MA345 Differential Equations & Matrix Method

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COAS.301.12

MODULE I - 1ST ORDER ODE

1.1 Background

Assignment: Homework 1

- 1.2 Solutions and Initial Value Problems
- 2.2 Separable Equations

1st order ODE:

Assignment: Homework 2

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1st order ODE: Substitutions / summary

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- 2.5 Special Integrating Factors
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Definition 1.1.1 Differential Equation

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation (DE)**.

In order to talk about them, we will classify a differential equation by **type**, **order**, and **linearity**.

Classification by Type If a differential equation contains only ordinary derivatives of one or more functions with respect to a *single* independent variable it is said to be an **ordinary differential equation (ODE)**. An equation involving only partial derivatives of one or more functions of two or more independent variables is called a **partial differential equation (PDE)**.

An example of a nonhomogeneous linear ODE is

$$y'' + 25y = e^{-x} \cos x,$$

and a homogeneous linear ODE is

$$xy'' + y' + xy = 0, \qquad \text{written in standard form} \qquad y'' + \frac{1}{x} y' + y = 0.$$

Finally, an example of a nonlinear ODE is

$$y''y + y'^2 = 0.$$

The functions p and q in (1) and (2) are called the **coefficients** of the ODEs. **Solutions** are defined similarly as for first-order ODEs in Chap. 1. A function

$$y = h(x)$$

is called a *solution* of a (linear or nonlinear) second-order ODE on some open interval I if h is defined and twice differentiable throughout that interval and is such that the ODE becomes an identity if we replace the unknown y by h, the derivative y' by h', and the second derivative y'' by h''. Examples are given below.

Example 1 Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to the linear equation

$$\frac{d^2y}{dx^2} - \frac{2}{x^2}y = 0, \tag{3}$$

but $\psi(x) = x^3$ is not.

