Topic 建立相似标准型的几何理论.

§1. 向量的极小多项式 循环(子)空间.

Recall 矩阵. φ. 线变换 $f(\cdot)$

$m(\lambda)$ $m(\lambda)$ | $\boxed{\text{不可约}}$

Def 1.1 若 $\dim_F V=n$, $\varphi\in\mathcal{L}(V)$. 若对 $\alpha\in V$, $f(\varphi)\alpha=0 \Rightarrow f$ 为 α 的一个零化多项式

记. $\Omega=\{f\in F[\lambda] \mid f(\varphi)\alpha=0\}$ f_φ. $m_\varphi\in\Omega$. Ω 关于加法构成

Abelian Group. $\boxed{\text{ideal}}$! $\forall g\in F[\lambda]$, $f\in\Omega$, $\underline{\underline{gf}}=\underline{\underline{fg}}\in\Omega$.

$\Omega=<p(\lambda)>$ 首一, 定义 $p(\lambda)=m_\alpha(\lambda)$ 为 α 的极小多项式.

Note: $p(\lambda) \mid f(\lambda)$.

$\alpha_1,\cdots,\alpha_n$ 为 V 的一组基, $m_\varphi(\lambda)=[m_{\alpha_1}(\lambda),\cdots,m_{\alpha_n}(\lambda)]$

pf: $\forall\alpha\in V$, $\alpha=\sum_i c_i\alpha_i \Rightarrow [m_{\alpha_1},\cdots,m_{\alpha_n}](\varphi)\left(\sum_i c_i\alpha_i\right)$

$$=\sum_i c_i\,[m_{\alpha_1},\cdots,m_{\alpha_n}](\varphi)\,\alpha_i$$

$$=\sum_i c_i\, m_{\alpha_i}u_i(\varphi)\,\alpha_i=\sum_i^{c_i} u_i(\varphi)\left(m_{\alpha_i}(\varphi)\alpha_i\right)$$

$$=0$$

$\Rightarrow m_\varphi \mid [\qquad\qquad]$

$m_\varphi(\varphi)\alpha_i = \mathscr{O}\alpha_i = 0. \Rightarrow m_\varphi \in \langle m_{\alpha_i}(\lambda)\rangle.$

\mathscr{O}\alpha_i $\Rightarrow m_{\alpha_i} \mid m_\varphi.$

$\Rightarrow [\quad] \mid m_\varphi.$

Def 1.2 V. φ 记号同上(后续同理) $\exists\ \alpha \in V$. s.t. $\alpha, \varphi\alpha, \cdots, \varphi^{r-1}\alpha$ 线性无关

$\alpha, \varphi\alpha, \cdots, \varphi^{r-1}\alpha, \varphi^r\alpha$ 线性相关

i.e. $\varphi^r\alpha$ 能被 $\alpha, \varphi\alpha, \cdots, \varphi^{r-1}\alpha$ 唯一线性表出

$\varphi^r\alpha \in \mathcal{L}(\alpha, \varphi(\alpha), \cdots, \varphi^{r-1}\alpha)$.

记 $F[\varphi]\alpha := \{ f(\varphi)\alpha \mid f(\lambda) \in F[\lambda] \}$ 为 φ-子空间，称为 α 的循环子空间，特别地，若 $F[\varphi]\alpha = V$，则称 V 成为循环空间，上述 $\alpha, \varphi\alpha, \cdots, \varphi^{n-1}\alpha$ 成为循环基。

e.g. 1. $\varphi(\alpha) = \lambda\alpha$ $\quad \langle\alpha\rangle$ φ-循环空间 ✓ $\quad r=1$

2. $\varphi^n = \mathscr{O}, \varphi^{n-1} \neq \mathscr{O}$，$V$ 成为循环空间

$\alpha \notin \mathrm{Ker}\,\varphi^{n-1}$. $\mathrm{Ker}\,\varphi^{n-1} \neq V$. $\varphi^{n-1}(\alpha) \neq 0 \Rightarrow \varphi(\alpha), \cdots, \varphi^{n-1}(\alpha) \neq 0$

且线性无关 $\Rightarrow V = \mathcal{L}(\alpha, \varphi\alpha, \cdots, \varphi^{n-1}\alpha)$

Thm 1. $\dim_F V = n$. $\varphi \in \mathcal{L}(V)$, $0 \neq \alpha \in V$, 记

$$F[\varphi]\alpha = \{ f(\varphi)\alpha \mid f(\lambda) \in F[\lambda] \}$$

(1) $F[\varphi]\alpha$ 为 φ-不变子空间, 且 $\dim F[\varphi]\alpha = \deg m_\alpha(\lambda)$

(2) 记 $\varphi_1 := \varphi|_{F[\varphi]\alpha}$, 则 $m_\alpha(\lambda) = m_{\varphi_1}(\lambda) = f_{\varphi_1}(\lambda)$

(3) 记 $m_\alpha(\lambda) = \lambda^r + a_{r-1}\lambda^{r-1} + \cdots + a_1\lambda + a_0$, 则

$$\varphi(\alpha, \varphi\alpha, \cdots, \varphi^{r-1}\alpha) = (\alpha, \varphi\alpha, \cdots, \varphi^{r-1}\alpha)\begin{bmatrix} 0 & & & -a_0 \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & \vdots \\ & & 1 & -a_{r-1} \end{bmatrix}$$

pf: (1) $\deg m_\alpha(\lambda) = r$ Claim: $F[\varphi]\alpha = \mathcal{L}(\alpha, \varphi(\alpha), \cdots, \varphi^{r-1}\alpha)$

$\forall c_i$, $\sum_{i=0}^{r-1} c_i\varphi^i\alpha = 0 \Rightarrow \left(\sum_{i=0}^{r-1} c_i\varphi^i\right)\alpha = 0$ 令 $f(\varphi) = \sum_{i=0}^{r-1} c_i\varphi^i$

$\Rightarrow f(\varphi)\alpha = 0$ 但 $\deg f < \deg m_\alpha(\lambda) \Rightarrow f \equiv 0$

$\Rightarrow c_i \equiv 0.$

$\forall f(\varphi)\alpha \in F[\varphi]\alpha$, $f(\varphi) = g(\varphi)m_\alpha(\varphi) + r(\varphi)$, 其中 $\deg r < r$.

$\Rightarrow f(\varphi)\alpha = r(\varphi)\alpha \in \mathcal{L}(\alpha, \varphi(\alpha), \cdots, \varphi^{r-1}\alpha) \Rightarrow \dim F[\varphi]\alpha = r.$

(2) $\forall \alpha \in F[\varphi]\alpha$, $m_\alpha(\lambda) \mid m_{\varphi_1}(\lambda)$

$m_{\varphi_1}(\varphi_1)\alpha = 0 \Rightarrow m_{\varphi_1}(\lambda) \mid m_\alpha(\lambda)$

$\Big\} \Rightarrow m_\alpha = m_{\varphi_1}$

$m_{\varphi_1} \mid f_{\varphi_1}$ $\quad \deg m_{\varphi_1} = r$, $\quad \deg f_{\varphi_1} =$ 阶数 $\left[\begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \end{bmatrix}\right] = r \Rightarrow f_{\varphi_1} = m_{\varphi_1}$

$\boxed{\text{||}}$

Prf 1

(1) $(m_{\alpha_1}(\lambda), m_{\alpha_2}(\lambda)) = 1$, 则 $m_{\alpha_1+\alpha_2}(\lambda) = m_{\alpha_1}(\lambda)\, m_{\alpha_2}(\lambda)$

