Case I: $f(x) = (b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0)e^{\lambda x}, b_i \in \mathbf{R}$.

特解可以选取为

$$y^* = x^k(B_m x^m + \cdots + B_1 x + B_0)e^{\lambda x}.$$

其中 $B_1, \cdots, B_m \in \mathbf{R}$，$k$ 为 $L[y] = 0$ 对应特征方程中 λ 的重数。

(1) γ

Case II: $f(x) = [\underline{P(x)}\,\underline{\cos\beta x} + Q(x)\sin\beta x]e^{\alpha x}, P(x), Q(x) \in \mathbf{R}[x]$.

$+$

0 $\sin\gamma x$

记 $\max\{\deg P(x), \deg Q(x)\} = n$，则特解可以选取为

(2) $e^{\alpha x}\cos\gamma x$ $\mathrm{Re}(e^{(\alpha+i\gamma)x})$

$$y^* = x^k\{A_1(x)\cos\beta x + B(x)\sin\beta x\}e^{\alpha x}.$$

其中 $\deg A(x) = \deg B(x) = n$，$A(x), B(x) \in \mathbf{R}[x]$，$k$ 为 $L[y] = 0$ 对应特征方程中 $\lambda = \alpha + i\beta$ 的重数。

HW-4

① 齐次 $\lambda^2+1=0 \Rightarrow \lambda=\pm i$

$C_1e^{ix}+C_2e^{-ix}$ $\quad 0\pm i$

$C_1(x) \quad C_2(x)$

$$\begin{cases} C_1'(x)e^{ix}+C_2'(x)e^{-ix}=0\\ iC_1'(x)e^{ix}-iC_2'(x)e^{-ix}=x \end{cases}$$

例 1

求微分方程

$$\begin{cases} e^0\cos 1x \\ e^0\sin 1x \end{cases}$$

xe^{0x}

$$\begin{cases} \boxed{y''+y=x,} & x\le \dfrac{\pi}{2},\\ y''+4y=0, & x>\dfrac{\pi}{2} \end{cases}$$

$\Rightarrow -C_1'(x)e^{2ix}=C_2'(x)$

$i[C_1'(x)e^{ix}+C_1'(x)e^{ix}]=x$

$\Rightarrow C_1'(x)=-ixe^{-ix}$

$y^*=(ax+b)=x$

满足条件 $y|_{x=0}=0$，$y'|_{x=0}=0$，且在 $x=\dfrac{\pi}{2}$ 处可导的解。

$C_2'(x)=ixe^{ix}$

Hint：隐含的初值条件是什么？

$\Rightarrow y=x+C_1\cancel{\cos x}+C_2\sin x$

$\Rightarrow C_1(x)=-\int ixe^{-ix}\,dx$

$y|_{x=0}=C_1=0$

$=i\int -ixe^{-ix}\,d(-ix)$

$y' = 1 - C_1 \sin x + C_2 \cos x$

$\Big|_{x=0} = 1 + C_2 = 0 \Rightarrow C_2 = -1$

$y = x - \sin x, \quad x \leq \frac{\pi}{2}$

$\Rightarrow y\Big|_{x=\frac{\pi}{2}} = \frac{\pi}{2} - 1 \quad y'\Big|_{x=\frac{\pi}{2}} = 1 - 0 = 1$

$= i \int m e^{m} \, dm \qquad m = -ix$

$= i \, \underline{e^{m}(m-1)} + C,$

$= i e^{-ix}(-ix-1) + C.$

$= e^{-ix}(x-i) + C.$

$C_2(x) = \int ix e^{ix} \, dx$

$= -i \int ix e^{ix} \, dix = -i \int m e^{m} \, dm$

$= -i e^{ix}(ix-1)$

$= e^{ix}(i+x)$

$y' = p, \quad \frac{dp}{dx} = y'' = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$

$p\frac{dp}{dy} + 4y = 0 \qquad p\,dp = -4y\,dy \Rightarrow \frac{1}{2}p^2 = -2y^2.$

$C_1\cos 2x + C_2\sin 2x \Leftarrow \frac{\pi}{2}$ $\quad -C_1 = \frac{\pi}{2} - 1$ ✓

$2C_1(-\sin 2x) + 2C_2\cos 2x \Leftarrow \frac{\pi}{2}$ $\quad -2C_2 = 1$ ✓.

