Review III of Chapter 4: Linear Mapping

illusion

Especially made for smy

School of Mathematical Science

XMU

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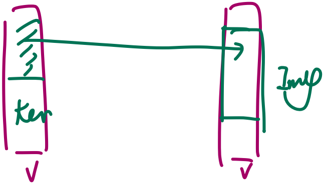
Outline



U 为线性空间 V 的子空间，$\varphi \in \mathcal{L}(V)$，那么

$$\varphi(\varphi^{-1}(U)) = U \cap \mathrm{Im}\varphi, \quad \varphi^{-1}(\varphi(U)) = U + \mathrm{Ker}\varphi.$$

Goal:

- $\varphi(\varphi^{-1}(U)) \subseteq U \cap \mathrm{Im}\varphi$

$\forall \alpha \in \varphi(\varphi^{-1}(U))$, $\boxed{\exists \beta \in \varphi^{-1}(U) \Leftrightarrow \varphi(\beta) \in U}$ s.t. $\alpha = \varphi(\beta) \in U \cap \mathrm{Im}\varphi$

- $U \cap \mathrm{Im}\varphi \subseteq \varphi(\varphi^{-1}(U))$

$\forall \alpha \in U \cap \mathrm{Im}\varphi$, $\exists \beta \in V$, $\alpha = \varphi(\beta) \in U \Leftrightarrow \beta \in \varphi^{-1}(U) \Rightarrow \alpha = \varphi(\beta) \in \varphi(\varphi^{-1}(U))$ ∎

U 为线性空间 V 的子空间，$\varphi \in \mathcal{L}(V)$，那么

$$\varphi(\varphi^{-1}(U)) = U \cap \mathrm{Im}\varphi, \varphi^{-1}(\varphi(U)) = U + \mathrm{Ker}\varphi.$$

- $\forall \alpha \in \varphi^{-1}(\varphi(U))$, $\varphi(\alpha) \in \varphi(U)$ $\exists \beta \in U$, $\varphi(\alpha) = \varphi(\beta)$

  $\Rightarrow \alpha = \beta + \gamma$, $\exists \gamma \in \mathrm{Ker}\varphi$

  $\Rightarrow \alpha \in U + \mathrm{Ker}\varphi$

- $\forall \alpha \in U + \mathrm{Ker}\varphi$, $\alpha = \beta + \gamma$, $\beta \in U$, $\gamma \in \mathrm{Ker}\varphi$

  $\Rightarrow \varphi(\alpha) = \varphi(\beta) \in \varphi(U) \Leftrightarrow \alpha \in \varphi^{-1}(\varphi(U))$

Practice to review

例1： 设 $\varphi, \psi \in \mathcal{L}(V)$，$U$ 是 V 的子空间，满足 $\operatorname{Im}\varphi = \operatorname{Im}\psi$，则

A. $\varphi = \psi$

B. $\operatorname{Ker}\varphi = \operatorname{Ker}\psi$

C. 若 $\varphi(U) \subseteq U$，则 $\psi(U) \subseteq U$

D. φ, ψ 在 V 的任意基下的矩阵相抵

例2： 设 U 是 V 的子空间，$\varphi \in \mathcal{L}(V)$，记 $\varphi^{-1}(U) = \{\alpha \in V \mid \varphi(\alpha) \in U\}$, 则

A. $\dim \varphi^{-1}(U) = \dim U$

B. $\dim \varphi^{-1}(U) \leq \dim U$

C. $\dim \varphi^{-1}(U) \geq \dim U$

D. $\dim \varphi^{-1}(U)$ 和 $\dim U$ 没有关系

例3： 设 V_1, V_2, W 都是 n 维线性空间 V 的子空间，若 $V_1 \subseteq V_2$ 且 $V_1 \oplus W = V_2 \oplus W$，则（选填“必”、“未必”）有 $V_1 = V_2$。

例4： 设 V_1, V_2, W 是 n 维线性空间 V 的非平凡子空间，且 $V = V_1 \oplus V_2$，$W \cap V_1 = \{\mathbf{0}\}$，$W \cap V_2 = \{\mathbf{0}\}$. 求 $(W + V_1) \cap (W + V_2)$ 的基和维数。

