§5.4—§5.5 Standard Factorization and Polynomial Functions

illusion

Especially made for smy

School of Mathematical Science, XMU

Monday 3rd March, 2025

http://illusion-hope.github.io/25-Spring-SMY-Discussion-Session/

HW-2

例 1

设 $f(x), g(x) \in F[x]$，$(f(x), g(x)) = d(x)$，求证：对于任意的正整数 n，

$$(f^n(x), f^{n-1}(x)g(x), \ldots, g^n(x)) = d^n(x).$$

例 2

设非零多项式 $f(x), g(x) \in F[x]$. 证明： $(f(x), g(x)) \neq 1$ 的充分必要条件是存在 $p(x), q(x) \in F[x]$，使得

$$p(x)f(x) = q(x)g(x),$$

其中 $0 \leq \deg p(x) < \deg g(x), 0 \leq \deg q(x) < \deg f(x)$.

Examples

例 3

Assume $f(x), g(x), h(x) \in F[x]$, prove that

(1) $(f(x), g(x), h(x)) = ((f(x), g(x)), h(x))$;

(2) There exists $a(x), b(x), c(x), u(x), v(x), r(x) \in F[x]$ such that

$$(f(x), g(x), h(x)) = \det \begin{bmatrix} f(x) & g(x) & h(x) \\ a(x) & b(x) & c(x) \\ u(x) & v(x) & r(x) \end{bmatrix}.$$

Notes:

- 一般地，(1) 可以推广为
$$((f_1(x), f_2(x), \cdots, f_{n-1}(x)), f_n(x)) = (f_1(x), f_2(x), \cdots, f_{n-1}(x), f_n(x)).$$
- 存在 $u_i(x) \in F[x]$ 使 $\sum_{i=1}^{n} u_i(x) f_i(x) = (f_1(x), \cdots, f_n(x))$.

Properties of Relatively Prime

The following statements are equivalent with $(f(x), g(x)) = 1$:

(1) $u(x)f(x) + v(x)g(x) = 1 \rightsquigarrow 1 \mid f(x), 1 \mid g(x)$ is trivial;

(2) We can conclude $f(x) \mid h(x)$ from $f(x) \mid g(x)h(x)$, for all $h(x) \in F[x]$;

(3) We can conclude $f(x)g(x) \mid h(x)$ from $f(x) \mid h(x),\ g(x) \mid h(x)$, for all $h(x) \in F[x]$;

(4) $(f(x^n), g(x^n)) = 1$ for any given positive integer n;

(5) $(f^n(x), g^n(x)) = 1$ for any given positive integer n;

(6) $(f(x) + g(x), f(x)g(x)) = 1$;

(7) For any $n_1, n_2 \in \mathbf{N}^*$, we hold $(f^{n_1}, g^{n_2}) = (f, g)$;

(8) f, g do not have common irreducible factors in their standard factorization;

(9) f, g do not have common roots over $\mathbf{C}$.

Properties of Relatively Prime

The following statements are equivalent with $(f(x), g(x)) = 1$:

(1) $u(x)f(x) + v(x)g(x) = 1 \rightsquigarrow 1 \mid f(x), 1 \mid g(x)$ is trivial;

(2) We can conclude $f(x) \mid h(x)$ from $f(x) \mid g(x)h(x)$, for all $h(x) \in F[x]$;

(3) We can conclude $f(x)g(x) \mid h(x)$ from $f(x) \mid h(x),\ g(x) \mid h(x)$, for all $h(x) \in F[x]$;

(4) $(f(x^n), g(x^n)) = 1$ for any given positive integer n;

(5) $(f^n(x), g^n(x)) = 1$ for any given positive integer n;

(6) $(f(x) + g(x), f(x)g(x)) = 1$;

(7) For any $n_1, n_2 \in \mathbf{N}^*$, we hold $(f^{n_1}, g^{n_2}) = (f, g)$;

(8) f, g do not have common irreducible factors in their standard factorization;

(9) f, g do not have common roots over $\mathbf{C}$.

