Discussion Session 2.   illusion   2024.12.12.

Example

设V_1为F上n维线性空间V的一个真子空间，且$\dim V_1 = r$，证明：若存在k个$n-1$维子空间满足$U_1,\cdots,U_k$使得

$$\bigcap_{i=1}^{k} U_i = V_1$$

则$k \geqslant n-r$

Notes:

- F^n上的真子空间一定可以看成某个线性方程组$AX=O$的解空间，其中A不可逆
- 如何用$V \cong F^n$给出一个更具体的构造？$\rightsquigarrow$ 将V_1看成线性方程组的解空间

法1：由V_1为真子空间，记 $V \cong F^n$，$\varphi: V \to F^n$，将α映为基下的坐标，只证 $\varphi(V)$ 可以看作是某个 $AX=0$ 的解空间。设 $\varphi(V)$ 的一组基为 $\xi_1,\cdots,\xi_r$，则令 $B=(\xi_1,\cdots,\xi_r)_{n\times r}$，$r(B)=r$

$\underset{r\times n}{B^T}X=0$ 的基础解系中有 $n-r$ 个向量，记为 $\eta_1,\cdots,\eta_{n-r}$，记 $A=(\eta_1,\cdots,\eta_{n-r})_{n\times (n-r)}$

$\Rightarrow$ $A^TX=0$ 的基础解系中有 r 个向量，且由 $B^T\eta_i = \begin{pmatrix}\xi_1^T\\ \vdots \\ \xi_r^T\end{pmatrix}\eta_i = 0 \Rightarrow \xi_j^T\eta_i=0$ $\Rightarrow \eta_i^T\xi_j = 0$

即 $\begin{pmatrix}\eta_1^T\\ \vdots \\ \eta_{n-r}^T\end{pmatrix}\xi_j = A^T\xi_j = 0 \Rightarrow \xi_1,\cdots,\xi_r$ 为 $A^TX=0$ 的一个基础解系

构造 $\eta_i^TX=0$ 的解空间为 U_i $\Rightarrow V_1 = \bigcap_{i=1}^{n-r} U_i$

法2：将 $U_1,\cdots,U_k$ 对应到 $C_1X=0,\cdots,C_kX=0$，考虑 $\begin{pmatrix}C_1\\ \vdots \\ C_k\end{pmatrix}X=0$ 的解空间 $\bigcap_{i=1}^{k} U_i$

$r(C_1)=\cdots=r(C_k)=1 \Rightarrow r\begin{pmatrix}C_1\\ \vdots \\ C_k\end{pmatrix} \leqslant k$

$\Rightarrow \dim\left(\bigcup_{i=1}^{k} U_i\right) \geqslant n-k$ □

Example

设数域$F \subseteq K \subseteq L$，则在下面两个运算下$K$成为$F$上的线性空间：$F$中元素与$K$中元素的数量乘法，$K$中元素的加法。同理$L$也是$K$上的线性空间，如果$\dim_F K < \infty$，则记$\dim_F K = [K : F]$，证明：

$$[L : K][K : F] = [L : F]$$

Notes:

- Revisit your homework: $\dim_{\mathbf{Q}} \mathbf{Q}(\sqrt{2}) = 2$
- $\dim_{\mathbf{Q}} \mathbf{Q}(\sqrt{2}, \sqrt{3}) = ?$

pf: 记 $[K:F]=m$, $[L:K]=n$

${}_F K$ 的一组基为 $\alpha_1 - - \alpha_m$, ${}_K L$ 的一组基为 $\beta_1 - - \beta_n$

· $\forall x \in {}_F L$, $x = c_1\beta_1 + \cdots + c_n\beta_n \quad (c_i \in K)$

$$= (c_{11}\alpha_1 + \cdots + c_{1m}\alpha_m)\beta_1 + \cdots + (c_{n1}\alpha_1 + \cdots + c_{nm}\alpha_m)\beta_n$$

$$= \sum_{i,j} c_{ji}\alpha_i\beta_j$$

· $\{\alpha_i\beta_j\}$ are linearly independent.

