Lecture 2.1 (Super) Lie algebras

Last time:

Tried to write super versions of various mathematical objects that are physically relevant. In particular: super vector spaces, super hilbert space, super algebras, super commutator ($[A, B] = AB - (-)^{\sigma_A \sigma_B} BA$, with $\sigma = 0$ for bosons and 1 for fermions and $(-)^{\sigma_A \sigma_B}$ is the Koszul sign). The super commutator then satisfies the super Jacobi identity:

$$(-)^{\sigma_A \sigma_C}[[A, B], C] + (-)^{\sigma_A \sigma_B}[[B, C], A] + (-)^{\sigma_C \sigma_B}[[C, A], B] \tag{1}$$

This time: The (super) Jacobi identity for (super) Lie algebras

First of all, why is the Jacobi identity important? Very often it happens that I have the concept of a bracket but I don't have a product, this structure is called a Lie algebra. It's important to note in fact that Lie algebras are not algebras.

Example: sl(2), three dimensional. Can think of it as spanned by the Pauli matrices. We have $[L_x, L_y] = L_z$ and cyclic permutations of that. (written like this it's real Lie algebra)

I can represent that algebra by real matrices. Classically $\vec{L} = \vec{r} \times \vec{p}$, so we have

$$L_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \tag{2}$$

$$L_y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \tag{3}$$

$$L_z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \tag{4}$$

The commutator tells us someting about rotations in space: the difference between doing first one rotation and then another or switching their order is just a rotation through a third axis. Let's calculate it explicitly:

$$L_x L_y = \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \tag{5}$$

$$= y\frac{\partial}{\partial x} + yz\frac{\partial^2}{\partial x \partial z} - z^2\frac{\partial^2}{\partial y \partial x} - yx\frac{\partial^2}{\partial z^2} + zx\frac{\partial^2}{\partial y \partial z} \tag{6}$$

$$L_y L_x = x\frac{\partial}{\partial y} + yz\frac{\partial^2}{\partial x \partial z} - z^2\frac{\partial^2}{\partial y \partial x} - yx\frac{\partial^2}{\partial z^2} + zx\frac{\partial^2}{\partial y \partial z} \tag{7}$$

so if i just apply one after the other I get something quite complicated, but taking the commutator I get again a vector field. This means that they're not an algebra but they're a Lie algebra.

So more abstractly I don't have a commutator but I have a bracket and it's important to make sure that it satisfies the Jacobi identity.

Definition 1. *A super Lie algebra is a super vector space g with a bracket operation $[\cdot, \cdot] : g \oplus g \to g$ that:*

1. preserves parity

2. is super antisymmetric

3. satisfies the super Jacobi identity

(i.e. exactly like a Lie algebra but with everything super and parity preservation)

Let's unpack that a bit. First of all, the bracket is actually three different maps:

$$g_0 \otimes g_0 \to g_0 g_0 \otimes g_1 \to g_1 g_1 \otimes g_1 \to g_0 \tag{8}$$

Let's take $g = g_0 \oplus g_1$ and let's consider the different cases of the commutator:

1. $+++$: g_0 is a normal Lie algebra

2. $++$-: $g, h \in g_0, \psi \in g_1$, so the super Jacobi is:

   $$[[g,h],\psi] + [[h,\psi],g] + [[\psi,g],h] = 0 \tag{9}$$

   or equivalently:

   $$[[g,h],\psi] = [g,[h,\psi]] - [h,[g,\psi]] \tag{10}$$

   i have two bosonic symmetries and I'm considering their action on the fermionic symmetries: on the right I'm considering the action of g and h on ψ taken in different orders and subtracting the two, on the left I'm saying that that is equal to the action of $[g,h]$ on ψ.

   $[g_0, g_1] \to g_1$ makes g_1 into a representation of g_0.

3. $+$ - -: $g \in g_0, \psi, \xi \in g_1$, so we have

   $$[[\psi,\xi],g] - [[\xi,g],\psi] + [[g,\psi],\xi] = 0 \tag{11}$$

   or also

   $$[g,[\psi,\xi]] = [\psi,[\xi,g]] + [[g,\psi],\xi] \tag{12}$$

   The bracket of fermions is compatible with the g_0 symmetry, so we're essentially in the situation of the representation of the tensor product $g_1 \otimes g_1$ and this relation is what we wrote for the representation of the tensor product.

4. - - -: $\psi, \xi, \lambda \in g_1$, so we have

   $$[[\psi,\xi],\lambda] + [[\xi,\lambda],\psi] + [[\lambda,\psi],\xi] = 0 \tag{13}$$

   equivalently:

   $$[[\psi,\psi],\psi] = 0, \quad \forall \psi \in g_1 \tag{14}$$

which would be trivial for bosons since already the commutator gives 0. Instead here $[\psi, \psi]$ is in general nontrivial, however it shouldn't act nontrivially on the fermion itself. Or said differently "no fermionic symmetry can only generate one bosonic symmetry but not more than that", or "a fermionic symmetry commutes with its own square".

What is a representation? When you think of symmetries, you can consider them as objects on their own, that's basically the idea of a group. Given a symmetry we can then ask what kind of objects have this symmetry: that's the idea of a representation. It basically tells us what kind of objects can I have in space. For example with systems of a certain spin we have that the same symmetry (rotations) is "represented" in different ways.

Definition 2. *A representation of a Lie algebra g on a vector space V is a map $\rho : g \otimes V \to V$ (equivalently, $g \to End(V)$) with the property*

$$\rho_g(\rho_h(v)) - \rho_h(\rho_g(v)) = \rho_{[g,h]}(v) \tag{15}$$

i.e. "The matrix $\rho_{[g,h]}$ is the commutator of the matrices ρ_g and ρ_h."

For example if we have a representation of V and of W, then for $V \otimes W$ we have $\rho_{V\otimes W} = \rho_V \otimes 1 + 1 \otimes \rho_W$.

Recipe: take a Lie algebra g_0, take a representation and call it g_1 and check that every symmetry commutes with its own square.

1 Four ways of looking at gl(1,1)

1. In general given V a vector space, $gl(V)=$ linear maps $V \to V =$ matrices acting in V. So $gl(n) = gl(C^n)$ or $gl(R^n)$. So then $gl(n|m) = gl(C^{n|m})$ with n dimensional even part and m dimensional odd.

   Take matrices:
   $$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{16}$$
   and we have $[E, F] = P$, $[R, E] = E$, $[R, F] = -F$ and P commutes with everything.

   In the bosonic case I could take $sl(2) \subseteq gl(2)$ which is the traceless part. Instead here the identity is on the RHS of a commutator, so I can't throw it away. Same goes for all the others. So $gl(1|1)$ is NOT a product, also it is not simple, there are no invariant subalgebras, so I can't take any quotients. Semisimple means if there are invariant subalgebras then....

   In susy qm we have that P is the hamiltonian, it generates time translations, but it has a fermionic square root, which is a general feature. But then the commutator that we need to check to have a super lie algebras is trivial, since translations commute with everything