Supersymmetry lecture notes

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Contents

1 Introduction and review of useful concepts

This course used to be about the minimally supersymmetric standard model, now it is more general, giving the tools to build supersymmetric systems in general.

Idea: supersymmetry is relevant to any theory with fermions in it. $a, a^\dagger$ are part of a superalgebra. We'll see how supersymmetry is linked to fermions, dynamics, symmetries and so on.

We'll talk about supersymmetric field theories, some of which are exactly solvable and can be used as toy models for quantum field theory. Susy QFT is an important source of exact results.

In this course lots of "scary", subtle, physical concepts show up naturally, such as spinors, fermions, spacetime symmetries.

In addition, we have the opportunity of going depper into the underlying concepts of QFT which are not as emphasized in regular QFT courses.

In general what we mean by supersymmetry is any symmetry of a theory that doesn't fix a splitting into bosonic and fermionic states.

To begin we'll start by recalling the concepts of observables, states, symmetries and dynamics. It's important to note that in any multiparticle system with fermions, everything is automatically super, meaning we get a decomposition of operators into bosonic and fermionic operators. So we're developing structures which are really already there.

NOTIONS, RELATIONS among them

Classical: Hamiltonian generates time translation because of $\frac{df}{dt} = \{H, f\}$

Quantum: H with $\frac{d|\psi\rangle}{dt} = H|\psi\rangle$

meas: assigning number to observables

	Observables	States	Symmetries: finite	Symmetries: infinitesimal	Dynamics
Classical	(Comm.) algebra* A of functions on a phase space (polynom. $\mathbb{R}[x,p]$) * with a Poisson bracket	• A map from obs. to numbers $f: \mathbb{R}[x,p] \to \mathbb{R}$ (evaluate on points) • A phase space $\mathbb{R}^2$ (symplectic space) — Correspondence between algebra and phase space	Group of "Canonical transformations" (simplectic form preserving) of the phase space.	A Lie algebra of symplectic form vector fields (exponentiates to something preserving sympl. form) $d\underbrace{\omega(X,-)}_{=df} = 0$: derivative of sympl. form in direction of vector field vanishes	Choice of H
Quantum	(Algebra of) (Hermitian) operators (on a Hilbert space)	• Hilbert space • I can only meas. exp. vae $\langle\psi\|A\|\psi\rangle = \langle A\rangle_\psi$ → as above see state space as collection of meas.	Group of unitary transformations of Hilbert space. We want it to preserve the data that lets us compute probabilities i.e. Hilbert space inner product!	A Lie Alg. of Herm. ops: the Lie bracket is $[A,B] = AB - BA$	Choice of H

In the table it's useful to recall that given the symplectic form

$$\omega = dx \wedge dp \tag{1.1}$$

and a generic vector field on $\mathbb{R}^2$

$$X = f(x,p)\frac{\partial}{\partial x} + g(x,p)\frac{\partial}{\partial p} \tag{1.2}$$

we can contract ω with X and get the following one form

$$\omega(X, -) = f(x,p)dp - g(x,p)dx \tag{1.3}$$

and setting its differential to 0 we get

$$0 = d\omega(X, -) = \frac{\partial f}{\partial x} dx \wedge dp - \frac{\partial g}{\partial p} dp \wedge dx \implies \frac{\partial f}{\partial x} + \frac{\partial g}{\partial p} = 0 \tag{1.4}$$

So the condition $d\omega(X, -) = 0$ is simply the condition that the divergence of the vector field vanishes.

The table refers to a theory of quantum mechanics with only fermions. In order to include fermions as well, everything should be extended to its super form, super Hilbert space, super algebra and so on.

Note also that there are two important correspondences:

- State space and algebra of observables
- Lie algebra of infinitesimal symmetries and algebra of observables

1.1 Fock Space

Let $\mathscr{H}$ be a one particle Hilbert space. In order to describe a system with an arbitrary number of particles we promote states to creation operators in the following way. Define

$$|0\rangle \tag{1.5}$$

to be the vacuum state. Now given $|\psi\rangle \in \mathscr{H}$, we define $a_\psi^\dagger$ the operator that creates a particle in the state ψ, namely

$$\psi = a_\psi^\dagger \tag{1.6}$$

Now, if I have two creation operators $a_1^\dagger, a_2^\dagger$, I should know how they commute, since both $a_1^\dagger a_2^\dagger |0\rangle$ and $a_2^\dagger a_1^\dagger |0\rangle$ should give rise to the same two particle system, meaning that the two should only differ by a phase α:

$$a_1^\dagger a_2^\dagger |0\rangle = \alpha a_2^\dagger a_1^\dagger |0\rangle = \alpha^2 a_1^\dagger a_2^\dagger |0\rangle \tag{1.7}$$

so $\alpha^2 = 1, \alpha = \pm 1$, therefore I can set

$$a_1^\dagger a_2^\dagger |0\rangle = \pm a_2^\dagger a_1^\dagger |0\rangle \tag{1.8}$$

If I choose $+$, the particles I'm describing are called bosons, while for $-$ they are fermions. In other words, bosons commute, while fermions anticommute. This then already gives rise to the Pauli exclusion principle, which states that I can't have two fermions in the same quantum state, so if I apply $a^\dagger$ twice to the vaccum I should get nothing:

$$a^\dagger a^\dagger |0\rangle = -a^\dagger a^\dagger |0\rangle = 0 \tag{1.9}$$

These two possibilities give rise to two kinds of Fock space:

- $\mathrm{Sym}(\mathscr{H}) = \{|0\rangle, a^\dagger|0\rangle, \dots, a^{\dagger n}|0\rangle, \dots\}$, bosons, infinite dimensional
- $\Lambda(\mathscr{H}) = \{|0\rangle, a^\dagger|0\rangle\}$, fermions, two dimensional

To build a Fock space with both bosons and fermions we will need both of these, which will lead to the notion of a super Hilbert space.

Lecture 1.2 (19/4): Super structures

In order to build a multiparticle system with both bosons and fermions, we need to start with a Hilbert space with can have both bosonic and fermionic states. In other words, the Hilbert space should have a splitting:

$$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \tag{1.10}$$

where $\mathscr{H}_0$ represents the bosonic states and $\mathscr{H}_1$ the fermionic ones. Note that a splitting is always possible, but a *given* splitting is extra information because it tells us what is bosonic and what is fermionic.

Now we can generalize the procedure we did in the case that $\mathscr{H}$ was 1 dimensional. Let $|\psi_1\rangle, \dots, |\psi_n\rangle \dots$ form a basis for $\mathscr{H}$. We then obtain creation operators $a_1^\dagger, \dots, a_n^\dagger$ by saying that when these act on the vacuum we obtain the states above.

We then obtain the Fock space by letting these operators act on the vaccum. However, these should not act freely (without any extra relations), but should satisfy some commutation relations. We said that bosonic creation operators commute and fermionic ones anticommute. This is not enough because I also need to fix commutation relations for bosons with fermions. The standard rule is the following

$$a_i^\dagger a_j^\dagger - (-)^{|a_i^\dagger||a_j^\dagger|} a_j^\dagger a_i^\dagger = 0 \tag{1.11}$$

where

$$|a_i| = \begin{cases} 0, & \text{if } |\psi_i\rangle \in \mathscr{H}_0 \\ 1, & \text{if } |\psi_i\rangle \in \mathscr{H}_1 \end{cases} \tag{1.12}$$

In the formula above we see that bosons commute with bosons and also with fermions, instead fermions anticommute with fermions.

The formula above defines the supercommutator:

$$[a_i^\dagger, a_j^\dagger] := a_i^\dagger a_j^\dagger - (-)^{|a_i^\dagger||a_j^\dagger|} a_j^\dagger a_i^\dagger \tag{1.13}$$

and the equation states that all creation operators supercommute.

In addition, the sign $(-)^{|a_i^\dagger||a_j^\dagger|}$ is called Koszul sign and will appear in many of the coming formulas and can be derived using the so called Koszul sign rule: when two operators are interchanged we have to insert the Koszul sign.

2 Super structures

In this section we define and study super analogues of mathematical structures that we need in quantum mechanics. In general *super* is a synonym for $\mathbb{Z}_2$-graded, which means that we have two copies of the "normal" structure.

A super vector space V is simply a vector space with a splitting $V = V_0 \oplus V_1$. In the following the first part is called even and the second odd.

A super algebra A is an algebra with a splitting $A = A_0 \oplus A_1$, where again $a_0 \in A_0$ is called even and $a_1 \in A_1$ is odd. However here there's a subtlety: the multiplication must respect the grading (or *respect parity*). Recall that in an algebra we have a multiplication $A \times A \to A$ and "respecting the grading" simply means that when restricting to the different degrees we have a multiplication multiplication $A_i \times A_j \to A_{i+j \mod 2}$ which is simply a fancy way of saying the following

- $even \cdot even = even$
- $even \cdot odd = odd$
- $odd \cdot odd = even$

The definition above can also be stated as: a superalgebra is a super vector space with a parity respecting multiplication. This definition is quite similar to the one of an algebra: an algebra is a vector space with a multiplication.

2.1 Super Lie algebras

We now want to define a super Lie algebra. Recall that a Lie algebra is a vector space with a Lie bracket. The definition of a super Lie algebra will be almost identical but adding *super* in the right places. However, we need to first define what is a super Lie bracket. Let's start by recalling what a Lie bracket is, which simply captures what the idea of a commutator is:

- antisymmetric: $[A, B] = -[B, A]$
- satisfies the Jacobi identity: $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$

The supercommutator has similar properties, but with the Koszul sign for every interchange:

- super antisymmetric: $[A, B] = -(-)^{|A||B|}[B, A]$
- satisfies the super Jacobi identity:

$$[[A, B], C] + (-)^{|A|(|B|+|C|)}[[B, C], A] + (-)^{|C|(|A|+|B|)}[[C, A], B] = 0$$

Lecture 2.1 (24/5) (Super) Lie algebras

Last time we tried to write super versions of various mathematical objects that are physically relevant. In particular: super vector spaces, super Hilbert space, super algebras and super commutator. Let us recall the definition of the supercommutator:

$$[A, B] = AB - (-)^{\sigma_A \sigma_B} BA \tag{2.1}$$

with $\sigma = 0$ for bosons and $\sigma = 1$ for fermions and $(-)^{\sigma_A \sigma_B}$ is the Koszul sign. The super commutator then satisfies the super Jacobi identity:

$$(-)^{\sigma_A \sigma_C} [[A, B], C] + (-)^{\sigma_A \sigma_B} [[B, C], A] + (-)^{\sigma_C \sigma_B} [[C, A], B] = 0 \tag{2.2}$$

First of all, why is the Jacobi identity important? Very often it happens that we have the concept of a bracket but we don't have a product, this structure is called a Lie algebra. It's important to note in fact that Lie algebras are not algebras.

Example. $\mathfrak{sl}(2)$: it's three dimensional, we can think of it as spanned by the Pauli matrices. We have $[L_x, L_y] = L_z$ and cyclic permutations of that. (written like this it's a real Lie algebra so I can represent it by real matrices)

Classically $\vec{L} = \vec{r} \times \vec{p}$, so (considering p as the generator of translations) we have

$$L_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \tag{2.3}$$

$$L_y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \tag{2.4}$$

$$L_z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \tag{2.5}$$

The commutator tells us someting about rotations in space: the difference between doing first a rotation A through one axis and then another B through another axis, or doing B and then A is just a rotation through the third axis. Let's calculate it explicitly:

$$L_x L_y = \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \tag{2.6}$$

$$= y\frac{\partial}{\partial x} + yz\frac{\partial^2}{\partial x \partial z} - z^2\frac{\partial^2}{\partial y \partial x} - yx\frac{\partial^2}{\partial z^2} + zx\frac{\partial^2}{\partial y \partial z} \tag{2.7}$$

$$L_y L_x = x\frac{\partial}{\partial y} + yz\frac{\partial^2}{\partial x \partial z} - z^2\frac{\partial^2}{\partial y \partial x} - yx\frac{\partial^2}{\partial z^2} + zx\frac{\partial^2}{\partial y \partial z} \tag{2.8}$$

so if i just apply one after the other I get something quite complicated, but taking the commutator

$$[L_x, L_y] = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = L_z \tag{2.9}$$

I get another rotation. This means that they're not an algebra but they're a Lie algebra. Actually this works more generally for vector fields: their product is not a vector field, but their commutator is again a vector field.

So more abstractly I don't have a commutator but I have a bracket and it's important to make sure that it satisfies the Jacobi identity.

Definition 1. A super Lie algebra is a super vector space $\mathfrak{g}$ with a bracket operation $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ that:

1. preserves parity
2. is super antisymmetric
3. satisfies the super Jacobi identity

(i.e. exactly like a Lie algebra but with everything super and parity preservation) Let's unpack that a bit. First of all, let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and note that the bracket is actually three different maps:

$$\mathfrak{g}_0 \otimes \mathfrak{g}_0 \to \mathfrak{g}_0 \tag{2.10}$$
$$\mathfrak{g}_0 \otimes \mathfrak{g}_1 \to \mathfrak{g}_1 \tag{2.11}$$
$$\mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathfrak{g}_0 \tag{2.12}$$

Now for the let's consider the different cases of the super Jacobi identity:

1. $+++$: $\mathfrak{g}_0$ is a normal Lie algebra
2. $++-$: $g, h \in \mathfrak{g}_0, \psi \in \mathfrak{g}_1$, so the super Jacobi is:

   $$[[g,h],\psi] + [[h,\psi],g] + [[\psi,g],h] = 0 \tag{2.13}$$

   or equivalently:

   $$[[g,h],\psi] = [g,[h,\psi]] - [h,[g,\psi]] \tag{2.14}$$

   i have two bosonic symmetries and I'm considering their action on the fermionic symmetries: on the right I'm considering the action of g and h on ψ taken in different orders and subtracting the two, on the left I'm saying that that is equal to the action of $[g,h]$ on ψ.

   Essentially this is saying that we have a representation of $\mathfrak{g}_0$ over $\mathfrak{g}_1$ (see appendix A.2): we have the map $\rho_{\mathfrak{g}_1} : \mathfrak{g}_0 \to \mathrm{End}(\mathfrak{g}_1)$ given by $g \mapsto [g,\cdot]$ and it's supposed to satisfy

   $$\rho_{[g,h]}(\psi) = \rho_g(\rho_h\psi) - \rho_h(\rho_g\psi) \tag{2.15}$$

   in order to be a Lie algebra homomorphism. But this equality is exactly the one above.

3. $+--$: $g \in \mathfrak{g}_0, \psi, \xi \in \mathfrak{g}_1$, so we have

$$[[\psi, \xi], g] - [[\xi, g], \psi] + [[g, \psi], \xi] = 0 \tag{2.16}$$

   or also

$$[g, [\psi, \xi]] = [\psi, [\xi, g]] + [[g, \psi], \xi] \tag{2.17}$$

   The bracket of fermions is compatible with the $\mathfrak{g}_0$ symmetry, so we're essentially in the situation of the representation of the tensor product $\mathfrak{g}_1 \otimes \mathfrak{g}_1$ and this relation is what we wrote for the representation of the tensor product.

4. $---$: $\psi, \xi, \lambda \in \mathfrak{g}_1$, so we have

$$[[\psi, \xi], \lambda] + [[\xi, \lambda], \psi] + [[\lambda, \psi], \xi] = 0 \tag{2.18}$$

   equivalently:

$$[[\psi, \psi], \psi] = 0, \quad \forall \psi \in g_1 \tag{2.19}$$

   which would be trivial for bosons since already the commutator gives 0. Instead here $[\psi, \psi]$ is in general nontrivial, however it shouldn't act nontrivially on the fermion itself. Or said differently "no fermionic symmetry can only generate one bosonic symmetry but not more than that", or "a fermionic symmetry commutes with its own square".

Recipe: take a Lie algebra $\mathfrak{g}_0$, take a representation and call it $\mathfrak{g}_1$ and check that every symmetry commutes with its own square.

2.2 Four ways of looking at $\mathfrak{gl}$(1,1)

In general given V a vector space, $\mathfrak{gl}(V)$= linear maps $V \to V$ = matrices acting in V. So $\mathfrak{gl}(n) = \mathfrak{gl}(\mathbb{C}^n)$ or $\mathfrak{gl}(\mathbb{R}^n)$. So then $\mathfrak{gl}(n|m) = \mathfrak{gl}(\mathbb{C}^{n|m})$ with n dimensional even part and m dimensional odd.

Take matrices:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad R = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.20}$$

and we have $[E, F] = P$, $[R, E] = E$, $[R, F] = -F$ and P commutes with everything.

In the bosonic case I could take $\mathfrak{sl}(2) \subseteq \mathfrak{gl}(2)$ which is the traceless part. Instead here the identity is on the RHS of a commutator, so I can't throw it away. Same goes for all the others. So $\mathfrak{gl}(1|1)$ is NOT a product, also it is not simple, there are no invariant subalgebras, so I can't take any quotients. Semisimple means if there are invariant subalgebras then....

In supersymmetric quantum mechanics we have that P is the hamiltonian, it generates time translations, but it has a fermionic square root, which is a general feature. But then the commutator that we need to check to have a super Lie algebras is trivial, since translations commute with everything.

Lecture 2.2 (26/5) Four ways of looking at $\mathfrak{gl}(1|1)$

$\mathfrak{gl}(1|1)$ = matrices acting on a $1|1$ dimensional super vector space $V = V_0 \oplus V_1$ with a convenient basis written above.

four ways of looking at $\mathfrak{gl}(1|1)$:

1. linear trasformations of a $1|1$ dim vector space
2. differential operators on the odd line $\mathbb{C}^{0|1}$
3. the algebra of creation and annihilation operators for one fermionic state (we talked about the creation operators in general terms already). The Fock space is $\mathbb{C}^{1|1}$ and te singlle particle Hilbert space is $\mathbb{C}^{0|1}$. Essentially this says "$\mathfrak{gl}(1|1)$ is the simplest Clifford algebra".
4. the algebra of spacetime symmetries as "$\mathcal{N} = 2$ supersymmetric quantum mechanics". Essentially this says "$\mathfrak{gl}(1|1)$ is the simplest supersymmetry algebra".

We'll also see that it's in some ways like representation theory of SU(2) but there will be some key differences: we'll have only two representations spin 0 and spin 1/2, in this sense it'll be simpler.

The logical order would be to do first 3. and then 4. because 3. introduces Clifford algebra which is needed for 4. But we'll start with 4. to examine the physical supersymmetric system and get a feel for it, to then go back to 3.

2.2.1 Linear algebra

In the normal case, $\mathfrak{gl}(2)$ is a product of Lie algebras:

- $\mathfrak{sl}(2)$ = traceless part
- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ = center (commutes with everything)

We can study them separately, find representations of the two factors and then use those representations to get representations of $\mathfrak{gl}(2)$.

In the super case this doesn't quite work. The trace should vanish on commutators $\mathrm{tr}([A, B]) = 0$, so the traceless matrices form a subalgebra.

We would like for the same to be true for the super trace, but $P = [E, F]$ is not traceless with the normal definition. So let's consider odd matrices and see the result:

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 & B' \\ C' & 0 \end{pmatrix} = \begin{pmatrix} BC' & 0 \\ 0 & CB' \end{pmatrix} \tag{2.21}$$

so the anticommutator (anti since we have two odd matrices) is

$$\begin{pmatrix} BC' + B'C & 0 \\ 0 & CB' + C'B \end{pmatrix} \tag{2.22}$$

so that suggests the following formula for the supertrace:

$$\mathrm{str} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{tr}(A) - \mathrm{tr}(D) = \mathrm{tr}\left((-)^F \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \tag{2.23}$$

where $(-)^F$ is the parity operator

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.24}$$

on the supervector space.

So now let $\mathfrak{sl}(1|1) = \{X \in \mathfrak{gl}(1|1) : \mathrm{str}(X) = 0\}$ which is spanned by E, F, P since $\mathrm{str}(R) = 1 \neq 0$. For now it looks similar to $\mathfrak{sl}(2)$ since we have three matrices, but it's different since we have the identity, meaning we still have a central element. While what is taken out, R, is NOT central.

Another element of superlinear algebra which is different is the transpose. So the key property of the trace was that it vanishes on commutators, what is then the key property of the transpose? Normally we have $(XY)^T = Y^T X^T$, but because we're moving the matrices past one another, then for the super transpose we need to insert a sign

$$(XY)^T = (-)^{\sigma_x \sigma_y} Y^T X^T \tag{2.25}$$

Let's take a product of odd matrices:

$$\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 & B' \\ C' & 0 \end{pmatrix} \right]^T = \begin{pmatrix} BC' & 0 \\ 0 & CB' \end{pmatrix}^T = \begin{pmatrix} C'^T B^T & 0 \\ 0 & B'^T C^T \end{pmatrix} \tag{2.26}$$

$$= - \begin{pmatrix} 0 & -C'^T \\ B'^T & 0 \end{pmatrix} \begin{pmatrix} 0 & -C^T \\ B^T & 0 \end{pmatrix} \tag{2.27}$$

We can see that the following choice works:

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -C^T \\ B^T & 0 \end{pmatrix} \tag{2.28}$$

But this seems arbitrary, I could have put the $-$ on B^T. To see why this is the better choice, let's examine another property which is what happens in the euclidean scalar product: $g(v, Xw) = g(X^T v, w)$. Here also we need a sign for the super case:

$$g(v, Xw) = (-)^{\sigma_v \sigma_X} g(X^T v, w) \tag{2.29}$$

Let's consider v and X odd and w even:

$$g\left(\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0 & B\\ C & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right)=C=C^T=-g\left(\begin{pmatrix}0 & -C^T\\ B^T & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right) \tag{2.30}$$

So here we see that the above choice works.

Note: it's no longer true that $(X^T)^T = X$, I actually get a $-$, but applying it four times I get back the original matrix. This is typical of fermions! When doing a certain operation gives the identity in the normal case, in the case of fermions it gives a $-$.

2.2.2 Differential operators on the super line

Let's consider $\mathbb{C}^{0|1}$. How is it different from a line? We see it by considering polynomial functions on the line $V = \mathbb{C}^{1|0}$: we have $1, x, x^2, \cdots =: \mathbb{C}[x] =$ algebra of creation operators, considering x as a creation operator. But where does x live? Well it's a functional on a vector space so it lives in the dual. So the algebra is generated by $V^\vee$ (the dual vector space which is simply made up of the linear coordinates).

So now what are the functions on $V = \mathbb{C}^{0|1}$? Well we have the algebra of fermionic creation operators generated by $V^\vee$ ("Grassmann numbers"). It is two dimensional, I only have $1, \theta$ with $\theta^2 = 0$ by supercommutativity.

For the normal line, a "point on the line" is a map from functions to numbers $\mathbb{C}[x] \to \mathbb{C}$, it sends x to some number that is it's coordinate.

Instead, the odd line has only one point! We only have the map $\mathbb{C}[\theta] \to \mathbb{C}$ which sends θ to 0.

Recall: if I want a map $\mathbb{R}[x]/(x^2+1) \to \mathbb{R}$ there are no possibilities, since on the left $x^2 = -1$ and on the right nothing squares to -1. This is way of saying that the space on the left has "no real points" and the solution is to use complex points.

The same reasoning applies above! The odd line has only one *even* point, but we should actually be looking for odd points!

Now let's move to the concept of differential operators. We have $\frac{\partial}{\partial\theta}$. So in total we can construct the following:

$$1, \theta, \frac{\partial}{\partial\theta}, \theta\frac{\partial}{\partial\theta} \tag{2.31}$$

of which the first and the last are even while the middle ones are odd. These have the following commutators:

$$[1, \cdot] = 0 \tag{2.32}$$
$$\left[\frac{\partial}{\partial\theta}, \theta\right] = 1 = \left[\theta, \frac{\partial}{\partial\theta}\right] \tag{2.33}$$
$$\left[\theta\frac{\partial}{\partial\theta}, \theta\right] = \theta \tag{2.34}$$
$$\left[\theta\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right] = -\frac{\partial}{\partial\theta} \tag{2.35}$$

which are exactly the commutators we have on $\mathfrak{gl}(1|1)$ with the following identifications!

$$1 = P, \quad \theta = E, \quad \frac{\partial}{\partial\theta} = F, \quad \theta\frac{\partial}{\partial\theta} = R \tag{2.36}$$

2.2.3 Creation and annihilation

We can see the action of F:

$$F : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.37}$$

and I can rename the two states and call the first one $|0\rangle$ "vacuum" and call the second $|\psi\rangle$ "one particle fermionic state". We then have:

$$F : |0\rangle \mapsto |\psi\rangle \tag{2.38}$$
$$E : |\psi\rangle \mapsto |0\rangle\,, \quad |0\rangle \mapsto 0 \tag{2.39}$$

So E does the opposite of F. Instead $-R$ somehow "counts fermions", but in order to understand this we actually have to define

$$\tilde{R} = R - \frac{1}{2}P = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.40}$$

which has the same commutation relations as R since P is central. Now we have:

$$\tilde{R}\,|0\rangle = 0 \tag{2.41}$$
$$\tilde{R}\,|\psi\rangle = \tilde{R}F\,|0\rangle = (F\tilde{R} - F)\,|0\rangle = -F\,|0\rangle = -\psi \tag{2.42}$$

So $-\tilde{R}$ tells us that $|0\rangle$ has 0 fermionic particles and $|\psi\rangle$ has 1.
Lecture 3.1/2 (3/5) Unitarity (and other things)

3 Supersymmetric Quantum Mechanics

3.1 Unitarity

Now we want to consider $\mathfrak{gl}(1|1)$ as $\mathcal{N} = 2$ supersymmetric quantum mechanics. But in order to have a quantum theory we need to have unitarity.

Let us first clear up the difference between observables and infinitesimal symmetries. The first are Hermitian operators, while infinitesimal symmetries are represented by antihermitian operator. And clearly I can get a Hermitian operator from an antihermitian one and vice versa, but the way of doing this is not unique: I can multiply by i or $-i$. Typically in physics I multiply the observable by i and then I exponentiate it to get a non infinitesimal symmetry, i.e. a unitary operator. We'll see that the relation between superhermitian and superantihermitian operators is not as simple. Let us first study an example.

Example (Real Lie algebras)**.** First of all note that the observables (e.g. Hermitian operators) form a real algebra, not a complex one. Symmetry algebras usually have a real form, normally dictated by unitarity.

