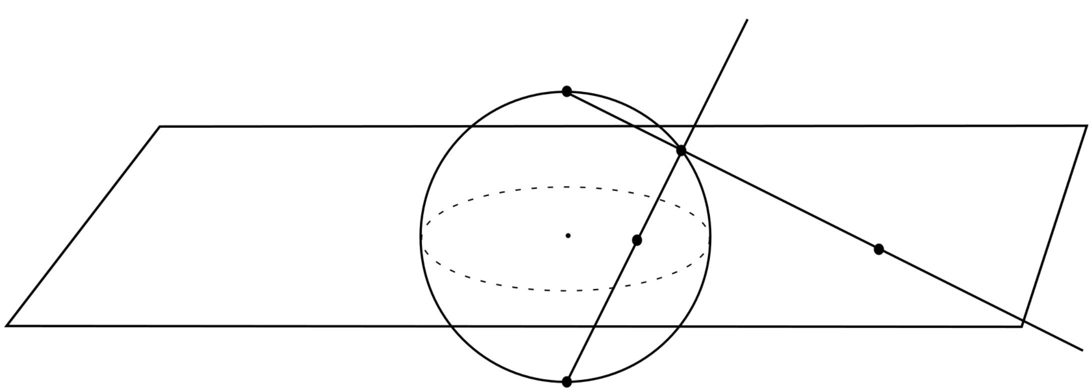
(b) Show that the sphere S^2 is an orientable manifold.

(c) Show that the total space of the tangent bundle of a smooth n-manifold is an orientable manifold.

A way of proving a manifold is orientable is finding a local trivialisation of the tangent bundle. To do this we first pick an open cover and then prove that the transition functions are orientation preserving (we understand what orientation means in Euclidean space).

(a) For S^1 we can use an open cover inspired by the universal cover. We choose $U_1 = (-\pi, \pi)$ and $U_2 = (0, 2\pi)$ along with the maps $\varphi_1 : x \mapsto e^{ix}$ and $\varphi_2 : x \mapsto e^{ix}$, respectively. Looking at the transition map $\varphi_1^{-}1 \circ \varphi_2$ we get a map between $(0,\pi) \cup (\pi, 2\pi) \to (0,\pi) \cup (-\pi, 0)$ acting like the identity on the first constituent interval and like the identity plus 2π on the second one. By symmetry, we have a similar situation for the other transition function. Therefore, the differential acts by the identity on the tangent spaces, and so this has determinant $1 > 0$. Thus we conclude that S^1 is orientable, as required.

(b) **Attempt 1:** We try doing this using the standard stereographic projections from the north and south poles, respectively, as an atlas. Then, it is well known that the (both!) transition maps are simply circle inversions on $\mathbb{R}^2 \setminus \{0\}$. See the picture below for clarification (this is a simple exercise in standard Euclidean geometry).



In other words, the map is given by $(x, y) \mapsto (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. If we do the computations, we find that the Jacobian is

$$J = \frac{1}{(x^2+y^2)^4} \det \begin{pmatrix} y^2 - x^2 & -2yx \\ -2yx & x^2 - y^2 \end{pmatrix} = -\frac{1}{(x^2+y^2)^2} < 0$$

so this approach doesn't work...

Attempt 2: S^2 can be viewed as the level set of a smooth function $f : (x, y, z) \mapsto x^2 + y^2 + z^2$ with non-zero gradient and is therefore an orientable manifold.

Attempt 3: For those of you that would (for good reasons) argue that attempt 2 is somehow cheating because is doesn't use our definition, but rather a theorem we haven't presented, we will now present a proof using first principles. This will use a similar idea as in exercise (a). If we describe the sphere as the set of points $\{(\sin\phi \sin\theta, \sin\phi \cos\theta, \cos\phi) \in \mathbb{R}^3\}$. Let (U_1, φ_1) be the chart where...

(c) Show that the total space of the tangent bundle of a smooth n-manifold is an orientable manifold.

Exercise 6.6.4. ***Computation of*** Ω_0^{or}
Compute the oriented bordism group Ω_0^{or}.