Properties of Relatively Prime

The following statements are equivalent with $(f(x), g(x)) = 1$:

(1) $u(x)f(x) + v(x)g(x) = 1 \rightsquigarrow 1 \mid f(x), 1 \mid g(x)$ is trivial;

(2) We can conclude $f(x) \mid h(x)$ from $f(x) \mid g(x)h(x)$, for all $h(x) \in F[x]$;

(3) We can conclude $f(x)g(x) \mid h(x)$ from $f(x) \mid h(x),\ g(x) \mid h(x)$, for all $h(x) \in F[x]$;

(4) $(f(x^n), g(x^n)) = 1$ for any given positive integer n;

(5) $(f^n(x), g^n(x)) = 1$ for any given positive integer n;

(6) $(f(x) + g(x), f(x)g(x)) = 1$;

(7) For any $n_1, n_2 \in \mathbf{N}^*$, we hold $(f^{n_1}, g^{n_2}) = (f, g)$;

(8) f, g do not have common irreducible factors in their standard factorization;

(9) f, g do not have common roots over $\mathbf{C}$.

Properties of Relatively Prime

Try

Just check them:

(1) $(f(x), h(x)) = 1, (g(x), h(x)) = 1 \Leftrightarrow (f(x)g(x), h(x)) = 1.$

(2) Assume $d(x) \neq 1$ is a common divisor of $f(x)$ and $g(x)$, let $f(x) = f_1(x)d(x), g(x) = g_1(x)d(x)$, prove that $(f_1, g_1) = 1 \Leftrightarrow (f, g) = d$. (Important!)

Properties of Relatively Prime $\rightsquigarrow (f, g) = d \neq 1$ Case

Slogan: $d(x) \neq 1$ is a common divisor, $(f_1, g_1) = 1 \Leftrightarrow (f, g) = d$.

例 4

(1) $(f(x), g(x)) = d(x)$, then $(f(x)h(x), g(x)h(x)) = d(x)h(x)$;

(2) $(f(x), g(x)) = 1$, then $(f(x)g(x), h(x)) = (f(x), h(x))(g(x), h(x))$;

(3) Only use two conditions from $(f_i(x), g_j(x)) = 1$ $(i, j = 1, 2)$ to prove that

$$(f_1(x)g_1(x), f_2(x)g_2(x)) = (f_1(x), f_2(x))(g_1(x), g_2(x)).$$

例 5

Prove that $(x^n - 1, x^m - 1) = x^{(m,n)} - 1$, where m, n are given positive integers.

Hint: For $m, n \in \mathbf{Z}$, there always exists $u, v \in \mathbf{Z}$ such that $um + vn = (m, n)$.

Chinese Reminder Theorem (CRT)

Thm 6

Given polynomials $f_1(x), \cdots, f_n(x)$, any two of which are relatively prime, then for $g_1(x), \cdots, g_n(x)$ such that $\deg g_i(x) < \deg f_i(x)$, there exists a unique polynomial $g(x)$ such that

$$g(x) \equiv g_i(x) \pmod{f_i(x)},$$

where $\deg g(x) < \sum_{i=1}^{n} \deg f_i(x)$.

(1) Create one $g(x)$ satisfying all the conditions $\rightsquigarrow \sum_{i=1}^{n} \left\{ u_i(x) \prod_{j \neq i} f_j(x) \right\} \cdot g_i(x)$

(2) Unique $\rightsquigarrow$ Relatively Prime

Lagrange Interpolation Formula

例 7

Suppose $a_1, \cdots, a_m \in F$ are different, for $b_1, \cdots, b_m \in F$, there exists a unique polynomial $L(x)$ such that $L(a_i) = b_i \ (i = 1, 2, \cdots, m)$:

$$L(x) = \sum_{j=1}^{m} b_j \prod_{i \neq j} \frac{x - a_i}{a_j - a_i},$$

where $\deg L(x) < m$.

Hint: $L(a_i) = b_i \Leftrightarrow L(x) \equiv b_i \pmod{x - a_i}$.

Outline of Chapter 5: Polynomial

Polynomial Algebra $\rightsquigarrow$ $F[x]$ is a PID (Principal Ideal Domain)

- Division Theorem, Divisibility
- The (Not A!) Greatest Common Divisor (GCD) and Relatively Prime
- $\rightsquigarrow$ Chinese Reminder Theorem (CRT) $\rightsquigarrow$ Lagrange Interpolation Formula
- In PID, Irreducible $\Leftrightarrow$ Prime
- (PID $\Rightarrow$ UFD) Unique Factorization
- Repeated Factor $\rightsquigarrow$ $(f(x), f'(x)) = 1$?

