Notes:

- $\boxed{F^n}$上的真子空间一定可以看成某个线性方程组$AX = O$的解空间，其中A不可逆
- 如何用$V \cong F^n$给出一个更具体的构造？ $\rightsquigarrow$ 将V_1看成线性方程组的解空间

$V \overset{\varphi}{\cong} F^n$ $\qquad V_1 \xrightarrow{\varphi} U_1 \subseteq F_n$

$U_1 = \langle \alpha_1 - - \alpha_r \rangle \quad \alpha_i \in F^n$ 列向量

Let $B = (\alpha_1 - - \alpha_r)_{n\times r}$ $\qquad \boxed{B^T X = 0} \in F^n$

$r\times n$ 基础解系

$\alpha_i^T \beta_j = 0 \Leftarrow \begin{pmatrix} \alpha_1^T \\ \vdots \\ \alpha_r^T \end{pmatrix} (\beta_1 - - \beta_{n-r}) = 0$

$(\forall 1 \le i \le r,\ 1 \le j \le n-r)$

$n - r(B^T) = n - r(B) = n - r$

$\boxed{\beta_j^T \alpha_i = 0}$ $\qquad$ Let $A^T = (\beta_1 - - \beta_{n-r})_{n\times n-r} \Leftarrow \beta_1 - - \beta_{n-r}$

$A = \begin{pmatrix} \beta_1^T \\ \vdots \\ \beta_{n-r}^T \end{pmatrix}_{n-r\times n}$ $\qquad \boxed{AX = 0} \rightsquigarrow$ Check! $\alpha_1 - - \alpha_r$?

$r(A) = n - r$.

Let U_i 为 $\beta_i^T x = 0$ 的解空间 $\Rightarrow$ $\underline{V_1 = \bigcap_{i=1}^{n-r} U_i}$ ✓

$\geq$:

$\underline{U_i: \ C_i X = 0}$ $\quad \forall\, 1 \leq i \leq k,\ r(C_i) = 1.$

$\Rightarrow \dim U_i = n-1$

$$\begin{pmatrix} C_1 \\ \vdots \\ C_k \end{pmatrix} X = 0 \rightsquigarrow \boxed{\dim \bigcap_{i=1}^{k} U_k \geq n-k} \ ?$$

$$= n - r\begin{pmatrix} C_1 \\ \vdots \\ C_k \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} C_1 \\ \vdots \\ C_k \end{pmatrix} \leq k \ ?$$

✓

Example

设数域$F \subseteq K \subseteq L$，则在下面两个运算下$K$成为$F$上的线性空间：$F$中元素与$K$中元素的数量乘法，$K$中元素的加法。同理$L$也是$K$上的线性空间，如果$\dim_F K < \infty$，则记$\dim_F K = [K : F]$，证明：

$$[L : K][K : F] = [L : F]$$

Notes:

- Revisit your homework: $\dim_{\mathbf{Q}} \mathbf{Q}(\sqrt{2}) = 2$
- $\dim_{\mathbf{Q}} \mathbf{Q}(\sqrt{2}, \sqrt{3}) = ?$

$F \subseteq K$

K_F

$ms = \dim_F L$

$\alpha_1 - - - \alpha_m$

$\alpha = c_1\alpha_1 + \cdots + c_m\alpha_m$

$K \subseteq L$

L_K

$\beta_1 - - - \beta_s$

$\forall \gamma \in L$,

$\gamma = d_1\beta_1 + \cdots + d_s\beta_s \quad (d_i \in K)$

$= \left(\sum c_{1i}\alpha_i\right)\beta_1 + \cdots + \left(\sum c_{si}\alpha_i\right)\beta_1$

K_F

$= \sum c_{ji}\,\alpha_i\beta_j \qquad c_{ji} \in F$

$$\sum a_{ij}\alpha_i\beta_j = 0 \quad \Rightarrow \quad \text{Goal: } a_{ij} \equiv 0$$

$$\Big(\underbrace{\sum a_{i1}\alpha_i}_{K}\Big)\underline{\beta_1} + \cdots + \Big(\underbrace{\sum a_{is}\alpha_i}_{K}\Big)\underline{\beta_s} = 0$$

$$\Rightarrow \sum a_{ij}\alpha_i \equiv 0 \quad \forall j$$

$$\Rightarrow \underset{F}{a_{1j}}\,\underset{K}{\alpha_1} + \cdots + a_{mj}\alpha_m = 0 \qquad \underset{F}{K}$$

$$\Rightarrow a_{1j} = \cdots = a_{mj} = 0$$

类比 $V_1 \times \cdots \times V_m$ 的直和

Example

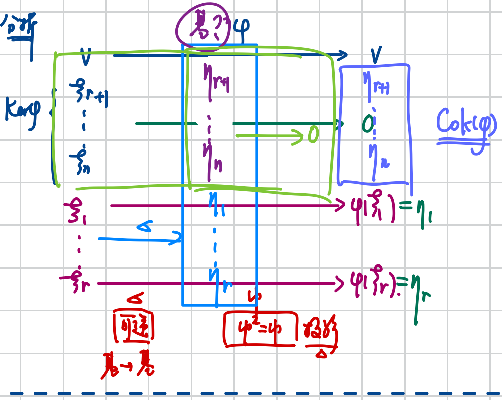
设 $\dim V = n$, $\varphi \in \mathrm{End}_F(V)$，则存在 $\psi,\sigma \in \mathrm{End}_F(V)$ 使得 $\varphi = \psi\sigma$，其中 $\psi^2 = \psi$，σ 是可逆变换。

