例 1

Solve $y'' + 4y = 0$. If we let $y' = p$, we will have $p\,\mathrm{d}p = -4y\,\mathrm{d}y \rightsquigarrow p^2 = -4y^2 + C$.

$$p = \pm\sqrt{C - 4y^2} = \frac{dy}{dx}$$

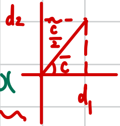
$$x + \hat{C} = \int \frac{1}{2 \pm \sqrt{C - 4y^2}}\, d(2y) = \int \frac{d(2y)}{\pm 2\sqrt{(\sqrt{C})^2 - (2y)^2}} = \pm\frac{1}{2}\arcsin\frac{2y}{\sqrt{C}}$$

$$= \frac{1}{2}\arcsin\frac{2y}{\tilde{C}}, \quad \tilde{C} \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \arcsin\frac{x}{a} + C$$

$$\Rightarrow \frac{2y}{\tilde{C}} = \sin(2x + 2\hat{C}) = \sin(2x + \bar{C}) = \sin 2x \cos\bar{C} + \cos 2x \sin\bar{C}$$

$$\Rightarrow y = \underbrace{\left(\frac{\tilde{C}}{2} \cdot \cos\bar{C}\right)}_{d_1} \sin 2x + \underbrace{\left(\frac{\tilde{C}}{2} \cdot \sin\bar{C}\right)}_{d_2} \cos 2x$$



例 2

设右手直角坐标系 $\{O;\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$. 给定 $\boldsymbol{a}=2\boldsymbol{i}+3\boldsymbol{j}-5\boldsymbol{k}$, $\boldsymbol{b}=3\boldsymbol{i}-4\boldsymbol{j}+\boldsymbol{k}$. 求向量 $\boldsymbol{c}$，使得 $|\boldsymbol{c}|=\sqrt{3}$，且由 $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ 三向量所张成的平行六面体的体积最大.

Note: 也可使用 Lagrange 乘数法来解决.

$$V=|(a,b,c)|=\left|\det\begin{pmatrix} c_1 & c_2 & c_3 \\ 2 & 3 & -5 \\ 3 & -4 & 1 \end{pmatrix}\right|$$

$$\frac{c_1+c_2+c_3}{3}\le\sqrt[3]{\frac{c_1^2+c_2^2+c_3^2}{3}} \checkmark$$

★目标.

$$=\left|-17c_1-17c_2-17c_3\right|=17\left|c_1+c_2+c_3\right|$$ 起.

max. $(c_1,c_2,c_3$ 同号$)$

★ $c_1^2+c_2^2+c_3^2=3$

约束

· 多元 极值　$f: \boxed{(x,y)} \mapsto f\boxed{(x,y)}$　$\forall$方向 $l \in \mathbb{R}^2$ 单位. $f(x+tl) \leq f(x)$　$\forall t \in \mathbb{R}$.

$x. \quad \mathbb{R}^2 \to \mathbb{R}$　$t \to \begin{cases} x_1+tl_1 \\ x_2+tl_2 \end{cases} \to f$　$\varphi(t) =$

$$f(x_1+tl_1, x_2+tl_2) = \varphi(t).$$

$\varphi(0)$为极值.

$$\varphi'(t) = \frac{df}{dt} = \frac{\partial f}{\partial(x_1+tl_1)} \cdot \frac{\partial(x_1+tl_1)}{\partial t} + \frac{\partial f}{\partial(x_2+tl_2)} \cdot \frac{\partial(x_2+tl_2)}{\partial t}.$$

$$= f_x \cdot l_1 + f_y \cdot l_2 = \underset{\text{grad}}{\nabla f} \cdot l = 0$$

$\nabla f = (f_x, f_y)$

$\forall l$ 成立　$\vec{e_1}, \vec{e_2}$　$\nabla f \cdot \vec{e_1} = 0$. $\underset{\in \mathbb{R}^2}{\nabla f} \cdot \vec{e_2} = 0 \Rightarrow \forall a \in \mathbb{R}^2. \nabla f \cdot a = 0$

$(\nabla f)^2 = \nabla f \cdot \nabla f = a \Rightarrow \boxed{\nabla f = 0}$

一元 Fermat. $f(x)$ 在 (a,b) 上有极值. 可导. $\exists \xi \in (a,b)$, $f'(\xi) = 0$. 中值定理

$f(x,y)$ 目标　　条件约束极值　　Lagrange 乘数法

约束 $\Phi(x,y)=0$. 假设: (x_0,y_0) 为极值点，$\Phi_y(x_0,y_0)\neq 0$.

