Example

在 $F^{2\times 2}$ 中，证明

$$A=\begin{pmatrix}3&1\\2&1\end{pmatrix},\quad B=\begin{pmatrix}6&2\\1&2\end{pmatrix}$$

线性无关，并扩为 $F^{2\times 2}$ 的一个基。

$$\det\begin{pmatrix}3&6&0&0\\1&2&1&0\\2&1&0&0\\1&2&0&1\end{pmatrix}=(-1)^{1+2+1+3}\det\begin{pmatrix}3&6\\2&1\end{pmatrix}\det E_2\neq 0$$

Example

设V_1为F上n维线性空间V的一个真子空间，且$\dim V_1=r$，证明：若存在k个$n-1$维子空间满足$U_1,\cdots,U_k$使得

$$\bigcap_{i=1}^{k}U_i=V_1$$

则$k\geqslant n-r$

pf: Claim: $\dim\left(\bigcap_{i=1}^{k}U_i\right)\geqslant n-k$

1° $\dim U_1=n-1\geqslant n-1$ ✓

2° $\dim\left(\bigcap_{i=1}^{k}U_i\right)=\dim\left(\left(\bigcap_{i=1}^{k-1}U_i\right)\cap U_k\right)=\dim\left(\bigcap_{i=1}^{k-1}U_i\right)+\dim U_k-\dim\left(\bigcap_{i=1}^{k-1}U_i+U_k\right)$

只需 $\bigcap_{i=1}^{k-1}U_i\neq U_k$

即有 $\bigcap_{i=1}^{k-1}U_i+U_k=V$ $\geqslant n-k+1+n-1-n=n-k$ (II)

(由 $\dim U_k=n-1$)

则 $n-k\leqslant r\Rightarrow k\geqslant n-r$

下面给出 $k=n-r$ 的构造. 设V_1的一组基 $\xi_1 \cdots \xi_r$ 扩为V的一组基

$\xi_1 \cdots \xi_r\ \xi_{r+1} \cdots \xi_n$

取 $U_i = \langle \{\xi_1 \cdots \xi_n\} \setminus \{\xi_{r+i}\} \rangle$ 即可

$\forall \alpha \in U_1 \cap U_2$,

$\alpha = a_1\xi_1 + \cdots a_r\xi_r + a_{r+2}\xi_{r+2} + \cdots + a_n\xi_n = b_1\xi_1 + \cdots + b_r\xi_r + b_{r+1}\xi_{r+1} + b_{r+3}\xi_{r+3} + \cdots + b_n\xi_n$

$\Rightarrow b_{r+1}=0,\ a_{r+2}=0$

所依次限制 α 位于 $U_1\cap U_2\cap U_3, \cdots \bigcap_{i=1}^{n-r} U_i \Rightarrow \xi_{r+3} \cdots \xi_n$ 上分量为0

$\Rightarrow \alpha = a_1\xi_1+\cdots+a_r\xi_r$ 即 $\bigcap_{i=1}^{n-r} U_i = \langle \xi_1 \cdots \xi_r\rangle = V_1$

另1: 由 V_1 为其子空间，记 $V\cong F^n$，$\varphi: V\to F^n$，将 α 映为其下的坐标，只证 $\varphi(V_1)$ 可以都写是某个 $AX=0$ 的解空间　设 $\varphi(V)$ 的一组基为 $\xi_1\cdots\xi_r$，则令 $B=(\xi_1\cdots\xi_r)_{n\times r}$，$r(B)=r$

$\underset{r\times n}{B^T}X=0$ 的基础解系中有 $n-r$ 个向量，记为 $\eta_1\cdots\eta_{n-r}$，记 $A=(\eta_1,\cdots,\eta_{n-r})_{n\times (n-r)}$

$\Rightarrow A^TX=0$ 的基础解系中有 r 个向量，且由 $B^T\eta_i = \begin{pmatrix}\xi_1^T\\ \vdots\\ \xi_r^T\end{pmatrix}\eta_i = 0 \Rightarrow \xi_j^T\eta_i=0$

$\Rightarrow \eta_i^T\xi_j=0$

即 $\begin{pmatrix}\eta_1^T\\ \vdots\\ \eta_{n-r}^T\end{pmatrix}\xi_j = A^T\xi_j=0 \Rightarrow \xi_1\cdots\xi_r$ 为 $A^TX=0$ 的一个基础解系

