1. 绝大多数比赛都使用“$(2n-1)$ 局 $n$ 胜制”的规则，这种比赛往往可以假定赛满 $2n-1$ 局来处理。如果甲，乙两人比赛，甲每局比赛获胜的概率均为 $p$ $(0<p<1)$ 且各局比赛的结果相互独立。求甲获胜的概率 $p_{甲}$.

**Case I** 如果正常求解，一旦甲累计胜利 $n$ 局则停止比赛，那么停止比赛前的最后一局**肯定是甲胜利**。得到

$$p_{甲}=C_{n-1}^{n-1}(1-p)^0p^n+C_{n}^{n-1}(1-p)^1p^n+\cdots+C_{2n-2}^{n-1}(1-p)^{n-1}p^n=\sum_{k=0}^{n-1}C_{n-1+k}^{n-1}(1-p)^kp^n. \tag{1}$$

**Case II** 如果**假定赛满** $2n-1$ **局**，那么此时设甲胜利的总次数为随机变量 $X\sim B(2n-1,p)$，那么

$$p_{甲}=P\{X\geq n\}=\sum_{k=n}^{2n-1}C_{2n-1}^{k}p^k(1-p)^{2n-1-k}. \tag{2}$$

为了说明上述两种解法的**等价性**，也即证明恒等式

$$\sum_{k=0}^{n-1}C_{n-1+k}^{n-1}(1-p)^kp^n=\sum_{k=n}^{2n-1}C_{2n-1}^{k}p^k(1-p)^{2n-1-k}.$$

下面用数学归纳法，首先 $n=1$ 时，所证结论等价为

$$\sum_{k=0}^{0}C_{k}^{0}(1-p)^kp^1=\sum_{k=1}^{1}C_{1}^{k}p^k(1-p)^{1-k}=p.$$

假设结论对 $n\in\mathbb{N}^*$ 成立，那么对 $n+1$ 的情形有

$$\begin{aligned}\sum_{k=n+1}^{2n+1}C_{2n+1}^{k}p^k(1-p)^{2n+1-k}&=\sum_{k=n+1}^{2n+1}(C_{2n-1}^{k}+2C_{2n-1}^{k-1}+C_{2n-1}^{k-2})p^k(1-p)^{2n+1-k}\\&=\sum_{k=n+1}^{2n-1}C_{2n-1}^{k}p^k(1-p)^{2n+1-k}+2\sum_{k=n+1}^{2n}C_{2n-1}^{k-1}p^k(1-p)^{2n+1-k}+\sum_{k=n+1}^{2n+1}C_{2n-1}^{k-2}p^k(1-p)^{2n+1-k}\end{aligned}$$

分别求这三项:

(a)

$$\begin{aligned}\sum_{k=n+1}^{2n-1} C_{2n-1}^{k} p^k (1-p)^{2n+1-k} &= (1-p)^2 \left\{ \sum_{k=n}^{2n-1} C_{2n-1}^{k} p^k (1-p)^{2n-1-k} - C_{2n-1}^{n} p^n (1-p)^{n-1} \right\} \\ &= (1-p)^2 \sum_{k=0}^{n-1} C_{n-1+k}^{n-1} (1-p)^k p^n - C_{2n-1}^{n} p^n (1-p)^{n+1}.\end{aligned}$$

(b)

$$\begin{aligned}\sum_{k=n+1}^{2n} C_{2n-1}^{k-1} p^k (1-p)^{2n+1-k} &= p(1-p) \left\{ \sum_{k-1=n}^{k-1=2n-1} C_{2n-1}^{k-1} p^{k-1} (1-p)^{2n-1-(k-1)} \right\} \\ &= p(1-p) \sum_{k=0}^{n-1} C_{n-1+k}^{n-1} (1-p)^k p^n.\end{aligned}$$

(c)

$$\begin{aligned}\sum_{k=n+1}^{2n+1} C_{2n-1}^{k-2} p^k (1-p)^{2n+1-k} &= p^2 \left\{ \sum_{k-2=n}^{k-2=2n-1} C_{2n-1}^{k-2} p^{k-2} (1-p)^{2n-1-(k-2)} + C_{2n-1}^{n-1} p^{n-1} (1-p)^n \right\} \\ &= p^2 \sum_{k=0}^{n-1} C_{n-1+k}^{n-1} (1-p)^k p^n + C_{2n-1}^{n-1} p^{n+1} (1-p)^n.\end{aligned}$$