We have that 0-dimensional oriented manifolds are classified by a finite collection of points, each with sign. By connecting points of different signs with oriented lines we get a cobordism to a finite collection of points all of the same sign. The sum of the signs in this sense is quite obviously invariant under cobordisms, and so we realise that $\Omega_0^{or} \cong \mathbb{Z}$.

Exercise 6.6.5. ***Computation of*** Ω_2^{or}

(a) Work through the argument in detail showing that Σ_g is cobordant to the empty set.

(b) Recall that the disjoint union is cobordant to the connected sum. Work through the details for an example different from what was shown in the lecture.

(c) Conclude that $\Omega_2^{\mathrm{or}} = 0$. (Here, you may omit details about the orientations of the 3-dimensional cobordisms.)

Let us now try to understand this cobordism group in dimension 2 a bit better.

(a) To understand that Σ_g is cobordant to the empty set we will use the "standard" embedding in $\mathbb{R}^3$. The manifold with boundary that is the interior of this surface in $\mathbb{R}^3$ forms a cobordism between Σ_g and $\emptyset$.

(b) Push along the normal bundle...

(c) This follows from classification of surfaces...

Exercise 6.6.6. ***Computation of*** Ω_2^{unor}***:***

(a) Show that the Klein bottle K is cobordant to the empty set.

(b) (Poincaré duality and Euler characteristic) Show that $\mathbb{RP}^2$ is non-zero in Ω_2^{unor}, i.e. that there is no compact 3-manifold X with boundary $\partial X = \mathbb{RP}^2$.
Hint: Consider the double $D = X \underset{\mathbb{RP}^2}{\cup} X$. What does Poincaré duality imply about the Euler characteristic of 3-dimensional closed manifolds?

(c) Conclude from the above that $\Omega_2^{\text{unor}} = \mathbb{Z}/2\mathbb{Z}$.

Sheet 2.

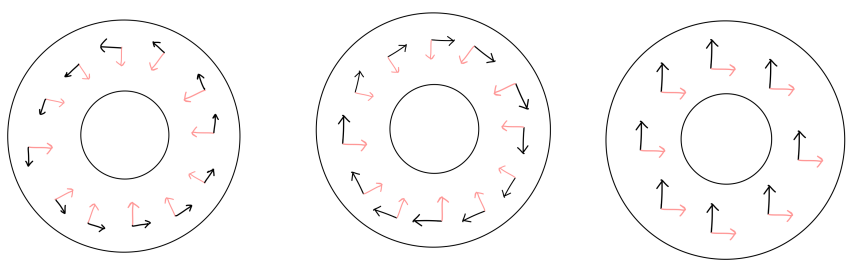
Exercise 6.6.7. ***Framings***
Justify your answers to the following questions:

(a) Can a Klein bottle be framed?

(b) Can S^2 be framed?

(c) An **isotopy between the framings** is a deformation given by a family of framings parameterized by the interval. Are any of the framed cylinders below isotopic?

(d) Which of the framings induce the same orientations?



(a) A Klein bottle cannot be framed because it is unoriented (existence of a framing is more restrictive).

(b) S^2 can also not be framed even though it is orientable. This is a consequence of the hairy ball theorem.

(c) An **isotopy between the framings** is a deformation given by a family of framings parameterized by the interval. Are any of the framed cylinders below isotopic? ...

(d) The two rightmost framings induce the same orientations.

Exercise 6.6.8. ***Attaching handles***
Definition. Let B^n denote the n-dimensional ball as a manifold with boundary and S^n the n-dimensional sphere.

Given a 2-dimensional manifold M, we *attach a j-handle* $H^j :- B^j \times B^{2-j}$, for $j \in \{0,1,2\}$ via and a smooth embedding $f : S^{j-1} \times D^{2-j} \hookrightarrow \partial M$ as follows:

$$M \cup_f H^j :- \left(M \sqcup (B^j \times B^{2-j})\right) / \sim$$

where for $(p,x) \in S^{j-1} \times B^{2-j} \subset B^j \times B^{2-j}$, we set $f(p,x) \sim (p,x)$.

(a) Convince yourself that there is a smooth structure on $M \cup_f H^j$.

