Prepare Notes for DS-4

U 为线性空间 V 的子空间，$\varphi \in \mathcal{L}(V)$，那么

$$\varphi(\varphi^{-1}(U)) = U \cap \mathrm{Im}\varphi, \varphi^{-1}(\varphi(U)) = U + \mathrm{Ker}\varphi.$$

1° $\forall \alpha \in \varphi(\varphi^{-1}(U))$, $\exists \beta \in \varphi^{-1}(U)$, $\varphi(\beta)=\alpha$, $\alpha = \varphi(\beta) \in U \Rightarrow \alpha \in U \cap \mathrm{Im}\varphi$

$\forall \alpha \in U \cap \mathrm{Im}\varphi$, $\exists \gamma \in V$, $\varphi(\gamma) = \alpha \in U \Rightarrow \gamma \in \varphi^{-1}(U) \Rightarrow \alpha = \varphi(\gamma) \in \varphi[\varphi^{-1}(U)]$

2° $\forall \alpha \in \varphi^{-1}(\varphi(U)) \Rightarrow \varphi(\alpha) \in \varphi(U)$ $\exists \beta \in U$, $\varphi(\alpha) = \varphi(\beta) \Rightarrow \alpha = \beta + \delta$, $\delta \in \mathrm{Ker}\varphi$ $\in U + \mathrm{Ker}\varphi$

$\forall \alpha \in U + \mathrm{Ker}\varphi$, $\exists \beta \in U$, $\tau \in \mathrm{Ker}\varphi$, $\alpha = \beta + \tau \Rightarrow \varphi(\alpha) = \varphi(\beta) \in \varphi(U)$

$\Rightarrow \alpha \in \varphi^{-1}(\varphi(U))$

$V = \langle \xi_1, \xi_2, \xi_3 \rangle$

	φ	ψ
ξ_1	$\to \xi_1$	$\to 0$
ξ_2	$\to \xi_2$	$\to \xi_1$
ξ_3	$\to 0$	$\to \xi_2$

例1： 设 $\varphi, \psi \in \mathcal{L}(V)$，$U$ 是 V 的子空间，满足 $\mathrm{Im}\,\varphi = \mathrm{Im}\,\psi$，则 D

A. $\varphi = \psi$ ×　　　B. $\mathrm{Ker}\,\varphi = \mathrm{Ker}\,\psi$ ×

C. 若 $\varphi(U) \subseteq U$，则 $\psi(U) \subseteq U$ ×　　　D. φ, ψ 在 V 的任意基下的矩阵相抵 ✓ 　相抵：秩相等 → $r = \dim \mathrm{Im}\varphi$

$\varphi(\langle \xi_2 \rangle) \subseteq \langle \xi_2 \rangle$ 但 $\psi(\langle \xi_2 \rangle) = \xi_1$

例2： 设 U 是 V 的子空间，$\varphi \in \mathcal{L}(V)$，记 $\varphi^{-1}(U) = \{\alpha \in V \mid \varphi(\alpha) \in U\}$，则 D

A. $\dim \varphi^{-1}(U) = \dim U$　　　B. $\dim \varphi^{-1}(U) \le \dim U$

C. $\dim \varphi^{-1}(U) \ge \dim U$　　　D. $\dim \varphi^{-1}(U)$ 和 $\dim U$ 没有关系 ✓

$\xi_1 \to \xi_1$; $\xi_2, \xi_3, \dots, \xi_n \to 0$

取 $U = \langle \xi_1, \dots, \xi_i \rangle$ 　$\forall i$, $\varphi^{-1}(U) = \xi_1$

例3： 设 V_1, V_2, W 都是 n 维线性空间 V 的子空间，若 $V_1 \subseteq V_2$ 且 $V_1 \oplus W = V_2 \oplus W$，则（选填"必"、"未必"）有 $V_1 = V_2$。 ✓

$\dim V_1 = \dim V_2$

例4： 设 V_1, V_2, W 是 n 维线性空间 V 的非平凡子空间，且 $V = V_1 \oplus V_2$，$W \cap V_1 = \{\mathbf{0}\}$，$W \cap V_2 = \{\mathbf{0}\}$. 求 $(W + V_1) \cap (W + V_2)$ 的基和维数。

- $\dim[(W+V_1) \cap (W+V_2)] = 2\dim W + \dim V_1 + \dim V_2 - \dim V = 2\dim W$

- $V_1 = \langle \xi_1, \dots, \xi_r \rangle$, $V_2 = \langle \xi_{r+1}, \dots, \xi_n \rangle$

