Review II of Chapter 4: Linear Mapping

illusion

Especially made for smy

School of Mathematical Science

XMU

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例1： $F^{n\times n}$ 按矩阵的数乘和以下定义的加法，有 __0__ 个构成 F 上的线性空间？

(1) $A\oplus B=AB$; ~> 交换 ×

(2) $A\oplus B=A+B^T$; ~> 交换结合 ×

(3) $A\oplus B=AB-BA$; ~> 交换 ×

(4) $A\oplus B=A^T+B$. ~> 交换结合 ×

保 保
Linear + multi + 1-1 + onto
数乘
‖
Algebraic Isomorphism

? $\boxed{H(\mathrm{id}_V)=E_n}$?

例2： 设 V 是 n 维线性空间，$\xi_1,\xi_2,\dots,\xi_n$ 和 $\xi_1',\xi_2',\dots,\xi_n'$ 分别是 V 的两个基，且从 $\xi_1,\xi_2,\dots,\xi_n$ 到 $\xi_1',\xi_2',\dots,\xi_n'$ 的过渡矩阵为 P。设 V 上可逆线性变换 φ 在 $\xi_1,\xi_2,\dots,\xi_n$ 下的矩阵为 A，则 $\varphi^3+3\varphi^{-1}+\mathrm{id}_V$ 在 $\xi_1',\xi_2',\dots,\xi_n'$ 下的矩阵为 $P^{-1}(A^3+3A^{-1}+E_n)P$ $\xi_1\cdots\xi_n$

$A^3+3A^{-1}+E_n$

$\varphi(\xi_1\cdots\xi_n)=(\xi_1\cdots\xi_n)A$

$(\xi_1'\cdots\xi_n')=(\xi_1'\cdots\xi_n')P^{-1}P$

Try

设 $\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ 是 F^m 的一组基，$\{\beta_1, \beta_2, \cdots, \beta_n\}$ 是 F^n 的一组基。证明：

$$\{\alpha_i \beta_j^\mathsf{T} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$$

是 $F^{m\times n}$ 的一组基。

Notes:

- (Chapter 4 Review B Ex.6) 设 $A, B \in F^{n\times n}$ 是取定的矩阵。对任意的 $X \in F^{n\times n}$，令 $\sigma(X) = AXB$. 求证 σ 可逆 $\Leftrightarrow \det(AB) \neq 0$.
- (FDU 2024) 若修改 $A \in F^{m\times n}, X \in F^{n\times q}, B \in F^{q\times l}$，上述充要条件该如何修改呢？
- 你能给出几种证明方式？如何求 σ^{-1}？

Try

设 $\{\alpha_1,\alpha_2,\cdots,\alpha_m\}$ 是 F^m 的一组基，$\{\beta_1,\beta_2,\cdots,\beta_n\}$ 是 F^n 的一组基。证明：

$$\{\alpha_i\beta_j^{\mathsf{T}} \mid 1\le i\le m, 1\le j\le n\}$$

是 $F^{m\times n}$ 的一组基。

√ 1° 线性无关

2° 线性表出 ? ✗

⟺ Way2: $\boxed{\dim F^{m\times n} = \text{个数}}$ √ 已介绍过

线性无关 ? √

pf:

Note: $\dim V=n$，$\alpha_1,\cdots,\alpha_n\in V$ 线性无关 $\Rightarrow$ $\alpha_1,\cdots,\alpha_n$ 是基

pf: $\forall\beta\in V$，若 β 不可被 $\alpha_1,\cdots,\alpha_n$ 线性表出

$$c_0\beta+c_1\alpha_1+\cdots+c_n\alpha_n=0$$

Claim: $c_0=0$ 否则 $c_0\neq0$，$\beta=-\frac{c_1}{c_0}\alpha_1-\cdots-\frac{c_n}{c_0}\alpha_n$ 与假设矛盾

$\Rightarrow c_1\alpha_1+\cdots+c_n\alpha_n=0 \Rightarrow c_1=\cdots=c_n=0$

$\Rightarrow \beta,\alpha_1,\cdots,\alpha_n$ 线性无关 $\Rightarrow \dim V\ge n+1$ 矛盾

从而 β 可被 $\alpha_1,\cdots,\alpha_n$ 线性表出 Way1 ⟺ Way2 □

pf: $c_{11}\alpha_1\beta_1^T + c_{12}\alpha_1\beta_2^T + \cdots\cdots c_{mn}\alpha_m\beta_n^T = 0$