Considering the typical example of rotations, the commutator of the observables is the following:

$$[L_i, L_j] = i\epsilon_{ijk}L_k \tag{3.1}$$

while for infinitesimal symmetries we have:

$$[R_i, R_j] = \epsilon_{ijk}R_k \tag{3.2}$$

but mathematically they're just different conventions and both are REAL Lie algebras. It's clear that there should be this difference: the commutator of Hermitian or antihermitian operators is antihermitian, so considering infinitesimal symmetries I'll just have another antihermitian operator on the right, while for observables i need a factor of i (or $-i$) in order to get an antihermitian operator from a hermitian one.

The same complex Lie algebra can have different real forms! Take for example $\mathfrak{su}(2) = \mathfrak{so}(3)$ represented by $\sigma_x, \sigma_y, \sigma_z$, but there's also $\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{so}(2,1)$ that can be represented by:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.3}$$

with commutators:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H \tag{3.4}$$

which look exactly like ladder operators. This means that if I can make a complex change of basis of L_x, L_y, L_z I could get the same algebra: I simply take $L_z, L_\pm = L_x \pm iL_y$. Note however that $\mathfrak{so}(2,1)$ is not compact, while $\mathfrak{so}(3)$ IS. (Essentially compact means if I do the non infinitesimal transformation I eventually get back to the same point, which is true for rotations, but not for boosts)

Before getting to a super Hilbert space and superhermitian matrices, let us recall the definition of Hilbert space

Definition 2. A Hilbert space is a complex vector space $\mathscr{H}$ with a Hermitian inner product $\langle\cdot,\cdot\rangle : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$ with the following properties:

1. linear in the second argument: $\langle u, \alpha v\rangle = \alpha\langle u, v\rangle$
2. conjugate symmetric: $\langle u, v\rangle = \overline{\langle v, u\rangle}$
3. positive definite: $\langle u, u\rangle > 0$

A non infinitesimal symmetry M preserves the inner product if

$$\langle Mu, Mv\rangle = \langle u, v\rangle \quad \forall u, v \in \mathscr{H} \tag{3.5}$$

Recalling the definition of the adjoint:

$$\langle u, Mv\rangle = \langle M^\dagger u, v\rangle \tag{3.6}$$

we can then write

$$\langle M^\dagger M u, v\rangle = \langle u, v\rangle \implies M^\dagger M = \mathbb{1} \tag{3.7}$$

which is what we call unitarity.

For infinitesimal symmetries we have $M = \mathbb{1} + \epsilon X$, which inserted above gives:

$$(\mathbb{1} + \epsilon X^\dagger)(\mathbb{1} + \epsilon X) = \mathbb{1} + \epsilon(X^\dagger + X) = \mathbb{1} \implies X^\dagger = -X \tag{3.8}$$

$$(\text{or } \langle u, Xv\rangle + \langle Xu, v\rangle = 0) \tag{3.9}$$

This is what was meant when we said that infinitesimal symmetries are represented by antihermitian operators.

Definition 3. A super Hilbert space structure is a super vector space $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$ with an inner product $g : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$ with the properties:

1. linear in the second argument: $g(u, \alpha v) = \alpha g(u, v)$
2. conjugate super symmetric: $g(u, v) = (-)^{|u||v|}\overline{g(v, u)}$
   ($|u| = 0$ for $u \in \mathscr{H}_+$, $|u| = 1$ for $u \in \mathscr{H}_-$)
3. positive definite over $\mathscr{H}_+$: $g(u, u) > 0$ for $u \in \mathscr{H}_+$

4. σ-positive-definite over $\mathscr{H}_-$: $i\sigma g(\psi,\psi) > 0$ for $\psi \in \mathscr{H}_-$ with $\sigma = \pm 1$ is a sign that we can choose.

But we should be worried since there's not a canonical choice for σ.

If $\left(\mathscr{H}, \langle , \rangle\right)$ is a Hibert space with a splitting $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$, then $\mathscr{H}$ with the inner product $g_\sigma(\cdot,\cdot) = \langle \cdot,\cdot\rangle_0 - i\sigma\langle\cdot,\cdot\rangle_1$ is a σ-positive-definite Hilbert space (I can also go in the opposite direction).

Following the normal case, we can then define a superantihermitian operator by

$$g(Xu, v) + (-)^{|X||u|} g(u, Xv) = 0 \tag{3.10}$$

and it may also be called an infinitesimal supersymmetry. By taking a basis for $\mathscr{H}$ we can write X as

$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{3.11}$$

by decomposing its action on $\mathscr{H}_+$ and on $\mathscr{H}_-$. Here $\alpha, \beta, \gamma, \delta$ are generally operators. We then have four cases to study for the superantihermitian matrix:

1. u and v even
$$\langle \alpha u, v\rangle_0 + \langle u, \alpha v\rangle_0 = 0 \tag{3.12}$$

2. ψ and ξ odd
$$-i\sigma(\langle \delta\psi, \xi\rangle_1 + \langle \psi, \delta\xi\rangle_1) = 0 \tag{3.13}$$
these two basically say that α and δ are antihermitian matrices, which is what we expected since they're the even parts which typically follow the normal rules.

3. ψ odd and u even:
$$\langle \beta\psi, u\rangle_0 - (-)i\sigma\langle \psi, \gamma u\rangle_1 = 0 \tag{3.14}$$
but they can also be in opposite order:
$$-i\sigma\langle \gamma u, \psi\rangle_1 + \langle u, \beta\psi\rangle_0 = 0 \tag{3.15}$$
In fact these are the same condition, since we can first rewrite the last equation as
$$\langle i\sigma\gamma u, \psi\rangle_1 + \langle \beta^\dagger u, \psi\rangle_1 = 0 \tag{3.16}$$
and then we get that
$$\beta^\dagger = -i\sigma\gamma \tag{3.17}$$
or equivalently
$$\gamma^\dagger = -i\sigma\beta \tag{3.18}$$

So this is what it means for a matrix to be super anti Hermitian, where it's important to note that β and γ are not independent.

Let us study the case $\mathfrak{sl}(1|1)$.

In particular we now know what it means to take all super unitary $1|1$ matrices, i.e. $\mathfrak{u}(1|1)$ starting from $\mathfrak{gl}(1|1,\mathbb{C})$. So it's relatively simple since $\alpha,\beta,\gamma,\delta$ are just numbers. In particular α and δ have to be anti Hermitian, which for numbers means they're purely imaginary $\alpha,\delta\in i\mathbb{R}$. We can then start writing a basis:

$$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{3.19}$$

while for γ and β we have $\bar{\gamma}=-i\sigma\beta$. We then get

$$q_1 := \begin{pmatrix} 0 & 1 \\ i\sigma & 0 \end{pmatrix}, \quad q_2 := \begin{pmatrix} 0 & i \\ \sigma & 0 \end{pmatrix} \tag{3.20}$$

we see that they have the following commutators:

$$[q_1,q_1] = 2q_1^2 = 2\sigma\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \tag{3.21}$$

$$[q_2,q_2] = 2q_2^2 \tag{3.22}$$

$$[q_1,q_2] = 0 \tag{3.23}$$

and these are different commutation relations than the ones we had for $\mathfrak{gl}(1|1)$, but I can make them the same by a complex change of coordinates $(q_1 \pm iq_2)$ which of course is not allowed in the real Lie algebra because it messes up the unitary structure. So this is its unitary real form, but as a complex Lie algebra it's the same one we were writing before.

Same idea as in $\mathfrak{su}(2)$! We first complexify it and study the representations of $\mathfrak{sl}(2,\mathbb{C})$ which is easier because we can build ladder operators. At the end of the day you then demand that the lowering operator is the adjoint of the raising operator which makes it a real representation of the thing you started with.

The reason it's called $\mathcal{N}=2$ supersymmetric quantum mechanics is that if we think of

$$p = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \tag{3.24}$$

as the Hamiltonian, then we now have 2 anticommuting square roots of the time translation operator. So $\mathcal{N}=2$ supersymmetric quantum mechanics is really just the unitary real form of $\mathfrak{gl}(1|1)$, but we have to be careful with the unitarity constraints. First, remember that an infinitesimal bosonic symmetry is represented by an antihermitian matrix and we could get an observable out of it by dividing by $\pm i$. So hopefully there's a similar way of associating an observable matrix to something that generates a fermionic symmetry.

If X is odd superantihermitian we have the constraint

$$\langle u, X\psi\rangle - i\sigma\langle Xu, \psi\rangle = 0 \tag{3.25}$$

and we want to use this to say that some matrix related to X is Hermitian (and therefore an observable). Let $\sigma = +1$ and let j be a square root of i (or $i\sigma$ more generally).

...

$$\begin{aligned} \langle u, X\psi\rangle &= j^2\langle Xu, \psi\rangle && (3.26) \\ j^{-1}\langle u, X\psi\rangle &= j\langle Xu, \psi\rangle && (3.27) \\ \langle u, j^{-1}X\psi\rangle &= \langle j^{-1}Xu, \psi\rangle && (3.28) \end{aligned}$$

where in the last step we used the fact that since j is a square root of i then its inverse is just its complex conjugate. So $j^{-1}X$ is Hermitian and observable.

So just as in the standard case there's a correspondance between generators of odd symmetries (i.e. superantihermitian things) and observables, this time with a fourth root of -1 instead of a square root. And this is why when you're not careful of thinking when an observable is an observable and when it's generating a symmetry you get confused.

So let's see what observables correspond to the two matrices we got before. Also, traditionally we use Q to represent supercharges, meaning Hermitian operators, so that's why before we used lowercase letters.

$$\begin{aligned} Q_1 &:= \frac{q_1}{j} = j^{-1}\begin{pmatrix} 0 & 1 \\ j^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & j^{-1} \\ j & 0 \end{pmatrix} && (3.29) \\ Q_2 &:= \frac{q_2}{j} = j^{-1}\begin{pmatrix} 0 & j^2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & j \\ j^{-1} & 0 \end{pmatrix} && (3.30) \end{aligned}$$

Which are in fact Hermitian since $\bar{j} = j^{-1}$. If I square them I then get the identity, whereas if I squared the previous ones I would get i times the identity, but now the i is now absorbed in the Qs. This is why in the physics convention if I write down the structure constants for a real super Lie algebra then I get is for the bosonic things and no i for the fermionic things: commutators of Hermitian matrices are antihermitian, while anticommutators of Hermitian matrices are Hermitian. Also we get the weird result that the structure constants of a real super Lie algebra are all purely real or purely imaginary.

We can then pick $j = \frac{1}{\sqrt{2}}(1+i)$ so we can write the Qs in terms of Pauli matrices (since every 2 by 2 Hermitian matrix is a real linear combination of Pauli matrices and since they have zero diagonal there's no σ_z):

$$Q_1 = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y) \tag{3.31}$$

$$Q_2 = \frac{1}{\sqrt{2}}(\sigma_x - \sigma_y) \tag{3.32}$$

So this is an explanation of why unitarity is strange for superalgebras: because it's the correspondance in what you mean with an observable and what you mean by a generator of a symmetry, which is more subtle than in the normal case.

Why did we choose $\sigma = +1$? Because we wanted the Hamiltonian to have a positve spectrum. We define

$$P := \frac{p}{i} = \mathbb{1} = Q_1^2 \tag{3.33}$$

This implies $P \geq 0$ since for a state $|u\rangle$ we have

$$\langle u| P |u\rangle = \langle u| Q_1^\dagger Q_1 |u\rangle = ||Q_1 |u\rangle ||^2 \geq 0 \tag{3.34}$$

So not only is P a operator with real eigenvalues, but also its real eigenvalues are the squares of the real eigenvalues of Q_1, and this is called the *BPS bound*. So $\sigma = +1$ is a convention to get the spectrum of P to be positive (doing the opposite would be like reversing the sign of the time).

Lecture 4.1 (8/5) Supersymmetric quantum mechanics

The following was the answer to a question. Why are tangent vectors the same as derivations? Let M be a space (manifold), a one parameter family of symmetries of M is a map $\phi : M \times \mathbb{R}_t \to M$ that maps $(p,t) \mapsto \phi_t p$. Such a map gives me a way of taking derivatives. When the space is more than one dimensional we need a way of knowing "how to move on the manifold", which is given by a vector. If f is a funciton on M, the "derivative of f along ϕ at p" is

$$X_\phi(f)(p) := \lim_{t\to 0} \frac{f(\phi_t(p)) - f(p)}{t} \tag{3.35}$$

This defines a derivation of the algebra of functions on M, meaning

$$X_\phi(fg) = X_\phi(f)g + fX_\phi(g) \tag{3.36}$$

(which is obeyed since it's just a normal derivative) But this is really defining a tangent vector at p.

That's why in general it's helpful to have multiple descriptions of the same object, for example one description may be more intuitive but another may be easier to generalize.

Good news: once you understand what it means to be unitary, you can then safely ignore it since there's a way of passing between symmetry generators and observables. It's fine then to just work with observables.

We therefore get the following table in which to go from right to left, above I divide by i, below I divide by j.

	Observables	Symmetry generators
Even	$P = \mathbb{1},\ R = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	$p = \begin{pmatrix}i & 0\\ 0 & i\end{pmatrix},\ r = \frac{1}{2}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$
Odd	$Q_1 = \begin{pmatrix}0 & j^{-1}\\ j & 0\end{pmatrix},\ Q_2 = \begin{pmatrix}0 & j\\ j^{-1} & 0\end{pmatrix}$	$q_1 = \begin{pmatrix}0 & 1\\ i & 0\end{pmatrix},\ q_2 = \begin{pmatrix}0 & i\\ 1 & 0\end{pmatrix}$

3.2 Representation of a super Lie algebra

Let us now use the Hermitian operators. Computing a few commutators we get:

$$\{Q_1, Q_1\} = 2P \tag{3.37}$$

$$[R, Q_1] = \begin{pmatrix}0 & j^{-1}\\ -j & 0\end{pmatrix} = -iQ_2 \tag{3.38}$$

$$[R, Q_2] = +iQ_1 \tag{3.39}$$

so the commutators have an i while the anticommutators do not (If we chose the convention of working with the symmetry generators we would get real structure constants and the matrices would have a "nicer" form since there are no js. In the end though it's simply a matter of preference, we can just choose one and work with it but it's important to remember this difference).

If I want to represent a super Lie algebra I need to start with a representation of the bosonic part, which has two commuting Hermitian operators P, R. So we have joint eigenspaces labeled by two numbers (p, r), respectively *energy* and *R charge*. What do we do in representation theory? Take commuting operators and use other operators as ladder operators (e.g. take L_z and L^2 which commute and then $L_\pm = L_x \pm L_y$ as ladder operators).

Now in $\mathfrak{u}(1|1)$ we should think of raising and lowering operators in terms of the odd things

$$Q = \frac{1}{\sqrt{2}}(Q_1 + iQ_2) \tag{3.40}$$

$$Q^\dagger = \frac{1}{\sqrt{2}}(Q_1 - iQ_2) \tag{3.41}$$

which have commutators

$$\{Q, Q^\dagger\} = 2P \tag{3.42}$$

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0 \tag{3.43}$$

$$[Q, P] = 0 \implies p \text{ stays the same} \tag{3.44}$$

$$[R, Q] = -Q \implies r \text{ lowered by one} \tag{3.45}$$

so its similar to the situation with angular momentum.

So now lets work as in quantum mechanics: start with a spin, act with a lowering operator until I reach the null state and then I'll know how to construct all states.

Let's start with a state of "highest R charge" $|p,r,\pm\rangle$, meaning that if I try to increase it I get 0: $Q^\dagger |p,r,\pm\rangle = 0$. (We assume there is such a state since we're considering finite dimensional representations) Acting with Q instead gives:

$$Q|p,r,\pm\rangle = \alpha|p,r-1,\mp\rangle \tag{3.46}$$

$$Q^\dagger Q|p,r,\pm\rangle = \alpha Q^\dagger|p,r-1,\mp\rangle = \alpha\bar{\alpha}|p,r,\pm\rangle \tag{3.47}$$

$$\{Q^\dagger, Q\}|p,r,\pm\rangle = 2P|p,r,\pm\rangle = 2p|p,r,\pm\rangle \implies |\alpha|^2 = 2p \tag{3.48}$$

So we have $\alpha = \sqrt{2p}$ up to a phase which is not important as it can be absorbed into the operators. Also, we now have another confirmation that $p \geq 0$.

Q annihilates $|p,r,\pm\rangle$ if and only if $p = 0$. This is interesting because it tells us that the spectrum of the Hamiltonian determines whether the representation is trivial or not. What do we get if we keep lowering?

$$Q|p,r-1,\mp\rangle = \frac{1}{\alpha}Q^2|p,r,\pm\rangle = \frac{1}{2\alpha}\{Q,Q\}|p,r,\pm\rangle = 0 \tag{3.49}$$

So I can't lower more than once! We have exactly two kinds of representations:

1. "Short", one dimensional: $|p=0,r,\pm\rangle$
2. "Long", two dimensional: $|p,r,\pm\rangle, |p,r-1,\mp\rangle$

So there's a lower bound of the energy which can call 0 and it can't be shifted because it's on the right side of the commutator, so it's really zero energy. We also have positive energy states which come in pairs of opposite parity.

If I wanted to calculate a partition function, only the vacuum would contribute, since there's always a boson and a fermion at the same energy level. So in some cases we only need zero energy states, i.e. states on the *BPS bound*.

3.3 Supersymmetric free particle

For a free particle on a line we have the Hilbert space $\mathscr{H} = L^2(\mathbb{R}) \ni \psi(x)$ and the Hamiltonian

$$P = -\frac{\partial^2}{\partial x^2} = \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^2 = \hat{p}^2 \tag{3.50}$$

simplest way to make it supersymmetric: tensor with $\mathbb{C}^{1|1}$. So we have $\mathscr{H}_0 = L^2(\mathbb{R})$ and $\mathscr{H}_1 = L^2(\mathbb{R})$ which can be interpreted as a particle with an internal switch so that it can be either bosonic or fermionic. An example of such a system is an hydrogen atom which can be either ionized or not. So this model is not completely unreasonable.

$$P = -\frac{\partial^2}{\partial x^2} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.51}$$
$$Q_1 = \frac{1}{i}\frac{\partial}{\partial x} \otimes \begin{pmatrix} 0 & j^{-1} \\ j & 0 \end{pmatrix} \tag{3.52}$$
$$Q_2 = \frac{1}{i}\frac{\partial}{\partial x} \otimes \begin{pmatrix} 0 & j \\ j^{-1} & 0 \end{pmatrix} \tag{3.53}$$
$$R = 1 \otimes \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.54}$$

Now recall $Q^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, so we have

$$Q^\dagger \begin{pmatrix} \psi(x) \\ \phi(x) \end{pmatrix} \sim \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ 0 \end{pmatrix}, \quad Q \begin{pmatrix} \psi(x) \\ \phi(x) \end{pmatrix} \sim \begin{pmatrix} 0 \\ \frac{\partial \psi}{\partial x} \end{pmatrix} \tag{3.55}$$

which gives the de Rham complex of $\mathbb{R}$.

Lecture 4.2 (10/5) Supersymmetric particles and interactions

So Q and $Q^\dagger$ in a different bases can be written as:

$$Q = \begin{pmatrix} 0 & 0 \\ \frac{\partial}{\partial x} & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & -\frac{\partial}{\partial x} \\ 0 & 0 \end{pmatrix} \tag{3.56}$$

while P and R are given by:

$$P = \begin{pmatrix} -\frac{\partial^2}{\partial x^2} & 0 \\ 0 & -\frac{\partial^2}{\partial x^2} \end{pmatrix}, \quad R = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.57}$$

And this is a unitary representation of $\mathfrak{u}(1|1)$, i.e. supersymmetric quantum mechanics with $\mathcal{N} = 2$.

3.4 General properties of supersymmetric systems

$\psi(x)$ is a map from position to "probability amplitudes". To do time dependent quantum mecanics we would instead need $\Psi(x,t)$. These two are simply related by the separation of variable method of solving differential equations.

$$\psi(x) = \sum_E c_E |E\rangle \implies \Psi(x,t) = \sum_E c_E |t\rangle e^{-iEt} \tag{3.58}$$

Classically the degree of freedom is $x(t)$. So thinking of this as a field theory, the x is the degree of freedom, the field, and the spacetime is whatever the field depends on, which here is just the time. So as a field teory it's only 0 + 1 dimensional (independently of how many components x has).

In general, supersymmetry algebras enhance the spacetime symmetries of a field theory.

For example in classical electromagnetism we have translations and Lorentz symmetries. The first is represented by $\mathbb{R}^{n-1,1}$, $n-1$ for the spatial dimensions and 1 for the time dimension (and generators will be the P_μ). Instead for the Lorentz symmetries we have $\mathrm{SO}(n-1,1)$ which are transformations that preserve the scalar product $ds^2 = -dt^2 + dx_1^2 + \dots dx_{n-1}^2$. Supersymmetry enhances the symmetries in two ways we will have a supersymmetry algebra that extends the Poincaré algebra and we will get R symmetries which commutes with the Ps.

$\mathfrak{u}(1|1)$ fits this pattern: the Lorentz part is SO(0,1) which is empty and for translations we have $\mathbb{R}^{0,1}$. These are extended and we have Q_1 and Q_2 and we also get R which commutes with translations but not with the Qs. We call this $\mathcal{N} = 2$ but we can get easily get any $\mathcal{N}$.

We can take no Lorentz symmetry, translations $\mathbb{R}^{0,1} = \langle P \rangle$ and now as many supercharges as we want: $Q_1, \dots, Q_\mathcal{N}$ with the following relations:

$$Q_i^2 = \frac{1}{2}\{Q_i, Q_i\} = P \tag{3.59}$$

$$\{Q_i, Q_j\} = 0, i \neq j \tag{3.60}$$

what structure do I have? I have $\mathbb{R}^\mathcal{N}$ with an inner product! $(\mathbb{R}^\mathcal{N}, g)$ a vector space with a Euclidean inner product which I interpret as a bracket. This may be a better description since it's basis independent. So now what is the R symmetry? It's always something that acts on the supercharges, but commutes with translations. Also we don't want the brackets to change, so we get $\mathrm{SO}(\mathcal{N})$.

3.5 Supersymmetric interacting particle

Can I make the free supersymmetric particle interacting? Normally I would turn on a potential term, so I would add something to the Hamiltonian and continue from there. However, since we want $\{Q, Q^\dagger\} = 2P$ we would also have to change the Qs and if we want to preserve supersymmetry their commutation relations shouldn't change:

$$\{Q, Q\} = 0 \tag{3.61}$$

$$Q^\dagger = (Q)^\dagger \tag{3.62}$$

$$\{Q^\dagger, Q^\dagger\} = 0 \tag{3.63}$$

$$\{Q, Q^\dagger\} = 2P \tag{3.64}$$

This suggests another way of proceding: instead of starting with a Hamiltonian I can do the opposite! Choose a Q that satisfies the first equation, then choose $Q^\dagger = (Q)^\dagger$ and define $P = \frac{1}{2}\{Q, Q^\dagger\}$. Then I'm done: that'll be a supersymmetric quantum mechanical theory. So essentially there's a one-to-one dictionary between $\mathcal{N} = 2$ susy QM (without R symmetry) and Hilbert spaces with an operator Q with $Q^2 = 0$. In order to also have the U(1) R symmetry we also need to specify the operator R, and in addition Q should have R charge -1.

It turns out that using this method for a given Q we get many systems, not just one. To see this let M be a self adjoint invertible operator. Then define $Q_M = M^{-1}QM$, so we have $Q_M^\dagger = MQ^\dagger M^{-1}$ and $Q^2 = 0$. So if I have one susy QM system I can obtain another like this. This is useful because I tipically don't want to change the kinetic term.

For the free superparticle: take $M = e^\lambda W(x)$, with $W(x)$ any real function (called *superpotential*) and λ a real coupling constant. Start with

$$Q = \begin{pmatrix} 0 & 0 \\ \frac{\partial}{\partial x} & 0 \end{pmatrix} \tag{3.65}$$

and then set

$$Q_M = e^{-\lambda W} \begin{pmatrix} 0 & 0 \\ \frac{\partial}{\partial x} & 0 \end{pmatrix} e^{\lambda W} = \begin{pmatrix} 0 & 0 \\ \frac{\partial}{\partial x} + \lambda W' & \end{pmatrix} \tag{3.66}$$

so for the adjoint we have

$$Q_M^\dagger = \begin{pmatrix} 0 & -\frac{\partial}{\partial x} + \lambda W' \\ 0 & 0 \end{pmatrix} \tag{3.67}$$

and the Hamiltonian is given by:

$$\begin{aligned} P_M &= \frac{1}{2}\{Q_M, Q_M^\dagger\} \\ &= \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \left(\frac{\partial}{\partial x} + \lambda W'\right)\left(-\frac{\partial}{\partial x} + \lambda W'\right) \end{pmatrix} + \begin{pmatrix} \left(-\frac{\partial}{\partial x} + \lambda W'\right)\left(\frac{\partial}{\partial x} + \lambda W'\right) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -\frac{\partial^2}{\partial x^2} + \lambda^2 W'^2 - \lambda W'' & 0 \\ 0 & -\frac{\partial^2}{\partial x^2} + \lambda^2 W'^2 + \lambda W'' \end{pmatrix} \end{aligned} \tag{3.68}$$

so we got a different, more interesting Hamiltonian! There is a difference in the bosonic and the fermionic part! Also, we know that although they are two different potentials they have the same spectrum! So W gives rise to two so called *partner potentials*:

$$V_\pm(x) = \lambda^2 W'^2 \pm \lambda W'' \tag{3.69}$$

which have the same spectrum, other than the vacua. It actually tells us soemthing about the NOT supersymmetric quantum mechanics potential!

Example. Let's take $W(x) = \frac{1}{2}x^2$, so $W'(x) = x$ and $W''(x) = 1$. The partner potentials are then

$$V_{\pm} = \lambda^2 x^2 \pm \lambda \tag{3.70}$$

and the Hamiltonian is

$$H_{\pm} = \frac{1}{2}(-\frac{\partial^2}{\partial x^2} + \lambda^2 x^2 \pm \lambda) = H_{\text{SHO}} \pm \frac{\lambda}{2} \tag{3.71}$$

where H_{SHO} is the Hamiltonian of the simple harmonic oscillator, for which we recall the spectrum is given by $\lambda(n + \frac{1}{2}), n \geq 0$.

