Lecture 2.1 (Super) Lie algebras

Last time:

Tried to write super versions of various mathematical objects that are physically relevant. In particular: super vector spaces, super Hilbert space, super algebras and super commutator. Let us recall the definition of the supercommutator:

$$[A, B] = AB - (-)^{\sigma_A \sigma_B} BA \tag{0.1}$$

with $\sigma = 0$ for bosons and $\sigma = 1$ for fermions and $(-)^{\sigma_A \sigma_B}$ is the Koszul sign. The super commutator then satisfies the super Jacobi identity:

$$(-)^{\sigma_A \sigma_C}[[A, B], C] + (-)^{\sigma_A \sigma_B}[[B, C], A] + (-)^{\sigma_C \sigma_B}[[C, A], B] = 0 \tag{0.2}$$

This time:

1 The (super) Jacobi identity for (super) Lie algebras

First of all, why is the Jacobi identity important? Very often it happens that we have the concept of a bracket but we don't have a product, this structure is called a Lie algebra. It's important to note in fact that Lie algebras are not algebras.

Example. $\mathfrak{sl}(2)$: it's three dimensional, we can think of it as spanned by the Pauli matrices. We have $[L_x, L_y] = L_z$ and cyclic permutations of that. (written like this it's a real Lie algebra so I can represent it by real matrices)

Classically $\vec{L} = \vec{r} \times \vec{p}$, so (considering p as the generator of translations) we have

$$L_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \tag{1.1}$$

$$L_y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \tag{1.2}$$

$$L_z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \tag{1.3}$$

The commutator tells us someting about rotations in space: the difference between doing first a rotation A through one axis and then another B through another axis, or doing B and then A is just a rotation through the third axis. Let's calculate it explicitly:

$$L_x L_y = \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \tag{1.4}$$

$$= y\frac{\partial}{\partial x} + yz\frac{\partial^2}{\partial x \partial z} - z^2\frac{\partial^2}{\partial y \partial x} - yx\frac{\partial^2}{\partial z^2} + zx\frac{\partial^2}{\partial y \partial z} \tag{1.5}$$

$$L_y L_x = x\frac{\partial}{\partial y} + yz\frac{\partial^2}{\partial x \partial z} - z^2\frac{\partial^2}{\partial y \partial x} - yx\frac{\partial^2}{\partial z^2} + zx\frac{\partial^2}{\partial y \partial z} \tag{1.6}$$

so if i just apply one after the other I get something quite complicated, but taking the commutator

$$[L_x, L_y] = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = L_z \tag{1.7}$$

I get another rotation. This means that they're not an algebra but they're a Lie algebra. Actually this works more generally for vector fields: their product is not a vector field, but their commutator is again a vector field.

So more abstractly I don't have a commutator but I have a bracket and it's important to make sure that it satisfies the Jacobi identity.

Definition 1. A super Lie algebra is a super vector space $\mathfrak{g}$ with a bracket operation $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ that:

1. preserves parity
2. is super antisymmetric
3. satisfies the super Jacobi identity

(i.e. exactly like a Lie algebra but with everything super and parity preservation) Let's unpack that a bit. First of all, let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and note that the bracket is actually three different maps:

$$\mathfrak{g}_0 \otimes \mathfrak{g}_0 \to \mathfrak{g}_0 \tag{1.8}$$
$$\mathfrak{g}_0 \otimes \mathfrak{g}_1 \to \mathfrak{g}_1 \tag{1.9}$$
$$\mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathfrak{g}_0 \tag{1.10}$$

Now for the let's consider the different cases of the super Jacobi identity:

1. $+++$: $\mathfrak{g}_0$ is a normal Lie algebra
2. $++-$: $g, h \in \mathfrak{g}_0, \psi \in \mathfrak{g}_1$, so the super Jacobi is:

   $$[[g,h],\psi] + [[h,\psi],g] + [[\psi,g],h] = 0 \tag{1.11}$$

   or equivalently:

   $$[[g,h],\psi] = [g,[h,\psi]] - [h,[g,\psi]] \tag{1.12}$$

   i have two bosonic symmetries and I'm considering their action on the fermionic symmetries: on the right I'm considering the action of g and h on ψ taken in different orders and subtracting the two, on the left I'm saying that that is equal to the action of $[g,h]$ on ψ.

