

§7.1–§7.4 Ordinary Differential Equations I

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Especially made for zqc

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<http://illusion-hope.github.io/25-Spring-ZQC-Calculus/>

Outline of Chapter 7: Ordinary Differential Equations

初等积分方法:

- 可分离变量的微分方程 \rightsquigarrow 一阶线性微分方程
- 变量代换法: 齐次方程, Bernoulli 方程
- 全微分方程, 积分因子法 \rightsquigarrow (Chapter 9) 多元函数微分学
- 几类可降阶的高阶微分方程

高阶微分方程:

- 齐次线性微分方程解的结构和性质
- 非齐次线性微分方程: 常数变易法
- 常系数齐次线性微分方程 \rightsquigarrow 变系数: Euler 方程
- 两类特殊的常系数非齐次线性微分方程

Separable Differential Equations

假设 $P(x), Q(y) \in C(-\infty, +\infty)$:

(1) $\frac{dy}{dx} = f(x)g(y) \Rightarrow \frac{1}{g(y)}dy = f(x)dx \rightsquigarrow$ 是否存在 $g(y_0) = 0$?

(2) $\int \frac{1}{g(y)}dy = \int f(x)dx + C \rightsquigarrow G(y) = F(x) + C.$

(3) $H(x, y) = G(y) - F(x) - C = 0 \rightsquigarrow H_y(x, y) = 1/g(y) \neq 0$. 由隐映射定理, 上述方程在满足方程的某一个点 (\tilde{x}, \tilde{y}) 附近唯一确定了隐函数 $y = \Phi(x)$.

(4) $f|_{(\tilde{x}, \tilde{y}), \tilde{y} \neq y_0} \neq 0 \Rightarrow y'|_{(\tilde{x}, \tilde{y}), \tilde{y} \neq y_0} = f(\tilde{x})g(\tilde{y}) \neq 0$. 由逆映射定理, 点 (\tilde{x}, \tilde{y}) 附近确定的隐函数 $y = \Phi(x)$ 局部可逆.

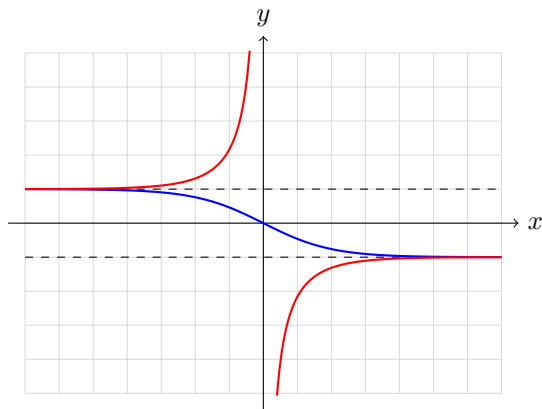
Try

分别求 $\frac{dy}{dx} = \frac{y^2 - 1}{2}$ 的通过 $(0, 0)$, $(\ln 2, 3)$ 的解的存在区间。

An Example

Try

分别求 $\frac{dy}{dx} = \frac{y^2 - 1}{2}$ 的通过 $(0, 0)$, $(\ln 2, 3)$ 的解的存在区间。



- 特解: $y = \pm 1$;
- $(0, 0) : y = \frac{1 - e^x}{1 + e^x}$;
- $(\ln 2, 3) : y = \frac{1 + e^x}{1 - e^x}$;
- 存在唯一性定理: 无限接近 $y = \pm 1$ 但不相交;
- 延拓定理: 解要么延拓到 $[x_0, +\infty)$ 要么延拓到 $[x_0, d)$.