Now we can ask: what is the bosonic ground state? To answer the question we need to solve the following differential equation:

$$Q \begin{pmatrix} \psi(x) \\ 0 \end{pmatrix} = 0 = \left(\frac{\partial}{\partial x} + \lambda x \right) \psi(x) \tag{3.72}$$

$$\frac{d\psi}{\psi} = -\lambda x dx \tag{3.73}$$

$$\ln \psi = -\frac{\lambda x^2}{2} \tag{3.74}$$

$$\psi(x) = e^{-\frac{\lambda x^2}{2}} \tag{3.75}$$

In general then the procedure to count the ground states is to solve

$$\left(\pm \frac{\partial}{\partial x} + \lambda W'(x) \right) \psi(x) = 0 \tag{3.76}$$

where $+$ is for the bosonic vacuum and $-$ is for the fermionic one.

Lecture 5.1 (15/5) Formulations of Supersymmetric particle mechanics, both classical and quantum

Last time: Hilbert-space formulation of SQM with superpotential interactions. We saw that the superpotential gives rise to two ordinary potentials given by

$$V_{\pm}(x) = \lambda^2 W'(x)^2 \pm \lambda W''(x) \tag{3.77}$$

So just as we had that supercharges (supersymmetry generators) were square roots of the time translation operator, we now have that the superpotential is kind of the square root of the potential. The Hilbert space space was then given by $\mathscr{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^{1|1}$ and the Hamiltonian was given by

$$\begin{pmatrix} -\frac{\partial^2}{\partial x^2} + V_+ & 0 \\ 0 & -\frac{\partial^2}{\partial x^2} + V_- \end{pmatrix} \tag{3.78}$$

Setting $\lambda = 1$ the supercharges were

$$Q = \begin{pmatrix} 0 & \frac{\partial}{\partial x} + W' \\ 0 & 0 \end{pmatrix} \tag{3.79}$$

$$Q^{\dagger} = \begin{pmatrix} 0 & 0 \\ -\frac{\partial}{\partial x} + W' & 0 \end{pmatrix} \tag{3.80}$$

Now, if we just took the bosonic part, i.e. we restricted to $\mathscr{H}_+$ the Hamiltonian would simply be

$$p^2 + V_+ \tag{3.81}$$

and we know that classically this corresponds to a phase space with coordinates (p, x) with Poisson brackets $\{p, x\} = 1$.

Instead, for the entire system there's a problem. Is this the quantization of any classical system? Secretly, for the purpose of interpretation we hope that there is a corresponding classical system.

This is a real issue. We can think up any Hilbert space and any Hamiltonian we want, but if we don't know what system this corresponds to it's not really useful. For a random example we could choose our Hilbert space to be $\mathbb{C}^7$ and our Hamiltonian to be some 7-by-7 Hermitian matrix, does that correspond to any classical system? Well we don't even know which things correspond to measurements I can make, I don't have a clear "position", "momentum" or something else.

Let's first review what is classical quantization so that we'll be prepared to quantize a system with both bosonic and fermionic observables.

3.6 Review of Canonical quantization

In classical mechanics we also have a Lagrangian formulation:

$$S = \int dt \mathscr{L}, \quad \mathscr{L} = \dot{x}^2 - V_+(x) \tag{3.82}$$

and we also get the equation of motion of x from

$$\frac{\delta S}{\delta x(t)} = 0 \tag{3.83}$$

and in general the conjugate momentum p_x is given by

$$p_x(t) := \frac{\delta S}{\delta \dot{x}(t)} = \dot{x}(t) \tag{3.84}$$

Equivalently, one may use the Hamiltonian formulation, in which we have a phase space such as $\mathbb{R}^2 \ni (x, p)$, along with a symplectic form. The symplectic form then allows us to define a Poisson bracket such as $\{p, x\} = 1$, which is just a Lie bracket which also has a compatibility condition between the product and bracket operations: the bracket is also a derivation on both arguments

$$\{f, g\} = \{f, g\}h + g\{f, h\} \tag{3.85}$$

(there's no need to check the second argument since we have antisymmetry). Sometimes the Poisson bracket is written as

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} \tag{3.86}$$

To canonically quantize we use the Hamiltonian formalism and we have the following starting point:

- we have classical observables x and p and our measurements could be any function of these. For simplicity let's say our measurement is only polynomial in x and p so the space of possible measurements is $\mathbb{C}[x, p]$.
- these observables have a Poisson bracket $\{p, x\} = 1$. This specific bracket is enough to calculate all Poisson brackets in the algebra.

In order to quantize it we want the following things:

- a noncommutative algebra $A_\hbar$ such that $\lim_{\hbar \to 0} A_\hbar = A = \mathbb{C}[x, p]$, where we call $k : A_\hbar \to A$ the map that associates to an operator its *classical limit*.
- The operators should satisfy: $[\hat{f}, \hat{g}] = i\hbar\widehat{\{f, g\}} + O(\hbar^2)$, $\forall \hat{f} \in A_\hbar$ with $k(\hat{f}) = f$. (The map $\hat{}$ need not be unique)
- the operators should act on a Hilbert space. (I don't want just an abstract algebra but I want operators on a Hilbert space)

Normally we just assume that we have all of this but that's not trivial. Essentially what we're doing is then looking for a representation of the noncommutative algebra. But as we know from representation theory, the representations are usually not unique.

The typical procedure is then the following:

1. pick a complete set of (Poisson)-commuting observables $\mathbb{C}[x]$
2. take the L^2 functions on the space they coordinatize, this is $\mathscr{H}$.
3. Then because the Poisson bracket is nondegenerate, x is supposed to have a partner and together they should generate the whole algebra of observables. The partners are called *conjugate momenta* p_x and are represented by $\frac{\hbar}{i}\frac{\partial}{\partial x}$.

but the procedure is clearly nonunique, for example we could have started with p instead of x. Which is actually something we often do and we obtain the momentum representation of the Hilbert space. I could have also chosen $p + x$ or many other things. So we need something extra to talk about *the* Hilbert space.

For another example, what if we started with L_x, L_y, L_z? Well we're essentially working with the Lie algebra $\mathfrak{so}(3)$ (or $\mathfrak{su}(2)$), for which we know that there are infinite representations, one for every spin: the trivial spin 0 representation; the spin $\frac{1}{2}$ representation; the vector, or spin 1, representation... And all these are inequivalent. So the fact of having a unique choice of Hilbert space when starting with the canonical commutation relations is astonishing.

Kind of amazingly, there is (up to equivalence) exactly one unitary irreducible representation of the canonical commutation relations. This is to emphasize that there is a big difference between the typical algebras we use in physics and the Poisson algebra we started with, since a lot of them have infinitely many representations while here we get only one. Also note that since we have the parameter $\hbar$ so this could hypotethically mean we have a $\mathfrak{u}(1)$ subalgebra, but we don't consider that since we assume $\hbar$ has some fixed numerical value.

The uniqueness of the irreducible representation of the Poisson algebra is the Stone-von Neumann theorem. This is kind of implicit in the way quantum mechanics is done. In particular we often say that the position and momentum representations are equivalent, which is true and the isomorphism is given by the Fourier transform. But we don't worry about checking if there are other representations and the reason is that we know there's a unique one.

This result is physically very deep because otherwise you might ask why aren't there different ways of quantizing with the same correspondence principle? A priori it is not at all clear that there shouldn't be multiple ways, but the fact that there is only one is very important. In particular going back to $\mathfrak{su}(2)$, the fact that it has multiple representations directly corresponds to the fact that the particle may have *internal degrees of freedom* that are related to angular momentum: I start with $\mathfrak{su}(2)$ being the classical algebra of angular momentum and I discover that there are in a sense "multiple ways of quantizing it" so the particle I'm considering will live in one of its representations and have this extra degree of freedom that is classically inexistent. Instead for the position degree of freedom this can't happen, so I can't have an *internal position* degree of freedom.

Here's the idea of the proof of the Stone-von Neumann theorem:

- The position operator is Hermitian so it has at least one eigenstate

$$\hat{x}\,|\lambda\rangle = \lambda\,|\lambda\rangle\,, \quad \text{for some } \lambda \in \mathbb{R} \tag{3.87}$$

- Then $\hat{x}e^{i\hat{p}a}\,|\lambda\rangle = (\lambda+a)e^{i\hat{p}a}\,|\lambda\rangle$, so $e^{i\hat{p}a}\,|\lambda\rangle = |\lambda + a\rangle$. Therefore there's one eigenvector for all positions in $\mathbb{R}$. That's already an irreducible subrepresentation isomorphic to $L^2(\mathbb{R})$, in the sense that $L^2(\mathbb{R})$ is spanned by the eigenstates of the position operator.

- By iterating this procedure for all eigenstates of $\hat{x}$ we find that the Hilbert space is just given by the direct sum of $L^2(\mathbb{R})$ with itself multiple times. So the only irreducible representation is in fact $L^2(\mathbb{R})$.

More details on canonical quantization, including a proof of the Stone-von Neumann theorem, can be found in [1].

3.7 Classical fermionic degree of freedom

So now we want to understand quantization in cases where I have both bosonic and fermionic observables. So we want to understand what algebra replaces the canonical commutation relations and then we want to study its representation theory in the hope that it's analogous to the bosonic case.

So now that we have both bosonic and fermionic observables, the algebra of classical observables should be super commuting and they should have a super Poisson bracket. We could simply try to write down such an algebra or we could try to imagine what a Lagrangian picture might look like. In particular we want that quantizing gives the Hilbert space $L^2(\mathbb{R}) \otimes C^{1|1}$. $L^2(\mathbb{R})$ came from quantizing $x(t) \in \mathrm{Map}(\mathbb{R} \to \mathbb{R}^{1|0})$, so a guess would be to think about $\theta(t) \in \mathrm{Map}(\mathbb{R} \to \mathbb{R}^{0|1})$, a function of the time which is odd, or a *Grassmann number*. For the first part we have

$$S = \int dt \frac{1}{2} \dot{x}^2 \tag{3.88}$$

but surely we can't have $\dot{\theta}^2$ since that would be zero. I also don't want higher derivatives otherwise the theory becomes weird. Instead we could have

$$S = \int dt \frac{1}{2} \theta \dot{\theta} \tag{3.89}$$

Why don't we write such a term for the even part? The reason is that such a term wouldn't contribute to the dynamics as it's a total derivative $x\dot{x} = \frac{1}{2}\frac{dx^2}{dt}$, while $\theta^2 = 0$ so $\theta\dot{\theta}$ is not the derivative of anything. Also, it's important to have something quadratic in θ otherwise we would have to integrate something odd.

Before we proceed, let's explain what is meant by observables. If we simply consider all paths in phase space (also called *histories*) then there are many things that could be measured: to start with we have position and momentum, all their powers and all their products; then we also have all derivatives as well, which gives rise to a very large set of possible measurements. However what we mean by observables is actually *on shell* observables, meaning observables for points in phase space along histories that satisfy the equations of motion. In that case for the lagrangian $\frac{1}{2}\dot{x}^2$ we don't care about derivatives higher than the second, since the equation of motion is $\ddot{x} = 0$. So the space of observables is much smaller. In particular since $\ddot{x} = 0$ if we plot x over t it would simply be a straight line, so it would be entirely determined by two values, which explains why the observables are simply x and p.

So if I want to understand the observables I should look at the equations of motion for θ:

$$\dot{\theta} = 0 \tag{3.90}$$

So the algebra of observables is simply $\mathbb{C}[\theta], \theta^2 = 0$. But normally the conjugate momentum is also an observable, so I should it as well. However we find that the conjugate momentum is

$$p_\theta = \frac{\delta \mathscr{L}}{\delta \dot{\theta}} - \frac{1}{2}\theta \tag{3.91}$$

so it's already included. In fact just like $\ddot{x} = 0$ leaves two degrees of freedom, $\dot{\theta} = 0$ leaves only one. Now the bracket structure can be constructed in analogy to $\{x, p\} = 1$: we would write $\{\theta, p_\theta\} = 1$, which recalling $p_\theta = -\frac{1}{2}\theta$ gives

$$\{\theta, \theta\} = -2 \tag{3.92}$$

But of course this bracket is not antisymmetric otherwise we would have $\{\theta, \theta\} = 0$. Instead it's super antisymmetric and it's easy to check that this gives a super Poisson algebra.

So what is this extra structure? Well, normally we have a symplectic phase space with position and momenta and a symplectic form which gives me the bracket. Instead now the symplectic form is not antisymmetric but super antisymmetric, which for the odd part means symmetric. So essentially it's just an inner product.

So just as in canonical quantization I start with an algebra, give it a bracket and then make it noncommuting according to the bracket, now I start with a Grassmann algebra and then make it noncommuting according to an inner product. This is the construction of a Clifford algebra. So just as canonical quantization corresponds to finding the representation of the Poisson algebra, to quantize a supersymmetric system we want to find representations of a Clifford algebra. In particular we'll find that the situation is almost as nice: namely we either have one or two possible representations.

Lecture 5.2 (17/5) Clifford algebras and the spinning particle

The following was the answer to a question.

$$\delta S = \frac{\partial \mathscr{L}}{\delta \theta}\delta\theta + \frac{\delta \mathscr{L}}{\delta \dot{\theta}}\delta\dot{\theta} = \text{EOM }\delta\theta + \text{total derivative} \tag{3.93}$$

$$= \left(\frac{\delta \mathscr{L}}{\delta \theta} - \frac{d}{dt}\frac{\delta \mathscr{L}}{\delta \dot{\theta}}\right)\delta\theta + \frac{d}{dt}\left(\frac{\delta \mathscr{L}}{\delta \dot{\theta}}\delta\theta\right) \tag{3.94}$$

$$\delta S = \int dt \left[\frac{1}{2}\delta\theta\dot{\theta} + \frac{1}{2}\theta\delta\dot{\theta}\right] = \int dt \left[\frac{1}{2}\delta\theta\dot{\theta} - \frac{1}{2}\dot{\theta}\delta\theta + \frac{1}{2}\frac{d}{dt}(\theta\delta\theta)\right] \tag{3.95}$$

$$= \int dt \left[\delta\theta(\dot{\theta}) - \frac{1}{2}\frac{d}{dt}(\theta\delta\theta)\right] \tag{3.96}$$

this last part is called the *variational 1-form*, the thing that tells you what is conjugate to what and gives you a symplectic form.

3.8 Quantization with fermionic observables

The quantization procedure should be clear by now: we start with a (super)commutative algebra with a (super)Poisson bracket and we construct a non(super)commutative algebra whose (super) commutators are given by the (super)Poisson brackets.

In the classical case we start with $\mathbb{C}[p, x]$ with $px = xp$ and $\{p, x\} = 1$. From this we get an algebra generated by $\hat{p}, \hat{x}$ with commutation relations $[\hat{p}, \hat{x}]_\hbar = -i\hbar$. This is called the *Weyl algebra.*

The (semi)classical algebra of observables is instead given by $\mathbb{C}[\theta]$, with $\theta^2 = 0$ and $\{\theta, \theta\} = -2$. Quantizing, we get the algebra generated by $\hat{\theta}$ with supercommutator $[\hat{\theta}, \hat{\theta}]_\hbar = 2\hat{\theta}^2 = -2\hbar$, meaning we now have $\hat{\theta}^2 = -\hbar$. This is called a *Clifford algebra.* Note that the i in the bosonic case was an artifact of the fact that we were taking commutators of Hermitian operators so the result was antihermitian, whereas now we have the anticommutator of Hermitian operators which is still Hermitian.

So normally we start with a Weyl algebra and then we look for a representation of it and we find there's only one possibility that leads to quantum mechanics. We now want to do the same for the Clifford algebra, which will give rise to spinor representations.

3.9 The supersymmetric (or *spinning*) classical particle

We can now put the ingredients together. We now have $(x(t), \theta(t)) \in \mathrm{Map}(\mathbb{R}_t \to R^{1|1})$ with the action

$$S = \int dt \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\theta\dot{\theta} \right) \tag{3.97}$$

Claim. This theory classically has $\mathcal{N} = 1$ supersymmetry.

Proof. I should therefore have one supercharge Q which is odd and has $Q^2 = P \propto \frac{\partial}{\partial t}$. Simply set Q to be the following odd transformations:

$$x \mapsto \theta \tag{3.98}$$

$$\theta \mapsto -\dot{x} \tag{3.99}$$

odd since it sends even fields to odd ones and viceversa. Let's check what happens if we apply it twice:

$$x \mapsto \theta \mapsto -\frac{\partial x}{\partial t} \tag{3.100}$$

$$\theta \mapsto -\dot{x} \mapsto -\frac{\partial \theta}{\partial t} \tag{3.101}$$

So we get $Q^2 = -\frac{\partial}{\partial t}$ which is what we wanted. What else do we need for it to be a classical symmetry? S should be invariant:

$$QS = \int dt \left[\dot{x}\dot{\theta} - \frac{1}{2}\dot{x}\dot{\theta} - \frac{1}{2}\theta(-\ddot{x}) \right] \tag{3.102}$$

in which the $-$ in the last expression is because Q moves past θ.

$$= \int dt \left[\frac{1}{2}\dot{x}\dot{\theta} + \frac{1}{2}\ddot{x}\theta\right] = \int dt \frac{d}{dt}\left[\frac{1}{2}\dot{x}\theta\right] \tag{3.103}$$

meaning $\frac{1}{2}\dot{x}\theta$ is the conserved current (also conserved charge since there are no spatial coordinates). So we have an odd charge that is the generator of the symmetry. □

However, the quantum system we were studying was an $\mathcal{N} = 2$ supersymmetric system. But it's not hard to obtain a supersymmetry with greater $\mathcal{N}$: since an odd variable gives me a way to get a square root of the time translation operator, I can just add more odd variables

$$S = \int dt \left(\frac{1}{2}\dot{x}^2 + \sum_{a=1}^{\mathcal{N}} \frac{1}{2}\theta_a\dot{\theta}_a\right) \tag{3.104}$$

Quantizing this system would then give $L^2(\mathbb{R})$ and an odd part.

If we want more spatial dimensions we could then give x an index i and then I have to give an index i to the θs. We could then get the most general such model: an "$\mathcal{N}$ extended supersymmetric particle in d dimensions":

$$S = \int dt \left(\sum_{i=1}^{d} \frac{1}{2}\dot{x}_i^2 + \sum_{a=1}^{\mathcal{N}} \frac{1}{2}\theta_{ia}\dot{\theta}_{ia}\right) \tag{3.105}$$

Note however that giving θ and x indices corresponds to increasing the dimension of the vector space they're in, and in that vector space we also want a way to multiply vectors to get a number, so we want them to have an inner product. In addition, we actually need two vector spaces: the one corresponding to the $\mathcal{N}$ supersymmetry $R = (\mathbb{R}^{\mathcal{N}}, h)$ (which is going to be the R symmetry space) and the space where the particle actually lives $V = (\mathbb{R}^d, g)$.

So a field is given by $(x_i, \theta_{ia}) \in \mathrm{Map}(\mathbb{R}_t \to V \oplus \Pi(V \otimes R))$, where Π is for the parity shift, in order for $V \otimes R$ to be odd. The action above can then be given in a basis independent form:

$$S = \int dt \left(\frac{1}{2}g(\dot{x}, \dot{x}) + \frac{1}{2}g \otimes h(\theta, \dot{\theta})\right) \tag{3.106}$$

where, to be clear, in coordinates $g \otimes h(\theta, \theta) = \theta_{ia}\theta_{jb}g^{ij}h^{ab}$. Now what is the supersymmetric algebra? $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where the first part is given by $\mathbb{R}_t$, $\mathrm{SO}(R)$ and $\mathrm{SO}(V)$ while the odd part is R.

If we go back to $\mathcal{N} = 1$

$$S = \int dt \left(\frac{1}{2}g(\dot{x}, \dot{x}) + \frac{1}{2}g(\theta, \dot{\theta})\right) \tag{3.107}$$

what phase space do we get? The bosonic part is T^*V with the normal symplectic form $dp^i \wedge dx_i$ which is invariant under g. For the fermionic part ΠV we need to be careful: the conjugate momentum of θ is given by:

$$p^i_\theta = \frac{\delta \mathscr{L}}{\delta \dot{\theta}_i} = -\frac{1}{2} g^{ij}\theta_j \tag{3.108}$$

so if we write the analogous symplectic form we get $p^i_\theta \wedge \theta_i = -\frac{1}{2} g^{ij}\theta_j \wedge \theta_i$ and we don't care about the constant in front so we have $g^{ij} d\theta_i \wedge d\theta_j$. While the bosonic form was independent of g, the fermionic one actually is g. So in the end we have

$$T^*V \oplus \Pi V, \text{ with symplectic form } \omega = dp^i \wedge dx_i + g^{ij} d\theta_i \wedge d\theta_j \tag{3.109}$$

which is actually symmetric in the odd part but that's okay since that's exactly what a super symplectic form should be. In particular $d\theta^i \wedge d\theta_j$ is not 0.

So when quantizing the algebra of observables is given by $\mathrm{Weyl}(T^*V) \otimes \mathrm{Clifford}(V, g)$, where $\mathrm{Weyl}(T^*V)$ is again the algebra generated by x_i and p_i with $[x_i, p_j] = -i\hbar\delta_{ij}$ and $\mathrm{Clifford}(V, g)$ is again the Clifford algebra but now with more generators θ_i and with $\{\theta_i, \theta_j\} = -g_{ij}$.

Now the Hilbert space is given by $L^2(V) \otimes S$ where S is somewhere the Clifford algebra acts, which is sort of an internal degree of freedom. In particular we'll find that S is a spinor of $\mathrm{SO}(V)$.

So what we're going to do is write down all the Clifford algebras and classify them and their representations. We'll find that there are either exactly one or exactly two irreducible representations.

The phase space in the most general case is given by

$$T^*V \oplus \Pi(V \otimes R), \text{ with symplectic form } \omega = dp^i \wedge dx_i + g \otimes h \tag{3.110}$$

The following was the answer to a question. The Poisson algebra can be obtained from the commutator of the quantum algebra of observables by taking a limit:

$$\hat{x}\hat{p} - \hat{p}\hat{x} = O(h) \tag{3.111}$$

$$\lim_{\hbar \to 0} \frac{\hat{x}\hat{p} - \hat{p}\hat{x}}{\hbar} = \text{ Poisson bracket} \tag{3.112}$$

So if you want the reason there's a Poisson bracket in classical mechanics is that classical mechanics has to sit in a family of noncommutative things called quantum mechanics and in such a family there's a first order deviation from being commutative and that's given by the Poisson bracket.

Lecture 6.1 (22/5) Clifford algebras and Clifford modules

4 Clifford algebras and Clifford modules

We now want give a definition of what a Clifford algebra is and then we want to classify all Clifford algebras. In order to give the definition we first need to define the tensor algebra $T(V)$, which is just the sum of all the tensor powers of V, or in other words it's a noncommutative analogue of the ring of polynomials.

$$T(V) = \bigoplus_{k\geq 0} V^{\otimes k} = k \oplus V \oplus (V \otimes V) \oplus \dots \tag{4.1}$$

Let's start with the bosonic case, i.e. the Weyl algebra. Let the phase space be $V = (\mathbb{R}^{2n}, \omega = dp^i \wedge dx_i)$, then $\mathrm{Weyl}(V) = {T(V)}/{(x\otimes y - y \otimes x - \pi(x,y)=0)} = {\mathbb{C}\langle \hat{q}_i, \hat{p}_i\rangle}/{[\hat{q}_i,\hat{p}_j]=\delta_{ij}}$. This is exactly the algebra we were describing when talking about canonical quantization: it's a noncommutative algebra whose commutator is given by the Poisson bracket $\{x, p\} = 1$.

The Clifford algebra is the odd analogue: we start with $V = (\Pi\mathbb{C}^n, g)$ and $\mathrm{Cl}(V) = {T(V)}/{(x\otimes y + y\otimes x - 2g(x,y)=0)}$. Let g be the standard bilinear product, then if $e_i, 1 \leq i \leq n$ is an orthonormal basis we have $g(e_i, e_j) = \delta_{ij}$. In this case then we're quotienting out by $e_i \otimes e_j + e_j \otimes e_i = -2\delta_{ij}$, which gives $e_i \otimes e_i = -1$ and $e_i \otimes e_j = -e_j \otimes e_i$ for $i \neq j$. We give a name to these algebras: $\mathrm{Cl}(n, \mathbb{C})$ = the algebra obtained from $\mathbb{C}$ with n anticommuting square roots of -1 (or equivalently $+1$ since $(ie_j) \otimes (ie_j) = +1$). From now on we'll omit the $\otimes$ symbol, so we just have $e_i^2 = 1$ and $e_ie_j = -e_je_i$.

The dimension of this space is 2^n since it's spanned by $1, e_i, e_ie_j, \dots$ meaning basically all the subsets of $\{e_1, e_2, \dots\}$ which are 2^n (the order of the e_is doesn't matter since we can anticommute them).

Claim. One square root of $+1$ it defines a pair of commuting projection operators i.e. $p\pm$ with $p_\pm^2 = p_\pm$.

Proof. We can simply define $p_\pm = \frac{1\pm e}{2}$ and also we have $p_\pm p_\mp = 0$ so they're actually orthogonal. So $\mathrm{Cl}(1, \mathbb{C}) = \mathbb{C} \oplus \mathbb{C}$. □

A word of caution though: the clifford algebra is $\mathbb{Z}_2$ graded, it's a super algebra by taking e_i to be odd. However the splitting that we got is not homogeneous. So this is NOT a decomposition as superalgebras.

Furthermore: we now have two possible meanings of tensor product as algebras or as superalgebras which are not the same.

Let A and B be algebras over $\mathbb{R}$ (or some other field), then $A \otimes_\mathbb{R} B$ is also an algebra by taking $(a\otimes b)(a' \otimes b') = aa' \otimes bb'$. If instead they are super algebras $A = A_+ \oplus A_-$ and $B = B_+ \oplus B_-$, then we have $(a\hat{\otimes} b)(a'\hat{\otimes} b') = (-)^{|b||a'|} aa' \otimes bb'$. Normally I have two phase spaces V and W and the composite phase space $V \oplus W$ which corresponds to tensoring algebras of observables. To have the same for a super algebra I need the super tensor product. What we have is $\mathrm{Cl}(V)\hat{\otimes}\mathrm{Cl}(W) = \mathrm{Cl}(V \oplus W)$.

