§7.1—§7.4 Ordinary Differential Equations I

illusion

Especially made for zqc

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http://illusion-hope.github.io/25-Spring-ZQC-Calculus/

Separable Differential Equations

假设 $P(x), Q(y) \in C(-\infty, +\infty)$:

(1) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y) \Rightarrow \dfrac{1}{g(y)}\mathrm{d}y = f(x)\mathrm{d}x \rightsquigarrow$ 是否存在 $g(y_0) = 0$?

(2) $\displaystyle\int \frac{1}{g(y)}\mathrm{d}y = \int f(x)\mathrm{d}x + C \rightsquigarrow G(y) = F(x) + C.$

(3) $H(x, y) = G(y) - F(x) - C = 0 \rightsquigarrow H_y(x, y) = 1/g(y) \neq 0.$ 由隐映射定理，上述方程在满足方程的某一个点 $(\tilde{x}, \tilde{y})$ 附近唯一确定了隐函数 $y = \Phi(x)$.

(4) $f|_{(\tilde{x},\tilde{y}),\tilde{y}\neq y_0} \neq 0 \Rightarrow y'|_{(\tilde{x},\tilde{y}),\tilde{y}\neq y_0} = f(\tilde{x})g(\tilde{y}) \neq 0.$ 由逆映射定理，点 $(\tilde{x}, \tilde{y})$ 附近确定的隐函数 $y = \Phi(x)$ 局部可逆.

Try

分别求 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y^2 - 1}{2}$ 的通过 $(0, 0)$, $(\ln 2, 3)$ 的解的存在区间。

$$\sum \frac{1}{n^2-1} = \sum \frac{1}{(n-1)(n+1)} = \frac{1}{2}\left[\frac{1}{n-1} - \frac{1}{n+1}\right]$$

$$\int \frac{2}{y^2-1}\,dy = \int dx = \int \left(\frac{1}{y-1} - \frac{1}{y+1}\right) dy$$

$$x + c = \ln\left|\frac{y-1}{y+1}\right|$$

$$\frac{y-1}{y+1} = \pm e^{x+c} = \boxed{\pm e^{c}}\, e^{x}$$

$$= C \cdot e^{x}, \ \forall C \in \mathbb{R}.$$

$$\Rightarrow \ y-1 = (Ce^{x})(y+1) \rightarrow (1-ce^{x})y = ce^{x}+1$$

$$\Rightarrow y = \frac{1+ce^{x}}{1-ce^{x}}.$$

$$\frac{dy}{dx} = P(x)y + Q(x)$$

$$\frac{dy}{dx} = P(x)y, \quad Q(x)=0 \qquad \text{Duhamel}$$

$$, \; y(x_0) = y_0$$

$\boxed{y=0}$

$y \neq 0$ $\Rightarrow$ $\frac{1}{y}dy = P(x)\,dx \Rightarrow \int_{y_0}^{y} \frac{1}{y}dy = \int_{x_0}^{x} P(x)dx$

$$\Rightarrow \ln\left|\frac{y}{y_0}\right| = \int_{x_0}^{x} P(x)dx$$

$$\Rightarrow \frac{y}{y_0} = \pm \exp\left\{\int_{x_0}^{x} P(x)dx\right\}$$

$$\Rightarrow y = y_0 \exp\left\{\int_{x_0}^{x} P(x)dx\right\}$$

$y(x_0)=y_0$

$$\frac{dy}{dx} = P(x)y + Q(x)$$

$$y = u(x)\,v(x)$$

$$\frac{d(uv)}{dx} = u'v + uv' = P(x)uv + Q(x)$$

$$= Puv + Q$$

$$\Rightarrow u(v' - Pv) + (u'v - Q) = 0$$

$$\begin{cases} v' = Pv \Rightarrow v \checkmark \\ u' = \frac{Q}{v} \Rightarrow u \checkmark \end{cases}$$

$$y = c(x)\exp\left\{\int_{x_0}^{x} p(s)\,ds\right\}$$

$$\frac{dy}{dx} = c'(x)\exp\left\{\int_{x_0}^{x} p(s)\,ds\right\} + \cancel{c(x)\,p(x)\cdot\exp\left\{\int_{x_0}^{x} p(s)\,ds\right\}}$$

$$= \cancel{P(x)\,c(x)\exp\left\{\quad\right\}} + Q(x)$$

$$c'(x) = Q(x)\exp\left\{\int_{x}^{x_0} p(s)\,ds\right\}$$

$$\int_{x_0}^{x} c'(\tau)\,d\tau = \int_{x_0}^{x} Q(\tau)\exp\left\{\int_{\tau}^{x_0} p(s)\,ds\right\}d\tau.$$

$$\Rightarrow c(x) = c(x_0) + \int_{x_0}^{x} Q(\tau)\exp\left\{\int_{\tau}^{x_0} p(s)\,ds\right\}d\tau.$$