例 2

求微分方程的通解：$x^3y''' + x^2y'' - 4xy' = 3x^2$.

$y^* = e^{2t} \cdot a$

$(D-(k+1)) \cdots D\,y.$

$D = d/dt$

$\Rightarrow (D-2) \cdots Dy + (D-1)Dy + 4Dy = 3e^{2t}$

$\Rightarrow (\lambda-2)(\lambda-1)\lambda + (\lambda-1)\lambda + 4\lambda = 0.$

$\lambda^3 - 3\lambda^2 + 2\lambda + \lambda^2 - \lambda + 4\lambda = \lambda^3 - 2\lambda^2 + 5\lambda = \lambda\left(\boxed{\lambda^2 - 2\lambda + 5}\right) = 0$

$$\frac{d^3}{dt^3}y - 2\frac{d^2}{dt^2}y + 5\frac{d}{dt}y = 3e^{2t}.$$

$$\lambda^2 - \lambda + 5 = 0 \qquad \frac{1 \pm \sqrt{+16}\, i}{2}$$

$$\underbrace{\frac{1}{2} \pm 2i}$$

$$\downarrow y^* = e^{2t} a.$$

$$C_1 \cdot 1 + C_2 \cdot \underline{e^{\frac{1}{2}x} \cos 2x} + C_3 \cdot \underline{e^{\frac{1}{2}x} \sin 2x}$$

$$8a - 8a + 10a = 3 \Rightarrow \underline{a = \frac{3}{10}}.$$

$$t \to \ln|x|$$

$$e^t \to |x|$$

例 3

给定微分方程

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} - xy = xe^{x^2}, \\ y(0) = 1; \end{cases}$$

$$[y' - xy]$$

$$e^{-\int p(x)\mathrm{d}x}\,[y' - p(x)y] = \left(e^{-\int p(x)\mathrm{d}x}\,y\right)'$$

$$e^{-\int x\mathrm{d}x} = e^{-\frac{x^2}{2}}$$

设 $y = f(x)$ 为上述方程的解，证明

$$\lim_{n\to\infty}\int_0^1 \frac{n}{n^2x^2+1} f(x)\mathrm{d}x = \frac{\pi}{2}.$$

$$\frac{dy}{dx} = xy + xe^{x^2}$$

$$e^{\int x\,dx}\left[\int xe^{x^2}\cdot e^{-\int x\,dx}\,dx + c\right]$$

$$e^{\frac{1}{2}x^2}\cdot\left[\int xe^{x^2}\, e^{-\frac{1}{2}x^2}\,dx + c\right] = \left[e^{\frac{1}{2}x^2}\, d\tfrac{1}{2}x^2 + c\right]$$

$$= e^{\frac{1}{2}x^2}\left[e^{\frac{1}{2}x^2} + c\right] = e^{x^2} + ce^{\frac{1}{2}x^2} \xrightarrow{y(0)=1} e^{x^2}.$$

prove that

$$\lim_{n\to\infty}\int_0^1 \frac{n}{n^2x^2+1}e^{x^2}\,dx = \frac{\pi}{2} = \lim_{n\to\infty}\int_0^1 \frac{n}{n^2x^2+1}\,dx.$$

$$\frac{\pi}{2} = \arctan(+\infty) - \arctan(0)$$

$$\lim_{n\to\infty}\int_0^1 \frac{n}{n^2x^2+1}\,dx = \lim_{n\to\infty}\int_0^1 \frac{n}{(nx)^2+1}\,dx = \int_0^{+\infty\,(n)} \frac{1}{t^2+1}\,dt$$

$$\Leftrightarrow \lim_{n\to\infty}\int_0^1 \frac{n}{n^2x^2+1}\left(e^{x^2}-1\right)dx = 0$$

$$e^{x^2}-1 = e^{x^2}-e^0 = \boxed{2\xi e^{\xi^2}}\,x^2,\quad \xi\in[0,1], \qquad \le 2ex^2.$$

$$\leq \underbrace{\int_0^1 \frac{2en\,x^2}{n^2x^2+1}\,dx}. \leq \frac{e}{n}\underbrace{\int_0^1 \frac{1}{n^2x^2+1}\,d\,n^2x^2}. = \frac{e\ln(n^2+1)}{n}$$

$$\int_0^{n^2} \frac{1}{t+1}\,dt. \qquad \to 0 \ (n\to+\infty)$$

$$\ln(n^2+1) - \ln 1$$

$\ln x^\alpha \ll x^\beta \ (\beta>0)$

$$\frac{x^{0.001}}{\ln x^{10000}} \to +\infty \ (x\to+\infty)$$

L'Hospital.