$\boxed{\text{Goal: } c_{ij} = 0}$ 令 $\beta_j^T = (b_{1j}, b_{2j}, \cdots, b_{nj})$

$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\alpha_i\beta_j^T = 0 \rightsquigarrow \Big(\underbrace{\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}b_{1j}\alpha_i, \cdots, \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}b_{nj}\alpha_i}_{=0}\Big)$

$\forall\ 1 \le k \le n,\quad \sum_{i=1}^{m}\sum_{j=1}^{n} \boxed{c_{ij}b_{kj}}\,\alpha_i = 0$

对 i

$\Rightarrow \sum_{j=1}^{n} c_{1j}b_{kj}\alpha_1 + \sum_{j=1}^{n} c_{2j}b_{kj}\alpha_2 + \cdots + \sum_{j=1}^{n} c_{mj}b_{kj}\alpha_m = 0$

$\alpha_1 \cdots \alpha_m$ 线性无关 $\Rightarrow \forall\ 1 \le k \le n,\ 1 \le i \le m,\ \sum_{j=1}^{n} c_{ij}b_{kj} = 0$

Let $C = (c_{ij})_{m\times n}$ $\quad B = (b_{kj})_{n\times n} = \begin{pmatrix}\beta_1^T \\ \vdots \\ \beta_n^T\end{pmatrix}$

$CB=0$ $\quad \beta_1 \cdots \beta_n$ 线性无关 $\Rightarrow B$ 可逆

$\Rightarrow C=0$. 即 $c_{ij}=0$. $\square$.

- (Chapter 4 Review B Ex.6) 设 $A, B \in F^{n\times n}$ 是取定的矩阵。对任意的 $X \in F^{n\times n}$，令 $\sigma(X) = AXB$. 求证 σ 可逆 $\Leftrightarrow \det(AB) \neq 0$.

transformation

•[Way I] 用上述结论 $\quad$ σ 可逆 $\Leftrightarrow$ 把基变为基

σ 同构 + 线性

$$\{\varepsilon_{ij}\} \xrightarrow{\sigma} \underbrace{\{\alpha_i\beta_j^T\}}_{\sigma}$$

$\varepsilon_{ij}=\varepsilon_i\varepsilon_j^T$ $\quad A=(\alpha_1 \cdots \alpha_n)$, $B=(\beta_1 \cdots \beta_n)^T$

$\sigma(\varepsilon_{ij}) = A(\varepsilon_i\varepsilon_j^T)B = (A\varepsilon_i)(\varepsilon_j^TB) = \alpha_i\beta_j^T$

$\Leftarrow$ A, B 可逆

$\{\alpha_1 \cdots \alpha_n\}$ $\{\beta_1 \cdots \beta_n\}$ F^n 一组基

$\Rightarrow \{\alpha_i\beta_j^T\}$ 为 $F^{n\times n}$ 一组基.

$\Rightarrow \sigma$ 把基变成基

$\Rightarrow \sigma$ 可逆

$\Rightarrow$ 已知 $\{\alpha_i\beta_j^T\}$ 为基

$\{\alpha_i\}$? ✓ | $\{\beta_j\}$? ✓

$\varphi: F^n \to F^{n\times n}$

$X \mapsto X\beta_1^T$

$\varphi(\alpha_1) \cdots \varphi(\alpha_1)$ 线性无关

$\Rightarrow \alpha_1 \cdots \alpha_n$ 线性无关

$\alpha_1\beta_1^T, \cdots \alpha_n\beta_1^T$ 线性无关

$c_1\alpha_1 + \cdots + c_n\alpha_n = 0$

$(c_1\alpha_1 + \cdots + c_n\alpha_n)\beta_1^T = 0$

$\Rightarrow c_1 = \cdots = c_n = 0.$

· [way II] 映射.