   Essentially this is saying that we have a representation of $\mathfrak{g}_0$ over $\mathfrak{g}_1$ (see appendix A.2): we have the map $\rho_{\mathfrak{g}_1} : \mathfrak{g}_0 \to \mathrm{End}(\mathfrak{g}_1)$ given by $g \mapsto [g,\cdot]$ and it's supposed to satisfy

   $$\rho_{[g,h]}(\psi) = \rho_g(\rho_h\psi) - \rho_h(\rho_g\psi) \tag{1.13}$$

in order to be a Lie algebra homomorphism. But this equality is exactly the one above.

3. $+--$: $g \in \mathfrak{g}_0, \psi, \xi \in \mathfrak{g}_1$, so we have

$$[[\psi, \xi], g] - [[\xi, g], \psi] + [[g, \psi], \xi] = 0 \tag{1.14}$$

or also

$$[g, [\psi, \xi]] = [\psi, [\xi, g]] + [[g, \psi], \xi] \tag{1.15}$$

The bracket of fermions is compatible with the $\mathfrak{g}_0$ symmetry, so we're essentially in the situation of the representation of the tensor product $\mathfrak{g}_1 \otimes \mathfrak{g}_1$ and this relation is what we wrote for the representation of the tensor product.

4. $---$: $\psi, \xi, \lambda \in \mathfrak{g}_1$, so we have

$$[[\psi, \xi], \lambda] + [[\xi, \lambda], \psi] + [[\lambda, \psi], \xi] = 0 \tag{1.16}$$

equivalently:

$$[[\psi, \psi], \psi] = 0, \quad \forall \psi \in g_1 \tag{1.17}$$

which would be trivial for bosons since already the commutator gives 0. Instead here $[\psi, \psi]$ is in general nontrivial, however it shouldn't act nontrivially on the fermion itself. Or said differently "no fermionic symmetry can only generate one bosonic symmetry but not more than that", or "a fermionic symmetry commutes with its own square".

Recipe: take a Lie algebra $\mathfrak{g}_0$, take a representation and call it $\mathfrak{g}_1$ and check that every symmetry commutes with its own square.

2 Four ways of looking at $\mathfrak{gl}(1,1)$

In general given V a vector space, $\mathfrak{gl}(V)$= linear maps $V \to V$ = matrices acting in V. So $\mathfrak{gl}(n) = \mathfrak{gl}(\mathbb{C}^n)$ or $\mathfrak{gl}(\mathbb{R}^n)$. So then $\mathfrak{gl}(n|m) = \mathfrak{gl}(C^{n|m})$ with n dimensional even part and m dimensional odd.

Take matrices:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad R = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.1}$$

and we have $[E, F] = P$, $[R, E] = E$, $[R, F] = -F$ and P commutes with everything.

In the bosonic case I could take $\mathfrak{sl}(2) \subseteq \mathfrak{gl}(2)$ which is the traceless part. Instead here the identity is on the RHS of a commutator, so I can't throw it away. Same goes for all the others. So $\mathfrak{gl}(1|1)$ is NOT a product, also it is not simple, there are no invariant subalgebras, so I can't take any quotients. Semisimple means if there are invariant subalgebras then....

In supersymmetric quantum mechanics we have that P is the hamiltonian, it generates time translations, but it has a fermionic square root, which is a general feature. But then the commutator that we need to check to have a super Lie algebras is trivial, since translations commute with everything.