(Cauchy 定理) $\lim a_n = a \ (+\infty)$ $\quad \lim\limits_{n\to\infty} \dfrac{a_1+\cdots+a_n}{n} = a.$

Note: $\ln n^\alpha \ll n^\beta \ll a^n \ll n! \ll n^n. \ (n\to\infty)$

例 4

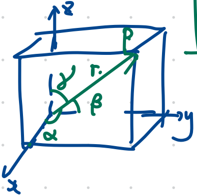
设右手直角坐标系 $\{O; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. 给定 $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} - 5\boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$.

(1) 求向量 $\boldsymbol{a}$ 的方向余弦和向量 $\boldsymbol{a}$ 在向量 $\boldsymbol{b}$ 上的投影;

(2) 求向量 $\boldsymbol{c}$，使得 $|\boldsymbol{c}| = \sqrt{3}$，且由 $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ 三向量所张成的平行六面体的体积最大。

$\longrightarrow \dfrac{\vec a\cdot\vec b}{|\vec b|}$ $\quad \dfrac{\vec a\cdot\vec b}{|\vec b|^2}\vec b$

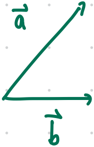
$$\boxed{\frac{\vec a\times\vec b}{|\vec a\times\vec b|}|\vec c| = \vec c}$$



$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$P(r\cos\alpha,\ r\cos\beta,\ r\cos\gamma)$

方向余弦



$\vec a\cdot\vec b = |\vec b|\cdot$

$\vec a$在$\vec b$上投影

例 5

已知 O 为正多边形 $A_1A_2\ldots A_n$ 的中心，P 为其外接圆上一点，证明：

(1) $\overrightarrow{OA_1}+\overrightarrow{OA_2}+\cdots+\overrightarrow{OA_n}=\mathbf{0}$;

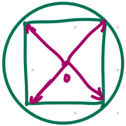
(2) $|\overrightarrow{PA_1}+\overrightarrow{PA_2}+\cdots+\overrightarrow{PA_n}|$ 为常数。$=nR.$

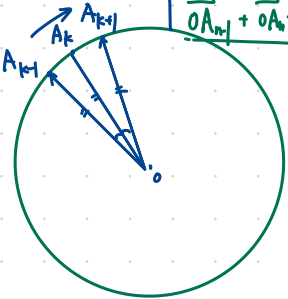
$2I=\lambda I.$　$(\lambda\neq 2)$? 不给

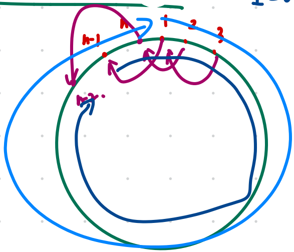
Hint: $\overrightarrow{OA_{k-1}}+\overrightarrow{OA_{k+1}}=\lambda\overrightarrow{OA_k}$, $2\le k\le n-1$.　$(\lambda\neq 0)$

$\overrightarrow{OA_1}+\cdots+\overrightarrow{OA_n}=I.$ $\Rightarrow (\lambda-2)I=\vec{0}$

$\overrightarrow{OA_{n-1}}+\overrightarrow{OA_n}+\cdots+\overrightarrow{OA_{k-2}}$ $\Rightarrow I=\vec{0}$







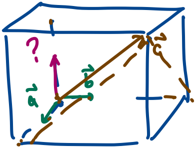
Projection and Schmidt Orthogonalization Process

Q: 在空间 $E^3 = (\mathbf{R}^3, \langle\cdot,\cdot\rangle)$ 中有三个不共面的向量 $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$，如何操作能让它们两两正交?