已知 $\det(AB) \neq 0$

A, B 可逆

$F^{n\times n} \to F^{n\times n}$

待证 $\sigma: X \mapsto AXB$ 可逆

(a) 单+满

(b) 直接构造逆

Check: $\sigma^{-1}\sigma = \sigma\sigma^{-1} = id_{F^{n\times n}}$ $\quad \sigma^{-1}: X \mapsto A^{-1}XB^{-1}$

⇒ 已知可逆，待证 A，B可逆

反证 若 A，B之一不可逆　　$\sigma: X \mapsto AXB$

$\hookrightarrow F^{n\times n}$

- A不可逆　　$AX=0$ 有非零解 $x_0 \in F^n$

令 $X_1 = (x_0, 0, \cdots, 0) \in F^{n\times n}$　$X_1 \neq 0$

$\sigma(X_1) = 0 = \sigma(0)$ 与单射矛盾！

- B不可逆　　$B^T X = 0$ 有非零解 $x_2 \Rightarrow B^T x_2 = 0$　$x_2 \in F^n$

注意B转置！　　$\Rightarrow x_2^T B = 0$　$x_2^T \in F^n$

令 $X_3 = \begin{pmatrix} x_2^T \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in F^{n\times n} \Rightarrow \sigma(X_3) = \sigma(0) = 0$ 矛盾！

$0 \neq$

$\Rightarrow$ 只能 A，B均可逆　∎.

- (Chapter 4 Review B Ex.6) 设 $A, B \in F^{n\times n}$ 是取定的矩阵。对任意的 $X \in F^{n\times n}$，令 $\sigma(X) = AXB$. 求证 σ 可逆 $\Leftrightarrow \det(AB) \neq 0$.
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- 你能给出几种证明方式？如何求 σ^{-1}?

?

σ可逆 $\Leftrightarrow r(A)=n,\ r(B)=q.$

$m \geq n$

[Way I] 同上

[Way II]

$\Leftarrow$ 构造σ^{-1}?

Chapter I Review C Ex

列满秩矩阵能通过行初等变换变为标准形.

$$A = P\begin{pmatrix} E_n \\ 0 \end{pmatrix} Q = P\begin{pmatrix} Q \\ 0 \end{pmatrix} = \underbrace{P_{m\times m}\begin{pmatrix} Q & 0 \\ 0 & E_{m-n} \end{pmatrix}}_{\text{可逆}} \begin{pmatrix} E_n \\ 0 \end{pmatrix} \triangleq \tilde{P}\begin{pmatrix} E_n \\ 0 \end{pmatrix}$$

$\tilde{P}$ 可逆

$\blacksquare$

$$A = \tilde{P}\begin{pmatrix} \varepsilon_n \\ 0 \end{pmatrix}$$

Goal: U? $UA = \varepsilon_n$

$$U = (\varepsilon_n \ 0)\tilde{P}^{-1}$$

$$\sigma^{-1}: \underset{F^{m\times l}}{X} \to \underbrace{\underset{F^{n\times g}}{UXV}}_{m\times n}$$

$\underline{U_{n\times m}}$ $\quad V_{l\times g}$.

$$\sigma: \underset{F^{n\times g}}{X} \to \underset{F^{m\times l}}{\boxed{AXB}}$$

$$\sigma^{-1}\sigma(X) = \underbrace{UA}_{\varepsilon_n}X\underbrace{BV}_{\varepsilon_m} = X$$

V $\quad$ Goal: V? $BV = \varepsilon_m$

$$B = (\varepsilon_m \ 0)\tilde{Q} \qquad V = \tilde{Q}^{-1}\begin{pmatrix} \varepsilon_m \\ 0 \end{pmatrix}$$

$\Rightarrow$ $\underset{m\times n}{A}$ 非列满秩 $\quad A\underset{\in F^n}{x_n} = 0$ 仍有非0解

基础解系个数 ≥ 1.

