§5.5—§5.7 Polynomial Functions and Polynomials Over $\mathbf{C}, \mathbf{R}, \mathbf{Q}$ and $\mathbf{Z}$

illusion

Especially made for smy

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http://illusion-hope.github.io/25-Spring-SMY-Discussion-Session/

Quiz 1

例 1

设 $f(x), g(x) \in F[x]$, $a, b, c, d \in F$ 且满足 $ad - bc \neq 0$，证明:

$$(af(x) + bg(x), cf(x) + dg(x)) = (f(x), g(x)).$$

Hint: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0.$

例 2

设 $f(x) \in \mathbf{Z}[x]$，证明：

(1) 若 $f(x) - 1$ 在 $\mathbf{Z}$ 上有至少四个互异整数根，则 $f(x) + 1$ 在 $\mathbf{Z}$ 上无根;

(2) 若存在一个偶数 m 和奇数 n 使得 $f(m)$, $f(n)$ 均为奇数，则 $f(x)$ 在 $\mathbf{Z}$ 上无根。

HW-3

例 3

设 $f(x) = x^3 + 3x + 1$，求满足同余方程 $v(x)f'(x) \equiv 1 \pmod{f(x)}$ 且次数最小的多项式 $v(x)$.

例 4

设 $(f(x), g(x)) = 1$，证明：$f^2(x) + g^2(x)$ 的重根必是 $[f'(x)]^2 + [g'(x)]^2$ 的根.

例 5

设 $f(x)$ 为 $\mathbf{R}[x]$ 上的任一实系数多项式. 证明：存在唯一实系数多项式 $g(x)$ 使得

$$\left((x^2 + 3x - 5)g(x)\right)'' = f(x).$$

Examples

例 6

Assume that an irreducible polynomial $p(x)$ is a $(k-1)$-multiple factor of $f'(x)$, then the following statements are equivalent:

(1) $p(x)$ is a $(k-1)$-multiple factor of (f, f');

(2) $p(x) \mid f(x)$;

(3) $p(x)$ is a k-multiple factor of $f(x)$.

(3') $f(x) = p^k(x)h(x),\ (p(x), h(x)) = 1.$

Note: When we talk about repeated factors, $p(x)$ should firstly be irreducible.

Lecture 4

Polynomial Functions:

- Remainder Theorem $\rightsquigarrow$ Roots
- Relationship Between Numbers of Distinct Roots and Degree

Further Discussion Mainly Over Number Fields:

- $\mathbf{C}[x]$: Factorization, Viéta Theorem
- $\mathbf{R}[x]$: Factorization, Complex Roots Come in Pairs
- $\mathbf{Q}[x]$: Rational Root Theorem, Eisenstein's Criterion
- $\mathbf{Z}[x]$: Gauss's Lemma, Factorization

Multiple Roots

- If $f(x)$ has a multiple root a over F, then it must have repeated factors. But the converse is wrong.
- If a is a k-multiple root of $f(x)$, then it's a $(k-1)$-multiple root of $f'(x)$. But the converse is wrong.
- If a is a $(k-1)$-multiple root of (f', f), then it's a k-multiple root of $f(x)$.

Examples

例 7

Assume $f(x) \in F[x]$ and $\deg f(x) = n$. If $f' \mid f$, prove that f has a n-multiple root over F.

Hint: $f/(f, f')$ has the same irreducible factors as f.

例 8

Assume $f(x) \in F[x]$, then $x - b$ is a k-multiple factor of $f(x)$ if and only if $f(b) = f'(b) = \cdots = f^{(k-1)}(b) = 0,\ f^{(k)}(b) \neq 0$.

Hint: $p(x)$ is a k-multiple factor of $f(x) \Rightarrow p(x)$ is a $(k-1)$-multiple factor of $f'(x)$.

(1) 设 $f(x) \in F[x]$，若 $p(x) \in F[x]$ 在 F 上不可约，且 $p(x), f(x)$ 在 $\mathbb{C}$ 上有公共根，那么必定有 $p(x) \mid f(x)$;

(2) 设 $f(x), g(x) \in F[x]$，若 $(f(x), g(x)) = 1$，那么 $f(x), g(x)$ 在 $\mathbb{C}$ 上必定没有公共根;

(3) 若 $p(x) \in F[x]$ 在 F 上不可约，给定数域扩张 $F \subseteq K$，那么 $p(x)$ 在 K 上有可能可约，设 $p(x)$ 在 $K[x]$ 中的标准分解式为

$$p(x) = q_1^{e_1}(x) \cdots q_s^{e_s}(x), \ e_i > 0.$$

其中 $q_1(x), \cdots, q_s(x)$ 为 K 上两两互素的首一不可约多项式，那么我们必定有 $e_1 = \cdots = e_s = 1$.