(b) Which surface is obtained from attaching a 1-handle to a disk?

(c) Which surface is obtained from attaching two 1-handles to a disk, i.e. from attaching an additional 1-handle to the surface obtained in part (a)?

(d) Build the torus by successively attaching handles to a disk.

(a) "I can explain it to you, but I can't understand it for you."

(b) A cylinder (or a Moebius strip).

(c) A so-called pair of pants.

(d) To a disk we successively add a 1-handle, another 1-handle, and finally cap everything off with a 2-handle.

Exercise 6.6.9. ***Properties of the connected sum of manifolds***

(a) Given n-manifolds M, M', and M'', show that the connected sum satisfies the following properties.

   (i) $M\#S^n \cong M$, *(neutral element)*

   (ii) $M\#M' \cong M'\#M$, and *(commutativity)*

   (iii) $(M\#M')\#M'' \cong M\#(M'\#M'')$. *(associativity)*

(b) If M and M' are smooth n-manifolds, construct a smooth structure on the connected sum $M\#M'$. Note that this is not unique but defines a well-defined diffeomorphism class. You may like to read more details using isotopies in Chapter 8, Section 2 in Hirsch, Differential Topology[18].

Exercise 6.6.10. ***Reading exercise***
Below is a list of several proofs of the classification theorem of 1-dimensional manifolds using different tools. Read through one (or several) of them, or find your own.

(i) `https://pnp.mathematik.uni-stuttgart.de/igt/eiserm/lehre/2014/Topologie/Gale%20-%201-manifolds.pdf`

(ii) Appendix of `https://www.maths.ed.ac.uk/~v1ranick/papers/milnortop.pdf`, starting at p.55.

[18] Can e.g. be accessed at `https://www.researchgate.net/publication/268035774_Differential_Topology`.

Sheet 3.

Exercise 6.6.11. ***Morse functions***

(a) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto x^3 - 3xy^2$. Find all the critical points and check if f is a Morse function. If it does not meet the criteria, perturb it in such a way that it becomes a Morse function.

(b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $(x, y) \mapsto x^2 y^2$. Find all the critical points and check if f is a Morse function. If it does not meet the criteria, perturb it in such a way that it becomes a Morse function.

(c) Show that if $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ are Morse functions, then $f + g : M \times N \to \mathbb{R}$ is also a Morse function, and the critical points are pairs of critical points of f and g. Visualize this for $M = N = S^1$ and $f : S^1 \subset \mathbb{R}^2 \to \mathbb{R}$ the projection onto the first coordinate.

(a) We have that $\operatorname{grad} f = (3x^2 - 3y^2, -6xy)$, and so the only critical point is the origin. The Hessian determinant is

$$\mathcal{H} = \det\begin{pmatrix} 6x & -6y \\ -6y & -6x \end{pmatrix}$$

and hence degenerate at the origin. This means f is not Morse. We can perturb it by adding $\varepsilon(x^2 + y^2)$ for a sufficiently small $\varepsilon > 0$ such that it becomes Morse. This has the origin as its only critical point and adds $2\varepsilon I_2$ to the Hessian matrix, thus making the determinant not vanish at the origin.

(b) By, again, computing the gradient $\operatorname{grad} f = (2xy^2, 2yx^2)$, we realise that the set of critical points of f consists precisely of the x- and y-axis. Therefore, f is obviously not Morse.

(c) Show that if $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ are Morse functions, then $f + g : M \times N \to \mathbb{R}$ is also a Morse function, and the critical points are pairs of critical points of f and g. Visualize this for $M = N = S^1$ and $f : S^1 \subset \mathbb{R}^2 \to \mathbb{R}$ the projection onto the first coordinate.

Exercise 6.6.12. ***Handle decomposition***

Definition 1. Let M be a compact 2-manifold. A **handle decomposition** of M is a finite sequence of manifolds

$$\emptyset = W_{-1} \subseteq W_0 \subseteq W_1 \subseteq W_2 = M$$

such that each W_i is obtained from W_{i-1} by attaching i-handles.