且 $F[\varphi]\alpha_1 \oplus F[\varphi]\alpha_2 = F[\varphi](\alpha_1+\alpha_2)$

(2) $m_\alpha(\lambda) = p(\lambda) g(\lambda)$, $\beta := p(\varphi)\alpha$, $m_\beta(\lambda) = g(\lambda)$

(3) $\forall \alpha_1, \alpha_2 \in V$, $\exists \beta \in V$, s.t. $m_\beta(\lambda) = [m_{\alpha_1}(\lambda), m_{\alpha_2}(\lambda)]$

↓ 归纳

特别地，上式可推广为 $\exists \alpha \in V$, $m_\varphi(\lambda) = [m_{\alpha_1}(\lambda), \cdots, m_{\alpha_n}(\lambda)] = m_\alpha(\lambda)$.

pf: (1) · $F[\varphi]\alpha_1 \cap F[\varphi]\alpha_2 = \{0\}$ $\qquad (m_{\alpha_1}, m_{\alpha_2}) = 1 \Rightarrow u_1 m_{\alpha_1} + u_2 m_{\alpha_2} = 1$.

$$\Rightarrow \underset{\substack{\uparrow \\ F[\varphi]\alpha_1 \cap F[\varphi]\alpha_2}}{\alpha} = u_1(\varphi)\, m_{\alpha_1}(\varphi)\alpha + u_2(\varphi)\, m_{\alpha_2}(\varphi)\alpha = u_1(\varphi) f(\varphi)\big(m_{\alpha_1}(\varphi)\alpha_1\big) + u_2(\varphi) g(\varphi)\big(m_{\alpha_2}(\varphi)\alpha_2\big) = 0 + 0 = 0.$$

$\alpha = f(\varphi)\alpha_1 = g(\varphi)\alpha_2$.

· $F[\varphi](\alpha_1+\alpha_2) = F[\varphi](\alpha_1) \oplus F[\varphi](\alpha_2)$

$m_{\alpha_1+\alpha_2}(\varphi)(\alpha_1+\alpha_2) = m_{\alpha_1+\alpha_2}(\varphi)\alpha_1 + m_{\alpha_1+\alpha_2}(\varphi)\alpha_2 = 0$

$\Rightarrow \underbrace{m_{\alpha_1+\alpha_2}(\varphi)(\alpha_1)}_{=0} = -\underbrace{m_{\alpha_1+\alpha_2}(\varphi)(\alpha_2)}_{=0} \in F[\varphi]\alpha_1 \cap F[\varphi]\alpha_2 = \{0\}$

$\Rightarrow m_{\alpha_1} \mid m_{\alpha_1+\alpha_2}$. $m_{\alpha_2} \mid m_{\alpha_1+\alpha_2}$ $\Rightarrow (m_{\alpha_1}, m_{\alpha_2}) = 1$ 知 $m_{\alpha_1}m_{\alpha_2} \mid m_{\alpha_1+\alpha_2}$.

$m_{\alpha_1}m_{\alpha_2}$ 可零化 $\alpha_1+\alpha_2$ 即可

Check $m_{\alpha_1}^{(\varphi)} m_{\alpha_2}^{(\varphi)}(\alpha_1+\alpha_2) = \underbrace{m_{\alpha_2}(\varphi)}\left(m_{\alpha_1}(\varphi)(\alpha_1)\right) + m_{\alpha_1}(\varphi)\left(m_{\alpha_2}(\varphi)(\alpha_2)\right)$

$= 0.$

$\Rightarrow m_{\alpha_1+\alpha_2} = m_{\alpha_1}m_{\alpha_2} \Rightarrow \deg m_{\alpha_1+\alpha_2} = \deg m_{\alpha_1} + \deg m_{\alpha_2}$.

$\Rightarrow \dim F[\varphi](\alpha_1+\alpha_2) = \dim F[\varphi]\alpha_1 + \dim F[\varphi]\alpha_2$.

(2) $g(\varphi)\beta = 0$ $m_\beta \mid g$.

$m_\beta(\varphi)p(\varphi)\alpha = m_\beta(\varphi)\beta = 0 \Rightarrow \underbrace{g(\varphi)p(\varphi)}_{m_\alpha} \mid m_\beta p(\varphi) \Rightarrow g \mid m_\beta$ ✓.

(3). 设

$$m_{\alpha_1}(\lambda) = p_1^{a_1}(\lambda)\cdots p_s^{a_s}(\lambda)\,p_{s+1}^{a_{s+1}}(\lambda)\cdots p_t^{a_t}(\lambda)$$

$$m_{\alpha_2}(\lambda) = p_1^{b_1}(\lambda)\cdots p_s^{b_s}(\lambda)\,p_{s+1}^{b_{s+1}}(\lambda)\cdots p_t^{b_t}(\lambda)$$

即为 m_{α_1}. m_{α_2} 的标准分解 其中 $a_i, b_i \geq 0$ 且 $a_i, b_i \in \mathbb{N}$. $a_i + b_i > 0$

对 $1\le i\le s$ 有 $a_i \ge b_i$　　$s+1\le i\le t$, $a_i < b_i$

$$[m_{\alpha_1}, m_{\alpha_2}] = p_1^{a_1}(\lambda)\cdots p_s^{a_s}(\lambda)\, p_{s+1}^{b_{s+1}}(\lambda)\cdots p_t^{b_t}(\lambda)$$

令 $\alpha_3 = p_{s+1}^{a_{s+1}}(\varphi)\cdots p_t^{a_t}(\varphi)\,\alpha_1$,　$\alpha_4 = p_1^{b_1}(\varphi)\cdots p_s^{b_s}(\varphi)\,\alpha_2$

由(1) $(m_{\alpha_3}, m_{\alpha_4}) = 1$　由(1)　$m_{\alpha_3+\alpha_4} = [m_{\alpha_1}, m_{\alpha_2}]$

Thm 2. (循环空间的准素分解定理)　$\varphi\in\mathcal{L}(V)$ 同上, $0\ne\alpha\in V$, 若

$$m_\alpha(\lambda) = p_1^{e_1}(\lambda)\cdots p_t^{e_t}(\lambda) \text{ 为标准分解}$$

记 $\varphi_0 = \varphi|_{F[\varphi]\alpha}$, 则 $\exists\ \alpha_1,\dots,\alpha_t\in V$,

(1)　$F[\varphi]\alpha_i = \mathrm{Ker}\, p_i^{e_i}(\varphi_0)$,　$\alpha = \alpha_1+\cdots+\alpha_t$.

(2)　$F[\varphi]\alpha = F[\varphi]\alpha_1 \oplus\cdots\oplus F[\varphi]\alpha_t$

(3)　$\varphi_i = \varphi_0|_{F[\varphi]\alpha_i}$,　$m_{\alpha_i}(\lambda) = m_{\varphi_i} = f_{\varphi_i} = p_i^{e_i}(\lambda)$, $1\le i\le t$

(4)　$F[\varphi]\alpha_i$ 不能再分解为两个真 φ-子空间的直和.

