§7.5, 7.6, 7.10 Ordinary Differential Equations II

illusion

Especially made for zqc

School of Mathematical Science, XMU

Wednesday 26th February, 2025

http://illusion-hope.github.io/25-Spring-ZQC-Calculus/

HW-1

例 1

(1)
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^6 - 2x^2}{2xy^5 + x^2y^2};$$

(1') (Similar!)
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x^3 + 3xy^2 + x}{3x^2y + 2y^3 - y};$$

(2)
$$y' = y^2 + 2(\sin x - 1)y + \sin^2 x - 2\sin x - \cos x + 1$$
.

例 2

已知微分方程 $y'+y=f(x),\,f(x)\in C(-\infty,+\infty).$ 若 f(x) 是以周期为 T 的函数,证明:方程存在唯一以 T 为周期的解。



(1') (Similar!)
$$\frac{dy}{dx} = \frac{2x^3 + 3xy^2 + x}{3x^2y + 2y^3 - y};$$

$$\frac{dy^2}{dx^2} = \frac{y}{x} \frac{dy}{dx} = \frac{2x^2 + 3y^2 + 1}{3x^2 + 2y^2 - 1}$$

$$\frac{du}{dx^2} = \frac{y}{x} \frac{dy}{dx} = \frac{2x^2 + 3y^2 + 1}{3x^2 + 2y^2 - 1}$$

$$\frac{du}{dx} = \frac{y}{y} \frac{dy}{dx} = \frac{2(y - 1) + 3(u + 1)}{3(u + 1) + 3(u + 1)} = \frac{d(u + 1)}{d(u + 1)}$$

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$$\frac{du}{dx} = \frac{du}{dx} = \frac{du}{dx}$$

$$\frac{d\omega}{d\hat{v}} \cdot \hat{v} = \frac{1}{3} + 2\omega^{2}$$

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$$\frac{1}{3}$$

$$(1) \frac{dy}{dx} = \frac{y^6 - 2x^2}{2xy^5 + x^2y^2};$$

$$\frac{y^2}{dx} = \frac{y^6 - 2x^2}{2xy^5 + x^2y^2};$$

$$\frac{y^3}{dx} = \frac{y^6 - 2x^2}{x^3}$$

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$$\frac{y^6}{x^5} = \frac{y^6}{x^5}$$

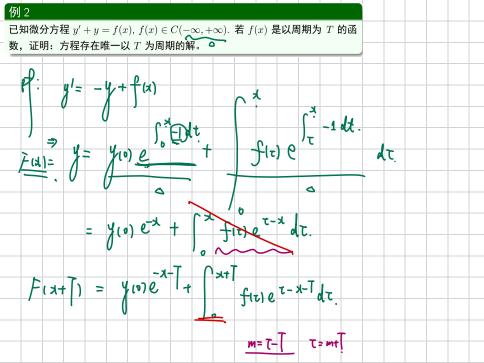
$$\frac{y^6}{x^5} = \frac{y^6}{x^$$

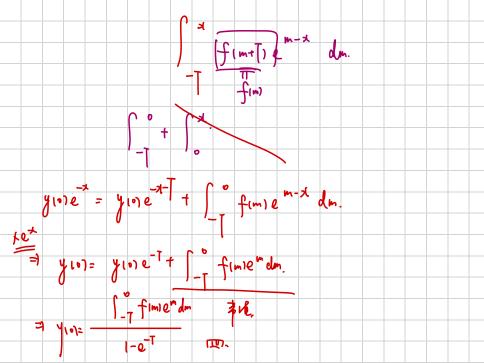
$$\frac{1}{1} \frac{du}{dx} = \frac{1}{1} \frac{1}{1}$$