H: $\varphi \rightsquigarrow A$ $\quad$ d: $\mathrm{Im}\varphi = r(A)$

例1： 设 $\varphi,\psi\in\mathcal{L}(V)$，$U$ 是 V 的子空间，满足 $\mathrm{Im}\,\varphi=\mathrm{Im}\,\psi$，则

$\psi \rightsquigarrow B$ $\quad \Rightarrow r(A)=r(B)$

~~A.~~ $\varphi=\psi$

~~B.~~ $\mathrm{Ker}\,\varphi=\mathrm{Ker}\,\psi$

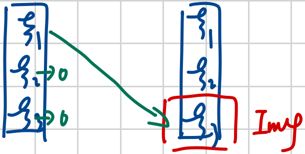
~~C.~~ 若 $\varphi(U)\subseteq U$，则 $\psi(U)\subseteq U$

D. φ,ψ 在 V 的任意基下的矩阵相抵 ✓

$\boxed{\text{秩不变量}}$

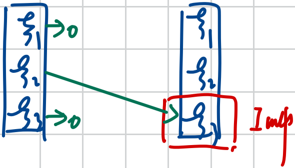
r

φ



$\varphi(\langle\xi_2\rangle)=\{0\}\subseteq\langle\xi_2\rangle$

ψ



$\psi(\langle\xi_2\rangle)=\langle\xi_3\rangle\not\subseteq\langle\xi_2\rangle$

例2： 设 U 是 V 的子空间，$\varphi \in \mathcal{L}(V)$，记 $\varphi^{-1}(U) = \{\alpha \in V \mid \varphi(\alpha) \in U\}$, 则

A. $\dim \varphi^{-1}(U) = \dim U$

B. $\dim \varphi^{-1}(U) \leq \dim U$

C. $\dim \varphi^{-1}(U) \geq \dim U$

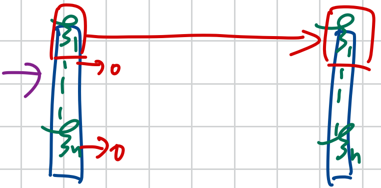
D. $\dim \varphi^{-1}(U)$ 和 $\dim U$ 没有关系

D

1-1 and onto

1-1: $\dim \varphi^{-1}(U) \leq \dim(\operatorname{Im}\varphi \cap U)$ 也不可保证不变

非1-1



$\dim \varphi^{-1}(\langle \xi_1, \cdots, \xi_n \rangle) = n$

$\forall 1 \leq r \leq n$

$\dim(\operatorname{Ker}\varphi + \langle \xi_r \rangle)$

例4： 设 V_1, V_2, W 是 n 维线性空间 V 的非平凡子空间，且 $V = V_1 \oplus V_2$，$W \cap V_1 = \{0\}$，$\boxed{W \cap V_2 = \{0\}}$. 求 $(W+V_1)\cap(W+V_2)$ 的基和维数。

$$\dim\left[(W+V_1)\cap(W+V_2)\right] = \dim(W+V_1) + \dim(W+V_2) - \dim\underbrace{(\cancel{W+V_1+W+V_2})}_{n}$$

$$= 2\dim W + n - n = 2\dim W$$

$W \rightsquigarrow \xi_1, \dots, \xi_r \in V = V_1 \oplus V_2$　设 $\xi_i = \alpha_i + \beta_i$，其中 $\alpha_i \in V_1$，$\beta_i \in V_2$

Check: $\underbrace{\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r}$ 为 $(W+V_1)\cap(W+V_2)$ 的一组基.