Polynomial Functions:

- Remainder Theorem
- Roots

Irreducible $\Leftrightarrow$ Prime

Def 8

In $F[x]$, a polynomial $f(x)$ is called prime if for all $g(x), h(x) \in F[x]$, we can conclude $f(x) \mid g(x)$ or $f(x) \mid h(x)$ from $f(x) \mid g(x)h(x)$.

Def 9

In $F[x]$, a polynomial $f(x)$ is called irreducible if

(1) $f(x)$ is not a unit, i.e., $\deg f(x) > 0$;

(2) If we hold $f(x) = g(x)h(x)$ in $F[x]$, then $g(x)$ or $h(x)$ must be a unit.

Notes:

- (Irreducible $\Rightarrow$ Prime) If $f \mid gh$ and $f \nmid g$, we set $(f, g) = d$. Then, d is a divisor of $f \rightsquigarrow d = 1$ or $d = cf$. But we have $f \nmid g$, which implies $(f, g) = 1$ $\rightsquigarrow uf + vg = 1 \rightsquigarrow f \mid ufh = h - vgh \rightsquigarrow f \mid h$. (c is the reciprocal of the leading coefficient of $f(x)$.)

Irreducible $\Leftrightarrow$ Prime

- (Prime $\Rightarrow$ Irreducible) If $f = gh \mid gh$. We have $f \mid g$ or $f \mid h$. WLOG, $f \mid g$. But we also hold $g \mid f \rightsquigarrow g, f$ are associate. Then, h can only be a unit.
- (Cor.1) Assume $p(x)$ is irreducible, then for all $f(x) \in F[x]$, we hold either $p(x) \mid f(x)$ or $(p(x), f(x)) = 1$.
- (Cor.2) Assume $p(x)$ is irreducible, and we have $p(x) \mid f_1(x) f_2(x) \cdots f_n(x)$. Then, there must exists i such that $p(x) \mid f_i(x)$.

例 10

Prove that $f(x), g(x) \in F[x]$ are irreducible at the same time:

(1) $g(x) = f(ax+b),\ a, c \in F, a \neq 0.$

(2) $f(x) = \sum_{k=0}^{n} a_k x^k, g(x) = \sum_{k=0}^{n} a_{n-k} x^k,\ a_i \in F,\ a_0 a_n \neq 0.$

Unique Factorization

Thm 11

Assume $f(x) \in F[x]$ is nonzero and not a unit.

(1) $f(x)$ can be expressed as a product of irreducible polynomials, i.e.,

$$f(x) = p_1(x) \cdots p_n(x),$$

where $p_i(x)$ are irreducible polynomials that are pairwise relatively prime.

(2) In any two such factorizations

$$f(x) = p_1(x) \cdots p_n(x) = q_1(x) \cdots q_m(x).$$

We have $n = m$ and it is possible to rearrange the factors so that $p_i(x)$ and $q_i(x)$ are associate.

Note: $F[x]$ satisfies ACC (Ascending Chain Condition) on ideals.

Unique Factorization

We write the common standard factorization of $f(x)$ and $g(x) \in F[x]$:

$$f(x) = c_1 p_1^{e_1}(x) \cdots p_n^{e_n}(x),\ g(x) = c_2 p_1^{t_1}(x) \cdots p_n^{t_n}(x),$$

where

- $p_i(x)$ are irreducible polynomials that are pairwise relatively prime and have leading coefficients of 1;
- The constant $c_1, c_2 \in F$ represent the leading coefficients $f(x)$ and $g(x)$;
- $e_i, t_j \in \mathbf{Z}$ satisfy $e_i, t_j \geq 0,\ e_i + t_j > 0$.

(1) $(f(x), g(x)), [f(x), g(x)]$, when $f(x) \mid g(x)$? $\checkmark$

(2) $F \subseteq K \rightsquigarrow p_1(x) = q_1^{m_1}(x) \cdots q_k^{m_k}(x)$ is the standard factorization of $p_1(x)$ in $K[x]$. $\rightsquigarrow$ Does there exist $m_s \geq 2$?

Examples

例 12

Assume $f(x), g(x) \in F[x]$ and n is a given positive integer,

(1) $f(x) \mid g(x) \Leftrightarrow f^n(x) \mid g^n(x)$;

(2) $(f(x), g(x)) = d(x) \Leftrightarrow (f^n(x), g^n(x)) = d^n(x)$.

Note: (§5.2) Recall that $x \mid f(x) \Leftrightarrow x^2 \mid f^2(x)$.