$\forall \alpha_i \in W,\quad \alpha_i = (c_1\xi_1 + \cdots + \underbrace{c_r\xi_r) + (c_{r+1}\xi_{r+1}}_{\text{不全为0}} + \cdots + c_n\xi_n) = \beta_i + \gamma_i,\ \beta_i \in V_1,\ \gamma_i \in V_2$

取 W 的一组基 $\alpha_1, \cdots, \alpha_s$, $\alpha_i = \beta_i + \gamma_i$

$\Rightarrow\ W + V_1 = L(\xi_1, \cdots, \xi_r, \gamma_1, \cdots, \gamma_s)$

$W + V_2 = L(\beta_1, \cdots, \beta_s, \xi_{r+1}, \cdots, \xi_n)$

$\forall \delta \in (W+V_1) \cap (W+V_2)$, $\delta = c_1\xi_1 + \cdots + c_r\xi_r + d_1\gamma_1 + \cdots + d_s\gamma_s = h_1\beta_1 + \cdots + h_s\beta_s + c_{r+1}\xi_{r+1} + \cdots + c_n\xi_n$

$\Rightarrow\ c_1\xi_1 + \cdots + c_r\xi_r - (h_1\beta_1 + \cdots + h_s\beta_s) = c_{r+1}\xi_{r+1} + \cdots + c_n\xi_n - (d_1\gamma_1 + \cdots + d_s\gamma_s)$

$\in V_1 \cap V_2 = \{0\}$

$\Rightarrow\ c_1\xi_1 + \cdots + c_r\xi_r = h_1\beta_1 + \cdots + h_s\beta_s$

即 $(W+V_1) \cap (W+V_2)$ 中向量可由 $\beta_1, \cdots, \beta_s, \gamma_1, \cdots, \gamma_s$ 线性表出

又 $V_1 \cap V_2 = \{0\}$ 只证 $\beta_1, \cdots, \beta_s$ 线性无关, $\gamma_1, \cdots, \gamma_s$ 线性无关

反证, 若 $\exists$ 不全为0 $b_1, \cdots, b_s$ s.t. $b_1\beta_1 + \cdots + b_s\beta_s = 0$

$\Rightarrow\ b_1\alpha_1 + \cdots + b_s\alpha_s = 0 + b_1\gamma_1 + \cdots + b_s\gamma_s \in V_2 \cap W$

但 $b_1\alpha_1 + \cdots + b_s\alpha_s \neq 0$ 与 $V_2 \cap W = \{0\}$ 矛盾!

即一组基为 $\beta_1, \cdots, \beta_s, \gamma_1, \cdots, \gamma_s$.

$\square$

例： 设

$\varepsilon_{12}-\varepsilon_{21}$ $\varepsilon_{13}-\varepsilon_{31}$ $\varepsilon_{23}-\varepsilon_{32}$

$$V=\{A\in F^{3\times 3}\mid A^\top=A\},\quad U=\{A\in F^{3\times 3}\mid A^\top=-A\},$$

$\boxed{V\oplus U=F^{3\times 3}}$

ε_{11} $\varepsilon_{12}+\varepsilon_{21}$
ε_{22} $\varepsilon_{13}+\varepsilon_{31}$
ε_{33} $\varepsilon_{23}+\varepsilon_{32}$

$$W=\left\{\begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix}\middle|\, a,b,c\in F\right\}.\rightarrow \varepsilon_{11},\ \varepsilon_{22},\ \varepsilon_{33} \subseteq V.$$

(1) 分别写出 V,U 和 W 的基；

(2) 证明：存在 $\varphi\in\mathcal{L}(V,U)$，使得 $\mathrm{Ker}\varphi=W$。

ε_{11} $\qquad\varepsilon_{12}+\varepsilon_{21}\to\varepsilon_{12}-\varepsilon_{21}$

$\varepsilon_{22}\to 0$ $\qquad\varepsilon_{13}+\varepsilon_{31}\to\varepsilon_{13}-\varepsilon_{31}$

ε_{33} $\qquad\varepsilon_{23}+\varepsilon_{32}\to\varepsilon_{23}-\varepsilon_{32}$.

Note: 对 $\varphi(A)=A^T$，$\exists F^{3\times 3}$ 的一组基 使 $\varphi(\quad)=(\quad)\begin{pmatrix}E_6&\\&-E_3\end{pmatrix}$

$\hookrightarrow \varphi^2=id_V \Rightarrow F^{3\times 3}=\mathrm{Ker}(\varphi-id)\oplus\mathrm{Ker}(\varphi+id)$ 可相似对角化.