$$a_{11}\alpha_1\beta_1 + \cdots + a_{1n}\alpha_1\beta_n + \cdots + a_{m1}\alpha_m\beta_1 + \cdots + a_{mn}\alpha_m\beta_n = 0, \quad a_{ij} \in F$$

$$\Rightarrow \begin{cases} \underbrace{a_{11}\alpha_1 + \cdots + a_{1n}\alpha_1}_{\in K} = 0 \\ \vdots \\ \underbrace{a_{m1}\alpha_m + \cdots + a_{mn}\alpha_m}_{K} = 0 \end{cases} \Rightarrow a_{ij} = 0$$

∎

Notes: $[\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}] = 4$

or $\mathbb{Q}(\sqrt{2},\sqrt{3}) = \{a_0 + a_1\sqrt{2} + a_2\sqrt{3} + a_3\sqrt{6} \mid a_i \in \mathbb{Q}\}$ ✓ Got it!

Try

设 $V=\{z\in\mathbb{C}\mid |z|=1, z\neq -1\}$，$V$ 中两个数 z_1,z_2 的运算定义为

$$z_1\oplus z_2=\frac{-1+z_1+z_2+3z_1z_2}{3+z_1+z_2-z_1z_2}\quad k\odot z=\frac{(1-k)+(1+k)z}{(1+k)+(1-k)z}.$$

证明：$(V,\oplus,\odot)$ 是实数域 $\mathbb{R}$ 上的线性空间

$z=a+bi$ $\bar{z}=a-bi$

(Hint: $U=\{ai\mid a\in\mathbb{R}\}$, $\varphi:V\to U, z\mapsto\frac{1-z}{1+z}$ 给出一个双射)

$$\frac{1-z}{1+z}=\frac{(1-z)(1+\bar{z})}{|1+z|^2}=\frac{1-|z|^2+\bar{z}-z}{|1+z|^2}=\frac{\bar{z}-z}{|1+z|^2}=\frac{-2b}{|1+z|^2}i\in U$$

$$z_1\oplus z_2=\varphi^{-1}(\varphi(z_1)+\varphi(z_2))=\frac{1-\left(\frac{1-z_1}{1+z_1}+\frac{1-z_2}{1+z_2}\right)}{1+\left(\frac{(1-z_1)(1+z_2)+(1-z_2)(1+z_1)}{(1+z_1)(1+z_2)}\right)}$$

$$z\longmapsto\frac{1-z}{1+z}$$

$$\frac{1-t}{1+t}\longleftarrow t$$

$$=\frac{1+z_1z_2+z_1+z_2-(2-2z_1z_2)}{1+z_1z_2+z_1+z_2+(2-2z_1z_2)}$$

$$=\frac{-1+z_1+z_2+3z_1z_2}{3-z_1z_2+z_1+z_2}$$

$$c\odot z_1=\varphi^{-1}(c\varphi(z))$$

同理

$\square$

Example

设$\dim V=n$, $\varphi\in \mathrm{End}_F(V)$，且$\varphi^2=\mathscr{O}$，求证：存在$V$的一组基$\xi_1,\cdots,\xi_n$满足

$$\varphi(\xi_1,\cdots,\xi_n)=(\xi_1,\cdots,\xi_n)\begin{pmatrix} O & E_r & O\\ O & O & O\end{pmatrix}.$$

pf: 矩阵语言: $A^2=0$, $r(A)=r$, 求证: $A\sim\begin{pmatrix} 0 & E_r & 0\\ 0 & 0 & 0\end{pmatrix}$

由 $A=P\begin{pmatrix} E_r & 0\\ 0 & 0\end{pmatrix}Q$, P,Q可逆

$=P\begin{pmatrix} E_r & 0\\ 0 & 0\end{pmatrix}QPP^{-1}$

Let $H=QP=\begin{pmatrix} \overbrace{H_1}^{r} & \overbrace{H_2}^{n-r}\\ H_3 & H_4\end{pmatrix}\rightsquigarrow A\sim\begin{pmatrix} E_r & 0\\ 0 & 0\end{pmatrix}H$ (H可逆)