(5)　取每一个 $F[\varphi]\alpha_i$ 的循环基凑成 $F[\varphi]\alpha$ 的一组基, φ在这组基下的矩阵为准对角

$$\mathrm{diag}\left\{ F(p_1^{e_1}(\lambda)), \cdots, F(p_t^{e_t}(\lambda)) \right\}.$$

Notes: 搞定. $F(p_1^{e_1}(\lambda)) \sim J(p_1^{e_1}(\lambda))$.

pf: (1). 令 $g_i = \prod_{j \neq i} p_j^{e_j}(\lambda)$, $(g_1, \cdots, g_t) = 1$.

$$\Rightarrow \quad u_1 g_1 + \cdots + u_t g_t = 1. \Rightarrow \alpha = \sum_i u_i(\varphi) g_i(\varphi) \alpha_i = \sum_i \alpha_i$$

$$\alpha_i = u_i(\varphi_0) g_i(\varphi_0) \alpha \in F[\varphi]\alpha \Rightarrow F[\varphi]\alpha_i \subseteq F[\varphi]\alpha.$$

$$\underbrace{F[\varphi]\alpha \subseteq}_{\boxed{\text{天然包含}}} F[\varphi]\alpha_1 + \cdots + F[\varphi]\alpha_t \subseteq F[\varphi]\alpha$$

$$\Rightarrow F[\varphi]\alpha = F[\varphi]\alpha_1 + \cdots + F[\varphi]\alpha_t$$

· $F[\varphi]\alpha_j = \mathrm{Ker}\, p_j^{e_j}(\varphi_0)$

$$\forall f(\varphi)\alpha_j \in F[\varphi]\alpha_j, \quad f(\varphi)\alpha_j = f(\varphi) u_j(\varphi_0) g_j(\varphi_0) \alpha \in \mathrm{Ker}\, p_j^{e_j}(\varphi_0).$$

$$\forall f(\varphi)\alpha \in \mathrm{Ker}\, p_j^{e_j}(\varphi_0) \subseteq F[\varphi]\alpha \Rightarrow g_k(\varphi_0) f(\varphi)\alpha = 0, \quad k \neq j$$

$$f(\varphi)\alpha = f(\varphi)(\alpha_1 + \cdots + \alpha_t) = f(\varphi)\left(\sum_k g_k(\varphi_0) u_k(\varphi_0) \alpha\right)$$

$$= f(\varphi) g_j(\varphi_0) u_j(\varphi_0) \alpha = f(\varphi)\alpha_j \in F[\varphi]\alpha_j.$$

$0 = \sum_i f_i(\varphi)\alpha_i \Rightarrow$ Goal: $f_i(\varphi)\alpha_i = 0.$

$g_j(\varphi_0)(0) = 0 = \sum_i g_j(\varphi_0) f_i(\varphi)\boxed{\alpha_i} = \boxed{g_i(\varphi_0)} f_i(\varphi)\alpha_i = \alpha$ 不化素

$1 = u_1 g_1 + \cdots + u_t g_t$ 即可. $\quad g_j(\varphi_0) f_i(\varphi)\alpha_i = 0,\ j \neq i$

天然截之.

$$\overset{1}{(u_1 g_1 + \cdots + u_t g_t)} f_i(\varphi)\alpha_i = 0.$$

$\Rightarrow f_i(\varphi)\alpha_i = 0$ $\qquad$ 0元素分解唯一.

(3) $F[\varphi]\alpha_j = \operatorname{Ker} p_j^{e_j}(\varphi_0)$ $\qquad m_{\alpha_j} = m_{\varphi_j} = p_j^{e_j}(\lambda)$. 显然

$m_\alpha = m_{\alpha_1 + \cdots + \alpha_t} \overset{\text{Prt.1}}{=\!=} [p_1^{l_1}(\lambda) \cdots p_t^{l_t}(\lambda)] = p_1^{l_1}(\lambda) \cdots p_t^{l_t}(\lambda)$

$= p_1^{e_1}(\lambda) \cdots p_t^{e_t}(\lambda).$

$\Rightarrow l_i = e_i.$

(4) $F[\varphi]\alpha_j = U_1 \oplus U_2$, U_i为φ-子空间, 记 $\varphi|_{U_i} = \varphi_i$

$m_{\varphi_i} \mid m_{\varphi_j} = p_j^{e_j}(\lambda)$ $\quad$ 设 $m_{\varphi_1} = p_j^{\tilde{e}_1}(\lambda)$, $m_{\varphi_2} = p_j^{\tilde{e}_2}(\lambda)$, $\tilde{e}_1 + \tilde{e}_2 = e_j$

不妨设 $\tilde{e}_1 \geq \tilde{e}_2 \geq 1.$

$$\Rightarrow m_{\varphi_j} = [m_{\varphi_1}, m_{\varphi_2}] = p_j^{\max(\tilde{e}_1, \tilde{e}_2)}(\lambda) \neq p_j^{e_j}(\lambda).$$

$\square$.

· 与 $J(\lambda_0, n)$ 可交换的必可表为 $J(\lambda_0, n)$ 的多项式，次数 $\le n-1$

$A \sim$ Frobenius 块　　$AB=BA$　　$\exists \deg f \le n-1$　$B=f(A)$.

§2. 不变因子与 Frobenius 标准形

idea: $V = F[\varphi]\alpha_1 \oplus \cdots \oplus F[\varphi]\alpha_k$.

取 $0 \ne \alpha_1 \in V$, $F[\varphi]\alpha_1$，取 $\alpha_2 \notin F[\varphi]\alpha_1$ 但 $\alpha_2 \in V$，$F[\varphi]\alpha_2, \cdots$

Q: 1° 先选循环空间？　2° 直和分解？

Def 2.1 $\dim_F V = n$. $\varphi \in \mathcal{L}(V)$，$W$ 为 V 的真子空间，称 W 为 φ-相容子空间，若

(1) W 为 φ-子空间　(2) 对 $f(\lambda) \in F[\lambda]$, $\beta \in V$，若 $f(\varphi)\beta \in W$，则必 $\exists$ $\alpha \in W$, s.t. $f(\varphi)\beta = f(\varphi)\alpha$.

Note: $S(\beta, W) := \{ f(\lambda) \in F[\lambda] \mid f(\varphi)\beta \in W \}$

· $S(\beta, W) \ne \emptyset$　　$f_\varphi(\varphi)\beta = 0 \in W$.

· $(S(\beta, W), +)$ 为 Group ✓　· $\forall g \in F$, $f \in S(\ )$，$gf \in S(\ )$ ✓　W 为 φ-子空间　$= fg$.

$\Rightarrow S(\beta, W)$ 为 ideal $:= \langle p(\lambda) \rangle$

$p(\varphi)\beta \in W \Rightarrow \exists \gamma \in W$, $p(\varphi)\beta = p(\varphi)\gamma \Rightarrow p(\varphi)(\beta - \gamma) = 0$. 令 $\alpha = \beta - \gamma$.

$p(\varphi)$ 零化 $\alpha \Rightarrow m_\alpha \mid p$　　$m_\alpha(\beta) = m_\alpha(\alpha + \gamma) = m_\alpha(\gamma) + 0 \in W$

$\Rightarrow m_\alpha \in S(\beta, W) = \langle p \rangle$

$\Rightarrow p \mid m_\alpha \Rightarrow p = m_\alpha$

$$\underbrace{S(\beta, W)} = \langle m_\alpha \rangle = \underbrace{S(\alpha, W)}$$

$f(\varphi)\beta \in W \Rightarrow \boxed{\underline{f(\varphi)}(\alpha+\underline{\gamma}) \in W} \Leftrightarrow f(\varphi)\alpha \in W$

eg. 1. 0空间为 φ-相容的. $f(\varphi)\beta = f(\varphi)0 = 0$.

