Supersymmetry lecture notes

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Lecture 1.1
Lecture 1.2
Maybe these will be filled in at some point
Lecture 2.1 (24/5) (Super) Lie algebras
Last time:

Tried to write super versions of various mathematical objects that are physically relevant. In particular: super vector spaces, super Hilbert space, super algebras and super commutator. Let us recall the definition of the supercommutator:

$$[A, B] = AB - (-)^{\sigma_A \sigma_B} BA \tag{0.1}$$

with $\sigma = 0$ for bosons and $\sigma = 1$ for fermions and $(-)^{\sigma_A \sigma_B}$ is the Koszul sign. The super commutator then satisfies the super Jacobi identity:

$$(-)^{\sigma_A \sigma_C}[[A, B], C] + (-)^{\sigma_A \sigma_B}[[B, C], A] + (-)^{\sigma_C \sigma_B}[[C, A], B] = 0 \tag{0.2}$$

This time:

1 The (super) Jacobi identity for (super) Lie algebras

First of all, why is the Jacobi identity important? Very often it happens that we have the concept of a bracket but we don't have a product, this structure is called a Lie algebra. It's important to note in fact that Lie algebras are not algebras.

Example. $\mathfrak{sl}(2)$: it's three dimensional, we can think of it as spanned by the Pauli matrices. We have $[L_x, L_y] = L_z$ and cyclic permutations of that. (written like this it's a real Lie algebra so I can represent it by real matrices)

Classically $\vec{L} = \vec{r} \times \vec{p}$, so (considering p as the generator of translations) we have

$$L_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \tag{1.1}$$

$$L_y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \tag{1.2}$$

$$L_z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \tag{1.3}$$

The commutator tells us someting about rotations in space: the difference between doing first a rotation A through one axis and then another B through another axis, or doing B and then A is just a rotation through the third axis.

Let's calculate it explicitly:

$$L_xL_y = \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \tag{1.4}$$

$$= y\frac{\partial}{\partial x} + yz\frac{\partial^2}{\partial x\partial z} - z^2\frac{\partial^2}{\partial y\partial x} - yx\frac{\partial^2}{\partial z^2} + zx\frac{\partial^2}{\partial y\partial z} \tag{1.5}$$

$$L_yL_x = x\frac{\partial}{\partial y} + yz\frac{\partial^2}{\partial x\partial z} - z^2\frac{\partial^2}{\partial y\partial x} - yx\frac{\partial^2}{\partial z^2} + zx\frac{\partial^2}{\partial y\partial z} \tag{1.6}$$

so if i just apply one after the other I get something quite complicated, but taking the commutator

$$[L_x, L_y] = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = L_z \tag{1.7}$$

I get another rotation. This means that they're not an algebra but they're a Lie algebra. Actually this works more generally for vector fields: their product is not a vector field, but their commutator is again a vector field.

So more abstractly I don't have a commutator but I have a bracket and it's important to make sure that it satisfies the Jacobi identity.

Definition 1. A super Lie algebra is a super vector space $\mathfrak{g}$ with a bracket operation $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ that:

1. preserves parity
2. is super antisymmetric
3. satisfies the super Jacobi identity

(i.e. exactly like a Lie algebra but with everything super and parity preservation) Let's unpack that a bit. First of all, let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and note that the bracket is actually three different maps:

$$\mathfrak{g}_0 \otimes \mathfrak{g}_0 \to \mathfrak{g}_0 \tag{1.8}$$

$$\mathfrak{g}_0 \otimes \mathfrak{g}_1 \to \mathfrak{g}_1 \tag{1.9}$$

$$\mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathfrak{g}_0 \tag{1.10}$$

Now for the let's consider the different cases of the super Jacobi identity:

1. $+++$: $\mathfrak{g}_0$ is a normal Lie algebra
2. $++-$: $g, h \in \mathfrak{g}_0, \psi \in \mathfrak{g}_1$, so the super Jacobi is:

$$[[g,h],\psi] + [[h,\psi],g] + [[\psi,g],h] = 0 \tag{1.11}$$

or equivalently:

$$[[g,h],\psi] = [g,[h,\psi]] - [h,[g,\psi]] \tag{1.12}$$

i have two bosonic symmetries and I'm considering their action on the fermionic symmetries: on the right I'm considering the action of g and h on ψ taken in different orders and subtracting the two, on the left I'm saying that that is equal to the action of $[g,h]$ on ψ.

Essentially this is saying that we have a representation of $\mathfrak{g}_0$ over $\mathfrak{g}_1$ (see appendix A.2): we have the map $\rho_{\mathfrak{g}_1} : \mathfrak{g}_0 \to \mathrm{End}(\mathfrak{g}_1)$ given by $g \mapsto [g, \cdot]$ and it's supposed to satisfy

$$\rho_{[g,h]}(\psi) = \rho_g(\rho_h \psi) - \rho_h(\rho_g \psi) \tag{1.13}$$

in order to be a Lie algebra homomorphism. But this equality is exactly the one above.

3. $+--$: $g \in \mathfrak{g}_0, \psi, \xi \in \mathfrak{g}_1$, so we have

$$[[\psi,\xi],g] - [[\xi,g],\psi] + [[g,\psi],\xi] = 0 \tag{1.14}$$

or also

$$[g,[\psi,\xi]] = [\psi,[\xi,g]] + [[g,\psi],\xi] \tag{1.15}$$

The bracket of fermions is compatible with the $\mathfrak{g}_0$ symmetry, so we're essentially in the situation of the representation of the tensor product $\mathfrak{g}_1 \otimes \mathfrak{g}_1$ and this relation is what we wrote for the representation of the tensor product.

4. $---$: $\psi, \xi, \lambda \in \mathfrak{g}_1$, so we have

$$[[\psi,\xi],\lambda] + [[\xi,\lambda],\psi] + [[\lambda,\psi],\xi] = 0 \tag{1.16}$$

equivalently:

$$[[\psi,\psi],\psi] = 0, \quad \forall \psi \in g_1 \tag{1.17}$$

which would be trivial for bosons since already the commutator gives 0. Instead here $[\psi,\psi]$ is in general nontrivial, however it shouldn't act nontrivially on the fermion itself. Or said differently "no fermionic symmetry can only generate one bosonic symmetry but not more than that", or "a fermionic symmetry commutes with its own square".

Recipe: take a Lie algebra $\mathfrak{g}_0$, take a representation and call it $\mathfrak{g}_1$ and check that every symmetry commutes with its own square.

2 Four ways of looking at $\mathfrak{gl}(1,1)$

In general given V a vector space, $\mathfrak{gl}(V)$= linear maps $V \to V$ = matrices acting in V. So $\mathfrak{gl}(n) = \mathfrak{gl}(\mathbb{C}^n)$ or $\mathfrak{gl}(\mathbb{R}^n)$. So then $\mathfrak{gl}(n|m) = \mathfrak{gl}(C^{n|m})$ with n dimensional even part and m dimensional odd.

Take matrices:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad R = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.1}$$

and we have $[E, F] = P$, $[R, E] = E$, $[R, F] = -F$ and P commutes with everything.

In the bosonic case I could take $\mathfrak{sl}(2) \subseteq \mathfrak{gl}(2)$ which is the traceless part. Instead here the identity is on the RHS of a commutator, so I can't throw it away. Same goes for all the others. So $\mathfrak{gl}(1|1)$ is NOT a product, also it is not simple, there are no invariant subalgebras, so I can't take any quotients. Semisimple means if there are invariant subalgebras then....

In supersymmetric quantum mechanics we have that P is the hamiltonian, it generates time translations, but it has a fermionic square root, which is a general feature. But then the commutator that we need to check to have a super Lie algebras is trivial, since translations commute with everything.

Lecture 2.2 (26/5) Four ways of looking at $\mathfrak{gl}(1|1)$

$\mathfrak{gl}(1|1)$ = matrices acting on a $1|1$ dimensional super vector space $V = V_0 \oplus V_1$ with a convenient basis written above.

four ways of looking at $\mathfrak{gl}(1|1)$:

1. linear trasformations of a $1|1$ dim vector space
2. differential operators on the odd line $\mathbb{C}^{0|1}$
3. the algebra of creation and annihilation operators for one fermionic state (we talked about the creation operators in general terms already). The Fock space is $\mathbb{C}^{1|1}$ and te singlle particle Hilbert space is $\mathbb{C}^{0|1}$. Essentially this says "$\mathfrak{gl}(1|1)$ is the simplest Clifford algebra".
4. the algebra of spacetime symmetries as "$\mathcal{N} = 2$ supersymmetric quantum mechanics". Essentially this says "$\mathfrak{gl}(1|1)$ is the simplest supersymmetry algebra".

We'll also see that it's in some ways like representation theory of SU(2) but there will be some key differences: we'll have only two representations spin 0 and spin 1/2, in this sense it'll be simpler.

The logical order would be to do first 3. and then 4. because 3. introduces Clifford algebra which is needed for 4. But we'll start with 4. to examine the physical supersymmetric system and get a feel for it, to then go back to 3.

2.1 Linear algebra

In the normal case, $\mathfrak{gl}(2)$ is a product of Lie algebras:

- $\mathfrak{sl}(2)$ = traceless part

- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ = center (commutes with everything)

We can study them separately, find representations of the two factors and then use those representations to get representations of $\mathfrak{gl}(2)$.

In the super case this doesn't quite work. The trace should vanish on commutators $\mathrm{tr}([A,B]) = 0$, so the traceless matrices form a subalgebra.

We would like for the same to be true for the super trace, but $P = [E,F]$ is not traceless with the normal definition. So let's consider odd matrices and see the result:

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 & B' \\ C' & 0 \end{pmatrix} = \begin{pmatrix} BC' & 0 \\ 0 & CB' \end{pmatrix} \tag{2.2}$$

so the anticommutator (anti since we have two odd matrices) is

$$\begin{pmatrix} BC' + B'C & 0 \\ 0 & CB' + C'B \end{pmatrix} \tag{2.3}$$

so that suggests the following formula for the supertrace:

$$\mathrm{str} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{tr}(A) - \mathrm{tr}(D) = \mathrm{tr} \left((-)^F \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \tag{2.4}$$

where $(-)^F$ is the parity operator

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.5}$$

on the supervector space.

So now let $\mathfrak{sl}(1|1) = \{X \in \mathfrak{gl}(1|1) : \mathrm{str}(X) = 0\}$ which is spanned by E, F, P since $\mathrm{str}(R) = 1 \neq 0$. For now it looks similar to $\mathfrak{sl}(2)$ since we have three matrices, but it's different since we have the identity, meaning we still have a central element. While what is taken out, R, is NOT central.

Another element of superlinear algebra which is different is the transpose. So the key property of the trace was that it vanishes on commutators, what is then the key property of the transpose? Normally we have $(XY)^T = Y^T X^T$, but because we're moving the matrices past one another, then for the super transpose we need to insert a sign

$$(XY)^T = (-)^{\sigma_x \sigma_y} Y^T X^T \tag{2.6}$$

Let's take a product of odd matrices:

$$\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 & B' \\ C' & 0 \end{pmatrix} \right]^T = \begin{pmatrix} BC' & 0 \\ 0 & CB' \end{pmatrix}^T = \begin{pmatrix} C'^T B^T & 0 \\ 0 & B'^T C^T \end{pmatrix} \tag{2.7}$$

$$= - \begin{pmatrix} 0 & -C'^T \\ B'^T & 0 \end{pmatrix} \begin{pmatrix} 0 & -C^T \\ B^T & 0 \end{pmatrix} \tag{2.8}$$

We can see that the following choice works:

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -C^T \\ B^T & 0 \end{pmatrix} \tag{2.9}$$

But this seems arbitrary, I could have put the $-$ on B^T. To see why this is the better choice, let's examine another property which is what happens in the euclidean scalar product: $g(v, Xw) = g(X^T v, w)$. Here also we need a sign for the super case:

$$g(v, Xw) = (-)^{\sigma_v \sigma_X} g(X^T v, w) \tag{2.10}$$

Let's consider v and X odd and w even:

$$g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = C = C^T = -g\left(\begin{pmatrix} 0 & -C^T \\ B^T & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \tag{2.11}$$

So here we see that the above choice works.

Note: it's no longer true that $(X^T)^T = X$, I actually get a $-$, but applying it four times I get back the original matrix. This is typical of fermions! When doing a certain operation gives the identity in the normal case, in the case of fermions it gives a $-$.

2.2 Differential operators on the super line

Let's consider $\mathbb{C}^{0|1}$. How is it different from a line? We see it by considering polynomial functions on the line $V = \mathbb{C}^{1|0}$: we have $1, x, x^2, \cdots =: \mathbb{C}[x] =$ algebra of creation operators, considering x as a creation operator. But where does x live? Well it's a functional on a vector space so it lives in the dual. So the algebra is generated by $V^\vee$ (the dual vector space which is simply made up of the linear coordinates).

So now what are the functions on $V = \mathbb{C}^{0|1}$? Well we have the algebra of fermionic creation operators generated by $V^\vee$ ("Grassmann numbers"). It is two dimensional, I only have $1, \theta$ with $\theta^2 = 0$ by supercommutativity.

For the normal line, a "point on the line" is a map from functions to numbers $\mathbb{C}[x] \to \mathbb{C}$, it sends x to some number that is it's coordinate.

Instead, the odd line has only one point! We only have the map $\mathbb{C}[\theta] \to \mathbb{C}$ which sends θ to 0.

Recall: if I want a map $\mathbb{R}[x]/(x^2+1) \to \mathbb{R}$ there are no possibilities, since on the left $x^2 = -1$ and on the right nothing squares to -1. This is way of saying that the space on the left has "no real points" and the solution is to use complex points.

The same reasoning applies above! The odd line has only one *even* point, but we should actually be looking for odd points!