不够

$$\nabla\Phi\big|_{(x_0,y_0)}\neq\vec{0}$$

[证] $\Phi_y\big|_{(x_0,y_0)}\neq 0 \Rightarrow$ 局部确定 $y=\varphi(x)$.

$$f(x,y)=\underset{g(x)}{\underset{\|}{f(x,\varphi(x))}} \longrightarrow g'(x_0)=0 \Rightarrow \left(f_x+f_y\varphi'(x)\right)\Big|_{(x_0,y_0)}=0$$

$$g'=\frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{d\varphi(x)}{dx}=f_x+f_y\varphi'(x)$$

新　　注意

$$\frac{\partial f(x^2,y^2)}{\partial x}$$

$(x,y)\to(x^2,y^2)$

$(u,v)\to f(u,v)$

$$\frac{df}{dx}=\boxed{\frac{\partial f}{\partial u}}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=f_x\,/\,f_1\,/\,\partial_x f\,/\,\partial_1 f$$

$$\frac{d\left(\sin x^2\right)}{dx} = \frac{d(\sin x^2)}{dx^2} \cdot \frac{dx^2}{dx}$$

$$\Phi(x, \underset{y.}{\varphi(x)}) = 0 \qquad \Phi_x + \varphi' \Phi_y = 0. \Rightarrow \underbrace{\varphi' = -\frac{\Phi_x}{\Phi_y}}$$

$$f_x + \varphi' f_y = 0. = f_x - \frac{\Phi_x}{\Phi_y} f_y = 0$$

$$\Rightarrow f_x - \left(\frac{f_y}{\Phi_y}\right)\Big|_{(x_0, y_0)} \Phi_x = 0.$$

$$\lambda = \frac{f_y}{\Phi_y}. \qquad \Rightarrow f_x - \lambda \Phi_x = 0. = F_x$$

$$f_y - \lambda \Phi_y = 0 = F_y \qquad \boxed{\Phi = 0}$$

$$\Phi=0 \Rightarrow \big(\underbrace{f-\lambda\Phi}\big)_\lambda = -\Phi = 0 = F_\lambda$$

$$\underbrace{F(x,y,\lambda)=f(x,y)-\lambda\Phi(x,y)} \Rightarrow \boxed{\nabla F=0} \text{ 求解.}$$

e.g. $x+y+z$ 在 $x^2+y^2+z^2=3$ 约束下的 max. $(x,y,z\geq 0)$

$$\underbrace{F(x,y,z,\lambda)} = (x+y+z)-\lambda\left(x^2+y^2+z^2-3\right)$$

$$\nabla F = \left(1-2\lambda x,\ 1-2\lambda y,\ 1-2\lambda z,\ x^2+y^2+z^2-3\right)=0.$$

$$\Rightarrow \underbrace{x=\frac{1}{2\lambda}=y=z}=1$$

$$\frac{1}{4\lambda^2}\cdot 3=3 \Rightarrow \lambda=\frac{1}{2}$$

$\boxed{\text{说明之示性}}$

$S=\{(x,y,z)\mid x^2+y^2+z^2=3\}$ 上有界闭集.

$f(x,y,z)=x+y+z$ 必在 S 上取 max, min.

$\Rightarrow$ $(1,1,1)$ 必定为极值 ~~且~~ 为最值. $\Rightarrow (x+y+z)_{max}=3$.

