§5.5—§5.6 Polynomial Functions and Polynomials Over $\mathbb{C}, \mathbb{R}$

illusion

Especially made for smy

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Monday 3rd March, 2025

http://illusion-hope.github.io/25-Spring-SMY-Discussion-Session/

HW-3

例 1

设 $f(x)=x^3+3x+1$，求满足同余方程 $v(x)f'(x)\equiv 1 \pmod{f(x)}$ 且次数最小的多项式 $v(x)$.

例 2

设 $(f(x),g(x))=1$，证明：$f^2(x)+g^2(x)$ 的重根必是 $[f'(x)]^2+[g'(x)]^2$ 的根.

Hint: Consider over $\mathbf{C}$.

例 3

设 $f(x)$ 为 $\mathbf{R}[x]$ 上的任一实系数多项式. 证明：存在唯一实系数多项式 $g(x)$ 使得

$$\big((x^2+3x-5)g(x)\big)''=f(x).$$

Examples

例 4

Assume that an irreducible polynomial $p(x)$ is a $(k-1)$-multiple factor of $f'(x)$, then the following statements are equivalent:

(1) $p(x)$ is a $(k-1)$-multiple factor of (f, f');

(2) $p(x) \mid f(x)$;

(3) $p(x)$ is a k-multiple factor of $f(x)$.

(3') $f(x) = p^k(x)h(x),\ (p(x), f(x)) = 1$.

Note: When we talk about repeated factors, $p(x)$ should firstly be irreducible.

例 5

Assume $f(x) \in F[x]$ and $\deg f(x) = n$. If $f' \mid f$, prove that f has a n-multiple root over F.

Multiple Roots

- If $f(x)$ has a multiple root a over F, then it must have repeated factors. But the converse is wrong.
- If a is a k-multiple root of $f(x)$, then it's a $(k-1)$-multiple root of $f'(x)$. But the converse is wrong.
- If a is a $(k-1)$-multiple root of (f', f), then it's a k-multiple root of $f(x)$.

例 6

(1) $p(x)$ is irreducible over $F \Rightarrow p(x)$ has no multiple roots over $\mathbf{C}$;

(2) $p(x)$ is irreducible over F and has common roots with $f(x)$ over $\mathbf{C} \Rightarrow p \mid f$.

Examples

Slogan: $p(x)$ 在 F 上不可约且在 $\mathbf{C}$ 上与 $f(x)$ 有公共根 $\Rightarrow p \mid f$.

例 7

(1) $f(x) \in \mathbf{R}[x],\ f(a+b\mathrm{i}) = 0\ (a, b \in \mathbf{R}) \rightsquigarrow f(a-b\mathrm{i}) = 0$;

(2) $f(x) \in \mathbf{Q}[x],\ f(\sqrt{2}+\sqrt{3}) = 0 \rightsquigarrow f(\sqrt{2}-\sqrt{3}) = f(-\sqrt{2}+\sqrt{3}) = f(-\sqrt{2} -\sqrt{3}) = 0$.

Hint: What is the standard factorization of $x^4 - 10x + 1$ in $\mathbf{R}[x]$?

例 8

$f(x) \in F[x]$ 在 F 上不可约，若非零常数 a, a^{-1}, b 为 $f(x)$ 在 $\mathbf{C}$ 上的根，证明：$f(b^{-1}) = 0$.

Hint: Consider $f(x) = \sum_{k=0}^{n} a_k x^k, g(x) = \sum_{k=0}^{n} a_{n-k} x^k,\ a_i \in F,\ a_0 a_n \neq 0$.

Examples

Slogan: $p(x)$ 在 F 上不可约且在 $\mathbf{C}$ 上与 $f(x)$ 有公共根 $\Rightarrow p \mid f$.

例 9

$f(x) \in F[x]$ 在 F 上不可约，对 $g(x) \in F[x]$，有 $\alpha \in \mathbf{C}$ 满足 $f(c) = 0, g(c) \neq 0$.

(1) 存在 $h(x) \in F[x]$ 满足 $h(c)g(c) = 1$;

(2) 求一个多项式 $h(x) \in \mathbf{Q}[x]$ 满足

$$h(\sqrt[3]{2}) = \frac{1}{3 + 2\sqrt[3]{2} + \sqrt[3]{4}}.$$

Hint: $p \mid f \Leftrightarrow$ All the roots of $p(x)$ are roots of $f(x)$.

New View of Divisibility

Slogan: $p \mid f \Leftrightarrow$ All the roots of $p(x)$ are roots of $f(x)$.

Revisit some examples:

(1) $x^2+x+1 \,\Big|\, \sum_{i\in I} x^{a_i} \Leftrightarrow a_i$ 除 3 余数为 0，1，2 的个数相等；

(2) $a \in F,\ x^d - a^d \mid x^n - a^n \Leftrightarrow d \mid n$.

例 10

设多项式 $f(x), g(x), h(x), k(x)$ 之间有关系式

$$\begin{cases} (x+1)f(x)+(x+2)g(x)+(x^2+1)h(x)=0, \\ (x-1)f(x)+(x-2)g(x)+(x^2+1)k(x)=0. \end{cases}$$

证明：$(x^2+1) \mid (f,g)$.

Synthetic Division

$$f(x) = \sum_{k=0}^{n} a_k x^k = (x-b)\left\{\sum_{s=0}^{n-1} b_s x^s\right\} + f(b) \rightsquigarrow a_k = -b \cdot b_k + b_{k-1}\ , 1 \le k \le n-1.$$

Try

$f(x) = x^5 - 2020x^4 - 2019x^3 - 4041x^2 - 2020x - 100$, figure out $f(2021)$.

- $f(x) = (x - 2021)(x^4 + b_3 x^3 + \cdots + b_0) + f(2021)$;
- $-2020 = -2021b_3 + b_4 = -2021b_3 + 1 \rightsquigarrow b_3 = 1$;
- $-2019 = -2021b_3 + b_2 \rightsquigarrow b_2 = 2021 - 2019 = 2$;
- $-4041 = -2021b_2 + b_1 \rightsquigarrow b_1 = 1$;
- $-2020 = -2021b_1 + b_0 \rightsquigarrow b_0 = 1$;
- $-100 = -2021b_0 + f(2021) \rightsquigarrow f(2021) = 1921$.

Polynomial Functions

Lemma 11

Let $f(x)$ be a polynomial over a field F with degree $n > 0$. Then, $f(x)$ has at most n distinct roots in F.

Note: Any polynomial of finite degree has only a finite number of roots. If a polynomial has infinitely many roots, it must be the zero polynomial.

Thm 12

Set $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,\ g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0,\ a_i, b_j \in F$. Then, the following two statements are equivalent:

(A) For all $c \in F$, we have $f(c) = g(c)$;

(A') For $n+1$ distinct numbers $c_1, \cdots, c_{n+1} \in F$, we have $f(c) = g(c)$;

(B) $n = m, a_i = b_i\ (1 \leq i \leq n)$.

Distinct Roots $>$ Degree $\rightsquigarrow$ Zero!

Slogan: If a polynomial has infinitely many roots $\Rightarrow$ zero polynomial.

例 13

$f(x) = \sin x$ is not a polynomial.

例 14

Figure out each $f(x)$ satisfying the following conditions:

(1) $f(x) = f(x + c),\ 0 \neq c \in F$;

(2) $f(a + b) = f(a) + f(b)$, for all $a, b \in F$.

例 15

$\deg f(x) = n > 0, f(k) = k/(k+1),\ k = 0, 1, 2, \cdots, n$. Find $f(n+1)$.