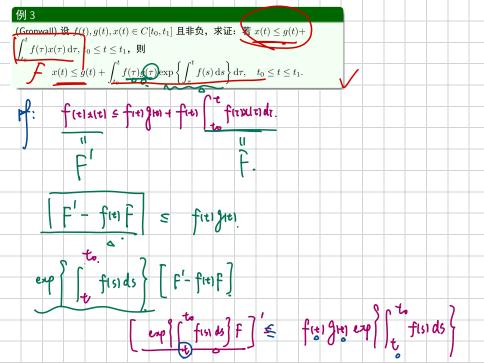
$$| \frac{1}{1+k(2)} - \frac{$$

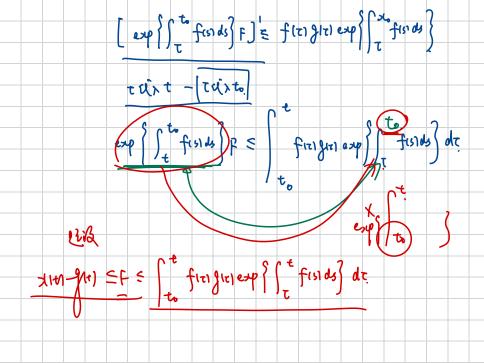
(2) $y' = y^2 + 2(\sin x - 1)y + \sin^2 x - 2\sin x - \cos x + 1$.

$$\frac{8^{2}t}{1} + \frac{28-2\sin 4}{2} = \frac{2\sin 4}{2} + \frac{2\sin 4}{2} + \frac{2\sin 4}{2} + \frac{2\sin 4}{2} = \frac{2\sin 4}{2} + \frac{2\sin 4}{2} + \frac{2\sin 4}{2} = \frac{2\sin 4}{2} + \frac{2\sin 4}{2} = \frac{2\sin 4}{2} + \frac{2\sin 4}{2} = \frac{2\sin 4}{2}$$









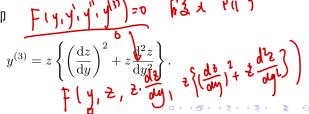
Differential Equations That Can Be Reduced In Order

- (1) $y^{(n)} = f(x)$;
- (2) y'' = f(x, y'):
- (3) 自治/驻定方程 $F(y,y',\cdots,y^{(n)})=0.$

前两类是基本的,我们只看第三类,考虑 $rac{\mathrm{d} y}{rac{\mathrm{d} x}{2}}=z$,那么

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y} \frac{\mathrm{d}y}{\mathrm{d}x} = z \frac{\mathrm{d}z}{\mathrm{d}y} \rightsquigarrow y^{(3)} = \frac{\mathrm{d}}{\mathrm{d}x} \left\{ z \frac{\mathrm{d}z}{\mathrm{d}y} \right\} = z \frac{\mathrm{d}}{\mathrm{d}y} \left\{ z \frac{\mathrm{d}z}{\mathrm{d}y} \right\}.$$
用微分的运算性质,即

利用微分的运算性质,即



Examples

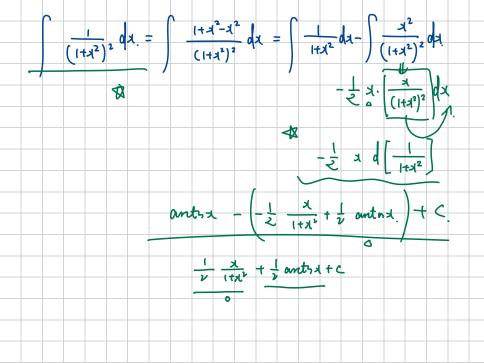
$$\frac{1}{(x^2+1)^2} = \frac{A}{|x-1|^2} + \frac{B}{|x-1|^2} + \frac{Cx+0}{|x^2+1|^2} + \frac{Ex+F}{|x^2+1|^2} + \frac{Gx+H}{|x^2+1|^2}$$

例 5

(1)
$$y'' = \frac{1}{(1+x^2)^2}$$
; $y'' = \frac{1}{dx} \cdot y' = \frac{dy}{dx} y' = 2 \cdot \frac{dz}{dy}$

(3) (22-23 Midterm) $yy'' + (y')^2 - 2yy' = 0$.