2. W与U为φ-子空间 $V = W \oplus U \Rightarrow W$与$U$为$\varphi$-相容的.

pf: 不妨设 W为φ-相容的. $f(\varphi)\beta \in W$ $\quad \beta \in V = W \oplus U \Rightarrow \beta = \underset{W}{\underset{\cap}{\alpha}} + \underset{U}{\underset{\cap}{\gamma}}$

Goal: $f(\varphi)\gamma = 0$ $\quad$ Note: $\underset{U}{\underset{\cap}{\boxed{f(\varphi)\gamma}}} = f(\varphi)(\beta-\alpha) = \underset{W}{\underset{\cap}{\underline{f(\varphi)\beta}}} - \underset{W}{\underset{\cap}{\boxed{f(\varphi)\alpha}}}$

$\in U \cap W = \{0\}$

$\Rightarrow f(\varphi)\gamma = 0$. ▭

Lem1 $\quad W$为φ-相容的, $\exists \alpha \notin W$, $\alpha \in V$, $\quad W \cap F[\varphi]\alpha = 0$

pf: Goal: $S(\alpha, W) = \langle m_\alpha \rangle$

取 $\beta \in V$, $S(\beta, W) = \langle p(\lambda) \rangle$ $\quad p(\varphi)\beta \in W \quad \exists\, p(\varphi)\beta = p(\varphi)\gamma$.

令 $\beta - \gamma = \alpha$ $\quad$ 由上述 Note 知 $S(\beta, W) = S(\alpha, W) = \langle m_\alpha \rangle$ ▭

$\underline{\mathcal{L}em_2}$. W 为 φ-相容的，若 $\exists\, \alpha\in V$. s.t.

(1) $W\cap F[\varphi]\alpha=\{0\}$ (2) α 满足(1) 且 $\dim F[\varphi]\alpha$ 极大

i.e. 对 $\forall \tilde{\alpha}$, s.t. $W\cap F[\varphi]\tilde{\alpha}=\{0\}$. 都有 $\dim F[\varphi]\tilde{\alpha}\leq \dim F[\varphi]\alpha$.

则 $W\oplus F[\varphi]\alpha$ 为 φ-相容的.

pf: · $W\oplus F[\varphi]\alpha$ 为 φ-子空间 ✓.

· $\forall f(\lambda)\in F[\lambda]$. $\beta\in V$, $f(\varphi)\beta\in W\oplus F[\varphi]\alpha$.

Q: $\exists ?\ \alpha_1\in W\oplus F[\varphi]\alpha$, $\underbrace{f(\varphi)}_{\delta}\beta=f(\varphi)\alpha_1$.

$S(\beta, W\oplus F[\varphi]\alpha)=\langle p(\lambda)\rangle$ 下证 $p(\varphi)\beta=p(\varphi)\alpha$. $\exists\alpha$.

$p(\varphi)\beta\in W\oplus F[\varphi]\alpha := \gamma+g(\varphi)\alpha$, $\gamma\in W$, $g\in F[\lambda]$

$\boxed{\underset{\sim}{\text{若 } p\mid g.}}$, $\exists\, g=g=pg$ $p(\varphi)\beta=\gamma+p(\varphi)g(\varphi)\alpha$

$\Rightarrow p(\varphi)(\underbrace{\beta-g(\varphi)\alpha}_{\in W})=\gamma\in W$ 为 φ-相容.

$\Rightarrow \exists\gamma_1$, $p(\varphi)(\beta-g(\varphi)\alpha)=p(\varphi)\gamma_1$

$\Rightarrow$ $p(\varphi)\beta=p(\varphi)\underbrace{(\gamma_1+g(\varphi)\alpha)}_{=\ \alpha_1\ \in W\oplus F[\varphi]\alpha}$.

只证 $p \mid g$ 成立. 条件: $p(\varphi)\beta = \underbrace{\gamma}_{\in W} + g(\varphi)\alpha$, $\dim F[\varphi]\alpha$ 极大.

反证. $g = qp + r$, $\deg r < \deg p$ $\qquad p(\varphi)\beta = \gamma + q(\varphi)p(\varphi)\alpha + r(\varphi)\alpha$

$$\Rightarrow p(\varphi)[\beta - q(\varphi)\alpha] = \underbrace{\gamma + r(\varphi)\alpha}_{W \oplus F[\varphi]\alpha}$$

令 $\beta_1 = \beta - q(\varphi)\alpha$

$$S(\beta_1, W) \subseteq S(\beta_1, W \oplus F[\varphi]\alpha) = S(\beta, W \oplus F[\varphi]\alpha) = \langle p(\lambda) \rangle$$

设 $S(\beta_1, W) = \langle p(\lambda)h(\lambda) \rangle \Rightarrow p(\varphi)h(\varphi)\beta_1 \in W$

$$\underbrace{p(\varphi)h(\varphi)\beta_1}_{\in W} = \underbrace{h(\varphi)\gamma}_{\in W} + \underbrace{h(\varphi)r(\varphi)\alpha}_{\in F[\varphi]\alpha}$$

$$\Rightarrow p(\varphi)h(\varphi)\beta_1 - h(\varphi)\gamma = h(\varphi)r(\varphi)\alpha \in W \cap F[\varphi]\alpha = \{0\}.$$

$$\Rightarrow \begin{cases} p(\varphi)h(\varphi)\beta_1 = h(\varphi)\gamma \in W \\ h(\varphi)r(\varphi)\alpha = 0 \Rightarrow m_\alpha(\lambda) \mid h(\lambda)r(\lambda) \end{cases}$$

$$\Rightarrow \boxed{\dim F[\varphi]\alpha} \leq \deg h(\lambda)r(\lambda) \leq$$

W φ-相容 $\Rightarrow$ $\exists \gamma_1 \in W$ $\quad p(\varphi)h(\varphi)\beta_1 = p(\varphi)h(\varphi)\gamma_1$

令 $\beta_2 = \beta_1 - \gamma_1 \Rightarrow p(\varphi)h(\varphi)\beta_2 = 0 \qquad m_{\beta_2} \mid h(\lambda)p(\lambda)$

$$m_{\beta_2}(\varphi)\beta_1 = m_{\beta_2}(\varphi)(\gamma_1 + \beta_2) = m_{\beta_2}(\varphi)\gamma_1 \in W$$

$\Rightarrow m_{\beta_2} \in S(\beta_1, W) = \langle p(\lambda)h(\lambda)\rangle$

$\Rightarrow ph \mid m_{\beta_2} \Rightarrow ph = m_{\beta_2}$ $\left[\,\underline{\deg m_{\beta_2}} = \deg ph\,\right]$

$\Downarrow$ $> \deg rh = \underline{\deg m_\alpha}$.

$\underline{\dim F[\varphi]\beta_2 \text{ 更大}}$ 矛盾!

$\boxed{///}$

Cor $S(\beta, W \oplus F[\varphi]\alpha) = \langle p(\lambda)\rangle$ 且 $\dim F[\varphi]\alpha$ 满足 Lem$_2$ 假设

则对 $p(\beta) = \gamma + g(\varphi)\alpha$ 这个唯一分解 必有 $p \mid g$.