(1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2}; = 0$

$\underbrace{0<}_{\Delta}|(x,y)-(0,0)|<\delta$

pf [way I] $\forall \varepsilon>0,\ \exists \delta>0.$ 当 $0<|x|<\delta^{(\varepsilon)}.\ 0<|y|<\delta(\varepsilon),$

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| \le \frac{|x^2|}{\underbrace{|x^2+y^2|}_{\Delta}}|x| + \frac{|y^2|}{\underbrace{|x^2+y^2|}_{\Delta}}|y|$$

$$< |x|+|y| < 2\delta < \varepsilon \text{ for } \delta = \varepsilon/2.$$

[way II]. $x=\rho\cos\theta.\ y=\rho\sin\theta.$

$$\frac{\rho^3(\cos^3\theta+\sin^3\theta)}{\rho^2} = \underbrace{\boxed{\rho}}_{\text{趋0}}\underbrace{\boxed{(\cos^3\theta+\sin^3\theta)}}_{\text{有界}} \to 0.$$

$\underline{(x,y)\to(0,0)}$ $\underline{\rho\to 0}$ $\underline{\forall\theta}$.

(3) $$\lim_{(x,y)\to(+\infty,+\infty)}\left(\frac{xy}{x^2+y^2}\right)^{x^2} = 0$$

$$\frac{xy}{x^2+y^2} = \frac{\rho^2 \sin\theta\cos\theta}{\rho^2} = \sin\theta\cos\theta \leq \frac{1}{2}$$

$\frac{1}{2}\sin 2\theta$.

$$0 \leq \left(\frac{xy}{x^2+y^2}\right)^{x^2} \leq \left(\frac{1}{2}\right)^{x^2} \to 0.$$

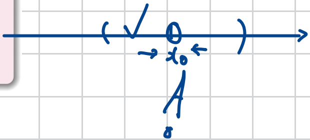
四.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}; = \boxed{\cos\theta\sin\theta}$$

Thm 5

(一元) 设函数 f 在 x_0 的一个空心开邻域内有定义，则 f 在 x_0 处的极限为 A 当且仅当对于任何收敛点列 $x_n \to x_0$ $(n \to \infty)$ 且 $x_n \neq x_0$ $(\forall n)$，均有

$$\lim_{n\to\infty} f(x_n) = A.$$



$\Rightarrow$ ✓ 记 $\lim\limits_{x\to x_0} f = A$.

$\forall \varepsilon > 0$, $\exists \delta > 0$, $0 < |x - x_0| < \delta$, $|f(x) - A| < \varepsilon$.

$\forall x_n \to x_0$, $x_n \neq x_0 \Rightarrow 0 < |x_n - x_0|$, $\forall n$.

对于这 δ, $\exists N > 0$, $\forall n > N$, $0 < |x_n - x_0| < \delta$

$\Rightarrow |f(x_n) - A| < \varepsilon$.

$\Rightarrow f(x_n) \to A$ $(n \to \infty)$.

Thm 5

(一元）设函数 f 在 x_0 的一个空心开邻域内有定义，则 f 在 x_0 处的极限为 A 当且仅当对于任何收敛点列 $x_n \to x_0$ $(n \to \infty)$ 且 $x_n \neq x_0$ $(\forall n)$，均有

$$\lim_{n\to\infty} f(x_n) = A.$$

⇐ 反证. 若 $f \not\to A$ $(x \to x_0)$

$(f \to A)$ $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall\, 0 < |x - x_0| < \delta$, $|f(x) - A| < \varepsilon$

$(f \not\to A)$ $\exists \varepsilon_0 > 0$, $\boxed{\forall \delta > 0}$, $\exists\, 0 < |\tilde{x} - x_0| < \delta$, $|f(\tilde{x}) - A| \geq \varepsilon_0$.

$\delta = 1$, $\exists\, 0 < |\tilde{x}_{(1)} - x_0| < 1$, $| \quad | \geq \varepsilon_0$

$\delta = \frac{1}{2}$ $\quad \tilde{x}_{(2)}$ $\quad \frac{1}{2}$ $\quad \geq \varepsilon_0$

$\delta = \frac{1}{3}$ $\quad \tilde{x}_{(3)}$ $\quad \frac{1}{3}$ $\quad \geq \varepsilon_0$

$$\delta = \frac{1}{n} \downarrow 0$$

$$0 < |\tilde{x}(n) - x_0| < \frac{1}{n}. \quad \geq \varepsilon_0$$

$$\{\tilde{x}_n\} \to x_0.$$

$$\{f(\tilde{x}_n)\} \not\to A$$

矛盾!

$\square$.