Now let's move to the concept of differential operators. We have $\frac{\partial}{\partial\theta}$. So in total we can construct the following:

$$1, \theta, \frac{\partial}{\partial\theta}, \theta\frac{\partial}{\partial\theta} \tag{2.12}$$

of which the first and the last are even while the middle ones are odd. These have the following commutators:

$$[1, \cdot] = 0 \tag{2.13}$$
$$\left[\frac{\partial}{\partial\theta}, \theta\right] = 1 = \left[\theta, \frac{\partial}{\partial\theta}\right] \tag{2.14}$$
$$\left[\theta\frac{\partial}{\partial\theta}, \theta\right] = \theta \tag{2.15}$$
$$\left[\theta\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right] = -\frac{\partial}{\partial\theta} \tag{2.16}$$

which are exactly the commutators we have on $\mathfrak{gl}(1|1)$ with the following identifications!

$$1 = P, \quad \theta = E, \quad \frac{\partial}{\partial\theta} = F, \quad \theta\frac{\partial}{\partial\theta} = R \tag{2.17}$$

2.3 Creation and annihilation

We can see the action of F:

$$F : \begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}0\\1\end{pmatrix} \tag{2.18}$$

and I can rename the two states and call the first one $|0\rangle$ "vacuum" and call the second $|\psi\rangle$ "one particle fermionic state". We then have:

$$F : |0\rangle \mapsto |\psi\rangle \tag{2.19}$$
$$E : |\psi\rangle \mapsto |0\rangle, \quad |0\rangle \mapsto 0 \tag{2.20}$$

So E does the opposite of F. Instead $-R$ somehow "counts fermions", but in order to understand this we actually have to define

$$\tilde{R} = R - \frac{1}{2}P = \begin{pmatrix}0 & 0\\0 & -1\end{pmatrix} \tag{2.21}$$

which has the same commutation relations as R since P is central. Now we have:

$$\tilde{R}\,|0\rangle = 0 \tag{2.22}$$
$$\tilde{R}\,|\psi\rangle = \tilde{R}F\,|0\rangle = (F\tilde{R} - F)\,|0\rangle = -F\,|0\rangle = -\psi \tag{2.23}$$

So $-\tilde{R}$ tells us that $|0\rangle$ has 0 fermionic particles and $|\psi\rangle$ has 1.
Lecture 3.1/2 (3/5) Unitarity (and other things)

3 Supersymmetric Quantum Mechanics

3.1 Unitarity

Now we want to consider $\mathfrak{gl}(1|1)$ as $\mathcal{N} = 2$ supersymmetric quantum mechanics. But in order to have a quantum theory we need to have unitarity.

Let us first clear up the difference between observables and infinitesimal symmetries. The first are Hermitian operators, while infinitesimal symmetries are represented by antihermitian operator. And clearly I can get a Hermitian operator from an antihermitian one and vice versa, but the way of doing this is not unique: I can multiply by i or $-i$. Typically in physics I multiply the observable by i and then I exponentiate it to get a non infinitesimal symmetry, i.e. a unitary operator. We'll see that the relation between superhermitian and superantihermitian operators is not as simple. Let us first study an example.

Example (Real Lie algebras)**.** First of all note that the observables (e.g. Hermitian operators) form a real algebra, not a complex one. Symmetry algebras usually have a real form, normally dictated by unitarity.

Considering the typical example of rotations, the commutator of the observables is the following:

$$[L_i, L_j] = i\epsilon_{ijk}L_k \tag{3.1}$$

while for infinitesimal symmetries we have:

$$[R_i, R_j] = \epsilon_{ijk}R_k \tag{3.2}$$

but mathematically they're just different conventions and both are REAL Lie algebras. It's clear that there should be this difference: the commutator of Hermitian or antihermitian operators is antihermitian, so considering infinitesimal symmetries I'll just have another antihermitian operator on the right, while for observables i need a factor of i (or $-i$) in order to get an antihermitian operator from a hermitian one.

The same complex Lie algebra can have different real forms! Take for example $\mathfrak{su}(2) = \mathfrak{so}(3)$ represented by $\sigma_x, \sigma_y, \sigma_z$, but there's also $\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{so}(2,1)$ that can be represented by:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.3}$$

with commutators:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H \tag{3.4}$$

which look exactly like ladder operators. This means that if I can make a complex change of basis of L_x, L_y, L_z I could get the same algebra: I simply take $L_z, L_\pm = L_x \pm iL_y$. Note however that $\mathfrak{so}(2,1)$ is not compact, while $\mathfrak{so}(3)$ IS. (Essentially compact means if I do the non infinitesimal transformation I eventually get back to the same point, which is true for rotations, but not for boosts)

Before getting to a super Hilbert space and superhermitian matrices, let us recall the definition of Hilbert space

Definition 2. A Hilbert space is a complex vector space $\mathscr{H}$ with a Hermitian inner product $\langle\cdot,\cdot\rangle : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$ with the following properties:

1. linear in the second argument: $\langle u, \alpha v\rangle = \alpha\langle u, v\rangle$
2. conjugate symmetric: $\langle u, v\rangle = \overline{\langle v, u\rangle}$
3. positive definite: $\langle u, u\rangle > 0$

A non infinitesimal symmetry M preserves the inner product if

$$\langle Mu, Mv\rangle = \langle u, v\rangle \quad \forall u, v \in \mathscr{H} \tag{3.5}$$

Recalling the definition of the adjoint:

$$\langle u, Mv\rangle = \langle M^\dagger u, v\rangle \tag{3.6}$$

we can then write

$$\langle M^\dagger M u, v\rangle = \langle u, v\rangle \implies M^\dagger M = \mathbb{1} \tag{3.7}$$

which is what we call unitarity.

For infinitesimal symmetries we have $M = \mathbb{1} + \epsilon X$, which inserted above gives:

$$(\mathbb{1} + \epsilon X^\dagger)(\mathbb{1} + \epsilon X) = \mathbb{1} + \epsilon(X^\dagger + X) = \mathbb{1} \implies X^\dagger = -X \tag{3.8}$$

$$(\text{or } \langle u, Xv\rangle + \langle Xu, v\rangle = 0) \tag{3.9}$$

This is what was meant when we said that infinitesimal symmetries are represented by antihermitian operators.

Definition 3. A super Hilbert space structure is a super vector space $\mathscr{H} = \mathscr{H}_+ \oplus H_-$ with an inner product $g : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$ with the properties:

1. linear in the second argument: $g(u, \alpha v) = \alpha g(u, v)$
2. conjugate super symmetric: $g(u, v) = (-)^{|u||v|}\overline{g(v, u)}$
   ($|u| = 0$ for $u \in \mathscr{H}_+$, $|u| = 1$ for $u \in \mathscr{H}_-$)
3. positive definite over $\mathscr{H}_+$: $g(u, u) > 0$ for $u \in \mathscr{H}_+$

4. σ-positive-definite over $\mathscr{H}_-$: $i\sigma g(\psi, \psi) > 0$ for $\psi \in \mathscr{H}_-$ with $\sigma = \pm 1$ is a sign that we can choose.

But we should be worried since there's not a canonical choice for σ.

If $\left(\mathscr{H}, \langle , \rangle\right)$ is a Hibert space with a splitting $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$, then $\mathscr{H}$ with the inner product $g_\sigma(\cdot, \cdot) = \langle \cdot, \cdot \rangle_0 - i\sigma \langle \cdot, \cdot \rangle_1$ is a σ-positive-definite Hilbert space (I can also go in the opposite direction).

Following the normal case, we can then define a superantihermitian operator by

$$g(Xu, v) + (-)^{|X||u|} g(u, Xv) = 0 \tag{3.10}$$

and it may also be called an infinitesimal supersymmetry. By taking a basis for $\mathscr{H}$ we can write X as

$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{3.11}$$

by decomposing its action on $\mathscr{H}_+$ and on $\mathscr{H}_-$. Here $\alpha, \beta, \gamma, \delta$ are generally operators. We then have four cases to study for the superantihermitian matrix:

1. u and v even
$$\langle \alpha u, v \rangle_0 + \langle u, \alpha v \rangle_0 = 0 \tag{3.12}$$

2. ψ and ξ odd
$$-i\sigma (\langle \delta\psi, \xi \rangle_1 + \langle \psi, \delta\xi \rangle_1) = 0 \tag{3.13}$$
these two basically say that α and δ are antihermitian matrices, which is what we expected since they're the even parts which typically follow the normal rules.

3. ψ odd and u even:
$$\langle \beta\psi, u \rangle_0 - (-) i\sigma \langle \psi, \gamma u \rangle_1 = 0 \tag{3.14}$$
but they can also be in opposite order:
$$-i\sigma \langle \gamma u, \psi \rangle_1 + \langle u, \beta\psi \rangle_0 = 0 \tag{3.15}$$
In fact these are the same condition, since we can first rewrite the last equation as
$$\langle i\sigma\gamma u, \psi \rangle_1 + \langle \beta^\dagger u, \psi \rangle_1 = 0 \tag{3.16}$$
and then we get that
$$\beta^\dagger = -i\sigma\gamma \tag{3.17}$$
or equivalently
$$\gamma^\dagger = -i\sigma\beta \tag{3.18}$$

So this is what it means for a matrix to be super anti Hermitian, where it's important to note that β and γ are not independent.

Let us study the case $\mathfrak{sl}(1|1)$.

In particular we now know what it means to take all super unitary $1|1$ matrices, i.e. $\mathfrak{u}(1|1)$ starting from $\mathfrak{gl}(1|1,\mathbb{C})$. So it's relatively simple since $\alpha,\beta,\gamma,\delta$ are just numbers. In particular α and δ have to be anti Hermitian, which for numbers means they're purely imaginary $\alpha,\delta \in i\mathbb{R}$. We can then start writing a basis:

$$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{3.19}$$

while for γ and β we have $\bar{\gamma} = -i\sigma\beta$. We then get

$$q_1 := \begin{pmatrix} 0 & 1 \\ i\sigma & 0 \end{pmatrix}, \quad q_2 := \begin{pmatrix} 0 & i \\ \sigma & 0 \end{pmatrix} \tag{3.20}$$

we see that they have the following commutators:

$$[q_1, q_1] = 2q_1^2 = 2\sigma \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \tag{3.21}$$

$$[q_2, q_2] = 2q_2^2 \tag{3.22}$$

$$[q_1, q_2] = 0 \tag{3.23}$$

and these are different commutation relations than the ones we had for $\mathfrak{gl}(1|1)$, but I can make them the same by a complex change of coordinates $(q_1 \pm iq_2)$ which of course is not allowed in the real Lie algebra because it messes up the unitary structure. So this is its unitary real form, but as a complex Lie algebra it's the same one we were writing before.

Same idea as in $\mathfrak{su}(2)$! We first complexify it and study the representations of $\mathfrak{sl}(2,\mathbb{C})$ which is easier because we can build ladder operators. At the end of the day you then demand that the lowering operator is the adjoint of the raising operator which makes it a real representation of the thing you started with.

The reason it's called $\mathcal{N}=2$ supersymmetric quantum mechanics is that if we think of

$$p = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \tag{3.24}$$

as the Hamiltonian, then we now have 2 anticommuting square roots of the time translation operator. So $\mathcal{N}=2$ supersymmetric quantum mechanics is really just the unitary real form of $\mathfrak{gl}(1|1)$, but we have to be careful with the unitarity constraints. First, remember that an infinitesimal bosonic symmetry is represented by an antihermitian matrix and we could get an observable out of it by dividing by $\pm i$. So hopefully there's a similar way of associating an observable matrix to something that generates a fermionic symmetry.

If X is odd superantihermitian we have the constraint

$$\langle u, X\psi\rangle - i\sigma\langle Xu, \psi\rangle = 0 \tag{3.25}$$

and we want to use this to say that some matrix related to X is Hermitian (and therefore an observable). Let $\sigma = +1$ and let j be a square root of i (or $i\sigma$ more generally).

...

$$\langle u, X\psi\rangle = j^2\langle Xu, \psi\rangle \tag{3.26}$$

$$j^{-1}\langle u, X\psi\rangle = j\langle Xu, \psi\rangle \tag{3.27}$$

$$\langle u, j^{-1}X\psi\rangle = \langle j^{-1}Xu, \psi\rangle \tag{3.28}$$

where in the last step we used the fact that since j is a square root of i then its inverse is just its complex conjugate. So $j^{-1}X$ is Hermitian and observable.

So just as in the standard case there's a correspondance between generators of odd symmetries (i.e. superantihermitian things) and observables, this time with a fourth root of -1 instead of a square root. And this is why when you're not careful of thinking when an observable is an observable and when it's generating a symmetry you get confused.

So let's see what observables correspond to the two matrices we got before. Also, traditionally we use Q to represent supercharges, meaning Hermitian operators, so that's why before we used lowercase letters.

$$Q_1 := \frac{q_1}{j} = j^{-1}\begin{pmatrix} 0 & 1 \\ j^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & j^{-1} \\ j & 0 \end{pmatrix} \tag{3.29}$$

$$Q_2 := \frac{q_2}{j} = j^{-1}\begin{pmatrix} 0 & j^2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & j \\ j^{-1} & 0 \end{pmatrix} \tag{3.30}$$

Which are in fact Hermitian since $\bar{j} = j^{-1}$. If I square them I then get the identity, whereas if I squared the previous ones I would get i times the identity, but now the i is now absorbed in the Qs. This is why in the physics convention if I write down the structure constants for a real super Lie algebra then I get is for the bosonic things and no i for the fermionic things: commutators of Hermitian matrices are antihermitian, while anticommutators of Hermitian matrices are Hermitian. Also we get the weird result that the structure constants of a real super Lie algebra are all purely real or purely imaginary.

We can then pick $j = \frac{1}{\sqrt{2}}(1 + i)$ so we can write the Qs in terms of Pauli matrices (since every 2 by 2 Hermitian matrix is a real linear combination of Pauli matrices and since they have zero diagonal there's no σ_z):

$$Q_1 = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y) \tag{3.31}$$

$$Q_2 = \frac{1}{\sqrt{2}}(\sigma_x - \sigma_y) \tag{3.32}$$

So this is an explanation of why unitarity is strange for superalgebras: because it's the correspondance in what you mean with an observable and what you mean by a generator of a symmetry, which is more subtle than in the normal case.

Why did we choose $\sigma = +1$? Because we wanted the Hamiltonian to have a positve spectrum. We define

$$P := \frac{p}{i} = \mathbb{1} = Q_1^2 \tag{3.33}$$

This implies $P \geq 0$ since for a state $|u\rangle$ we have

$$\langle u| P |u\rangle = \langle u| Q_1^\dagger Q_1 |u\rangle = ||Q_1 |u\rangle ||^2 \geq 0 \tag{3.34}$$

So not only is P a operator with real eigenvalues, but also its real eigenvalues are the squares of the real eigenvalues of Q_1, and this is called the *BPS bound*. So $\sigma = +1$ is a convention to get the spectrum of P to be positive (doing the opposite would be like reversing the sign of the time).

Lecture 4.1 (8/5) Supersymmetric quantum mechanics

The following was the answer to a question. Why are tangent vectors the same as derivations? Let M be a space (manifold), a one parameter family of symmetries of M is a map $\phi : M \times \mathbb{R}_t \to M$ that maps $(p,t) \mapsto \phi_t p$. Such a map gives me a way of taking derivatives. When the space is more than one dimensional we need a way of knowing "how to move on the manifold", which is given by a vector. If f is a funciton on M, the "derivative of f along ϕ at p" is

$$X_\phi(f)(p) := \lim_{t\to 0} \frac{f(\phi_t(p)) - f(p)}{t} \tag{3.35}$$

This defines a derivation of the algebra of functions on M, meaning

$$X_\phi(fg) = X_\phi(f)g + fX_\phi(g) \tag{3.36}$$

(which is obeyed since it's just a normal derivative) But this is really defining a tangent vector at p.

That's why in general it's helpful to have multiple descriptions of the same object, for example one description may be more intuitive but another may be easier to generalize.

Good news: once you understand what it means to be unitary, you can then safely ignore it since there's a way of passing between symmetry generators and observables. It's fine then to just work with observables.