构造 $\eta_i^TX=0$ 的解空间为 $U_i \Rightarrow V_1=\bigcap_{i=1}^{n-r}U_i$

另2: 将 $U_1,\cdots,U_k$ 对应到 $C_1X=0,\cdots C_kX=0$ 考虑 $\begin{pmatrix}C_1\\ \vdots\\ C_k\end{pmatrix}X=0$ 的解空间 $\bigcap_{i=1}^{k} U_i$

$r(C_1)=\cdots=r(C_k)=1 \Rightarrow r\begin{pmatrix}C_1\\ \vdots\\ C_k\end{pmatrix}\le k$

$\Rightarrow \dim\left(\bigcup_{i=1}^{k} U_i\right) \geqslant n-k$ ∎

Example

设数域$F \subseteq K \subseteq L$，则在下面两个运算下$K$成为$F$上的线性空间：$F$中元素与$K$中元素的数量乘法，$K$中元素的加法。同理$L$也是$K$上的线性空间，如果$\dim_F K < \infty$，则记$\dim_F K = [K:F]$，证明：

$$[L:K][K:F] = [L:F]$$

pf: 设 $[K:F]=m$, $[L:K]=n$

${}_F K$ 的一组基为 $\alpha_1 - - \alpha_m$, ${}_K L$ 的一组基为 $\beta_1 - - \beta_n$

· $\forall x \in {}_F L$, $x = c_1\beta_1 + \cdots + c_n\beta_n \quad (c_i \in K)$

$= (c_{11}\alpha_1 + \cdots + c_{1m}\alpha_m)\beta_1 + \cdots + (c_{n1}\alpha_1 + \cdots + c_{nm}\alpha_m)\beta_n$

$= \underbrace{\sum_{i,j} c_{ji}\alpha_i\beta_j}$

· $\{\alpha_i\beta_j\}$ are linearly independent.

$a_{11}\alpha_1\beta_1 + \cdots + a_{1n}\alpha_1\beta_n + \cdots + a_{m1}\alpha_m\beta_1 + \cdots + a_{mn}\alpha_m\beta_n = 0$, $a_{ij} \in F$

$\Rightarrow \begin{cases} \underbrace{a_{11}\alpha_1 + \cdots + a_{1n}\alpha_1}_{\in K} = 0 \\ \vdots \\ \underbrace{a_{m1}\alpha_m + \cdots + a_{mn}\alpha_m}_{K} = 0 \end{cases} \Rightarrow a_{ij} = 0$

∎

Note: $[\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}] = 4$

or $\mathbb{Q}(\sqrt{2},\sqrt{3}) = \{a_0 + a_1\sqrt{2} + a_2\sqrt{3} + a_3\sqrt{6} \mid a_i \in \mathbb{Q}\}$ ✓ Got it!

Example

设$A \in M_n(F)$不可逆，$V_1 = \{X \in F^n \mid AX = O\}$，将$A$按列分块，得到$A = (A_1, \cdots, A_n)$，设$V_2 = \mathcal{L}(A_1, \cdots, A_n)$，证明：

$$\dim V_1 + \dim V_2 = \dim F^n = n$$

(Notes: 若使用 $\dim V_1 = n - r(A)$, $\dim V_2 = r(A)$ 此题为 trivial 的)

pf: 设 V_1 的一组基为 $\xi_1, \cdots, \xi_r$ 扩为 F^n 的一组基 $\xi_1 \cdots \xi_r \, \xi_{r+1} \cdots \xi_n$

Claim: $A\xi_{r+1}, \cdots A\xi_n$ 为 V_2 的一组基

- 首先 $V_2 = \{x_1 A_1 + \cdots + x_n A_n \mid \forall x_i \in F\} = \{AX \mid \forall X \in F^n\}$

  则 $\forall \alpha \in V_2$, $\alpha = A(c_1\xi_1 + \cdots + c_n\xi_n) = c_{r+1}A\xi_1 + \cdots + c_n A\xi_n$

- $\{A\xi_i\}_{i=r+1}^{n}$ are linearly independent.

  Consider $a_{r+1} A\xi_{r+1} + \cdots + a_n A\xi_n = 0 \Rightarrow A(a_{r+1}\xi_{r+1} + \cdots + a_n\xi_n) = 0$

  $\Rightarrow a_{r+1}\xi_{r+1} + \cdots + a_n\xi_n = a_1\xi_1 + \cdots + a_r\xi_r \Rightarrow a_i \equiv 0$ $\boxed{\text{证}}$

Notes:

- $V_1 \oplus V_2$ 一般不成立.