Claim. 2 anticommuting square roots of $+1$ give a copy of 2-by-2 matrices. So we have $\mathrm{Cl}(2, \mathbb{C}) = M_2(\mathbb{C})$ (Here $M_n(A)$=$\{n$-by-n matrices with coefficients in $A\}$).

Proof. Having e, f with $e^2 = f^2 = 1$ and $ef = -fe$, we can write the matrices as:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad ef = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -fe \tag{4.2}$$

(which also works with real coefficients). $\square$

Now, from what we said above $\mathrm{Cl}(3, \mathbb{C})$ is just the super tensor product $\mathrm{Cl}(2, \mathbb{C}) \hat{\otimes} \mathrm{Cl}(1, \mathbb{C})$ and similarly we can obtain $\mathrm{Cl}(n, \mathbb{C})$ for any n.

Over $\mathbb{R}$, however, the situation is not so simple. In the real case I need to worry about the "signature", meaning I can have a vector space $\mathbb{R}^{p,q} = \mathbb{R}^{p+q}$ which has p negative orthonormal basis vectors and q positive vectors. So $\mathrm{Cl}(p, q) =$ real algebra generated by p roots of $+1$ and q roots of -1 which are all anticommuting. It turns out that what we've done up to now is enough to find the first Clifford algebras:

- $\mathrm{Cl}(0, 1) = \mathbb{C}$: we simply have $\mathbb{R}$ and then an additional generator that is a square root of -1. That's simply the complex algebra.
- $\mathrm{Cl}(0, 2) = \mathbb{H}$ (quaternions): we now have two anticommuting generators whose square is -1 and we can call them i and j. Now we also have $(ij)^2 = ijij = -iijj = -1$ so (ij) is also a square root of -1 and if we call it k it's clear that we're working with the algebra of quaternions.
- $\mathrm{Cl}(1, 0) = \mathbb{R} \oplus \mathbb{R}$: we saw above that $\mathrm{Cl}(1, \mathbb{C}) = \mathbb{C} \oplus \mathbb{C}$, but working with $\mathbb{R}$ instead of $\mathbb{C}$ makes no difference in the proof.
- $\mathrm{Cl}(2, 0) = M_2(\mathbb{R})$: just like before, works the same way as $\mathrm{Cl}(2, \mathbb{C}) = M_2(\mathbb{C})$.
- $\mathrm{Cl}(1, 1) = M_2(\mathbb{R}) = \mathrm{Mat}(\mathbb{R}^{1|1})$: using the matrices from the last claim, we can set $g = ef$ and note that $g^2 = -1$, so e and g are now respectively square roots of $+$ and -1. To be sure we span the whole space, note that we can get back f from g and e since $f = eg$. We wrote $\mathrm{Mat}(\mathbb{R}^{1|1})$ since that is the actual super algebra structure that we obtain, which is different from the previous case.

Claim. $\mathrm{Cl}(0, q) \otimes \mathrm{Cl}(2, 0) = \mathrm{Cl}(q+2, 0)$ and $\mathrm{Cl}(p, 0) \otimes \mathrm{Cl}(0, 2) = \mathrm{Cl}(0, p+2)$

Proof. In $\mathrm{Cl}(0, q)$ I have e_i with $e_i^2 = -1$ while in $\mathrm{Cl}(2, 0)$ I have f_1, f_2 with $f_i^2 = 1$. Consider the elements $1 \otimes f_1, 1 \otimes f_2, e_i \otimes f_1 f_2$ it's clearly a generating set so we want to check that they anticommute and square to 1. The first two clearly square to 1 and for the others

$$(e_i \otimes f_1 f_2)^2 = e_i^2 \otimes (f_1 f_2)^2 = -1 \otimes (-1) = 1 \tag{4.3}$$

and one can check the anticommutativity.

The other equality is analogous by simply changing signs. $\square$

Claim. $\mathrm{Cl}(p,q) \otimes \mathrm{Cl}(1,1) = \mathrm{Cl}(p+1,q+1) = M_2(\mathrm{Cl}(p,q))$

Proof. Let the first have generators e_i, f_j with $e_i^2 = 1, f_j^2 = -1$ and the second E, F with $E^2 = 1, F^2 = -1$. Now simply take $1 \otimes E, 1 \otimes F, e_i \otimes EF, f_j \otimes EF$. For the last equality note that we have $\mathrm{Cl}(1,1) = M_2(\mathbb{R})$ and $M_2(\mathbb{R}) \otimes_{\mathbb{R}} A = M_2(A)$ for any algebra A. So $\mathrm{Cl}(p,q) \otimes \mathrm{Cl}(1,1) = M_2(\mathrm{Cl}(p,q))$. $\square$

In deriving the next Clifford algebras, it's useful to note the following properties of the tensor product and direct sum in relation to matrices with coefficients:

- $M_n(A) \otimes B = M_n(A \otimes B)$,
- $M_i(M_j(A)) = M_{ij}(A)$ which is clear since on the left hand side we're taking the i-by-i matrices with entries j-by-j matrices which are clearly ij-by-ij matrices,
- $M_n(A \oplus B) = M_n(A) \oplus M_n(B)$.

So we have a recipe for obtaining the gamma matrices in all dimensions. Let us explicitly calculate a few:

	$\mathrm{Cl}(k,0)$	$\mathrm{Cl}(0,k)$
$k=0$	$\mathbb{R}$	$\mathbb{R}$
$k=1$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{C}$
$k=2$	$M_2(\mathbb{R})$	$\mathbb{H}$
$k=3$	$M_2(\mathbb{C})$	$\mathbb{H} \oplus \mathbb{H}$
$k=4$	$M_2(\mathbb{H})$	$M_2(\mathbb{H})$
$k=5$	$M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$	$M_4(\mathbb{C})$
$k=6$	$M_4(\mathbb{H})$	$M_8(\mathbb{R})$
$k=7$	$M_8(\mathbb{C})$	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$
$k=8$	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})$

which are obtained with the following reasoning:

- $\mathrm{Cl}(3,0) = \mathrm{Cl}(2,0) \otimes \mathrm{Cl}(0,1) = M_2(\mathbb{R}) \otimes \mathbb{C} = M_2(\mathbb{C})$
- $\mathrm{Cl}(0,3) = \mathrm{Cl}(0,2) \otimes \mathrm{Cl}(1,0) = \mathbb{H} \otimes (\mathbb{R} \oplus \mathbb{R}) = (\mathbb{H} \otimes \mathbb{R}) \oplus (\mathbb{H} \otimes \mathbb{R})$

  And so on. We also show $\mathrm{Cl}(0,5)$ and $\mathrm{Cl}(0,6)$ since they're a little tricky:
- $M_2(\mathbb{C}) \otimes \mathbb{H} = M_2(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}) = M_2(M_2(\mathbb{C})) = M_4(\mathbb{C})$, where $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} = M_2(\mathbb{C})$ since when tensoring with $\mathbb{C}$ having square roots of $+1$ is the same as having square roots of -1, so $\mathbb{H}$ is the same as $M_2(\mathbb{R})$ and tensoring $\mathbb{C}$ with $M_2(\mathbb{R})$ gives $M_2(\mathbb{C})$.
- $M_2(\mathbb{H}) \otimes \mathbb{H} = M_2(\mathbb{H} \otimes \mathbb{H}) = M_2(M_4(\mathbb{R})) = M_8(\mathbb{R})$

There are now two important things to note:

1. There is a certain symmetry between left and right columns, namely $\mathrm{Cl}(4+k,0) = M_{2k}(\mathrm{Cl}(0,4-k))$ and analogously $\mathrm{Cl}(0,4+k) = M_{2k}(\mathrm{Cl}(4-k,0))$

2. $\mathrm{Cl}(8,0) = \mathrm{Cl}(0,8) = M_{16}(\mathrm{Cl}(1,0))$. This is a very important property because it means that the next algebras can be trivially derived from the ones in the table by simply taking the matrix algebra with some specific coefficients. So in general $\mathrm{Cl}(k+8,0) = M_{16}(\mathrm{Cl}(k,0))$ and $\mathrm{Cl}(0,k+8) = M_{16}(\mathrm{Cl}(0,k))$. For this reason a lot of the properties of fermions we will derive will be dependent on the dimension mod 8.

Lecture 6.2 (24/5) The spin groups

We could also make a table for the complex Lie algebras:

n	$\mathrm{Cl}(n,\mathbb{C})$
1	$\mathbb{C} \oplus \mathbb{C}$
2	$M_2(\mathbb{C})$
3	$M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$
4	$M_4(\mathbb{C})$
$\vdots$	$\vdots$

which is simpler since we just have a periodicity of 2. This is because the formula $\mathrm{Cl}(p,0) \otimes \mathrm{Cl}(0,2) = \mathrm{Cl}(0,p+2)$ becomes $\mathrm{Cl}(k,\mathbb{C}) \otimes \mathrm{Cl}(2,\mathbb{C}) = \mathrm{Cl}(k+2,\mathbb{C})$ in the complex case.

Note that both this and the real classification is a classification as *algebras* not as super algebras. So now we want to understand the classification as super algebras.

4.1 The spin group

The spin group is a subgroup of the set of invertible elements of a Clifford algebra.

Analogy: If I have a bosonic phase space $(\mathbb{R}^2, dp \wedge dx)$ where we have canonical transformations, i.e. endomorphisms of the phase space wich preserve the phase space. We can in particular choose linear canonical transformations. A canonical transformation is a transformation that is generated from an observable. So for linear ones I just need to write down observables of a particular polynomial degree, such that the resulting things looks like linear vector fields. It is the quadratic observables, since for a generic observable we have:

$$f \to df = \omega(X_f, -), \quad X_f(g) = \{f, g\} \tag{4.4}$$

If f is a quadratic observable, then df is linear and I get a linear canonical transformations. What quadratic observables do we have and what transformations do these generate?

$$\begin{array}{lll} x^2 \to & 2xdx = \omega(X_{x^2}, -) & X_{x^2} = 2x\frac{\partial}{\partial p} \\ p^2 \to & 2pdp = \omega(X_{p^2}, -) & X_{p^2} = -2p\frac{\partial}{\partial x} \\ xp \to & xdp + pdx = \omega(X_{xp}, -) & X_{xp} = p\frac{\partial}{\partial p} - x\frac{\partial}{\partial x} \end{array}$$

These should then generate some symmetry algebra. I can compute their Poisson brackets:

$$\{p^2, x^2\} = 2p\{p, x^2\} = 4px\{p, x\} = 4px \tag{4.5}$$

$$\{p^2, p^2\} = 0 \tag{4.6}$$

$$\{p^2, px\} = 2p^2 \tag{4.7}$$

$$\{x^2, px\} = -2x^2 \tag{4.8}$$

but if we change names:

$$E = \frac{p^2}{2}, \quad F = \frac{x^2}{2}, \quad H = px \tag{4.9}$$

So they generate the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. Generally $\mathfrak{sl}$ is supposed to be volume preserving transformations, while we expected to find transformations that preserve a symplectic form, but because we're only in a two dimensional space, a symplectic form is also a volume form. So we expected $\mathfrak{sp}(2, \mathbb{R})$, but it's okay since in two dimensions they're isomorphic.

This procedure could be repeated also in higher dimensions: we compute all quadratic polynomials in ps and xs and then compute their Poisson brackets.

What should happen in the odd case? In the odd case we have an inner product, we still have a Poisson bracket and we would like it to be closed in the quadratic terms like we found in the bosonic case and we expect to get the Lie algebra of orthogonal transformations $\mathfrak{so}(n)$ or $\mathfrak{so}(p, q)$ since this time we're preserving an inner product.

Also note that in the procedure above we could instead have started with the Weyl algebra using the commutators instead of the Poisson brackets since the two are the same. So for the bosonic case we can start with the commutators of quadratic elements in the Clifford algebra $\mathrm{Cl}(p, q)$ and we expect to reproduce the Lie bracket of $\mathfrak{so}(p, q)$.

Recall that the generators of $\mathfrak{so}$ are antisymmetric matrices, so we can write

$$\mathfrak{so}(V) = \Lambda^2 V \tag{4.10}$$

this is why for example Lorentz transformations have two lower indices which are antisymmetric.

But how do I think of it? Well if we have $e_i \wedge e_j$ it will be something like $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the i and j indices, meaning that applied to e_i it gives e_j and viceversa but with a minus sign.

Now to calculate their commutator $[e_i \wedge e_j, e_k \wedge e_l]$ simply note:

- if all indices are different then it's zero since $(e_i \wedge e_j)(e_k \wedge e_l)$ would first send e_k to e_l and viceversa but these are sent to 0 since neither is e_i or e_j
- it's nonzero only if exactly two indices are the same. If we had three equal indices then we would have something like $e_i \wedge e_i$ which is zero.
- it's nonzero only if two indices are the same and we have that $(e_i \wedge e_j)(e_j \wedge e_k)$ sends e_k to e_i, while $(e_j \wedge e_k)(e_i \wedge e_j)$ sends e_i to e_k. So we have $[e_i \wedge e_j, e_j \wedge e_k] = e_k \wedge e_i$

Putting these together gives:

$$[e_i \wedge e_j, e_k \wedge e_l] = +g_{il}e_k \wedge e_j - g_{ik}e_l \wedge e_j + g_{jl}e_i \wedge e_k - g_{jk}e_i \wedge e_l \tag{4.11}$$

Claim. The map $\Lambda^2 V \to \mathrm{Cl}(V)$ given by $v \wedge w \mapsto \frac{1}{4}(vw - wv)$, carries Lie brackets to commutators, i.e. it's a homomorpism of Lie algebras

Proof. Work with an orthonormal basis.

$$e_i \wedge e_j \mapsto \frac{1}{4}(e_i e_j - e_j e_i) = \frac{1}{2}(e_i e_j) \tag{4.12}$$

$$[e_i \wedge e_j, e_k \wedge e_l] \mapsto \delta_{il}\frac{e_k e_j}{2} - \delta_{ik}\frac{e_l e_j}{2} + \delta_{jl}\frac{e_i e_k}{2} - \delta_{jk}\frac{e_i e_l}{2} \tag{4.13}$$

In which the image of the Lie bracket is just taken from the result above. Now we want to calculate $\left[\frac{1}{2}e_i e_j, \frac{1}{2}e_k e_l\right]$ and make sure that it gives the same result. To do this we use the commutation relation:

$$e_a e_b + e_b e_a + 2\delta_{ab} = 0 \tag{4.14}$$

and using this relation multiple times we get

$$\left[\frac{1}{2}e_i e_j, \frac{1}{2}e_k e_l\right] = \frac{1}{4}(e_i e_j e_k e_l - e_k e_l e_i e_j) \tag{4.15}$$

$$= \frac{1}{4}(e_i e_j e_k e_l - e_i e_j e_k e_l + 2e_k e_j \delta_{il} - 2e_l e_j \delta_{ik} + 2e_i e_k \delta_{jl} - 2e_i e_l \delta_{jk}) \tag{4.16}$$

$$= + \delta_{il}\frac{e_k e_j}{2} - \delta_{ik}\frac{e_l e_j}{2} + \delta_{jl}\frac{e_i e_k}{2} - \delta_{jk}\frac{e_i e_l}{2} \tag{4.17}$$

which is exactly the image of the Lie bracket. $\square$

I can then exponentiate a quadratic term and I would still get something in the Clifford algebra, which should be isomorphic to some group with the Lie algebra $\mathfrak{so}(V)$. As usual, the exponential is given by the power series

$$\exp\left(\frac{e_i e_j}{2}\right) = 1 + \frac{e_i e_j}{2} + \frac{1}{2!}\left(\frac{e_i e_j}{2}\right)^2 + \dots \tag{4.18}$$

and we get a Lie group of non infinitesimal symmetries inside $\mathrm{Cl}(V)^\times_+$. $\times$ indicates that we only take invertible elements and $+$ indicates we're in the positive parity side, so with only even powers. But since we're starting with the Lie algebra $\mathfrak{so}(V)$ is it possible to get a different Lie group from SO(V)? Yes, for example SU(2) and SO(3) have the same Lie algebra, in fact there's a map which says that there are two SU(2) matrices for every SO(3) matrix. This can be written as the short exact sequence:

$$1 \to \mathbb{Z}_2 \to \mathrm{SU}(2) = Spin(3) \to \mathrm{SO}(3) \to 1 \tag{4.19}$$

which just says that there is a 2:1 mapping from SU(2) to SO(3). In addition, we found our first example of a spin group: SU(2) = $Spin(3)$. In general a spin group is going to be a 2:1 cover of the orthogonal group.

Why is it a 2:1 map?

How do I get the the 2:1 map in general? Consider the group in $\mathrm{Cl}(V)^\times$ generated by unit vectors. ($e_i^2 = \pm 1 \implies e_i^{-1} = \pm e_i$, so the unit vectors are in fact invertible and their products are also invertible because I can invert the unit vectors one by one) Note that $x \in V$ is also an element in $\mathrm{Cl}(V)^\times$, since we have $x\,x = -g(x,x)$ and therefore $x^{-1} = -\frac{1}{g(x,x)}x$ is the inverse.

Claim. This group acts on V and preserves the inner product. In other words there's a map from the group to the group of orthogonal transformations of V.

Proof. There's a "twisted" adjoint action of $\mathrm{Cl}(V)^\times$ on $\mathrm{Cl}(V)$, so I get a representation of $\mathrm{Cl}(V)^\times$ on $\mathrm{Cl}(V)$.

$$y \in \mathrm{Cl}(V), x \in \mathrm{Cl}(V)^\times \tag{4.20}$$

$$x \to (y \mapsto \pi(x) y x^{-1}) \tag{4.21}$$

where π is the parity operator on $\mathrm{Cl}(V)$. Now, if x is a unit vector in V, $x \in \mathrm{Cl}(V)^\times, \pi(x) = -x$, pick $y \in V \subseteq \mathrm{Cl}(V)$ so we get

$$y \mapsto -xyx^{-1} = +\frac{xyx}{g(x,x)} = \frac{x}{g(x,x)}(-xy - 2g(x,y)) = -\frac{x^2}{g(x,x)}y - 2\frac{g(x,y)x}{g(x,x)} \tag{4.22}$$

$$= y - 2\frac{g(x,y)}{g(x,x)}x \ \in V \tag{4.23}$$

so the adjoint action preserves the subspace V. But what is the action we found? Well if it were $y \mapsto y - \frac{g(x,y)}{g(x,x)}x$ then it would be projection onto the hyperplane orthogonal to x. Instead since we have the 2, we get a reflection through the hyperplane orthogonal to x. Now, how many unit vectors are there that have the same orthogonal hyperplane? Just two, the two that stick out on opposite sides of the hyperplane. So that means there's exactly two elements of the Clifford algebra that map to the same orthogonal transformation, namely reflection through the plane perpendicular to the unit vector.

It's simple to check that the transformation preserves the inner product (that's where the fact that x is a unit vector is used) □

Note however that this transformation doesn't preserve parity and is in the odd part of the algebra. But from this we can construct two groups:

$$Pin(V) = \text{generated by unit vectors in } V \text{ inside } \mathrm{Cl}(V)^\times \tag{4.24}$$

The (twisted) adjoint action restricts to $V \subseteq \mathrm{Cl}(V)$ and defines a 2:1 map

$$1 \to \mathbb{Z}_2 \to Pin(V) \to O_o(V) \to 1 \tag{4.25}$$

where $O_o(V)$ indicates the part of $O(V)$ generated by reflections, which may in general not be the whole orthogonal group. But when the inner product is normal, so it has positive definite signature, we get $O_o(V) = O(V)$. Now we can define the spin group, which corresponds to only the even part of the transformations generated by reflection:

$$Spin(V) = Pin(V) \cap \mathrm{Cl}(V)_+ \tag{4.26}$$

so we get the following 2:1 map

$$1 \to \mathbb{Z}_2 \to Spin(V) \to SO_o(V) \to 1 \tag{4.27}$$

since two reflections together preserve the parity, so I have the parity preserving part of the part of the rotation group generated by reflections. Again, if we have a positive definite signature then $SO_o(V) = SO(V)$.

Let's look at SO(2). A generator of the Lie algebra is $e_1 \wedge e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and it's exponential is given by:

$$\exp\left(t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \tag{4.28}$$

and where does this map in the Clifford algebra? $\frac{e_1 e_2}{2}$ whose exponential is:

$$\exp\left(t\frac{1}{2}e_1 e_2\right) = \cos\frac{t}{2} + e_1 e_2 \sin\frac{t}{2} \xrightarrow[t=2\pi]{} -1 \tag{4.29}$$

noting $(e_1 e_2)^2 = -1$. So just as SO(2) is a circle, so is Spin(2), and the map that we have simply wraps Spin(2) around SO(2) twice. Also in higher dimensions, whatever the Spin group does, when restricted to a plane then it simply rotates in the plane half as fast as in SO.

Why is then the spin group important in quantum mechanics, well there are fermions, so we expected there to be representations of half integer spin, meaning that if I rotate in a 2-plane it only goes half the way around, so half integer spin really just means that we're working with the Spin group.

To get reps of Spin(V) use $\mathrm{Cl}(V)_+$ modules.

Claim. $\mathrm{Cl}(p, q) = \mathrm{Cl}(p, q + 1)_+$ as algebras.

Proof. the map is $e_i \mapsto e_i f$, where f is the additional square root of -1. $e_i f$ squared is again e_i^2. So we have a map from the entire $\mathrm{Cl}(p,q)$ to quadratic elements in $\mathrm{Cl}(p,q+1)$, meaning even elements. But then the dimension is the same so we get that they're isomorphic. □

Lecture 7.2 (31/5) Properties of, and pairings on, spinors

Last time we constructed $\mathrm{Spin}(V)$ in $\mathrm{Cl}_+^\times$. The key idea was to associate to a unit vector in $\mathrm{Cl}(V)$ a reflection of V through a hyperlane orthogonal to that vector, from which we can easily see that it's a 2:1 cover.

4.2 Representations of Clifford algebras

We're moving towards a field teory, so we're interested in $Spin(d-1,1)$ for a d dimensional spacetime, so the spin cover of the group of lorentz transformations in a $(d-1,1)$ spacetime. We're going to have spinor fields, which have a certain spin index, which transform in one of the representations of the spin group. Now recall that

$$Spin(d-1,1) \subseteq \mathrm{Cl}(d-1,1)_+ \cong \mathrm{Cl}(d-1,0) \tag{4.30}$$

and we already computed the Clifford algebras of the form $\mathrm{Cl}(k,0)$. In the following table we collect information about the algebras $\mathrm{Cl}(k,0)$ and their representations, i.e. the *Clifford modules.*

d	k	$\mathrm{Cl}(k,0)$	Representations	Real dimension
2 (10)	1	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}_\pm$	1 (16)
3 (11)	2	$M_2(\mathbb{R})$	$\mathbb{R}^2$	2 (32)
4	3	$M_2(\mathbb{C})$	$\mathbb{C}^2, \overline{\mathbb{C}}^2$	4
5	4	$M_2(\mathbb{H})$	$\mathbb{H}^2$	8
6	5	$M_2(\mathbb{H})\oplus M_2(\mathbb{H})$	$\mathbb{H}^2_\pm$	8
7	6	$M_4(\mathbb{H})$	$\mathbb{H}^4$	16
8	7	$M_8(\mathbb{C})$	$\mathbb{C}^8$	16
9	8	$M_{16}(\mathbb{R})$	$\mathbb{R}^{16}$	16

in which the dimension is the number of indices in physics. In order to find the representations I use the fact that matrix algebras are simple: the only thing the matrices can do is act on row vectors or column vectors. See also section A.4 for more details. For example we have:

- $\mathbb{R}\oplus\mathbb{R}$ has representations $\mathbb{R}_+$ and $\mathbb{R}_-$ which correspond to the representations in which we act by multiplying by a number from one of the two $\mathbb{R}$s and trivially from the other $\mathbb{R}$. These are called Majorana-Weyl spinors.
- $M_2(\mathbb{R})$ has representation $\mathbb{R}^2$ which we can think equivalently as column or row vectors.

- $M_2(\mathbb{C})$ is similar but in this case column and row vectors are distinct because complex conjugation is not a linear map, so we have $\mathbb{C}^2$ and $\overline{\mathbb{C}}^2$. These are called Weyl spinors.

- $M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$ gives $\mathbb{H}^2_\pm$ just as in the first example and these are called pseudo Majorana-Weyl spinors.

There are a few things we can observe from the table. Note that for $k = 3$ the dimension jumps to 4, then for $k = 4$ it jumps to 8, for $k = 6$ to 16 and for $k = 10$ to 32, and the indices of these jumps are related to the dimensions of the four normed division algebras:

$$k = 2 + \dim \mathbb{K}, \quad \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \tag{4.31}$$

In addition we can see that for any spacetime dimension we always have either one or two representations. This is exactly what we anticipated when we were reviewing canonical quantization: quantization of the Poisson algebra gives the Weyl algebra which has a unique representation, while quantization of fermionic observables lead to the Clifford algebras which have either one or two representations, based on the spacetime dimension.

4.3 Real, complex and pseudoreal representations

So in general we can get many properties of the spin representations we want from that table. Although, in order to do that we need to understand what it means for a representation to be quaternionic.

It turns out that there is a threefold classification of representations of groups:

- complex, where I "cannot raise or lower indices"
- real, where I can raise indices with a symmetric pairing (e.g. by contracting with the matric)
- pseudoreal or quaternionic, where I can raise indices with an antisymmetric pairing (*pseudo* because I still have a way of raising and lowering indices, but not real because I don't do it with a symmetric object)

We have an example of the pseudoreal form with the symplectic form:

$$\omega = dp \wedge dx \tag{4.32}$$
$$\partial_x \vee \omega = -dp \tag{4.33}$$
$$\partial_p \vee \omega = +dx \tag{4.34}$$

so passing between vector fields and one forms corresponds to raising and lowering indices using an antisymmetric pairing.

Let W be a complex vector space with a unitary group representation $\rho : G \to U(W)$, i.e. the group is represented by unitary matrices. I then get a G representation on $\overline{W}$, the dual representation:

$$|w\rangle \to U_g |w\rangle \tag{4.35}$$

$$\langle w'| \to \langle w|' U_g^\dagger \tag{4.36}$$

where I act on the "row vector" $\langle w'|$ with $U^\dagger = U^{-1}$ because there is a pairing $\langle w'|w\rangle$ between vectors and dual vectors, and I want the pairing to be preserved.