Try

设 $\mathscr{O}\neq\varphi,\psi\in\mathcal{L}(V)$，$\dim V=2n+1\ (n\in\mathbf{N}^*)$，若 $\varphi\psi+\psi\varphi=\mathscr{O}$，求证：$\varphi$ 既有非平凡的 $\psi-$子空间，也有非平凡的 $\varphi-$子空间。

设 φ 无非平凡 $\varphi-$子空间 $\because 0\neq\varphi\Rightarrow\varphi$ 可逆 $\quad\varphi\rightsquigarrow A,\ \psi\rightsquigarrow B$

$\Rightarrow AB+BA=0$ 即 $B=-ABA^{-1}\Rightarrow \det B=(-1)^{2n+1}\det B\Rightarrow\det B=0$

即 ψ 不可逆 $\Rightarrow \mathrm{Ker}\psi$ 为非平凡 $\psi-$子空间 也是 $\varphi-$子空间 矛盾!

则 φ 有非平凡 $\varphi-$子空间 $\quad$ 同理 ψ 有非平凡 $\psi-$子空间 也为 $\varphi-$子空间 $\square$

Example

证明：若 $\varphi \in \mathcal{L}(V)$ 在一组基下的矩阵为

$$\boldsymbol{J}(\lambda, n) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

那么 V 不能分解为两个非平凡 $\varphi-$子空间的直和。

Note: 如何求所有的 $\varphi-$子空间？

设 $\varphi(\xi_1, \cdots, \xi_n) = (\xi_1, \cdots, \xi_n) J(\lambda, n)$

· 取 U 为非平凡 $\varphi-$子空间，Claim: $\xi_1 \in U$

由 $U \neq \{0\}$，则 $\exists\, 0 \neq \alpha = c_1\xi_1 + \cdots + c_n\xi_n \in U$，从右往左找第1个 $c_k \neq 0$

$$\Rightarrow \alpha = c_1\xi_1 + \cdots + c_k\xi_k \in U \Rightarrow \varphi(\alpha) = c_1(\lambda\xi_1) + c_2(\xi_1 + \lambda\xi_2) + \cdots + c_k(\xi_{k-1} + \lambda\xi_k)$$

$$= c_2\xi_1 + \cdots + c_k\xi_{k-1} + \underline{\underline{\lambda\alpha}}$$

$$\Rightarrow \varphi(\alpha) - \lambda\alpha = c_2\xi_1 + \cdots + c_k\xi_{k-1} \in U$$

$$\Rightarrow \varphi[\varphi(\alpha) - \lambda\alpha] - \lambda[\varphi(\alpha) - \lambda\alpha] = c_3\xi_1 + \cdots + c_k\xi_{k-2} \in U$$

$\cdots \to$ 反复 $k-1$ 次有 $c_k\xi_1 \in U \Rightarrow \xi_1 \in U$

另：由 $(\varphi - \lambda \mathrm{id}_V)(\xi_1, \cdots, \xi_n) = (\xi_1, \cdots, \xi_n) J(0, n)$

(此处为 $\Leftrightarrow$)

Goal 若空间能分解为 $\varphi-$子空间直和 $\Rightarrow$ $(\varphi - \lambda \mathrm{id}_V)-$子空间直和.

即若不能分解为 $(\varphi - \lambda \mathrm{id}_V)-$子空间直和 $\Rightarrow$ 不能分解为 $\varphi-$子空间直和.

若 U 为 $\varphi-$子空间，则 U 为 $f(\varphi)-$子空间，f 为多项式

反之当 $f(\varphi)=a\varphi+b\ (a\neq 0)$ ✓ $=0$ 显然不成立 对高次情形

$$\varphi(\xi_1\cdots\xi_3)=(\xi_1\cdots\xi_3)\begin{pmatrix}1&1&0\\2&-2&0\\0&0&1\end{pmatrix}$$

$(\varphi^2+\varphi)(\xi_1)=\varphi(\xi_1+2\xi_2)+\xi_1+2\xi_2=2(\xi_1+2\xi_2)+2(\xi_1-2\xi_2)=4\xi_1$

从而 $\langle\xi_1\rangle$ 为 $\varphi^2+\varphi$-子空间 但非 φ-子空间

Let $\psi=\varphi-\lambda\, id_V$, 非平凡 φ-子空间 $U\Longrightarrow$ 非平凡 ψ-子空间 U.