$$\Rightarrow y(x) = c(x)\cdot \exp\left\{\int_{x_0}^{x} p(s)\,ds\right\}$$

$$\begin{cases} \dfrac{dy}{dx} = P(x)y, \quad x > \tau \\ y(\tau) = Q(\tau) \end{cases}$$

$$= \underbrace{\cancel{c(x_0)}}_{y_0} \exp\left\{\int_{x_0}^{x} p(s)\,ds\right\} + \int_{x_0}^{x} Q(\tau)\exp\left\{\int_{\tau}^{x} p(s)\,ds\right\} d\tau.$$

$$y(x_0) = y_0 = c(x_0)$$

$\Downarrow$

齐次

非齐次

$$\begin{cases} \dfrac{dy}{dx} = p(x)y \\ y(x_0) = y_0. \end{cases} + \begin{cases} \dfrac{dy}{dx} = p(x)y + Q(x) \\ y(x_0) = 0 \end{cases}$$

(1) $y' - y = Q(x) \rightsquigarrow e^{-x}(y' - y) = e^{-x}Q(x) = [ye^{-x}]'$.

$\rightsquigarrow e^{-x}\mathrm{d}y - e^{-x}[y + Q(x)]\mathrm{d}x = 0.$

$$\frac{\partial(e^{-x})}{\partial x} = -\frac{\partial\{e^{-x}[y + Q(x)]\}}{\partial y} = -e^{-x}.$$

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ $\boxed{g' = P(x)g} \rightsquigarrow \exp\left\{\int_{x_0}^{x} P(s)\mathrm{d}s\right\}$

(2) $y' - P(x)y = Q(x) \rightsquigarrow$ Let $\mu(x) = \exp\left\{\int_{x}^{x_0} P(x)\mathrm{d}x\right\}$.

$\rightsquigarrow \mu(x)\mathrm{d}y - \mu(x)[P(x)y + Q(x)]\mathrm{d}x = 0.$

$$\frac{\partial\mu(x)}{\partial x} = -\frac{\partial\{\mu(x)[P(x)y + Q(x)]\}}{\partial y} = -\mu(x)P(x).$$

$$\left[y' - P(x) y \right] \underline{\underline{\exp\left\{ - \int_{x_0}^{x} P(s)\,ds \right\}}}$$

$$= \left[y \exp\left\{ - \int_{x_0}^{x \to \tau} P(s)\,ds \right\} \right]' = Q(x \to \tau) \exp\left\{ - \int_{x_0}^{x \to \tau} P(s)\,ds \right\}$$

例 1

(1) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1+y^2}{x-\arctan y}$;

(2) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = (x+1)^2 + (4y+1)^2 + 8xy + 1$.

$$\frac{dy}{dx} = P(x)y + Q(x)$$

$$e^{\int p(x)dx}\left[C + \int Q(x) e^{-\int p(x)dx} dx \right]$$

$$\frac{dx}{dy} = \frac{-x + \arctan y}{1+y^2}$$

$$= -\frac{1}{1+y^2}x + \frac{\arctan y}{1+y^2}$$

$$\Rightarrow x = e^{\int -\frac{1}{1+y^2}dy}\left[\int \frac{\arctan y}{1+y^2} \cdot e^{\int \frac{1}{1+y^2}dy} dy + C \right]$$

$$e^{-\arctan y}$$

$$\int \frac{\arctan y}{1+y^2} e^{\arctan y} dy$$

令 $t = \arctan y \Rightarrow \int t e^t dt$

$$= e^t(t-1)$$

例 1

(1) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{1+y^2}{x-\arctan y}$;

(2) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=(x+1)^2+(4y+1)^2+8xy+1$.

$$\frac{dy}{dx}=x^2+2x+1+16y^2+8y+1+8xy+1$$

$$=x^2+2x+16y^2+8y+8xy\ \underline{\underline{+3}}$$

$$=(\underbrace{x+4y+1}_{t})^2+2.$$

令 $t=x+4y+1$ $\quad \dfrac{dt}{dx}=1+4\dfrac{dy}{dx}=1+4(t^2+2)$

$$=\underline{4t^2+9}$$

$$\int\frac{1}{4t^2+9}\,dt=\int dx$$

$$\frac{1}{4}\int\frac{1}{t^2+\left(\frac{3}{2}\right)^2}\,dt=\frac{1}{4}\cdot\frac{1}{\frac{3}{2}}\cdot\arctan\frac{t}{\frac{3}{2}}+C=x$$

例 3

设可微函数 $f(x)$ 满足方程 $xy' - (2x^2+1)y = x^2\ (x \geq 1)$。试确定 $f(x)$ 在点 $x=1$ 处的值使得极限 $\lim\limits_{x\to+\infty} f(x)$ 存在，并求出此极限值。