Is $\overline{W}$ the same representation as W? Meaning, are they related by a change of basis? In general no, take for example the 1 dimensional complex space on which the circle acts as a change of phase, then we have

$$\overline{e^{i\theta}} = e^{-i\theta} \tag{4.37}$$

but the two are not equivalent since there is no change of basis from one to the other. We call these two distinct representations $+1$ and -1. On the other hand if I take the two dimensional representation built up as the direct sum of the two, meaning it has one $+1$ charge and one -1, then we have

$$\overline{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}} \cong \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \tag{4.38}$$

which are equivalent since we clearly have a change of basis to bring one into the other.

In this situation, in which $W \ncong \overline{W}$, then we say W is complex. Note that it does not just mean that we have a complex vector space. In particular for the two dimensional representation above, there's a change of basis that brings it into

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{4.39}$$

which has only real entries. Instead for the first case there's no change of basis that leads to it having real entries. So in a complex representation the matrix entries are complex no matter what basis I consider.

Suppose now that I can identify W with $\overline{W} = W^\vee$, meaning I have a map $\phi : W \to W^\vee$, or analogously I have a map $\phi : \mathbb{C} \to W^\vee \otimes W^\vee$ which is a 2 index tensor. In particular if the first is invariant for the group action, then so is the second one, so ϕ is a G invariant two index tensor. Therefore there's a copy of the trivial G representation inside of 2 index tensors.

So ϕ is the map to raise and lower indices. In particular the tensor ϕ has some symmetry property, since the G action on this space of tensors commutes with symmetrizing and antisymmetrizing, so the irreducible G representations are either totally symmetric (real representation) or totally antisymmetric (pseudoreal representation).

Take for example SU(2) in which we start with the spin $\frac{1}{2}$ doublet in $\mathbb{C}$. Then if I tensor this representation with itself then I should get a copy of the trivial representation which I do get since I get the triplet and the singlet, respectively symmetric and antisymmetric parts. Since we want the trivial representation, we're interested in the latter and have an antisymmetric 2 index invariant tensor, so I can raise or lower indices with something antisymmetric like the Levi Civita symbol. That's kind of why the Pauli matrices have complex entries, while if it were a real representation we would have some change of basis to give it real entries. Note that SU(2) also has real representations, for example the vector representation on which it acts simply as spatial rotations.

In addition, one can show that if we have a pseudoreal structure, then the representation can be written in terms of quaternionic matrices.

So we can just read from the table what kind of representation we have: real, pseudoreal (also called pseudo Majorana) or complex.

4.4 Clifford module construction

In section 4.2 we saw some properties of the representations of Clifford algebras, i.e. the Clifford modules. These are important since they're essentially the fermionic version of the Hilbert space. Recall in fact that the Clifford algebra is the fermionic analogue of the Weyl algebra and the representation of the Weyl algebra is the Hilbert space. We would like to push the analogy further. The normal trick to build the Hilbert space is the following:

- split phase space as $\langle x\rangle \oplus \langle p\rangle$
- take a complete set of commuting observables, i.e. x or p
- take Hilbert space $\mathscr{H} = L^2(\mathbb{R}_x)$ with x and p acting as x and $i\frac{\partial}{\partial x}$ (or analogously take $\mathscr{H} = L^2(\mathbb{R}_p)$ ...) which automatically has an inner product which is respected by x and p, meaning they're self adjoint. The inner product is given by the integral over $\mathbb{R}$.

We now want to be able to do the same for the odd part: take a complete set of supercommuting observables, think about the functions they generate, let the observables from the set act by multiplication and the other ones by derivatives and we should have a way of integrating as well. The analogy works best for even dimensional clifford algebras over $\mathbb{C}$.

Set $V = \mathbb{R}^{2n}$ with coordinates $x_1, \dots, x_n, y_1, \dots, y_n$ and standard inner product $ds^2 = \sum_i dx_i^2 + dy_i^2$. Consider the complexification $V \otimes \mathbb{C}$, we have $\mathrm{Cl}(V_\mathbb{C}) \cong \mathrm{Cl}(V) \otimes \mathbb{C}$. Why do I do that? Well I want a complete set of supercommuting observables, meaning anticommuting since they're fermionic. For them to be anticommuting, they should have zero length, but in the real case I can't get a nonzero vector of zero length. Instead I can set

$$z_i = x_i + iy_i \tag{4.40}$$
$$\bar{z}_i = x_i - iy_i \tag{4.41}$$

now $ds^2 = \sum_i \frac{1}{2} dz^i d\bar{z}^i$, so z_i and $\bar{z}_i$ have zero square but nontrivial inner product.

The complete set of supercommuting observables is then a maximal isotropic subspace. Isotropic meaning I have a bilinear inner product which is zero when I restrict it to my subspace. In this case the complete set may be the set of all z_i since here the inner product is identically zero (analogously I could choose $\bar{z}_i$). This is analogous to the fact that if I take a symplectic form on the normal phase space and I restrict it to the space spanned by the xs then it vanishes.

So essentially we're decomposing $V_\mathbb{C}$ as $V_\mathbb{C} = L \oplus L^\vee$, the first with coords z_i and the second with $\bar{z}_i$, so we have something like the separation in x and p. However it's important to note that to do it we had to complexify.

So we've built a Clifford module, since now the Hilbert space is functions on L and this is the same as $\Lambda^*(L^\vee)$, the exterior algebra over $L^\vee$, since functions on L are generated by elements in $L^\vee$, meaning linear coordinates of L, and these are anticommuting so they generate the exterior algebra.

$$\mathscr{H} = \Lambda^*(L^\vee) \tag{4.42}$$

However this notation is not typical. The elements in the fermionic Hilbert space are called *spinors* and the Hilbert space is indicated by S:

$$S = \Lambda^*(L^\vee) \tag{4.43}$$

Let's call θ_i the anticommuting coordinates of the exterior algebra. Now what's the representation? It's given by

$$\bar{z}_i \mapsto \theta_i \tag{4.44}$$
$$z_i \mapsto \frac{\partial}{\partial \theta_i} \tag{4.45}$$

The map above is also interesting for another reason. It gives me a Clifford map $\mathrm{Cl}(V_\mathbb{C}) \to \mathrm{DiffOps}(L) \subseteq \mathrm{End}(\Lambda^*(L^\vee))$ since commutators are sent to commutators. Recall we had seen $\mathfrak{gl}(1|1) = \mathrm{DiffOps}(\mathbb{C}^{0|1}) = \mathrm{Cl}(2, \mathbb{C})$, now we have something more general:

$$\mathrm{Cl}(2n, \mathbb{C}) \cong \mathrm{DiffOps}(\mathbb{C}^{0|n}) \cong \mathrm{End}(\Lambda^*(\mathbb{C}^{0|n})) \cong \mathfrak{gl}(n|n) \tag{4.46}$$

To do so we simply count dimensions:

- $\mathrm{Cl}(2n, \mathbb{C})$ has dimensions 2^{2n}

- the map above is injective so $\mathrm{DiffOps}(\mathbb{C}^{0|n})$ also has dimensions 2^{2n}
- $\Lambda^*(\mathbb{C}^{0|n})$ has dimensions 2^n, so $\mathrm{End}(\Lambda^*(\mathbb{C}^{0|n}))$ has dimension 2^{2n}
- all of them have the same amount of odd and even things, n for each. Therefore they can be identified with $\mathfrak{gl}(n|n)$.

So $\mathrm{Cl}(2n, \mathbb{C})$ is a matrix superalgebra.

4.5 Pairings on spinors

We built the space of spinors S and claimed it's a Hilbert space, however we still need an inner product. We now want to recover the pairings on spinors from an integral over $\Pi L = \mathbb{C}^{0|n}$. We're still considering $\mathrm{Cl}(2n, 0)$, corresponding to the cases with an odd dimensional spacetime. Looking at the table I see which of them have a real and which have a pseudoreal structure, so I expect the pairing to go symmetric antisymmetric antisymmetric symmetric in the four cases.

Let us recall the procedure up to now: V even dimensional vector space $d = 2n$, which I then split as $L \oplus L^\vee$ with $g|_L = 0$ (analog of splitting phase space into x and p). We then think of the spinors S as $\Lambda^*(L^\vee)$ and we sent $\overline{z}_i \in L^\vee \mapsto \theta_i$ and $z_i \in L \mapsto \frac{\partial}{\partial \theta}$.

Now, $\psi \in S$ is a function on L, just like a wave function is a function on the space whose coordinates are position. We would like to have a pairing with any other element $\phi \in S$ and our guess for the pairing could be

$$(\psi, \phi) = \int_L \psi^t \phi \tag{4.47}$$

Where ^{t} reverses the order in $\Lambda^*(L^\vee)$. We need to then check that it is a Clifford invariant pairing and we need to understand what it means to integrate over L.

The key property of integration is the fundamental theorem of algebra, so total derivatives integrate to 0,so I want integral over fermionic variables to have that. Consider $L = \Pi\mathbb{R}$, a function over it is $\psi \in \Lambda^*(L^\vee)$, $\psi = a + b\theta, a \in \mathbb{R}, b \in \mathbb{R}$, so $\frac{\partial \psi}{\partial \theta} = b$. We want

$$\int d\theta \frac{\partial \psi}{\partial \theta} = 0 \tag{4.48}$$

so we get that the integration of a number is just 0:

$$\int d\theta b = 0 \tag{4.49}$$

so to have a linear map, the only possibility is to have:

$$\int d\theta (a + b\theta) = b \tag{4.50}$$

in which we set the normalization

$$\int d\theta\theta = 1 \tag{4.51}$$

In more coordinates we have:

$$\int_{\Pi\mathbb{R}^n} d^n\theta\psi = [\psi]_{\theta_1\ldots\theta_n} \tag{4.52}$$

where $[\psi]_{\theta_1\ldots\theta_n}$ is the coefficient of the $\theta_1\ldots\theta_n$ term in ψ which is its top form. We only get that term since every integral picks up the top form in the respective variable, so taking the integral over all θs then gives the overall top form. For example with two variables we would have:

$$\int d\theta_1 d\theta_2(a + b\theta_1 + c\theta_2 + d\theta_1\theta_2) = d \tag{4.53}$$

If we take as a pairing on spinors the integration over L, then we get a form with a definite parity (positive parity if it pairs even with even and odd with odd; negative parity if it pairs even with odd) and symmetry (symmetric or antisymmetric under the exchange of the inputs).

Let us now check that the pairing above reproduces the symmetry results we expect from the table, namely $+--+$. In the first case the space we're integrating over is $\Pi L = \mathbb{R}^{0|1}$ so we have a single odd variable θ. Therefore there's just one pairing

$$(1, \theta) = \int d\theta\theta = (\theta, 1) \tag{4.54}$$

which is clearly symmetric so it works. Now, for $k = 2$ we have two odd variables θ_1 and θ_2:

$$(1, \theta_1\theta_2) = \int d\theta_1 d\theta_2\theta_1\theta_2 = -\int d\theta_1 d\theta_2\theta_2\theta_1 = \int d\theta_1 d\theta_2\overline{\theta_1\theta_2} = -(\theta_1\theta_2, 1) \tag{4.55}$$

$$(\theta_1, \theta_2) = \int d\theta_1 d\theta_2\theta_1\theta_2 = -\int d\theta_1 d\theta_2\theta_2\theta_1 = -(\theta_2, \theta_1) \tag{4.56}$$

so we have an antisymmetric pairing. For three odd variables:

$$\overline{\theta_1\theta_2\theta_3} = \theta_3\theta_2\theta_1 = -\theta_1\theta_2\theta_3 \tag{4.57}$$

Again antisymmetric. What if we do four? Well for $\theta_4\theta_3\theta_2\theta_1$ the sign of the permutation is even, so $\overline{\theta_1\theta_2\theta_3\theta_4} = \theta_1\theta_2\theta_3\theta_4$ and again we get a symmetric pairing. These results are summarized in the following table:

$n = \dim L$	Parity	Symmetry
1	$-$	$+$
2	$+$	$-$
3	$-$	$-$
4	$+$	$+$

So we get a pairing which we'll prove to be invariant and which is symmetric or antisymmetric in order to reproduce the real or quaternionic structure from the table. It just has to do with the sign of the permutation to reverse the order of the θs which is given by $(-1)^{n(n-1)/2}$ which has period four, so the pattern $+--+$ is repeated. The parity, instead, is simply alternating.

Lecture 8.1 (5/6) Pairings on spinors, gamma matrices and supersymmetry algebras

4.6 Gamma matrices and supersymmetry algebras

Reminder: we're interested in Lorentz invariant super Lie algebras with square roots of d dimensional translations. $P, Q, \{Q, Q\} = 2P$. In higher dimensions, Q needs to carry a spinor Lorentz representation, but the generalization is not simply

$$P_\mu, Q_\mu,\ \{Q_\mu, Q_\mu\} = 2P_\mu \tag{4.58}$$

since it's not Lorentz invariant.

The objective of this lecture is then to obtain the super Poincaré algebra.

In particular that means that we need the lorenz transformations to act on the supercharges in a way that's compatible with the bracket, meaning we want the supercharges to trasnform in a representation which is in a sense the square root of the vector representation (i.e. its tensor squared contains the vector rep) That's the spin representation! We already constructed it!

Why? For spin statistics Q is a fermion so it should have half integer spin, so it should transform in a half integer spin representation. Another way of thinking about it is that if you take any tensor representation of the Lorentz group and square it, you never get something that includes the vector representation. This is because in dimension d the tensor square decomposes into symmetric 2 index tensors and antisymmetric 2 index tensors, then I can also take the trace of the symmetric part and get a copy of the trivial representation, so we have

$$V^{\otimes 2} = \mathbb{C} + \mathrm{Sym}^2(V)_o + \Lambda^2(V) \tag{4.59}$$

which have dimensions $1 + (\frac{d(d+1)}{2} - 1) + \frac{d(d-1)}{2}$ which works out. However, none of these are in general of the dimension of the vector representation.

The question then is why is there always a map $S \otimes S \to V$?

S is not only a $Spin(V)$ representation but actually a $\mathrm{Cl}(V)$ module, since we have a map $V \otimes S \to S$ called Clifford multiplication. The Clifford multiplication is the same as a map $S \otimes S^\vee \to V^\vee$, but $V \cong V^\vee$ using the scalar product and if we have a (symmetric or antisymmetric) pairing $S \otimes S \to \mathbb{C}$ we can identify S with $S^\vee$ and then obtain the $S \otimes S \to V$ map.

We can then read off the properties of even dimensional gamma matrices from those of the pairing on spinors:

$n = \dim L$	Parity	Symmetry
1	$-$	$+$
2	$+$	$-$
3	$-$	$-$
4	$+$	$+$

For example let's work in four spacetime dimensions, meaning $\dim L = n = 2$. We have $S = \Lambda^*(L^\vee) = \langle 1, \theta_1, \theta_2, \theta_1\theta_2 \rangle$ which splits into two irreducible subspinors, the chiral spinors $S^+ = \Lambda^{even} = \langle 1, \theta_1\theta_2 \rangle$ and $S_- = \Lambda^{odd} = \langle \theta_1, \theta_2 \rangle$. From the table we see that we have an antisymmetric pairing of S_+ with S_+ and S_- with S_-, in the physics language this means I have two types of spinor indices, undotted (S_+) and dotted (S_-), which refer to the two chiral spinor representations, respectively called left and right chiral spinors. We indicate an element in S_+ by ψ^α and an element in S_- by $\bar{\chi}_{\dot{\beta}}$ and their indices can be raised or lowered with an antisymmetric tensor

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \tag{4.60}$$

$$\bar{\chi}_{\dot{\beta}} = \epsilon_{\dot{\beta}\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \tag{4.61}$$

These are the Weyl spinors. What about the gamma matrix? I want to take a spinor and multiply it by a vector using Clifford multiplication and then pair it with another spinor. Multiplying by a vector changes the parity so we have $S_- \xrightarrow{V\cdot} S_+$ which then pairs with S_+. So we have a map $S_- \otimes S_+ \otimes V \to \mathbb{C}$ which is given by

$$\langle s_-, v \cdot s_+ \rangle \tag{4.62}$$

where $s_- \in S_-, s_+ \in S_+, v \in V$ and $\cdot$ is Clifford multiplication. This results in a tensor with two spinor indices and one vector index which we indicate by $(\gamma_\mu)_{\alpha\dot{\beta}}$.

This allows us to rewrite a vector index as a bispinor index, one dotted and one undotted. That's like saying that the vector index comes from one left chiral and one right chiral spinor.

Let's check that it works. V acts on S with the following operators:

$$\theta_1, \theta_2, \frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2} \tag{4.63}$$

Now take the vector θ_1 and acting on the spinor S_+ we only get $\theta_1 \in S_-$. This only pairs with $\theta_2 \in S_-$ giving -1. In formulas:

$$\langle \theta_2, \theta_1 \cdot 1 \rangle = -1, \quad \theta_2 \in S_-, \theta_1 \in V \text{ and } 1 \in S_+ \tag{4.64}$$

Note that the map we constructed also gives a map between S_+ and S_- when the element of V is fixed. In particular for the first example, for $\theta_1 \in V$ we have that $1 \mapsto -\theta_2$ which can be written as:

$$\rho(\theta_1) = \begin{array}{c c} & \begin{array}{cc} 1 & \theta_1\theta_2 \end{array} \\ \begin{array}{c} \theta_1 \\ \theta_2 \end{array} & \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \end{array} \tag{4.65}$$

and for $\frac{\partial}{\partial\theta_1} \in V$, we have that $\theta_1 \in S_- \mapsto \theta_1\theta_2 \in S_+$, which can be written as

$$\rho\left(\frac{\partial}{\partial\theta_1}\right) = \begin{array}{cc} & \begin{array}{cc}\theta_1 & \theta_2\end{array} \\ \begin{array}{c}1 \\ \theta_1\theta_2\end{array} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\end{array} \tag{4.66}$$

Since we get a map of S_+ with S_- we say that the gamma matrices have negative chirality. It can also happen that we simply get a map of S_- with S_- and S_+ with S_+, in that case we say they have positive chirality. This property is only due to the fact that multiplying with elements in V passes S_+ to S_- and viceversa and the pairing has positive parity for $n = 2$. So in general the chirality is simply the opposite of the parity of the pairing.

Instead the symmetry is the same as that of the pairing. Note however that when the chirality is negative, the symmetry is intended over $S_+ \oplus S_-$, not over only S_+ or S_- which wouldn't make sense.

This information is summarized in the following table:

n	chirality	symmetry
1	$+$	$+$
2	$-$	$-$ (on $S_+ \oplus S_-$)
3	$+$	$-$
4	$-$	$+$ (on $S_+ \oplus S_-$)

For $n = 1 \mod 2$ we say that the theory is chiral, otherwise it is nonchiral.

What we've said up to now is enough to build the even dimensional minimal susy algebras:

d	Odd part	dim	algebra	R symmetry
2	S_+ or S_-	1	$\{Q_\alpha, Q_\beta\} = \gamma^\mu_{\alpha\beta} P_\mu$	$\mathrm{SO}(\mathcal{N}_+) \times \mathrm{SO}(\mathcal{N}_-)$
4	$S_+ \oplus S_-$	4	$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = \gamma^\mu_{\alpha\dot\beta} P_\mu$	$\mathrm{U}(\mathcal{N})$
6	$S_+ \oplus \mathbb{C}^2$ or $S_- \oplus \mathbb{C}^2$	8	$\{Q^i_\alpha, Q^j_{\underline{\beta}}\} = \gamma^\mu_{\alpha\beta}\omega_{ij} P_\mu$	$\mathrm{Sp}(\mathcal{N}_+) \times \mathrm{Sp}(\mathcal{N}_-)$
8	$S_+ \oplus S_-$	16	$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = \gamma^\mu_{\alpha\dot\beta} P_\mu$	$\mathrm{U}(\mathcal{N})$
10	S_+ or S_-	16	$\{Q_\alpha, Q_\beta\} = \gamma^\mu_{\alpha\beta} P_\mu$	$\mathrm{SO}(\mathcal{N}_+) \times \mathrm{SO}(\mathcal{N}_-)$

The first three columns should be clear, the fourth is obtained by using the gamma pairing to obtain a bispinor from the vector P_μ. In particular for $d = 6$ the gamma matrices are antisymmetric in the spinor indices, but the commutator should be symmetric. So we need extra indices i and j with another antisymmetric tensor ω_{ij} in order for the commutator to be symmetric. The last column is explained below

The supersymmetry algebra can then be "extended" by taking multiple copies of the same thing. In the following we use $\mathcal{N}$ to denote the amount of copies taken, meaning $\mathcal{N} = 1$ corresponds to minimal supersymmetry. In particular for n odd we have a chiral theory, meaning we can separately choose how many copies of S_+ and of S_- we want, so we have two independent parameters $\mathcal{N}_+$ and $\mathcal{N}_-$.

In the $\mathcal{N}$ extended case we also have an R symmetry. For $d = 2$ we get supersymmetric quantum mechanics which we already studied and saw that its R symmetry is given by $\mathrm{SO}(\mathcal{N})$. Instead for $d = 6$ I want to preserve a symplectic pairing so that's why I get Sp instead of SO. For $d = 0 \mod 4$ I'm always pairing dotted and undotted indices, so I never need to raise and lower indices, so there's no form I need to preserve. Therefore we want transformations that don't preserve any pairing, meaning $\mathrm{GL}(\mathcal{N})$, or its unitary form $\mathrm{U}(\mathcal{N})$ (they have almost the same Lie algebra, the complexification of $\mathfrak{u}(n)$ is $\mathfrak{gl}(n)$).

Lecture 8.2 (7/6) Susy algebra in odd dimensions, supermultiplets and superfields

Da mettere a posto

Important: having constructed $S = \Lambda^*(L^\vee)$, it is important to emphasize that there is no algebraic structure on S itself:

$$\psi_1(x) \cdot \psi_2(x) \tag{4.67}$$

is not physically relevant, doesn't have a proper meaning.

Second quantization: In quantum mechanics we have a map $\mathbb{R}_{time} \to R_{space}$, $x(t)$. The action is given by $\int dt |\dot{x}|^2$ with phase space $\mathbb{R}^2_{x,p}$ and when quantizing we get the Hillbert space $L^2(\mathbb{R}_x)$. Instead in quantum field theory we have a map $\mathbb{R}_{space} \to \mathbb{R}_{field}$, $\phi(x)$ with action $\int dx |d\phi|^2$ which when quanitized gives the Fock space $\mathrm{Sym}^*(L^2(\mathbb{R}_x))$. This procedure is typically called second quantization.

Instead for fermionic quantum mechanics we have a map $\mathbb{R}_{time} \to \mathbb{R}_\theta$, $\theta(t)$ with action $\int dt \theta\dot{\theta}$ which when quantized gives phase space $\Pi\mathbb{R}_\theta$ and Hilbert space S.

Second quantization for fermions then is similar. We start with a map $\mathbb{R}_{space} \to S_{spinorfield}$, $\psi_\alpha(x)$ with action $\int dx \overline{\psi} \partial\!\!\!/ \psi$ which when quantized gives $\Lambda^*(S)$.

Note that the thing that we constructed last time, namely

$$S_+ = \Lambda^{even}(L^\vee) = \langle 1, \theta_1\theta_2 \rangle \quad S_- = \Lambda^{odd}(L^\vee) = \langle \theta_1, \theta_2 \rangle \tag{4.68}$$

are the "first quantized" variables. When we second quantize we get something like θ_a and $\overline{\theta}_{\dot{\alpha}}$.

Symmetry property of gamma matrices

Pairings of spinors: $\psi\chi \in S = \Lambda^*(L^\vee)$

$$(\psi, \chi) = \int_{\Pi L} \psi^t \chi = [\psi^t \chi]_{\theta_1 \dots \theta_n} \tag{4.69}$$

is symmetric when $\frac{n(n-1)}{2}$ is even and antisymmetric otherwise.

The defining relation for gamma matrices is:

$$\langle v, \Gamma(\psi, \chi) \rangle = (\psi, v\chi) = \pm(v\psi, \chi) = \langle v, \Gamma(\psi, \chi) \rangle \tag{4.70}$$

Claim: there is always a universal map

$$\Gamma : S \otimes S \to V \tag{4.71}$$

this is the gamma matrix. It's the same as Clifford multiplication, only that it

From the table of symmetries

These Γ matrices are exactly those of QFT.

Let $V = \mathbb{R}^{2n}$

$$\mathrm{Cl}(V) \hookrightarrow \Lambda^*(L^\vee) = \mathbb{C}^{2^{n-1}|2^{n-1}} \tag{4.72}$$

$$\mathrm{Cl}(V) \cong \mathrm{Mat}_{2^{n-1}|2^{n-1}}(\mathbb{C}) \cong S \otimes S^\vee \cong S \otimes S \tag{4.73}$$

Also note that $\mathrm{Cl}(V) \cong \Lambda^*(V)$ as a vector space. The map from $\Lambda^*(V)$ to $S \otimes S^\vee$ is $\Gamma^{\mu_1,\dots,\mu_n}_{\alpha\beta}$.

Let's continue with the classification of algebras in even dimension

d	Scalar pairing chirality	P chirality	Scalar pairing symmetry	Γ symmetry	Minimal super charges	R symmetry
1						
2	$-$	$+$	$+$	$+$	S_+ or S_-	$\mathrm{SO}(\mathcal{N}_+) \times \mathrm{SO}(\mathcal{N}_-)$
3	$+$		$-$	$+$	S	$\mathrm{SO}(\mathcal{N})$
4	$+$	$-$	$-$	$-$	$S_+ \oplus S_-$	$\mathrm{U}(\mathcal{N})$
5	$+$		$-$	$-$	$S \otimes \mathbb{C}^2$	$\mathrm{Sp}(\mathcal{N})$
6	$-$	$+$	$-$	$-$	$S_+ \otimes \mathbb{C}^2$ or $S_- \otimes \mathbb{C}^2$	$\mathrm{Sp}(\mathcal{N}_+)\times \mathrm{Sp}(\mathcal{N}_-)$
7	$+$		$+$	$-$	$S \otimes \mathbb{C}^2$	$\mathrm{Sp}(\mathcal{N})$
8	$+$	$-$	$+$	$+$	$S_+ \oplus S_-$	$\mathrm{U}(\mathcal{N})$

We find the odd cases from dimensional reducing from the even ones.

Recall $\mathrm{Cl}(2n) \cong \mathrm{Cl}(2n-1)$ and $\mathrm{Cl}(V_{\mathbb{C}}) \cong \mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}$.

