§7.1—§7.4 Ordinary Differential Equations I

illusion

Especially made for zqc

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http://illusion-hope.github.io/25-Spring-ZQC-Calculus/

Outline of Chapter 7: Ordinary Differential Equations

初等积分方法：

- 可分离变量的微分方程 $\rightsquigarrow$ 一阶线性微分方程
- 变量代换法：齐次方程，Bernoulli 方程
- 全微分方程，积分因子法 $\rightsquigarrow$ (Chapter 9) 多元函数微分学
- 几类可降阶的高阶微分方程

高阶微分方程：

- 齐次线性微分方程解的结构和性质
- 非齐次线性微分方程：常数变易法
- 常系数齐次线性微分方程 $\rightsquigarrow$ 变系数：Euler 方程
- 两类特殊的常系数非齐次线性微分方程

Separable Differential Equations

假设 $P(x), Q(y) \in C(-\infty, +\infty)$:

(1) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = P(x)y + Q(x) \Rightarrow \dfrac{1}{g(y)}\mathrm{d}y = f(x)\mathrm{d}x \rightsquigarrow$ 是否存在 $g(y_0) = 0$?

(2) $\displaystyle\int \frac{1}{g(y)}\mathrm{d}y = \int f(x)\mathrm{d}x + C \rightsquigarrow G(y) = F(x) + C.$

(3) $H(x, y) = G(y) - F(x) - C = 0 \rightsquigarrow H_y(x, y) = 1/g(y) \neq 0$. 由隐映射定理，上述方程在满足方程的某一个点 $(\tilde{x}, \tilde{y})$ 附近唯一确定了隐函数 $y = \Phi(x)$.

(4) $f|_{(\tilde{x},\tilde{y}),\tilde{y}\neq y_0} \neq 0 \Rightarrow y'|_{(\tilde{x},\tilde{y}),\tilde{y}\neq y_0} = f(\tilde{x})g(\tilde{y}) \neq 0$. 由逆映射定理，点 $(\tilde{x}, \tilde{y})$ 附近确定的隐函数 $y = \Phi(x)$ 局部可逆.

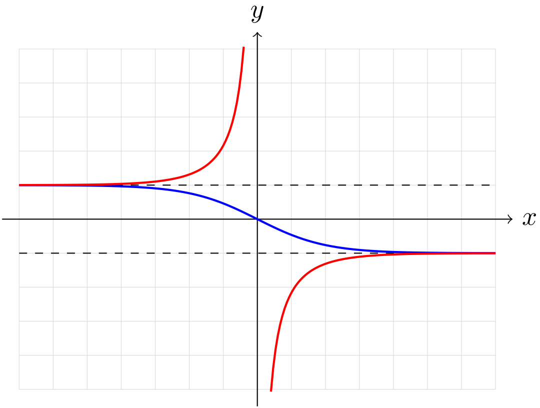
Try

分别求 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y^2 - 1}{2}$ 的通过 $(0, 0)$, $(\ln 2, 3)$ 的解的存在区间。

An Example

Try

分别求 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y^2-1}{2}$ 的通过 $(0,0)$, $(\ln 2, 3)$ 的解的存在区间。



- 特解：$y = \pm 1$;
- $(0,0) : y = \dfrac{1-e^x}{1+e^x}$;
- $(\ln 2, 3) : y = \dfrac{1+e^x}{1-e^x}$;
- 存在唯一性定理：无限接近 $y = \pm 1$ 但不相交;
- 延拓定理：解要么延拓到 $[x_0, +\infty)$ 要么延拓到 $[x_0, d)$.

First-Order Linear Differential Equations

假设 $P(x), Q(x) \in C(-\infty, +\infty)$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = P(x)y + Q(x)$, $y(x_0) = y_0$.

(1) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = P(x)y \Rightarrow y = y_0 \exp\left\{\int_{x_0}^{x} P(s)\mathrm{d}s\right\}$.

(2) Let $y = c(x) \exp\left\{\int_{x_0}^{x} P(s)\mathrm{d}s\right\} \rightsquigarrow c'(x) = Q(x) \exp\left\{\int_{x}^{x_0} P(s)\mathrm{d}s\right\}$.