  Let $A = \begin{pmatrix} O_{n-r,r} & E_{n-r} \\ O_{r,r} & O_{r,n-r} \end{pmatrix}$ 则 $\varepsilon_1 \cdots \varepsilon_r$ 为一个基础解系 $V_1 = \langle \varepsilon_1, \cdots \varepsilon_r \rangle$

  $V_2 = \{AX \mid \forall X \in F^n\} = \langle \varepsilon_1, \cdots \varepsilon_{n-r} \rangle$ 显然 $V_1 \cap V_2 \neq \{0\}$ $\boxed{\text{证}}$

Chapter 4 $\rightsquigarrow$ Generally $\operatorname{Im}\varphi \oplus \operatorname{Ker}\varphi$ 不成立

Try

设 A, B 分别是 $m \times n$, $n \times s$ 矩阵。求证： F^n 的子空间

$$W = \{BX \mid ABX = 0\}$$

的维数等于 $\mathrm{r}(B) - \mathrm{r}(AB)$。

pf: 记 $\ker B = \{x \mid Bx = 0\}$, $\ker AB = \{x \mid ABx = 0\}$

Notice: $\ker B \subseteq \ker AB$

设 $\ker B$ 的一组基为 $\xi_1, \cdots, \xi_r$，扩为 $\ker AB$ 的一组基 $\xi_1, \cdots, \xi_r, \xi_{r+1}, \cdots, \xi_s$

$\Rightarrow W = \langle B\xi_1, \cdots, B\xi_s \rangle$ (why?) $= \langle B\xi_{r+1}, \cdots, B\xi_s \rangle$

下证 $\{B\xi_i\}_{i=r+1}^{s}$ are linearly independent.

Consider $C_{r+1}B\xi_{r+1} + \cdots + C_sB\xi_s = B(C_{r+1}\xi_{r+1} + \cdots + C_s\xi_s) = 0$

$\Rightarrow C_{r+1}\xi_{r+1} + \cdots + C_s\xi_s = + C_1\xi_1 + \cdots + C_r\xi_r \Rightarrow C_i \equiv 0$

从而 $\dim W = r - s = \dim \ker AB - \dim \ker B$

$= n - r(AB) - n + r(B) = r(B) - r(AB)$

$\blacksquare$

Notes: Let $B|_{\ker AB}: \ker AB \longrightarrow \mathrm{Im} B$

$$\begin{cases} \ker\left(B|_{\ker AB}\right) = \ker B \subseteq \ker AB \\ \mathrm{Im}\left(B|_{\ker AB}\right) = \mathrm{Im} B \cap \ker A \end{cases}$$

$= W$

$\Rightarrow \dim(\ker AB) = \dim \ker B + \dim(\mathrm{Im} B \cap \ker A)$

Try

设 $\{\alpha_1, \alpha_2, \cdots, \alpha_p\}$ 是 $\mathbb{K}^m$ 中 m 个线性无关的 m 维列向量，$\{\beta_1, \beta_2, \cdots, \beta_q\}$ 是 $\mathbb{K}^n$ 中 n 个线性无关的 n 维列向量。证明：

$$\{\alpha_i \beta_j^\mathsf{T} \mid 1 \le i \le m, 1 \le j \le n\}$$

是$\mathbb{K}^{m\times n}$的一组基。

pf: Consider $\sum_{i,j} c_{ij} \alpha_i \beta_j^T = 0$

Let $\alpha_i = (a_{i1}, \cdots, a_{in})^T$, $\beta_j = (b_{j1} \cdots b_{jn})^T$

$\Rightarrow \left(\sum_{i,j} c_{ij} \alpha_i b_{j1}, \cdots, \sum_{i,j} c_{ij} \alpha_i b_{jn} \right) = 0$

$\Rightarrow \underbrace{\sum_{j=1}^{n} c_{ij} b_{jk} = 0}$, $\forall i, k$. $\Rightarrow$ Let $C = (c_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$

$\Rightarrow CB = 0$ $\because B$ 可逆 (why?) $\Rightarrow C = 0$

Example

Suppose $r = 2,\ t = 2$ and we figure out $X = k(1, 2, 3, 4)^\mathsf{T}$

Can you give a basis of $V_1 \cap V_2$? (Use the form of α_i or β_j)