$2r = 2\dim W$ $\Rightarrow$ 所求维数　$\boxed{\text{只需线性无关即可}}$

$$\underbrace{c_1\alpha_1 + \dots + c_r\alpha_r}_{\in V_1} + \underbrace{d_1\beta_1 + \dots + d_r\beta_r}_{\in V_2} = \underset{V_1 \oplus V_2}{0}$$

$$\Rightarrow \begin{cases} c_1\alpha_1 + \dots + c_r\alpha_r = 0 \\ d_1\beta_1 + \dots + d_r\beta_r = 0 \end{cases}$$

若 $c_1, \dots, c_r$ 不全为0　$\rightsquigarrow$ 只能 $c_1 = \dots = c_r = 0$

$c_1\alpha_1 + \dots + c_r\alpha_r = 0$

$\downarrow$

同理 $d_1 = \dots = d_r = 0$

$$\triangle\ \underbrace{c_1\xi_1+\cdots+c_r\xi_r}_{\in W}=\underbrace{(c_1\alpha_1+\cdots+c_r\alpha_r)}_{=0}+(c_1\beta_1+\cdots+c_r\beta_r)$$

$$=\underbrace{c_1\beta_1+\cdots+c_r\beta_r}_{\in V_2}\in V_2\cap W=\{0\}$$

$\downarrow$

$\xi_1\cdots\xi_r$ 线性无关

$c_1\cdots c_r$ 不全为 0　　$c_1\xi_1+\cdots+c_r\xi_r\neq 0$ 矛盾!

Practice to review

例：设

$$V = \{A \in F^{3\times 3} \mid A^\top = A\}, \quad U = \{A \in F^{3\times 3} \mid A^\top = -A\},$$

$$W = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \middle| \, a, b, c \in F \right\}.$$

(1) 分别写出 V, U 和 W 的基；

(2) 证明：存在 $\varphi \in \mathcal{L}(V, U)$，使得 $\mathrm{Ker}\varphi = W$。

例：设 ε_{11} ε_{22} ε_{33} $\varepsilon_{12}-\varepsilon_{21}$ $\varepsilon_{13}-\varepsilon_{31}$ $\varepsilon_{23}-\varepsilon_{32}$

$$V=\{A\in F^{3\times 3}\mid A^{\top}=A\},\quad U=\{A\in F^{3\times 3}\mid A^{\top}=-A\},$$

$\varepsilon_{12}+\varepsilon_{21}$ $\varepsilon_{13}+\varepsilon_{31}$ $\varepsilon_{23}+\varepsilon_{32}$

$$W=\left\{\begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix}\middle|\,a,b,c\in F\right\}.$$ ε_{11} ε_{22} ε_{33}

(1) 分别写出 V,U 和 W 的基；

(2) 证明：存在 $\varphi\in\mathcal{L}(V,U)$，使得 $\mathrm{Ker}\varphi=W$。

$\varepsilon_{ii}\mapsto 0$

$\varepsilon_{ij}+\varepsilon_{ji}\mapsto\varepsilon_{ij}-\varepsilon_{ji}$ $\quad\varphi(\xi_i)=\xi_i$

$$\begin{bmatrix}\boxed{E_6}&0\\0&\boxed{-E_3}\end{bmatrix}$$

(3) $\underline{\varphi(A)=A^{T}}$ $F^{3\times3}$ $\varphi(\xi_1\cdots\xi_9)=(\xi_1\cdots\xi_9)\begin{bmatrix}E_6&0\\0&-E_3\end{bmatrix}$

$\varphi(\varepsilon_{ii})=\varepsilon_{ii}$ 3

Ch6 → [illegible]

$\varphi(\xi_j)=-\xi_j$

$\varphi(\varepsilon_{ij}+\varepsilon_{ji})=\varepsilon_{ij}+\varepsilon_{ji}$ 3 $\quad\varphi(\varepsilon_{ij}-\varepsilon_{ji})=-(\varepsilon_{ij}-\varepsilon_{ji})$ 3

$$\rightsquigarrow F^{3\times3} = \langle \xi_1, \dots, \xi_6\rangle \oplus \langle \xi_7, \dots, \xi_9\rangle$$

$\varphi(\cdot) = 1(\cdot)$ $\quad$ φ-子空间 $\quad$ $\varphi(\cdot) = -1(\cdot)$

$(\varphi - id_{F^{3\times3}})(\cdot) = 0$ $\qquad$ $(\varphi + id_{F^{3\times3}})(\cdot) = 0$

