#### §7.1—§7.4 Ordinary Differential Equations I

illusion

Especially made for zqc

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http://illusion-hope.github.io/25-Spring-ZQC-Calculus/

# Separable Differential Equations

假设  $P(x), Q(y) \in C(-\infty, +\infty)$ :

(1) 
$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y) \Rightarrow \frac{1}{g(y)}\mathrm{d}y = f(x)\mathrm{d}x \leadsto$$
 是否存在  $g(y_0) = 0$  ?

(2) 
$$\int \frac{1}{g(y)} dy = \int f(x) dx + C \rightsquigarrow G(y) = F(x) + C.$$

- (3)  $H(x,y) = G(y) F(x) C = 0 \leftrightarrow H_y(x,y) = 1/g(y) \neq 0$ . 由隐映射定理, 上述方程在满足方程的某一个点  $(\tilde{x},\tilde{y})$  附近唯一确定了隐函数  $y = \Phi(x)$ .
- (4)  $f|_{(\tilde{x},\tilde{y}),\tilde{y}\neq y_0} \neq 0 \Rightarrow y'|_{(\tilde{x},\tilde{y}),\tilde{y}\neq y_0} = f(\tilde{x})g(\tilde{y}) \neq 0$ . 由逆映射定理,点  $(\tilde{x},\tilde{y})$  附近确定的隐函数  $y = \Phi(x)$  局部可逆.

#### Try

分别求  $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2-1}{2}$  的通过  $(0,0), (\ln 2,3)$  的解的<u>存在区间</u>。

$$\frac{y+1}{y+1} = \pm e^{\frac{1}{4}c} = \frac{1}{4}e^{\frac{1}{6}}e^{\frac{1}{4}}$$

$$= \frac{e^{\frac{1}{4}}}{e^{\frac{1}{4}}} + \frac{e^{\frac{1}{4}c}}{e^{\frac{1}{4}}}$$

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$$= \frac{e^{\frac{1}{4}}}{e^{\frac{1}{4}}} + \frac{e^{\frac{1}{4}}}{e^{\frac{1}{$$

$$\frac{dy}{dx} = P(x)y + Q(x)$$

$$\frac{dy}{dx} = P(x)y +$$

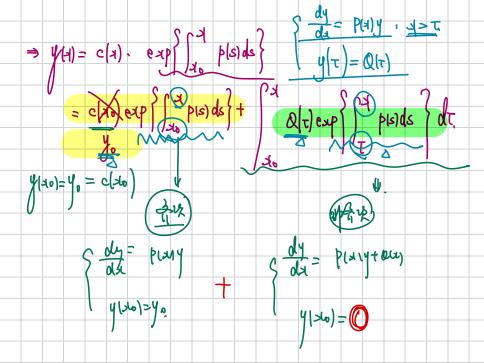
$$\frac{dy}{dx} = c(x) \exp\left\{ \int_{x_0}^{x_0} p(s) ds \right\}$$

$$= P(x) c(x) \exp\left\{ \int_{x_0}^{x_0} p(s) ds \right\} + c(x) p(x) \exp\left\{ \int_{x_0}^{x_0} p(s) ds \right\}$$

$$= c(x) = Q(x) \exp\left\{ \int_{x_0}^{x_0} p(s) ds \right\} d\tau.$$

$$= c(x) = c(x_0) + \int_{x_0}^{x_0} Q(x) \exp\left\{ \int_{x_0}^{x_0} p(s) ds \right\} d\tau.$$

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## Integrating Factor → Chapter 9

(1) 
$$y' - y = Q(x) \leadsto e^{-x}(y' - y) = e^{-x}Q(x) = [ye^{-x}]'.$$
  
 $\leadsto e^{-x}dy - e^{-x}[y + Q(x)]dx = 0.$ 

$$\frac{\partial(e^{-x})}{\partial x} = -\frac{\partial\{e^{-x}[y+Q(x)]\}}{\partial y} = -e^{-x}.$$

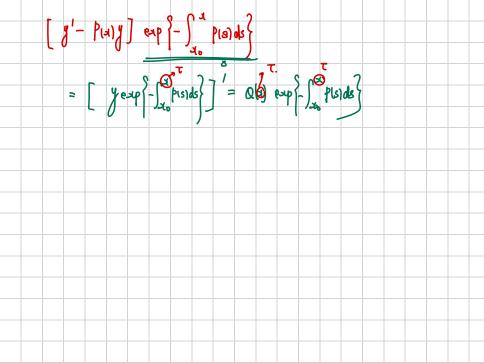
$$(2) \ y' - P(x)y = Q(x) \Rightarrow \text{Let } \mu(x) = \exp\left\{\int_{x}^{x_0} P(x) dx\right\}.$$

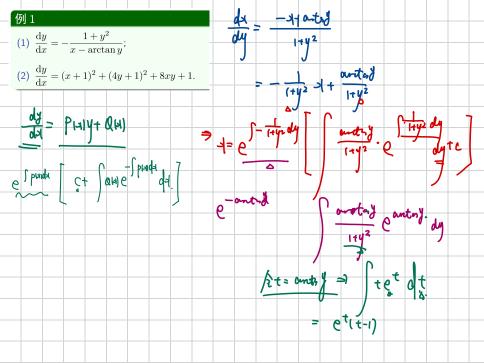
$$\Rightarrow \mu(x)dy - \mu(x)[P(x)y + Q(x)]dx = 0.$$

$$\frac{\partial \mu(x)}{\partial x} = -\frac{\partial \{\mu(x)[P(x)y + Q(x)]\}}{\partial y} = -\mu(x)P(x).$$



