§7.5, 7.6, 7.10 Ordinary Differential Equations II

illusion

Especially made for zqc

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http://illusion-hope.github.io/25-Spring-ZQC-Calculus/

HW-1

例 1

(1) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y^6 - 2x^2}{2xy^5 + x^2y^2}$;

(1') (Similar!) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x^3 + 3xy^2 + x}{3x^2y + 2y^3 - y}$;

(2) $y' = y^2 + 2(\sin x - 1)y + \sin^2 x - 2\sin x - \cos x + 1$.

例 2

已知微分方程 $y' + y = f(x)$, $f(x) \in C(-\infty, +\infty)$. 若 $f(x)$ 是以周期为 T 的函数，证明：方程存在唯一以 T 为周期的解。

(1') (Similar!) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x^3+3xy^2+x}{3x^2y+2y^3-y}$;

$$\frac{dy^2}{dx^2} = \frac{y\,dy}{x\,dx} = \frac{2x^2+3y^2+1}{3x^2+2y^2-1}$$

$y^2=u$, $x^2=v$.

$$\frac{du}{dv} = \frac{2v+3u+1}{3v+2u-1} = \frac{2(v-1)+3(u+1)}{3(v-1)+2(u+1)} = \frac{d(u+1)}{d(v+1)}$$

$u+1=\tilde{u}$, $v+1=\tilde{v}$.

$\dfrac{\tilde{u}}{\tilde{v}} = w$ $\quad \tilde{u}=w\tilde{v}$

$$\frac{d\tilde{u}}{d\tilde{v}} = \frac{2\tilde{v}+3\tilde{u}}{3\tilde{v}+2\tilde{u}} = \frac{2+3\left(\frac{\tilde{u}}{\tilde{v}}\right)}{3+2\left(\frac{\tilde{u}}{\tilde{v}}\right)}$$

$$\Rightarrow \frac{d\tilde{u}}{d\tilde{v}} = \frac{dw}{d\tilde{v}}\cdot\tilde{v}+1\cdot w$$

$$= \frac{2+3w}{3+2w}$$

$$\Rightarrow \frac{dw}{d\tilde{v}} \cdot \tilde{v} = \frac{2+3w-3w-2w^2}{3+2w}$$

$$= \frac{2-2w^2}{3+2w}$$

$$\Rightarrow \frac{(3+2w)\,dw}{2(1-w^2)} = \frac{1}{\tilde{v}}\,d\tilde{v}$$

$$\Rightarrow \int \frac{3}{2(1-w^2)}\,dw \;\cancel{+}\; - \int \frac{2w}{2(1-w^2)}\,dw \qquad d(-w^2+1)$$

$$\frac{1}{(1-w)(1+w)}$$

$$\frac{3}{4}\ln\left|\frac{1+w}{1-w}\right| - \frac{1}{2}\ln\left|1-w^2\right| = \ln\left|\tilde{v}\right| + C$$

$$\Rightarrow \frac{\left|\frac{1+w}{1-w}\right|^{\frac{3}{4}}}{\left|1-w^2\right|^{\frac{1}{2}}} = c\tilde{v} \quad , \quad c \neq 0.$$

(1) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y^6 - 2x^2}{2xy^5 + x^2y^2}$;

$$y^2\frac{dy}{dx} = \frac{y^6 - 2x^2}{2xy^3 + x^2} = \frac{(y^3)^2 - 2x^2}{2xy^3 + x^2} = \frac{\left(\frac{y^3}{x^2}\right)^2 - 2}{2\left(\frac{y^3}{x}\right) + 1}$$

令 $y^3 = xu \qquad \dfrac{dy^3}{dx} = u + x\dfrac{du}{dx}$

$$= \frac{3(u^2-2)}{2u+1} = \frac{3u^2 - 6}{2u+1}$$

$$x\frac{du}{dx} = \frac{3u^2-6-2u^2-u}{2u+1} = \frac{u^2-u-6}{2u+1}$$

$$\Rightarrow \frac{(2u+1)\,du}{u^2-u-6} = \frac{1}{x}\,dx$$

$$\Rightarrow \frac{2u+1\quad du}{(u+2)(u-3)}$$

$$\Rightarrow \frac{1}{u+2} + \frac{1}{u-3} + \frac{2}{(u+2)(u-3)}$$

$$\frac{2}{5}\left(\frac{1}{u-3} - \frac{1}{u+2}\right)$$

$$\Rightarrow \frac{7}{5}\,\frac{1}{u-3} + \frac{3}{5}\,\frac{1}{u+2}$$

$$\ln\left(|u-3|^{\frac{7}{5}}\,|u+2|^{\frac{3}{5}}\right) = \ln\left(C|x|\right)$$