	Observables	Symmetry generators
Even	$P = \mathbb{1},\ R = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$p = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},\ r = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$
Odd	$Q_1 = \begin{pmatrix} 0 & j^{-1} \\ j & 0 \end{pmatrix},\ Q_2 = \begin{pmatrix} 0 & j \\ j^{-1} & 0 \end{pmatrix}$	$q_1 = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix},\ q_2 = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}$

To go from right to left, above I divide by i, below I divide by j.

3.2 Representation of a super Lie algebra

Let us now use the Hermitian operators. Computing a few commutators we get:

$$\{Q_1, Q_1\} = 2P \tag{3.37}$$

$$[R, Q_1] = \begin{pmatrix} 0 & j^{-1} \\ -j & 0 \end{pmatrix} = -iQ_2 \tag{3.38}$$

$$[R, Q_2] = +iQ_1 \tag{3.39}$$

so the commutators have an i while the anticommutators do not (If we chose the convention of working with the symmetry generators we would get real structure constants and the matrices would have a "nicer" form since there are no js. In the end though it's simply a matter of preference, we can just choose one and work with it but it's important to remember this difference).

If I want to represent a super Lie algebra I need to start with a representation of the bosonic part, which has two commuting Hermitian operators P, R. So we have joint eigenspaces labeled by two numbers (p, r), respectively *energy* and *R charge*. What do we do in representation theory? Take commuting operators and use other operators as ladder operators (e.g. take L_z and L^2 which commute and then $L_\pm = L_x \pm L_y$ as ladder operators).

Now in $\mathfrak{u}(1|1)$ we should think of raising and lowering operators in terms of the odd things

$$Q = \frac{1}{\sqrt{2}}(Q_1 + iQ_2) \tag{3.40}$$

$$Q^\dagger = \frac{1}{\sqrt{2}}(Q_1 - iQ_2) \tag{3.41}$$

which have commutators

$$\{Q, Q^\dagger\} = 2P \tag{3.42}$$

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0 \tag{3.43}$$

$$[Q, P] = 0 \implies p \text{ stays the same} \tag{3.44}$$

$$[R, Q] = -Q \implies r \text{ lowered by one} \tag{3.45}$$

so its similar to the situation with angular momentum.

So now lets work as in quantum mechanics: start with a spin, act with a lowering operator until I reach the null state and then I'll know how to construct all states.

Let's start with a state of "highest R charge" $|p, r, \pm\rangle$, meaning that if I try to increase it I get 0: $Q^\dagger |p, r, \pm\rangle = 0$. (We assume there is such a state since we're considering finite dimensional representations) Acting with Q instead gives:

$$Q |p, r, \pm\rangle = \alpha |p, r-1, \mp\rangle \tag{3.46}$$

$$Q^\dagger Q |p, r, \pm\rangle = \alpha Q^\dagger |p, r-1, \mp\rangle = \alpha\bar{\alpha} |p, r, \pm\rangle \tag{3.47}$$

$$\{Q^\dagger, Q\} |p, r, \pm\rangle = 2P |p, r, \pm\rangle = 2p |p, r, \pm\rangle \implies |\alpha|^2 = 2p \tag{3.48}$$

So we have $\alpha = \sqrt{2p}$ up to a phase which is not important as it can be absorbed into the operators. Also, we now have another confirmation that $p \geq 0$.

Q annihilates $|p, r, \pm\rangle$ if and only if $p = 0$. This is intereseting because it tells us that the spectrum of the Hamiltonian determines whether the representation is trivial or not. What do we get if we keep lowering?

$$Q |p, r-1, \mp\rangle = \frac{1}{\alpha} Q^2 |p, r, \pm\rangle = \frac{1}{2\alpha} \{Q, Q\} |p, r, \pm\rangle = 0 \tag{3.49}$$

So I can't lower more than once! We have exactly two kinds of representations:

1. "Short", one dimensional: $|p = 0, r, \pm\rangle$

2. "Long", two dimensional: $|p, r, \pm\rangle, |p, r-1, \mp\rangle$

So there's a lower bound of the energy which can call 0 and it can't be shifted because it's on the right side of the commutator, so it's really zero energy. We also have positive energy states which come in pairs of opposite parity.

If I wanted to calculate a partition function, only the vacuum would contribute, since there's always a boson and a fermion at the same energy level. So in some cases we only need zero energy states, i.e. states on the *BPS bound*.

3.3 Supersymmetric free particle

For a free particle on a line we have the Hilbert space $\mathscr{H} = L^2(\mathbb{R}) \ni \psi(x)$ and the Hamiltonian

$$P = -\frac{\partial^2}{\partial x^2} = \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^2 = \hat{p}^2 \tag{3.50}$$

simplest way to make it supersymmetric: tensor with $\mathbb{C}^{1|1}$. So we have $\mathscr{H}_0 = L^2(\mathbb{R})$ and $\mathscr{H}_1 = L^2(\mathbb{R})$ which can be interpreted as a particle with an internal switch so that it can be either bosonic or fermionic. An example of such a system is an hydrogen atom which can be either ionized or not. So this model is not completely unreasonable.

$$P = -\frac{\partial^2}{\partial x^2} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.51}$$

$$Q_1 = \frac{1}{i}\frac{\partial}{\partial x} \otimes \begin{pmatrix} 0 & j^{-1} \\ j & 0 \end{pmatrix} \tag{3.52}$$

$$Q_2 = \frac{1}{i}\frac{\partial}{\partial x} \otimes \begin{pmatrix} 0 & j \\ j^{-1} & 0 \end{pmatrix} \tag{3.53}$$

$$R = 1 \otimes \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.54}$$

Now recall $Q^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, so we have

$$Q^\dagger \begin{pmatrix} \psi(x) \\ \phi(x) \end{pmatrix} \sim \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ 0 \end{pmatrix}, \quad Q \begin{pmatrix} \psi(x) \\ \phi(x) \end{pmatrix} \sim \begin{pmatrix} 0 \\ \frac{\partial \psi}{\partial x} \end{pmatrix} \tag{3.55}$$

which gives the de Rham complex of $\mathbb{R}$.

Lecture 4.2 (10/5) Supersymmetric particles and interactions

So Q and $Q^\dagger$ in a different bases can be written as:

$$Q = \begin{pmatrix} 0 & 0 \\ \frac{\partial}{\partial x} & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & -\frac{\partial}{\partial x} \\ 0 & 0 \end{pmatrix} \tag{3.56}$$

while P and R are given by:

$$P = \begin{pmatrix} -\frac{\partial^2}{\partial x^2} & 0 \\ 0 & -\frac{\partial^2}{\partial x^2} \end{pmatrix}, \quad R = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.57}$$

And this is a unitary representation of $\mathfrak{u}(1|1)$, i.e. supersymmetric quantum mechanics with $\mathcal{N} = 2$.

3.4 General properties of supersymmetric systems

$\psi(x)$ is a map from position to "probability amplitudes". To do time dependent quantum mecanics we would instead need $\Psi(x,t)$. These two are simply related by the separation of variable method of solving differential equations.

$$\psi(x) = \sum_E c_E \left|E\right\rangle \implies \Psi(x,t) = \sum_E c_E \left|t\right\rangle e^{-iEt} \tag{3.58}$$

Classically the degree of freedom is $x(t)$. So thinking of this as a field theory, the x is the degree of freedom, the field, and the spacetime is whatever the field depends on, which here is just the time. So as a field teory it's only 0 + 1 dimensional (independently of how many components x has).

In general, supersymmetry algebras enhance the spacetime symmetries of a field theory.

For example in classical electromagnetism we have translations and Lorentz symmetries. The first is represented by $\mathbb{R}^{n-1,1}$, $n-1$ for the spatial dimensions and 1 for the time dimension (and generators will be the P_μ). Instead for the Lorentz symmetries we have $\mathrm{SO}(n-1,1)$ which are transformations that preserve the scalar product $ds^2 = -dt^2 + dx_1^2 + \dots dx_{n-1}^2$. Supersymmetry enhances the symmetries in two ways we will have a supersymmetry algebra that extends the Poincaré algebra and we will get R symmetries which commutes with the Ps.

$\mathfrak{u}(1|1)$ fits this pattern: the Lorentz part is SO(0,1) which is empty and for translations we have $\mathbb{R}^{0,1}$. These are extended and we have Q_1 and Q_2 and we also get R which commutes with translations but not with the Qs. We call this $\mathcal{N} = 2$ but we can get easily get any $\mathcal{N}$.

We can take no Lorentz symmetry, translations $\mathbb{R}^{0,1} = \langle P \rangle$ and now as many supercharges as we want: $Q_1, \dots, Q_{\mathcal{N}}$ with the following relations:

$$Q_i^2 = \frac{1}{2}\{Q_i, Q_i\} = P \tag{3.59}$$

$$\{Q_i, Q_j\} = 0, i \neq j \tag{3.60}$$

what structure do I have? I have $\mathbb{R}^{\mathcal{N}}$ with an inner product! $(\mathbb{R}^{\mathcal{N}}, g)$ a vector space with a Euclidean inner product which I interpret as a bracket. This may be a better description since it's basis independent. So now what is the R symmetry? It's always something that acts on the supercharges, but commutes with translations. Also we don't want the brackets to change, so we get $\mathrm{SO}(\mathcal{N})$.

3.5 Supersymmetric interacting particle

Can I make the free supersymmetric particle interacting? Normally I would turn on a potential term, so I would add something to the Hamiltonian and continue from there. However, since we want $\{Q, Q^\dagger\} = 2P$ we would also have to change the Qs and if we want to preserve supersymmetry their commutation relations shouldn't change:

$$\{Q, Q\} = 0 \tag{3.61}$$

$$Q^\dagger = (Q)^\dagger \tag{3.62}$$

$$\{Q^\dagger, Q^\dagger\} = 0 \tag{3.63}$$

$$\{Q, Q^\dagger\} = 2P \tag{3.64}$$

This suggests another way of proceding: instead of starting with a Hamiltonian I can do the opposite! Choose a Q that satisfies the first equation, then choose $Q^\dagger = (Q)^\dagger$ and define $P = \frac{1}{2}\{Q, Q^\dagger\}$. Then I'm done: that'll be a supersymmetric quantum mechanical theory. So essentially there's a one-to-one dictionary between $\mathcal{N} = 2$ susy QM (without R symmetry) and Hilbert spaces with an operator Q with $Q^2 = 0$. In order to also have the U(1) R symmetry we also need to specify the operator R, and in addition Q should have R charge -1.

It turns out that using this method for a given Q we get many systems, not just one. To see this let M be a self adjoint invertible operator. Then define $Q_M = M^{-1}QM$, so we have $Q_M^\dagger = MQ^\dagger M^{-1}$ and $Q^2 = 0$. So if I have one susy QM system I can obtain another like this. This is useful because I tipically don't want to change the kinetic term.

For the free superparticle: take $M = e^\lambda W(x)$, with $W(x)$ any real function (called *superpotential*) and λ a real coupling constant. Start with

$$Q = \begin{pmatrix} 0 & 0 \\ \frac{\partial}{\partial x} & 0 \end{pmatrix} \tag{3.65}$$

and then set

$$Q_M = e^{-\lambda W} \begin{pmatrix} 0 & 0 \\ \frac{\partial}{\partial x} & 0 \end{pmatrix} e^{\lambda W} = \begin{pmatrix} 0 & 0 \\ \frac{\partial}{\partial x} + \lambda W' & \end{pmatrix} \tag{3.66}$$

so for the adjoint we have

$$Q_M^\dagger = \begin{pmatrix} 0 & -\frac{\partial}{\partial x} + \lambda W' \\ 0 & 0 \end{pmatrix} \tag{3.67}$$

and the Hamiltonian is given by:

$$\begin{aligned} P_M &= \frac{1}{2}\{Q_M, Q_M^\dagger\} \\ &= \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \left(\frac{\partial}{\partial x} + \lambda W'\right)\left(-\frac{\partial}{\partial x} + \lambda W'\right) \end{pmatrix} + \begin{pmatrix} \left(-\frac{\partial}{\partial x} + \lambda W'\right)\left(\frac{\partial}{\partial x} + \lambda W'\right) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -\frac{\partial^2}{\partial x^2} + \lambda^2 W'^2 - \lambda W'' & 0 \\ 0 & -\frac{\partial^2}{\partial x^2} + \lambda^2 W'^2 + \lambda W'' \end{pmatrix} \end{aligned} \tag{3.68}$$

so we got a different, more interesting Hamiltonian! There is a difference in the bosonic and the fermionic part! Also, we know that although they are two different potentials they have the same spectrum! So W gives rise to two so called *partner potentials*:

$$V_\pm(x) = \lambda^2 W'^2 \pm \lambda W'' \tag{3.69}$$

which have the same spectrum, other than the vacua. It actually tells us soemthing about the NOT supersymmetric quantum mechanics potential!

Example. Let's take $W(x) = \frac{1}{2}x^2$, so $W'(x) = x$ and $W''(x) = 1$. The partner potentials are then

$$V_\pm = \lambda^2 x^2 \pm \lambda \tag{3.70}$$

and the Hamiltonian is

$$H_\pm = \frac{1}{2}(-\frac{\partial^2}{\partial x^2} + \lambda^2 x^2 \pm \lambda) = H_{\text{SHO}} \pm \frac{\lambda}{2} \tag{3.71}$$

where H_{SHO} is the Hamiltonian of the simple harmonic oscillator, for which we recall the spectrum is given by $\lambda(n + \frac{1}{2}), n \geq 0$.

Now we can ask: what is the bosonic ground state? To answer the question we need to solve the following differential equation:

$$Q \begin{pmatrix} \psi(x) \\ 0 \end{pmatrix} = 0 = \left(\frac{\partial}{\partial x} + \lambda x\right) \psi(x) \tag{3.72}$$

$$\frac{d\psi}{\psi} = -\lambda x dx \tag{3.73}$$

$$\ln \psi = -\frac{\lambda x^2}{2} \tag{3.74}$$

$$\psi(x) = e^{-\frac{\lambda x^2}{2}} \tag{3.75}$$

In general then the procedure to count the ground states is to solve

$$\left(\pm\frac{\partial}{\partial x} + \lambda W'(x)\right) \psi(x) = 0 \tag{3.76}$$

where $+$ is for the bosonic vacuum and $-$ is for the fermionic one.