$\lim_{(x,y)\to(0,0)}$ $\lim_{x\to 0}\lim_{y\to 0}$ $\lim_{y\to 0}\lim_{x\to 0}$

一般情况下，重极限和累次极限的存在没有什么关系.就算两个累次极限都存在，也不一定相等(需要一致性条件)

例 7

讨论下列函数在 $(0,0)$ 处的两个累次极限和重极限:

$\lim_{(x,y)\to(0,0)}$ $\lim_{x\to 0}\lim_{y\to 0}$ $\lim_{y\to 0}\lim_{x\to 0}$

(1) $$f(x,y)=\begin{cases}(x+y)\sin\dfrac{1}{x}\sin\dfrac{1}{y}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$

0　不存　不存

(2) $$f(x,y)=\frac{x^2-y^2}{x^2+y^2}.$$

$\dfrac{\cancel{r^2}(\cos^2\theta-\sin^2\theta)}{\cancel{r^2}}$

不存.　1　-1

设 $f(x,y)$ 在点 (x_0,y_0) 的一个去心邻域上有定义，且重极限

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=\ell$$

存在，则

- 如果 $y\neq y_0$ 时，$\lim_{x\to x_0} f(x,y)$ 存在，则 $\lim_{y\to y_0}\lim_{x\to x_0} f(x,y)=\ell$;
- 如果 $x\neq x_0$ 时，$\lim_{y\to y_0} f(x,y)$ 存在，则 $\lim_{x\to x_0}\lim_{y\to y_0} f(x,y)=\ell$.

$\forall \varepsilon>0,\ \exists\delta>0,\ 0<|x-x_0|<\delta,\ 0<|y-y_0|<\delta,$

$$\ell-\varepsilon< f(x,y)<\ell+\varepsilon.$$

上式中令 $x\to x_0$, $\ell-\varepsilon\le \lim_{x\to x_0} f(x,y)\le \ell+\varepsilon.$

$$\Rightarrow \underbrace{\left|\lim_{x\to x_0} f-\ell\right|\le\varepsilon<2\varepsilon}$$

$$\Rightarrow \lim_{y\to y_0}\left(\lim_{x\to x_0} f\right)=\ell.$$

Prop. 11

函数 $f(x,y)$ 的两个偏导数 $\dfrac{\partial f}{\partial x}$ 和 $\dfrac{\partial f}{\partial y}$ 在 (x_0,y_0) 点的某个邻域内存在且连续，则 f 在 (x_0,y_0) 点连续.

$$|f(x_0+h_1, y_0+h_2)-f(x_0,y_0)|$$

$$=|f(x_0+h_1, y_0+h_2)-f(x_0+h_1,y_0)|+|f(x_0+h_1,y_0)-f(x_0,y_0)|$$

$$=|f_y(x_0+h_1, y_0+\theta_2 h_2)\cdot h_2|+|f_x(x_0+\theta_1 h_1, y_0)||h_1|$$

$\theta_2\in(0,1)$ $\theta_1\in(0,1)$.

连续

$$\leq M_1|h_2|+M_2|h_1| < \varepsilon.$$

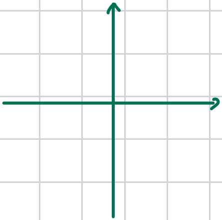
取 $|h_2|, |h_1| \in \left(0, \dfrac{1}{M_1+M_2}\right)$.

$\| \delta$.

$$f(a+h) = f(a) + \lambda \cdot h + o(|h|), \quad (|h| \to 0),$$

$$f(a+h^{(1)}) = f(a) + \lambda_1 h_1 + o(h_1)$$

$$h^{(1)} = (h_1, 0, \ldots, 0)$$



$$\frac{f(a_1+h_1, a_2, \ldots, a_n) - f(a_1, \ldots, a_n)}{h_1} = \lambda_1 + \underline{o(1)}$$

$$\underline{h_1 \to 0} \qquad \lambda_1 = f_1(a)$$

$$\Rightarrow \lambda_i = f_i(a).$$

$$\frac{\partial f}{\partial l} = \lim_{t\to 0^+} \frac{f(a+tl) - f(a)}{t} = \frac{\lambda \cdot t\vec{l}}{t} = \lambda \cdot \vec{l} = \underbrace{\nabla f(a) \cdot \vec{l}}$$

l ↳ 单位