$C \neq 0$

(2) $y' = y^2 + 2(\sin x - 1)y + \sin^2 x - 2\sin x - \cos x + 1.$

$$(1-\sin x)' = -\cos x = \cancel{1 - 2\sin x + \sin^2 x} - 2(\sin x - 1)^2 + \cancel{\sin^2 x - 2\sin x - \cos x + 1}$$

$$\cancel{-2\sin^2 x + 4\sin x - 2}$$

令 $y = z + 1 - \sin x$

$$\frac{dy}{dx} = \frac{dz}{dx} - \cancel{\cos x} = (z+1-\sin x)^2 + 2(\sin x - 1)(z + 1 - \sin x) + \sin^2 x - 2\sin x$$

$$- \cancel{\cos x} + 1$$

$$z^2 + 1 + \sin^2 x + 2z - 2\sin x\, z - 2\sin x + 2\left(z\sin x + \sin x - \cancel{\sin^2 x} - z - 1 + \sin x\right)$$

$$+ \sin^2 x - 2\sin x + 1$$

$$z^2 + \cancel{1} + \cancel{\sin^2 x} + \cancel{2z} - \underline{2\sin x\, z} - \cancel{2\sin x} + \underline{2z\sin x} + \cancel{2\sin x} - \cancel{2\sin^2 x} - \cancel{2z}$$

$$\cancel{-2x} + \cancel{2\sin x} + \cancel{\sin^2 x} - \cancel{2\sin x} + \cancel{1}$$

$$z^2 = \frac{dz}{dx} \qquad dx = \frac{1}{z^2}\, dz$$

$$\boxed{x + c = -\frac{1}{z}}$$

例 2

已知微分方程 $y'+y=f(x)$, $f(x)\in C(-\infty,+\infty)$. 若 $f(x)$ 是以周期为 T 的函数，证明：方程存在唯一以 T 为周期的解。

解: $y' = -y + f(x)$

$\Rightarrow$ $F(x) = y = y(0)\,e^{\int_0^x (-1)\,dt} + \int_0^x f(\tau)\, e^{\int_\tau^x -1\,dt}\, d\tau$

$= y(0)\,e^{-x} + \int_0^x f(\tau)\, e^{\tau - x}\, d\tau$

$F(x+T) = y(0)\,e^{-x-T} + \int_0^{x+T} f(\tau)\, e^{\tau-x-T}\, d\tau$

$m = \tau - T \qquad \tau = m + T$

$$\int_{-T}^{x} \underbrace{\boxed{f(m+T)}}_{=\,f(m)} e^{m-x}\, dm.$$

$$\int_{-T}^{0} + \cancel{\int_{0}^{x}}$$

$$y(0)e^{-x} = y(0)e^{-x-T} + \int_{-T}^{0} f(m)e^{m-x}\, dm.$$

$\underline{\underline{\times e^{x}}}$

$$\Rightarrow \quad y(0) = y(0)e^{-T} + \underline{\int_{-T}^{0} f(m)e^{m}\, dm.}$$

常值.

$$\Rightarrow \quad y(0) = \frac{\int_{-T}^{0} f(m)e^{m}\, dm}{1-e^{-T}}$$

四、

例 3

(Gronwall) 设 $f(t), g(t), x(t) \in C[t_0, t_1]$ 且非负，求证：若 $x(t) \le g(t) + \int_{t_0}^{t} f(\tau)x(\tau)\,\mathrm{d}\tau,\ t_0 \le t \le t_1$，则

$$x(t) \le g(t) + \int_{t_0}^{t} f(\tau)g(\tau)\exp\left\{\int_{\tau}^{t} f(s)\,\mathrm{d}s\right\}\mathrm{d}\tau, \quad t_0 \le t \le t_1.$$

F

pf:

$$\underbrace{f(t)x(t)}_{=F'} \le f(t)g(t) + f(t)\underbrace{\int_{t_0}^{t} f(\tau)x(\tau)\,d\tau}_{=F}.$$

$$\boxed{F' - f(t)F} \le f(t)g(t)$$

$$\exp\left\{\int_{t}^{t_0} f(s)\,ds\right\}\left[F' - f(t)F\right]$$

$$\left[\exp\left\{\int_{t}^{t_0} f(s)\,ds\right\}F\right]' \le f(t)g(t)\exp\left\{\int_{t_0}^{t_0} f(s)\,ds\right\}$$

$$\left[\exp\left\{\int_\tau^{t_0} f(s)\,ds\right\} F\right]' \le f(\tau)\,g(\tau)\exp\left\{\int_\tau^{t_0} f(s)\,ds\right\}$$