Lecture 5.1 (15/5) Formulations of Supersymmetric particle mechanics, both classical and quantum

Last time: Hilbert-space formulation of SQM with superpotential interactions. We saw that the superpotential gives rise to two ordinary potentials given by

$$V_\pm(x) = \lambda^2 W'(x)^2 \pm \lambda W''(x) \tag{3.77}$$

So just as we had that supercharges (supersymmetry generators) were square roots of the time translation operator, we now have that the superpotential is kind of the square root of the potential. The Hilbert space space was then given by $\mathscr{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^{1|1}$ and the Hamiltonian was given by

$$\begin{pmatrix} -\frac{\partial^2}{\partial x^2} + V_+ & 0 \\ 0 & -\frac{\partial^2}{\partial x^2} + V_- \end{pmatrix} \tag{3.78}$$

Setting $\lambda = 1$ the supercharges were

$$Q = \begin{pmatrix} 0 & \frac{\partial}{\partial x} + W' \\ 0 & 0 \end{pmatrix} \tag{3.79}$$

$$Q^\dagger = \begin{pmatrix} 0 & 0 \\ -\frac{\partial}{\partial x} + W' & 0 \end{pmatrix} \tag{3.80}$$

Now, if we just took the bosonic part, i.e. we restricted to $\mathscr{H}_+$ the Hamiltonian would simply be

$$p^2 + V_+ \tag{3.81}$$

and we know that classically this corresponds to a phase space with coordinates (p, x) with Poisson brackets $\{p, x\} = 1$.

Instead, for the entire system there's a problem. Is this the quantization of any classical system? Secretly, for the purpose of interpretation we hope that there is a corresponding classical system.

This is a real issue. We can think up any Hilbert space and any Hamiltonian we want, but if we don't know what system this corresponds to it's not really useful. For a random example we could choose our Hilbert space to be $\mathbb{C}^7$ and our Hamiltonian to be some 7-by-7 Hermitian matrix, does that correspond to any classical system? Well we don't even know which things correspond to measurements I can make, I don't have a clear "position", "momentum" or something else.

Let's first review what is classical quantization so that we'll be prepared to quantize a system with both bosonic and fermionic observables.

3.6 Canonical quantization

In classical mechanics we also have a Lagrangian formulation:

$$S = \int dt \mathscr{L}, \quad \mathscr{L} = \dot{x}^2 - V_+(x) \tag{3.82}$$

and we also get the equation of motion of x from

$$\frac{\delta S}{\delta x(t)} = 0 \tag{3.83}$$

and in general the conjugate momentum p_x is given by

$$p_x(t) := \frac{\delta S}{\delta \dot{x}(t)} = \dot{x}(t) \tag{3.84}$$

Equivalently, one may use the Hamiltonian formulation, in which we have a phase space such as $\mathbb{R}^2 \ni (x, p)$, along with a symplectic form. The symplectic form then allows us to define a Poisson bracket such as $\{p, x\} = 1$, which is just a Lie bracket which also has a compatibility condition between the product and bracket operations: the bracket is also a derivation on both arguments

$$\{f, g\} = \{f, g\}h + g\{f, h\} \tag{3.85}$$

(there's no need to check the second argument since we have antisymmetry). Sometimes the Poisson bracket is written as

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} \tag{3.86}$$

To canonically quantize we use the Hamiltonian formalism and we have the following starting point:

- we have classical observables x and p and our measurements could be any function of these. For simplicity let's say our measurement is only polynomial in x and p so the space of possible measurements is $\mathbb{C}[x,p]$.
- these observables have a Poisson bracket $\{p,x\} = 1$. This specific bracket is enough to calculate all Poisson brackets in the algebra.

In order to quantize it we want the following things:

- a noncommutative algebra $A_\hbar$ such that $\lim_{\hbar\to 0} A_\hbar = A = \mathbb{C}[x,p]$, where we call $k : A_\hbar \to A$ the map that associates to an operator its *classical limit.*
- The operators should satisfy: $[\hat{f},\hat{g}] = i\hbar\widehat{\{f,g\}} + O(\hbar^2)$, $\forall \hat{f} \in A_\hbar$ with $k(\hat{f}) = f$. (The map $\hat{}$ need not be unique)
- the operators should act on a Hilbert space. (I don't want just an abstract algebra but I want operators on a Hilbert space)

Normally we just assume that we have all of this but that's not trivial. Essentially what we're doing is then looking for a representation of the noncommutative algebra. But as we know from representation theory, the representations are usually not unique.

The typical procedure is then the following:

1. pick a complete set of (Poisson)-commuting observables $\mathbb{C}[x]$
2. take the L^2 functions on the space they coordinatize, this is $\mathscr{H}$.
3. Then because the Poisson bracket is nondegenerate, x is supposed to have a partner and together they should generate the whole algebra of observables. The partners are called *conjugate momenta* p_x and are represented by $\frac{\hbar}{i}\frac{\partial}{\partial x}$.

but the procedure is clearly nonunique, for example we could have started with p instead of x. Which is actually something we often do and we obtain the momentum representation of the Hilbert space. I could have also chosen $p + x$ or many other things. So we need something extra to talk about *the* Hilbert space.

For another example, what if we started with L_x, L_y, L_z? Well we're essentially working with the Lie algebra $\mathfrak{so}(3)$ (or $\mathfrak{su}(2)$), for which we know that there are infinite representations, one for every spin: the trivial spin 0 representation; the spin $\frac{1}{2}$ representation; the vector, or spin 1, representation... And all these are inequivalent. So the fact of having a unique choice of Hilbert space when starting with the canonical commutation relations is astonishing.

Kind of amazingly, there is (up to equivalence) exactly one unitary irreducible representation of the canonical commutation relations. This is to emphasize that there is a big difference between the typical algebras we use in physics and the Poisson algebra we started with, since a lot of them have infinitely many representations while here we get only one. Also note that since we have the parameter $\hbar$ so this could hypotethically mean we have a $\mathfrak{u}(1)$ subalgebra, but we don't consider that since we assume $\hbar$ has some fixed numerical value.

The uniqueness of the irreducible representation of the Poisson algebra is the Stone-von Neumann theorem. This is kind of implicit in the way quantum mechanics is done. In particular we often say that the position and momentum representations are equivalent, which is true and the isomorphism is given by the Fourier transform. But we don't worry about checking if there are other representations and the reason is that we know there's a unique one.

This result is physically very deep because otherwise you might ask why aren't there different ways of quantizing with the same correspondence principle? A priori it is not at all clear that there shouldn't be multiple ways, but the fact that there is only one is very important. In particular going back to $\mathfrak{su}(2)$, the fact that it has multiple representations directly corresponds to the fact that the particle may have *internal degrees of freedom* that are related to angular momentum: I start with $\mathfrak{su}(2)$ being the classical algebra of angular momentum and I discover that there are in a sense "multiple ways of quantizing it" so the particle I'm considering will live in one of its representations and have this extra degree of freedom that is classically inexistent. Instead for the position degree of freedom this can't happen, so I can't have an *internal position* degree of freedom.

Here's the idea of the proof of the Stone-von Neumann theorem:

- The position operator is Hermitian so it has at least one eigenstate

$$\hat{x}\left|\lambda\right\rangle = \lambda\left|\lambda\right\rangle, \quad \text{for some } \lambda \in \mathbb{R} \tag{3.87}$$

- Then $\hat{x}e^{i\hat{p}a}\left|\lambda\right\rangle = (\lambda+a)e^{i\hat{p}a}\left|\lambda\right\rangle$, so $e^{i\hat{p}a}\left|\lambda\right\rangle = \left|\lambda+a\right\rangle$. Therefore there's one eigenvector for all positions in $\mathbb{R}$. That's already an irreducible subrepresentation isomorphic to $L^2(\mathbb{R})$, in the sense that $L^2(\mathbb{R})$ is spanned by the eigenstates of the position operator.

- By iterating this procedure for all eigenstates of $\hat{x}$ we find that the Hilbert space is just given by the direct sum of $L^2(\mathbb{R})$ with itself multiple times. So the only irreducible representation is in fact $L^2(\mathbb{R})$.

More details on canonical quantization, including a proof of the Stone-von Neumann theorem, can be found in [1].

3.7 Classical fermionic observables

So now we want to understand quantization in cases where I have both bosonic and fermionic observables. So we want to understand what algebra replaces the canonical commutation relations and then we want to study its representation theory in the hope that it's analogous to the bosonic case.

So now that we have both bosonic and fermionic observables, the algebra of classical observables should be super commuting and they should have a super Poisson bracket. We could simply try to write down such an algebra or we could try to imagine what a Lagrangian picture might look like. In particular we want that quantizing gives the Hilbert space $L^2(\mathbb{R}) \otimes C^{1|1}$. $L^2(\mathbb{R})$ came from quantizing $x(t) \in \mathrm{Map}(\mathbb{R} \to \mathbb{R}^{1|0})$, so a guess would be to think about $\theta(t) \in \mathrm{Map}(\mathbb{R} \to \mathbb{R}^{0|1})$, a function of the time which is odd, or a *Grassmann number*. For the first part we have

$$S = \int dt \frac{1}{2}\dot{x}^2 \tag{3.88}$$

but surely we can't have $\dot{\theta}^2$ since that would be zero. I also don't want higher derivatives otherwise the theory becomes weird. Instead we could have

$$S = \int dt \frac{1}{2}\theta\dot{\theta} \tag{3.89}$$

Why don't we write such a term for the even part? The reason is that such a term wouldn't contribute to the dynamics as it's a total derivative $x\dot{x} = \frac{1}{2}\frac{dx^2}{dt}$, while $\theta^2 = 0$ so $\theta\dot{\theta}$ is not the derivative of anything. Also, it's important to have something quadratic in θ otherwise we would have to integrate something odd.

Before we proceed, let's explain what is meant by observables. If we simply consider all paths in phase space (also called *histories*) then there are many things that could be measured: to start with we have position and momentum, all their powers and all their products; then we also have all derivatives as well, which gives rise to a very large set of possible measurements. However what we mean by observables is actually *on shell* observables, meaning observables for points in phase space along histories that satisfy the equations of motion. In that case for the lagrangian $\frac{1}{2}\dot{x}^2$ we don't care about derivatives higher than the second, since the equation of motion is $\ddot{x} = 0$. So the space of observables is much smaller. In particular since $\ddot{x} = 0$ if we plot x over t it would simply be a straight line, so it would be entirely determined by two values, which explains why the observables are simply x and p.

So if I want to understand the observables I should look at the equations of motion for θ:

$$\dot{\theta} = 0 \tag{3.90}$$

So the algebra of observables is simply $\mathbb{C}[\theta], \theta^2 = 0$. But normally the conjugate momentum is also an observable, so I should it as well. However we find that the conjugate momentum is

$$p_\theta = \frac{\delta \mathscr{L}}{\delta \dot{\theta}} - \frac{1}{2}\theta \tag{3.91}$$

so it's already included. In fact just like $\ddot{x} = 0$ leaves two degrees of freedom, $\dot{\theta} = 0$ leaves only one. Now the bracket structure can be constructed in analogy to $\{x, p\} = 1$: we would write $\{\theta, p_\theta\} = 1$, which recalling $p_\theta = -\frac{1}{2}\theta$ gives

$$\{\theta, \theta\} = -2 \tag{3.92}$$

But of course this bracket is not antisymmetric otherwise we would have $\{\theta, \theta\} = 0$. Instead it's super antisymmetric and it's easy to check that this gives a super Poisson algebra.

So what is this extra structure? Well, normally we have a symplectic phase space with position and momenta and a symplectic form which gives me the bracket. Instead now the symplectic form is not antisymmetric but super antisymmetric, which for the odd part means symmetric. So essentially it's just an inner product.

So just as in canonical quantization I start with an algebra, give it a bracket and then make it noncommuting according to the bracket, now I start with a Grassmann algebra and then make it noncommuting according to an inner product. This is the construction of a Clifford algebra. So just as canonical quantization corresponds to finding the representation of the Poisson algebra, to quantize a supersymmetric system we want to find representations of a Clifford algebra. In particular we'll find that the situation is almost as nice: namely we either have one or two possible representations.

Lecture 5.2 (17/5) Clifford algebras and the spinning particle

The following was the answer to a question.

$$\delta S = \frac{\partial \mathscr{L}}{\delta \theta}\delta\theta + \frac{\delta \mathscr{L}}{\delta \dot{\theta}}\delta\dot{\theta} = \text{EOM } \delta\theta + \text{total derivative} \tag{3.93}$$

$$= \left(\frac{\delta \mathscr{L}}{\delta \theta} - \frac{d}{dt}\frac{\delta \mathscr{L}}{\delta \dot{\theta}}\right)\delta\theta + \frac{d}{dt}\left(\frac{\delta \mathscr{L}}{\delta \dot{\theta}}\delta\theta\right) \tag{3.94}$$

$$\delta S = \int dt \left[\frac{1}{2}\delta\theta\dot{\theta} + \frac{1}{2}\theta\delta\dot{\theta}\right] = \int dt \left[\frac{1}{2}\delta\theta\dot{\theta} - \frac{1}{2}\dot{\theta}\delta\theta + \frac{1}{2}\frac{d}{dt}(\theta\delta\theta)\right] \tag{3.95}$$

$$= \int dt \left[\delta\theta(\dot{\theta}) - \frac{1}{2}\frac{d}{dt}(\theta\delta\theta)\right] \tag{3.96}$$

this last part is called the *variational 1-form*, the thing that tells you what is conjugate to what and gives you a symplectic form.

3.8 Quantization with fermionic observables

The quantization procedure should be clear by now: we start with a (super)commutative algebra with a (super)Poisson bracket and we construct a non(super)commutative algebra whose (super) commutators are given by the (super)Poisson brackets.

In the classical case we start with $\mathbb{C}[p,x]$ with $px = xp$ and $\{p,x\} = 1$. From this we get an algebra generated by $\hat{p}, \hat{x}$ with commutation relations $[\hat{p}, \hat{x}]_\hbar = -i\hbar$. This is called the *Weyl algebra*.

The (semi)classical algebra of observables is instead given by $\mathbb{C}[\theta]$, with $\theta^2 = 0$ and $\{\theta, \theta\} = -2$. Quantizing, we get the algebra generated by $\hat{\theta}$ with supercommutator $[\hat{\theta}, \hat{\theta}]_\hbar = 2\hat{\theta}^2 = -2\hbar$, meaning we now have $\hat{\theta}^2 = -\hbar$. This is called a *Clifford algebra*. Note that the i in the bosonic case was an artifact of the fact that we were taking commutators of Hermitian operators so the result was antihermitian, whereas now we have the anticommutator of Hermitian operators which is still Hermitian.

So normally we start with a Weyl algebra and then we look for a representation of it and we find there's only one possibility that leads to quantum mechanics. We now want to do the same for the Clifford algebra, which will give rise to spinor representations.

4 The supersymmetric (or *spinning*) classical particle

We can now put the ingredients together. We now have $(x(t), \theta(t)) \in \mathrm{Map}(\mathbb{R}_t \to R^{1|1})$ with the action

$$S = \int dt \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\theta\dot{\theta} \right) \tag{4.1}$$

Claim. This theory classically has $\mathcal{N} = 1$ supersymmetry.