- $\boldsymbol{e}_1 = \dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}$;
- $\boldsymbol{b}' = \boldsymbol{b} - \dfrac{\boldsymbol{b}\cdot\boldsymbol{a}}{|\boldsymbol{a}|^2}\,\boldsymbol{a} \rightsquigarrow \boldsymbol{e}_2 = \dfrac{\boldsymbol{b}'}{|\boldsymbol{b}'|}$;
- $\boldsymbol{c}' = \boldsymbol{c} - \dfrac{\boldsymbol{c}\cdot\boldsymbol{a}}{|\boldsymbol{a}|^2}\,\boldsymbol{a} - \dfrac{\boldsymbol{c}\cdot\boldsymbol{b}'}{|\boldsymbol{b}'|^2}\,\boldsymbol{b}' \rightsquigarrow \boldsymbol{e}_3 = \dfrac{\boldsymbol{c}'}{|\boldsymbol{c}'|}$.

Def. $\boxed{QQ^T = E}$ $\Leftrightarrow [e_1\ e_2\ e_3]\begin{bmatrix} e_1^T \\ e_2^T \\ e_3^T \end{bmatrix} = \begin{bmatrix} \boxed{e_1e_1^T} & e_1e_2^T & e_1e_3^T \\ & \vdots & \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

$= Q^TQ$ $\longrightarrow$ Q 行向量也是标正基.



Notes:

det Q=1 或 det Q=-1

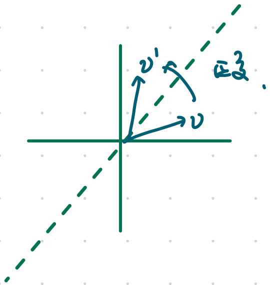
- $\boldsymbol{Q} = [\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3]$ 是正交矩阵 $\rightsquigarrow$ 未必右手系，即未必 $\det Q = 1$；
- QR 分解：设 $\boldsymbol{P}$ 可逆，那么必定存在一个正交矩阵 $\boldsymbol{Q}$，和一个上三角矩阵 $\boldsymbol{R}$，使得 $\boldsymbol{P} = \boldsymbol{Q}\boldsymbol{R}$，且这种分解是唯一的。

$\boxed{\text{保定向的}}$

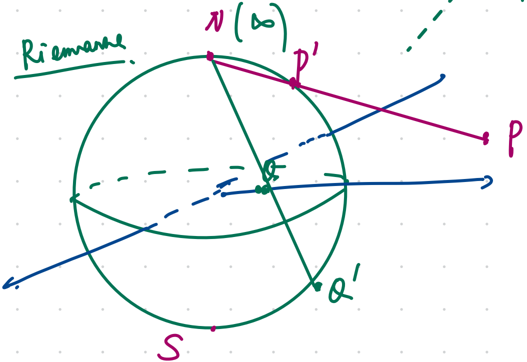
$\det Q = 1$ $\quad v \mapsto Qv$ $\quad$ 旋转

$\downarrow$

$-1.$ $\quad v \mapsto Qv$ $\quad \to$ + 反射



注: Riemann.



Common Operators On Vectors

给定 $\boldsymbol{a} = (x_1, y_1, z_1), \boldsymbol{b} = (x_2, y_2, z_2), \boldsymbol{c} = (x_3, y_3, z_3) \in E^3$.

叉乘(外积，向量积)：

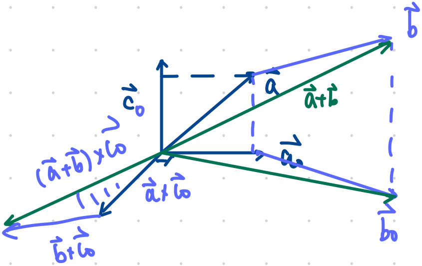
- 运算律 I：$(\lambda \boldsymbol{a}) \times \boldsymbol{b} = \lambda(\boldsymbol{a} \times \boldsymbol{b}), \boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$;
- 运算律 II：$(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c}$;
- $\rightsquigarrow \left(\sum_{i=1}^{m} \lambda_i \boldsymbol{a}_i\right) \times \left(\sum_{j=1}^{m} \mu_j \boldsymbol{b}_j\right) = \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_i \mu_j (\boldsymbol{a}_i \times \boldsymbol{b}_j)$;
- $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0} \Leftrightarrow \boldsymbol{a} \parallel \boldsymbol{b}$;
- $\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$.

