MA345 Differential Equations & Matrix Method

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MODULE I - 1ST ORDER ODE

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QUIZ 2

2.6 Substitutions and Transformations

When the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is not a separable, exact, or linear equation, it may still be possible to transform it into one that we know how to solve. This was in fact our approach in Section 2.5, where we used an integrating factor to transform our original equation into an exact equation.

Substitution Procedure

(a) Identify the type of equation and determine the appropriate substitution or transformation.
(b) Rewrite the original equation in terms of new variables.
(c) Solve the transformed equation.
(d) Express the solution in terms of the original variables.

Homogeneous Equations

Homogeneous Equation

Definition 4. If the right-hand side of the equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

can be expressed as a function of the ratio y/x alone, then we say the equation is **homogeneous.**

For example, the equation

(2) $$(x-y)\,dx + x\,dy = 0$$

$$\frac{dy}{dx} = -\frac{x-y}{x} = -1 + \frac{y}{x}$$

$$V = \frac{y}{x} \qquad \frac{dy}{dx} = -1 + \frac{y}{x}$$

$$y = Vx$$

$$y' = V + V'x$$

$$\cancel{V} + \frac{dV}{dx}x = -1 + \cancel{V}$$

$$\int dV = \int -\frac{1}{x}\,dx$$

$$V = -\ln|x| + C$$

$$\boxed{y = Vx = -x\ln|x| + Cx}$$

$$\underbrace{\frac{dy}{dx}}\ \overbrace{-\frac{1}{x}}^{P(x)}\underbrace{y} = -1$$

$$g(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{x}$$

$$\int\left(\frac{1}{x}y\right)'dx = \int -\frac{1}{x}\,dx$$

$$\frac{1}{x}y = -\ln|x| + C$$

$$\boxed{y = -x\ln|x| + Cx}$$

Example 1 Solve

$$\textbf{(6)} \qquad (xy + y^2 + x^2)\,dx - x^2\,dy = 0\,.$$

Linear?
Exact? No
Separable?

$$\frac{dy}{dx} = \frac{(xy + y^2 + x^2)}{x^2} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 + 1.$$

$$V = \frac{y}{x}$$

$$y = Vx$$

$$\frac{dy}{dx} = V + V'x$$

$$\cancel{V} + V'x = \cancel{V} + V^2 + 1$$

$$x\frac{dV}{dx} = V^2 + 1$$

$$\int \frac{dV}{V^2+1} = \int \frac{1}{x}\,dx$$

$$\arctan V = \ln|x| + c$$

$$V = \tan(\ln|x| + c)$$

$$\boxed{y = Vx = x \cdot \tan(\ln|x| + c)}$$

Equations of the Form $\dfrac{dy}{dx} = G(ax + by)$

When the right-hand side of the equation $dy/dx = f(x, y)$ can be expressed as a function of the combination $ax + by$, where a and b are constants, that is,

$$\frac{dy}{dx} = G(ax + by) \; ,$$

then the substitution

$$z = ax + by$$

transforms the equation into a separable one. The method is illustrated in the next example.

Example 2 Solve ~~G(ab~~ $G(ax+by)$

$$\text{(8)} \qquad \frac{dy}{dx} = y - x - 1 + (x - y + 2)^{-1}.$$

Let

$z = x - y$

$\frac{dz}{dx} = 1 - \frac{dy}{dx}$

$\frac{dy}{dx} = 1 - \frac{dz}{dx}$

$\frac{dy}{dx} = -(x-y) - 1 + ((x-y)+2)^{-1}$

$1 - \frac{dz}{dx} = -z - 1 + \frac{1}{z+2}$

$\frac{dz}{dx} = (z+2) - \frac{1}{z+2} = \frac{(z+2)^2 - 1}{z+2}$

$u = (z+2)^2 - 1$

$du = 2(z+2)\,dz$

$\frac{1}{2} du = (z+2)\,dz$

$\int \frac{z+2}{(z+2)^2 - 1} dz = \int dx$

$\frac{1}{2} \ln|(z+2)^2 - 1| = x + C$

$e^{\ln|(z+2)^2 - 1|} = e^{(2x+2C)}$

$(z+2)^2 - 1 = A e^{2x}$

$\boxed{(x-y+2)^2 = Ae^{2x} + 1}$

Bernoulli Equations

Bernoulli Equation

Definition 5. A first-order equation that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \tag{9}$$

where $P(x)$ and $Q(x)$ are continuous on an interval (a, b) and n is a real number, is called a **Bernoulli equation.**†