Proof. I should therefore have one supercharge Q which is odd and has $Q^2 = P \propto \frac{\partial}{\partial t}$. Simply set Q to be the following odd transformations:

$$x \mapsto \theta \tag{4.2}$$

$$\theta \mapsto -\dot{x} \tag{4.3}$$

odd since it sends even fields to odd ones and viceversa. Let's check what happens if we apply it twice:

$$x \mapsto \theta \mapsto -\frac{\partial x}{\partial t} \tag{4.4}$$

$$\theta \mapsto -\dot{x} \mapsto -\frac{\partial \theta}{\partial t} \tag{4.5}$$

So we get $Q^2 = -\frac{\partial}{\partial t}$ which is what we wanted. What else do we need for it to be a classical symmetry? S should be invariant:

$$QS = \int dt \left[\dot{x}\dot{\theta} - \frac{1}{2}\dot{x}\dot{\theta} - \frac{1}{2}\theta(-\ddot{x}) \right] \tag{4.6}$$

in which the $-$ in the last expression is because Q moves past θ.

$$= \int dt \left[\frac{1}{2}\dot{x}\dot{\theta} + \frac{1}{2}\ddot{x}\theta\right] = \int dt \frac{d}{dt}\left[\frac{1}{2}\dot{x}\theta\right] \tag{4.7}$$

meaning $\frac{1}{2}\dot{x}\theta$ is the conserved current (also conserved charge since there are no spatial coordinates). So we have an odd charge that is the generator of the symmetry. □

However, the quantum system we were studying was an $\mathcal{N} = 2$ supersymmetric system. But it's not hard to obtain a supersymmetry with greater $\mathcal{N}$: since an odd variable gives me a way to get a square root of the time translation operator, I can just add more odd variables

$$S = \int dt \left(\frac{1}{2}\dot{x}^2 + \sum_{a=1}^{\mathcal{N}} \frac{1}{2}\theta_a\dot{\theta}_a\right) \tag{4.8}$$

Quantizing this system would then give $L^2(\mathbb{R})$ and an odd part.

If we want more spatial dimensions we could then give x an index i and then I have to give an index i to the θs. We could then get the most general such model: an "$\mathcal{N}$ extended supersymmetric particle in d dimensions":

$$S = \int dt \left(\sum_{i=1}^{d} \frac{1}{2}\dot{x}_i^2 + \sum_{a=1}^{\mathcal{N}} \frac{1}{2}\theta_{ia}\dot{\theta}_{ia}\right) \tag{4.9}$$

Note however that giving θ and x indices corresponds to increasing the dimension of the vector space they're in, and in that vector space we also want a way to multiply vectors to get a number, so we want them to have an inner product. In addition, we actually need two vector spaces: the one corresponding to the $\mathcal{N}$ supersymmetry $R = (\mathbb{R}^{\mathcal{N}}, h)$ (which is going to be the R symmetry space) and the space where the particle actually lives $V = (\mathbb{R}^d, g)$.

So a field is given by $(x_i, \theta_{ia}) \in \mathrm{Map}(\mathbb{R}_t \to V \oplus \Pi(V \otimes R))$, where Π is for the parity shift, in order for $V \otimes R$ to be odd. The action above can then be given in a basis independent form:

$$S = \int dt \left(\frac{1}{2}g(\dot{x}, \dot{x}) + \frac{1}{2}g \otimes h(\theta, \dot{\theta})\right) \tag{4.10}$$

where, to be clear, in coordinates $g \otimes h(\theta, \theta) = \theta_{ia}\theta_{jb}g^{ij}h^{ab}$. Now what is the supersymmetric algebra? $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where the first part is given by $\mathbb{R}_t$, $\mathrm{SO}(R)$ and $\mathrm{SO}(V)$ while the odd part is R.

If we go back to $\mathcal{N} = 1$

$$S = \int dt \left(\frac{1}{2}g(\dot{x}, \dot{x}) + \frac{1}{2}g(\theta, \dot{\theta})\right) \tag{4.11}$$

what phase space do we get? The bosonic part is T^*V with the normal symplectic form $dp^i \wedge dx_i$ which is invariant under g. For the fermionic part ΠV we need to be careful: the conjugate momentum of θ is given by:

$$p_\theta^i = \frac{\delta \mathscr{L}}{\delta \dot{\theta}_i} = -\frac{1}{2} g^{ij}\theta_j \tag{4.12}$$

so if we write the analogous symplectic form we get $p_\theta^i \wedge \theta_i = -\frac{1}{2}g^{ij}\theta_j \wedge \theta_i$ and we don't care about the constant in front so we have $g^{ij}d\theta_i \wedge d\theta_j$. While the bosonic form was independent of g, the fermionic one actually is g. So in the end we have

$$T^*V \oplus \Pi V, \text{ with symplectic form } \omega = dp^i \wedge dx_i + g^{ij} d\theta_i \wedge d\theta_j \tag{4.13}$$

which is actually symmetric in the odd part but that's okay since that's exactly what a super symplectic form should be. In particular $d\theta^i \wedge d\theta_j$ is not 0.

So when quantizing the algebra of observables is given by $\mathrm{Weyl}(T^*V) \otimes \mathrm{Clifford}(V, g)$, where $\mathrm{Weyl}(T^*V)$ is again the algebra generated by x_i and p_i with $[x_i, p_j] = -i\hbar\delta_{ij}$ and $\mathrm{Clifford}(V, g)$ is again the Clifford algebra but now with more generators θ_i and with $\{\theta_i, \theta_j\} = -g_{ij}$.

Now the Hilbert space is given by $L^2(V) \otimes S$ where S is somewhere the Clifford algebra acts, which is sort of an internal degree of freedom. In particular we'll find that S is a spinor of $\mathrm{SO}(V)$.

So what we're going to do is write down all the Clifford algebras and classify them and their representations. We'll find that there are either exactly one or exactly two irreducible representations.

The phase space in the most general case is given by

$$T^*V \oplus \Pi(V \otimes R), \text{ with symplectic form } \omega = dp^i \wedge dx_i + g \otimes h \tag{4.14}$$

The following was the answer to a question. The Poisson algebra can be obtained from the commutator of the quantum algebra of observables by taking a limit:

$$\hat{x}\hat{p} - \hat{p}\hat{x} = O(h) \tag{4.15}$$

$$\lim_{\hbar \to 0} \frac{\hat{x}\hat{p} - \hat{p}\hat{x}}{\hbar} = \text{ Poisson bracket} \tag{4.16}$$

So if you want the reason there's a Poisson bracket in classical mechanics is that classical mechanics has to sit in a family of noncommutative things called quantum mechanics and in such a family there's a first order deviation from being commutative and that's given by the Poisson bracket.

Lecture 6.1 (22/5) Clifford algebras and Clifford modules

5 Clifford algebras and Clifford modules

We now want give a definition of what a Clifford algebra is and then we want to classify all Clifford algebras. In order to give the definition we first need to define the tensor algebra $T(V)$, which is just the sum of all the tensor powers of V, or in other words it's a noncommutative analogue of the ring of polynomials.

$$T(V) = \bigoplus_{k\geq 0} V^{\otimes k} = k \oplus V \oplus (V \otimes V) \oplus \dots \tag{5.1}$$

Let's start with the bosonic case, i.e. the Weyl algebra. Let the phase space be $V = (\mathbb{R}^{2n}, \omega = dp^i \wedge dx_i)$, then $\mathrm{Weyl}(V) = {}^{T(V)}/_{(x\otimes y - y\otimes x - \pi(x,y)=0)} = {}^{\mathbb{C}\langle \hat{q}_i, \hat{p}_i\rangle}/_{[\hat{q}_i,\hat{p}_j]=\delta_{ij}}$. This is exactly the algebra we were describing when talking about canonical quantization: it's a noncommutative algebra whose commutator is given by the Poisson bracket $\{x, p\} = 1$.

The Clifford algebra is the odd analogue: we start with $V = (\Pi\mathbb{C}^n, g)$ and $\mathrm{Cl}(V) = {}^{T(V)}/_{(x\otimes y + y\otimes x - 2g(x,y)=0)}$. Let g be the standard bilinear product, then if $e_i, 1 \leq i \leq n$ is an orthonormal basis we have $g(e_i, e_j) = \delta_{ij}$. In this case then we're quotienting out by $e_i \otimes e_j + e_j \otimes e_i = -2\delta_{ij}$, which gives $e_i \otimes e_i = -1$ and $e_i \otimes e_j = -e_j \otimes e_i$ for $i \neq j$. We give a name to these algebras: $\mathrm{Cl}(n, \mathbb{C})$ = the algebra obtained from $\mathbb{C}$ with n anticommuting square roots of -1 (or equivalently $+1$ since $(ie_j) \otimes (ie_j) = +1$). From now on we'll omit the $\otimes$ symbol, so we just have $e_i^2 = 1$ and $e_ie_j = -e_je_i$.

The dimension of this space is 2^n since it's spanned by $1, e_i, e_ie_j, \dots$ meaning basically all the subsets of $\{e_1, e_2, \dots\}$ which are 2^n (the order of the e_is doesn't matter since we can anticommute them).

Claim. One square root of $+1$ it defines a pair of commuting projection operators i.e. $p\pm$ with $p_\pm^2 = p_\pm$.

Proof. We can simply define $p_\pm = \frac{1\pm e}{2}$ and also we have $p_\pm p_\mp = 0$ so they're actually orthogonal. So $\mathrm{Cl}(1, \mathbb{C}) = \mathbb{C} \oplus \mathbb{C}$. □

A word of caution though: the clifford algebra is $\mathbb{Z}_2$ graded, it's a super algebra by taking e_i to be odd. However the splitting that we got is not homogeneous. So this is NOT a decomposition as superalgebras.

Furthermore: we now have two possible meanings of tensor product as algebras or as superalgebras which are not the same.

Let A and B be algebras over $\mathbb{R}$ (or some other field), then $A \otimes_{\mathbb{R}} B$ is also an algebra by taking $(a\otimes b)(a' \otimes b') = aa' \otimes bb'$. If instead they are super algebras $A = A_+ \oplus A_-$ and $B = B_+ \oplus B_-$, then we have $(a\hat{\otimes} b)(a'\hat{\otimes} b') = (-)^{|b||a'|} aa' \otimes bb'$. Normally I have two phase spaces V and W and the composite phase space $V \oplus W$ which corresponds to tensoring algebras of observables. To have the same for a super algebra I need the super tensor product. What we have is $\mathrm{Cl}(V)\hat{\otimes}\mathrm{Cl}(W) = \mathrm{Cl}(V \oplus W)$.

Claim. 2 anticommuting square roots of $+1$ give a copy of 2-by-2 matrices. So we have $\mathrm{Cl}(2, \mathbb{C}) = M_2(\mathbb{C})$ (Here $M_n(A)$=$\{n$-by-n matrices with coefficients in $A\}$).

Proof. Having e, f with $e^2 = f^2 = 1$ and $ef = -fe$, we can write the matrices as:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad ef = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -fe \tag{5.2}$$

(which also works with real coefficients). □

Now, from what we said above $\mathrm{Cl}(3, \mathbb{C})$ is just the super tensor product $\mathrm{Cl}(2, \mathbb{C}) \hat{\otimes} \mathrm{Cl}(1, \mathbb{C})$ and similarly we can obtain $\mathrm{Cl}(n, \mathbb{C})$ for any n.

Over $\mathbb{R}$, however, the situation is not so simple. In the real case I need to worry about the "signature", meaning I can have a vector space $\mathbb{R}^{p,q} = \mathbb{R}^{p+q}$ which has p negative orthonormal basis vectors and q positive vectors. So $\mathrm{Cl}(p, q) =$ real algebra generated by p roots of $+1$ and q roots of -1 which are all anticommuting. It turns out that what we've done up to now is enough to find the first Clifford algebras:

- $\mathrm{Cl}(0, 1) = \mathbb{C}$: we simply have $\mathbb{R}$ and then an additional generator that is a square root of -1. That's simply the complex algebra.

- $\mathrm{Cl}(0, 2) = \mathbb{H}$ (quaternions): we now have two anticommuting generators whose square is -1 and we can call them i and j. Now we also have $(ij)^2 = ijij = -iijj = -1$ so (ij) is also a square root of -1 and if we call it k it's clear that we're working with the algebra of quaternions.

- $\mathrm{Cl}(1, 0) = \mathbb{R} \oplus \mathbb{R}$: we saw above that $\mathrm{Cl}(1, \mathbb{C}) = \mathbb{C} \oplus \mathbb{C}$, but working with $\mathbb{R}$ instead of $\mathbb{C}$ makes no difference in the proof.

- $\mathrm{Cl}(2, 0) = M_2(\mathbb{R})$: just like before, works the same way as $\mathrm{Cl}(2, \mathbb{C}) = M_2(\mathbb{C})$.

- $\mathrm{Cl}(1, 1) = M_2(\mathbb{R}) = \mathrm{Mat}(\mathbb{R}^{1|1})$: using the matrices from the last claim, we can set $g = ef$ and note that $g^2 = -1$, so e and g are now respectively square roots of $+$ and -1. To be sure we span the whole space, note that we can get back f from g and e since $f = eg$. We wrote $\mathrm{Mat}(\mathbb{R}^{1|1})$ since that is the actual super algebra structure that we obtain, which is different from the previous case.

Claim. $\mathrm{Cl}(0, q) \otimes \mathrm{Cl}(2, 0) = \mathrm{Cl}(q+2, 0)$ and $\mathrm{Cl}(p, 0) \otimes \mathrm{Cl}(0, 2) = \mathrm{Cl}(0, p+2)$

Proof. In $\mathrm{Cl}(0, q)$ I have e_i with $e_i^2 = -1$ while in $\mathrm{Cl}(2, 0)$ I have f_1, f_2 with $f_i^2 = 1$. Consider the elements $1 \otimes f_1, 1 \otimes f_2, e_i \otimes f_1 f_2$ it's clearly a generating set so we want to check that they anticommute and square to 1. The first two clearly square to 1 and for the others

$$(e_i \otimes f_1 f_2)^2 = e_i^2 \otimes (f_1 f_2)^2 = -1 \otimes (-1) = 1 \tag{5.3}$$

and one can check the anticommutativity.

The other equality is analogous by simply changing signs. □

Claim. $\mathrm{Cl}(p,q) \otimes \mathrm{Cl}(1,1) = \mathrm{Cl}(p+1,q+1) = M_2(\mathrm{Cl}(p,q))$

Proof. Let the first have generators e_i, f_j with $e_i^2 = 1, f_j^2 = -1$ and the second E, F with $E^2 = 1, F^2 = -1$. Now simply take $1 \otimes E, 1 \otimes F, e_i \otimes EF, f_j \otimes EF$. For the last equality note that we have $\mathrm{Cl}(1,1) = M_2(\mathbb{R})$ and $M_2(\mathbb{R}) \otimes_{\mathbb{R}} A = M_2(A)$ for any algebra A. So $\mathrm{Cl}(p,q) \otimes \mathrm{Cl}(1,1) = M_2(\mathrm{Cl}(p,q))$. □

In deriving the next Clifford algebras, it's useful to note the following properties of the tensor product and direct sum in relation to matrices with coefficients:

- $M_n(A) \otimes B = M_n(A \otimes B)$,
- $M_i(M_j(A)) = M_{ij}(A)$ which is clear since on the left hand side we're taking the i-by-i matrices with entries j-by-j matrices which are clearly ij-by-ij matrices,
- $M_n(A \oplus B) = M_n(A) \oplus M_n(B)$.