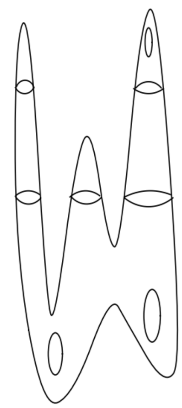
(a) Find two different handle decompositions of S^2.

(b) Find a handle decomposition of $\mathbb{RP}^2$.

(c) Find a handle decomposition of the Klein bottle.

(d) Explain why, for any non-empty closed connected surface, we can start a handle decomposition with a single 0-handle. *Hint*: The key argument was mentioned in lectures as "*handle cancellation*".

(e) Using the idea of handle cancellation, bring the surface below into *normal form*, i.e. such that read from bottom to top, the index of the critical points are non-decreasing.



Exercise 6.6.13. ***Reading exercise - Classification of closed 1-manifolds***
Prove the following theorem using Morse theory and/or read through the proof in `https://www.math.csi.cuny.edu/~abhijit/papers/classification.pdf` [Theorem 15].

Theorem 6.6.14. *Any closed 1-manifold is homeomorphic to S^1.*

Bibliography

[ADA78] J. F. ADAMS. *Infinite Loop Spaces (AM-90): Hermann Weyl Lectures, The Institute for Advanced Study. (AM-90)*. Princeton University Press, 1978.

[AF15] David Ayala and John Francis. Factorization homology of topological manifolds. *Journal of Topology*, 8(4):1045–1084, October 2015.

[AF17] David Ayala and John Francis. The cobordism hypothesis, 2017.

[AF19] David Ayala and John Francis. A factorization homology primer, 2019.

[AFR20] David Ayala, John Francis, and Nick Rozenblyum. Factorization homology i: higher categories, 2020.

[ART50] EMIL ARTIN. The theory of braids. *American Scientist*, 38(1):112–119, 1950.

[Bae04] John C. Baez. Quantum quandaries: a category-theoretic perspective, 2004.

[BD95] John C. Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *Journal of Mathematical Physics*, 36(11):6073–6105, November 1995.

[Ber06] Julia E. Bergner. A survey of $(\infty, 1)$-categories, 2006.

[BHLS23] Robert Burklund, Jeremy Hahn, Ishan Levy, and Tomer M. Schlank. k-theoretic counterexamples to ravenel's telescope conjecture, 2023.

[CG23] Kevin Costello and Owen Gwilliam. Factorization algebra, 2023.

[Cos06] Kevin J. Costello. Topological conformal field theories and calabi-yau categories, 2006.

[CR18] Nils Carqueville and Ingo Runkel. Introductory lectures on topological quantum field theory. *Banach Center Publications*, 114:9–47, 2018.

[CS19] Damien Calaque and Claudia Scheimbauer. A note on the (∞, n)–category of cobordisms. *Algebraic & Geometric Topology*, 19(2):533–655, March 2019.

[Dav23] David Ben-Zvi, Yiannis Sakellaridis, Akshay Venkatesh. Relative langlands duality. `https://www.math.ias.edu/~akshay/research/BZSVpaperV1.pdf`, (Work in progress) 2023.

[DBZ21] (notes by Jackson Van Dyke) David Ben-Zvi. Between electric-magnetic duality and the Langlands program. `https://web.ma.utexas.edu/users/vandyke/notes/langlands_sp21/langlands.pdf`, 2021.

[Den20] Denis Nardin. Introduction to stable homotopy theory. https://homepages.uni-regensburg.de/~nad22969/stable-homotopy-2020/stable-homotopy.pdf, 2020.

[FH21] Daniel S Freed and Michael J Hopkins. Reflection positivity and invertible topological phases. *Geometry & Topology*, 25(3):1165–1330, May 2021.

[FM13] Daniel S. Freed and Gregory W. Moore. Twisted equivariant matter. *Annales Henri Poincaré*, 14(8):1927–2023, March 2013.

[FR19] Christopher J. Fewster and Kasia Rejzner. Algebraic quantum field theory – an introduction, 2019.

[Fre94] Daniel S. Freed. Higher algebraic structures and quantization. *Communications in Mathematical Physics*, 159(2):343–398, January 1994.