First-Order Linear Differential Equations

假设 $P(x), Q(x) \in C(-\infty, +\infty)$, $\frac{dy}{dx} = P(x)y + Q(x)$, $y(x_0) = y_0$.

$$(1) \quad \frac{dy}{dx} = P(x)y \Rightarrow y = y_0 \exp \left\{ \int_{x_0}^x P(s) ds \right\}.$$

$$(2) \quad \text{Let } y = c(x) \exp \left\{ \int_{x_0}^x P(s) ds \right\} \rightsquigarrow c'(x) = Q(x) \exp \left\{ \int_x^{x_0} P(s) ds \right\}.$$

$$(3) \quad \int_{x_0}^x c'(t) dt = c(x) - c(x_0) = c(x) - y_0 = \int_{x_0}^x Q(t) \exp \left\{ \int_t^{x_0} P(s) ds \right\} dt.$$

$$\rightsquigarrow (\textbf{Duhamel}) \quad y(x) = y_0 \exp \left\{ \int_{x_0}^x P(s) ds \right\} + \int_{x_0}^x Q(t) \exp \left\{ \int_t^x P(s) ds \right\} dt.$$

Duhamel's Principle

$$\leadsto (\textbf{Duhamel}) \quad y(x) = y_0 \exp \left\{ \int_{x_0}^x P(s) ds \right\} + \int_{x_0}^x Q(t) \exp \left\{ \int_t^x P(s) ds \right\} dt.$$

$$\begin{cases} \frac{dy}{dx} = P(x)y + Q(x) \\ y(x_0) = y_0 \end{cases} \Leftrightarrow \begin{cases} \frac{dy}{dx} = P(x)y \\ y(x_0) = y_0 \end{cases} + \begin{cases} \frac{dy}{dx} = P(x)y + Q(x) \\ y(x_0) = 0 \end{cases}$$

Observation: 非齐次方程的解可以由齐次方程的解导出!

Is The Method of Variation of Parameters Correct?

Assume $y(x) = u(x)v(x)$, then $y'(x) = u'(x)v(x) + u(x)v'(x)$.

$$\rightsquigarrow u'(x)v(x) + u(x)v'(x) = P(x)u(x)v(x) + Q(x).$$

$$\rightsquigarrow [u'(x)v(x) - Q(x)] + u(x)[v'(x) - P(x)v(x)] = 0.$$

$$\rightsquigarrow \text{We set } \begin{cases} v'(x) - P(x)v(x) = 0, \\ u'(x)v(x) - Q(x) = 0. \end{cases}$$

Integrating Factor \rightsquigarrow Chapter 9

$$(1) \quad y' - y = Q(x) \rightsquigarrow e^{-x}(y' - y) = e^{-x}Q(x) = [ye^{-x}]'.$$

$$\rightsquigarrow e^{-x}dy - e^{-x}[y + Q(x)]dx = 0.$$

$$\frac{\partial(e^{-x})}{\partial x} = -\frac{\partial\{e^{-x}[y + Q(x)]\}}{\partial y} = -e^{-x}.$$

$$(2) \quad y' - P(x)y = Q(x) \rightsquigarrow \text{Let } \mu(x) = \exp \left\{ \int_x^{x_0} P(x)dx \right\}.$$

$$\rightsquigarrow \mu(x)dy - \mu(x)[P(x)y + Q(x)]dx = 0.$$

$$\frac{\partial\mu(x)}{\partial x} = -\frac{\partial\{\mu(x)[P(x)y + Q(x)]\}}{\partial y} = -\mu(x)P(x).$$

例 1

$$(1) \quad \frac{dy}{dx} = -\frac{1+y^2}{x - \arctan y};$$

$$(2) \quad \frac{dy}{dx} = (x+1)^2 + (4y+1)^2 + 8xy + 1.$$

例 2

求微分方程 $(x - 2y)dx + xdy = 0$ 的一个解 $y = y(x)$, 使得曲线 $y = y(x)$ 与直线 $x = 1, x = 2$ 以及 x 轴所围成的平面图形绕 x 轴旋转而成的旋转体体积最小。

Example

例 3

设可微函数 $f(x)$ 满足方程 $xy' - (2x^2 + 1)y = x^2$ ($x \geq 1$)。试确定 $f(x)$ 在点 $x = 1$ 处的值使得极限 $\lim_{x \rightarrow +\infty} f(x)$ 存在, 并求出此极限值。

$$\text{Recall: (Gauss)} \quad \Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-1/2} e^{-t} dt = 2 \int_0^{+\infty} e^{-t^2} dt = \sqrt{\pi}.$$

