Lecture 2
Fundamental Principles of Counting, Permutation & Combination

Sample space = {1, 2, 3, 4, 5, 6 }

e = rolling a odd number = $\underbrace{\{1, 3, 5\}}$

Counting

Event e **:** an action or outcome meeting a success criteria.

Atomic event: an event that contains only a single outcome in a sample space (sample point)

$\Omega = \{\underline{11}, 12, 13, 14, \ldots\}$

$6^2 = 36 = \Omega$

Composite event: an event is made up of group of atomic events.

Size of an event $|e|$**:** $|e| = 3$

Addition Principle: m ways to perform a task

n ways to perform 2^{nd} task

$m+n$ ways to perform either task.

** task cannot be performed simultaneously.

Ex 1: A college library has 40 textbooks on sociology and 50 textbooks dealing with anthropology. How many textbooks about sociology or anthropology do we have to choose from this library?

+

and

×

$$\boxed{40 + 50 = 90 \text{ textbooks}}$$

Multiplication Principle:
* tasks cannot be performed simultaneously.

* m ways to do 1st task
* n ways to do 2nd task

$m \times n$ ways to do both tasks.

Ex 2: A restaurant offers 10 appetizers and 15 main courses. In how many ways can you order a two-course meal?

1 appetizer and (x) 1 main course.

$$\boxed{10 \times 15 = 150 \text{ ways.}}$$

$\times$ 26 $\qquad$ 10

<u>Ex 3</u>: Consider the manufacturing of license plates consisting of $\boxed{\text{two letters}}$ followed by $\boxed{\text{four digits}}$. Find the number of possible plates under the following conditions:

a. No letter or digit can be repeated.

$$\underline{26} \times \underline{25} \times \underline{10} \times \underline{9} \times \underline{8} \times \underline{7} = 3{,}276{,}000 \text{ possibilities.}$$

*

b. Repetition of letters and digits is allowed.

$$\underline{26} \times \underline{26} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{10} = 26^2 \times 10^4$$
$$= 6{,}760{,}000 \text{ possibilities}$$

c. If repetitions are allowed and only vowels and even digits.

AEIOU = 5

0, 2, 4, 6, 8 = 5

$$\underline{5} \times \underline{5} \times \underline{5} \times \underline{5} \times \underline{5} \times \underline{5}$$
$$= 5^6 = \boxed{15{,}625 \text{ possibilities}}.$$

Ex 4: At the AWL corporation Mrs. Foster operates the Quick Snack Coffee Shop. The menu at her shop is limited: six kinds of muffins, eight kinds of sandwiches, and five beverages (hot coffee, hot tea, iced tea, cola, and orange juice). Ms. Dodd, an editor at AWL, sends her assistant Carl to the shop to get her lunch-either a muffin and a hot beverage or a sandwich and cold beverage. How many ways can Carl purchase Ms. Dodd's lunch?

muffin & hot beverage Or Sandwich & cold beverage.

$$= (6 \overset{(\times)}{\times} 2) \overset{(+)}{+} (8 \overset{(\times)}{\times} 3) = 12 + 24 = 36 \text{ ways.}$$

Permutations $_nP_k$, $P(n,k)$: a permutation of items occur when
- no repetition
- order makes a difference.

$4\ 2\ 1 \neq 2\ 1\ 4$

n factorial $n!$: $!$ $\quad n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120 = 5 \times 4!$ or $\boxed{5 \times 4 \times 3!}$

Ex 5: A class of 10 students are to be seated in a row for a picture. How many such linear arrangements are possible?

$\underline{10} \times \underline{9} \times \underline{8} \times \underline{7} \times \underline{6} \times \underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1} = \boxed{10!}$

$n! = {}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$

Permutation Formula: n of n unique items

$${}_nP_n = n!$$

Ex 6 : In a class of 10 students, five are to be chosen and seated in a row for a picture. How many such linear arrangements are possible?

$_{10}P_5$ $\qquad$ $_nP_k$

$$\underline{10} \times \underline{9} \times \underline{8} \times \underline{7} \times \underline{6} = 30{,}240.$$

$$_{10}P_5 = \frac{10!}{(10-5)!} = \frac{10!}{5!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5!} = 30{,}240.$$

Permutation Formula: k of n unique items

$$* \quad _nP_k = \frac{n!}{(n-k)!} \quad *$$

$$n! = 10 \times 9 \times 8 \times 7 \times 6 \times \underbrace{5 \times 4 \times 3 \times 2 \times 1}$$

5 $\qquad 5 \times 4 \times 3 \times 2 \times 1 = 5! \quad (n-k)!$

Ex 7: A corporation has seven members on its board of directors. In how many different ways can it elect a president, vice-president, secretary, and treasurer?

$n = 7$

$k = 4$

$${}_7P_4 = \frac{7!}{(7-4)!} = \frac{7!}{3!} = \frac{7\times6\times5\times4\times\cancel{3!}}{\cancel{3!}} \boxed{= 840 \text{ Ways.}}$$

Permutation Formula: n of n items, with $r_1, r_2, \ldots r_k$ repetitions

$$\frac{n!}{r_1! \times r_2! \times \ldots \times r_k!}$$

Ex 8: In how many ways can the letters in the word DATABASES be arranged?