$A^2=0\iff\begin{pmatrix} E_r & 0\\ 0 & 0\end{pmatrix}H\begin{pmatrix} E_r & 0\\ 0 & 0\end{pmatrix}H=0\iff\begin{pmatrix} E_r & 0\\ 0 & 0\end{pmatrix}H\begin{pmatrix} E_r & 0\\ 0 & 0\end{pmatrix}=0$

即 $\begin{pmatrix} H_1 & H_2\\ 0 & 0\end{pmatrix}\begin{pmatrix} E_r & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix} H_1 & 0\\ 0 & 0\end{pmatrix}=0\Rightarrow H_1=0$

则 $A\sim\begin{pmatrix} E_r & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} 0 & H_2\\ H_3 & H_4\end{pmatrix}=\begin{pmatrix} 0 & \overbrace{H_2}^{n-r}\\ 0 & 0\end{pmatrix}\}r$ $(r(A)=r(H_2)=r)$

$\star$ $n-r\geqslant r\Rightarrow r\leqslant n/2$

从而 H_2行满秩 $\Rightarrow H_2=(E_r\ \ 0)K_{n-r}$ (K可逆)

即 $A\sim\begin{pmatrix} E_r & 0\\ 0 & K_{n-r}^{-1}\end{pmatrix}\begin{pmatrix} 0 & \overbrace{E_r\ \ 0}^{n-r}\\ 0 & 0\ \ 0\end{pmatrix}\begin{pmatrix} E_r & 0\\ 0 & K_{n-r}\end{pmatrix}$

$\sim\begin{pmatrix} 0 & E_r & 0\\ 0 & 0 & 0\end{pmatrix}$ $\square$

例1:

$$
\begin{array}{llll}
\operatorname{Ker}\varphi & \left\{\begin{array}{l}\xi_r \\ \vdots \\ \xi_{r+t}\end{array}\right. & \longrightarrow & 0 \\
 & \xi_1 & \longrightarrow & \varphi(\xi_1) \\
 & \vdots & & \\
 & \xi_r & \longrightarrow & \varphi(\xi_r)
\end{array}
$$

· Claim $\langle \varphi(\xi_1), \cdots \varphi(\xi_r) \rangle \subseteq \operatorname{Ker}\varphi$

即 $r \leq n-r \Rightarrow r \leq n/2.$

将 $\varphi(\xi_1) - - \varphi(\xi_r)$ 扩为 $\operatorname{Ker}\varphi$ 一组基

$\varphi(\xi_1) - - \varphi(\xi_r)\ \delta_1 - - \delta_t$

Claim $\xi_1 - - \xi_r\ \varphi(\xi_1) - - \varphi(\xi_r)\ \delta_1 - - \delta_t$ 线性无关

$$c_1\xi_1 + \cdots + c_r\xi_r + \tilde{c}_1\varphi(\xi_1) + \cdots + \tilde{c}_r\varphi(\xi_r) + d_1\delta_1 + \cdots + d_t\delta_t = 0$$

φ左作用

$$c_1\varphi(\xi_1) + \cdots + c_r\varphi(\xi_r) = 0 \qquad c_1 = \cdots = c_r = 0$$

$$\Rightarrow \tilde{c}_1\varphi(\xi_1) + \cdots + \tilde{c}_r\varphi(\xi_r) + d_1\delta_1 + \cdots + d_t\delta_t = 0$$

$$\Rightarrow \tilde{c}_1 = \cdots = \tilde{c}_r = d_1 = \cdots = d_t = 0$$

从而 $\varphi(\varphi(\xi_1) - - \varphi(\xi_r)\ \xi_1 - - \xi_r\ \delta_1 - - \delta_t) =$

$$(\varphi(\xi_1) - - \varphi(\xi_r)\ \xi_1 - - \xi_r\ \delta_1 - - \delta_t)\begin{pmatrix} 0 & E_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{(Chapter 7)}$$

$r(A) + r(A^3) - 2r(A^2)$ ☆

注：调换顺序后得到 $N(J_2(0)) = r(A)$ ☆

其中 $J_2(0) = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}.$

$$A \sim \operatorname{diag}\{\underbrace{J_2(0), - - , J_2(0)}_{r个}, \underbrace{J_1(0) - - , J_1(0)}_{n-2r个}\} \qquad J_1(0) = (0)$$