$$(\vec{a}+\vec{b})\times\vec{C} = \vec{a}\times\vec{C} + \vec{b}\times\vec{C}$$

$\downarrow$ $\vec{c}$ 换成 $\vec{c_0} := \dfrac{\vec{c}}{|\vec{c}|}$

Step $\vec{a}\times\vec{c_0} = ?$

pf:



$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\tau_1"'] & U \arrow[d, "\tau_2"] \\ X \arrow[r, "A"] & Ax \end{tikzcd}$$

$\square$.

- Q: 若 $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ 不共面，那么任意 $\boldsymbol{r} \in E^3$ 都有分解

$$\boldsymbol{r} = \frac{(\boldsymbol{r}, \boldsymbol{b}, \boldsymbol{c})\boldsymbol{a} + (\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{c})\boldsymbol{b} + (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{r})\boldsymbol{c}}{\underset{\Delta}{(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})}}.$$

det

Cramer法则

$$\vec{r} = c_1\vec{a} + c_2\vec{b} + c_3\vec{c}$$

$$\underline{\underline{\vec{r}\cdot(\vec{b}\times\vec{c})}} = c_1\vec{a}\cdot(\vec{b}\times\vec{c}) + 0 \Rightarrow c_1 = \frac{(\vec{r}, \vec{b}, \vec{c})}{(\vec{a}, \vec{b}, \vec{c})}$$

$$\begin{cases} r_1 = x_1 a_1 + x_2 b_1 + x_3 c_1 \\ r_2 = x_1 a_2 + x_2 b_2 + x_3 c_2 \\ r_3 = x_1 a_3 + x_2 b_3 + x_3 c_3 \end{cases}$$

$$x_1 = \frac{\det(\vec{r}, \vec{b}, \vec{c})}{\det(\vec{a}, \vec{b}, \vec{c})}$$

例 6

证明：对任意的向量 $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\in E^3$，有

$(a_1c_1+a_2c_2+a_3c_3)\,b_1-(b_1c_1+b_2c_2+b_3c_3)\,a_1$

$$\vec{d}=(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a},\quad \boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}.$$

$a_2c_2b_1+a_3c_3b_1-\underline{b_2c_2a_1}-\underline{b_3c_3a_1}$

pf: $\vec{a}=(a_1,a_2,a_3)\quad \vec{b}=(b_1,b_2,b_3)\quad \vec{c}=(c_1,c_2,c_3)$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}=(a_2b_3-a_3b_2)\vec{i}+(a_3b_1-a_1b_3)\vec{j}+(a_1b_2-a_2b_1)\vec{k}$$

$$\begin{vmatrix}\ &\ &\ \\0&0&0\\c_1&c_2&c_3\end{vmatrix}$$

$c_3(a_3b_1-a_1b_3)-c_2(a_1b_2-a_2b_1)$

Suppose two planes π_1 and π_2 are given by the equations:

$$\pi_1: \quad A_1x + B_1y + C_1z + D_1 = 0,$$

$$\pi_2: \quad A_2x + B_2y + C_2z + D_2 = 0.$$

解的构成 → 基础 (1)　$n - r(P)$　$3 - r(P) = 1 \Rightarrow r(P) = 2.$

(1) **Intersection:** $A_1 : B_1 : C_1 \neq A_2 : B_2 : C_2$;

(2) **Parallel:** $\dfrac{A_1}{A_2} = \dfrac{B_1}{B_2} = \dfrac{C_1}{C_2} \boxed{\neq \dfrac{D_1}{D_2}}$;

无解

(3) **Coincidence:** $\dfrac{A_1}{A_2} = \dfrac{B_1}{B_2} = \dfrac{C_1}{C_2} = \dfrac{D_1}{D_2}$.

(2)

$r[P] = r[P; \beta] = 1$

$$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{bmatrix} = P$$

$$P\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -D_1 \\ -D_2 \end{bmatrix} = \beta$$

$r[P] \neq r[P; \beta]$

$1 \leq \underset{\shortparallel\\ 1}{r[P]} \leq r[P; \beta] \leq 2.$　(2)