$$n - \underline{r(A)} \geq n - (n-1) = 1 \checkmark$$

Review I: Properties of Homomorphic Mapping

设 $\varphi \in \mathrm{Hom}_F(V, U)$，且 V_1, V_2为V的子空间，U_1, U_2 为U的子空间，则

- V中线性相关，线性表出 $\Rightarrow$ $\mathrm{Im}\varphi \subseteq U$中线性相关，线性表出
- $\mathrm{Im}\ \varphi|_{V_1} = \varphi(V_1)$ 是U的子空间
- $\rightsquigarrow V_1 = V,\ \mathrm{Im}\ \varphi|_{V_1} = \mathrm{Im}\ \varphi$ ✓
- $\varphi^{-1}(U_1) := \{\alpha \in V \mid \varphi(\alpha) \in U_1\}$ 为V的子空间
- $\rightsquigarrow U_1 = \{\mathbf{0}\}, \varphi^{-1}(U_1) = \mathrm{Ker}\varphi$ ✓
- $\varphi(V_1 + V_2) = \varphi(V_1) + \varphi(V_2)$
- $\varphi^{-1}(U_1 \cap U_2) = \varphi^{-1}(U_1) \cap \varphi^{-1}(U_2)$

$\forall \alpha \in \varphi^{-1}(U_1 \cap U_2)$

$\Rightarrow \varphi(\alpha) \in U_1 \cap U_2 \Rightarrow \varphi(\alpha) \in U_1$ 且 $\varphi(\alpha) \in U_2$

$\Rightarrow \alpha \in \varphi^{-1}(U_1)$, $\alpha \in \varphi^{-1}(U_2)$

反之 $\forall \beta \in \varphi^{-1}(U_1) \cap \varphi^{-1}(U_2)$

$\varphi(\beta) \in U_1$ 且 $\varphi(\beta) \in U_2$

$\Rightarrow \varphi(\beta) \in U_1 \cap U_2 \Rightarrow \beta \in \varphi^{-1}(U_1 \cap U_2)$

Try

When $\varphi^{-1}(U_1 + U_2) = \varphi^{-1}(U_1) + \varphi^{-1}(U_2)$? $\Leftrightarrow$ $\rightsquigarrow$ onto ✓

Try

When $\varphi^{-1}(U_1+U_2)=\varphi^{-1}(U_1)+\varphi^{-1}(U_2)$?

Goal: $\rightsquigarrow$ onto ✓

pf:

- $\varphi^{-1}(U_1)+\varphi^{-1}(U_2)\subseteq\varphi^{-1}(U_1+U_2)$

$\forall \alpha_1+\alpha_2\in\varphi^{-1}(U_1)+\varphi^{-1}(U_2)$, $\alpha_i\in\varphi^{-1}(U_i)\Rightarrow\varphi(\alpha_i)\in U_i$

$\Rightarrow\varphi(\alpha_1)+\varphi(\alpha_2)\in U_1+U_2$ $\quad \varphi\in\mathcal{L}(V,U)$

$\Rightarrow\varphi(\alpha_1+\alpha_2)\in U_1+U_2\Rightarrow\alpha_1+\alpha_2\in\varphi^{-1}(U_1+U_2)$ ✓

$\boxed{\text{反例}}$

- $\varphi^{-1}(U_1+U_2)\subseteq\varphi^{-1}(U_1)+\varphi^{-1}(U_2)$?

$\forall\gamma\in\varphi^{-1}(U_1+U_2)$, $\varphi(\gamma)\in U_1+U_2$

分解 $\varphi^{-1}(U_1)+\varphi^{-1}(U_2)$?

φ 的分解 $\varphi(\gamma)=\gamma_1+\gamma_2$ 【且 $\gamma_1,\gamma_2\notin\mathrm{Im}\varphi$】

满射 √ $\quad \mathrm{Im}\,\varphi = V \qquad \varphi(\alpha) = \gamma_1 \quad \varphi(\beta) = \gamma_2$.

$$\varphi(\gamma) = \varphi(\alpha) + \varphi(\beta) = \varphi(\alpha + \beta)$$

$$\varphi(\alpha) \in U_1 \Rightarrow \alpha \in \varphi^{-1}(U_1)$$

$$\varphi(\beta) \in U_2 \Rightarrow \beta \in \varphi^{-1}(U_2)$$

$$\Rightarrow \exists\, \delta \in \mathrm{Ker}\,\varphi, \quad \gamma = \alpha + \beta + \delta$$

$$= \alpha + (\beta + \delta)$$

$\alpha \to \varphi^{-1}(U_1)$ √ $\qquad \beta + \delta \to \varphi^{-1}(U_2)$, [子空间]