Some Important Propositions

(3') 换言之，若 $p(x) \in F[x]$ 在 F 上不可约，那么 $p(x)$ 在 $\mathbb{C}$ 上无重根。进一步地，在给定任意一个数域扩张 $F \subseteq K$ 下，都有 $p(x)$ 在 K 上无重因式;

(4) 作为对比，我们给出命题：若 $p(x) \in F[x]$，给定数域扩张 $F \subseteq K$，且 $p(x)$ 在 K 上不可约，那么 $p(x)$ 在 F 上必定不可约。

Examples

Slogan: $p(x)$ 在 F 上不可约且在 $\mathbf{C}$ 上与 $f(x)$ 有公共根 $\Rightarrow p \mid f$.

例 9

(1) $f(x) \in \mathbf{R}[x],\ f(a+b\mathrm{i}) = 0\ (a, b \in \mathbf{R}) \rightsquigarrow f(a-b\mathrm{i}) = 0$;

(2) $f(x) \in \mathbf{Q}[x],\ f(\sqrt{2}+\sqrt{3}) = 0 \rightsquigarrow f(\sqrt{2}-\sqrt{3}) = f(-\sqrt{2}+\sqrt{3}) = f(-\sqrt{2} -\sqrt{3}) = 0$.

Hint: What is the standard factorization of $x^4 - 10x + 1$ in $\mathbf{R}[x]$?

例 10

$f(x) \in F[x]$ 在 F 上不可约，若非零常数 a, a^{-1}, b 为 $f(x)$ 在 $\mathbf{C}$ 上的根，证明：$f(b^{-1}) = 0$.

Hint: Consider $f(x) = \sum_{k=0}^{n} a_k x^k, g(x) = \sum_{k=0}^{n} a_{n-k} x^k,\ a_i \in F,\ a_0 a_n \neq 0$.

Examples

Slogan: $p(x)$ 在 F 上不可约且在 $\mathbf{C}$ 上与 $f(x)$ 有公共根 $\Rightarrow p \mid f$.

例 11

$f(x) \in F[x]$ 在 F 上不可约，对 $g(x) \in F[x]$，有 $\alpha \in \mathbf{C}$ 满足 $f(c) = 0, g(c) \neq 0$.

(1) 存在 $h(x) \in F[x]$ 满足 $h(c)g(c) = 1$;

(2) 求一个多项式 $h(x) \in \mathbf{Q}[x]$ 满足

$$h(\sqrt[3]{2}) = \frac{1}{3 + 2\sqrt[3]{2} + \sqrt[3]{4}}.$$

Hint: $p \mid f \Leftrightarrow$ All the roots of $p(x)$ are roots of $f(x)$ and **their multiplicities do not exceed those in $f(x)$.**

New View of Divisibility

Slogan: $p \mid f \Leftrightarrow$ All the roots of $p(x)$ are roots of $f(x)$ and **their multiplicities do not exceed those in** $f(x)$**.**

Revisit some examples:

(1) $x^2+x+1 \,\Big|\, \sum\limits_{i\in I} x^{a_i} \Leftrightarrow a_i$ 除 3 余数为 0，1，2 的个数相等；

(2) $a \in F,\ x^d - a^d \mid x^n - a^n \Leftrightarrow d \mid n$.

例 12

设多项式 $f(x), g(x), h(x), k(x)$ 之间有关系式

$$\begin{cases} (x+1)f(x) + (x+2)g(x) + (x^2+1)h(x) = 0, \\ (x-1)f(x) + (x-2)g(x) + (x^2+1)k(x) = 0. \end{cases}$$

证明：$(x^2+1) \mid (f, g)$.

Synthetic Division

$$f(x) = \sum_{k=0}^{n} a_k x^k = (x-b)\left\{\sum_{s=0}^{n-1} b_s x^s\right\} + f(b) \rightsquigarrow a_k = -b \cdot b_k + b_{k-1}\,, 1 \leq k \leq n-1.$$

Try

$f(x) = x^5 - 2020x^4 - 2019x^3 - 4041x^2 - 2020x - 100$, figure out $f(2021)$.