例 13

Given $f(x), h(x) \in F[x]$ such that $f^{27} \mid h^{29}$, if $\deg h(x) \leq 13$, then $f(x) \mid h(x)$. For cases that we hold $\deg h(x) \geq 14$, check that the conclusion may be wrong.

Hint: Divisibility remains unchanged under number field extensions.

Repeated Factors

$$f = c_1 p_1^{e_1}(x) \cdots p_n^{e_n}(x) \rightsquigarrow f' = c_1 p_1^{e_1 - 1}(x) \cdots p_n^{e_n - 1}(x) \sum_{i=1}^{n} \left\{ e_i p_i'(x) \prod_{j \neq i} p_j(x) \right\}.$$

Slogan: $(f, f') = p_1^{e_1-1}(x) \cdots p_n^{e_n-1}(x), \dfrac{f}{(f, f')} = p_1(x) \cdots p_n(x).$

Notes:

(1) $e_i \equiv 1 \Leftrightarrow (f(x), f'(x)) = 1$; $\rightsquigarrow$不随数域扩大而改变!

(2) $\dfrac{f}{(f, f')}$ 与 $f(x)$ 有完全相同的不可约因式且无重因式;

(2') $g(x) = \dfrac{f}{(f, f')} \rightsquigarrow (g, g') = 1.$

Examples

例 14

Assume that an irreducible polynomial $p(x)$ is a $(k-1)$-multiple factor of $f'(x)$, then the following statements are equivalent:

(1) $p(x)$ is a $(k-1)$-multiple factor of (f, f');

(2) $p(x) \mid f(x)$;

(3) $p(x)$ is a k-multiple factor of $f(x)$.

(3') $f(x) = p^k(x)h(x), \ (p(x), f(x)) = 1$.

Note: When we talk about repeated factors, $p(x)$ should firstly be irreducible.

例 15

Assume $f(x) \in F[x]$ and $\deg f(x) = n$. If $f' \mid f$, prove that f has a n-multiple root over F.

Multiple Roots

- If $f(x)$ has a multiple root a over F, then it must have repeated factors. But the converse is wrong.
- If a is a k-multiple root of $f(x)$, then it's a $(k-1)$-multiple root of $f'(x)$. But the converse is wrong.
- If a is a $(k-1)$-multiple root of (f', f), then it's a k-multiple root of $f(x)$.

例 16

(1) $p(x)$ is irreducible over $F \Rightarrow p(x)$ has no multiple roots over $\mathbf{C}$;

(2) $p(x)$ is irreducible over F and has common roots with $f(x)$ over $\mathbf{C} \Rightarrow p \mid f$.

Examples

Slogan: $p(x)$ 在 F 上不可约且在 $\mathbf{C}$ 上与 $f(x)$ 有公共根 $\Rightarrow p \mid f$.

例 17

(1) $f(x) \in \mathbf{R}[x],\ f(a+b\mathrm{i}) = 0\ (a, b \in \mathbf{R}) \rightsquigarrow f(a-b\mathrm{i}) = 0$;

(2) $f(x) \in \mathbf{Q}[x],\ f(\sqrt{2}+\sqrt{3}) = 0 \rightsquigarrow f(\sqrt{2}-\sqrt{3}) = f(-\sqrt{2}+\sqrt{3}) = f(-\sqrt{2} -\sqrt{3}) = 0$.

Hint: What is the standard factorization of $x^4 - 10x + 1$ in $\mathbf{R}[x]$?

例 18

$f(x) \in F[x]$ 在 F 上不可约，若非零常数 a, a^{-1}, b 为 $f(x)$ 在 $\mathbf{C}$ 上的根，证明：$f(b^{-1}) = 0$.

Hint: Consider $f(x) = \sum_{k=0}^{n} a_k x^k, g(x) = \sum_{k=0}^{n} a_{n-k} x^k,\ a_i \in F,\ a_0 a_n \neq 0$.

Examples

Slogan: $p(x)$ 在 F 上不可约且在 $\mathbf{C}$ 上与 $f(x)$ 有公共根 $\Rightarrow p \mid f$.

例 19

$f(x) \in F[x]$ 在 F 上不可约，对 $g(x) \in F[x]$，有 $\alpha \in \mathbf{C}$ 满足 $f(c) = 0, g(c) \neq 0$.