Recall: (Gauss) $\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-1/2}e^{-t}\mathrm{d}t = 2\int_0^{+\infty} e^{-t^2}\mathrm{d}t = \sqrt{\pi}.$

$$y' = \left(2x+\frac{1}{x}\right)y + x$$

$$y = y(1)\cdot e^{\int_1^x \left(2s+\frac{1}{s}\right)ds} + \int_1^x \tau\cdot e^{\int_\tau^x \left(2s+\frac{1}{s}\right)ds}\, d\tau$$

$$= y(1)\cdot e^{x^2+\ln x-1} + \int_1^x \tau\cdot e^{\left(s^2+\ln s-1\right)\big|_\tau^x}\, d\tau$$

$$y(1)\frac{xe^{x^2}}{e} + \int_1^x \tau\cdot\left[e^{x^2+\ln x\cancel{-1}-\tau^2-\ln\tau\cancel{+1}}\right] d\tau$$

$$y(1)\cdot\frac{xe^{x^2}}{e}+xe^{x^2}\cdot\int_1^x \tau\cdot e^{-\tau^2}/\tau\, d\tau$$

$$=\lim_{x\to+\infty} xe^{x^2}\left[\frac{y(1)}{e}+\int_1^x e^{-\tau^2}d\tau\right]=A.$$

$$\lim_{x\to+\infty}\frac{y(1)}{e}+\int_1^x e^{-\tau^2}d\tau=\lim_{x\to+\infty}\underline{\qquad}\cdot\lim_{x\to+\infty}\frac{1}{xe^{x^2}}$$

$$=A\cdot 0=0.$$

$$\frac{y(1)}{e}+\int_1^{+\infty}e^{-\tau^2}d\tau=\frac{y(1)}{e}-\underbrace{\int_0^1 e^{-\tau^2}d\tau}_{0}+\frac{\sqrt{\pi}}{2}=0.$$

$$y(1)=e\left[\int_0^1 e^{-\tau^2}d\tau-\frac{\sqrt{\pi}}{2}\right]$$

$$\lim_{x\to+\infty} \frac{\boxed{\int_0^x e^{-\tau^2}\,d\tau} - \frac{\sqrt{\pi}}{2}}{\frac{1}{x}e^{-x^2}} \quad \rightsquigarrow \quad e^{-x^2}$$

$$\rightsquigarrow -\frac{1}{x^2}e^{-x^2} + \frac{1}{x}\cdot(-2x)\cdot e^{-x^2}$$

$$= \lim_{x\to+\infty} \frac{1}{-\frac{1}{x^2} - 2} = -\frac{1}{2}$$

$$= A.$$

Bernoulli Differential Equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = P(x)y + Q(x)y^n, n \in \mathbf{N}^*, n \neq 1, \overset{0}{\cancel{2}}.$$

- $\dfrac{1}{y^n}\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{y^{n-1}}P(x) + Q(x).$
- Let $t = 1/y^{n-1} = y^{1-n} \rightsquigarrow \dfrac{\mathrm{d}t}{\mathrm{d}x} = y' \dfrac{1-n}{y^n}$
- $\rightsquigarrow \dfrac{1}{1-n} \cdot \dfrac{\mathrm{d}t}{\mathrm{d}x} = P(x)t + Q(x).$
- Note: Of course, the Bernoulli differential equation has a **particular solution** $y = 0$.

More Examples

(1) (Riccati) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = P(x)y + Q(x)y^2 + R(x) \rightsquigarrow$ Attention is all you need!

$\rightsquigarrow$ If you find one solution $\tilde{y}$, then let $y = z + \tilde{y} \rightsquigarrow$ Bernoulli.

(2) $\dfrac{x\mathrm{d}y}{y\mathrm{d}x} = f(xy) \rightsquigarrow u = xy,\ \dfrac{\mathrm{d}u}{\mathrm{d}x} = y + x\dfrac{\mathrm{d}y}{\mathrm{d}x}$

(3) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(ax + by + c) \rightsquigarrow u = ax + by + c,\ \dfrac{\mathrm{d}u}{\mathrm{d}x} = a + b\dfrac{\mathrm{d}y}{\mathrm{d}x}$

(4) $\dfrac{x^2\mathrm{d}y}{\mathrm{d}x} = f(xy)$

(5) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = xf\left(\dfrac{y}{x^2}\right)$

(6) $\cdots$

(1) (Riccati) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = P(x)y + Q(x)y^2 + R(x)$ $\rightsquigarrow$ Attention is all you need!