- $f(x) = (x-2021)(x^4 + b_3x^3 + \cdots + b_0) + f(2021)$;
- $-2020 = -2021b_3 + b_4 = -2021b_3 + 1 \rightsquigarrow b_3 = 1$;
- $-2019 = -2021b_3 + b_2 \rightsquigarrow b_2 = 2021 - 2019 = 2$;
- $-4041 = -2021b_2 + b_1 \rightsquigarrow b_1 = 1$;
- $-2020 = -2021b_1 + b_0 \rightsquigarrow b_0 = 1$;
- $-100 = -2021b_0 + f(2021) \rightsquigarrow f(2021) = 1921$.

Polynomial Functions

Lemma 13

Let $f(x)$ be a polynomial over a field F with degree $n > 0$. Then, $f(x)$ has at most n distinct roots in F.

Note: Any polynomial of finite degree has only a finite number of roots. If a polynomial has infinitely many roots, it must be the zero polynomial.

Thm 14

Set $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,\ g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0,\ a_i, b_j \in F$. Then, the following two statements are equivalent:

(A) For all $c \in F$, we have $f(c) = g(c)$;

(A') For $n+1$ distinct numbers $c_1, \cdots, c_{n+1} \in F$, we have $f(c) = g(c)$;

(B) $n = m, a_i = b_i\ (1 \leq i \leq n)$.

Distinct Roots > Degree $\rightsquigarrow$ Zero!

Slogan: If a polynomial has infinitely many roots $\Rightarrow$ zero polynomial.

例 15

Figure out each $f(x) \in F[x]$ satisfying the following conditions:

(1) $f(x) = f(x+c),\ 0 \neq c \in F$;

(2) $f(a+b) = f(a) + f(b)$, for all $a, b \in F$.

例 16

$\deg f(x) = n > 0, f\,(k) = k/(k+1),\ k = 0, 1, 2, \cdots, n.$ Find $f(n+1)$.

Further Discussion: $\mathbf{C}[x]$

- $0 \neq f(x) \in \mathbf{C}[x], \deg f(x) = n > 1$, 则 $f(x)$ 在 $\mathbf{C}$ 中恰有 n 个复根(计重数)，且
$$f(x) = c(x-a_1)^{e_1}(x-a_2)^{e_2}\cdots(x-a_m)^{e_m}, e_i \in \mathbf{N}^*, \sum_{i=1}^{m} e_i = n.$$
其中 $a_i \in \mathbf{C}$ 两两互异，$c \in \mathbf{C}$ 为 $f(x)$ 的首项系数;
- (Viéta) 记 $f(x) = (x-c_1)(x-c_2)\cdots(x-c_n) \in \mathbf{C}[x]$，其中未必 c_i 两两不同，则
$$\sum_{i=1}^{n} c_i = -a_{n-1}, \sum_{1\leqslant i<j\leqslant n} c_i c_j = a_{n-2}, \cdots, \prod_{i=1}^{n} c_i = (-1)^n a_0.$$
- $x^n - \rho e^{\mathrm{i}\theta} = \prod_{k=1}^{n}\left(x - \rho^{\frac{1}{n}} e^{\mathrm{i}\frac{\theta+2k\pi}{n}}\right)$，其中 $\rho > 0$.

Examples

例 17

(Quiz 1) 已知 $f(x)=x^3+px^2+qx+r\in\mathbf{R}[x]$ 的三个根是 $x_1,x_2,x_3\in\mathbf{R}$.

(1) 证明：$p^2\geq 3q$;

(2) 求首一的三次实系数多项式 $g(x)$，使得 x_1^2,x_2^2,x_3^2 为其所有根。（要求 $g(x)$ 的系数用 p,q,r 表示）

例 18

若 $x^2+x+1\mid((x+1)^n-x^n-1)$，求 $n\in\mathbf{N}^*$ 的所有可能取值。

Further Discussion: $\mathbf{R}[x]$

- $0 \neq f(x) \in \mathbf{R}[x], \deg f(x) = n > 1$，则 $f(x)$ 在 $\mathbf{R}$ 上的标准分解式为

$$f(x) = d(x-a_1)^{l_1} \cdots (x-a_m)^{l_m} (x+b_1x+c_1)^{h_1} \cdots (x+b_rx+c_r)^{h_r},$$

其中 $d, a_i, b_j, c_j \in \mathbf{R}, l_i, h_j \in \mathbf{N}^*, b_j^2 - 4c_j < 0, \sum_{i=1}^{m} l_i + 2\sum_{j=1}^{r} h_j = \deg f(x)$，$a_i$ 两两互异，$x^2 + b_jx + c_j$ 两两互素。

- 设 $f(x) \in \mathbf{R}[x]$，若 $a, b \in \mathbf{R}$ 且 $f(a+b\mathrm{i}) = 0 \rightsquigarrow f(a-b\mathrm{i}) = 0.$