Lecture 2.2 Four ways of looking at $\mathfrak{gl}(1|1)$

$\mathfrak{gl}(1|1)$ = matrices acting on a $1|1$ dimensional super vector space $V = V_0 \oplus V_1$ with a convenient basis written above.

four ways of looking at $\mathfrak{gl}(1|1)$:

1. linear trasformations of a $1|1$ dim vector space

2. differential operators on the odd line $\mathbb{C}^{0|1}$

3. the algebra of creation and annihilation operators for one fermionic state (we talked about the creation operators in general terms already). The Fock space is $\mathbb{C}^{1|1}$ and te singlle particle Hilbert space is $\mathbb{C}^{0|1}$. Essentially this says "$\mathfrak{gl}(1|1)$ is the simplest Clifford algebra".

4. the algebra of spacetime symmetries as "$N = 2$ supersymmetric quantum mechanics". Essentially this says "$\mathfrak{gl}(1|1)$ is the simplest supersymmetry algebra".

We'll also see that it's in some ways like representation theory of SU(2) but there will be some key differences: we'll have only two representations spin 0 and spin 1/2, in this sense it'll be simpler.

The logical order would be to do first 3. and then 4. because 3. introduces Clifford algebra which is needed for 4. But we'll start with 4. to examine the physical supersymmetric system and get a feel for it, to then go back to 3.

2.1 Linear algebra

In the normal case, $\mathfrak{gl}(2)$ is a product of Lie algebras:

- $\mathfrak{sl}(2)$ = traceless part
- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ = center (commutes with everything)

We can study them separately, find representations of the two factors and then use those representations to get representations of $\mathfrak{gl}(2)$.

In the super case this doesn't quite work. The trace should vanish on commutators $\mathrm{tr}([A, B]) = 0$, so the traceless matrices form a subalgebra.

We would like for the same to be true for the super trace, but $P = [E, F]$ is not traceless with the normal definition. So let's consider odd matrices and see the result:

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 & B' \\ C' & 0 \end{pmatrix} = \begin{pmatrix} BC' & 0 \\ 0 & CB' \end{pmatrix} \tag{2.2}$$

so the anticommutator (anti since we have two odd matrices) is

$$\begin{pmatrix} BC' + B'C & 0 \\ 0 & CB' + C'B \end{pmatrix} \tag{2.3}$$

so that suggests the following formula for the supertrace:

$$\mathrm{str} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{tr}(A) - \mathrm{tr}(D) = \mathrm{tr} \left((-)^F \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \tag{2.4}$$

where $(-)^F$ is the parity operator

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.5}$$

on the supervector space.

So now let $\mathfrak{sl}(1|1) = \{X \in \mathfrak{gl}(1|1) : \mathrm{str}(X) = 0\}$ which is spanned by E, F, P since $\mathrm{str}(R) = 1 \neq 0$. For now it looks similar to $\mathfrak{sl}(2)$ since we have three matrices, but it's different since we have the identity, meaning we still have a central element. While what is taken out, R, is NOT central.

Another element of superlinear algebra which is different is the transpose. So the key property of the trace was that it vanishes on commutators, what is then the key property of the transpose? Normally we have $(XY)^T = Y^T X^T$, but because we're moving the matrices past one another, then for the super transpose we need to insert a sign

$$(XY)^T = (-)^{\sigma_x \sigma_y} Y^T X^T \tag{2.6}$$

Let's take a product of odd matrices:

$$\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 & B' \\ C' & 0 \end{pmatrix} \right]^T = \begin{pmatrix} BC' & 0 \\ 0 & CB' \end{pmatrix}^T = \begin{pmatrix} C'^T B^T & 0 \\ 0 & B'^T C^T \end{pmatrix} \tag{2.7}$$

$$= - \begin{pmatrix} 0 & -C'^T \\ B'^T & 0 \end{pmatrix} \begin{pmatrix} 0 & -C^T \\ B^T & 0 \end{pmatrix} \tag{2.8}$$

We can see that the following choice works:

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -C^T \\ B^T & 0 \end{pmatrix} \tag{2.9}$$

But this seems arbitrary, I could have put the $-$ on B^T. To see why this is the better choice, let's examine another property which is what happens in the euclidean scalar product: $g(v, Xw) = g(X^T v, w)$. Here also we need a sign for the super case:

$$g(v, Xw) = (-)^{\sigma_v \sigma_X} g(X^T v, w) \tag{2.10}$$

Let's consider v and X odd and w even:

$$g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = C = C^T = -g\left(\begin{pmatrix} 0 & -C^T \\ B^T & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \tag{2.11}$$

So here we see that the above choice works.