$\rightsquigarrow$ If you find one solution $\tilde{y}$, then let $y = z + \tilde{y}$ $\rightsquigarrow$ Bernoulli.

$$\frac{dz}{dx} = \frac{dy}{dx} - \frac{d\tilde{y}}{dx}$$

$$= P(x)y + Q(x)y^2 - P(x)\tilde{y} - Q(x)\tilde{y}^2.$$

$$= P(x)(y-\tilde{y}) + Q(x)(y-\tilde{y})(y+\tilde{y})$$

$$= P(x)\,z + Q(x)\,z\cdot(z+2\tilde{y})$$

$$= (P(x)+2\tilde{y}Q(x))z + Q(x)z^2.$$

(4) $y' + \dfrac{y}{x} = y^2 - \dfrac{4}{x^2}.$

[1]

$z - \dfrac{2}{x} = y.$

$$\frac{dz}{dx} = \frac{dy}{dx} - \frac{2}{x^2} = -\frac{y}{x} + y^2 - \frac{4}{x^2} - \frac{2}{x^2}$$

$$= -\frac{z - \frac{2}{x}}{x} + \left(z - \frac{2}{x}\right)^2 - \frac{4}{x^2} - \frac{2}{x^2}$$

$$= -\frac{z}{x} + \cancel{\frac{2}{x^2}} + z^2 - \frac{4}{x}z + \cancel{\frac{4}{x^2}} - \cancel{\frac{4}{x^2}} - \cancel{\frac{2}{x^2}}$$

$$= -\frac{5}{x}z + z^2$$

令 $\dfrac{1}{z} = t.$

$$\frac{1}{z^2}\frac{dz}{dx} = -\frac{5}{x}\cdot\frac{1}{z} + 1$$

$$\frac{dt}{dx} = -\frac{z'}{z^2}$$

$$-\frac{dt}{dx} = -\frac{5}{x}\cdot t + 1$$

$$\Rightarrow \frac{dt}{dx} = \frac{5}{x}t - 1$$

$$\frac{dt}{dx} = \frac{5}{x}t - 1$$

$$t = e^{\int \frac{5}{x} dx} \left[C + \int -1 \cdot e^{-\int \frac{5}{x} dx} dx \right]$$

$$= x^5 \left[C + \int \underset{\Delta}{\boxed{-x^{-5}}} dx \right]$$

$$= x^5 \left(C + \frac{1}{4x^4} \right) = Cx^5 + \frac{x}{4}, \ \forall C \in \mathbb{R}.$$

(4) $y' + \frac{y}{x} = y^2 - \frac{4}{x^2}$.

$$\left(\frac{1}{xy}\right)' = \frac{-(xy)'}{(x^2y^2)}$$

$$\boxed{\frac{\underbrace{xy'+y}_{(xy)'}}{x}} = y^2 - \frac{4}{x^2}$$

两边除 xy^2.

$$-\left(\frac{1}{xy}\right)' = \frac{1}{x} - \frac{4}{x^3y^2} = \frac{1}{x}\left[1 - \frac{4}{(xy)^2}\right]$$

令 $t = \frac{1}{xy}$

$$-\frac{dt}{dx} = \frac{1}{x}\left(1 - \frac{4}{t^2}\right)$$

(2) $\frac{x\mathrm{d}y}{y\mathrm{d}x} = f(xy) \rightsquigarrow u = xy,\ \frac{\mathrm{d}u}{\mathrm{d}x} = y + x\frac{\mathrm{d}y}{\mathrm{d}x}$

$$\frac{du}{dx} = y + x\left(\frac{y}{x} f(u)\right)$$

$$= y\,(1+f(u)) = \frac{u}{x}(1+f(u))$$

$$\frac{du}{u(1+f(u))} = \frac{1}{x}dx$$

(2) $y(1+x^2y^2)\,\mathrm{d}x = x\,\mathrm{d}y$; ✓ ✓

$\frac{x}{2}$ $t = xy$ $\qquad \frac{dt}{dx} = y + x\frac{dy}{dx}$

$$1+t^2 = \frac{dy}{dx}\cdot\frac{x}{y} = \frac{\frac{dt}{dx} - y}{x}\cdot\frac{x}{y}$$

$$= \frac{1}{y}\frac{dt}{dx} - 1 = \frac{x}{t}\frac{dt}{dx} - 1$$

$$\frac{dt}{dx} = (t^2+2)\cdot\frac{t}{x}$$

$$\Rightarrow \int \frac{1}{t(t^2+2)}\,dt = \int \frac{1}{x}\,dx.$$

$$\int \frac{1}{2}\left[\frac{1}{t} + \frac{-t}{t^2+2}\right]dt = \ln|x| + c.$$

$$C|x|$$

$$\|$$

$$\frac{1}{2}\ln|t| - \frac{1}{4}\int \frac{\cancel{t\,dt}\ d(t^2+2)}{t^2+2}. \qquad \frac{1}{2}\ln|t| - \frac{1}{4}\ln(t^2+2) = \ln\left|\frac{t^{\frac{1}{2}}}{(t^2+2)^{\frac{1}{4}}}\right|$$