现在只需证明

$$\begin{aligned}&\sum_{k=0}^{n-1} C_{n-1+k}^{n-1} (1-p)^k p^n + C_{2n-1}^{n-1} p^{n+1} (1-p)^n - C_{2n-1}^{n} p^n (1-p)^{n+1} = \sum_{k=0}^{n} C_{n+k}^{n} (1-p)^k p^{n+1} \\ \Leftrightarrow &\sum_{k=0}^{n-1} C_{n-1+k}^{n-1} (1-p)^k + C_{2n-1}^{n-1} p (1-p)^n - C_{2n-1}^{n} (1-p)^{n+1} = \sum_{k=0}^{n} C_{n+k}^{n} (1-p)^k p \\ \Leftrightarrow &\sum_{k=0}^{n-1} C_{n-1+k}^{n-1} (1-p)^k + 2C_{2n-1}^{n-1} p (1-p)^n - C_{2n-1}^{n} (1-p)^{n} = \sum_{k=0}^{n} C_{n+k}^{n} (1-p)^k p \\ \Leftrightarrow &\sum_{k=0}^{n-1} C_{n-1+k}^{n-1} (1-p)^k - C_{2n-1}^{n} (1-p)^{n} = -\sum_{k=0}^{n-1} C_{n+k}^{n} (1-p)^{k+1} + \sum_{k=0}^{n-1} C_{n+k}^{n} (1-p)^k \\ \Leftrightarrow &\sum_{k=0}^{n-1} \left\{ C_{n-1+k}^{n-1} (1-p)^k - C_{n+k}^{n} (1-p)^k \right\} = -\sum_{k=0}^{n-1} C_{n+k}^{n} (1-p)^{k+1} + C_{2n-1}^{n} (1-p)^n. \\ \Leftrightarrow &\sum_{k=1}^{n-1} -C_{n+k-1}^{n} (1-p)^k = -\sum_{k=0}^{n-2} C_{n+k}^{n} (1-p)^{k+1}. \\ \Leftrightarrow &\sum_{k-1=0}^{k-1=n-2} C_{n+(k-1)}^{n} (1-p)^{(k-1)+1} = \sum_{k=0}^{n-2} C_{n+k}^{n} (1-p)^{k+1}.\end{aligned}$$

最后一项是显然成立的。 □

2. 但也有一些项目，比如冰壶运动，其整个比赛通常是进行偶数局。现在有甲、乙两名同学进行一项趣味项目的比赛，两人约定比赛规则为：共进行 $2n$ ($n \in \mathbb{N}^*$) 局，谁赢的局数大于 $n$，谁就获得最终胜利。已知每局比赛中，甲的获胜概率均为 $p$ ($0 < p < 1/2$)，乙获胜的概率均为 $1-p$，记甲赢得比赛的概率为 $P_{2n}$，讨论数列 $\{P_{2n}\}$ 的单调性。

我们考虑建立 $P_{2n+2}$ 与 $P_{2n}$ 之间的**递推关系**，设甲在共进行 $2(n+1)$ 局的比赛中前 $2n$ 局胜利的局数为 $X$，最后两局胜利的局数为 $Y$，自然有

$$P_{2n+2} = P\{Y \geq 1\}P\{X = n+1\} + 1 \cdot P\{X > n+1\} + P\{Y = 2\}P\{X = n\}.$$

带入即

$$P_{2n+2} = [1-(1-p)^2] \cdot C_{2n}^{n+1}p^{n+1}(1-p)^{n-1} + [P_{2n} - C_{2n}^{n+1}p^{n+1}(1-p)^{n-1}] + p^2 \cdot C_{2n}^{n}p^n(1-p)^n.$$

化简得到

$$\begin{aligned} P_{2n+2} - P_{2n} &= C_{2n}^{n}p^{n+2}(1-p)^n - C_{2n}^{n+1}p^{n+1}(1-p)^{n+1} \\ &= p^{n+1}(1-p)^n[C_{2n}^{n}p - C_{2n}^{n+1}(1-p)] \\ &= \frac{p^{n+1}(1-p)^n C_{2n}^{n}}{n+1}\{[2p-1]n + p\}. \end{aligned}$$

由于 $2p-1<0$ 从而上式在 $\mathbb{R}_{\geq 0}$ 上有根 $n_0 = p/(1-2p)$. 若 $n_0 \in \mathbb{Z}$，那么 $\{P_{2n}\}$ 满足

$$P_2 < P_4 < \cdots < P_{2n_0} = P_{2n_0+2} > P_{2n_0+4} > \cdots$$

如果 $n_0 \notin \mathbb{Z}$，那么取 $n_1 = [n_0]$，此处 $[\cdot]$ 表示整数部分，$\{P_{2n}\}$ 满足

$$P_2 < P_4 < \cdots < P_{2n_1} < P_{2n_1+2} > P_{2n_1+4} > \cdots$$

□