Homogeneous Differential Equations

$$(1) \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Let $y = xu$, we have $\frac{dy}{dx} = u + x \frac{du}{dx}$.

$$(2) \frac{dy}{dx} = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right), \mathbf{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

- $r(\mathbf{A}) = 1 = r(\mathbf{A}, \boldsymbol{\beta})$, dy/dx is a constant \rightsquigarrow a line.
- $r(\mathbf{A}) = 1 \neq r(\mathbf{A}, \boldsymbol{\beta})$, Let $t = a_1x + b_1y = \lambda(a_2x + b_2y) \rightsquigarrow \frac{dt}{dx} = a_1 + b_1 \frac{dy}{dx}$.
- $r(\mathbf{A}) = 2$, then $\mathbf{A}\mathbf{X} = \boldsymbol{\beta}$ has a nonzero solution $\mathbf{X}_0 = [m, n]'$, which implies

$$\frac{dy}{dx} = \frac{d(y+m)}{d(x+m)} = f\left[\frac{a_1(x+m) + b_1(y+n)}{a_2(x+m) + b_2(y+n)}\right].$$

Bernoulli Differential Equations

$$\frac{dy}{dx} = P(x)y + Q(x)y^n, n \in \mathbf{N}^*, n \neq 0, 1.$$

- $\frac{1}{y^n} \frac{dy}{dx} = \frac{1}{y^{n-1}} P(x) + Q(x).$
- Let $t = 1/y^{n-1} = y^{1-n} \rightsquigarrow \frac{dt}{dx} = y' \frac{1-n}{y^n}$
- $\rightsquigarrow \frac{1}{1-n} \cdot \frac{dt}{dx} = P(x)t + Q(x).$
- Note: Of course, the Bernoulli differential equation has a **particular solution** $y = 0$.

More Examples

(1) (Riccati) $\frac{dy}{dx} = P(x)y + Q(x)y^2 + R(x) \rightsquigarrow$ **Attention is all you need!**

\rightsquigarrow If you find one solution \tilde{y} , then let $y = z + \tilde{y} \rightsquigarrow$ Bernoulli.

(2) $\frac{xdy}{ydx} = f(xy) \rightsquigarrow u = xy, \frac{du}{dx} = y + x \frac{dy}{dx}$

(3) $\frac{dy}{dx} = f(ax + by + c) \rightsquigarrow u = ax + by + c, \frac{du}{dx} = a + b \frac{dy}{dx}$

(4) $\frac{x^2 dy}{dx} = f(xy)$

(5) $\frac{dy}{dx} = xf\left(\frac{y}{x^2}\right)$

(6) ...

例 4

$$(1) \quad xy' = \sqrt{x^2 - y^2} + y;$$

$$(2) \quad y(1 + x^2y^2) dx = x dy;$$

$$(3) \quad \frac{dy}{dx} = \frac{y^6 - 2x^2}{2xy^5 + x^2y^2};$$

$$(4) \quad y' + \frac{y}{x} = y^2 - \frac{4}{x^2}.$$

Examples

例 5

(Gronwall) 设 $f(t), g(t), x(t) \in C[t_0, t_1]$ 且非负, 求证: 若 $x(t) \leq g(t) + \int_{t_0}^t f(\tau)x(\tau) d\tau, t_0 \leq t \leq t_1$, 则

$$x(t) \leq g(t) + \int_{t_0}^t f(\tau)g(\tau) \exp \left\{ \int_{\tau}^t f(s) ds \right\} d\tau, \quad t_0 \leq t \leq t_1.$$

例 6

设可微函数 $y = f(x)$ 对于任意 $x, h \in (-\infty, +\infty)$, 恒满足关系式:

$$f(x+h) = \frac{f(x) + f(h)}{1 + f(x)f(h)}.$$

已知 $f'(0) = 1$, 试求 $f(x)$.