D	1
A	3 = 3!
T	1
B	1
S	2 = 2!
E	1
	9

$$= \frac{9!}{3! \times 2!} = \frac{9\times8\times7\times6\times5\times\overset{2}{\cancel{4}}\times\cancel{3!}}{\cancel{3!}\times\cancel{2}\times1}$$

$= 9\times8\times7\times6\times5\times2 = 30,240$ arrangements

Combinations nC_k, $C(n,k)$, $\binom{n}{k}$ "n choose k" A combination occurs when

* no repetation
* order make no difference

Combination Formula: Choose k objects of n unique objects

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

Ex 9: You volunteer to pet-sit for your friend who has $\boxed{\text{seven different}}$ animals. How many different pet combinations are possible if you take three of the seven pets?

$$\binom{7}{3} = \frac{7!}{(7-3)! \times 3!} = \frac{7!}{4! \times 3!} = \frac{7 \times \cancel{6} \times 5 \times \cancel{4!}}{\cancel{4!} \times \cancel{3 \times 2} \times 1} = 35$$

combinations

Ex 10: A zoo has six male bears and seven female bears. Two male bears and three female bears will be selected for an animal exchange program with another zoo. How many five-bear collections are possible?

$$\binom{6}{2} \text{ and } (\times) \binom{7}{3} = \frac{6!}{4! \times 2!} \times \frac{7!}{4! \times 3!}$$

$$= \frac{\overset{3}{\cancel{6}} \times 5 \times \cancel{4!}}{\cancel{4!} \times \cancel{2}} \times 35$$

$$= 15 \times 35$$

$$\boxed{= 525 \text{ five-bear combinations}}$$

<u>Ex 11</u>: A student taking a history exam is directed to answer 7 of 10 essay questions. Find the number of combinations with the following restrictions:

a. answer any 7 of the 10 questions.

$$\binom{10}{7} = \frac{10!}{3! \times 7!} = \frac{10 \times \cancel{9}^{3} \times \cancel{8}^{4} \times \cancel{7!}}{\cancel{3} \times \cancel{2} \times 1 \times \cancel{7!}} = 120 \text{ ways.}$$

b. answer three questions from the first five $\boxed{\text{and}}$ four questions from the last five.

$$\binom{5}{3} \text{ and } \binom{5}{4} = \frac{5!}{2! \times 3!} \times \frac{5!}{1! \times 4!} = \frac{5 \times \cancel{4}^{2} \times \cancel{3!}}{\cancel{2} \times \cancel{3!}} \times \frac{5 \times \cancel{4!}}{\cancel{4!}}$$

$$\boxed{= 10 \times 5 = 50 \text{ ways}}$$

c. at least three questions are selected from the first five.

$$\binom{5}{3} \text{ and } \binom{5}{4} \text{ or } \binom{5}{4} \text{ and } \binom{5}{3} \text{ or } \binom{5}{5} \text{ and } \binom{5}{2}$$

$$= \frac{5!}{3! \times 2!} \times \frac{5!}{1! \times 4!} + \frac{5!}{4!} \times \frac{5!}{3! \times 2!} + 1 \times \frac{5!}{3! \times 2!} = 10 \times 5 + 5 \times 10 + 10$$

$$= 110 \text{ ways.}$$

Pascal's Identity

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

Proof: $LHS = \binom{n-1}{k} + \binom{n-1}{k-1}$

$$= \frac{(n-1)!}{(n-1-k)!\,k!} + \frac{(n-1)!}{(n-1-(k-1))!\,(k-1)!}$$

$$= \frac{(n-1)!}{(n-k-1)!\,k(k-1)!} + \frac{(n-1)!}{(n-k)!\,(k-1)!}$$

$$= \frac{(n-1)!}{(n-k-1)!\,k\,(k-1)!} + \frac{(n-1)!}{(n-k)(n-k-1)!\,(k-1)!}$$

$$= \frac{(n-1)!\,(n-k)}{(n-k-1)!\,(n-k)\,k\,(k-1)!} + \frac{(n-1)!\,k}{(n-k)(n-k-1)!\,(k-1)!\,k}$$

$$= \frac{(n-1)!\,(n-k) + (n-1)!\,k}{(n-k-1)!\,(n-k)\;k\,(k-1)!}$$

$$= \frac{(n-1)!\,\{n-\cancel{k}+\cancel{k}\}}{(n-k)!\;k!}$$

$$= \frac{(n-1)!\,n}{(n-k)!\,k!}$$

$$= \frac{n!}{(n-k)!\,k!} = \binom{n}{k} = RHS$$

Hence Prooved

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + \binom{n}{n}x^0 y^n$$

Binomial Theorem Exploration

$$(x+y)^0 = 1$$

$$(x+y)^1 = 1x^1y^0 + 1x^0y^1$$

$$(x+y)^2 = 1x^2y^0 + 2x^1y^1 + 1x^0y^2$$

$$(x+y)^3 = 1x^3y^0 + 3x^2y^1 + 3x^1y^2 + 1x^0y^3$$

$$(x+y)^4 = 1x^4y^0 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + 1x^0y^4$$

$$\binom{4}{0} = \frac{4!}{4! \times 0!} = 1 \qquad \binom{4}{1} = \frac{4!}{3! \times 1!} = 4 \qquad \binom{4}{2} = \frac{4!}{2! \times 2!} = 6$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Pascal's Triangle

$n=0$: 1

$n=1$: 1 1

$n=2$: 1 2 1

$n=3$: 1 3 3 1

$n=4$: 1 4 6 4 1

Combination Formula: n objects taken r at a time with repetition

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{(n+\cancel{r}-1-\cancel{r})!\,r!} = \frac{(n+r-1)!}{(n-1)!\,r!}$$

☆☆

stars and bars

Ex 12: A donut shop offers $\boxed{20}^{\,n}$ kinds of donuts. Assuming that there are at least a dozen of each kind, how many different ways can we select $\boxed{12}_{\,r}$ donuts? 12 donuts of same kind.

$$\binom{20+12-1}{12} = \binom{31}{12} = \frac{31!}{19!\,12!} = \boxed{141{,}120{,}525 \text{ ways}}$$