[Fre08] Daniel S. Freed. Remarks on chern-simons theory, 2008.

[Fre12] Daniel S. Freed. The cobordism hypothesis, 2012.

[Fre13] Daniel S. Freed. Bordism: Old and new, 2013.

[Fre14] Daniel S. Freed. Short-range entanglement and invertible field theories, 2014.

[GO13] Nick Gurski and Angélica M. Osorno. Infinite loop spaces, and coherence for symmetric monoidal bicategories. *Advances in Mathematics*, 246:1–32, October 2013.

[GP22] Daniel Grady and Dmitri Pavlov. The geometric cobordism hypothesis, 2022.

[Gro15] Moritz Groth. A short course on ∞-categories, 2015.

[Gro21] Alexander Grothendieck. Pursuing stacks, 2021.

[GS18] Owen Gwilliam and Claudia Scheimbauer. Duals and adjoints in higher morita categories, 2018.

[Har19] Yonatan Harpaz. Little cube algebras and factorization homology (course notes, in progress), 2019.

[Hir76] Morris W. Hirsch. *Differential Topology*, pages 1–6. Springer New York, New York, NY, 1976.

[HK64] Rudolf Haag and Daniel Kastler. An Algebraic approach to quantum field theory. *J. Math. Phys.*, 5:848–861, 1964.

[HM06] Hans Halvorson and Michael Mueger. Algebraic quantum field theory, 2006.

[HW] Fabian Hebestreit and Ferdinand Wagner. Lecture notes for algebraic and hermitian k-theory. https://florianadler.github.io/AlgebraBonn/KTheory.pdf.

[Jac09] Jacob Lurie. $(\infty,2)$-categories and the goodwillie calculus i. https://www.math.ias.edu/~lurie/papers/GoodwillieI.pdf, 2009.

[Jac17] Jacob Lurie. Higher algebra. `https://www.math.ias.edu/~lurie/papers/HA.pdf`, 2017.

[Jor21] D. Jordan. Topological Field Theories, Lecture Notes of the Working Seminar at The Hodge Institute. `https://www.maths.ed.ac.uk/~djordan/lecture_notes_tft_seminar.pdf`, 2021.

[Kap95] M. M. Kapranov. *Analogies between the Langlands Correspondence and Topological Quantum Field Theory*, pages 119–151. Birkhäuser Boston, Boston, MA, 1995.

[Kim16] Minhyong Kim. Arithmetic chern-simons theory i, 2016.

[KLF09] Alexei Kitaev, Vladimir Lebedev, and Mikhail Feigel'man. Periodic table for topological insulators and superconductors. In *AIP Conference Proceedings*. AIP, 2009.

[Koc03] Joachim Kock. *Frobenius Algebras and 2-D Topological Quantum Field Theories*. London Mathematical Society Student Texts. Cambridge University Press, 2003.

[Kos13] A.A. Kosinski. *Differential Manifolds*. Dover Books on Mathematics. Dover Publications, 2013.

[KRT97] Christian Kassel, Marc Rosso, and Vladimir Turaev. Quantum groups and knot invariants. *HAL*, 1997(0), 1997.

[KW07] Anton Kapustin and Edward Witten. Electric-magnetic duality and the geometric langlands program, 2007.

[Lan71] Saunders Mac Lane. *Categories for the Working Mathematician*. Springer New York, New York, NY, 1971.

[Lan21] M. Land. *Introduction to Infinity-Categories*. Compact Textbooks in Mathematics. Springer International Publishing, 2021.

[Lee12] John M. Lee. *Smooth Manifolds*. Springer New York, New York, NY, 2012.

[Liu02] Q. Liu. *Algebraic Geometry and Arithmetic Curves*. Oxford graduate texts in mathematics. Oxford University Press, 2002.

[LP08] Aaron D. Lauda and Hendryk Pfeiffer. Open–closed strings: Two-dimensional extended tqfts and frobenius algebras. *Topology and its Applications*, 155(7):623–666, March 2008.

[Luk20] Luke Trujillo. A coherent proof of mac lane's coherence theorem. `https://scholarship.claremont.edu/hmc_theses/243/`, 2020.