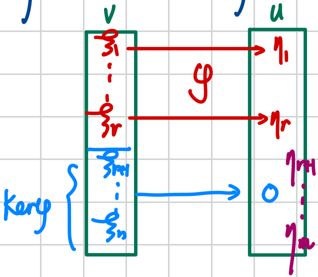
$\varphi^{-1}(0) = \mathrm{Ker}\,\varphi$

$\varphi^{-1}(0) \subseteq \varphi^{-1}(U_2)$

$\delta \in \mathrm{Ker}\,\varphi \subseteq \varphi^{-1}(U_2)$

$\beta + \delta \in \varphi^{-1}(U_2)$

反例 $\varphi^{-1}(U_1+U_2) \subseteq \varphi^{-1}(U_1)+\varphi^{-1}(U_2)$ 不一定成立.



$\ker\varphi$

φ^{-1} $U_1 = \langle \eta_1+\eta_{r+1}\rangle$

$\varphi^{-1}(U_2) = \langle \eta_2-\eta_{r+1}\rangle$

$\varphi^{-1}(U_1)+\varphi^{-1}(U_2) = \ker\varphi$

$\boxed{U_1+U_2}$ $\langle \eta_1+\eta_{r+1},\ \eta_2-\eta_{r+1}\rangle$

$\eta_1+\eta_2$

$c_1(\eta_1+\eta_{r+1})+c_2(\eta_2-\eta_{r+1}) \Rightarrow c_1(\eta_1+\eta_2)$

$c_1=c_2$

将 $\eta_1+\eta_2$ 视为 U_1+U_2 内一元素

$\eta_1+\eta_2 \neq c\eta_{r+1}$

$\dim \varphi^{-1}(U_1+U_2) \geq \dim \ker\varphi + 1.$

$\xi_1+\xi_2.$

Review II: Properties of Isomorphic Mapping

Slogan

$$\varphi \in \mathrm{Hom}_F(V,U), V/\mathrm{Ker}\varphi \cong \mathrm{Im}\varphi$$

φ is **1-1** $\rightsquigarrow V \cong \mathrm{Im}\varphi$

- V中线性相关，无关，表出，基 $\Leftrightarrow$ $\mathrm{Im}\varphi$中线性相关，无关，表出，基
- V_1为r维子空间 $\Rightarrow$ $\varphi(V_1)$为r维子空间
- $\varphi(V_1 \cap V_2) = \varphi(V_1) \cap \varphi(V_2)$
- $V = V_1 \oplus V_2$, $\mathrm{Im}\varphi = \varphi(V_1) \oplus \varphi(V_2)$

$V \longrightarrow U$

$\varphi: \varphi^{-1}(U_1) \longrightarrow U$ $\left\{ \begin{array}{l} \mathrm{Ker}\varphi \ \checkmark \\ \mathrm{Im}\varphi \cap U_1 \end{array} \right.$

Try

φ is **1-1**, U_1为U的r维子空间，则 $\dim \varphi^{-1}(U_1) = \dim(\mathrm{Im}\varphi \cap U_1) \leqslant r$

$\dim \varphi^{-1}(U_1) = \dim \mathrm{Ker}\varphi + \dim(\mathrm{Im}\varphi \cap U_1) = 0 + \dim(\mathrm{Im}\varphi \cap U_1) \leqslant \dim U_1 = r.$

Review II: Properties of Isomorphic Mapping

Slogan

$$\varphi \in \mathrm{Hom}_F(V, U), V/\mathrm{Ker}\varphi \cong \mathrm{Im}\varphi$$

φ is **1-1 and onto** $\rightsquigarrow V \cong U$

- V中线性相关，无关，表出，基 $\Leftrightarrow$ U中线性相关，无关，表出，基
- $V = V_1 \oplus V_2,\ U = \varphi(V_1) \oplus \varphi(V_2)$
- U_1为U的r维子空间，则$\varphi^{-1}(U_1)$也是U的r维子空间
- $\varphi^{-1}(U_1 + U_2) = \varphi^{-1}(U_1) + \varphi^{-1}(U_2)$
- $\rightsquigarrow \varphi^{-1}(U_1 \oplus U_2) = \varphi^{-1}(U_1) \oplus \varphi^{-1}(U_2)$

$U_1 \oplus U_2$

$U_1 \cap U_2 = \{0\}$

onto

Note: $\varphi^{-1}(U_1) \cap \varphi^{-1}(U_2) = \varphi^{-1}(U_1 \cap U_2)$ $= \mathrm{Ker}\varphi$ $= \{0\}$

isomorphic!