(3) $\int_{x_0}^{x} c'(t)\mathrm{d}t = c(x) - c(x_0) = c(x) - y_0 = \int_{x_0}^{x} Q(t) \exp\left\{\int_{t}^{x_0} P(s)\mathrm{d}s\right\}\mathrm{d}t.$

$\rightsquigarrow$ (**Duhamel**) $y(x) = y_0 \exp\left\{\int_{x_0}^{x} P(s)\mathrm{d}s\right\} + \int_{x_0}^{x} Q(t) \exp\left\{\int_{t}^{x} P(s)\mathrm{d}s\right\}\mathrm{d}t.$

Duhamel's Principle

$\rightsquigarrow$ **(Duhamel)** $y(x) = y_0 \exp\left\{\int_{x_0}^{x} P(s)\mathrm{d}s\right\} + \int_{x_0}^{x} Q(t) \exp\left\{\int_{t}^{x} P(s)\mathrm{d}s\right\}\mathrm{d}t.$

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = P(x)y + Q(x) \\ y(x_0) = y_0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = P(x)y \\ y(x_0) = y_0 \end{cases} + \begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = P(x)y + Q(x) \\ y(x_0) = 0 \end{cases}$$

Observation: 非齐次方程的解可以由齐次方程的解导出!

Is The Method of Variation of Parameters Correct?

Assume $y(x) = u(x)v(x)$, then $y'(x) = u'(x)v(x) + u(x)v'(x)$.

$\rightsquigarrow u'(x)v(x) + u(x)v'(x) = P(x)u(x)v(x) + Q(x)$.

$\rightsquigarrow [u'(x)v(x) - Q(x)] + u(x)[v'(x) - P(x)v(x)] = 0$.

$\rightsquigarrow$ We set $\begin{cases} v'(x) - P(x)v(x) = 0, \\ u'(x)v(x) - Q(x) = 0. \end{cases}$

(1) $y' - y = Q(x) \rightsquigarrow e^{-x}(y' - y) = e^{-x}Q(x) = [ye^{-x}]'$.

$\rightsquigarrow e^{-x}\mathrm{d}y - e^{-x}[y + Q(x)]\mathrm{d}x = 0.$

$$\frac{\partial(e^{-x})}{\partial x} = -\frac{\partial\{e^{-x}[y + Q(x)]\}}{\partial y} = -e^{-x}.$$

(2) $y' - P(x)y = Q(x) \rightsquigarrow$ Let $\mu(x) = \exp\left\{\int_x^{x_0} P(x)\mathrm{d}x\right\}$.

$\rightsquigarrow \mu(x)\mathrm{d}y - \mu(x)[P(x)y + Q(x)]\mathrm{d}x = 0.$

$$\frac{\partial\mu(x)}{\partial x} = -\frac{\partial\{\mu(x)[P(x)y + Q(x)]\}}{\partial y} = -\mu(x)P(x).$$

Try

例 1

(1) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-1+y^2}{x-\arctan y}$;

(2) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = (x+1)^2 + (4y+1)^2 + 8xy + 1$.

例 2

求微分方程 $(x-2y)\mathrm{d}x + x\mathrm{d}y = 0$ 的一个解 $y=y(x)$，使得曲线 $y=y(x)$ 与直线 $x=1, x=2$ 以及 x 轴所围成的平面图形绕 x 轴旋转而成的旋转体体积最小。

Example

例 3

设可微函数 $f(x)$ 满足方程 $xy' - (2x^2+1)y = x^2\ (x \geq 1)$。试确定 $f(x)$ 在点 $x=1$ 处的值使得极限 $\lim\limits_{x\to+\infty} f(x)$ 存在，并求出此极限值。

Recall: (Gauss) $\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-1/2}e^{-t}\mathrm{d}t = 2\int_0^{+\infty} e^{-t^2}\mathrm{d}t = \sqrt{\pi}$.

Homogeneous Differential Equations

(1) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f\left(\dfrac{y}{x}\right)$

Let $y = xu$, we have $\dfrac{\mathrm{d}y}{\mathrm{d}x} = u + x\,\dfrac{\mathrm{d}u}{\mathrm{d}x}$.