Q: $F^{3\times3} = \mathrm{Ker}(\varphi - id_{F^{3\times3}}) \oplus \mathrm{Ker}(\varphi + id_{F^{3\times3}})$ 怎么证?

$$\varphi(A) = A^T \rightsquigarrow \varphi^2(A) = \varphi(A^T) = A \rightsquigarrow \varphi^2 = id_{F^{3\times3}}$$

$\rightsquigarrow$ 矩阵 $\underset{9\times9}{\boxed{A^2 = E}}$, 证 $F^9 = \mathrm{Ker}(A-E) \oplus \mathrm{Ker}(A+E)$

$$\rightarrow r(A-E) + r(A+E) = \underset{r(E_9)}{9} + \underset{0}{\underline{r[A^2-E]}}$$

$$\begin{pmatrix} A-E & E \\ & A+E \end{pmatrix} \to \begin{pmatrix} A-E & A-E \\ & A+E \end{pmatrix} \to \begin{pmatrix} A-E & -2E \\ A+E & \end{pmatrix} \quad r_2 \times \tfrac{1}{2}(A+E)$$

$\times\frac{1}{2}$ $\times\frac{1}{2}$

$$E = \frac{1}{2}\left[(A+E)-(A-E)\right] \to \begin{pmatrix} A-E & -2E \\ \frac{1}{2}(A+E)(A-E) & 0 \end{pmatrix}$$

$(A+E)(A-E)=0$

$$\Rightarrow F^n = \operatorname{Ker}(A+E) \oplus \operatorname{Ker}(A-E) \to \begin{pmatrix} A-E & -2E \\ \frac{1}{2}(A^2-E) & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} A-E & -2E \\ 0 & 0 \end{pmatrix}$$

Bezout

$f(A)g(A)=0.$ $\quad u(A)f(A)+v(A)g(A)=E$ $\quad \to \begin{pmatrix} 0 & E_q \\ 0 & 0 \end{pmatrix}$ ✓

$F^n = \operatorname{Ker} f(A) \oplus \operatorname{Ker} g(A)$ $\quad \boxed{\gcd(f,g)=1}$.

· $\operatorname{Ker}(A-E) \cap \operatorname{Ker}(A+E) = \{0\}$?

Q: $\bar{2} \overset{?}{\neq} \bar{0}$ $\quad \mathbb{Z}_2$, 4

$\forall \alpha \in \operatorname{Ker}(A-E) \cap \operatorname{Ker}(A+E) \rightsquigarrow (A-E)\alpha=0,\ (A+E)\alpha=0.$

char$(\mathbb{Z}_2)$ = ?

$$\alpha = \frac{1}{2}\left[(A+E)-(A-E)\right]\alpha = 0-0 = 0 \quad \boxed{(2\neq 0)}$$

$$\rightsquigarrow \varepsilon = \frac{1}{2}\left[(A+\varepsilon) - (A-\varepsilon)\right]$$

$$F^q \overset{?}{=} \operatorname{Ker}(A-\varepsilon) + \operatorname{Ker}(A+\varepsilon)$$

$$\forall \alpha \in F^q, \quad \alpha = \frac{1}{2}\Big[\underbrace{(A+\varepsilon)\alpha}_{\in \operatorname{Ker}(A-\varepsilon)} + \underbrace{-(A-\varepsilon)\alpha}_{\operatorname{Ker}(A+\varepsilon)}\Big]. \checkmark$$

$\varepsilon\alpha$

$$\underset{\delta}{\underline{\varphi^2 = id_V}} \rightsquigarrow \underline{V = \operatorname{Ker}(\varphi + id_V) \oplus \operatorname{Ker}(\varphi - id_V)}$$

$$\varphi \leftrightsquigarrow \begin{bmatrix} E_p & 0 \\ 0 & -E_q \end{bmatrix}$$

φ 相似对角化

Review of §4.6 Invariant Subspace

- 导出变换的维数公式：设 $\varphi \in \mathcal{L}(V,U), V_1 \subseteq V, U_1 \subseteq U$

  (1) $\dim \varphi(V_1) + \dim(\mathrm{Ker}\varphi \cap V_1) = \dim V_1 \rightsquigarrow \varphi|_{V_1}$