$y = x^2 - x^{-1}$

$y' = 2x + x^{-2}$

$y'' = 2 - 2x^{-3}$

$y'' - \frac{2}{x^2}y = 0$

$(2 - 2x^{-3}) - \frac{2}{x^2}(x^2 - x^{-1}) = 0$

$(\cancel{2 - 2x^{-3}}) - (\cancel{2 - 2x^{-3}}) = 0$

$0 = 0$

$y = x^3$

$y' = 3x^2$

$y'' = 6x$

$y'' - \frac{2}{x^2}y = 0$

$6x - \frac{2}{x^2}x^3 = 0$

$6x - 2x \neq 0$

$y' - 2x = 0$

$y' = 2x$

$y = x^2$

$$\boxed{y = x^2 + c}$$

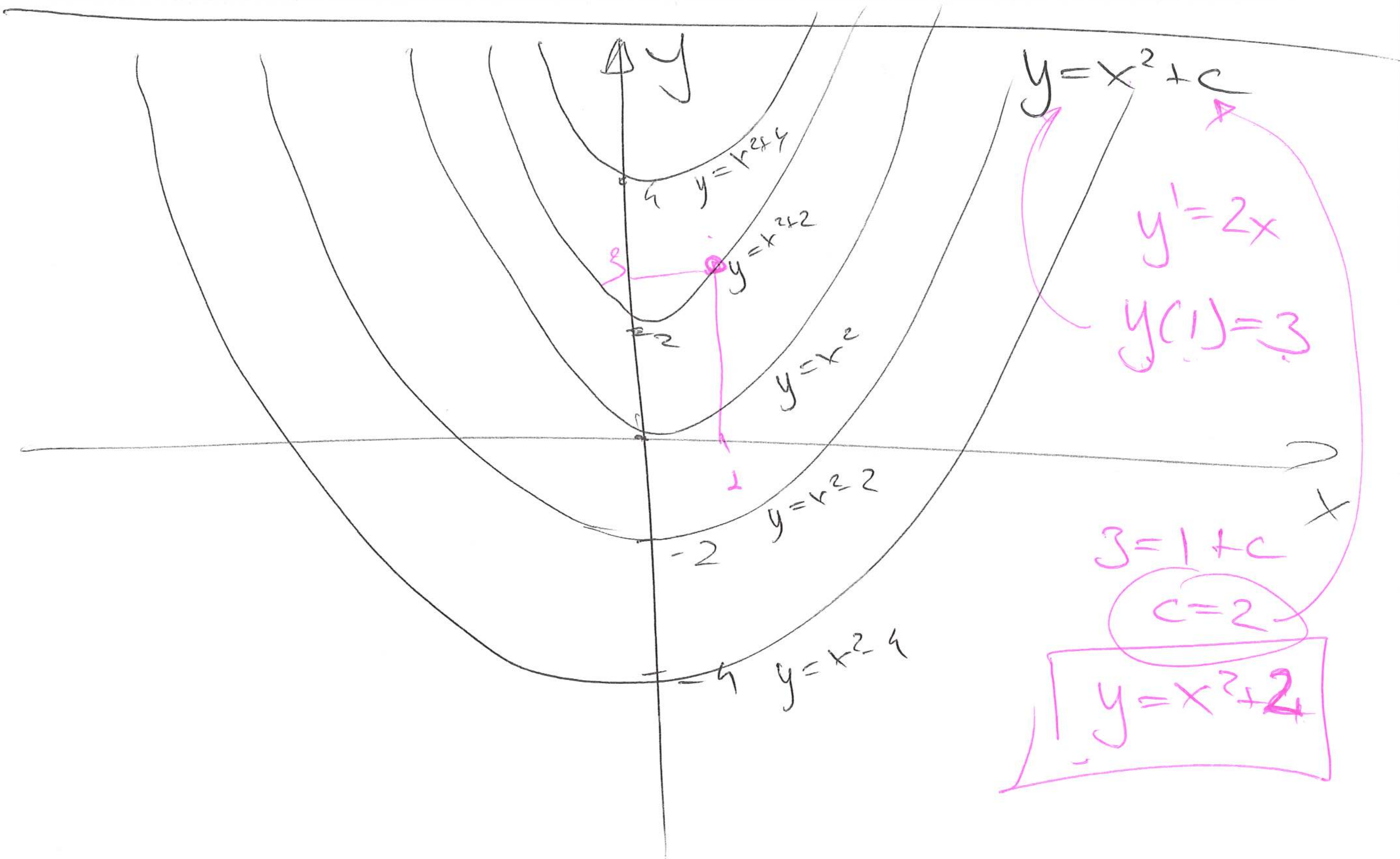
$y - x^2 + c = 0$

$y = x^2 + 10$

$y = x^2 + 23$

Implicit Solution

Definition 2. A relation $G(x, y) = 0$ is said to be an **implicit solution** to equation (1) on the interval I if it defines one or more explicit solutions on I.



IVP - initial value problem

IC: initial condition

$$y(x_0) = y_0$$

General solution: $\rightarrow y = \ldots + c$

Particular Solution IC

Existence and Uniqueness of Solution

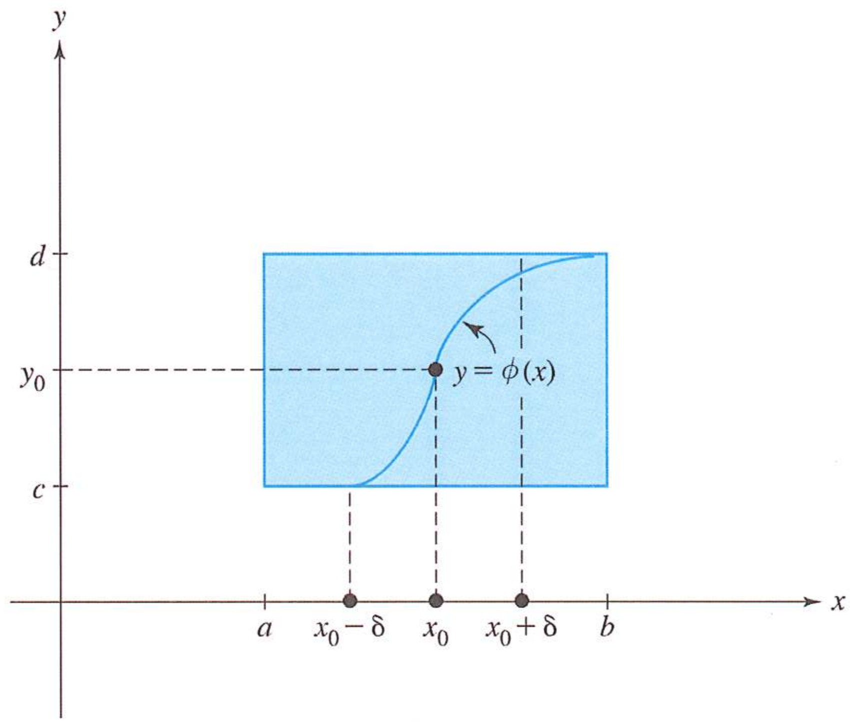
Theorem 1. Consider the initial value problem