And we aldo know how to build representations of these things:

$$\text{Spinor: } = \Lambda^*(\mathbb{C}^n) = \Lambda^*(L^\vee) \tag{4.74}$$

decomposes into $S_+ = \Lambda^{even}$ and $S_- = \Lambda^{odd}$

How does e_0 act? Pick one of the θs and use

$$\theta_0 + \frac{\partial}{\partial \theta_0} \tag{4.75}$$

From this we have the reduction of the dimension in the table.

Think about $\Lambda^*(L^\vee)$ in four dimensions: $\mathbb{C}[\theta_0, \theta_1]$, $S_+ = \langle 1, \theta_1\theta_2\rangle, S_- = \langle \theta_1, \theta_2\rangle$. Now reduce along θ_0. S in 3d $=$ functions of θ_1. In 4d add θ_0 to odd powers to get $\Lambda^{even}(L^\vee) = S_+$ and add θ_0 to even powers to get $\Lambda^{odd}(L^\vee) = S_-$.

Scalar pairing: for $\alpha, \beta \in \mathbb{C}[\theta_1]$

$$\int_{\Pi L} \alpha^t \theta_0 \beta \tag{4.76}$$

we get

$$(1, \theta_1) \mapsto 1 \tag{4.77}$$
$$(\theta_1, 1) \mapsto -1 \tag{4.78}$$
$$\tag{4.79}$$

In 6d we get

$$(1, \theta_1\theta_2) = \int \theta_0\theta_1\theta_2 = 1 \tag{4.80}$$
$$(\theta_1\theta_2, 1) = \int \theta_2\theta_1\theta_0 = -1 \tag{4.81}$$
$$\tag{4.82}$$

What about the Γ matrices themselves? $v = \theta_1$. In 3d we have

$$\int_{\Pi L} \alpha^t \theta_1 \left(\theta_0 + \frac{\partial}{\partial \theta_0}\right) \cdot 1 = (\alpha, v\beta) = \int_{\Pi L} 1 \cdot \theta_1\theta_0 \cdot 1 = -1 \tag{4.83}$$

$(\alpha, v \cdot \beta) = (1, \theta_1 \cdot 1)$. v changes the parity, so there is a swap of sign. The same is true in 5 dimensions.

Lecture 9.1 (12/6) The "functor of points" and superfield formalisms

In the odd dimensional examples there is no chirality. In the even dimensional examples $d = 2n$ we either had a scalar pairing which was chiral (pairing $S_+ \times S_+ \to \mathbb{C}$) for n even. While the gamma pairing was chiral if n is odd, while for n even we had a pairing $S_+ \times S_- \to V$.

For odd dimensional cases S_+ or S_- becomes the unique spinor of $Spin(2n-1)$. For n even pull back the scalar pairing, while for n odd we pull back the gamma pairing. So either the scalar pairing matches the one one dimension higher, or the vector pairing does, but not both. These then will come back in examples.

5 Supersymmetric field theory

We now want to start constructing representations of the supersymmetry algebras we've found. We want to come up with actual systems whose spacetime symmetries consist of these supersymmetric extensions of Poincare symmetry that we've built. We sort of already did this for supersymmetric quantum mechanics: we described its superalgebra, saying we need a square root of time translations and then we started to write down Hilbert spaces where that algebra was actually represented. Then we actually wrote down a dynamical model:

$$x \in \mathrm{Map}(\mathbb{R}_t \to V), \quad \theta \in \mathrm{Map}(\mathbb{R}_t \to \Pi V) \tag{5.1}$$

bosonic and fermionic degrees of freedom, with action

$$S = \int dt \left(\frac{1}{2}|\dot{x}|^2 + \frac{1}{2}(\psi, \dot{\psi}) \right) \tag{5.2}$$

This was classical $\mathcal{N} = 1$ supersymmetric quantum mechanics, whose phase space was $T^*V \oplus \Pi V$ with symplectic structure $dp_i \wedge dx^i + g_{ij} d\psi^i d\psi^j$. When quantizing, the observables were then given by $Weyl(T^*V) \otimes \mathrm{Cl}(V)$.

This is what we were doing before getting into Clifford algebras.

Want to come up with a way of describing this system that gives me an idea for how to build higher dimensional analogs. This was a $0+1$ dimensional field theory. We now want to generalize this: in quantum field theory the spacetime is the set of parameters on which the fields depend, which are the actual degrees of freedom. In that sense, in quantum mechanics, the space coordinates are the fields.

5.1 Superfield formalism

We've built the supersymmetry algebras in all dimensions, so we now want to know what field theory looks like in more spacetime dimensions, on which the supersymmetric algebras will act. By doing this we'll get into supersymmetric dynamics and supersymmetric field theory. One way of doing this is to introduce superfields.

Idea: In special relativity, Lorentz/Poincaré summetry is entirely geometric, meaning it acts on the spacetime. For $X = \mathbb{R}^{1,d-1}$ with the flat metric $\eta = -dt^2 + \Sigma dx_i^2$, then the Poincaré group is the group of all transformations of X preserving η. A nice thing about relativity is that it tells you that a lot of the structure of what's going on is controlled by spacetime geometry. General relativity is the theory of the geometry of spacetime. In particular if the metric is flat then we get a global symmetry from the symmetries of the flat metric. Now that we have supersymmetric extensions of those algebras, so if we have a theory on flat spacetime we know what it means for it to be supersymmetric. We know what it means for there to be square roots of translations, those are the spinors, we know their algebra is a nontrivial extension of the Poincaré algebra. It would be really nice if there were some geometric object of which that super Poincaré algebra is the geometric group of symmetries. That object we would then call super spacetime.

There are such objects. You can sometimes formulate the fields of a supersymmetric theory, not separately as we did before, but as a map from spacetime into V.

In other words, I would like to combine x and θ into $\Phi \in \mathrm{Map}(\mathbb{R}^{1|1} \to V)$. This is then a superparticle moving in V where the supersymmetry algebra acts geometrically on its super spacetime. This can be done but there are two subtleties: we need the right notion of a map and then we need to understand how to get back θ and x back from Φ.

To work out the answer, we need to revisit some abstract nonsense from earlier in the term, in particular there was a connection between states and observables.

5.1.1 Functor of points

Idea: At least locally, a space is defined by the algebra of functions on it.

There should be a way of translating back and forth between properties of the two or operations on them.

I can define a point in the space by saying: there's one point for every map that goes from the observables to numbers.

The state consists of complete information: knowing which state you're in means knowing everything eabout the system, so in particular the numerical value of every observable. A state should be something that for each observables tells me what number I get when I measure it in that state. This corresponds to complete information, or a complete set of measurements. Note however that there are infinitely many observables but the number of measurements I actually need to do is finite, just x and p for phase space $\mathbb{R}^2_{x,p}$.

However, there's really no reason to privilege complete information. For example it happens often in physics that I measure some observables and I don't know what the others are. So I'm not on a zero dimensional subspace of the phase space, but some other subspace. So more generally I can have the notion of a parametrized subspace, like a path of states parametrized by P. Now, a complete measurement was a map from the algebra of observables to the algebra of numbers, so now a partial measurement is from the algebra of observables to A where A is the algebra of functions on P. We then have the following table

Space	Algebra
phase space	observables = functions on phase space
$\mathbb{R}^2_{x,p}$	$Obs = \mathbb{C}[x,p]$
point: $* \to \mathbb{R}^2_{x,p}$	$\mathbb{C}[x,p] \to \mathbb{C}$
$P \to \mathbb{R}^2_{x,p}$	$Obs \to A$

So if I'm interested in getting an algebraic perspective on the space of states then I'm interested in understanding how other spaces map into it, what every parametrized family of subspaces looks like.

This is actually how we often conceive of phase space: we say phase space is the set of states, in other words it's the collection of all its points. So remembering the space should be roughly the same information as remembering every possible parametrized family of points in this space.

Once I've convinced myself to accept any space of parameters, described by any algebra of functions over it, then I can do some cool stuff, for example take $A = \mathbb{C}[t]$, so $Obs = \mathbb{C}[x_i, p^i] \to \mathbb{C}[t]$ sending $x_i \mapsto f_i(t)$, $p^i \mapsto g^i(t)$, which I can interpret as a path in phase space, so $P =$ a line. If I then want points, I can maps from there into $\mathbb{C}$ by just setting the time to a specific value.

Now take $A = \mathbb{C}[t]/t^2$ (the dual numbers). Now we have the following types of maps $\mathbb{C}[x_i, p^i] \to \mathbb{C}[t]/t^2$, $x_i \mapsto a_i + tb_i$ and $p^i \mapsto c_i + td_i$. If we then want its points, the only map into $\mathbb{C}$ is to evaluate at $t = 0$ so it only has one point. But in the parametrized subspace there was more information, we didn't just have a_i and c_i, but also b_i and d_i, so we had a point plus a tangent vector. This concept is called a *fat point*, so $P =$ a fat point. We can think of this as a path where the first taylor coefficient is defined but everything higher is set to 0.

We call this concept in general the *functor of A points*, the functor sends X to $\mathrm{Map}(X, A)$ and as we said above, the idea of an A point in X is the same as that of a map from X into A. The *normal* points are simply $\mathbb{R}$ or $\mathbb{C}$ points, depending on the context. Recalling an old example, we had $\mathbb{R}[x]/(x^2 + 1)$ which has no $\mathbb{R}$ points since I have no maps from this algebra into $\mathbb{R}$, but I have two complex points, since the only two maps into $\mathbb{C}$ I can build are $x \mapsto \pm i$.

Then in the example above, the A points are given by a normal $\mathbb{C}$ point and a deformation around said point.

So in general the idea is that I need to allow myself to probe the space not just by points but by abstract families described by arbitrary algebras.

There's in fact a very general, category theoretic, result which is Yoneda's lemma which states that knowing the obect is the same as knowing how other objects map into that one. In formulas, if I know what the space of P points of X is for every P, that is equivalent to knowing X.

This is generalizing the idea that if I have a set, one way to say what the set is is to give a list of points, since a list of points is actually a list of maps from the point into that set.

5.1.2 Points of a supermanifold

This tells me how to generalize the notion of a point to a supermanifold, i.e. something whose functions are a superalgebra. So now *Obs* is a supercommutative algebra. For example we had $\mathbb{R}^{1|1}$ and we said it had one bosonic and one fermionic coordinate, so its algebra of functions is $\mathbb{C}[x, \theta], \theta$ odd, $\theta^2 = 0$. Now, standard points of $\mathbb{R}^{1|1}$ are given by maps $\mathbb{C}[x, \theta] \to \mathbb{C}$ which can only be of the type $x \mapsto a$ and $\theta \mapsto 0$. Recall that $\mathbb{R}^{1|1}$ was the phase space of supersymmetric quantum mechanics. So its observables were a superalgebra which we wrote down. Now what does it mean to take a measurement in such a space, since it's no longer true that i can reconstruct the super phase space by mapping to numbers. Instead now this space of points is too narrow since I completely forget about θ, meaning $\mathbb{R}^{1|1}$ has the same $\mathbb{C}$ points as $\mathbb{R}^{1|0}$.

Now let $A = \mathbb{C}[\eta]$ with η odd. There are "new points" of $\mathbb{R}^{1|1}$: $\mathbb{C}[x, \theta] \to \mathbb{C}[\eta]$, with $x \mapsto a$ and $\theta \mapsto b\eta$. a and b are then coordinates on the even and odd parts of $\mathbb{R}^{1|1}$. We won't prove it but in general it's enough to think about "superpoints": $A = \mathbb{C}[\eta_1, \eta_2 \dots]$ with arbitrarily many odd parameters η.

In the normal case, it's enough to know the points, because P points are just going to be given by parametrized paths, which can be built up from the points. Instead for supermanifolds, the points don't give us enough information, since we lose all information about the odd part. In other words, a supermanifold is not determined by its points, but it is determined by its superpoints.

However, what we're doing is not that strange, since it is the same procedure that lead to the complex numbers: in general you would like a space to be determined by its points, but you find for example that $\mathbb{R}[x]/(x^2 + 1)$ has no points, so then you generalize the concept of points by mapping not into $\mathbb{R}$ but into $\mathbb{C}$. In general the idea is that if a space is not determined by its points, it's a sign that you have the wrong notion of a point.

But what does that mean for actual measurements? Well the thing is we're doing something weird because we're trying to build a classical theory of fermions, while any measurement on a fermion is actually quantum mechanical. So we're just discussing the abstract notion of what a classical measurement would be.

5.1.3 Superfield

Second problem: we're trying to define a superfield Φ to be a map from $\mathbb{R}^{1|1}$ to V, i.e. a map $Fun(V) \to Fun(\mathbb{R}^{1|1})$ with $Fun(V) = \mathbb{C}[x_1, \dots, x_n]$ and $Fun(\mathbb{R}^{1|1}) = \mathbb{C}[t, \theta]$. Such a map could be $x_i \mapsto f_i(t)$, but θ can't appear, so it's as if we just have a map from $\mathbb{R}^{1|0}$ to V.

We need to think about parametrized families of maps to get a sensible answer. In particular its important to think of odd parameters.

How do I think of a parametrized family of maps?

In general we have something like $\mathrm{Map}(P, \mathrm{Map}(A, B)) = \mathrm{Map}(P \times A, B)$. On the right we have functions $f : (p, a) \mapsto f(p, a) \in B$, while on the left we have $f : f \mapsto f(p, -) \in \mathrm{Map}(A, B)$, which are the same since in general if we have a function of two variables we can also think of it as a function of the first variable, which takes as values functions on the second. Note that on the left we have the functor of P points of $\mathrm{Map}(A, B)$.

Example: a structure preserving map of two supervector spaces is a degree preserving linear map. However we have that $\mathrm{Hom}(A, B) \neq A^\vee \hat{\otimes} B$, where $A^\vee \hat{\otimes} B$ is called internal Hom and gives maps from A to B which are not degree preserving. We have instead $\mathrm{Hom}(A, B) = (A^\vee \otimes B)_+$ and sometimes $A^\vee \otimes B$ is what we actually want, since $\mathrm{Hom}(P, \mathrm{Hom}(A, B)) = Hom(P \otimes A, B)$ works if $\mathrm{Hom}(A, B)$ is defined by $A^\vee \otimes B$.

Now let's think of a parametrized family now and take P with algebra $A = \mathbb{C}[\eta]$. We then want a P parametrized family of maps in $\mathrm{Map}(\mathbb{R}^{1|1}, V)$, meaning an element in $\mathrm{Map}(P, \mathrm{Map}(\mathbb{R}^{1|1}, V)) = \mathrm{Map}(P \times \mathbb{R}^{1|1}, V))$, so on the algebras we have a map $\mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[t, \theta, \eta]$. Now we can do something interesting and take the following map

$$x_i \to f_i(t) + g_i(t)\theta\eta \tag{5.3}$$

since $\theta\eta$ is another even element. But since $(\theta\eta)^2 = 0$ we actually get the dual numbers. Often we write $\Phi(t, \theta) = f_i(t) + \psi_i(t)\theta$ with an even field $f_i(t)$ and an odd one ψ_i, thinking of it as a Taylor expansion in the odd variable θ, which ends at the first degree since $\theta^2 = 0$. A superfield is not an element of the algebra of functions on the superspace, since that would be something like $f_i(t) + g_i(t)\theta$ where $f_i(t)$ and $g_i(t)$ are both even.

Next time: work out this example and write down the most general superfield action functional which will be defined as an integral over a superspace such as $\mathbb{R}^{1|1}$.

Lecture 9.2 (14/6) Superfields in one and four spacetime dimensions

Last time we talked about superfields and the motto was: a superfield is an element of the (internal) mapping space between a "superspacetime" and some target space.

We explained last time all the different concepts: superspacetime is an ordinary spacetime with an algebra of functions that includes fermions, maybe something like $\mathbb{C}[x_\mu, \theta_\alpha]$ which is supercommuting with x bosonic and θ fermionic. Instead the target space is simply where the field takes values. For example for a real scalar field the target space is $\mathbb{R}$, for a complex scalar it's $\mathbb{C}$, for a "circular" field it would be S^1.

Example: Supersymmetric quantum mechanics. We wanted to write it as $\mathrm{Map}(\mathbb{R}^{1|1} \to \mathbb{R}^{1|0})$. If I normally have a function on spacetime I can evaluate the function and its derivatives at a point, therefore I get the Taylor expansion and I get the map locally in a neighborhood of the point. In superspacetime, we write $\Phi(x, \theta) = \phi(x) + \psi(x)\theta = \phi(0) + x\phi'(0) + \dots$ so I can expand both in the even and odd variables. But the algebra we saw did not obtain elements of the form $\psi(x)\theta$, instead we had something like $f(x) + g(x)\theta \in \mathbb{C}[x, \theta]$ which is not a real number and in particular not a superfield, so a superfield is not simply a function over superspacetime. We had to think of families that depend on odd parameters to get the description above.

5.2 Superfield lagrangian for Supersymmetric quantum mechanics

Goals for today:

- write down Superfield Lagrangian for SQM. If I write an action for the component fields, then I can choose them independently so a priori I don't know whether the action is actually supersymmetric and I don't know what the supersymmetric transformations should be. Just like I don't have to always check Lorentz invariance, I simply always use an integral over spacetime of a Lorentz invariant lagrangian in which all indices are contracted. The point of the superfield formulation is to make supersymmetry just as easy.

- general superspacetimes. We want to enhance the normal spacetime with some fermionic coordinates. That's suppposed to be a space on which the supersymmetry algebra acts geometrically.
- $4d, \mathcal{N} = 1$ (minimal supersymmetry) superfields. Here we had two 2D spinors with dotted and undotted indices. The gamma matrices paired them with one another, so the minimal supersymmetry algebra contains one S_+ and one S_- and that's four supercharges. We have $Q_\alpha, \overline{Q}_{\dot\beta}$ respectively in S_+ and S_- with bracket $\{Q_\alpha, \overline{Q}_{\dot\beta}\} = \gamma^\mu_{\alpha\dot\beta} P_\mu$.

A $d = 1, \mathcal{N} = 1$ superfield is an element $\Phi \in \mathrm{Map}(\mathbb{R}^{1|1}_{t,\theta} \to \mathbb{R}^{1|0})$. Formally, I should think of it as an element in $\mathrm{Map}(P_\eta \times \mathbb{R}^{1|1}_{t,\theta}, \mathbb{R}^{1|0}_\phi)$ for any space of superparameters. Such a map is the same as an element of $\mathrm{Hom}(\mathbb{C}[\phi] \to \mathbb{C}[t,\theta,\eta])$, the most general of which is given by $\phi \mapsto f(t) + \eta\theta g(t)$. I can then define a superfield as

$$\Phi(t,\theta) = f(t) + \psi(t)\theta, \text{ with } \psi(t) = \eta g(t) \tag{5.4}$$

The supersymmetry algebra is simply given by $\{Q, Q\} = 2P$, with $P \mapsto \frac{\partial}{\partial t}$ since it generates time translations. Now we want some other odd vector field such that its anticommutator is $2\frac{\partial}{\partial t}$. The most general vector field is $x(t)\frac{\partial}{\partial\theta} + y(t)\theta\frac{\partial}{\partial t}$, and if I square it I get

$$x(t)y(t)\frac{\partial}{\partial t} + y(t)\theta\frac{\partial x}{\partial t}\frac{\partial}{\partial\theta} \tag{5.5}$$

now I can pick x and y in order to get something that squares to $\frac{\partial}{\partial t}$. I can simply set $x = y = 1$, so $Q \mapsto \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial t}$, so in particular I'm not just translating with respect to the odd component. Also the choice was not unique, I only needed $xy = 1$ and $x = \text{const}$. If we did this correclty we should have rediscovered the transformation rules that we had already found. Now we can work out the supersymmetry transformations on the superfield and we could check if it reproduces the transformations we had before.

Just to fix notation, we're writing

$$\Phi(t,\theta) = x(t) + \psi(t)\theta \tag{5.6}$$

and we had the action

$$S = \int dt \frac{1}{2}|\dot{x}|^2 - \frac{1}{2}\psi\dot{\psi} \tag{5.7}$$

Let's now work out $Q\Phi$:

$$Q\Phi = \left(\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial t}\right)\Phi = \psi(t) + \theta\dot{x}(t) \tag{5.8}$$

so under supersymmetry we have:

$$x \mapsto \psi \tag{5.9}$$

$$\psi \mapsto \frac{\partial x}{\partial t} \tag{5.10}$$

which is exactly what we had said, but we were just guessing last time.

So now we want to build an action for the superfield which is totally supersymmetry invariant, therefore I should just use derivative operators that are covariant, meaning they should commute with supersymmetry transformations. Let's derive what they should be. A general vector field is given by:

$$a(t)\frac{\partial}{\partial t} + b(t)\theta\frac{\partial}{\partial\theta} + x(t)\frac{\partial}{\partial\theta} + y(t)\theta\frac{\partial}{\partial t} \tag{5.11}$$

the first two even and the last two odd. In order to have an operator which commutes with translations, we just need a, b, x, y to be constant. But then we also want it to commute with supersymmetry, so for the even part we want

$$\left[\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial t}, a\frac{\partial}{\partial t} + b\theta\frac{\partial}{\partial\theta}\right] = 0 \tag{5.12}$$

meaning

$$b\left(\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial t}\right) = 0 \tag{5.13}$$

so I need $b = 0$. So the even covariant derivative is simply given by:

$$\frac{\partial}{\partial t} \tag{5.14}$$

Now for the odd part:

$$\left[\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial t}, x\frac{\partial}{\partial\theta} + y\theta\frac{\partial}{\partial t}\right] = (y + x)\frac{\partial}{\partial t} \tag{5.15}$$

so we need $x = -y$ and the odd covariant derivative is given by

$$D = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial t} \tag{5.16}$$

which satisfies $\{D, Q\} = 0$ and $\{D, P\} = 0$.

The even covariant derivative is just ∂_t since we're working in a flat background, instead even in flat superspace the odd derivatives don't behave the way we may expect. This can be stated as "flat superspace has torsion". In general relativity we assume torsion is 0 and curvature is interesting, while here its the opposite.

The upshot now is that I can get a supersymmetric action by forming a lagrangian density out of $\Phi, \partial_t\Phi$ and $D\Phi$, and integrating it over super-spacetime.

$$S = \int dt d\theta \mathscr{L}(\Phi, D\Phi, \partial_t \Phi) \tag{5.17}$$

which will automatically work, although we didn't prove that it's the most general. Since we're integrating over θ, we're taking out the top component, so $\mathscr{L}$ should be fermionic otherwise we have a trivial action. But Φ and $\partial_t\Phi$ are even, while only $D\Phi$ is odd, so I definitely need $D\Phi$ and if I want a quadratic action I need $D\Phi$ and something else, and if I want it to be second order in time derivatives, then the something else should be $\partial_t\Phi$:

$$D\Phi\partial_t\Phi = (\psi(t) - \theta\dot{x}(t))(\dot{x}(t) + \dot{\psi}(t)\theta) \tag{5.18}$$

and I want the top component so I get

$$[D\Phi\partial_t\Phi]_{top} = -\dot{x}^2 + \psi\dot{\psi} \tag{5.19}$$

which is exactly what we wrote in the beginning. We already checked it was a supersymmetric action, now it's automatic.

For interactions we could have something like

$$\begin{aligned} [D\Phi\Phi^n]_{top} &= [(\psi(t) - \theta\dot{x}(t))(x(t)^n + nx(t)^{n-1}\theta\psi(t))]_{top} && (5.20) \\ &= -\dot{x}x^n + nx^{n-1}\psi^2 = -\dot{x}x^n && (5.21) \end{aligned}$$

but it looks like I turn on a potential just for x, but actually this is trivial because it's a total derivative $-\frac{d}{dt}(x^{n+1})$.

That doesn't prove that there are no supersymmetric interactions for this theory, but it shows that any supersymmetric interaction term you can write coming from superfields is trivial.

5.3 Superfield lagrangian for $4d, \mathcal{N} = 1$

This procedure is now easy to generalize. Let's now study the $d = 4, \mathcal{N} = 1$ case. The observables are given by $\mathbb{C}[x^\mu, \theta^\alpha, \overline{\theta}^{\dot{\alpha}}]$, respectively Lorentz vector, S_+ spinor and S_- spinor. The time translation operator is as before and we can imagine that we'll have a Q for every θ and the index structure comes naturally:

$$P_\mu = \frac{\partial}{\partial x^\mu} \tag{5.22}$$

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} + \overline{\theta}^{\dot\beta}\gamma^\mu_{\alpha\dot\beta}\frac{\partial}{\partial x^\mu} \tag{5.23}$$

$$\overline{Q}_{\dot\alpha} = \frac{\partial}{\partial \overline{\theta}^{\dot\alpha}} + \theta^\beta\gamma^\mu_{\beta\dot\alpha}\frac{\partial}{\partial x^\mu} \tag{5.24}$$

and the odd covariant derivatives are given by

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} - \overline{\theta}^{\dot\beta}\gamma^\mu_{\alpha\dot\beta}\frac{\partial}{\partial x^\mu} \tag{5.25}$$

$$\overline{D}_{\dot\alpha} = \frac{\partial}{\partial \overline{\theta}^{\dot\alpha}} - \theta^\beta\gamma^\mu_{\beta\dot\alpha}\frac{\partial}{\partial x^\mu} \tag{5.26}$$

Then the most general action will be built from Φ, $\partial_\mu\Phi$ and $D_\alpha\Phi$. Now Φ has a total of $2^4 = 16$ component fields (8 even + 8 odd) which are a lot. We may then wonder if we can cut this down somewhat, if there are smaller representations. The answer is yes and we'll see how in the following sections.

Note that there was nothing special about $d = 4$, the construction above, using the correct gamma matrices and index strucure, works in general.

Lecture 10.1 (18/6) Four dimensional $\mathcal{N} = 1$ superfields

Note on supersymmetry transformations of superspacetime

We've constructed every possible supersymmetry algebra. We constructed superspacetime because we want to understand supersymmetries as transformations of some superspace and the odd transformations should be some kind of odd tranlsations, so we get odd directions.