  (2) $\dim(U_1 \cap \mathrm{Im}\varphi) + \dim \mathrm{Ker}\varphi = \dim \varphi^{-1}(U_1) \rightsquigarrow \varphi|_{\varphi^{-1}(U_1)}$

- 设 $\varphi \in \mathcal{L}(V,U)$ 非零非可逆，则 $\mathrm{Im}\varphi, \mathrm{Ker}\varphi$ 为非平凡 $\varphi-$子空间
- $\rightsquigarrow$ 若 $\mathscr{O} \neq \varphi \in \mathcal{L}(V,U)$ 只有平凡 $\varphi-$子空间，则 φ 可逆，反之不成立

- 设 $\varphi, \psi \in \mathcal{L}(V,U)$，且 $\varphi\psi = \psi\varphi$，则 $\mathrm{Ker}\varphi, \mathrm{Im}\varphi$ 均为 $\psi-$子空间
- $\rightsquigarrow \varphi\psi + \psi\varphi = \mathscr{O}$ 上述结论也成立

Try

设 $\mathscr{O} \neq \varphi, \psi \in \mathcal{L}(V)$，$\dim V = 2n+1$ $(n \in \mathbf{N}^*)$，若 $\varphi\psi + \psi\varphi = \mathscr{O}$，求证：$\varphi$ 既有非平凡的 $\psi-$子空间，也有非平凡的 $\varphi-$子空间。

Try

设 $\mathscr{O} \neq \varphi, \psi \in \mathcal{L}(V)$，$\dim V = 2n+1$ $(n \in \mathbf{N}^*)$，若 $\varphi\psi + \psi\varphi = \mathscr{O}$，求证：$\varphi$ 既有非平凡的 ψ–子空间，也有非平凡的 φ–子空间。

反证. φ 若没有非平凡的 φ-子空间, $\varphi \neq 0 \rightsquigarrow \varphi$ 可逆 $\rightsquigarrow A$ 可逆

$B \rightsquigarrow \psi$.

$AB + BA = 0 \Rightarrow B = -A^{-1}BA$

$\det B = \det(-A^{-1}BA)$

$= \det(-B) = (-1)^{\boxed{2n+1}} \det B = -\det B$

$\Rightarrow \det B = 0 \rightsquigarrow B$ 不可逆. $\rightsquigarrow \psi$ 非0 非可逆

$\rightsquigarrow \operatorname{Ker}\psi, \operatorname{Im}\psi$ 为非平凡 ψ 子空间 $\rightsquigarrow$ φ-子空间

$\varphi\psi + \psi\varphi = 0.$

$\rightsquigarrow \varphi$ 必有非平凡 φ-子空间

φ 反证 $\rightsquigarrow$ φ 若有非平凡 $\boxed{\varphi\text{-子空间}}$ 也是 φ 子空间

Review of §4.6 Invariant Subspace

设 $\varphi \in \mathcal{L}(V)$ 且 U 为 $\varphi-$ 子空间，取 U 的一组基为 $\xi_1, \cdots, \xi_r$，扩为 V 的一组基 $\xi_1, \cdots, \xi_r, \xi_{r+1}, \cdots, \xi_n$，由于 $\varphi(U) \subseteq U$，这说明

$$
\begin{cases}
\varphi(\xi_1) = a_{11}\xi_1 + \cdots + a_{r1}\xi_r + 0 \cdot \xi_{r+1} + \cdots + 0 \cdot \xi_n \\
\cdots \\
\varphi(\xi_r) = a_{1r}\xi_1 + \cdots + a_{rr}\xi_r + 0 \cdot \xi_{r+1} + \cdots + 0 \cdot \xi_n \\
\varphi(\xi_{r+1}) = b_{1,r+1}\xi_1 + \cdots + b_{r,r+1}\xi_r + c_{r+1,r+1}\xi_{r+1} + \cdots + c_{n,r+1}\xi_n \\
\cdots \\
\varphi(\xi_n) = b_{1n}\xi_1 + \cdots + b_{rn}\xi_r + c_{r+1,n}\xi_{r+1} + \cdots + c_{nn}\xi_n
\end{cases}
$$