$\exists\ \alpha=c_1\xi_1+\cdots+c_k\xi_k+0\xi_{k+1}+\cdots+0\xi_n\in U$, $\varphi^{k-1}(\alpha)=c_k\xi_1\in U\Rightarrow\xi_1\in U$

Note: 求所有 φ-子空间 $\Leftrightarrow$ $\psi=\varphi-id_V$-子空间

设 U 为 ψ-子空间, 记 $m=\max\{k\mid c_1\xi_1+\cdots+c_k\xi_k\in U,\ c_k\neq 0,\ 1\leq k\leq n\}$

记 $\alpha=c_1\xi_1+\cdots+c_m\xi_m$, $c_m\neq 0$, 则 $\psi^{m-2}(\alpha)=c_{m-1}\xi_1+c_m\xi_2\in U$ 而 $\xi_1\in U$, $c_m\neq 0\Rightarrow\xi_2\in U$

$\psi^{m-3}(\alpha)=c_{m-2}\xi_1+c_{m-1}\xi_2+c_m\xi_3$ 又 $\xi_1,\xi_2\in U$. $c_m\neq 0\Rightarrow\xi_3\in U$ 以此类推, $\xi_4,\cdots,\xi_m\in U$

由 m 的取法知 U 中任意向量均可由 $\xi_1\cdots\xi_m$ 线性表出

即 $\dim U=m$, $U=\langle\xi_1,\cdots,\xi_m\rangle$

而 $m=1,\cdots,n$ 共有 n 种取法 $\rightsquigarrow$ n 个 U $+$ $U=\{0\}$ 共 $n+1$ 个 φ-子空间

▥

Try

$$\varphi^2 = 2\varphi \Leftrightarrow \varphi \leftrightsquigarrow \begin{bmatrix} 2E_r & O \\ O & O \end{bmatrix} ?$$

1° $V = \operatorname{Ker}\varphi \oplus \operatorname{Ker}(\varphi - 2\mathrm{id}_V)$ ✓

2° $\operatorname{Ker}\varphi$ $\xi_{r+1}, \cdots, \xi_n$ 扩充为 V $\xi_1, \cdots, \xi_r, \xi_{r+1}, \cdots, \xi_n$

$\operatorname{Im}\varphi$: $\varphi(\xi_1) \cdots \varphi(\xi_r)$，而 $\varphi^2(\xi_i) = 2\varphi(\xi_i)$

Claim $\varphi(\xi_1) \cdots \varphi(\xi_r), \xi_{r+1}, \cdots, \xi_n$ 为 V 的一组基 $\Rightarrow$ $c_1\varphi(\xi_1) + \cdots + c_r\varphi(\xi_r) + c_{r+1}\xi_{r+1} + \cdots + c_n\xi_n = 0$

$\Rightarrow \varphi\left[c_1\varphi(\xi_1) + \cdots + c_r\varphi(\xi_r) + c_{r+1}\xi_{r+1} + \cdots + c_n\xi_n\right] = 0 \Rightarrow 2\left(c_1\varphi(\xi_1) + \cdots + c_r\varphi(\xi_r)\right) = 0$

$\Rightarrow c_1 = \cdots = c_r = 0 \Rightarrow c_{r+1} = \cdots = c_n = 0$

(Note: 上述其实证明了 $V = \operatorname{Ker}\varphi \oplus \operatorname{Im}\varphi$)

则 $\varphi(\varphi(\xi_1) \cdots \varphi(\xi_r), \xi_{r+1}, \cdots, \xi_n) = (\varphi(\xi_1) \cdots \varphi(\xi_r), \xi_{r+1}, \cdots, \xi_n)\begin{bmatrix} 2E_r & 0 \\ 0 & 0 \end{bmatrix}$

3° 矩阵相抵标准型 $\quad A^2 = 2A \Rightarrow A \sim \begin{bmatrix} 2E_r & 0 \\ 0 & 0 \end{bmatrix}$

$$A = P\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}Q = P\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}QPP^{-1} \quad \text{Let } QP = M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

$$A^2 = 2A \Leftrightarrow \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}M\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}M = 2\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}M \quad \text{& } M \text{ 可逆}$$

即 $\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}M\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2E_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} M_1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow M_1 = 2E_r$

$A \sim \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2E_r & M_2 \\ M_3 & M_4 \end{pmatrix} = \begin{pmatrix} 2E_r & M_2 \\ 0 & 0 \end{pmatrix}$ $\underset{\substack{\text{第1列} \times -\frac{M_2}{2} \xrightarrow{+} \text{第2列} \\ \text{第2行} \times \frac{M_2}{2} \xrightarrow{+} \text{第1行}}}{\underline{\text{作相似初等变换}}}$ $\sim \begin{pmatrix} 2E_r & 0 \\ 0 & 0 \end{pmatrix}$

$\blacksquare$