Notes:

- 线性同构：转化为矩阵语言用相抵标准型
- 映射手段：从Kerφ开始扩基

[话一] 扩基

分析



Goal: $\boxed{\psi(\xi_i)=\xi_i}$

$\psi^2=\psi$

$\xi_1 \xrightarrow{\psi} \xi_1$

$\vdots$

$\xi_r \longrightarrow \xi_r$

$\left.\begin{matrix}\xi_{r+1}\\ \vdots \\ \xi_n\end{matrix}\right| \longrightarrow 0$

pf: Kerφ $\xi_{r+1},\cdots,\xi_n$，扩为 V 的一组基 $\xi_1,\cdots,\xi_r,\xi_{r+1},\cdots,\xi_n$.

则 $\varphi(\xi_1),\cdots,\varphi(\xi_r)$ 为 Imφ 的一组基

Let $\varphi(\xi_i)=\eta_i$ 再扩 $\eta_1,\cdots,\eta_r$ 扩为 V 的一组基

$\eta_1,\cdots,\eta_r,\eta_{r+1},\cdots,\eta_n$

构造映射 $\sigma: V\to V,\ \xi_i \longmapsto \eta_i \quad 1\le i\le n.$

$$\psi: V\to V,\quad \psi(\eta_i)=\begin{cases}0, & r+1\le i\le n\\ \eta_i, & 1\le i\le r.\end{cases}$$

则 验证 $\forall \alpha\in V,\ \varphi(\alpha)=\psi\sigma(\alpha)$ □

$$\alpha=\sum c_i\xi_i \qquad \varphi(\alpha)=\sum_{i=1}^{r} c_i\varphi(\xi_i)=\sum_{i=1}^{r} c_i\eta_i=$$

$$\underline{\varphi(\xi_1 \cdots \xi_n) = (\xi_1 \cdots \xi_n) A}$$

$$A = P\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} Q = \underbrace{P\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} P^{-1}}_{B^2=B} \underbrace{\boxed{PQ}}_{可}$$

BC, $\underline{B^2 = B}$, C 可逆

$\downarrow \ \downarrow$

$\varphi \ \ \sigma$

取 $\varphi(\xi_1 \cdots \xi_n) = (\xi_1 \cdots \xi_n) \underline{P\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} P^{-1}}$

$$\underline{\sigma(\xi_1 \cdots \xi_n) = (\xi_1 \cdots \xi_n) PQ.}$$

$$\Rightarrow \varphi\sigma(\xi_1 \cdots \xi_n) = (\xi_1 \cdots \xi_n) \underbrace{P\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} Q}_{A}$$

Example

设$\dim V=n$，$\varphi\in \mathrm{End}_F(V)$，且$\varphi^2=\mathscr{O}$，求证：存在$V$的一组基$\xi_1,\cdots,\xi_n$满足

$$\varphi(\xi_1,\cdots,\xi_n)=(\xi_1,\cdots,\xi_n)\begin{pmatrix} O & E_r & O \\ O & O & O \end{pmatrix}.$$

Note:

- 证明：$r\leqslant \dfrac{n}{2}$

$(r\leq n/2)$

$\dim \quad r \quad \leqslant \quad n-r$

- $\langle \varphi(\xi_1),\cdots,\varphi(\xi_r)\rangle \subseteq \mathrm{Ker}\,\varphi$

$(\mathrm{Ker}\,\varphi^2=V)$ $\quad \varphi^2\to\varphi|_{\mathrm{Im}\,\varphi}$

pf: [扩基]

- Claim: $\xi_1,\cdots,\xi_r,\ \varphi(\xi_1),\cdots,\varphi(\xi_r),\ \delta_1,\cdots,\delta_t$ 线性无关

$$\begin{array}{ccc}
\mathrm{Ker}\,\varphi & & \\
\xi_{r+1} & & \\
\vdots & & \\
\xi_n & & \\
\xi_1 & \longrightarrow & \varphi(\xi_1) \\
\vdots & & \vdots \\
\xi_r & \longrightarrow & \varphi(\xi_r) \\
V & & V
\end{array}$$

$\xrightarrow{\text{扩}}$ Kerφ的一组基.

$\varphi(\xi_1)$
$\vdots$
$\varphi(\xi_r)$
δ_1
$\vdots$
δ_t

$$\varphi(\xi_1, \underline{\varphi(\xi_1)}, \cdots, \xi_r, \varphi(\xi_r), \underbrace{\delta_1, \cdots, \delta_t}) = (\qquad\qquad)$$

$$\begin{pmatrix} \boxed{\begin{matrix} 0 & \\ 1 & 0 \end{matrix}} & & & \\ & \begin{matrix} 0 & \\ 1 & 0 \end{matrix} & & \\ & & \begin{matrix} 0 & \\ 1 & 0 \end{matrix} & \\ & & & \ddots \ 0 \ \ddots \ 0 \end{pmatrix}$$

$\begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix} = J_2(0)$ 2阶Jordan块

分块对角阵

$$\varphi[\langle \xi_1, \varphi(\xi_1) \rangle] \subseteq \langle \xi_1, \varphi(\xi_1) \rangle \rightsquigarrow \langle \xi_1, \varphi(\xi_1) \rangle \text{ 是 } \varphi\text{-子空间不变}$$