Let $A = (a_{ij})_{r \times r}, B = (b_{i,j+r})_{r \times n-r}, C = (c_{i+r,j+r})_{n-r \times n-r}$

Review of §4.6 Invariant Subspace

就有

$$\varphi(\xi_1,\cdots,\xi_r,\xi_{r+1},\cdots,\xi_n)=(\xi_1,\cdots,\xi_r,\xi_{r+1},\cdots,\xi_n)\begin{bmatrix}A & B\\ O & C\end{bmatrix} \tag{1}$$

调换基的位置有

$$\varphi(\xi_{r+1},\cdots,\xi_n,\xi_1,\cdots,\xi_r)=(\xi_{r+1},\cdots,\xi_n,\xi_1,\cdots,\xi_r)\begin{bmatrix}C & O\\ B & A\end{bmatrix} \tag{2}$$

Q: 当基的位置调换为 $\xi_{r+1},\cdots,\xi_{r+s},\xi_1,\cdots,\xi_r,\xi_{r+s+1},\cdots,\xi_n$，其中 $s<n-r$，你能对 C,B 适当分块写出在这组基下的表示矩阵吗？

可以使用三类相似初等变换： $E(i,j)AE(i,j)$, $E(i(c))AE(i(c^{-1}))$, $E(i,j(c))AE(i,j(-c))$.

$$\varphi(\xi_1,\cdots,\xi_r,\xi_{r+1},\cdots,\xi_n)=(\xi_1,\cdots,\xi_r,\xi_{r+1},\cdots,\xi_n)\begin{bmatrix}A & B\\ O & C\end{bmatrix}$$

$r+s$ $r+s+1$ s $\}s$

$B=(\underbrace{B_1}_{s},\ \underbrace{B_2}_{n-r-s})$

$$C=\begin{array}{c|cc} & s & n-r-s \\ \hline s & C_1 & C_2 \\ n-r-s & C_3 & C_4 \end{array}$$

Q: 当基的位置调换为 $\xi_{r+1},\cdots,\xi_{r+s},\xi_1,\cdots,\xi_r,\xi_{r+s+1},\cdots,\xi_n$，其中 $s<n-r$，你能对 C,B 适当分块写出在这组基下的表示矩阵吗？

$$\varphi(\qquad)=(\qquad)\begin{bmatrix}0 & C_1 & C_2\\ A & B_1 & B_2\\ 0 & C_3 & C_4\end{bmatrix}$$

$$\begin{bmatrix}A & B_1 & B_2\\ 0 & C_1 & C_2\\ 0 & C_3 & C_4\end{bmatrix}$$

1° $E(i,j)\, A\, [E(i,j)]^{-1} = E(i,j)\, A\, E(i,j) \rightsquigarrow \begin{pmatrix} & E \\ E & \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} & E \\ E & \end{pmatrix}$

$$\begin{pmatrix} E & \\ & E \end{pmatrix} = \begin{pmatrix} & E \\ E & \end{pmatrix}^2$$

2° $E(i(c))\, A\, E(i(\frac{1}{c})) \rightsquigarrow \begin{pmatrix} N & \\ & E \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} N^{-1} & \\ & E \end{pmatrix}$

3° $E(i,j(c))\, A\, E(i,j(-c)) \rightsquigarrow \begin{pmatrix} E & \\ N & E \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} E & \\ -N & E \end{pmatrix}$

A的第j行c倍+第i行

A的第i列-c倍+第j列

Example

(Multiple Choice) 设 $\varphi \in \mathcal{L}(V)$ 有非平凡的 $\varphi-$子空间，则必定存在 V 的某个基，使得 φ 在这组基下的矩阵为

A. $\begin{bmatrix} O & A \\ B & C \end{bmatrix}$ ✗ $\quad id_V \quad E = \begin{bmatrix} E_r & \\ & E_{n-r} \end{bmatrix}$ ≠0