Lem$_3$ W 为 φ-不变子空间, $F[\varphi]\beta \cap \underbrace{\boxed{W \oplus F[\varphi]\alpha}}_{W_1} = \{0\}$, Lem$_2$ 条件下

$\Rightarrow m_\beta \mid m_\alpha$.

pf: $S(\beta, W_1) = \langle m_\beta(\lambda)\rangle$ ✓

$\Rightarrow m_\beta(\beta) = 0 \in W_1 = W \oplus F[\varphi]\alpha := \gamma + g(\varphi)\alpha$

0 的分解唯一 $\Rightarrow \gamma = 0$, $g(\varphi)\alpha = 0 = m_\alpha(\varphi)\alpha$.

$\Rightarrow \underline{m_\beta(\varphi)\beta = \gamma + m_\alpha(\varphi)\alpha} \Rightarrow m_\beta \mid m_\alpha$.

$\boxed{///}$

Thm (Frobenius) $\dim_F V = n$, $\varphi \in \mathcal{L}(V)$, $\exists\, \alpha_1, \cdots, \alpha_k \in V$

$$V = F[\varphi]\alpha_1 \oplus \cdots \oplus F[\varphi]\alpha_k.$$

其中 $\dim F[\varphi]\alpha_i = \deg m_{\alpha_i}(\lambda)$ 且 $m_{\alpha_{i+1}}(\lambda) \mid m_{\alpha_i}(\lambda)$

不变因子.

pf: $\{0\}$ φ-相容 $\exists\, \alpha_1$. $0 \cap F[\varphi]\alpha_1 = \{0\}$ $\Omega_1 = \{\dim F[\varphi]\alpha_1 \mid 0 \cap F[\varphi]\alpha_1 = 0\}$

非空 $\Rightarrow$ 取极大元.

$\{0\} \oplus F[\varphi]\alpha_1$ 为 φ-相容的 $\exists\, \alpha_2$, $F[\varphi]\alpha_1 \cap F[\varphi]\alpha_2 = \{0\}$

$\Omega_2 := \{\alpha: F[\varphi]\alpha_2 \mid F[\varphi]\alpha_1 \cap F[\varphi]\alpha_2 = \{0\}\}$ 非空.

取α极大元 $\Rightarrow$ $m_{\alpha_2} \mid m_{\alpha_1}$

由 $\dim_F V$ 有限 上述构造有限步完成 $\square$.

$$m_\varphi = \underline{p_1}^{e_1} \cdots \underline{p_t}^{e_t}.$$

不变因子 $V = F[\varphi]\alpha_1 \oplus \cdots \oplus F[\varphi]\alpha_k$

准素分解

$$F[\varphi]\alpha_1 = F[\varphi]\alpha_{11} \oplus \cdots \oplus F[\varphi]\alpha_{1t}$$

$$\vdots$$

$$F[\varphi]\alpha_k = F[\varphi]\alpha_{k1} \oplus \cdots \oplus F[\varphi]\alpha_{kt}.$$

初等因子 $V = \operatorname{Ker} p_1^{e_1}(\varphi) \oplus \cdots \oplus \operatorname{Ker} p_t^{e_t}(\varphi)$

$F(p^e(\lambda))$ $\deg p(\lambda) = d$ $J(p^e(\lambda))$ p irreducible.

$$\begin{bmatrix} \alpha & & & \vdots \\ 1 & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ & & 1 & -a \end{bmatrix} \quad \begin{bmatrix} F(p(\lambda)) & & & \\ C & F(p(\lambda)) & & \\ & \ddots & \ddots & \\ & & C & F(p(\lambda)) \end{bmatrix}$$

循环不动了

$$(\alpha, \dots, \varphi^{ed-1}(\alpha))$$

循环

$$f\begin{bmatrix} F(p(\lambda)) & & & \\ E & F(p(\lambda)) & & \\ & \ddots & \ddots & \\ & & E & F(p(\lambda)) \end{bmatrix} = \begin{bmatrix} f(F(p(\lambda))) & & \\ f' & & \\ \frac{1}{2!}f'' & & \\ \vdots & & \\ \frac{1}{(n-1)!}f^{(n-1)} & \cdots & \end{bmatrix}$$

计算3段.

$$\varphi \mid \alpha, \dots, \varphi^{d-1}(\alpha), \ p(\varphi)(\alpha), \ p(\varphi)\varphi(\alpha), \dots, p(\varphi)\varphi^{d-1}(\alpha), \dots \quad) = (\quad)$$

f循环

$$\begin{bmatrix} 0 & & & & & & \\ 1 & \ddots & & & & & \\ & 1 & 0 & & & & \\ & & 1 & 0 & & & \\ & & & 1 & \ddots & & \\ & & & & 1 & 0 & \\ & & & & & 1 & \ddots \end{bmatrix}$$

$p^{e-1}(\varphi), \dots, p^{e-1}(\varphi)\varphi^{d-1}(\alpha)$

$$\bigcup_{j=0}^{e-1} p^j(\varphi)\{\alpha, \varphi\alpha, \dots, \varphi^{d-1}\alpha\}$$

$$\sum_{i=0}^{d-1}\sum_{j=0}^{e-1} p^j(\varphi)\, c_{ij}\, \varphi^i \alpha = 0 \xRightarrow{\text{同系}} p^{e-1}(\varphi)$$

$$\sum_{i=0}^{d-1} \underline{p^{e-1}}(\varphi)\, c_{i,e-1}\, \underline{\varphi^{i}(\alpha)} = 0$$

$$\boxed{\sum_{i=0}^{d-1} c_{i,e-1}\,\underline{\varphi^{i}}} \left[\underline{p^{e-1}(\varphi)(\alpha)}\right] = 0 \Rightarrow \boxed{\underline{c_{i,e-1}} = 0}$$

$$\underbrace{{}^{m}\, p^{e-1}(\varphi)(\alpha)}\ = p(\varphi).$$

$$\underline{F[\varphi]\left(p^{e-1}(\varphi)(\alpha)\right)} \sim \underline{F(p(\varphi))}$$

$\boxed{1\,1\,1}$

$$\begin{bmatrix} \lambda_0 & & \\ 1 & \ddots & \\ & \ddots & \lambda_0 \end{bmatrix}_n \longrightarrow$$ 第1类且的有2类了α Jordan块

$1 \to E$

$(\varphi - \lambda_0 id_V)$ 一个循不变空间

$\to C$

$$\bigcup_{i=0}^{n-1} \underline{(\varphi - \lambda_0 id_V)}^{i} \left\{ \alpha,\ \cancel{\varphi^{i-1}\alpha} \right\}$$

$$\downarrow$$

$$p^{i}(\lambda)$$

$$f\begin{bmatrix} \lambda_0 & & \\ 1 & \ddots & \\ & \ddots & \\ & 1 & \lambda_0 \end{bmatrix} = \begin{bmatrix} f(\lambda_0) & & & \\ f'(\lambda_0) & \ddots & & \\ \frac{1}{2!}f''(\lambda_0) & \ddots & \ddots & \\ \vdots & & & \\ \frac{1}{(m-1)!}f^{(m-1)}(\lambda_0) & \cdots & & \end{bmatrix}$$

$$\underline{J(\lambda_0, e)^{m} \sim J(\lambda_0^{m}, e)}, \quad \lambda_0 \neq 0$$

§3 准素分解定理

Thm 3.1 (空间的准素分解定理) 设 $\varphi \in \mathcal{L}(V)$ ($\mathrm{End}_F(V)$)，$m_\varphi(\lambda) = p_1^{e_1}(\lambda) \cdots p_t^{e_t}(\lambda)$，

$f_\varphi(\lambda) = p_1^{n_1}(\lambda) \cdots p_t^{n_t}(\lambda)$ 其中 $p_1 \cdots p_t$ 为两两互素的首一不可约。$n_i \geq e_i \geq 1$.