Note: it's no longer true that $(X^T)^T = X$, I actually get a $-$, but applying it four times I get back the original matrix. This is typical of fermions! When doing a certain operation gives the identity in the normal case, in the case of fermions it gives a $-$.

2.2 Differential operators on the super line

Let's consider $\mathbb{C}^{0|1}$. How is it different from a line? We see it by considering polynomial functions on the line $V = \mathbb{C}^{1|0}$: we have $1, x, x^2, \cdots =: \mathbb{C}[x] =$ algebra of creation operators, considering x as a creation operator. But where does x live? Well it's a functional on a vector space so it lives in the dual. So the algebra is generated by $V^\vee$ (the dual vector space which is simply made up of the linear coordinates).

So now what are the functions on $V = \mathbb{C}^{0|1}$? Well we have the algebra of fermionic creation operators generated by $V^\vee$ ("Grassmann numbers"). It is two dimensional, I only have $1, \theta$ with $\theta^2 = 0$ by supercommutativity.

For the normal line, a "point on the line" is a map from functions to numbers $\mathbb{C}[x] \to \mathbb{C}$, it sends x to some number that is it's coordinate.

Instead, the odd line has only one point! We only have the map $\mathbb{C}[\theta] \to \mathbb{C}$ which sends θ to 0.

Recall: if I want a map $\mathbb{R}[x]/(x^2+1) \to \mathbb{R}$ there are no possibilities, since on the left $x^2 = -1$ and on the right nothing squares to -1. This is way of saying that the space on the left has "no real points" and the solution is to use complex points.

The same reasoning applies above! The odd line has only one *even* point, but we should actually be looking for odd points!

Now let's move to the concept of differential operators. We have $\frac{\partial}{\partial\theta}$. So in total we can construct the following:

$$1, \theta, \frac{\partial}{\partial\theta}, \theta\frac{\partial}{\partial\theta} \tag{2.12}$$

of which the first and the last are even while the middle ones are odd. These have the following commutators:

$$[1, \cdot] = 0 \tag{2.13}$$

$$\left[\frac{\partial}{\partial\theta}, \theta\right] = 1 = \left[\theta, \frac{\partial}{\partial\theta}\right] \tag{2.14}$$

$$\left[\theta\frac{\partial}{\partial\theta}, \theta\right] = \theta \tag{2.15}$$

$$\left[\theta\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right] = -\frac{\partial}{\partial\theta} \tag{2.16}$$

which are exactly the commutators we have on $\mathfrak{gl}(1|1)$ with the following identifications!

$$1 = P, \quad \theta = E, \quad \frac{\partial}{\partial\theta} = F, \quad \theta\frac{\partial}{\partial\theta} = R \tag{2.17}$$

2.3 Creation and annihilation

We can see the action of F:

$$F : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.18}$$

and I can rename the two states and call the first one $|0\rangle$ "vacuum" and call the second $|\psi\rangle$ "one particle fermionic state". We then have:

$$F : |0\rangle \mapsto |\psi\rangle \tag{2.19}$$

$$E : |\psi\rangle \mapsto |0\rangle\,, \quad |0\rangle \mapsto 0 \tag{2.20}$$

So E does the opposite of F. Instead $-R$ somehow "counts fermions", but in order to understand this we actually have to define

$$\tilde{R} = R - \frac{1}{2}P = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.21}$$

which has the same commutation relations as R since P is central. Now we have:

$$\tilde{R}\,|0\rangle = 0 \tag{2.22}$$

$$\tilde{R}\,|\psi\rangle = \tilde{R}F\,|0\rangle = (F\tilde{R} - F)\,|0\rangle = -F\,|0\rangle = -\psi \tag{2.23}$$

So $-\tilde{R}$ tells us that $|0\rangle$ has 0 fermionic particles and $|\psi\rangle$ has 1.
Lecture 3.1

A Useful definitions

A.1 Algebraic structures

Let us recall a few useful definitions.