So we have a recipe for obtaining the gamma matrices in all dimensions. Let us explicitly calculate a few:

	$\mathrm{Cl}(k,0)$	$\mathrm{Cl}(0,k)$
$k=0$	$\mathbb{R}$	$\mathbb{R}$
$k=1$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{C}$
$k=2$	$M_2(\mathbb{R})$	$\mathbb{H}$
$k=3$	$M_2(\mathbb{C})$	$\mathbb{H} \oplus \mathbb{H}$
$k=4$	$M_2(\mathbb{H})$	$M_2(\mathbb{H})$
$k=5$	$M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$	$M_4(\mathbb{C})$
$k=6$	$M_4(\mathbb{H})$	$M_8(\mathbb{R})$
$k=7$	$M_8(\mathbb{C})$	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$
$k=8$	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})$

which are obtained with the following reasoning:

- $\mathrm{Cl}(3,0) = \mathrm{Cl}(2,0) \otimes \mathrm{Cl}(0,1) = M_2(\mathbb{R}) \otimes \mathbb{C} = M_2(\mathbb{C})$
- $\mathrm{Cl}(0,3) = \mathrm{Cl}(0,2) \otimes \mathrm{Cl}(1,0) = \mathbb{H} \otimes (\mathbb{R} \oplus \mathbb{R}) = (\mathbb{H} \otimes \mathbb{R}) \oplus (\mathbb{H} \otimes \mathbb{R})$

  And so on. We also show $\mathrm{Cl}(0,5)$ and $\mathrm{Cl}(0,6)$ since they're a little tricky:

- $M_2(\mathbb{C}) \otimes \mathbb{H} = M_2(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}) = M_2(M_2(\mathbb{C})) = M_4(\mathbb{C})$, where $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} = M_2(\mathbb{C})$ since when tensoring with $\mathbb{C}$ having square roots of $+1$ is the same as having square roots of -1, so $\mathbb{H}$ is the same as $M_2(\mathbb{R})$ and tensoring $\mathbb{C}$ with $M_2(\mathbb{R})$ gives $M_2(\mathbb{C})$.
- $M_2(\mathbb{H}) \otimes \mathbb{H} = M_2(\mathbb{H} \otimes \mathbb{H}) = M_2(M_4(\mathbb{R})) = M_8(\mathbb{R})$

There are now two important things to note:

1. There is a certain symmetry between left and right columns, namely $\mathrm{Cl}(4+k,0) = M_{2k}(\mathrm{Cl}(0,4-k))$ and analogously $\mathrm{Cl}(0,4+k) = M_{2k}(\mathrm{Cl}(4-k,0))$

2. $\mathrm{Cl}(8,0) = \mathrm{Cl}(0,8) = M_{16}(\mathrm{Cl}(1,0))$. This is a very important property because it means that the next algebras can be trivially derived from the ones in the table by simply taking the matrix algebra with some specific coefficients. So in general $\mathrm{Cl}(k+8,0) = M_{16}(\mathrm{Cl}(k,0))$ and $\mathrm{Cl}(0,k+8) = M_{16}(\mathrm{Cl}(0,k))$. For this reason a lot of the properties of fermions we will derive will be dependent on the dimension mod 8.

Lecture 6.2 (24/5) The spin groups
We could also make a table for the complex Lie algebras:

n	$\mathrm{Cl}(n,\mathbb{C})$
1	$\mathbb{C}\oplus\mathbb{C}$
2	$M_2(\mathbb{C})$
3	$M_2(\mathbb{C})\oplus M_2(\mathbb{C})$
4	$M_4(\mathbb{C})$
$\vdots$	$\vdots$

which is simpler since we just have a periodicity of 2. This is because the formula $\mathrm{Cl}(p,0)\otimes\mathrm{Cl}(0,2) = \mathrm{Cl}(0,p+2)$ becomes $\mathrm{Cl}(k,\mathbb{C})\otimes\mathrm{Cl}(2,\mathbb{C}) = \mathrm{Cl}(k+2,\mathbb{C})$ in the complex case.

Note that both this and the real classification is a classification as *algebras* not as super algebras. So now we want to understand the classification as super algebras.

5.1 The spin group

The spin group is a subgroup of the set of invertible elements of a Clifford algebra.

Analogy: If I have a bosonic phase space $(\mathbb{R}^2, dp\wedge dx)$ where we have canonical transformations, i.e. endomorphisms of the phase space wich preserve the phase space. We can in particular choose linear canonical transformations. A canonical transformation is a transformation that is generated from an observable. So for linear ones I just need to write down observables of a particular polynomial degree, such that the resulting things looks like linear vector fields. It is the quadratic observables, since for a generic observable we have:

$$f \to df = \omega(X_f, -), \quad X_f(g) = \{f,g\} \tag{5.4}$$

If f is a quadratic observable, then df is linear and I get a linear canonical transformations. What quadratic observables do we have and what transformations do these generate?

$$\begin{array}{lll} x^2 \to & 2xdx = \omega(X_{x^2}, -) & X_{x^2} = 2x\frac{\partial}{\partial p} \\ p^2 \to & 2pdp = \omega(X_{p^2}, -) & X_{p^2} = -2p\frac{\partial}{\partial x} \\ xp \to & xdp + pdx = \omega(X_{xp}, -) & X_{xp} = p\frac{\partial}{\partial p} - x\frac{\partial}{\partial x} \end{array}$$

These should then generate some symmetry algebra. I can compute their Poisson brackets:

$$\{p^2, x^2\} = 2p\{p, x^2\} = 4px\{p, x\} = 4px \tag{5.5}$$

$$\{p^2, p^2\} = 0 \tag{5.6}$$

$$\{p^2, px\} = 2p^2 \tag{5.7}$$

$$\{x^2, px\} = -2x^2 \tag{5.8}$$

but if we change names:

$$E = \frac{p^2}{2}, \quad F = \frac{x^2}{2}, \quad H = px \tag{5.9}$$

So they generate the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. Generally $\mathfrak{sl}$ is supposed to be volume preserving transformations, while we expected to find transformations that preserve a symplectic form, but because we're only in a two dimensional space, a symplectic form is also a volume form. So we expected $\mathfrak{sp}(2, \mathbb{R})$, but it's okay since in two dimensions they're isomorphic.

This procedure could be repeated also in higher dimensions: we compute all quadratic polynomials in ps and xs and then compute their Poisson brackets.

What should happen in the odd case? In the odd case we have an inner product, we still have a Poisson bracket and we would like it to be closed in the quadratic terms like we found in the bosonic case and we expect to get the Lie algebra of orthogonal transformations $\mathfrak{so}(n)$ or $\mathfrak{so}(p, q)$ since this time we're preserving an inner product.

Also note that in the procedure above we could instead have started with the Weyl algebra using the commutators instead of the Poisson brackets since the two are the same. So for the bosonic case we can start with the commutators of quadratic elements in the Clifford algebra $\mathrm{Cl}(p, q)$ and we expect to reproduce the Lie bracket of $\mathfrak{so}(p, q)$.

Recall that the generators of $\mathfrak{so}$ are antisymmetric matrices, so we can write

$$\mathfrak{so}(V) = \Lambda^2 V \tag{5.10}$$

this is why for example Lorentz transformations have two lower indices which are antisymmetric.

But how do I think of it? Well if we have $e_i \wedge e_j$ it will be something like $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the i and j indices, meaning that applied to e_i it gives e_j and viceversa but with a minus sign.

Now to calculate their commutator $[e_i \wedge e_j, e_k \wedge e_l]$ simply note:

- if all indices are different then it's zero since $(e_i \wedge e_j)(e_k \wedge e_l)$ would first send e_k to e_l and viceversa but these are sent to 0 since neither is e_i or e_j
- it's nonzero only if exactly two indices are the same. If we had three equal indices then we would have something like $e_i \wedge e_i$ which is zero.
- it's nonzero only if two indices are the same and we have that $(e_i \wedge e_j)(e_j \wedge e_k)$ sends e_k to e_i, while $(e_j \wedge e_k)(e_i \wedge e_j)$ sends e_i to e_k. So we have $[e_i \wedge e_j, e_j \wedge e_k] = e_k \wedge e_i$

Putting these together gives:

$$[e_i \wedge e_j, e_k \wedge e_l] = +g_{il}e_k \wedge e_j - g_{ik}e_l \wedge e_j + g_{jl}e_i \wedge e_k - g_{jk}e_i \wedge e_l \tag{5.11}$$

Claim. The map $\Lambda^2 V \to \mathrm{Cl}(V)$ given by $v \wedge w \mapsto \frac{1}{4}(vw - wv)$, carries Lie brackets to commutators, i.e. it's a homomorpism of Lie algebras

Proof. Work with an orthonormal basis.

$$e_i \wedge e_j \mapsto \frac{1}{4}(e_i e_j - e_j e_i) = \frac{1}{2}(e_i e_j) \tag{5.12}$$

$$[e_i \wedge e_j, e_k \wedge e_l] \mapsto \delta_{il}\frac{e_k e_j}{2} - \delta_{ik}\frac{e_l e_j}{2} + \delta_{jl}\frac{e_i e_k}{2} - \delta_{jk}\frac{e_i e_l}{2} \tag{5.13}$$

In which the image of the Lie bracket is just taken from the result above. Now we want to calculate $\left[\frac{1}{2}e_i e_j, \frac{1}{2}e_k e_l\right]$ and make sure that it gives the same result. To do this we use the commutation relation:

$$e_a e_b + e_b e_a + 2\delta_{ab} = 0 \tag{5.14}$$

and using this relation multiple times we get

$$\left[\frac{1}{2}e_i e_j, \frac{1}{2}e_k e_l\right] = \frac{1}{4}(e_i e_j e_k e_l - e_k e_l e_i e_j) \tag{5.15}$$

$$= \frac{1}{4}(e_i e_j e_k e_l - e_i e_j e_k e_l + 2e_k e_j \delta_{il} - 2e_l e_j \delta_{ik} + 2e_i e_k \delta_{jl} - 2e_i e_l \delta_{jk}) \tag{5.16}$$

$$= +\delta_{il}\frac{e_k e_j}{2} - \delta_{ik}\frac{e_l e_j}{2} + \delta_{jl}\frac{e_i e_k}{2} - \delta_{jk}\frac{e_i e_l}{2} \tag{5.17}$$

which is exactly the image of the Lie bracket. □

I can then exponentiate a quadratic term and I would still get something in the Clifford algebra, which should be isomorphic to some group with the Lie algebra $\mathfrak{so}(V)$. As usual, the exponential is given by the power series

$$\exp\left(\frac{e_i e_j}{2}\right) = 1 + \frac{e_i e_j}{2} + \frac{1}{2!}\left(\frac{e_i e_j}{2}\right)^2 + \dots \tag{5.18}$$

and we get a Lie group of non infinitesimal symmetries inside $\mathrm{Cl}(V)^\times_+$. $\times$ indicates that we only take invertible elements and $+$ indicates we're in the positive parity side, so with only even powers. But since we're starting with the Lie algebra $\mathfrak{so}(V)$ is it possible to get a different Lie group from $\mathrm{SO}(V)$? Yes, for example SU(2) and SO(3) have the same Lie algebra, in fact there's a map which says that there are two SU(2) matrices for every SO(3) matrix. This can be written as the short exact sequence:

$$1 \to \mathbb{Z}_2 \to \mathrm{SU}(2) = Spin(3) \to \mathrm{SO}(3) \to 1 \tag{5.19}$$

which just says that there is a 2:1 mapping from SU(2) to SO(3). In addition, we found our first example of a spin group: $\mathrm{SU}(2) = Spin(3)$. In general a spin group is going to be a 2:1 cover of the orthogonal group.

Why is it a 2:1 map?

How do I get the the 2:1 map in general? Consider the group in $\mathrm{Cl}(V)^\times$ generated by unit vectors. ($e_i^2 = \pm 1 \implies e_i^{-1} = \pm e_i$, so the unit vectors are in fact invertible and their products are also invertible because I can invert the unit vectors one by one) Note that $x \in V$ is also an element in $\mathrm{Cl}(V)^\times$, since we have $x\,x = -g(x,x)$ and therefore $x^{-1} = -\frac{1}{g(x,x)}x$ is the inverse.

Claim. This group acts on V and preserves the inner product. In other words there's a map from the group to the group of orthogonal transformations of V. The map is actually fairly intuitive.

There's a "twisted" adjoint action of $\mathrm{Cl}(V)^\times$ on $\mathrm{Cl}(V)$, so I get a representation of $\mathrm{Cl}(V)^\times$ on $\mathrm{Cl}(V)$.

$$y \in \mathrm{Cl}(V), x \in \mathrm{Cl}(V)^\times \tag{5.20}$$

$$x \to (y \mapsto \pi(x) y x^{-1}) \tag{5.21}$$

where π is the parity operator on $\mathrm{Cl}(V)$. Now, if x is a vector in V, $x \in \mathrm{Cl}(V)^\times, \pi(x) = -x$, pick $y \in V \subseteq \mathrm{Cl}(V)$ so we get

$$y \mapsto -xyx^{-1} = +\frac{xyx}{g(x,x)} = \frac{x}{g(x,x)}(-xy - 2g(x,y)) = -\frac{x^2}{g(x,x)}y - 2\frac{g(x,y)x}{g(x,x)} \tag{5.22}$$

$$= y - 2\frac{g(x,y)}{g(x,x)}x \in V \tag{5.23}$$

so the adjoint action preserves the subspace V. But what is the action we found? Well if it were $y \mapsto y - \frac{g(x,y)}{g(x,x)}x$ then it would be projection onto the hyperplane orthogonal to x. Instead since we have the 2, we get a reflection through the hyperplane orthogonal to x. Now, how many unit vectors are there that have the same orthogonal hyperplane? Just two, the two that stick out on opposite sides of the hyperplane. So that means there's exactly two elements of the Clifford algebra that map to the same orthogonal transformation, namely reflection through the plane perpendicular to the unit vector.

Note however that this transformation doesn't preserve parity and is in the odd part of the algebra. But from this we can construct two groups:

$$Pin(V) = \text{generated by unit vectors in } V \text{ inside } \mathrm{Cl}(V)^{\times} \tag{5.24}$$

The (twisted) adjoint action restricts to $V \subseteq \mathrm{Cl}(V)$ and defines a 2:1 map

$$1 \to \mathbb{Z}_2 \to Pin(V) \to O_o(V) \to 1 \tag{5.25}$$

where $O_o(V)$ indicates the part of $O(V)$ generated by reflections, which may in general not be the whole orthogonal group. But when the inner product is normal, so it has positive definite signature, we get $O_o(V) = O(V)$. Now we can define the spin group, which corresponds to only the even part of the transformations generated by reflection:

$$Spin(V) = Pin(V) \cap \mathrm{Cl}(V)_+ \tag{5.26}$$

so we get the following 2:1 map

$$1 \to \mathbb{Z}_2 \to Spin(V) \to SO_o(V) \to 1 \tag{5.27}$$

since two reflections together preserve the parity, so I have the parity preserving part of the part of the rotation group generated by reflections. Again, if we have a positive definite signature then $SO_o(V) = SO(V)$.