$\alpha_1 + 2\alpha_2$ 或 $3\beta_1 + 4\beta_2$

Try

设V是$n(n \geqslant 3)$维线性空间，U, W为V的两个子空间，且$\dim U = n-1$，$\dim \cancel{V}\ W = n-2$，则$\dim(U \cap W) = ?$

(19 Final)

- $W \subseteq U$ $\Rightarrow \dim(U+W) = n-1$ $\dim(U \cap W) = 2n-3-n+1 = n-2$
- $W \nsubseteq U$ $\Rightarrow \dim(U+W) = n$ $\dim(U \cap W) = n-3$

Example

设 $\alpha_1 = (1, 0, -1, 0)$，$\alpha_2 = (0, 1, 2, 1)$，$\alpha_3 = (2, 1, 0, 1)$ 是四维实向量空间 V 中的向量，它们生成的子空间为 V_1，又向量

$$\beta_1 = (-1, 1, 1, 1),\ \beta_2 = (1, -1, -3, -1),\ \beta_3 = (-1, 1, -1, 1)$$

生成的子空间为 V_2，求子空间 $\underline{V_1 + V_2}$ 和 $\underline{V_1 \cap V_2}$ 的基。

$\dim V_1 = \dim V_2 = 2$　　3　　1

$$\begin{pmatrix} 1 & 0 & 2 & -1 & 1 & -1 \\ 0 & 1 & 1 & 1 & -1 & 1 \\ \cancel{-1}\,0 & 2 & \cancel{0}\,2 & \cancel{1}\,0 & \cancel{-3}\,-2 & \cancel{-1}\,-2 \\ 0 & 1 & 1 & 1 & -1 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & \cancel{0}\,1 & \cancel{2}\,3 & \cancel{-1}\,0 & \cancel{1}\,0 & \cancel{-1}\,0 \\ 0 & 1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

✓　✓　　✓　✓

$\boxed{\alpha_1\ \alpha_2\ \beta_1\ (或\ \beta_3)}$

$\underbrace{V_1 \cap V_2}$

$$\begin{array}{cccc} c_1 & c_2 & -k_1 & -k_2 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & \cancel{1}\,0 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} c_1 + c_2 = 0 \\ c_2 = -k_2 \\ -k_1 = 0 \end{cases}$$

$\boxed{\alpha_1 - \alpha_2}$ 或 $\boxed{\beta_2}$

Try

在 $F^{2\times 2}$ 中，记

$$V_1 = \left\{ \begin{pmatrix} a+b & b \\ 0 & a+b \end{pmatrix} \middle| a, b \in F \right\}$$

$$V_2 = \left\{ \begin{pmatrix} a+b-c & a+b-c \\ -a+b+2c & a+b-c \end{pmatrix} \middle| a, b, c \in F \right\}.$$

写出 $V_1 + V_2$ 和 $V_1 \cap V_2$ 的一个基。

$\underline{\alpha_1\ \alpha_2\ \beta_1}$　α_2 或 $\underline{\beta_1+\beta_2}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 2 \\ 1 & 1 & 1 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{cccc} c_1 & c_2 & -k_1 & -k_2 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} c_1 = 0 \\ c_2 = k_1 + k_2 \\ \underbrace{k_1 = k_2} \end{array}$$

$$V_1+V_2+\cdots+V_n=V_1\oplus V_2\oplus\cdots\oplus V_n=\bigoplus_{i=1}^{n}V_i$$

任意元素分解唯一 $\Leftrightarrow$ 0元素分解唯一

$1^\circ \Leftrightarrow V_i\cap\left(\bigcup_{j\neq i}V_j\right)=\{\mathbf{0}\},i=1,\cdots,n$

直和 $\Rightarrow 1^\circ$ $\forall\alpha\in V_i\cap\left(\bigcup_{j\neq i}V_j\right)$ $\alpha=\alpha_i=\sum_{j\neq i}\alpha_j \Rightarrow \alpha_i-\sum_{j\neq i}\alpha_j=0\Rightarrow\alpha_i=0$

$2^\circ \Leftrightarrow V_i\cap\left(\bigcup_{j=1}^{i-1}V_j\right)=\{\mathbf{0}\},\ i=2,\cdots,n$