$1 \leq i \leq t$，$n_i, e_i \in \mathbb{N}^*$. 令 $g_i(\lambda) = \prod_{j \neq i} p_j^{e_j}(\lambda)$，$\varphi_i = \varphi|_{\mathrm{Ker}\, p_i^{e_i}(\lambda)}$

(1) $\mathrm{Im}\, g_i(\varphi) = \mathrm{Ker}\, p_i^{e_i}(\varphi) = \mathrm{Ker}\, p_i^{n_i}(\varphi)$

(2) $V = \mathrm{Ker}\, p_1^{e_1}(\varphi) \oplus \cdots \oplus \mathrm{Ker}\, p_t^{e_t}(\varphi)$

(3) $\dim \mathrm{Ker}\, p_i^{e_i}(\varphi) = n_i \deg p_i(\lambda)$，$m_{\varphi_i}(\lambda) = p_i^{e_i}(\lambda)$，$f_{\varphi_i}(\lambda) = p_i^{n_i}(\lambda)$

pf: (1) $\mathrm{Im}\, g_i(\varphi) \subseteq \mathrm{Ker}\, p_i^{e_i}(\varphi)$ ✓

$\forall \alpha \in \mathrm{Ker}\, p_i^{e_i}(\varphi)$，$g_j(\varphi)\alpha = 0$，对 $\forall j \neq i$. $\because (g_1, \cdots, g_t) = 1 \Rightarrow 1 = u_1 g_1 + \cdots + u_t g_t$.

$\Rightarrow \alpha = \sum_k u_k(\varphi) g_k(\varphi) \alpha = u_i(\varphi) g_i(\varphi) \alpha = g_i(\varphi)\big(u_i(\varphi)\alpha\big) \in \mathrm{Im}\, g_i(\varphi)$.

Goal: $\mathrm{Ker}\, p_i^{e_i}(\varphi) = \mathrm{Ker}\, p_i^{n_i}(\varphi)$.

$p_i^{n_i - e_i}(\varphi) \boxed{m_\varphi(\varphi)} = 0 \Rightarrow p_i^{n_i}(\varphi) g_i(\varphi) = 0 \Rightarrow$ 由上述同理 $\mathrm{Ker}\, p_i^{n_i}(\varphi) = \mathrm{Im}\, g_i(\varphi)$

$= \mathrm{Ker}\, p_i^{e_i}(\varphi)$.

(2) 由 $\alpha = \sum u_i(\varphi) g_i(\varphi) \alpha \Rightarrow V \subseteq \mathrm{Im}\, g_1 + \cdots + \mathrm{Im}\, g_t = \sum \mathrm{Im}\, g_i \subseteq V$

$$\Rightarrow V = \sum \mathrm{Im}\, g_i$$

$\forall \alpha \in V,\ 0 = \sum g_i \alpha_i,\ \alpha_i \in \mathrm{Im}\, g_i$

$\Rightarrow 0 = g_k \sum g_i \alpha_i = \boxed{g_k} g_k \alpha_k$，对 k 成立

又由 $0 = \boxed{g_j} g_k \alpha_k$ 对 $j \neq k$ 成立

$\Rightarrow \left(\underline{\sum_i u_i g_i}\right) g_k \alpha_k = 0$ ① / id

$\Rightarrow g_k \alpha_k = 0$. 0元素分解唯一.

$\Rightarrow V = \sum \mathrm{Im}\, g_i = \bigoplus \mathrm{Im}\, g_i = \bigoplus \mathrm{Ker}\, p_i^{e_i}(\varphi)$.

(3)

$V_i := \mathrm{Ker}\, p_i^{e_i}(\varphi) \quad \varphi_i = \varphi|_{V_i} \quad p_i^{e_i}(\varphi_i) = 0 \Rightarrow m_{\varphi_i} \mid p_i^{e_i}(\lambda)$

设 $m_{\varphi_i} = p_i^{l_i}(\lambda) \Rightarrow m_\varphi = [m_{\varphi_1} \cdots m_{\varphi_n}] \Rightarrow l_i = e_i$.

由 $m_{\varphi_i} \mid f_{\varphi_i}$ 设 $f_{\varphi_i} = p_i^{\tilde{l}_i}(\lambda)$, $\tilde{l}_i \geq e_i$ $\quad f_{\varphi_1} \cdots f_{\varphi_n} = f_\varphi \Rightarrow \tilde{l}_i = n_i$.

$\dim \mathrm{Ker}\, p_i^{n_i}(\varphi) = \varphi_i$ 的表示矩阵阶数 $= \deg f_{\varphi_i} = n_i \deg p_i(\lambda)$.

$\boxed{(1)}$

Thm 7.2 (根子空间的循环分解) $\varphi \in \mathcal{L}(V)$, $V = \mathrm{Ker}\, p^e(\varphi)$，其中 $p(\lambda)$ 为 F 上首一不可约.

则 $\exists\ \alpha_1, \cdots, \alpha_t \in V$，使 $V = \bigoplus_{i=1}^{t} F[\varphi]\alpha_i$

pf: 记 $\deg p(\lambda) = l$，对 e 用归纳法.

- $e=1$ 时. $V = \mathrm{Ker}\, p(\varphi) \Rightarrow m_\varphi(\lambda) = \underline{p(\lambda)}_{\hookrightarrow \text{不可约}}$. $\forall \alpha \in V,\ m_\alpha(\lambda) = p(\lambda)$.

$\underset{\Delta}{p(\varphi)=0}$, $\boxed{F[\varphi]}:=\{f(\varphi)\mid f\in F[\lambda]\}$ = 关于 φ 不超过 $l-1$ 次多项式

clai: 环 → 成为一个域

Group 无逆元 (非0) 有逆元

$\forall f(\varphi)\in F[\varphi]$. $f(\varphi)\neq 0$, 则 $(f,p)=1$ ⇒ $uf+\boxed{vp}=1$ ⇒ $u(\varphi)f(\varphi)=id$

⇒ $u(\varphi)=(f(\varphi))^{-1}$

$\dim_{F[\varphi]}V\leq \dim_F V=n$ 有限维

$$\Rightarrow\quad V=F[\varphi]\alpha_1\oplus\cdots\oplus F[\varphi]\alpha_t.$$

· $e\geq 2$ 时, 用第二小归纳假设 假设对 $<e$ 时成立, $=e$ 时,

$p(\varphi)$: $\operatorname{Ker}p^e(\varphi)\to \operatorname{Im}p(\varphi)=\operatorname{Ker}p^{e-1}(\varphi)$

在 $\operatorname{Im}p(\varphi)$ 中用归纳假设 $\exists\ \tilde\alpha_1,\cdots,\tilde\alpha_t\in \operatorname{Im}p(\varphi)\subseteq V$, 记 $m_{\tilde\alpha_i}(\lambda)=p^{\tilde e_i}(\lambda)$,

其中 $1\leq \tilde e_i\leq e-1$. 同时取 $\tilde\alpha_i=p(\varphi)\alpha_i$, $m_{\alpha_i}=p^{\tilde e_i+1}(\lambda):=p^{e_i}(\lambda)$, $2\leq e_i=\tilde e_i+1$

$\leq e$. 有 $\operatorname{Im}p(\varphi)=F[\varphi]\tilde\alpha_1\oplus\cdots\oplus F[\varphi]\tilde\alpha_t$.