Let's look at SO(2). A generator of the Lie algebra is $e_1 \wedge e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and it's exponential is given by:

$$\exp\left(t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \tag{5.28}$$

and where does this map in the Clifford algebra? $\frac{e_1 e_2}{2}$ whose exponential is:

$$\exp\left(t\frac{1}{2}e_1e_2\right) = \cos\frac{t}{2} + e_1e_2\sin\frac{t}{2} \xrightarrow[t=2\pi]{} -1 \tag{5.29}$$

noting $(e_1e_2)^2 = -1$. So just as SO(2) is a circle, so is Spin(2), and the map that we have simply wraps Spin(2) around SO(2) twice. Also in higher dimensions, whatever the Spin group does, when restricted to a plane then it simply rotates in the plane half as fast as in SO.

Why is then the spin group important in quantum mechanics, well there are fermions, so we expected there to be representations of half integer spin, meaning that if I rotate in a 2-plane it only goes half the way around, so half integer spin really just means that we're working with the Spin group.

To get reps of Spin(V) use $\mathrm{Cl}(V)_+$ modules.

Fact: $\mathrm{Cl}(p, q) = \mathrm{Cl}(p, q+1)_+$ as algebras.

Proof. the map is $e_i \mapsto e_i f$, where f is the additional square root of -1. $e_i f$ squared is again e_i^2. So we have a map from the entire $\mathrm{Cl}(p,q)$ to quadratic elements in $\mathrm{Cl}(p,q+1)$, meaning even elements. But then the dimension is the same so we get that they're isomorphic. $\square$

Lecture 7.2 (31/5) Properties of, and pairings on, spinors

Last time we constructed $\mathrm{Spin}(V)$ in $\mathrm{Cl}_+^\times$. The key idea was to associate to a unit vector in $\mathrm{Cl}(V)$ a reflection of V through a hyperlane orthogonal to that vector, from which we can easily see that it's a 2:1 cover.

We're moving towards a field teory, so we're interested in $Spin(d-1,1)$ for a d dimensional spacetime, so the spin cover of the group of lorentz transformations in a $(d-1,1)$ spacetime. We're going to have spinor fields, which have a certain spin index, which transform in one of the representations of the spin group. Now recall that

$$Spin(d-1,1) \subseteq \mathrm{Cl}(d-1,1)_+ \cong \mathrm{Cl}(d-1,0) \tag{5.30}$$

and we already computed the Clifford algebras of the form $\mathrm{Cl}(k,0)$:

d	k	$\mathrm{Cl}(k,0)$	Representations	Real dimension
2 (10)	1	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}_\pm$	1 (16)
3 (11)	2	$M_2(\mathbb{R})$	$\mathbb{R}^2$	2 (32)
4	3	$M_2(\mathbb{C})$	$\mathbb{C}^2, \overline{\mathbb{C}}^2$	4
5	4	$M_2(\mathbb{H})$	$\mathbb{H}^2$	8
6	5	$M_2(\mathbb{H})\oplus M_2(\mathbb{H})$	$\mathbb{H}^2_\pm$	8
7	6	$M_4(\mathbb{H})$	$\mathbb{H}^4$	16
8	7	$M_8(\mathbb{C})$	$\mathbb{C}^8$	16
9	8	$M_{16}(\mathbb{R})$	$\mathbb{R}^{16}$	16

in which the dimension is the number of indices in physics. In order to find the representations I use the fact that matrix algebras are simple: the only thing the matrices can do is act on row vectors or column vectors. See also section A.4 for more details. For example we have:

- $\mathbb{R}\oplus\mathbb{R}$ has representations $\mathbb{R}_+$ and $\mathbb{R}_-$ which correspond to the representations in which we act by multiplying by a number from one of the two $\mathbb{R}$s and trivially from the other $\mathbb{R}$. These are called Majorana-Weyl spinors.

- $M_2(\mathbb{R})$ has representation $\mathbb{R}^2$ which we can think equivalently as column or row vectors.

- $M_2(\mathbb{C})$ is similar but in this case column and row vectors are distinct because complex conjugation is not a linear map, so we have $\mathbb{C}^2$ and $\overline{\mathbb{C}}^2$. These are called Weyl spinors.

- $M_2(\mathbb{H})\oplus M_2(\mathbb{H})$ gives $\mathbb{H}^2_\pm$ just as in the first example and these are called pseudo Majorana-Weyl spinors.

There are a few things we can observe from the table. Note that for $k = 3$ the dimension jumps to 4, then for $k = 4$ it jumps to 8, for $k = 6$ to 16 and for $k = 10$ to 32, and the indices of these jumps are related to the dimensions of the four normed division algebras:

$$k = 2 + \dim \mathbb{K}, \quad \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \tag{5.31}$$

In addition we can see that for any spacetime dimension we always have either one or two representations. This is exactly what we anticipated when we were reviewing canonical quantization: quantization of the Poisson algebra gives the Weyl algebra which has a unique representation, while quantization of fermionic observables lead to the Clifford algebras which have either one or two representations, based on the spacetime dimension.

So in general we can get many properties of the spin representations we want from that table. Although, in order to do that we need to understand what it means for a representation to be quaternionic.

It turns out that there is a threefold classification of representations of groups:

- complex, where I "cannot raise or lower indices"
- real, where I can raise indices with a symmetric pairing (e.g. by contracting with the matric)
- pseudoreal or quaternionic, where I can raise indices with an antisymmetric pairing (*pseudo* because I still have a way of raising and lowering indices, but not real because I don't do it with a symmetric object)

We have an example of the pseudoreal form with the symplectic form:

$$\omega = dp \wedge dx \tag{5.32}$$

$$\partial_x \vee \omega = -dp \tag{5.33}$$

$$\partial_p \vee \omega = +dx \tag{5.34}$$

so passing between vector fields and one forms corresponds to raising and lowering indices using an antisymmetric pairing.

Let W be a complex vector space with a unitary group representation $\rho : G \to U(W)$, i.e. the group is represented by unitary matrices. I then get a G representation on $\overline{W}$, the dual representation:

$$|w\rangle \to U_g |w\rangle \tag{5.35}$$

$$\langle w'| \to \langle w|' U_g^\dagger \tag{5.36}$$

where I act on the "row vector" $\langle w'|$ with $U^\dagger = U^{-1}$ because there is a pairing $\langle w'|w\rangle$ between vectors and dual vectors, and I want the pairing to be preserved.

Is $\overline{W}$ the same representation as W? Meaning, are they related by a change of basis? In general no, take for example the 1 dimensional complex space on which the circle acts as a change of phase, then we have

$$\overline{e^{i\theta}} = e^{-i\theta} \tag{5.37}$$

but the two are not equivalent since there is no change of basis from one to the other. We call these two distinct representations $+1$ and -1. On the other hand if I take the two dimensional representation built up as the direct sum of the two, meaning it has one $+1$ charge and one -1, then we have

$$\overline{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}} \cong \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \tag{5.38}$$

which are equivalent since we clearly have a change of basis to bring one into the other.

In this situation, in which $W \ncong \overline{W}$, then we say W is complex. Note that it does not just mean that we have a complex vector space. In particular for the two dimensional representation above, there's a change of basis that brings it into

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{5.39}$$

which has only real entries. Instead for the first case there's no change of basis that leads to it having real entries. So in a complex representation the matrix entries are complex no matter what basis I consider.

Suppose now that I can identify W with $\overline{W} = W^\vee$, meaning I have a map $\phi : W \to W^\vee$, or analogously I have a map $\phi : \mathbb{C} \to W^\vee \otimes W^\vee$ which is a 2 index tensor. In particular if the first is invariant for the group action, then so is the second one, so ϕ is a G invariant two index tensor. Therefore there's a copy of the trivial G representation inside of 2 index tensors.

So ϕ is the map to raise and lower indices. In particular the tensor ϕ has some symmetry property, since the G action on this space of tensors commutes with symmetrizing and antisymmetrizing, so the irreducible G representations are either totally symmetric (real representation) or totally antisymmetric (pseudoreal representation).

Take for example SU(2) in which we start with the spin $\frac{1}{2}$ doublet in $\mathbb{C}$. Then if I tensor this representation with itself then I should get a copy of the trivial representation which I do get since I get the triplet and the singlet, respectively symmetric and antisymmetric parts. Since we want the trivial representation, we're interested in the latter and have an antisymmetric 2 index invariant tensor, so I can raise or lower indices with something antisymmetric like the Levi Civita symbol. That's kind of why the Pauli matrices have complex entries, while if it were a real representation we would have some change of basis to give it real entries. Note that SU(2) also has real representations, for example the vector representation on which it acts simply as spatial rotations.

In addition, one can show that if we have a pseudoreal structure, then the representation can be written in terms of quaternionic matrices.

So we can just read from the table what kind of representation we have: real, pseudoreal (also called pseudo Majorana) or complex.

However the real and pseudoreal structures come from some invariant pairing, we now consider a way of constructing such a pairing. Let's go back to the analogy between $\mathrm{Cl}(V)$ and canonical quantization. Clifford algebra was the algebra of the odd quantum observables associated to a phase space V. The Clifford module is supposed to be the Hilbert space! So Clifford algebras appear in two places: for spinor representations in field theory and as the quantization of the field space of an odd particle moving in V. We would like to push the analogy further. The normal trick to build the Hilbert space is the following:

- split phase space as $\langle x\rangle \oplus \langle p\rangle$
- take a complete set of commuting observables, i.e. x or p
- take Hilbert space $\mathscr{H} = L^2(\mathbb{R}_x)$ with x and p acting as x and $i\frac{\partial}{\partial x}$ (or analogously take $\mathscr{H} = L^2(\mathbb{R}_p)$ ...) which automatically has an inner product which is respected by x and p, meaning they're self adjoint. The inner product is given by the integral over $\mathbb{R}$.

We now want to be able to do the same for the odd part: take a complete set of supercommuting observables, think about the functions they generate, let the observables from the set act by multiplication and the other ones by derivatives and we should have a way of integrating as well. The analogy works best for even dimensional clifford algebras over $\mathbb{C}$.

Set $V = \mathbb{R}^{2n}$ with coordinates $x_1, \dots, x_n, y_1, \dots, y_n$ and standard inner product $ds^2 = \sum_i dx_i^2 + dy_i^2$. Consider the complexification $V \otimes \mathbb{C}$, we have $\mathrm{Cl}(V_{\mathbb{C}}) \cong \mathrm{Cl}(V) \otimes \mathbb{C}$. Why do I do that? Well I want a complete set of supercommuting observables, meaning anticommuting since they're fermionic. For them to be anticommuting, they should have zero length, but in the real case I can't get a nonzero vector of zero length. Instead I can set

$$z_i = x_i + iy_i \tag{5.40}$$
$$\bar{z}_i = x_i - iy_i \tag{5.41}$$

now $ds^2 = \sum_i \frac{1}{2} dz^i d\bar{z}^i$, so z_i and $\bar{z}_i$ have zero square but nontrivial inner product.

The complete set of supercommuting observables is then a maximal isotropic subspace. Isotropic meaning I have a bilinear inner product which is zero when I restrict it to my subspace. In this case the complete set may be the set of all z_i since here the inner product is identically zero (analogously

I could choose $\bar{z}_i$). This is analogous to the fact that if I take a symplectic form on the normal phase space and I restrict it to the space spanned by the xs then it vanishes.

So essentially we're decomposing $V_{\mathbb{C}}$ as $V_{\mathbb{C}} = L \oplus L^{\vee}$, the first with coords z_i and the second with $\bar{z}_i$, so we have something like the separation in x and p. However it's important to note that to do it we had to complexify. Then the Hilbert space is functions on L and this is the same as $\Lambda^*(L^{\vee})$, the exterior algebra over $L^{\vee}$, since functions on L are generated by elements in $L^{\vee}$, meaning linear coordinates of L, and these are anticommuting so they generate the exterior algebra. Let's call θ_i the anticommuting coordinates of the exterior algebra. Now what's the representation? It's given by

$$\bar{z}_i \mapsto \theta_i \tag{5.42}$$

$$z_i \mapsto \frac{\partial}{\partial \theta_i} \tag{5.43}$$

This gives me a Clifford map $\mathrm{Cl}(V_{\mathbb{C}}) \to \mathrm{DiffOps}(L) \subseteq \mathrm{End}(\Lambda^*(L^{\vee}))$ since commutators are sent to commutators.

Recall we had seen $\mathfrak{gl}(1|1) = \mathrm{DiffOps}(\mathbb{C}^{0|1}) = \mathrm{Cl}(2, \mathbb{C})$, now we have something more general:

$$\mathrm{Cl}(2n, \mathbb{C}) \cong \mathrm{DiffOps}(\mathbb{C}^{0|n}) \cong \mathrm{End}(\Lambda^*(\mathbb{C}^{0|n})) \cong \mathfrak{gl}(n|n) \tag{5.44}$$

To do so we simply count dimensions:

- $\mathrm{Cl}(2n, \mathbb{C})$ has dimensions 2^{2n}
- the map above is injective so $\mathrm{DiffOps}(\mathbb{C}^{0|n})$ also has dimensions 2^{2n}
- $\Lambda^*(\mathbb{C}^{0|n})$ has dimensions 2^n, so $\mathrm{End}(\Lambda^*(\mathbb{C}^{0|n}))$ has dimension 2^{2n}
- all of them have the same amount of odd and even things, n for each. Therefore they can be identified with $\mathfrak{gl}(n|n)$.

So $\mathrm{Cl}(2n, \mathbb{C})$ is a matrix superalgebra.

We now want to recover the pairings on spinors from an integral over $\Pi L = \mathbb{C}^{0|n}$. We're still considering $\mathrm{Cl}(2n, 0)$, corresponding to the cases with an odd dimensional spacetime. Looking at the table I see which of them have a real and which have a pseudoreal structure, so I expect the pairing to go symmetric antisymmetric antisymmetric symmetric in the four cases.

Firstly, note that the integral of fermions is defined by:

$$\int d\theta 1 = 0, \int d\theta\theta = 1 \tag{5.45}$$

Now, given $s, t \in \Lambda^*(L^{\vee})$ I can write the pairing

$$(s, t) \mapsto \int \bar{s} \wedge t \tag{5.46}$$

where $\bar{}$ simply means reversing the order.