Can build a superspace T for any supertranslation algebra $\mathfrak{t}$, such that $\mathfrak{t}$ acts on T by supertranslations. The supertranslation algebra was characterized by commuting bosonic generators P^μ with vector index and odd spinor generators Q^α that satisfy $\{Q, Q\} = 2P$. Given $a^\mu P_\mu \in \mathfrak{t} \to e^{a^\mu P_\mu} = e^{a^\mu\partial_\mu}$ which acts as

$$e^{a^\mu P_\mu} f(x^\mu) = f(x^\mu) + a^\mu\partial_\mu f(x^\mu) + ... = f(x^\mu + a^\mu) \tag{5.27}$$

In the susy case I want some kind of superfield such that when I act with the exponential of the odd generators I get a translation in an odd direction. In the super case, functions look like $\Phi(x^\mu, \theta^\mu)$ and we still have $P_\mu \to \frac{\partial}{\partial x^\mu}$ but we can't just have $Q_\alpha \to \frac{\partial}{\partial x^\alpha}$ since that wouldn't give the correct commutator. Instead we can take

$$Q_\alpha \to \frac{\partial}{\partial \theta^\alpha} + \theta^\beta\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial x^\mu} \tag{5.28}$$

however that is not exactly a translation, since for the transformation $e^{\eta^\alpha Q_\alpha}$ we get

$$e^{\eta^\alpha Q_\alpha}\Phi(x^\mu,\theta^\alpha) = \left(1+\eta^\alpha\frac{\partial}{\partial\theta^\alpha}+\eta^\alpha\theta^\beta\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial x^\mu}+\frac{1}{2}\eta^\alpha\eta^\beta\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta}+\ldots\right)\Phi$$

instead of

$$\Phi(x^\mu,\theta^\alpha+\eta^\alpha) = \Phi(x^\mu,\theta^\alpha)+\eta^\alpha\frac{\partial}{\partial\theta^\alpha}\Phi(x^\mu,\theta^\alpha)+\frac{1}{2}\eta^\alpha\eta^\beta\frac{\partial^2}{\partial\theta^\alpha\theta^\beta}+\ldots$$

so we have the extra piece $\eta^\alpha\theta^\beta\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial x^\mu}$ which looks llike the first piece of a Taylor series for

$$x^\mu \mapsto x^\mu+\eta^\alpha\theta^\beta\gamma^\mu_{\alpha\beta} \tag{5.29}$$

So we have

$$e^{\eta^\alpha Q_\alpha}\Phi(x^\mu,\theta^\alpha) = \Phi(x^\mu+\eta^\alpha\theta^\beta\gamma^\mu_{\alpha\beta},\theta^\alpha+\eta^\alpha) \tag{5.30}$$

In retrospect it's clear that it couldn't be only θ translations, because those trivially commute, but that's not the relation we want.

Normally for two elements A, B in a Lie algebra $\mathfrak{g}$ we have

$$e^{tA}e^{sB} = e^{tA+sB+\frac{1}{2}ts[A,B]+\ldots} \tag{5.31}$$

which is the Baker-Campbell-Hausdorff formula. In this case it's rather simple since any commutator of two elements commutes with everything else, so here we have the exact formula

$$e^Ae^B = e^{A+B+\frac{1}{2}[A,B]} \tag{5.32}$$

So we should have

$$e^{\eta' Q}e^{\eta Q} = e^{(\eta+\eta')Q+\frac{\eta\eta'}{2}2\gamma P} \tag{5.33}$$

which we can check explicitly:

$$\begin{aligned}\Phi(x,\theta) &\mapsto \Phi(x+\eta\gamma\theta,\theta+\eta) &(5.34)\\ &\mapsto \Phi(x+\eta\gamma\theta+\eta'\gamma(\theta+\eta),\theta+\eta+\eta') &(5.35)\\ &= \Phi(x+(\eta+\eta')\gamma\theta+\eta'\gamma\theta,\theta+\eta+\eta') &(5.36)\end{aligned}$$

Note that we used

$$[\eta Q,\eta' Q'] = \eta Q\eta' Q' - \eta' Q'\eta Q = \pm(\eta\eta' QQ' - \eta'\eta Q'Q) = \pm\eta\eta'\{Q,Q'\} \tag{5.37}$$

in which the $\pm$ is because we can *choose* whether η and Q commute or anticommute. However without loss of generality we can choose that they commute. The point here is that we actually start with the commutator which is what we want for the Baker Campbell Hausdorff formula but then we get the anticommutator which we know.

In this sense we're working with a group that is even easier than SU(2) since there iterated commutators are in general nontrivial.

Back to $4d$, $\mathcal{N}=1$ superfields

Now let us specialize to $d=4$ and $\mathcal{N}=1$, here the spin representation is $S_+ \oplus S_-$ and $S_+ \otimes S_- \cong V$, so I can think of a vector index as a bispinor index:

$$V_\mu \cong V_{\alpha\dot{\alpha}}\gamma_\mu^{\alpha\dot{\alpha}} \tag{5.38}$$

or equivalently

$$V_{\alpha\dot{\alpha}} \cong V_\mu \gamma^\mu_{\alpha\dot{\alpha}} \tag{5.39}$$

which we will often use. So in particular we can write the commutator as

$$\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = 2P_{\alpha\dot{\alpha}} \tag{5.40}$$

Now, the most general superfield is given by

$$\Phi(x_{\alpha\dot{\alpha}}, \theta_\alpha, \overline{\theta}_{\dot{\alpha}}) = \begin{array}{ccccc} & & \theta^2\overline{\theta}^2 D(x) & & \\ & \theta^2\overline{\theta}_{\dot{\alpha}}\overline{\lambda}^{\dot{\alpha}} & & \overline{\theta}^2\theta_\alpha\lambda^\alpha & \\ F(x)\theta^2 & & \theta_\alpha\overline{\theta}_{\dot{\alpha}}V^{\alpha\dot{\alpha}(x)} & & \overline{F}(x)\overline{\theta}^2 \\ & \psi^\alpha(x)\theta_\alpha & & \overline{\psi}^{\dot{\alpha}}(x)\overline{\theta}_{\dot{\alpha}} & \\ & & \phi(x) & & \end{array}$$

in which we expanded Φ in terms of its component fields as coefficients of a product of θs and $\overline{\theta}$s. Note that going up and to the left the number of θ is increased and going up and to the right the number of $\overline{\theta}$ is increased.

We can see that there are scalar fields $F, \overline{F}, \phi, D$, spinor fields $\psi, \overline{\psi}, \lambda, \overline{\lambda}$ and a vector field $V^{\alpha\dot{\alpha}}$ which are a lot of degrees of freedom. From this general superfield, we can now construct superfields which have a more reasonable amount of fields:

- Chiral superfield: impose a chiral constraint $\rightarrow$ Chiral multiplet (one spin 0 field and one spin $\frac{1}{2}$field)
- Gauge superfield: introduce a gauge equivalence $\rightarrow$ gauge multiplet (one spin 1 field and one spin $\frac{1}{2}$field)

Note that both the constraint and gauge equivalence must be consistent with supersymmetry.

5.3.1 Chiral superfield

We first consider the possibility of imposing a constraint. Recall the action of Q:

$$Q_\alpha = \partial_\alpha + \overline{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}} \tag{5.41}$$

$$\overline{Q}_\alpha = \overline{\partial}_{\dot{\alpha}} + \theta^\alpha\partial_{\alpha\dot{\alpha}} \tag{5.42}$$

where

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha}, \quad \partial_{\alpha\dot{\alpha}} \cong \partial_\mu \tag{5.43}$$

Now, to set the constraint we can look for an operator that commutes with the Qs, so that if the constraint is satisfied, then it holds also under a supersymmetry transformation. We already found such operators:

$$\partial_{\alpha\dot{\alpha}}, \quad D_\alpha = \partial_\alpha - \overline{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \quad \overline{D}_{\dot{\alpha}} = \overline{\partial}_{\dot{\alpha}} - \theta^\alpha \partial_{\alpha\dot{\alpha}} \tag{5.44}$$

we can then set any of these to kill the superfield, however setting $\partial_{\alpha\dot{\alpha}}\Phi = 0$ is uninteresting because we get a field which doesn't depend on the spacetime coordinates. Instead we can set the chiral constraint:

$$\overline{D}_{\dot{\alpha}}\Phi = 0 \tag{5.45}$$

or the antichiral constraint

$$D_\alpha \Phi = 0 \tag{5.46}$$

But not both because we have

$$\{D_\alpha, \overline{D}_{\dot{\alpha}}\} = 2\partial_{\alpha\dot{\alpha}} \tag{5.47}$$

and if D_α and $\overline{D}_{\dot{\alpha}}$ act trivially then so does their commutator, meaning $\partial_{\alpha\dot{\alpha}}\Phi = 0$.

Let us consider the chiral constraint (the antichiral one is analogous):

$$\overline{\partial}_{\dot{\alpha}}\Phi = \theta^\alpha \partial_{\alpha\dot{\alpha}}\Phi \tag{5.48}$$

The operator on the left just deletes a $\overline{\theta}$, so in the diagram we move down and to the left. Instead the operator on the right adds a θ so it goes up and to the left. So the equation determines a field above a given field in terms of a derivative of the field below it. More precisely, the field above with a $\overline{\theta}$ removed should be the same as the field below with a $\theta^\alpha\partial_{\alpha\dot{\alpha}}$ added.

From the diagram before, we now obtain the following

$$\begin{array}{ccccc}
 & & \theta^2\overline{\theta}^2\partial^2\phi & & \\
 & \theta^2\overline{\theta}_{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\psi^\alpha & & - & \\
F(x)\theta^2 & & \theta_\alpha\overline{\theta}_{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\phi & & - \\
 & \psi^\alpha(x)\theta_\alpha & & - & \\
 & & \phi(x) & &
\end{array}$$

In which we can also see that we only the left chiral field for ψ, which is why we call this the chiral multiplet. It was important that the superfield was complex, otherwise setting the chiral constraint would set everything to a constant. So it's kind of like a holomorphy condition, we're picking either a holomorphic field or the antiholomorphic one, and there are no real holomorphic functions (a part from constants).

We would now like to write down an action for that superfield. How do I get an action? Well, if K is a real superfield, then

$$\int d^2\theta d^2\overline{\theta} K \tag{5.49}$$

is a supersymmetric lagrangian density, just like we did for supersymmetric quantum mechanics. The simplest real superfield we can build is just $K = \overline{\Phi}\Phi$ and after integration we only get

$$[\overline{\Phi}\Phi]_{top} = \overline{\phi}\partial^2\phi + \partial\phi\partial\overline{\phi} + \psi^\alpha\partial_{\alpha\dot{\alpha}}\overline{\psi}^{\dot{\alpha}} + (\partial_{\alpha\dot{\alpha}}\psi^\alpha)\overline{\psi}^{\dot{\alpha}} + F\overline{F} \tag{5.50}$$

which is the standard second order action for a scalar field, the dirac lagrangian for a Weyl spinor field and an action for a complex scalar F which has no derivatives at all. We call F an auxiliary field, since its equations of motion just set it to 0.

We've constructed the free theory of a supersymmetric chiral multiplet. We did it in a way that is off shell since we've set the constraint before writing the action which imposes the equations of motion. Let's now count the degrees of freedom. Susy requires that the number of fermionic and bosonic degrees of freedom to be the same and this must be true both on shell and off shell (if susy acts off shell).

The *mass shell* is the locus where the free equations of motion are satisfied which are usually given by

$$p^2 = m^2 \tag{5.51}$$

so in general we talk of *on shell* if the equations of motion are imposed or *off shell* if they are not.

Let us count the real degrees of freedom:

Field	Type	off shell	on shell
ϕ	complex scalar	2	2
ψ	Weyl fermion	4	2
F	complex auxiliary field	2	0

To count off shell I just count how many functions of the spacetime coordinates there are, so the complex scalar and auxiliary field have 2 degrees of freedom, while the Weyl fermion ψ^α is made up of two complex functions, so 4 real functions.

To count on shell I need to impose the equations of motion. For the scalar the equations of motion do not restrict the degrees of freedom, while the Dirac equation sets half of the components to 0 and the auxiliary field is just set to 0. So again the number of bosonic and fermionic degrees of freedom are the same but the counting is different from the off shell case. This is why we actually need the auxiliary field.

It's connected to the idea of the little group, which is important to understand the difference between the particle description (on shell) and the wave description (off shell).

In the massive case we have $p^\mu p_\mu = 1$ and we can boost to the rest frame

$$p^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{5.52}$$

Now what residual Lorent symmetry do we have which preserves this rest frame? SO(3) since we can arbitrarily rotate the spatial dimensions. Instead in the massless case $p^\mu p_\mu = 0$ we can boost to the standard frame

$$p^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.53}$$

and now the Lorentz symmetries which preserve this frame turn out to be ISO(2) the set of isometries of the plane not including reflections. This is given by the semi-direct product SO(2)$\ltimes$T(2) in which the rotation part is clear, while the translational part is harder to interpret. See [2] section §2.5 for more on the little group.

So the representation on the Hilbert space of a single particle, that particle has a momentum and the only invariant in the momentum quantum numbers is the mass. There's another invariant thing which is the representation that i get when I put that particle at the origin and ask how rotations act on its internal Hilbert space. So that's why we talk about an SO(3) representation for massive particles, while for massless particles we actually talk about ISO(2) representations. Now, in this particular case since T(2) is abelian and SO(2) is a one parameter group, the representations of ISO(2) $\cong$ SO(2)$\ltimes$T(2) are simply given by those of SO(2), which is isomorphic to U(1) whose representations are labelled by helicity. So that's why the photon, although it is spin 1, it has two polarizations and not three, because it's representation of ISO(2), not SO(3).

The reason the fermion decomposes as we said is because the spin representation of the little group is only half as big as the spin representation of the group itself.

Lecture 10.2 (21/6)

5.3.2 Superpotential

In the last section we wrote the kinetic term for chiral superfields.

First we found that the chiral superfield can be written as

$$\Phi(x_{\alpha\dot{\alpha}}, \theta_\alpha, \overline{\theta}_{\dot{\alpha}}) = \phi(x) + \psi^\alpha(x)\theta_\alpha + F(x)\theta^2 + \dots \tag{5.54}$$

where the ... are terms with $\overline{\theta}$ which are entirely determined by the other terms because of the chirality constraint. Then we got the following lagrangian density

$$\int d^\theta d^2\overline{\theta}\Phi\overline{\Phi} = \partial_\mu\phi\partial^\mu\overline{\phi} + \overline{\psi}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\psi^\alpha + \overline{F}F \tag{5.55}$$

which contains the standard kinetic term for the scalar and spinor fields and has an auxiliary field F without derivatives and therefore without dynamics.

However, it is not a particularly interesting lagrangian since there are no mass or interaction terms. In order to get such terms, we could take a more general expression, such as

$$\int d^2\theta d^2\overline{\theta} K(\Phi, \overline{\Phi}) \tag{5.56}$$

where $K(\Phi, \overline{\Phi})$ is any real function. Unfortunately this still could never have a standard mass term for ϕ, i.e. $m^2\phi\overline{\phi}$ because the integration only keeps terms with $\theta^2\overline{\theta}^2$ and just multiplying the ϕ term we don't get any θs.

However, note that there are two ways of constructing terms for supersymmetric actions:

- integrate a real expression $K(\Phi, \overline{\Phi})$ over all of superspace (D term).

  It's supersymmmetric since

  $$[Q_\alpha K(\Phi, \overline{\Phi})]_{top} = [\partial_\alpha K + \overline{\theta}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} K]_{top} \tag{5.57}$$

  but both terms are zero since ∂_α reduces the number of θs so there's no top component left, while $\overline{\theta}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} K$ has a nonvanishing top form but it's just a total derivative. The $\overline{Q}_{\dot{\alpha}}$ works exactly the same.

  This is what we've considered up to now and the function K is called the Kähler potential.

- integrate a chiral expression W over the chiral half of superspacetime (F term).

  In fact, let $W(\Phi)$ be chiral, so $\overline{D}_{\dot{\alpha}} = 0$, so $(\overline{\partial}_{\dot{\alpha}} - \theta^\alpha \partial_{\alpha\dot{\alpha}})W = 0$, and since I'm integrating only over θ we have:

  $$[Q_\alpha W]_{\theta^2} = [\partial_\alpha W + \overline{\theta}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} W]_{\theta^2} = 0 \tag{5.58}$$

  since neither of the terms in parentheses has a θ^2 component. However now $\overline{Q}_{\dot{\alpha}}$ is different

  $$[\overline{Q}_{\dot{\beta}} W]_{\theta^2} = [\overline{\theta}_{\dot{\beta}} W + \theta^\alpha \partial_{\alpha\dot{\beta}} W]_{\theta^2} \tag{5.59}$$

  the one on the right can have a θ^2 component but it's a total derivative. Instead the one on the left is only a total derivative because of the chiral constraint $\overline{\partial}_{\dot{\beta}} W = \theta^\alpha \partial_{\alpha\dot{\beta}} W$, which is why we need W to be chiral.

  Now, this is not real but we can just add the conjugate:

  $$\int d^2\theta W + \int d^2\overline{\theta}\, \overline{W} \tag{5.60}$$

  Note that we should have $W = W(\Phi, \overline{\Phi})$, but because Φ and W are chiral we actually get that $W = W(\Phi)$.

From this term we could get a mass term or some interactions, but it's not trivial because we still don't have any quadratic ϕ term without any other field, instead we have something like $\phi F\theta^2$, but then imposing the equations of motion for F leads to interesting terms.

So the most general supersymmetric lagrangian density is given by

$$\mathscr{L} = \underbrace{\int d^2\theta d^2\overline{\theta} K(\Phi, \overline{\Phi})}_{\text{D-term}} + \underbrace{\int d^2\theta W(\Phi) + \int d^2\overline{\theta}\, \overline{W}(\overline{\Phi})}_{\text{F-term}} \tag{5.61}$$

Now, let's see what component field interactions the superpotential $W(\Phi)$ generates. It's easiest to think of polynomials:

$$[\Phi^n]_{\theta^2} = \left[n\theta^2 F\phi^{n-1} + \frac{n(n-1)}{2}(\theta\psi)^2\phi^{n-2} \right]_{\theta^2} \tag{5.62}$$

so in general we have

$$[W(\Phi)] = W'(\phi)F - \frac{1}{2}\psi^2 W''(\phi) \tag{5.63}$$

in which the $-$ is because $(\theta\psi)^2 = -\theta^2\psi^2$.

So my lagrangian is given by

$$\mathscr{L} = |\partial\phi|^2 + \overline{\psi}\partial\!\!\!/\psi + \overline{F}F + W'(\phi)F - \frac{1}{2}\psi^2 W''(\phi) + \overline{W}'(\overline{\phi})\overline{F} - \frac{1}{2}\overline{\psi}^2\overline{W}''(\overline{\phi}) \tag{5.64}$$

Now F is not a physical field because it has no derivatives, so to get a lagrangian for the physical fields I should eliminate it using its equations of motion:

$$\overline{F} = -W'(\phi) \tag{5.65}$$

$$F = -\overline{W}'(\overline{\phi}) \tag{5.66}$$

which if reinserted into the lagrangian gives

$$\mathscr{L} = |\partial\phi|^2 + \overline{\psi}\partial\!\!\!/\psi - |W'|^2 - \frac{1}{2}\psi^2 W''(\phi) - \frac{1}{2}\overline{\psi}^2\overline{W}''(\overline{\phi}) \tag{5.67}$$

which is interesting because we now have a potential for ϕ, which is nonnegative just as we had in supersymmetric quantum mechanics. Here we can already see that a self interaction of the scalar given by $|W'|^2$ will gives rise to an interaction of the scalar and the fermion because of the $\psi^2 W''(\phi)$ term.

Now to get a mass term we can choose $W(\Phi) = \frac{1}{2}m\Phi^2$ which actually gives rise to a mass term for both the scalar and the fermion: $m\phi\overline{\phi}$ and $-m\overline{\psi}\psi$ since

$$\psi^2 + \overline{\psi}^2 = 2\,\mathrm{Re}(\psi^2) = 2\overline{\psi}\psi \tag{5.68}$$

so the fermion and the boson both have mass m, just as we expect from a supersymmetric system.

Note that this means that in general we should have for every boson a corresponding fermion with the same mass. However, this is clearly not true for the standard model of particle physics, so if supersymmetry is in fact relevant, then it is certainly broken at some scale.

Instead more generally if we set $W(\Phi) \sim \Phi^n$ then we get

$$|W'|^2 \sim (\phi\overline{\phi})^{n-1} \tag{5.69}$$

$$\psi^2 W'' \sim \psi\psi\phi^{n-2} \tag{5.70}$$

so it relates a mass term to a mass term, but for example for $n = 3$ it relates a four point self interaction to a three point interaction between ϕ and ψ.

5.4 Gauge fields

We now study the possibility of imposing a gauge equivalence, just as in electrodynamics A_μ and $A_\mu + \partial_\mu \chi$ describe the same field. In particular, we want to obtain a vector (or *gauge*) field that has exactly that gauge invariance. This will allow us to describe a QED-like system with a spin 1 field and a spin $\frac{1}{2}$field.

Of course we can't use a chiral constraint because we saw that there the spin 1 component is simply set to some derivative of the scalar field. I should still impose some reality condition since normally we want real gauge fields and now we want some *super* gauge invariance that does to the vector field the same thing that the normal gauge invariance does. Recall that the vector field in the superfield was the following

$$A_{\alpha\dot{\alpha}}\theta^\alpha\overline{\theta}^{\dot{\alpha}} \tag{5.71}$$

and gauge invariance is given by

$$A_{\alpha\dot{\alpha}} \sim A_{\alpha\dot{\alpha}} + \partial_{\alpha\dot{\alpha}}\phi \tag{5.72}$$

Now we want a superfield, whose $\theta\overline{\theta}$ component looks like the derivative of a scalar. We already found something like this! Recall that for the chiral superfield we had

$$(\overline{\partial}_{\dot{\alpha}} - \theta^\alpha \partial_{\alpha\dot{\alpha}})\Phi = 0 \tag{5.73}$$

which is the same as

$$\overline{\partial}_{\dot{\alpha}}[\Phi]_{\theta\overline{\theta}} = \theta^\alpha \partial_{\alpha\dot{\alpha}}[\Phi]_1 \implies [\Phi]_{\theta\overline{\theta}} = \partial_{\alpha\dot{\alpha}}\phi \tag{5.74}$$

so a chiral superfield already has the property that we're looking for. However, we actually wanted the derivative of a *real* scalar but that can be solved by simply taking the real part. Therefore the idea becomes:

$$\text{vector superfield} = \frac{\text{real superfield}}{\text{real part of chiral superfield}} \tag{5.75}$$

In other words we have $V = V(x, \theta, \overline{\theta})$, $V = \overline{V}$, with the following gauge invariance:

$$V \sim V + (\Phi + \overline{\Phi}) \tag{5.76}$$

for any chiral Φ.

By construction we know how that the vector component of the superfield transforms by a derivative of a scalar field under super gauge invariance. Now let's study the transformation in general. So the question is what components the real part of a chiral superfield has at every order:

$$\begin{array}{ccccc}
 & & \theta^2\overline{\theta}^2\partial^2(\phi + \overline{\phi}) & & \\
 & \theta^2\overline{\theta}\partial\psi & & \overline{\theta}^2\theta\partial\overline{\psi} & \\
F(x)\theta^2 & & \theta\overline{\theta}(\partial\phi - \partial\overline{\phi}) & & \overline{F}(x)\overline{\theta}^2 \\
 & \psi\theta & & \overline{\psi}\,\overline{\theta} & \\
 & & \phi + \overline{\phi} & &
\end{array}$$

Note that we get a $-$ in the center because $\overline{(\theta\overline{\theta})} = \overline{\theta}\theta = -\theta\overline{\theta}$. So the gauge parameters are complex scalars ϕ and F and a Weyl fermion ψ. Gauge transformations shift the lowest part of the superfield by the real part of ϕ, they make the imaginary part of ϕ into the gauge parameter for the vector field, they shift the θ^2 component by F and the θ component by ψ.

Now we can gauge fix by choosing ϕ, ψ and F such as to cancel the lower component, the θ and the θ^2 components. We then are left with the following fields

$$\begin{array}{ccccc}
 & & \theta^2\overline{\theta}^2 D & & \\
 & \theta^2\overline{\theta}\,\overline{\lambda} & & \overline{\theta}^2\theta\lambda & \\
0 & & \theta^\alpha\overline{\theta}^{\dot{\alpha}} A_{\alpha\dot{\alpha}} & & 0 \\
 & 0 & & 0 & \\
 & & 0 & &
\end{array}$$

with a residual gauge invariance given by

$$A_{\alpha\dot{\alpha}} \sim A_{\alpha\dot{\alpha}} + \partial_{\alpha\dot{\alpha}} \operatorname{Im}\phi \tag{5.77}$$

which is exactly the gauge invariance we want for a vector field. Since we have a real superfield we have that $\overline{\lambda}$ is actually the conjugate of λ so we have a Majorana spinor. Instead D is simply a real scalar.

This choice, this partial gauge fixing, is called Wess-Zumino gauge and leads to the standard component form of the vector multiplet:

- one spin 1 gauge field with its gauge invariance
- one Majorana spinor field

- one real scalar which again will be auxiliary.

Lecture 11.2 (28/6) Supersymmetric gauge theories ($4d, \mathcal{N}=1$)

More restrictive than in the normal case. Nonsusy case: list of allowed fields (scalars, gauge fields, fermions), set of allowed interactions (mass terms, potentials, Yukawas, gauge groups)

I won't get to choose independently the yukawa interactions, instead they will be fixed by the gauge group repr. I get gauge fields from the vector multiplet and scalars from the chiral multiplet, instead the fermions just appear from the constraint that the thoery be supersymmetric.

Theory of gauge fields coupled to matter in some rep which may have additional interactions.

Choose vector multiplets, chiral multiplets in the repr and super potential and Fayev Lyiapolis term.

On the homework: supersymmetric field strength. Given a real superfield V we had super gauge transformations. From this we can build a gauge invariant field

$$W_\alpha = \frac{1}{4}\overline{D}^2 D_\alpha V \tag{5.78}$$

which is clearly a superfield since the Ds commute with susy trafos and it is also chiral. In addition it should contain $F = dA$ the normal field strength of the vector potential.