$1^\circ\Rightarrow$ 直和 $\alpha=\alpha_i+\sum_{j\neq i}\alpha_j=\alpha_i'+\sum_{j\neq i}\alpha_j'\ (\alpha_i\neq\alpha_i')$

$3^\circ \Leftrightarrow \dim\left(\sum_{i=1}^{n}V_i\right)=\sum_{i=1}^{n}\dim V_i$

$\Rightarrow \alpha_i-\alpha_i'=\sum_{j\neq i}\alpha_j'-\sum_{j\neq i}\alpha_j\in V_i\cap\left(\bigcup_{j\neq i}V_j\right)=\{0\}$

矛盾!

$$\rightsquigarrow\sum_{i=1}^{m}\dim V_i=\dim\left(\sum_{i=1}^{m}V_i\right)+\sum_{i=2}^{m}\dim\left(V_i\bigcap\sum_{j=1}^{i-1}V_j\right)$$

$1^\circ\Rightarrow2^\circ$ ✓ $2^\circ\Rightarrow1^\circ$ $\forall\alpha\in V_i\cap\left(\bigcup_{j\neq i}V_j\right)$, $\alpha=\alpha_i=\alpha_1+\alpha_2+\cdots+\alpha_{i-1}+\alpha_{i+1}+\cdots+\alpha_n$

$\Rightarrow \alpha_1+\cdots+\alpha_{i-1}-\alpha_i+\alpha_{i+1}+\cdots+\alpha_n=0$

从 α_n 到 α_1 选取第1个 $\alpha_k\neq0$ 不妨 $k>i$

$\Rightarrow \alpha_1+\cdots+\alpha_{i-1}-\alpha_i+\cdots+\alpha_k=0$

$\Rightarrow -\alpha_k=\alpha_1+\cdots+\alpha_{i-1}-\alpha_i+\cdots+\alpha_{k-1}\in V_k\cap\left(\bigcup_{i=1}^{k-1}V_i\right)=\{0\}$

矛盾!

$2^\circ\Leftrightarrow3^\circ$ 维数公式即可

pf: $$\rightsquigarrow\sum_{i=1}^{m}\dim V_i=\dim\left(\sum_{i=1}^{m}V_i\right)+\sum_{i=2}^{m}\dim\left(V_i\bigcap\sum_{j=1}^{i-1}V_j\right)$$

$m=1,2$ ✓ $m-1$ 时 ✓ 对 m 时,

$$\dim\left(\sum_{i=1}^{m-1}V_i+V_m\right)+\sum_{i=2}^{m-1}\dim\left(V_i\cap\sum_{j=1}^{i-1}V_j\right)+\dim\left(V_m\cap\sum_{j=1}^{m-1}V_j\right)$$

$$=\underline{\dim\left(\sum_{i=1}^{m-1}V_i\right)}+\dim V_m-\underbrace{\dim\left(\left(\sum_{i=1}^{m-1}V_i\right)\cap V_m\right)+\dim\left(V_m\cap\sum_{j=1}^{m-1}V_j\right)}_{0}+\underline{\sum_{i=2}^{m-1}\dim\left(V_i\cap\sum_{j=1}^{i-1}V_j\right)}$$

归纳 $=\dim V_1+\cdots+\dim V_{m-1}$

$\square$

Example

设 $A_i \in F^{m\times n}$，齐次线性方程组 $A_iX = O$ 的解空间是 V_i $(i = 1, 2)$。证明：

$$V_1 \oplus V_2 = F^n \Leftrightarrow \qquad r\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = r(A_1) + r(A_2) = n.$$

pf:

$V_1 \cap V_2 = 0 \Leftrightarrow \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} X = 0$ 只有0解 即 $r\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = n$

$\dim V_1 + \dim V_2 = n \Leftrightarrow n - r(A_1) + n - r(A_2) = n$

Try

$A \in F^{n\times n}$ 且 A 为可逆矩阵，$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$。齐次线性方程组 $A_iX = O$ 的解空间是 V_i $(i = 1, 2)$。证明：

$$F^n = V_1 \oplus V_2.$$

pf: $r\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = n$ ✓ 只证 $r(A_1) + r(A_2) = r\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ 由 A 为可逆阵 行向量线性无关即得 ✓ □

Example

设A为n阶方阵，证明：$A^2 - A - 2E = O \Leftrightarrow \cancel{V}\ F^n = \mathrm{Ker}(A + E) \oplus \mathrm{Ker}(A - 2E)$.

