§6.1—§6.2 Eigenvalues and Eigenvectors, Diagonalizability I

illusion

Especially made for smy

School of Mathematical Science, XMU

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http://illusion-hope.github.io/25-Spring-SMY-Discussion-Session/

例 1

设 $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$，非零数 $c_1,c_2,\ldots,c_n$ 是 $f(x)$ 的根。

(1) 证明 $a_0\neq 0$；

(2) 证明 $c_1,c_2,\ldots,c_n$ 两两互异，求 $\frac{1}{c_1},\frac{1}{c_2},\ldots,\frac{1}{c_n}$ 为根的 n 次多项式，并证明；

(3) 若 (2) 中的根不是全互异(重根按重数计)，求以 $\frac{1}{c_1},\frac{1}{c_2},\ldots,\frac{1}{c_n}$ 为根的 n 次多项式，并证明。

例 2

设 p 是素数，整数 $n \geq 3$，证明：

(1) $\sqrt[n]{p}$ 是无理数；

(2) 对任意不全为 0 的有理数 $a_1, a_2, \ldots, a_{n-1}$，都有

$$1 + a_1 \sqrt[n]{p} + a_2 \sqrt[n]{p^2} + \cdots + a_{n-1} \sqrt[n]{p^{n-1}} \notin \mathbb{Q}.$$

(2)* $\sqrt[n]{p}$ 在 $\mathbb{Q}$ 上的极小多项式为 n 次的，也即无理数 $\sqrt[n]{p}, \sqrt[n]{p^2}, \cdots, \sqrt[n]{p^{n-1}}$ 在 $\mathbb{Q}$ 上线性无关；

(3)* $\mathbb{Q}[\sqrt[n]{p}]$ 为一个数域，若将其看作 $\mathbb{Q}$ 上的线性空间，有 $\dim \mathbb{Q}[\sqrt[n]{p}] = n = [\mathbb{Q}[\sqrt[n]{p}] : \mathbb{Q}] = \deg m(x)$，其中 $m(x) = x^n - p$.

注记：我们在 §6.3 节还会继续讨论矩阵和向量的极小多项式。

例 3

证明多项式 $f(x)=(x-1)^2(x-2)^2\dots(x-2024)^2+2025$ 在 $\mathbb{Q}$ 上不可约。

例 4

设整数 $n>0, f(x), g(x)$ 是数域 F 上首一的 n 次多项式，定义不超过 $n-1$ 次多项式的线性空间 V 上的线性变换 φ_1, φ_2 分别为：

(i) 对任意 $h(x)\in V$，φ_1 表示 $h(x)f(x)$ 除 $g(x)$ 的余式；

(ii) 对任意 $h(x)\in V$，φ_2 表示 $h(x)g(x)$ 除 $f(x)$ 的余式。

证明：$\det\varphi_1=(-1)^n\det\varphi_2$.

Outline of Chapter 6: Eigenvalue and Eigenvectors

Goal

认识一类最简单的相似标准型：可对角化

基础概念的准备：

- 特征值和特征向量；
- 几何重数和代数重数；
- 矩阵，向量的的零化多项式和极小多项式；
- 相似初等变换，一些相似的必要条件.

一些相似标准型的刻画：

- 复上三角化 $\rightsquigarrow$ 观察特征值；
- 实分块上三角化；
- 对角矩阵.

交换矩阵带来的性质:

- 特征子空间互为不变子空间;
- 公共特征向量;
- 同时上三角,对角化;
- 多项式表示.

Outline of Chapter 6: Eigenvalue and Eigenvectors

可对角化的9大等价命题 I:

- n 个特征向量;
- 全空间成为特征子空间的直和;
- 有完全的特征向量系，i.e.，几何重数=代数重数;
- 极小多项式无重根;
- 在不变子空间上的限制仍然可对角化;
- 只有 0 级的广义特征向量 $\rightsquigarrow$ Jordan 块全是1阶的，初等因子组全为一次的.
- $\cdots$

§6.1 Preparation

Common Properties:

- $\lambda_1\lambda_2\cdots\lambda_n = \det A,\ \lambda_1+\cdots+\lambda_n = \operatorname{tr}(A)$;
- $f_A(\lambda) = f_{A^T}(\lambda),\ f_{P^{-1}AP}(\lambda) = f_A(\lambda)$;
- $A\in F^{m\times n}, B\in F^{n\times m}, m\geq n \rightsquigarrow f_{AB}(\lambda) = \lambda^{m-n}f_{BA}(\lambda)$;
- (Laplace) $A = \begin{bmatrix} B & D \\ O & C \end{bmatrix} \Rightarrow f_A(\lambda) = f_B(\lambda)f_C(\lambda)$.

Note: 上，下三角矩阵可以帮助我们看出特征值

$$A = \begin{bmatrix} \lambda_1 & * & * & * \\ & \lambda_2 & * & * \\ & & \ddots & * \\ & & & \lambda_n \end{bmatrix} \Rightarrow f_A(\lambda) = (\lambda-\lambda_1)\cdots(\lambda-\lambda_n).$$

Upper Triangularization Over $\mathbb{C}$

Thm 5

$\mathbb{C}$ 上任意方阵 A 都相似到一个上三角矩阵，且对角元为 A 的全部特征值.