Claim. Integrating over fermions extracts the top component, meaning

$$(s,t) \mapsto \int \bar{s} \wedge t = [\bar{s} \wedge t]_{top} \tag{5.47}$$

claim: symmetry of that pairing obeys same rules as the pairings we want from the table, namely symmetric antisymmetric antisymmetric symmetric. Let us check that the pairing above reproduces the results we expect from the table. In the first case the space we're integrating over is $\Pi L = \mathbb{C}^{0|1}$ so we have a single odd variable θ. Therefore there's just one pairing

$$(1,\theta) = \int d\theta\theta = (\theta, 1) \tag{5.48}$$

which is clearly symmetric so it works. Now, for $k = 2$ we have two odd variables θ_1 and θ_2:

$$(1, \theta_1\theta_2) = \int d\theta_1 d\theta_2 \theta_1\theta_2 = -\int d\theta_1 d\theta_2 \theta_2\theta_1 = \int d\theta_1 d\theta_2 \overline{\theta_1\theta_2} = -(\theta_1\theta_2, 1) \tag{5.49}$$

$$(\theta_1, \theta_2) = \int d\theta_1 d\theta_2 \theta_1\theta_2 = -\int d\theta_1 d\theta_2 \theta_2\theta_1 = -(\theta_2, \theta_1) \tag{5.50}$$

so we have an antisymmetric pairing. For three odd variables:

$$\overline{\theta_1\theta_2\theta_3} = \theta_3\theta_2\theta_1 = -\theta_1\theta_2\theta_3 \tag{5.51}$$

Again antisymmetric. What if we do four? Well for $\theta_4\theta_3\theta_2\theta_1$ the sign of the permutation is even, so $\overline{\theta_1\theta_2\theta_3\theta_4} = \theta_1\theta_2\theta_3\theta_4$ and again we get a symmetric pairing. So we get a pairing which we'll prove to be invariant and which is symmetric or antisymmetric in order to reproduce the real or quaternionic structure from the table. It just has to do with the sign of the permutation to reverse the order of the θs. So if $k = 2n$ I expect a pairing with a sign which has period four and the pattern $+--+$ is repeated.

Next time: clear where the pairing comes from, talk about the fact that it's an invariant pairing for the spin group and then talk about how to use the same trick to get pairings in the odd dimensional cases.

In the normal Hilbert space we have an invariant inner product $\bar{s}t$: take two wavefunctions and multiply one times the conjugate of the other. The difference is that in the odd case there are ordering ambiguities so there are minus signs that appear.

A Useful definitions

A.1 Algebraic structures

Let us recall a few useful definitions.

Definition 4 (Field)**.** A field $\mathbb{F}$ is a set with two binary operations called addition and multiplication satisfying the following field axioms:

- Associativity of addition and multiplication
- Commutativity of addition and multiplication
- Additive and multiplicative identity (resp. 0 and 1)
- Additive inverse
- Multiplicative inverse for every element except 0
- Distributivity of multiplication over addition.

or, more simply:

- Abelian group under addition
- nonzero elements are an abelian group under multiplication
- Distributivity of multiplication over addition.

Given a field we can define a vector space over it:

Definition 5 (Vector space)**.** A vector space V over a field $\mathbb{F}$ is a set with two operations:

1. addition $+ : V \times V \to V$,
2. multiplication by a scalar $\cdot : \mathbb{F} \times V \to V$

such that it is an abelian group with respect to addition and has the following properties: $(x, y \in \mathbb{F}, \mathbf{v}, \mathbf{w} \in V)$

- $x \cdot (\mathbf{v} + \mathbf{w}) = x \cdot \mathbf{v} + x \cdot \mathbf{w}$
- $(x + y) \cdot \mathbf{v} = x \cdot \mathbf{v} + y \cdot \mathbf{v}$
- $(xy) \cdot (\mathbf{v} + \mathbf{w}) = x \cdot (y \cdot \mathbf{v})$
- $1 \cdot \mathbf{v} = \mathbf{v}$

However there are more general concepts which can be useful.

A ring generalizes the concept of field, without requiring commutativity and inverses of multiplication.

Definition 6 (Ring)**.** A ring is a set with two binary operations called addition and multiplication satisfying the following ring axioms:

- Abelian group under addition
- Semigroup under multiplication (i.e. is only associative)
- Distributivity of multiplication from left and right over addition.

In addition, if it has a multiplicative identity (i.e. it is a monoid under multiplication) it's called a ring with unity.

We also define *division ring* a ring in which every nonzero element has a multiplicative inverse (i.e. a field in which multiplication may be noncommutative).

Now, just as one can define vector spaces over fields, one can define the analogous but more general concept of modules over rings.

Definition 7 (Module)**.** A left module over a ring R consists of an abelian group M and a "scalar multiplication" between elements of R and M that gives another element in M with the following properties: ($r, s \in R$, $x, y \in M$)

- $r \cdot (x + y) = r \cdot x + r \cdot y$
- $(r + s) \cdot x = r \cdot x + s \cdot x$
- $(rs) \cdot x = r \cdot (s \cdot x)$
- $1 \cdot x = x$

Then, it can happen that in a vector space we have a natural concept of a product between vectors that gives another vector, for example the the vector product $\times$ in $\mathbb{R}^3$ or the obvious product in the vector space of polynomials. This additional operation gives rise to the concept of an algebra.

Definition 8 (Algebra)**.** An algebra is a vector space V with an additional operation (multiplication) $\cdot : V \times V \to V$ which is bilinear and associative.

Given an algebra we can then define the commutator

$$[A, B] = AB - BA \tag{A.1}$$

which satisfies

$$[A, B] = -[B, A] \tag{A.2}$$

and is another bilinear operation $[\cdot, \cdot] : V \times V \to V$. This then raises the question: is V with the operation $[\cdot, \cdot]$ also an algebra? But the answer is no since the operation is not associative. Instead we have:

$$[[A, B], C] = [A, [B, C]] + [B, [C, A]] \tag{A.3}$$

in which the last term ruins the associativity. Usually the equality above is written as:

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0 \tag{A.4}$$

which is called the Jacobi identity.

However such an operation appears often enough that the resulting structure deserves a name:

Definition 9 (Lie algebra)**.** A Lie algebra is a vector space with a bilinear operation $[\cdot, \cdot] : V \times V \to V$ with the following properties: $(A, B \in V)$

1. $[A, B] = -[B, A]$
2. $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$

i.e. it is antisymmetric and satisfies the Jacobi identity. The operation is generally called *bracket*, but if it is constructed as above it is usually called *commutator*.

Note that a Lie algebra is not in fact an algebra, as it is not associative. In fact the Jacobi identity somehow measures the non-associativity of the bracket.

Also note that a bilinear operation $[\cdot, \cdot] : V \times V \to V$ is the same thing as a *linear* operation from the tensor product $[\cdot, \cdot] : V \otimes V \to V$. This is just the universal property defining the tensor product.

All these structures have their *super* counterpart which is constructed with the following guidelines:

- add a Koszul sign every time one operator is passed after another,
- preserve parity.

A.2 Representation theory

Given a group, the concept of its representation arises.

But what is a representation? When you think of symmetries, you can consider them as objects on their own, that's basically the idea of a group. Given a symmetry we can then ask what kind of objects have this symmetry: that's the idea of a representation. It basically tells us what kind of objects can I have in space. For example with systems of a certain spin we have that the same symmetry (rotations) is "represented" in different ways.

We can therefore give the following definition.

Definition 10. A representation of a group G on a vector space V is a map $\rho : G \to \mathrm{Aut}(V)$, where the automorphism group of a vector space V, $\mathrm{Aut}(V)$, is the group of invertible linear maps from V to V. The map must have the following property:

$$\rho(g \cdot g') = \rho(g) \circ \rho(g') \tag{A.5}$$

where $\circ$ is simply the composition.

In other words, we need a group homeomorphism from g to $\mathrm{Aut}(V)$.

It is also useful to define the same concept for algebras and Lie algebras. The only difference is that we will have algebra homomorphisms and Lie algebra homomorphisms. Let us start by defining those.

Definition 11. An algebra homomorphism is a map $\phi : A \to B$ between algebras over a field $\mathbb{F}$ that satisfies: $(x, y \in A, k \in \mathbb{F})$

1. $\phi(kx) = k\phi(x)$
2. $\phi(x + y) = \phi(x) + \phi(y)$
3. $\phi(xy) = \phi(x)\phi(y)$

i.e. it is a linear map that preserves the product.

Definition 12. A Lie algebra homomorphism is a linear map $\phi : A \to B$ between Lie algebras over a field $\mathbb{F}$ that satisfies: $(x, y \in A, k \in \mathbb{F})$

$$\phi([x, y]_A) = [\phi(x)\phi(y)]_B \tag{A.6}$$

We can then note that given a vector space V, the space of linear maps from V to itself, $\mathrm{End}(V)$, naturally has a (Lie) algebra structure (with "product" given by composition and bracket given by the commutator). So the following definition makes sense.

Definition 13. A representation of a (Lie) algebra A on a vector space V is a (Lie) algebra homomorphism $\rho : A \to \mathrm{End}(V)$.

More explicitly, for Lie algebras we have

$$\rho_g(\rho_h(v)) - \rho_h(\rho_g(v)) = \rho_{[g,h]}(v) \tag{A.7}$$

i.e. "The matrix $\rho_{[g,h]}$ is the commutator of the matrices ρ_g and ρ_h."

Furthermore, the following remarks are useful in practical situations:

1. A map $\rho : A \to \mathrm{End}(A)$ is equivalent to a map $\rho : A \otimes V \to V$.
2. Given representations over vector spaces V and W it's possible to construct one over the duals, over the direct sum and over the tensor product. In particular for the tensor product we have:

$$\rho_{V \otimes W} = \rho_V \otimes 1 + 1 \otimes \rho_W \tag{A.8}$$

A.3 Short exact sequences

Definition 14. Given groups A, B and C with maps $f : A \to B$, $g : B \to C$, the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{A.9}$$

is called a *exact at* B if $\operatorname{im} f = \ker g$.

Note that in particular we have $g \circ f = 0$, however when checking exactness this is not enough since it simply gives the inclusion $\operatorname{im} f \subseteq \ker g$.

Definition 15 (Short exact sequence)**.** Given groups A, B and C with maps $f : A \to B$, $g : B \to C$, the sequence

$$1 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 1 \tag{A.10}$$

is called a *short exact sequence* if it is exact at A, B and C.

Note that exactness at A is equivalent to injectiveness of f while exactness at C is equivalent to surjectiveness of g.

There is a typical situation in which we have an injective map into a group and a surjective map out of it such that the composition of the two is trivial:

$$1 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} B/A \longrightarrow 1 \tag{A.11}$$

where i is the inclusion of A in B and π is the projection to the quotient. Note that the quotient may be more accurately written as $B/\operatorname{im}(i)$ and it makes sense to take the quotient since $A \cong \operatorname{im}(i) = \ker(\pi)$ and the kernel is a normal subgroup. Any short exact sequence can be thought of in this way.

There is also another type of exact sequence which often appears:

$$1 \longrightarrow A \xrightarrow{i} A \oplus B \xrightarrow{\pi} B \longrightarrow 1 \tag{A.12}$$

where i is the inclusion of A in $A \oplus B$ and π is the projection onto the B part of the sum. i is clearly injective, π is clearly surjective and $\pi \circ i = 0$, so it is in fact exact. However in this situation there's not really a difference between A and B and we could just as easily have arrows going in the opposite direction. This does not always happen, but it's interesting enough to deserve a name:

Definition 16. A short exact sequence

$$1 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 1 \tag{A.13}$$

is called *split* if $B \cong A \oplus C$.

One can prove that this is equivalent to having a *left-splitting*, i.e. a map $j : B \to A$ with $j \circ f = id_A$. From the comment above, both of these lead to a map $k : C \to B$ with $g \circ k = id_C$ which is called a *right-splitting*. So a split sequence essentially "is exact in both directions". In addition, for abelian groups a right splitting also gives a left splitting and so the three properties are equivalent.

An example of a short exact sequence in physics is given by the Poincaré group of a vector space V which is generated by translations (given by V itself) and rotations ($\mathrm{SO}(V)$):

$$1 \longrightarrow V \xrightarrow{i} \text{Poincaré}(V) \xrightarrow{j} \mathrm{SO}(V) \longrightarrow 1 \tag{A.14}$$

i is the inclusion of translations and the map j is given by simply "forgetting" the translation part of the transformation. Again i is injective, j is surjective and $j \circ i = 0$. In addition, this sequence has a right splitting since we can include $\mathrm{SO}(V)$ in the Poincaré group but the groups are not abelian so that doesn't guarantee a left splitting. In fact we don't have a splitting since having the sequence in the opposite direction would mean that $\mathrm{SO}(V)$ is a normal subgroup of the Poincaré group, which it is not.

A.4 Simple rings

This section contains some results from [2, 3] that are useful in the section about Clifford modules. In particular there we said that algebras of the type $M_n(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ have unique representation $\mathbb{K}^n$ up to isomorphism. (Note that $\mathbb{C}$ had two representations which are *inequivalent* but still isomorphic)

That is a result about the representation theory of simple rings (which is a synonym of simple algebra). In fact the matrix rings $M_n(\mathbb{K})$ are simple:

Theorem 17 (Theorem 5.5 of [2])**.** Let $\mathbb{K}$ be a division ring and E a finite dimensional vector space over $\mathbb{K}$. Let $R = \mathrm{End}_{\mathbb{K}}(E)$, then R is simple and E is a simple R-module, i.e. a representation of R.

So we have that $R = M_n(\mathbb{K}) = \mathrm{End}_{\mathbb{K}}(\mathbb{K}^n)$ has $\mathbb{K}^n$ as a representation. The next theorem gives its uniqueness:

Theorem 18 (Corollary 4.6 of [2])**.** A simple ring has exactly one simple module up to isomorphism.

These results are collected in Theorem 5.6 of [3]:

Theorem 19. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and consider the ring $M_n(\mathbb{K})$. Then the natural representation ρ of $M_n(\mathbb{K})$ on the vector space $\mathbb{K}^n$ is, up to isomorphism, the only irreducible representation of $M_n(\mathbb{K})$.

In addition, the algebra $M_n(\mathbb{K}) \oplus M_n(\mathbb{K})$ has exactly two equivalence classes of irreducible representations, given by:

$$\rho_1(\phi_1, \phi_2) = \rho(\phi_1) \quad \text{and} \quad \rho_2(\phi_1, \phi_2) = \rho(\phi_2) \tag{A.15}$$

acting on $\mathbb{K}^n$.

In the section on Clifford modules, these two representations are labeled $\mathbb{K}^n_+$ and $\mathbb{K}^n_-$.

References

[1] Brian C. Hall. *Quantum Theory for Mathematicians*. Graduate Texts in Mathematics. Springer, 2013.

[2] Serge Lang. *Algebra*. Springer, 2002.

[3] H. Blaine Lawson and Marie-Louis Michelson. *Spin Geometry (PMS-38)*. Princeton University Press, 1989.