It is chiral since $\overline{D}_{\dot\alpha}\overline{D}^2 = 0$ for any $\dot\alpha$. It is gauge invariant because with gauge transformations we use a chiral superfield which disappears when acted on by $\overline{D}$.

$$\overline{D}^2 D_\alpha = (\frac{\partial}{\partial\overline{\theta}})^2\frac{\partial}{\partial\theta^\alpha} + (\frac{\partial}{\partial\overline{\theta}})^2\overline{\theta}^{\dot\alpha} + \frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}\theta^\beta\partial_{\beta\dot\alpha}\frac{\partial}{\partial\theta^\alpha} + \dots \tag{5.79}$$

$$\begin{array}{cccccc}
 & & & \text{x} & & \\
 & & \text{x} & & \text{x} & \\
W_\alpha = & \theta^2\partial_{\alpha\dot\alpha}\overline{\lambda}^{\dot\alpha} & & \text{x} & & \text{x} \\
 & & \theta_\alpha D + & & \text{x} & \\
 & & \theta^\beta\partial_{\beta\dot\alpha}A_{\alpha\dot\alpha} & & & \\
 & & & \lambda_\alpha & &
\end{array}$$

Maxwell action: (with complexified coupling constant)

$$[i\tau W^\alpha W_\alpha]_F + h.c. \tag{5.80}$$

there are two possible terms i can write down that look like F^2 which are not the same. One of them is the Yang Mills action, the other is a total derivative but it's important because it changes the topological nature.

F is a two form. For the action I could write $F \wedge F = dA \wedge dA$ but this is not the Yang Mills action, in particular it's a total derivative because

$d(A\wedge dA)$. We can instead use $*: k$-forms $\to (d-k)$-forms which in particular for F would work as

$$F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{5.81}$$

and the Yang Mill action id given by $F \wedge *F = (F^{\mu\nu}F_{\mu\nu})$d vol.

We wrote down a gauge theory with photon, a fermion lambda and an auxiliary field D. But we don't have interactions. Now we need to understand how to couple light to matter in a supersymmetric way.

Coupling super Maxwell to matter (SQED)

Have a super gauge parameter Λ, $\mathrm{Im}\,\Lambda|_{\theta=\bar\theta=0}$ = normal gauge parameter. Chiral matter Φ with charge q should transform under gauge transformations as $\phi \to e^{iq\alpha}\phi$, so I would guess $\Phi \to e^{q\Lambda}\Phi$ for Φ.

Kinetic term:

$$[\overline{\Phi}\Phi]_{\theta^2\bar\theta^2} \to [\overline{\Phi}e^{q(\Lambda+\overline{\Lambda}}\Phi]_{\theta^2\bar\theta^2} \tag{5.82}$$

which at first doesn't look good but we can see $e^{q(\Lambda+\overline{\Lambda}}$ which contains $\Lambda+\overline{\Lambda}$ which is how V transforms! so if we take

$$\overline{\Phi}e^{-qV}\Phi \tag{5.83}$$

we have something super gauge invariant. But actually, at least in Wess Zumino gauge, the exponential is very simple since already in V^2 all the terms disappear except for $(\theta\bar\theta A)^2$ because there are too many thetas. We have:

$$e^{-qV} = 1 - qV + \frac{1}{2}q^2V^2 + \dots \tag{5.84}$$

$$[\overline{\Phi}\Phi]_D - q[\overline{\Phi}V\Phi]_D + \frac{q^2}{2}[\overline{\Phi}V^2\Phi]_D \tag{5.85}$$

we already know the first term, the last term gives

$$\frac{q^2}{2}\overline{\phi}A^\mu A_\mu\phi \tag{5.86}$$

Instead from the middle term we have more terms:

$$-qD\overline{\phi}\phi - q(\overline{\lambda\psi}\phi + \overline{\phi}\psi\lambda) - q\phi A^\mu\partial_\mu\overline{\phi} + h.c - q\overline{\psi}A_\mu\psi \tag{5.87}$$

We can now define

$$D_\mu = \partial_\mu - qA_\mu \tag{5.88}$$

and we get

$$\overline{F}F + \psi \not{D}\psi + |D_\mu\phi|^2 - q(\phi\overline{\lambda\psi} + \overline{\phi}\lambda\psi) - qD\overline{\phi}\phi + D^2 + \overline{\lambda}\not{\partial}\lambda + F\wedge *F + \theta F\wedge F \tag{5.89}$$

e.o.m. for D

$$D = \frac{1}{2}q|\phi|^2 \tag{5.90}$$

which gives

$$-\frac{1}{4}(q|\phi|^2)^2 \tag{5.91}$$

new feature which has to do with supersymmetry! Normally I *could* add suc a term and the coupling would be independent of the charge. So we have another source of potential terms for scalars.

Most general action for super U(1) gauge theory:

- chiral multiplets Φ_i with charges q_i.
- vector field V.

$$S = [i\tau W_\alpha W^\alpha]_F + h.c. + [K(\overline{\Phi}_i e^{-qV}\Phi_i)]|_D + [W(\Phi_i)]|_F + h.c. + [\zeta V]|_D \tag{5.92}$$

Lecture 13.1 (10/7) Moduli spaces via local observables

3 ways to get to moduli spaces:

- Impose D-term equations which came from the D term part of the potential which came out of the vector multiplet (recall it has three fields A, λ, D) and then quotient by the gauge group (real group). EOM for D was
$$D \sim \sum q_i|\phi_i|^2 + \zeta \tag{5.93}$$
potential is then given by
$$U \sim \frac{1}{2}(\sum q_i|\phi_i|^2 + \zeta)^2 \tag{5.94}$$
imposing D term equation means imposing the minimum of the potential, so the vacuum state.
- froget D term relation but quotient by the complexified gauge group.
- Algebraic quotient

Example (Symplectic quotient)**.** Particle moving in $\mathbb{R}^2_{x,y}$ with phase space $\mathbb{R}^4_{x,y;p_x,p_y}$ with $\omega = dp_x \wedge dx + dp_y \wedge dy$. Now want to see y as a gauge degree of freedom: $(x,y) \sim (x',y')$ if $x = x'$. This is an $\mathbb{R}$ action by transalting y. This is also an action on the phase space but $\mathbb{R}^4/G \cong \mathbb{R}^3_{x;p_x,p_y}$ which is not a phase space since it's not even dimensional. So a symplectic quotient is suppose to fix this problem. This $\mathbb{R}$ action is actually generated by an observable which is $p_y =: \mu$. This is often called the *moment map*, it says for each symmetry, which momentum generates that symmetry. So the symplectic quotient then is

$$R^4//G = \frac{\mu^{-1}(0)}{G} = \mathbb{R}^2_{x;p_x} \tag{5.95}$$

so essentially we're fixing the momentum conjugate to the thing we're quotienting out.

Now want to think of it as an algebraic variety. This is computationally the simplest. This relates to the idea that one way to study a space is to think of it not as a geometry but to think abou tthe funcitons on the space and to let those determine what the space is. If the vacua are distinguishable there should be a possible measurement to distinguish them, so there's an observable that takes different values in one and the other.

This relates to the algebra of (chiral) observables on the moduli space. Chiral observables come from a chiral superfield, what is a modulus? a vacuum expectation value. A vacuum state is a constant value for the field which solves the equations of motion, so it's preceisely the zero modes of the (scalar parts of) chiral superfields. So if I have $X(x, \theta, \overline{\theta}), \overline{D}_{\dot{\alpha}} X = 0$, replace by X a single complex variable which represents the zero mode.

Don't have to think about D term and it's easier to take quotient! How do I implement the $G_{\mathbb{C}}$ quotient? $M = \hat{M}/G_{\mathbb{C}}$, so there's a map $\hat{M} \to M$, which sends a point to the equivalence class it lives in (and the map is surjective). Now le'ts think of the dual, functions on the space.

Moving to the dual I then have a map $\mathrm{Fun}(M) \to \mathrm{Fun}(\hat{M})$. In particular it's injective since the other was surjective. So I can think of $\mathrm{Fun}(M)$ as a subspace of $\mathrm{Fun}(\hat{M})$, which subspace is it? The set of functions invariant under the $G_{\mathbb{C}}$ action! The $G_{\mathbb{C}}$ invariant subspace.

Example (SQED)**.** SQED is a U(1) susy gauge theory. Because of anomaly cancellation the field comes in pairs of opposite charge, for example two field with charges ± 1. 1 flavor, so two chiral multiplets $\Phi_\pm$ with charges ± 1. $\mathrm{Fun}(\hat{M}) = \mathbb{C}[\Phi_+, \Phi_-]$, now what are the invariants? $\mathrm{Fun}(M)$ there are no linear ones because these have nonzero charge. Of the three degree two ones only $\Phi_+\Phi_-$ is invariant and also its powers are. So we can write $\mathrm{Fun}(M) = \mathbb{C}[\Phi_+\Phi_-]$ which is some one dimensional space. Might guess it looks like $\mathbb{C}$ but there's an issue with the grading. From the Kähler potential $|\Phi_+|^2 + |\Phi_-|^2$, while the D term relation gives $|\Phi_+|^2 - |\Phi_-|^2 = 0$, which then gives Kähler potential $2|\Phi_+|^2$.

Let's call $M = \Phi_+\Phi_-$ a *meson* in analogy to QCD. Let's write the Kähler potential in terms of M:

$$K = 2|\Phi_+|^2 = 2|M| = 2\sqrt{MM^\dagger} \tag{5.96}$$

and the Kähler metric is given by

$$g_{i\bar{j}} = \frac{\partial^2 K}{\partial M \partial M^\dagger} = \frac{1}{2}\frac{1}{\sqrt{MM^\dagger}} \tag{5.97}$$

We can write

$$\Phi_+ = re^{i\theta_+} \tag{5.98}$$

$$\Phi_- = re^{i\theta_-} \tag{5.99}$$

and the gauge action gives

$$\theta_+ \sim \theta_+ + \lambda \tag{5.100}$$

$$\theta_- \sim \theta_- - \lambda \tag{5.101}$$

if I want to identify the two fields I have two possibilities so I get

$$\mathbb{C}/\mathbb{Z}_2 \tag{5.102}$$

which identifies

$$re^{i\theta} \sim re^{i(\theta+\pi)} = -re^{i\theta} \tag{5.103}$$

The thing is there are two gradings, charge and degree of polynomial. $\mathbb{C}[z]/\mathbb{Z}_2$ which sends $z \mapsto -z$ which gives $\mathbb{C}[z^2]$.

Example (SQED with N flavors)**.** Now have chiral multiplets Φ_+^i, Φ_-^i, $1 < i \leq N$. Now $\mathrm{Fun}(\hat{M}) = \mathbb{C}[\Phi_+^i, \Phi_-^i]$ and $\mathrm{Fun}(M) = \mathbb{C}[\Phi_+^i \Phi_-^j =: M^{ij}]$ which are N^2 generators which are too many. In fact in higher degrees there are obvious redundancies such as

$$M^{ij}M^{kl} = M^{il}M^{kj} \tag{5.104}$$

which we need to quoteint out:

$$\mathrm{Fun}(M) = \mathbb{C}[M^{ij}]/(M^{ij}M^{kl} = M^{il}M^{kj}) \tag{5.105}$$

Do we recognize these relations?

Let's specialize to $N = 2$, so we have

$$\mathbb{C}[M^{ij}]/(M^{12}M^{21} = M^{11}M^{22}) \tag{5.106}$$

so we're thinking of matrices M quotienting out the zero determinant matrices, so $M =$ zero locus of determinant.

So the relation above says that all 2by2 subdeterminants vanish, so in particular the determinant vanishes. In other words, the matrices of rank 1. Matrix that's a product of a vector and a covector has to be of rank 1.

A generic matrix up to rescaling $\iff$ a line in $\mathbb{C}^{N^2}$. A rank 1 matrix up to rescaling $\iff$ a pair of lines in $\mathbb{C}^N$. Get an embedding $\mathbb{C}P^{N-1} \times \mathbb{C}P^{N-1} \to \mathbb{C}P^{N^2-1}$ which I could also think of as the subspace of pure states in $(\mathbb{C}^N)^{\otimes 2}$. Usually $2N - 2 << N^2 - 1$ which means that most states are entangled. *Segre embedding.*

What about the presence of a superpotential? Meaning, what if I have F term conditions. They come from a chiral field so they're automatically complex equations. Now I need to think about imposing the F term equation.

$$W \implies \bar{F} = \frac{\partial W}{\partial \Phi^i} \implies U \sim |F|^2 \tag{5.107}$$

meaning the potential is zero only when the F term is set to 0

$$\frac{\partial W}{\partial \Phi^i} = 0 \tag{5.108}$$

which is clearly a chiral condition. This means we're at a critical point of the superpotential (since every derivative vanishes). So we have $\mathrm{Crit}(W) \subseteq \hat{M}$ and $\mathrm{Crit}(W) \to \mathrm{Crit}(W)/G_{\mathbb{C}} = M$.

$$\mathrm{Fun}(Crit(W)) = \mathrm{Fun}(\hat{M})/\langle \frac{\partial W}{\partial \Phi^i} \rangle \tag{5.109}$$

Example. No gauge group, three fields X, Y, Z with $W = XYZ$. $\mathrm{Fun}(\hat{M}) = \mathbb{C}[X, Y, Z] \to \mathrm{Fun}(Crit(M))$. F term relations give

$$XY = YZ = ZX = 0 \tag{5.110}$$

so $\mathrm{Fun}(Crit(M)) = \mathbb{C}[X, Y, Z]/\langle XY, XZ, YZ \rangle =$ coordinate axes in $\mathbb{C}^3$

we generally expect in a qft that in th elimit of low energy the theory becomes just a theory of mpas into its moduli psace of vacuea. at very low energy sjould be close to vacuum state, so i can think of a map that takes spacetime .....

A Useful definitions

A.1 Algebraic structures

Let us recall a few useful definitions.

Definition 4 (Field)**.** A field $\mathbb{F}$ is a set with two binary operations called addition and multiplication satisfying the following field axioms:

- Associativity of addition and multiplication
- Commutativity of addition and multiplication
- Additive and multiplicative identity (resp. 0 and 1)
- Additive inverse
- Multiplicative inverse for every element except 0
- Distributivity of multiplication over addition.

or, more simply:

- Abelian group under addition
- nonzero elements are an abelian group under multiplication
- Distributivity of multiplication over addition.

Given a field we can define a vector space over it:

Definition 5 (Vector space)**.** A vector space V over a field $\mathbb{F}$ is a set with two operations:

1. addition $+ : V \times V \to V$,
2. multiplication by a scalar $\cdot : \mathbb{F} \times V \to V$

such that it is an abelian group with respect to addition and has the following properties: ($x, y \in \mathbb{F}$, $\mathbf{v}, \mathbf{w} \in V$)

- $x \cdot (\mathbf{v} + \mathbf{w}) = x \cdot \mathbf{v} + x \cdot \mathbf{w}$
- $(x + y) \cdot \mathbf{v} = x \cdot \mathbf{v} + y \cdot \mathbf{v}$
- $(xy) \cdot (\mathbf{v} + \mathbf{w}) = x \cdot (y \cdot \mathbf{v})$
- $1 \cdot \mathbf{v} = \mathbf{v}$

However there are more general concepts which can be useful.
A ring generalizes the concept of field, without requiring commutativity and inverses of multiplication.

Definition 6 (Ring)**.** A ring is a set with two binary operations called addition and multiplication satisfying the following ring axioms:

- Abelian group under addition
- Semigroup under multiplication (i.e. is only associative)
- Distributivity of multiplication from left and right over addition.

In addition, if it has a multiplicative identity (i.e. it is a monoid under multiplication) it's called a ring with unity.

We also define *division ring* a ring in which every nonzero element has a multiplicative inverse (i.e. a field in which multiplication may be noncommutative).

Now, just as one can define vector spaces over fields, one can define the analogous but more general concept of modules over rings.

Definition 7 (Module)**.** A left module over a ring R consists of an abelian group M and a "scalar multiplication" between elements of R and M that gives another element in M with the following properties: ($r, s \in R$, $x, y \in M$)

- $r \cdot (x + y) = r \cdot x + r \cdot y$
- $(r + s) \cdot x = r \cdot x + s \cdot x$
- $(rs) \cdot x = r \cdot (s \cdot x)$
- $1 \cdot x = x$

Then, it can happen that in a vector space we have a natural concept of a product between vectors that gives another vector, for example the the vector product $\times$ in $\mathbb{R}^3$ or the obvious product in the vector space of polynomials. This additional operation gives rise to the concept of an algebra.

Definition 8 (Algebra)**.** An algebra is a vector space V with an additional operation (multiplication) $\cdot : V \times V \to V$ which is bilinear and associative.

Given an algebra we can then define the commutator

$$[A, B] = AB - BA \tag{A.1}$$

which satisfies

$$[A, B] = -[B, A] \tag{A.2}$$

and is another bilinear operation $[\cdot, \cdot] : V \times V \to V$. This then raises the question: is V with the operation $[\cdot, \cdot]$ also an algebra? But the answer is no since the operation is not associative. Instead we have:

$$[[A, B], C] = [A, [B, C]] + [B, [C, A]] \tag{A.3}$$

in which the last term ruins the associativity. Usually the equality above is written as:

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0 \tag{A.4}$$

which is called the Jacobi identity.

However such an operation appears often enough that the resulting structure deserves a name:

Definition 9 (Lie algebra)**.** A Lie algebra is a vector space with a bilinear operation $[\cdot, \cdot] : V \times V \to V$ with the following properties: $(A, B \in V)$

1. $[A, B] = -[B, A]$
2. $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$

i.e. it is antisymmetric and satisfies the Jacobi identity. The operation is generally called *bracket*, but if it is constructed as above it is usually called *commutator*.

Note that a Lie algebra is not in fact an algebra, as it is not associative. In fact the Jacobi identity somehow measures the non-associativity of the bracket.

Also note that a bilinear operation $[\cdot, \cdot] : V \times V \to V$ is the same thing as a *linear* operation from the tensor product $[\cdot, \cdot] : V \otimes V \to V$. This is just the universal property defining the tensor product.

All these structures have their *super* counterpart which is constructed with the following guidelines:

- add a Koszul sign every time one operator is passed after another,
- preserve parity.

A.2 Representation theory

Given a group, the concept of its representation arises.

But what is a representation? When you think of symmetries, you can consider them as objects on their own, that's basically the idea of a group. Given a symmetry we can then ask what kind of objects have this symmetry: that's the idea of a representation. It basically tells us what kind of objects can I have in space. For example with systems of a certain spin we have that the same symmetry (rotations) is "represented" in different ways.

We can therefore give the following definition.

Definition 10. A representation of a group G on a vector space V is a map $\rho : G \to \mathrm{Aut}(V)$, where the automorphism group of a vector space V, $\mathrm{Aut}(V)$, is the group of invertible linear maps from V to V. The map must have the following property:

$$\rho(g \cdot g') = \rho(g) \circ \rho(g') \tag{A.5}$$

where $\circ$ is simply the composition.

In other words, we need a group homeomorphism from g to $\mathrm{Aut}(V)$.

It is also useful to define the same concept for algebras and Lie algebras. The only difference is that we will have algebra homomorphisms and Lie algebra homomorphisms. Let us start by defining those.

Definition 11. An algebra homomorphism is a map $\phi : A \to B$ between algebras over a field $\mathbb{F}$ that satisfies: $(x, y \in A, k \in \mathbb{F})$

1. $\phi(kx) = k\phi(x)$
2. $\phi(x + y) = \phi(x) + \phi(y)$
3. $\phi(xy) = \phi(x)\phi(y)$

i.e. it is a linear map that preserves the product.

Definition 12. A Lie algebra homomorphism is a linear map $\phi : A \to B$ between Lie algebras over a field $\mathbb{F}$ that satisfies: $(x, y \in A, k \in \mathbb{F})$

$$\phi([x, y]_A) = [\phi(x)\phi(y)]_B \tag{A.6}$$

We can then note that given a vector space V, the space of linear maps from V to itself, $\mathrm{End}(V)$, naturally has a (Lie) algebra structure (with "product" given by composition and bracket given by the commutator). So the following definition makes sense.

Definition 13. A representation of a (Lie) algebra A on a vector space V is a (Lie) algebra homomorphism $\rho : A \to \mathrm{End}(V)$.

More explicitly, for Lie algebras we have

$$\rho_g(\rho_h(v)) - \rho_h(\rho_g(v)) = \rho_{[g,h]}(v) \tag{A.7}$$

i.e. "The matrix $\rho_{[g,h]}$ is the commutator of the matrices ρ_g and ρ_h."

Furthermore, the following remarks are useful in practical situations:

1. A map $\rho : A \to \mathrm{End}(A)$ is equivalent to a map $\rho : A \otimes V \to V$.
2. Given representations over vector spaces V and W it's possible to construct one over the duals, over the direct sum and over the tensor product. In particular for the tensor product we have:

$$\rho_{V\otimes W} = \rho_V \otimes 1 + 1 \otimes \rho_W \tag{A.8}$$

A.3 Short exact sequences

Definition 14. Given groups A, B and C with maps $f : A \to B$, $g : B \to C$, the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{A.9}$$

is called a *exact at* B if $\operatorname{im} f = \ker g$.

Note that in particular we have $g \circ f = 0$, however when checking exactness this is not enough since it simply gives the inclusion $\operatorname{im} f \subseteq \ker g$.

Definition 15 (Short exact sequence)**.** Given groups A, B and C with maps $f : A \to B$, $g : B \to C$, the sequence

$$1 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 1 \tag{A.10}$$

is called a *short exact sequence* if it is exact at A, B and C.

Note that exactness at A is equivalent to injectiveness of f while exactness at C is equivalent to surjectiveness of g.

There is a typical situation in which we have an injective map into a group and a surjective map out of it such that the composition of the two is trivial:

$$1 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} B/A \longrightarrow 1 \tag{A.11}$$

where i is the inclusion of A in B and π is the projection to the quotient. Note that the quotient may be more accurately written as $B/\operatorname{im}(i)$ and it makes sense to take the quotient since $A \cong \operatorname{im}(i) = \ker(\pi)$ and the kernel is a normal subgroup. Any short exact sequence can be thought of in this way.

There is also another type of exact sequence which often appears:

$$1 \longrightarrow A \xrightarrow{i} A \oplus B \xrightarrow{\pi} B \longrightarrow 1 \tag{A.12}$$

where i is the inclusion of A in $A \oplus B$ and π is the projection onto the B part of the sum. i is clearly injective, π is clearly surjective and $\pi \circ i = 0$, so it is in fact exact. However in this situation there's not really a difference between A and B and we could just as easily have arrows going in the opposite direction. This does not always happen, but it's interesting enough to deserve a name:

Definition 16. A short exact sequence

$$1 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 1 \tag{A.13}$$

is called *split* if $B \cong A \oplus C$.

One can prove that this is equivalent to having a *left-splitting*, i.e. a map $j : B \to A$ with $j \circ f = id_A$. From the comment above, both of these lead to a map $k : C \to B$ with $g \circ k = id_C$ which is called a *right-splitting*. So a split sequence essentially "is exact in both directions". In addition, for abelian groups a right splitting also gives a left splitting and so the three properties are equivalent.

An example of a short exact sequence in physics is given by the Poincaré group of a vector space V which is generated by translations (given by V itself) and rotations ($\mathrm{SO}(V)$):

$$1 \longrightarrow V \xrightarrow{\ i\ } \text{Poincaré}(V) \xrightarrow{\ j\ } \mathrm{SO}(V) \longrightarrow 1 \tag{A.14}$$

i is the inclusion of translations and the map j is given by simply "forgetting" the translation part of the transformation. Again i is injective, j is surjective and $j \circ i = 0$. In addition, this sequence has a right splitting since we can include $\mathrm{SO}(V)$ in the Poincaré group but the groups are not abelian so that doesn't guarantee a left splitting. In fact we don't have a splitting since having the sequence in the opposite direction would mean that $\mathrm{SO}(V)$ is a normal subgroup of the Poincaré group, which it is not.

This case is also interesting: when in an exact sequence $0 \to A \to B \to C \to 0$ we have only a right splitting but not a left splitting we get the following isomorphism

$$B \cong A \rtimes C \cong C \ltimes A \tag{A.15}$$

where $\rtimes$ denotes the semi direct product. This makes sense because in this situation one can construct a map $C \to \mathrm{Aut}(A)$ meaning that there is an action of the group C on the group A and with such an action one can construct the direct product $A \rtimes C$ which turns out to be isomorphic to B. Therefore for the previous example we can write

$$\text{Poincaré}(V) \cong \mathrm{SO}(V) \ltimes V \tag{A.16}$$

and we of course have an action of $\mathrm{SO}(V)$ on V. From what we said before about the quotient, we can also write

$$V \cong \text{Poincaré}(V)/\mathrm{SO}(V) \tag{A.17}$$

A.4 Simple rings

This section contains some results from [3, 4] that are useful in the section about Clifford modules. In particular there we said that algebras of the type $M_n(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ have unique representation $\mathbb{K}^n$ up to isomorphism. (Note that $\mathbb{C}$ had two representations which are *inequivalent* but still isomorphic)

That is a result about the representation theory of simple rings (which is a synonym of simple algebra). In fact the matrix rings $M_n(\mathbb{K})$ are simple:

Theorem 17 (Theorem 5.5 of [3])**.** Let $\mathbb{K}$ be a division ring and E a finite dimensional vector space over $\mathbb{K}$. Let $R = \mathrm{End}_{\mathbb{K}}(E)$, then R is simple and E is a simple R-module, i.e. a representation of R.

So we have that $R = M_n(\mathbb{K}) = \mathrm{End}_{\mathbb{K}}(\mathbb{K}^n)$ has $\mathbb{K}^n$ as a representation. The next theorem gives its uniqueness:

Theorem 18 (Corollary 4.6 of [3])**.** A simple ring has exactly one simple module up to isomorphism.

These results are collected in Theorem 5.6 of [4]:

Theorem 19. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and consider the ring $M_n(\mathbb{K})$. Then the natural representation ρ of $M_n(\mathbb{K})$ on the vector space $\mathbb{K}^n$ is, up to isomorphism, the only irreducible representation of $M_n(\mathbb{K})$.

In addition, the algebra $M_n(\mathbb{K}) \oplus M_n(\mathbb{K})$ has exactly two equivalence classes of irreducible representations, given by:

$$\rho_1(\phi_1, \phi_2) = \rho(\phi_1) \quad \text{and} \quad \rho_2(\phi_1, \phi_2) = \rho(\phi_2) \tag{A.18}$$

acting on $\mathbb{K}^n$.

In the section on Clifford modules, these two representations are labeled $\mathbb{K}^n_+$ and $\mathbb{K}^n_-$.

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