[Lur08] Jacob Lurie. Higher topos theory, 2008.

[Lur09] Jacob Lurie. On the classification of topological field theories, 2009.

[Max21] Maxine E. Calle. A bit about infinite loop spaces. `https://bpb-us-w2.wpmucdn.com/web.sas.upenn.edu/dist/0/713/files/2022/07/Infinite_Loop_Spaces.pdf`, 2021.

[Maz73] Barry Mazur. Notes on étale cohomology of number fields. *Annales Scientifiques De L Ecole Normale Superieure*, 6:521–552, 1973.

[MF88] Atiyah Michael F. Topological quantum field theory. *Publication Mathematiques de l'I.H.É.S.*, 1988.

[MG24] Aaron Mazel-Gee. *Higher Algebra: Chapter 0*. Cambridge University Press, 2024?

[Mor11] Jack Morava. The cosmic galois group as koszul dual to waldhausen's a(pt), 2011.

[MRR23] Lyne Moser, Nima Rasekh, and Martina Rovelli. (∞, n)-limits i: Definition and first consistency results, 2023.

[MS05] John W. Milnor and James D. Stasheff. *Characteristic classes*. Texts Read. Math. New Delhi: Hindustan Book Agency, 2005.

[Mul97] F.A. Muller. The equivalence myth of quantum mechanics —part i. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 28(1):35–61, 1997.

[MW97] J.W. Milnor and D.W. Weaver. *Topology from the Differentiable Viewpoint*. Princeton Landmarks in Mathematics and Physics. Princeton University Press, 1997.

[Mü20] Lukas Müller. Extended functorial field theories and anomalies in quantum field theories, 2020.

[Rez21] Charles Rezk. Introduction to quasicategories, 2021.

[Rie17] E. Riehl. *Category theory in context*. Aurora: Dover modern math originals. Dover Publications, 2017.

[Rok19] Rok Gregoric. Spectra are your friends. `https://web.ma.utexas.edu/users/gregoric/Spectra%20Are%20Your%20Friends.pdf`, 2019.

[Run] Ingo Runkel. Algebra in braided tensor categories and conformal field theory. `https://www.math.uni-hamburg.de/home/runkel/PDF/alg.pdf`.

[Sch14] Claudia Isabella Scheimbauer. *Factorization Homology as a Fully Extended Topological Field Theory*. PhD thesis, Zurich, ETH, 2014.

[Sch23] Christoph Schweigert. Hopf algebras, quantum groups and topological field theory. `https://www.math.uni-hamburg.de/home/schweigert/skripten/hskript.pdf`, 2023.

[Shu08] Michael A. Shulman. Set theory for category theory, 2008.

[Shu10] Michael A. Shulman. Constructing symmetric monoidal bicategories, 2010.

[SP14] Christopher J. Schommer-Pries. The classification of two-dimensional extended topological field theories, 2014.

[Str22] Catharina Stroppel. Categorification: tangle invariants and tqfts, 2022.

[Tan20] Hiro Lee Tanaka. *Lectures on Factorization Homology, (∞, n)-Categories, and Topological Field Theories*. Springer International Publishing, 2020.

[Tan22] (Notes written by Peter Haine) Tanaka, Hiro Lee. Viva fukaya!, January 2022.

[Tin15] Peter Tingley. A minus sign that used to annoy me but now i know why it is there, 2015.

[TS04] Peter Teichner and Stephan Stolz. What is an elliptic object? *Topology, geometry and quantum field theory, 247-343 (2004)*, 308, 06 2004.

[Tur] Vladimir G. Turaev. Modular categories and 3-manifold invariants. *International Journal of Modern Physics B; (United States)*, 6.

[Tur16] Vladimir G. Turaev. *Quantum Invariants of Knots and 3-Manifolds*. De Gruyter, Berlin, Boston, 2016.

[Wei94] C.A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.

[Wit89] Edward Witten. Quantum field theory and the jones polynomial. *Communications in Mathematical Physics*, 121(3):351–399, Sep 1989.

[Wit91] Edward Witten. Introduction to cohomological field theories. *Int. J. Mod. Phys. A*, 6:2775–2792, 1991.