$V = y^{1-n}$

$y' + Py = Qy^n \quad \Big| \frac{1}{y^n}$

$\frac{dV}{dx} = (1-n)\underbrace{y^{-n}\frac{dy}{dx}}$

$\underbrace{y^{-n}y'} + P\underbrace{y^{1-n}} = Q$

$y^{-n}y' = \frac{1}{1-n}V'$

$\frac{1}{1-n}V' + PV = Q$

Example 3 Solve

$$\frac{dy}{dx} - 5y = -\frac{5}{2}xy^3 . \qquad (11)$$

$n=3$

$\frac{1}{y^3}$ $\quad V = y^{1-n} = y^{-2}$ $\quad y \neq 0 \quad y \equiv 0$

$y^{-3}y' - 5y^{-2} = -\frac{5}{2}x$

$\frac{dV}{dx} = -2y^{-3}y'$

$-\frac{1}{2}V' = y^{-3}y'$

$\boxed{-\frac{1}{2}V' - 5V = -\frac{5}{2}x}$

$V' + 10V = 5x$

$g(x) = e^{\int 10\,dx} = e^{10x}$

$P(x)$

$\int (Ve^{10x})'\,dx = \int 5xe^{10x}\,dx$

$u = x \quad dv = e^{10x}dx$

$du = dx \quad v = \frac{1}{10}e^{10x}$

$Ve^{10x} = 5\left(\frac{1}{10}e^{10x}\cdot x - \frac{1}{100}e^{10x} + c\right) = \frac{1}{2}xe^{10x} - \frac{1}{20}e^{10x} + C$

$V = \frac{1}{2}x - \frac{1}{20} + Ce^{-10x}$

$\boxed{y^{-2} = \frac{1}{2}x - \frac{1}{20} + Ce^{-10x}}$

In this chapter we have discussed various types of first-order differential equations. The most important were the separable, linear, and exact equations. Their principal features and method of solution are outlined below.

Separable Equations: $dy/dx = g(x)\,p(y)$. Separate the variables and integrate.

Linear Equations: $dy/dx + P(x)y = Q(x)$. The integrating factor $\mu = \exp[\int P(x)\,dx]$ reduces the equation to $d(\mu y)/dx = \mu Q$, so that $\mu y = \int \mu Q\,dx + C$.

Exact Equations: $dF(x,y) = 0$. Solutions are given implicitly by $F(x,y) = C$. If $\partial M/\partial y = \partial N/\partial x$, then $M\,dx + N\,dy = 0$ is exact and F is given by

$$F = \int M\,dx + g(y)\,, \quad \text{where} \quad g'(y) = N - \frac{\partial}{\partial y}\int M\,dx$$

or

$$F = \int N\,dy + h(x)\,, \quad \text{where} \quad h'(x) = M - \frac{\partial}{\partial x}\int N\,dy\,.$$

When an equation is not separable, linear, or exact, it may be possible to find an integrat ing factor or perform a substitution that will enable us to solve the equation.

Special Integrating Factors: $\mu M\,dx + \mu N\,dy = 0$ is exact. If $(\partial M/\partial y - \partial N/\partial x)/N$ depends only on x, then

$$\mu(x) = \exp\left[\int\left(\frac{\partial M/\partial y - \partial N/\partial x}{N}\right)dx\right]$$

is an integrating factor. If $(\partial N/\partial x - \partial M/\partial y)/M$ depends only on y, then

$$\mu(y) = \exp\left[\int\left(\frac{\partial N/\partial x - \partial M/\partial y}{M}\right)dy\right]$$

is an integrating factor.

Homogeneous Equations: $dy/dx = G(y/x)$. Let $v = y/x$. Then $dy/dx = v + x(dv/dx)$, and the transformed equation in the variables v and x is separable.

Equations of the Form: $dy/dx = G(ax + by)$. Let $z = ax + by$. Then $dz/dx = a + b(dy/dx)$, and the transformed equation in the variables z and x is separable.

Bernoulli Equations: $dy/dx + P(x)y = Q(x)y^n$. For $n \neq 0$ or 1, let $v = y^{1-n}$. Then $dv/dx = (1-n)y^{-n}(dy/dx)$, and the transformed equation in the variables v and x is linear.