Note: $\varphi \in \mathrm{End}_F(V), \varphi$ is **1-1** $\Leftrightarrow \varphi$ is **onto** $\Leftrightarrow \varphi$ is **isomorphic**.

设V, U为F上的有限维线性空间，$\varphi,\ \psi \in \mathrm{Hom}_F(V,U)$，设$\xi_1,\cdots,\xi_n$为$V$的一组基，$\eta_1,\cdots,\eta_m$为$U$的一组基，设

$$\varphi(\xi_1,\cdots,\xi_n)=(\eta_1,\cdots,\eta_m)A,\ \psi(\xi_1,\cdots,\xi_n)=(\eta_1,\cdots,\eta_m)B$$

$\mathcal{L}(V,U) \cong F^{m\times n}$

同一组基 取定基

- $A,\ B \in F^{m\times n}$
- φ由在V的基下的像唯一确定，即 $\varphi(\xi_i)\equiv\psi(\xi_i) \Leftrightarrow \varphi=\psi$

$\mathcal{L}(V) \cong F^{n\times n}$

- 在取定V和U的基的情况下，φ由在基下的表示矩阵唯一确定，即 $\varphi=\psi \Leftrightarrow A=B$

再设$(\xi_1',\cdots,\xi_n')=(\xi_1,\cdots,\xi_n)P,\ (\eta_1',\cdots,\eta_m')=(\eta_1,\cdots,\eta_m)Q$，$P,Q$可逆，且$\varphi(\xi_1',\cdots,\xi_n')=(\eta_1',\cdots,\eta_m')C$，那么

不同的基 取定的

- A与C相抵，且$C=Q^{-1}AP \rightsquigarrow$ 同一线性映射在不同基下的矩阵是相抵的
- 可以选取ξ_i', η_j'，使得$C=\mathrm{diag}\{E_r, O\}$，$r(A)=\dim \mathrm{Im}\varphi=r$

设V为F上的有限维线性空间，$\varphi \in \mathrm{End}_F(V)$，设$\xi_1, \cdots, \xi_n$和$\xi_1', \cdots, \xi_n'$分别为$V$的一组基且$(\xi_1', \cdots, \xi_n') = (\xi_1, \cdots, \xi_n)P$，设

$$\varphi(\xi_1, \cdots, \xi_n) = (\xi_1, \cdots, \xi_n)A, \ \varphi(\xi_1', \cdots, \xi_n') = (\xi_1', \cdots, \xi_n')C$$

- A与C**相似**，且$C = P^{-1}AP \rightsquigarrow$ 同一线性变换在不同基下的矩阵是相似的
- A与C相似可以记为$A \sim C$
- 相似的必要条件：$A \sim C \Rightarrow \det, \mathrm{tr}, \mathrm{rank}$ 相等，反之不成立

$P^{-1}AP = C$. $\det C = \det P^{-1} \cdot \det A \cdot \det P = \det A$

$\mathrm{tr}(C) = \mathrm{tr}(P^{-1}AP) = \mathrm{tr}(APP^{-1}) = \mathrm{tr}(A)$

$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $\rightsquigarrow \begin{pmatrix} 2024 & 0 \\ 0 & \frac{1}{2024} \end{pmatrix}$ 与 $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E_2$相似吗?

$C \sim A$

$\boxed{P^{-1}A^kP = C^k}$

- (Why?) 设f为多项式，则$f(C) = P^{-1}f(A)P \rightsquigarrow f(C) \sim f(A)$

数乘 加法 次幂

$\boxed{aC \sim aA,\ a \in F}$

$B \sim D \not\Rightarrow A+B \sim C+D$ (同时 — —) ☆ 💡