Note: 除了
$$y'=z$$
,你能给出类似积分因子法的思路吗? 注意 $(yy')'=?$
$$y\cdot z \frac{ds}{dy} + 2z^2 = 0 \text{ D} \frac{z}{z} = 0 \text{ D} \frac{$$



(2) $yy'' + 2(y')^2 = 0$;

From High-Order L.D.E. to Systems of L.D.E.

给定一个高阶线性微分方程

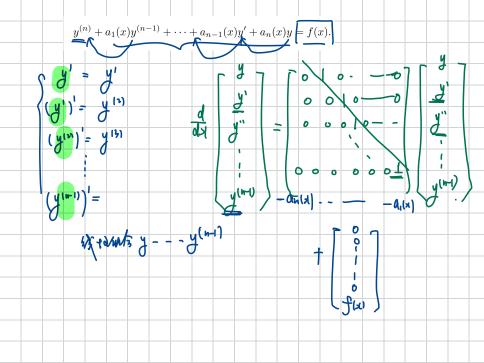
$$y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_{n-1}(x)y' + a_n(x)y = f(x).$$
 (2)

令
$$y_1 = y, y_2 = y', \dots, y_n = y^{(n-1)}$$
 那么有

$$\frac{d}{dx} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -a_n(x) & -a_{n-1}(x) & \cdots & -a_1(x) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f(x) \end{bmatrix}.$$
(3)

$$\rightsquigarrow \boldsymbol{y}' = \boldsymbol{A}(x)\boldsymbol{y} + \boldsymbol{f}(x).$$





From High-Order L.D.E. to Systems of L.D.E.

存在唯一性定理: 对线性微分方程组

$$y' = A(x)y + f(x), \ y(x_0) = y_0, \ x_0 \in (a, b)$$
 (4)

若 $A(x), f(x) \in C[a, b]$, 方程 (2) 满足初始条件 $y(x_0) = y_0$ 的解 y = y(x) 在 [a,b] 上存在且唯一。

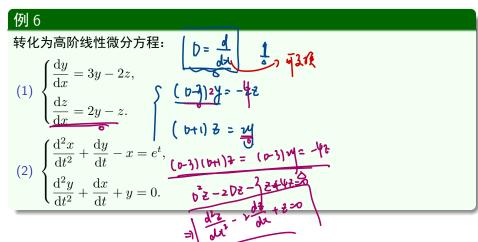
→ 对高阶线性微分方程

$$y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_{n-1}(x)y' + a_n(x)y = f(x).$$
 (5)

若 $a_i(x)$ \in C[a,b]. 则方程 (5) 在给定初值条件 $y(x_0)=y_0,y'(x_0)=y_0^{(1)},\cdots,y^{(n-1)}(x_0)=y_0^{(n-1)}$ 下存在唯一解 $y=y(x),\ x\in[a,b]$.

Examples

在求解某些线性微分方程组时,也可以将其化为高阶线性微分方程来处理。



Structure of Solutions to y' = A(x)y

我们首先引入线性相关和线性无关的定义。称向量函数 $m{y}_1(x),\cdots,m{y}_n(x)$ 在 [a,b] 上是线性相关的,若存在不全为 0 的常数 c_1,\cdots,c_n 使得

$$c_1 \mathbf{y}_1(x) + \cdots + c_n \mathbf{y}_n(x) = 0, \ \forall \ x \in [a, b].$$

否则称为线性无关, i.e.,

$$c_1 \mathbf{y}_1(x) + \dots + c_n \mathbf{y}_n(x) = 0, \ \forall \ x \in [a, b] \Rightarrow c_1 = \dots = c_n = 0.$$

记 Wronsky 行列式 $W(x) = \det[y_1(x), \cdots, y_n(x)]$.

Try

- (1) $y_1(x), \dots, y_n(x)$ 在 [a,b] 上线性相关 $\Rightarrow W(x) \equiv 0$.
- (2) 上述结论的逆定理不成立。