Definition 2 (Field)**.** A field $\mathbb{F}$ is a set with two binary operations called addition and multiplication satisfying the following field axioms:

- Associativity of addition and multiplication
- Commutativity of addition and multiplication
- Additive and multiplicative identity (resp. 0 and 1)
- Additive inverse
- Multiplicative inverse for every element except 0
- Distributivity of multiplication over addition.

or, more simply:

- Abelian group under addition
- nonzero elements are an abelian group under multiplication
- Distributivity of multiplication over addition.

Given a field we can define a vector space over it:

Definition 3 (Vector space)**.** A vector space V over a field $\mathbb{F}$ is a set with two operations:

1. addition $+ : V \times V \to V$,
2. multiplication by a scalar $\cdot : \mathbb{F} \times V \to V$

such that it is an abelian group with respect to addition and has the following properties: $(x, y \in \mathbb{F}$, $\mathbf{v}, \mathbf{w} \in V)$

- $x \cdot (\mathbf{v} + \mathbf{w}) = x \cdot \mathbf{v} + x \cdot \mathbf{w}$
- $(x + y) \cdot \mathbf{v} = x \cdot \mathbf{v} + y \cdot \mathbf{v}$
- $(xy) \cdot (\mathbf{v} + \mathbf{w}) = x \cdot (y \cdot \mathbf{v})$
- $1 \cdot \mathbf{v} = \mathbf{v}$

However there are more general concepts which can be useful.

A ring generalizes the concept of field, without requiring commutativity and inverses of multiplication.

Definition 4 (Ring)**.** A ring is a set with two binary operations called addition and multiplication satisfying the following ring axioms:

- Abelian group under addition
- Semigroup under multiplication (i.e. is only associative)
- Distributivity of multiplication from left and right over addition.

In addition, if it has a multiplicative identity (i.e. it is a monoid under multiplication) it's called a ring with unity.

We also define *division ring* a ring in which every nonzero element has a multiplicative inverse (i.e. a field in which multiplication may be noncommutative).

Now, just as one can define vector spaces over fields, one can define the analogous but more general concept of modules over rings.

Definition 5 (Module)**.** A left module over a ring R consists of an abelian group M and a "scalar multiplication" between elements of R and M that gives another element in M with the following properties: $(r, s \in R,\ x, y \in M)$

- $r \cdot (x + y) = r \cdot x + r \cdot y$
- $(r + s) \cdot x = r \cdot x + s \cdot x$
- $(rs) \cdot x = r \cdot (s \cdot x)$
- $1 \cdot x = x$

Then, it can happen that in a vector space we have a natural concept of a product between vectors that gives another vector, for example the the vector product $\times$ in $\mathbb{R}^3$ or the obvious product in the vector space of polynomials. This additional operation gives rise to the concept of an algebra.

Definition 6 (Algebra)**.** An algebra is a vector space V with an additional operation (multiplication) $\cdot : V \times V \to V$ which is bilinear and associative.

Given an algebra we can then define the commutator

$$[A, B] = AB - BA \tag{A.1}$$

which satisfies

$$[A, B] = -[B, A] \tag{A.2}$$

and is another bilinear operation $[\cdot, \cdot] : V \times V \to V$. This then raises the question: is V with the operation $[\cdot, \cdot]$ also an algebra? But the answer is no since the operation is not associative. Instead we have:

$$[[A, B], C] = [A, [B, C]] + [B, [C, A]] \tag{A.3}$$

in which the last term ruins the associativity. Usually the equality above is written as:

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0 \tag{A.4}$$

which is called the Jacobi identity.