(2) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f\left(\dfrac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right)$, $\boldsymbol{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $\boldsymbol{\beta} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

- $r(\boldsymbol{A}) = 1 = r(\boldsymbol{A}, \boldsymbol{\beta})$, $\mathrm{d}y/\mathrm{d}x$ is a constant $\rightsquigarrow$ a line.
- $r(\boldsymbol{A}) = 1 \neq r(\boldsymbol{A}, \boldsymbol{\beta})$, Let $t = a_1x + b_1y = \lambda(a_2x + b_2y) \rightsquigarrow \dfrac{\mathrm{d}t}{\mathrm{d}x} = a_1 + b_1\,\dfrac{\mathrm{d}y}{\mathrm{d}x}$.
- $r(\boldsymbol{A}) = 2$, then $\boldsymbol{AX} = \boldsymbol{\beta}$ has a nonzero solution $\boldsymbol{X}_0 = [m, n]'$, which implies

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}(y+m)}{\mathrm{d}(x+m)} = f\left[\frac{a_1(x+m) + b_1(y+n)}{a_2(x+m) + b_2(y+n)}\right].$$

Bernoulli Differential Equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = P(x)y + Q(x)y^n, n \in \mathbf{N}^*, n \neq 1, 2.$$

- $\displaystyle \frac{1}{y^n}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y^{n-1}}P(x) + Q(x).$
- Let $t = 1/y^{n-1} = y^{1-n} \rightsquigarrow \displaystyle \frac{\mathrm{d}t}{\mathrm{d}x} = y'\frac{1-n}{y^n}$
- $\rightsquigarrow \displaystyle \frac{1}{1-n} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} = P(x)t + Q(x).$
- Note: Of course, the Bernoulli differential equation has a **particular solution** $y = 0$.

More Examples

(1) (Riccati) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = P(x)y + Q(x)y^2 + R(x) \rightsquigarrow$ Attention is all you need!

$\rightsquigarrow$ If you find one solution $\tilde{y}$, then let $y = z + \tilde{y} \rightsquigarrow$ Bernoulli.

(2) $\dfrac{x\mathrm{d}y}{y\mathrm{d}x} = f(xy) \rightsquigarrow u = xy,\ \dfrac{\mathrm{d}u}{\mathrm{d}x} = y + x\dfrac{\mathrm{d}y}{\mathrm{d}x}$

(3) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(ax + by + c) \rightsquigarrow u = ax + by + c,\ \dfrac{\mathrm{d}u}{\mathrm{d}x} = a + b\dfrac{\mathrm{d}y}{\mathrm{d}x}$

(4) $\dfrac{x^2\mathrm{d}y}{\mathrm{d}x} = f(xy)$

(5) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = xf\left(\dfrac{y}{x^2}\right)$

(6) $\cdots$

Try

例 4

(1) $xy' = \sqrt{x^2 - y^2} + y$;

(2) $y(1 + x^2y^2)\,\mathrm{d}x = x\,\mathrm{d}y$;

(3) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y^6 - 2x^2}{2xy^5 + x^2y^2}$;

(4) $y' + \dfrac{y}{x} = y^2 - \dfrac{4}{x^2}$.

Examples

例 5

(Gronwall) 设 $f(t), g(t), x(t) \in C[t_0, t_1]$ 且非负，求证：若 $x(t) \leq g(t) + \int_{t_0}^{t} f(\tau)x(\tau)\,\mathrm{d}\tau$, $t_0 \leq t \leq t_1$，则

$$x(t) \leq g(t) + \int_{t_0}^{t} f(\tau)g(\tau)\exp\left\{\int_{\tau}^{t} f(s)\,\mathrm{d}s\right\}\mathrm{d}\tau, \quad t_0 \leq t \leq t_1.$$

例 6

设可微函数 $y = f(x)$ 对于任意 $x, h \in (-\infty, +\infty)$，恒满足关系式：

$$f(x+h) = \frac{f(x) + f(h)}{1 + f(x)f(h)}.$$

已知 $f'(0) = 1$，试求 $f(x)$.