Notes:

- (Cor.) 若 F 上一方阵 A 的特征值全落在 F 内，也可上三角化 $\checkmark$
- $A^{-1}, f(A), A^*$ 的全部特征值?
- 实方阵在 $\mathbb{R}$ 上也可以上三角化吗?
- (Chapter 8) 事实上，这个相似可以加强为酉相似.

Examples

例 6

Given $A \in M(3, F)$ such that $\det A = \det(A + E) = \det(A + 2E) = a \in F$, Figure out $\det g(A)$ under following conditions:

(1) $a = 0,\ g(x) = x(x+1)(x+2)$;

(2) $a \neq 0,\ g(x) = x + 2025$.

例 7

If $A \in GL(n, \mathbb{Z})$ has an eigenvalue λ in $\mathbb{Z}$, prove that A^* has at least an integer eigenvalue and try to find one.

Eigen Subspace

$$V_{\lambda_0} = \{X \in F^n \mid AX = \lambda_0 X\}.$$

- $A \rightsquigarrow \varphi \Rightarrow \dim V_{\lambda_0} = \dim \mathrm{Ker}(\lambda_0 \mathrm{id}_{F^n} - \varphi) = n - r(\lambda_0 E - A)$;
- $\varphi|_{V_{\lambda_0}} \in \mathcal{L}(V_{\lambda_0})$;
- $\mathrm{tr}(A|_{V_{\lambda_0}}) = \mathrm{tr}(\varphi|_{V_{\lambda_0}}) = \lambda_0 \cdot \dim V_{\lambda_0}$.

Thm 8

Suppose $\lambda_1, \cdots, \lambda_s$ are all the eigenvalues of a $n \times n$ matrix A, then it holds

$$V_{\lambda_1} + V_{\lambda_2} + \cdots + V_{\lambda_s} = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_s}.$$

When $V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_s} = V \rightsquigarrow$ what happens? Diagonalizability!

§6.2 Diagonalizability

If A has n distinct eigenvalues $\lambda_1, \cdots, \lambda_n$, it follows that

$$\dim V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_n} \geq n.$$

However, the sum of subspaces cannot exceed the dimension of the entire space V. In other words,

$$\dim V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_n} \leq n \rightsquigarrow \dim V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_n} = n.$$

Thus, from the equality of dimensions and the natural inclusion relationship, we obtain

$$V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_n} = V. \rightsquigarrow \text{Diagonalizability!}$$

The next question that arises is that what happens if the number of distinct eigenvalues is smaller than n?

§6.2 Diagonalizability

Goal: The criterion for diagonalizability in general case

Suppose $f_A(\lambda) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_t)^{n_t}, n_i \geq 1, t < n$. We have

$$F^n = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_t}.$$

Then, we denote $\dim V_{\lambda_i} = s_i$, which implies

$$n_1 + \cdots + n_t = s_1 + \cdots + s_t = n, \ t < n, \ n_i, s_i \geq 1.$$

Q: Could $n_i > s_i$ and $n_j < s_j$ hold at the same time for some i, j? $\rightsquigarrow$ No!

Lemma 9

For λ_i, we always have the geometric multiplicity s_i is smaller than the algebraic multiplicity n_i.

Then, we obtain $n = n_1 + \cdots + n_t \leq s_1 + \cdots + s_t = n \rightsquigarrow n_i \equiv s_i$.

§6.2 Diagonalizability

Thm 10

The following statements are equivalent:

(1) A is diagonalizable.

(2) A has n linearly independent eigenvectors.

(3) F^n becomes the direct sum of eigen subspace.

(4) For any eigenvalue λ_i, we always have $n_i \equiv s_i$.

Notes: Keep these statements in mind!

- Compute P such that $P^{-1}AP$ is diagonal matrix $\rightsquigarrow$ easy!
- Translate these statements into the language of linear transformations $\varphi \in \mathcal{L}(V) \rightsquigarrow$ a good exercise for you.

An Example

例 11

Suppose $A = \begin{bmatrix} B & D \\ O & C \end{bmatrix}$ is diagonalizable, where B, C, D are $n \times n$ matrices.

Prove that

(1) B, C are diagonalizable;

(2) $A \leftrightsquigarrow \varphi$, V_1 is $\varphi-$subspace, then $\varphi|_{V_1}, \varphi|_{V/V_1}$ are diagonalizable.

Notes:

- If φ has n distinct eigenvalues, find out all the $\varphi-$subspace.
- If B, C are diagonalizable, can we conclude that A is diagonalizable?
- Additionally, if B, C do not share the same eigenvalues, how does that affect diagonalizability?

Equivalent Characterizations of Diagonalizability

Thm 12

The following statements are equivalent:

(1) A is diagonalizable.

(2) A has n linearly independent eigenvectors.

(3) F^n becomes the direct sum of eigen subspace.

(4) For any eigenvalue λ_i, we always have $n_i \equiv s_i$.

(5) For any $\mathscr{A}-$subspace V, we have $\mathscr{A}|_V$ is diagonalizable.

(6) For any $\mathscr{A}-$subspace V, we can find another $\mathscr{A}-$subspace W such that

$$F^n = V \oplus W.$$

Notes: Keep these statements in mind!