B. $\begin{bmatrix} A & O \\ B & C \end{bmatrix}$ ✓ $\quad \begin{bmatrix} C & B \\ 0 & A \end{bmatrix}$ ✓

C. $\begin{bmatrix} A & O & O & B \\ O & C & D & O \\ O & E & F & O \\ G & O & O & H \end{bmatrix}$ $\quad \begin{matrix} 0 & E & F & 0 \\ 0 & C & D & 0 \\ A & 0 & 0 & B \\ G & 0 & 0 & H \end{matrix}$

D. $\begin{bmatrix} A & B & C & D \\ O & E & O & F \\ G & H & I & J \\ O & K & O & L \end{bmatrix}$ ✓ $\quad \rightsquigarrow \begin{bmatrix} \tilde{A} & \tilde{B} \\ 0 & \tilde{C} \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad n=1$ → 1阶的矩阵

1×1 → 不可能

$V = V_1 \oplus V_2$

V_1, V_2 均为 1 维的 φ-子空间

$\langle \xi_1 \rangle \quad \langle \xi_2 \rangle \quad \xi_1, \xi_2$ 为特征向量 λ_1, λ_2

$\varphi(\xi_1\ \xi_2) = (\xi_1\ \xi_2)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

Hint: 考虑调换基的顺序

$\begin{bmatrix} A & O \\ O & \tilde{B} \end{bmatrix}$

Examples

Example

(Multiple Choice) 设 $\varphi \in \mathcal{L}(\mathbf{R}^2)$ 定义为 $\varphi : X \mapsto AX$，若 φ 有非平凡 φ−子空间，则 A 不可能为

BC.

A. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 0 & -2 \\ 1 & -2 \end{bmatrix}$ Frobenius C. $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Hint: 二维空间的非平凡不变子空间必定为一维的，即特征子空间

Quickly Review

What we have: $\varphi^2 = \mathrm{id}_V \Leftrightarrow \varphi \rightsquigarrow \begin{bmatrix} E_r & \\ & -E_{n-r} \end{bmatrix} \Leftrightarrow V = \mathrm{Ker}(\varphi - \mathrm{id}_V) \oplus \mathrm{Ker}(\varphi + \mathrm{id}_V)$

- $\varphi^2 = \mathscr{O} \Leftrightarrow \varphi \rightsquigarrow \mathrm{diag}\{\boldsymbol{J}(0,2), \cdots, \boldsymbol{J}(0,2), 0, \cdots, 0\} \Leftrightarrow V = \mathrm{Ker}\varphi^2$
- $\varphi^2 = \varphi \Leftrightarrow \varphi \rightsquigarrow \begin{bmatrix} E_r & O \\ O & O \end{bmatrix} \Leftrightarrow V = \mathrm{Ker}\varphi \oplus \mathrm{Ker}(\varphi - \mathrm{id}_V)$

  ☆ $(\varphi - \mathrm{id}_V)^2 - (\varphi - 2\mathrm{id}_V)\varphi = \mathrm{id}_V$
- $(\varphi - \mathrm{id}_V)^2 \varphi = \mathscr{O} \Leftrightarrow \varphi \rightsquigarrow \mathrm{diag}\{\boldsymbol{J}(1,2), \cdots, \boldsymbol{J}(1,2), 1, \cdots, 1, 0, \cdots, 0\}$

  $V = \mathrm{Ker}(\varphi - \mathrm{id}_V)^2 \oplus \mathrm{Ker}\varphi$
- $\varphi^n = \mathscr{O}, \varphi^{n-1} \neq \mathscr{O} \Leftrightarrow \varphi \rightsquigarrow \boldsymbol{J}(0,n)$
- (HW-4) $\varphi^m = \mathscr{O}, \varphi^{m-1} \neq \mathscr{O}, \dim \mathrm{Im}\varphi = n-1 \Leftrightarrow \varphi \rightsquigarrow \boldsymbol{J}(0,n)$

  $\Rightarrow m = n$

Try

$$\varphi^2 = 2\varphi \Leftrightarrow \varphi \rightsquigarrow \begin{bmatrix} 2E_r & O \\ O & O \end{bmatrix} ?$$