取 $\operatorname{Im}p(\varphi)$ 的一组准循环基 $\displaystyle\bigcup_{i=1}^{t}\bigcup_{j=0}^{\tilde e_i-1}p^j(\varphi)\{\tilde\alpha_i,\varphi(\tilde\alpha_i),\cdots\varphi^{l-1}(\tilde\alpha_i)\}$

$$\Big\downarrow\ \tilde\alpha_i=p(\varphi)\alpha_i$$

$$\bigcup_{i=1}^{t}\bigcup_{j=1}^{\tilde e_i}p^j(\varphi)\{\alpha_i,\varphi(\alpha_i)\cdots\varphi^{l-1}(\alpha_i)\}$$

$\tilde{e}_{i-1}$ $\uparrow \tilde{e}_i$. 循环 既为了 $p(\varphi)$

不是V中那个循环： $\bigcup\limits_{i=1}^{t} \bigcup\limits_{j=0} p^{j}(\varphi)\{\alpha_i, \varphi(\alpha_i) \cdots \varphi^{l-1}(\alpha_i)\}$

α_i $m_{\alpha_i} = p^{\tilde{e}_{i+1}}$. linearly independent

$\boxed{l.\ ind.}$?

利用若 $\varphi(\alpha_1) \cdots \varphi(\alpha_s)$ l. ind. $\Rightarrow$ $\alpha_1 \cdots \alpha_s$ l. ind.

Notice $\bigcup\limits_{i=1}^{t} \underbrace{p^{\tilde{e}_i}(\varphi)}_{\Delta}\{\alpha_i, \varphi(\alpha_i) \cdots \varphi^{l-1}(\alpha_i)\} \subseteq \boxed{\operatorname{Ker} p(\varphi)}$

$\boxed{\text{将上述扩为}}$ $\operatorname{Ker} p(\varphi)$ 的一组基. (循环子空间的直和).

$\boxed{\operatorname{Ker} p(\varphi)}$ $F[\varphi] \to$ 模

扩出的每一个基 $\to$ 循环子空间

$$\operatorname{Ker} p(\varphi) = \left(\bigoplus_{i=1}^{t} \left(\underline{p^{\tilde{e}_i}(\varphi)\alpha_i} \right)_{F[\varphi]} \right) \oplus \left(\bigoplus_{i=t+1}^{s} F[\varphi]\alpha_i \right)$$

$\alpha_{t+1}, \cdots \alpha_s$

将上述两个集合在一起

$$\bigcup_{i=1}^{t} \bigcup_{j=0}^{\tilde{e}_i} p^{j}(\varphi)\{\alpha_i, \varphi(\alpha_i) \cdots \varphi^{l-1}(\alpha_i)\} \bigcup \bigcup_{i=t+1}^{s} \{\alpha_i, \varphi(\alpha_i) \cdots \varphi^{l}(\alpha_i)\}$$

上述集合 $= \dim \operatorname{Im} p(\varphi) + \dim \operatorname{Ker} p(\varphi) = \dim V$.

Check l. ind. 对于上述线性组合 $\sum C_i \square$ 左作用 $p(\varphi) \to \operatorname{Im} p(\varphi)$ 一组基 $\Rightarrow$ C_i 部分为0

剩下部分为 $\operatorname{Ker} p(\varphi)$ 一组基 $\Rightarrow C_i \equiv 0$.

α 取了每一条非循环链 $\to J(\ ) \to F[\varphi]\alpha$. 证毕 $\square$

§4 空间分解的唯一性讨论

Thm 在§2 & §3中得到的基下的分块对角阵在不计次序的意义下唯一.

pf: 只证给定 φ 之后, $J(p_i^{e_{ij}}(\lambda))$ 的个数唯一确定即可.

记 $\deg p_i(\lambda) = d_i$. 由空间的根子分解 $V = \bigoplus_{i=1}^{t} \operatorname{Ker} p_i^{e_i}(\varphi)$.

$J(p_i^{e_{ij}}(\lambda))$ 的个数 $=$ $\left.\begin{array}{l} \text{循环链} \subseteq \operatorname{Ker} p_i^{e_i}(\varphi) \\ \text{长度} = e_{ij} \end{array}\right\}$ 非循环链的个数.

只须从 $\operatorname{Ker} p_i^{e_i}(\varphi)$ 中寻找即可 $\to m_{\alpha_{ik}} = p_i^{m_k}(\lambda)$

↳ 基 $\bigcup_{k=1}^{s} \bigcup_{j=1}^{m_k - 1} p_i^{j}(\varphi)\{\alpha_{ik}, \varphi\alpha_{ik}, \ldots, \varphi^{d_i-1}\alpha_{ik}\}$

对 m_k 排序: 设 $m_1 = e_i \geq m_2 \geq \cdots m_s$

$$\{\alpha_{i1}\cdot\varphi^{d_i-1}(\alpha_{i1})\}$$

向下的构造

$p_i^{e_i}$, $p_i^{e_i-1}$, …, p_i^{l+1}, p_i^{l}, p_i^{l-1}, …, p_i

$$p_i(\varphi)\{\alpha_{i1}\cdot\varphi^{d_i-1}(\alpha_{i1})\} \qquad \{\alpha_{i2}\cdots\varphi^{d_i-1}(\alpha_{i2})\}$$

$$p_i^2(\varphi)\{\alpha_{i1}\cdot\varphi^{d_i-1}(\alpha_{i1})\} \qquad p_i(\varphi)\{\alpha_{i2}\cdots\varphi^{d_i-1}(\alpha_{i2})\}$$

$\vdots$ — — — … — — —

$$\underset{e_i}{p_i^{m_1-1}}(\varphi)\{\alpha_{i1}\cdot\varphi^{d_i-1}(\alpha_{i1})\} \qquad p_i^{m_2-1}(\varphi)\{\alpha_{i2}\cdots\varphi^{d_i-1}(\alpha_{i2})\} \qquad \{\alpha_{is}\cdots\varphi^{d_i-1}(\alpha_{is})\} \to \boxed{p_i}$$

一组基

dim $\operatorname{Ker} p_i^{l}(\lambda)$, $\boxed{1\le l\le m_1=e_i}$ $= l\,d_i$ (不论是否循环的小链) $\boxed{f=m}$

1 [Frobenius

$$J(p_i^{l}(\lambda))=\frac{1}{d}\left[2\dim\operatorname{Ker}p_i^{l}(\varphi)-\dim\operatorname{Ker}p_i^{l+1}(\varphi)-\dim\operatorname{Ker}p_i^{l-1}(\varphi)\right]$$

$n-r[p_i^{l}(\varphi)]$

$l+1 \to \#J(p_i^{\ge l+1}(\varphi))$

$l \to \#J(p_i^{\ge l}(\varphi))$

$l-1$

$\Rightarrow J(p_i^{l}(\lambda))$ 个数

$$=\frac{1}{d}\#J(p_i^{\ge l}(\lambda))-\#J(p_i^{\ge l+1}(\lambda))$$

$$=\frac{1}{d}\left(\dim\operatorname{Ker}p_i^{l}(\varphi)-\dim\operatorname{Ker}p_i^{l-1}(\varphi)\right)-\left(\dim\operatorname{Ker}p_i^{l+1}(\varphi)-\dim\operatorname{Ker}p_i^{l}(\varphi)\right)$$

证: $J(\lambda_0,k)$ 个数 $= \frac{1}{1}\left[2\dim\ker(\varphi-\lambda_0 id)^k - \dim\ker(\varphi-\lambda_0 id)^{k+1} - \dim\ker(\varphi-\lambda_0 id)^{k-1}\right]$

$$= 2\left[n-r(A-\lambda_0E)^k\right] - 2n + r(A-\lambda_0E)^{k+1} + r(A-\lambda_0E)^{k-1}$$

$$= r(A-\lambda_0E)^{k+1} + r(A-\lambda_0E)^{k-1} - 2r(A-\lambda_0E)^k.$$

例 1. $J(\lambda_0,n)$ 的所有 φ-不变子空间 个数 $n+1$ 个 $L(e_r,\cdots,e_n)$ $1\le r\le n$ 和 0.

$$\varphi(e_1,\cdots,e_n)=(e_1,\cdots,e_n)\begin{bmatrix}\lambda & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \lambda\end{bmatrix}$$

2. $\varphi(e_1,\cdots,e_n)=(e_1,\cdots,e_n)\operatorname{diag}\{\lambda_1,\cdots,\lambda_n\}$, λ_i 互不相同, φ-不变子空间个数 2^n.