τ代入t $-$ $\boxed{\tau\text{代入}t_0}$

$$\exp\left\{\int_t^{t_0} f(s)\,ds\right\} F \le \int_{t_0}^{t} f(\tau)\,g(\tau)\exp\left\{\int_\tau^{t_0} f(s)\,ds\right\} d\tau$$

$$\exp\left\{\int_{t_0}^{t} \right\}$$

这说

$$x(t) - g(t) \le F \le \int_{t_0}^{t} f(\tau)\,g(\tau)\exp\left\{\int_\tau^{t} f(s)\,ds\right\} d\tau$$

Differential Equations That Can Be Reduced In Order

(1) $y^{(n)} = f(x)$;

(2) $y'' = f(x, y')$;

(3) 自治/驻定方程 $F(y, y', \cdots, y^{(n)}) = 0$. $\boxed{\text{不含}x}$

前两类是基本的，我们只看第三类，考虑 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = z$，那么

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = z\frac{\mathrm{d}z}{\mathrm{d}y} \rightsquigarrow y^{(3)} = \frac{\mathrm{d}}{\mathrm{d}x}\left\{z\frac{\mathrm{d}z}{\mathrm{d}y}\right\} = z\frac{\mathrm{d}}{\mathrm{d}y}\left\{z\frac{\mathrm{d}z}{\mathrm{d}y}\right\}.$$

$F(y, y', y'', y^{(3)}) = 0$ 隐 x 阶 3

利用微分的运算性质，即

$$y^{(3)} = z\left\{\left(\frac{\mathrm{d}z}{\mathrm{d}y}\right)^2 + z\frac{\mathrm{d}^2 z}{\mathrm{d}y^2}\right\}.$$

$F\left(y, z, z\cdot\frac{dz}{dy}, z\left\{\left(\frac{dz}{dy}\right)^2 + z\frac{d^2z}{dy^2}\right\}\right)$

Examples

$$\frac{1}{(x-1)^2(x^2+1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2} + \frac{Gx+H}{(x^2+1)^3}$$

例 5

(1) $y'' = \dfrac{1}{(1+x^2)^2}$;

(2) $yy'' + 2(y')^2 = 0$;

$y' = z$

$y'' = \frac{d}{dx} \cdot y' = \frac{dy}{dx} \frac{d}{dy} y' = z \cdot \frac{dz}{dy}$

(3) (22-23 Midterm) $yy'' + (y')^2 - 2yy' = 0$.

Note: 除了 $y' = z$，你能给出类似积分因子法的思路吗？注意 $(yy')' = ?$

$y \cdot z \frac{dz}{dy} + 2z^2 = 0$ ① $\boxed{z=0}$ 特解 ② $\boxed{y^2 z = C}$

$y \frac{dz}{dy} + 2z = 0 \quad -\frac{1}{2z} dz = \frac{1}{y} dy \quad C - \frac{1}{2} \ln|z| = \ln|y|$

$\boxed{C|z|^{-\frac{1}{2}} = |y|}$

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1+x^2-x^2}{(1+x^2)^2}\,dx = \int \frac{1}{1+x^2}\,dx - \int \frac{x^2}{(1+x^2)^2}\,dx$$

$$-\frac{1}{2}\,x \cdot \boxed{\frac{x}{(1+x^2)^2}}\,dx$$

$$-\frac{1}{2}\,x\,d\left[\frac{1}{1+x^2}\right]$$

$$\arctan x - \left(-\frac{1}{2}\,\frac{x}{1+x^2} + \frac{1}{2}\arctan x\right) + C$$

$$\frac{1}{2}\,\frac{x}{1+x^2} + \frac{1}{2}\arctan x + C$$

(2) $yy'' + 2(y')^2 = 0;$

$(yy')' = yy'' + (y')^2$

$(e^{2y} y')' = 2(y')^2 \cdot e^{2y} + e^{2y} \cdot y''$

$(y^2 y')' = 2y(y')^2 + y^2 \cdot y''$

$= y[2(y')^2 + yy''] = 0$

$y^2 y' = C.$

From High-Order L.D.E. to Systems of L.D.E.