(illusion) Lecture 1

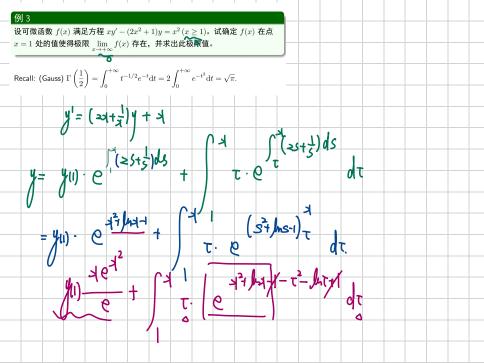




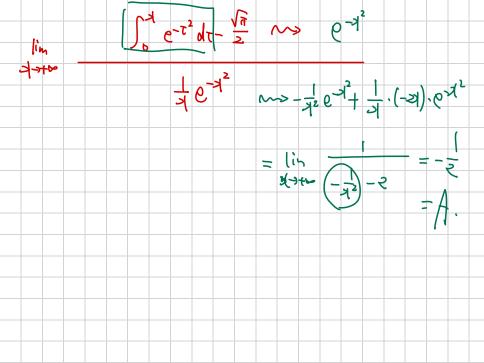
(1) 
$$\frac{dy}{dx} = -\frac{1+y^2}{x-\arctan y}$$
;  
(2)  $\frac{dy}{dx} = (x+1)^2 + (4y+1)^2 + 8xy + 1$ .  

$$\frac{dy}{dx} = \frac{1+y^2}{x^2+3x+1} + \frac{1}{1} + \frac{1}{$$

例 1



$$\frac{1}{2} \frac{1}{1} \frac{1}$$



### Bernoulli Differential Equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = P(x)y + Q(x)y^n, n \in \mathbf{N}^*, n \neq 1, 2.$$

- $\bullet \ \frac{1}{y^n} \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y^{n-1}} P(x) + Q(x).$
- Let  $t = 1/y^{n-1} = y^{1-n} \leadsto \frac{\mathrm{d}t}{\mathrm{d}x} = y'\frac{1-n}{y^n}$
- $\rightsquigarrow \frac{1}{1-n} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} = P(x)t + Q(x).$
- Note: Of course, the Bernoulli differential equation has a particular solution y=0.

#### More Examples

(1) (Riccati)  $\frac{\mathrm{d}y}{\mathrm{d}x} = P(x)y + Q(x)y^2 + R(x) \rightsquigarrow \text{Attention is all you need!}$ 

 $\leadsto$  If you find one solution  $\tilde{y}$ , then let  $y=z+\tilde{y} \leadsto$  Bernoulli.

(2) 
$$\frac{x dy}{y dx} = f(xy) \rightsquigarrow u = xy, \frac{du}{dx} = y + x \frac{dy}{dx}$$

(3) 
$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(ax + by + c) \rightsquigarrow u = ax + by + c, \ \frac{\mathrm{d}u}{\mathrm{d}x} = a + b\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$(4) \ \frac{x^2 \mathrm{d}y}{\mathrm{d}x} = f(xy)$$

(5) 
$$\frac{\mathrm{d}y}{\mathrm{d}x} = xf\left(\frac{y}{x^2}\right)$$

(6) ...



(1) (Riccati) 
$$\frac{\mathrm{d}y}{\mathrm{d}x} = P(x)y + Q(x)y^2 + R(x) \leadsto \text{Attention is all you need!}$$

 $\leadsto$  If you find one solution  $\tilde{y}$ , then let  $y=z+\tilde{y} \leadsto$  Bernoulli.

$$\frac{d \approx}{d \Rightarrow} = \frac{d \frac{1}{2}}{d \frac{1}{2}} - \frac{d \frac{1}{2}}{d \frac{1}{2}}$$

$$= P(\frac{1}{2}) y + Q(\frac{1}{2}) y^2 - P(\frac{1}{2}) y - Q(\frac{1}{2}) y^2$$

$$= P(\frac{1}{2}) (y - \frac{1}{2}) + Q(\frac{1}{2}) (y - \frac{1}{2}) (y + \frac{1}{2})$$

$$= p(a) (y-\widetilde{y}) + Q(a) (y-\widetilde{y}) (y+\widetilde{y})$$

$$\frac{dt}{dx} = \frac{s}{x}t - \frac{1}{x}dx$$

$$t = e^{-\frac{s}{x}}dx \left[ \frac{c}{c} + \frac{1}{x} - \frac{s}{x}dx \right]$$

$$= x^{\frac{s}{x}} \left[ \frac{c}{c} + \frac{1}{x} - \frac{1}{x} + \frac{1}{$$

$$(4) y' + \frac{y}{x} = y^{2} - \frac{4}{x^{2}}.$$

$$(3y) = (3^{2}y^{2})$$

(2) 
$$\frac{x \, dy}{y \, dx} = f(xy) \rightsquigarrow \underline{u} = xy, \quad \frac{du}{dx} = y + x \frac{dy}{dx}$$

$$\frac{du}{dx} = y + x \left( \frac{y}{x} + f(u) \right)$$

$$= y \left( \frac{y}{x} + f(u) \right)$$

$$= \frac{y}{x} + x \frac{dy}{dx}$$

$$=$$

$$\frac{dt}{dx} = (t^2t^2) \cdot \frac{t}{3}$$

$$\Rightarrow \int \frac{1}{t^3t^2} dt = \int \frac{1}{t^3} dt$$

$$\frac{1}{t^3} \cdot \frac{1}{t^2t^2} dt = \ln|x| + C.$$

$$\frac{1}{2} \cdot \frac{1}{t^3} + \frac{1}{t^2t^2} dt = \ln|x| + C.$$

$$\frac{1}{2} \cdot \ln|t| - \frac{1}{t^3} \cdot \frac{1}{t^3t^2} \cdot \frac{1}$$