$L(e_{i_1},\cdots,e_{i_r})$ $0\le r\le n$

$(i_1,\cdots,i_r)$ 为 $(1,\cdots,n)$ 中的 r-排列

Prop F V 存在有限个 φ-不变子空间 $\Leftrightarrow$ $f_\varphi=m_\varphi=p_1^{e_1}(\lambda)\cdots p_t^{e_t}(\lambda)$ 为标准分解 (F上)

则 φ-不变子空间共有 $(e_1+1)\cdots(e_t+1)$ 个, 具体而言 φ-不变子空间

$$U=\bigoplus \operatorname{Ker} p_i^{l_i}(\varphi) \quad (0\le l_i\le e_i).$$

pf: 由空间的准素分解 $V=\bigoplus \operatorname{Ker} p_i^{e_i}(\varphi) := \bigoplus V_i$

Claim: $\forall$ φ-不变子空间 U, $\exists$ $U_i\subseteq V_i$ 为 φ-不变子空间, $U=\bigoplus_{i=1}^{t} U_i$

取 $U_i = V_i \cap U$，显然 $\sum U_i \subseteq U$ 且 $\sum U_i = \oplus U_i$ （由 $\oplus V_i$） 只证 $U \subseteq \sum U_i$

$\forall u \in U$，$u = \sum u_i$，$u_i \in V_i$。若 $u_i \in U$，则 $u_i \in U \cap V_i = U_i$。
$\subseteq V$

构造 $g(\lambda) \in F[\lambda]$ 使 $\begin{cases} g \equiv 1 \pmod{p_i^{e_i}} \\ g \equiv 0 \pmod{p_j^{e_j}},\ j \neq i \end{cases}$ 由 CRT $\Rightarrow$ 得到 $g \equiv \tilde{g}$.

$\tilde{g}(\varphi)(u) = \tilde{g}(\varphi)\left(\sum U_i\right) = \tilde{g}(\varphi) u_i = u_i \in U$ （由 $u \in U$ 为 φ-子空间）

设 $\varphi|_{U_i} = \tilde{\varphi}_i$，$\varphi|_{V_i} = \varphi_i$ 　$f_{\tilde{\varphi}_i} \mid f_{\varphi_i} \xlongequal[\text{循环}]{\text{循环}} m_{\varphi_i} = p_i^{e_i}(\lambda)$

φ-子空间

设 $f_{\tilde{\varphi}_i} = p_i^{m_i}(\lambda)$，其中 $0 \le m_i \le e_i$ $\Rightarrow$ $\dim U_i = \deg f_{\tilde{\varphi}_i} = m_i \deg p_i$

由 Cayley-Hamilton Thm. $f_{\tilde{\varphi}_i}(U_i) = 0$ $\Rightarrow$ $\boxed{U_i \subseteq \operatorname{Ker} p_i^{m_i}(\varphi)}$

$f = m$ 循环空间 $\Rightarrow$ 由 $\dim \operatorname{Ker} p_i^{m_i}(\varphi) = m_i \deg p_i = \dim U_i$?

$\Rightarrow U_i = \operatorname{Ker} p_i^{m_i}(\varphi)$.

m_i 唯一确定 U_i 　$0 \le m_i \le e_i$ 　$\Rightarrow \prod_{i=1}^{t} (e_i + 1)$ 个.

下证 $f \neq m$ 时有无限多个 φ-子空间. 即 $\exists V_i$，在 V_i 中 $\sim$ 2个以上的广义 Jordan 块.

取这两个广义 Jordan 块对应的循环向量 $\bigcup_{k=1}^{2} \bigcup_{j=0}^{m_k - 1} p_i^j(\varphi)\left\{\alpha_{ik}, \varphi(\alpha_{ik}) - \cdots \varphi^{\deg p_i - 1}(\alpha_{ik})\right\}$

不能只看 $\operatorname{Ker} p_i(\varphi)$ 元素！

一组基

$$p_i^{m_1-1}(\varphi)\left\{\alpha_{i_1}, \cdots, \varphi^{\deg p_i - 1}(\alpha_{i_1})\right\} = F[\varphi]\left(p_i^{m_1-1}(\varphi)(\alpha_{i_1})\right) = V_{C_1}$$

$$p_i^{m_2-1}(\varphi)\left\{\alpha_{i_2}, \cdots, \varphi^{\deg p_i - 1}(\alpha_{i_2})\right\} = F[\varphi]\left(p_i^{m_2-1}(\varphi)(\alpha_{i_2})\right) = V_{C_2}$$

Claim: $\underbrace{V_\lambda}_{\text{元素的 }\varphi\text{-子空间}} = F[\varphi]\left(\underline{p_i^{m_1-1}(\varphi)(\alpha_{i_1}) + \lambda p_i^{m_2-1}(\varphi)(\alpha_{i_2})}\right)$

$\subseteq \operatorname{Ker} p_i(\varphi)$, $\lambda \in F$.

都是两两不同的 φ-子空间. i.e. $\lambda_1 \neq \lambda_2 \Rightarrow V_{\lambda_1} \neq V_{\lambda_2}$ 即可.

$p_i^{m_1-1}(\varphi)(\alpha_{i_1}) + \lambda_1 p_i^{m_2-1}(\varphi)(\alpha_{i_2})$ 不能被线性表出

$$= \sum_{x=0}^{\deg p_i - 1} c_x \varphi^x \left(p_i^{m_1-1}(\varphi)(\alpha_{i_1}) + \lambda_2 p_i^{m_2-1}(\varphi)(\alpha_{i_2})\right)$$

$$\Rightarrow \left[\left(\boxed{id} - \sum_{x=0}^{\deg p_i - 1} c_x \varphi^x\right)\left(p_i^{m_1-1}(\varphi)\alpha_{i_1}\right)\right] = \left[\left(-\lambda_1 id + \sum_{x=0}^{\deg p_i - 1} c_x \varphi^x\right)\left(p_i^{m_2-1}(\varphi)\alpha_{i_2}\right)\right]$$

$\in V_{C_1}$ 矛盾!

$\Rightarrow 1 = c_0,\ c_i = 0,\ \forall i \neq 0$

$\in V_{C_1} \cap V_{C_2} = \{0\}$.

$\lambda_1 = \lambda_2 c_0$, $\lambda_2 c_i = 0,\ \forall i \neq 0 \Leftarrow c_i = 0$

$= \lambda_2$, 但 $\lambda_1 \neq \lambda_2$ 矛盾!

$\square$