给定一个高阶线性微分方程

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x). \tag{2}$$

令 $y_1 = y, y_2 = y', \cdots, y_n = y^{(n-1)}$ 那么有

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -a_n(x) & -a_{n-1}(x) & \cdots & -a_1(x) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f(x) \end{bmatrix}. \tag{3}$$

$$\rightsquigarrow \boldsymbol{y}' = \boldsymbol{A}(x)\boldsymbol{y} + \boldsymbol{f}(x).$$

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = \boxed{f(x)}.$$

$$\begin{cases} y' = y' \\ (y')' = y^{(2)} \\ (y^{(2)})' = y^{(3)} \\ \quad\vdots \\ (y^{(n-1)})' = \end{cases}$$

$$\frac{d}{dx}\begin{bmatrix} y \\ y' \\ y'' \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 0 \; 1 \\ -a_n(x) & \cdots & & & -a_1(x) \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \\ \vdots \\ y^{(n-1)} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(x) \end{bmatrix}$$

[illegible] $y \cdots y^{(n-1)}$

存在唯一性定理：对线性微分方程组

$$\boldsymbol{y}' = \boldsymbol{A}(x)\boldsymbol{y} + \boldsymbol{f}(x),\ \boldsymbol{y}(x_0) = \boldsymbol{y}_0,\ x_0 \in (a,b) \tag{4}$$

若 $\boldsymbol{A}(x), \boldsymbol{f}(x) \in C[a,b]$，方程 (2) 满足初始条件 $\boldsymbol{y}(x_0) = \boldsymbol{y}_0$ 的解 $\boldsymbol{y} = \boldsymbol{y}(x)$ 在 $[a,b]$ 上存在且唯一。

$\rightsquigarrow$ 对高阶线性微分方程

$\boxed{f(x)}$

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x). \tag{5}$$

若 $a_i(x) \in C[a,b]$, 则方程 (5) 在给定初值条件 $y(x_0) = y_0, y'(x_0) = y_0^{(1)}, \cdots,$ $y^{(n-1)}(x_0) = y_0^{(n-1)}$ 下存在唯一解 $y = y(x),\ x \in [a,b]$.

Examples

在求解某些线性微分方程组时，也可以将其化为高阶线性微分方程来处理。

例 6

转化为高阶线性微分方程：

$$\boxed{D = \frac{d}{dx}} \qquad \frac{1}{D} \to \text{可移项}$$

(1) $\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = 3y - 2z, \\[2ex] \dfrac{\mathrm{d}z}{\mathrm{d}x} = 2y - z. \end{cases}$

$$\begin{cases} (D-3)2y = -4z \\ (D+1)z = 2y \end{cases}$$

$$(D-3)(D+1)z = (D-3)2y = -4z$$

$$D^2z - 2Dz - 3z + 4z = 0$$

$$\Rightarrow \boxed{\frac{d^2z}{dx^2} - 2\frac{dz}{dx} + z = 0}$$

(2) $\begin{cases} \dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + \dfrac{\mathrm{d}y}{\mathrm{d}t} - x = e^t, \\[2ex] \dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} + \dfrac{\mathrm{d}x}{\mathrm{d}t} + y = 0. \end{cases}$

Structure of Solutions to $\boldsymbol{y}' = \boldsymbol{A}(x)\boldsymbol{y}$

我们首先引入线性相关和线性无关的定义。称向量函数 $\boldsymbol{y}_1(x), \cdots, \boldsymbol{y}_n(x)$ 在 $[a, b]$ 上是线性相关的，若存在不全为 0 的常数 $c_1, \cdots, c_n$ 使得

$$c_1\boldsymbol{y}_1(x) + \cdots + c_n\boldsymbol{y}_n(x) = 0, \ \forall\, x \in [a, b].$$

否则称为线性无关，i.e.,

$$c_1\boldsymbol{y}_1(x) + \cdots + c_n\boldsymbol{y}_n(x) = 0, \ \forall\, x \in [a, b] \Rightarrow c_1 = \cdots = c_n = 0.$$

记 Wronsky 行列式 $W(x) = \det[\boldsymbol{y}_1(x), \cdots, \boldsymbol{y}_n(x)]$.

Try

(1) $\boldsymbol{y}_1(x), \cdots, \boldsymbol{y}_n(x)$ 在 $[a, b]$ 上线性相关 $\Rightarrow W(x) \equiv 0$.

(2) 上述结论的逆定理不成立。

$[-1,1]$ 2

$$f(x)=\begin{cases} x^2, & x<0 \\ 0, & x\geq 0 \end{cases} \qquad g(x)=\begin{cases} 0, & x<0 \\ x^2, & x\geq 0 \end{cases}$$

$$\det\begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{cases} \det\begin{pmatrix} x^2 & 0 \\ 2x & 0 \end{pmatrix}=0 & x<0 \\ \det\begin{pmatrix} 0 & x^2 \\ 0 & 2x \end{pmatrix}=0, & x\geq 0 \end{cases}$$

$$\det\begin{pmatrix} x^{100} & x^2 & x^2 \\ 0 & 1 & x^3 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

$$\cancel{0 \quad 0 \quad 0}$$