(1) 存在 $h(x) \in F[x]$ 满足 $h(c)g(c) = 1$;

(2) 求一个多项式 $h(x) \in \mathbf{Q}[x]$ 满足

$$h(\sqrt[3]{2}) = \frac{1}{3 + 2\sqrt[3]{2} + \sqrt[3]{4}}.$$

Hint: $p \mid f \Leftrightarrow$ All the roots of $p(x)$ are roots of $f(x)$.

New View of Divisibility

Slogan: $p \mid f \Leftrightarrow$ All the roots of $p(x)$ are roots of $f(x)$.

Revisit some examples:

(1) $x^2+x+1 \,\Big|\, \sum\limits_{i\in I} x^{a_i} \Leftrightarrow a_i$ 除 3 余数为 0，1，2 的个数相等；

(2) $a \in F,\ x^d - a^d \mid x^n - a^n \Leftrightarrow d \mid n.$

例 20

设多项式 $f(x), g(x), h(x), k(x)$ 之间有关系式

$$\begin{cases} (x+1)f(x) + (x+2)g(x) + (x^2+1)h(x) = 0, \\ (x-1)f(x) + (x-2)g(x) + (x^2+1)k(x) = 0. \end{cases}$$

证明：$(x^2+1) \mid (f, g)$.

More Examples

例 21

设 $f(x) = x^3 + 3x + 1$，求满足同余方程 $v(x)f'(x) \equiv 1 \pmod{f(x)}$ 且次数最小的多项式 $v(x)$.

例 22

设 $(f(x), g(x)) = 1$，证明：$f^2(x) + g^2(x)$ 的重根必是 $[f'(x)]^2 + [g'(x)]^2$ 的根.

Hint: Consider over $\mathbf{C}$.

例 23

设 $f(x)$ 为 $\mathbf{R}[x]$ 上的任一实系数多项式. 证明：存在唯一实系数多项式 $g(x)$ 使得

$$\big((x^2 + 3x - 5)g(x)\big)'' = f(x).$$

Synthetic Division

$$f(x) = \sum_{k=0}^{n} a_k x^k = (x-b)\left\{\sum_{s=0}^{n-1} b_s x^s\right\} + f(b) \rightsquigarrow a_k = -b \cdot b_k + b_{k-1}\,, 1 \le k \le n-1.$$

Try

$f(x) = x^5 - 2020x^4 - 2019x^3 - 4041x^2 - 2020x - 100$, figure out $f(2021)$.

- $f(x) = (x - 2021)(x^4 + b_3x^3 + \cdots + b_0) + f(2021)$;
- $-2020 = -2021b_3 + b_4 = -2021b_3 + 1 \rightsquigarrow b_3 = 1$;
- $-2019 = -2021b_3 + b_2 \rightsquigarrow b_2 = 2021 - 2019 = 2$;
- $-4041 = -2021b_2 + b_1 \rightsquigarrow b_1 = 1$;
- $-2020 = -2021b_1 + b_0 \rightsquigarrow b_0 = 1$;
- $-100 = -2021b_0 + f(2021) \rightsquigarrow f(2021) = 1921$.

Polynomial Functions

Lemma 24

Let $f(x)$ be a polynomial over a field F with degree $n > 0$. Then, $f(x)$ has at most n distinct roots in F.

Note: Any polynomial of finite degree has only a finite number of roots. If a polynomial has infinitely many roots, it must be the zero polynomial.

Thm 25

Set $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,\ g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0,\ a_i, b_j \in F$. Then, the following two statements are equivalent:

(A) For all $c \in F$, we have $f(c) = g(c)$;

(A') For $n+1$ distinct numbers $c_1, \cdots, c_{n+1} \in F$, we have $f(c) = g(c)$;

(B) $n = m, a_i = b_i\ (1 \leq i \leq n)$.

Distinct Roots > Degree $\rightsquigarrow$ Zero!

Slogan: If a polynomial has infinitely many roots $\Rightarrow$ zero polynomial.

例 26

$f(x) = \sin x$ is not a polynomial.

例 27

Figure out each $f(x)$ satisfying the following conditions:

(1) $f(x) = f(x + c),\ 0 \neq c \in F$;

(2) $f(a + b) = f(a) + f(b)$, for all $a, b \in F$.

例 28

$\deg f(x) = n > 0, f(k) = k/(k+1),\ k = 0, 1, 2, \cdots, n$. Find $f(n+1)$.