pf:

1° $\forall x \in F^n$, $x = \frac{1}{3}(A+E)x + \frac{1}{3}(2E-A)x$

而 $\frac{1}{3}(A+E)x \in \mathrm{Ker}(A-2E)$, $\frac{1}{3}(2E-A)x \in \mathrm{Ker}(A+E)$

即 $V = \mathrm{Ker}(A-2E) + \mathrm{Ker}(A+E)$

2° $\forall x \in \mathrm{Ker}(A-2E) \cap \mathrm{Ker}(A+E)$

$(A-2E)x = 0 \quad (A+E)x = 0 \quad 3x = (A+E)x + (2E-A)x = 0 \Rightarrow x = 0$

3° To show $r(A-2E) + r(A+E) = n$

$$\begin{pmatrix} A-2E & \\ & A+E \end{pmatrix} \to \begin{pmatrix} A-2E & \\ 3E & A+E \end{pmatrix} \to \begin{pmatrix} A-2E & (A-2E)\left[-\frac{1}{3}(A+E)\right] \\ 3E & 0 \end{pmatrix} \to \begin{pmatrix} 0 & (A-2E)(A+E) \\ E & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix} \quad ✓$$

法2: only to show $\exists u(x), v(x)$, $f(x)u(x) + v(x)g(x) = 1$.

By properties of gcd.

Bézout's Theorem. □

Thm 6

设V为n维线性空间，$V_i\ (i=1,\cdots,n)$为V的两两不同的非平凡子空间，求证：

(1) $\exists \alpha \in V, \alpha \notin \bigcup_{i=1}^{2} V_i$

(2) $\exists \beta \in V, \beta \notin \bigcup_{i=1}^{n} V_i$

Hint: Consider the vectors in the set

$$S=\{\xi_1+j\xi_2+j^2\xi_3+\cdots+j^{n-1}\xi_n \mid j=1,2,\cdots\}$$

pf: (1). 若 $V_1 \subseteq V_2$ 或 $V_2 \subseteq V_1$ 取 $\alpha \in V \setminus V_1$ 或 $V \setminus V_2$ 即可

· 若 V_1 与 V_2 无包含关系 取 $\alpha = \alpha_1 + \alpha_2$, $\alpha_1, \alpha_2 \neq 0$, $\alpha_1 \in V_1 \setminus V_2$, $\alpha_2 \in V_2 \setminus V_1$

Claim: $\alpha \notin V_1$, $\alpha \notin V_2$ 否则 若 $\alpha \in V_1 \Rightarrow \alpha_2 = \alpha - \alpha_1 \in V_1$ 矛盾!

(2) [法一] 归纳

[法二] Hint

Try

设 n 维空间 V 的两个子空间 V_1, V_2 的维数均为 m，且 $m < n$，求使得

$$V = V_1 \oplus U = V_2 \oplus U$$

的 U 的最大维数 k，并构造 U.

$n-m$　　取 $\alpha \notin V_1 \cup V_2$ 起始构造即可

Example

设 A 为数域 $\mathbb{K}$ 上的 $n(n>1)$ 阶方阵，$r(A) = n-1$，A^* 是 A 的伴随矩阵。记齐次线性方程组 $Ax = 0$ 的解空间为 V_A，$A^*x = 0$ 的解空间为 V_{A^*}。证明：

$$\mathbb{K}^n = V_A \oplus V_{A^*}$$

成立的充要条件是 $\operatorname{tr}(A^*) \neq 0$.

pf: ($\Rightarrow$) If $tr(A^*) = 0$　令 $A^* = \alpha\beta^T = (k_1\alpha, \cdots, k_n\alpha)$

则 $\beta^T\alpha = 0$　对 $x \in V_A$, $x = k\alpha$, $A^*x = k\alpha\beta^T\alpha = 0$

$\Rightarrow$ $V_A \cap V_{A^*} = \langle\alpha\rangle \neq \{0\}$

($\Leftarrow$) $\dim K^n = \dim V_A + \dim V_{A^*}$ ✓

$\forall x \in V_A \cap V_{A^*}$,　$x = k\alpha$,　$A^*x = k\alpha\beta^T\alpha = k\,tr(A^*)\alpha = 0$

又 $\alpha \neq 0 \Rightarrow k = 0$　即 $V_A \cap V_{A^*} = \{0\}$