However such an operation appears often enough that the resulting structure deserves a name:

Definition 7 (Lie algebra)**.** A Lie algebra is a vector space with a bilinear operation $[\cdot, \cdot] : V \times V \to V$ with the following properties: $(A, B \in V)$

1. $[A, B] = -[B, A]$
2. $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$

i.e. it is antisymmetric and satisfies the Jacobi identity. The operation is generally called *bracket*, but if it is constructed as above it is usually called *commutator*.

Note that a Lie algebra is not in fact an algebra, as it is not associative. In fact the Jacobi identity somehow measures the non-associativity of the bracket.

Also note that a bilinear operation $[\cdot, \cdot] : V \times V \to V$ is the same thing as a *linear* operation from the tensor product $[\cdot, \cdot] : V \otimes V \to V$. This is just the universal property defining the tensor product.

All these structures have their *super* counterpart which is constructed with the following guidelines:

- add a Koszul sign every time one operator is passed after another,
- preserve parity.

A.2 Representation

Given a group, the concept of its representation arises.

But what is a representation? When you think of symmetries, you can consider them as objects on their own, that's basically the idea of a group. Given a symmetry we can then ask what kind of objects have this symmetry: that's the idea of a representation. It basically tells us what kind of objects can I have in space. For example with systems of a certain spin we have that the same symmetry (rotations) is "represented" in different ways.

We can therefore give the following definition.

Definition 8. A representation of a group G on a vector space V is a map $\rho : G \to \mathrm{Aut}(V)$, where the automorphism group of a vector space V, $\mathrm{Aut}(V)$, is the group of invertible linear maps from V to V. The map must have the following property:

$$\rho(g \cdot g') = \rho(g) \circ \rho(g') \tag{A.5}$$

where $\circ$ is simply the composition.

In other words, we need a group homeomorphism from g to $\mathrm{Aut}(V)$.

It is also useful to define the same concept for algebras and Lie algebras. The only difference is that we will have algebra homomorphisms and Lie algebra homomorphisms. Let us start by defining those.

Definition 9. An algebra homomorphism is a map $\phi : A \to B$ between algebras over a field $\mathbb{F}$ that satisfies: $(x, y \in A, k \in \mathbb{F})$

1. $\phi(kx) = k\phi(x)$
2. $\phi(x + y) = \phi(x) + \phi(y)$
3. $\phi(xy) = \phi(x)\phi(y)$

i.e. it is a linear map that preserves the product.

Definition 10. A Lie algebra homomorphism is a linear map $\phi : A \to B$ between Lie algebras over a field $\mathbb{F}$ that satisfies: $(x, y \in A, k \in \mathbb{F})$

$$\phi([x, y]_A) = [\phi(x)\phi(y)]_B \tag{A.6}$$

We can then note that given a vector space V, the space of linear maps from V to itself, $\mathrm{End}(V)$, naturally has a (Lie) algebra structure (with "product" given by composition and bracket given by the commutator). So the following definition makes sense.

Definition 11. A representation of a (Lie) algebra A on a vector space V is a (Lie) algebra homomorphism $\rho : A \to \mathrm{End}(V)$.

More explicitly, for Lie algebras we have

$$\rho_g(\rho_h(v)) - \rho_h(\rho_g(v)) = \rho_{[g,h]}(v) \tag{A.7}$$

i.e. "The matrix $\rho_{[g,h]}$ is the commutator of the matrices ρ_g and ρ_h."

Furthermore, the following remarks are useful in practical situations:

1. A map $\rho : A \to \mathrm{End}(A)$ is equivalent to a map $\rho : A \otimes V \to V$.
2. Given representations over vector spaces V and W it's possible to construct one over the duals, over the direct sum and over the tensor product. In particular for the tensor product we have:

$$\rho_{V\otimes W} = \rho_V \otimes 1 + 1 \otimes \rho_W \tag{A.8}$$