$$\frac{dy}{dx} = f(x, y) \ , \qquad y(x_0) = y_0 \ .$$

If f and $\partial f/\partial y$ are continuous functions in some rectangle

$$R = \{(x, y) : a < x < b, c < y < d\}$$

that contains the point (x_0, y_0), then the initial value problem has a unique solution $\phi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$, where δ is a positive number.†



DEFINITION 2.2.1 Separable Equation

A first-order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be **separable** or to have **separable variables.**

$$\frac{dy}{dx} = g(x)$$

$$\int dy = \int g(x)\,dx$$

$$y = G(x) + C$$

$$\frac{dy}{dx} = g(x)h(y)$$

$$\int \frac{dy}{h(y)} = \int g(x)\,dx$$

$$\frac{dy}{dx} = 1 + e^{2x}$$

$$\int dy = \int (1 + e^{2x})\,dx$$

$$\boxed{y = x + \frac{e^{2x}}{2} + C}$$

Any constant

$$\frac{dy}{dx} = y^2 x e^{3x+4y}$$

$$\frac{dy}{dx} = y^2 x\, e^{3x} \cdot e^{4y}$$

$$\frac{dy}{dx} = (y^2 e^{4y})(xe^{3x})$$

$$\int \frac{dy}{y^2 e^{4y}} = \int (xe^{3x})\,dx$$

$$\frac{dy}{dx} = y + \sin x$$

$$dy = (y + \sin x)\,dx$$

nonseparable

$$a^{v+s} = a^v \cdot a^s$$

Method for Solving Separable Equations

To solve the equation

$$\frac{dy}{dx} = g(x)p(y) \tag{2}$$

multiply by dx and by $h(y) := 1/p(y)$ to obtain

$$h(y)\,dy = g(x)\,dx\,.$$

Then integrate both sides:

$$\int h(y)\,dy = \int g(x)\,dx\,,$$

$$H(y) = G(x) + C\,, \tag{3}$$

where we have merged the two constants of integration into a single symbol C. The last equation gives an implicit solution to the differential equation.

Example 1 Solve the nonlinear equation

$$\frac{dy}{dx} = \frac{x-5}{y^2}.$$

$$\int y^2\,dy = \int (x-5)\,dx$$

$$\frac{y^3}{3} = \frac{x^2}{2} - 5x + C$$

$$y^3 = \frac{3}{2}x^2 - 15x + 3C$$

$$\downarrow$$

$$k$$

$$y = \sqrt[3]{\tfrac{3}{2}x^2 - 15x + k}$$

EXAMPLE 1 Solving a Separable DE

Solve $(1 + x)\,dy - y\,dx = 0.$

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$e^{\ln x} = x$$

$$\ln|y| = \ln|1+x| + c$$

$$e^{\ln|y|} = e^{\ln|1+x| + c} \longrightarrow e^{\ln|1+x|} \cdot e^{c}$$

$$\downarrow$$

$|1+x| \cdot k$ still any constant

$$y = k\,|1+x| \qquad \begin{cases} |1+x| = 1+x & x > -1 \\ |1+x| = -(1+x) & x < -1 \end{cases}$$

$$y = \pm k(1+x)$$

$$y = A(1+x)$$