MA345 Differential Equations & Matrix Method

Lecture: 01

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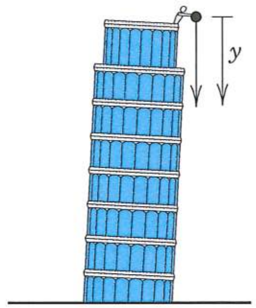
COAS.301.12

Definition 1.1.1 Differential Equation

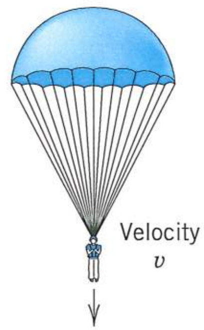
An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation (DE)**.

In order to talk about them, we will classify a differential equation by **type**, **order**, and **linearity**.



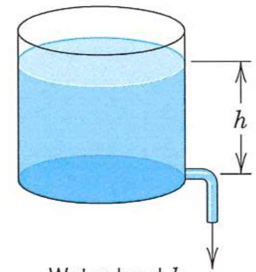
Falling stone

$y'' = g = const.$
(Sec. 1.1)



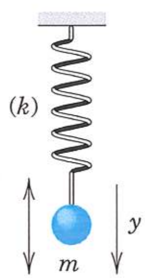
Parachutist

$mv' = mg - bv^2$
(Sec. 1.2)



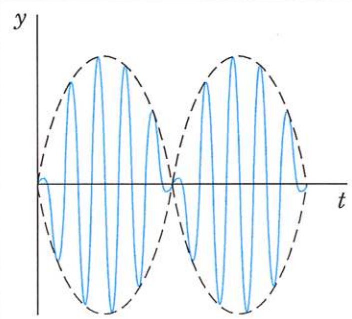
Water level h

Outflowing water
$h' = -k\sqrt{h}$
(Sec. 1.3)



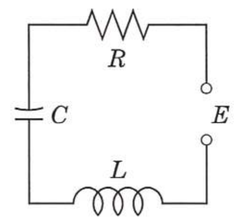
Displacement y

Vibrating mass
on a spring
$my'' + ky = 0$
(Secs. 2.4, 2.8)



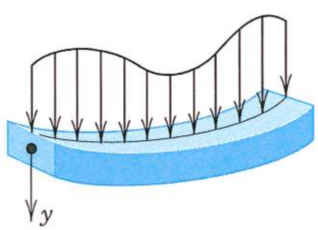
Beats of a vibrating
system

$y'' + \omega_0^2 y = \cos \omega t, \quad \omega_0 \approx \omega$
(Sec. 2.8)



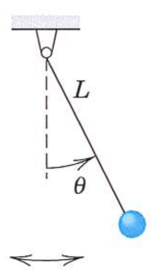
Current I in an
RLC circuit

$LI'' + RI' + \frac{1}{C}I = E'$
(Sec. 2.9)



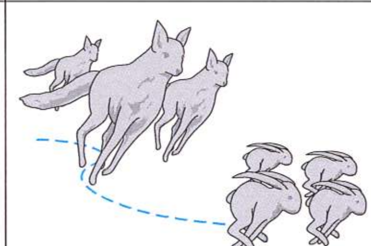
Deformation of a beam

$EIy^{iv} = f(x)$
(Sec. 3.3)



Pendulum

$L\theta'' + g \sin \theta = 0$
(Sec. 4.5)



Lotka–Volterra
predator–prey model

$y_1' = ay_1 - by_1y_2$
$y_2' = ky_1y_2 - ly_2$
(Sec. 4.5)

To begin our study of differential equations, we need some common terminology. If an equation involves the derivative of one variable with respect to another, then the former is called a **dependent variable** and the latter an **independent variable.**

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0$$

$x = f(t)$

coefficients

$$y'e^x + \cos x\, y'' = 0$$

$y = f(x)$

variable coefficients

Definition 1.1.1 Differential Equation

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation (DE)**.

In order to talk about them, we will classify a differential equation by **type**, **order**, and **linearity**.

Classification by Type If a differential equation contains only ordinary derivatives of one or more functions with respect to a *single* independent variable it is said to be an **ordinary differential equation (ODE)**. An equation involving only partial derivatives of one or more functions of two or more independent variables is called a **partial differential equation (PDE)**.

Leibniz: $\frac{dy}{dx}$; $\frac{d^2y}{dx^2}$

Prime: y'; y''

Newton's: $\ddot{s} = -32$; $\frac{d^2s}{dt^2} = -32$

Subscript: D_x / $\frac{\partial^2 u}{\partial x^2} = u_{xx}$ $u_{xx} = u_{tt} - u_t$ $u(x,t)$

an ODE can contain more
than one dependent variable

$$\downarrow \quad \downarrow$$

$$\frac{dy}{dx} + 6y = e^{-x}, \quad \frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0, \quad \text{and} \quad \frac{dx}{dt} + \frac{dy}{dt} = 3x + 2y$$

(b) The equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{3}$$

are examples of partial differential equations. Notice in the third equation that there are two

□ **Classification by Order** The **order of a differential equation** (ODE or PDE) is the order of the highest derivative in the equation.

The differential equations

$$\overset{\substack{\text{highest order}\\ \downarrow}}{\frac{d^2y}{dx^2}} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x, \qquad 2\overset{\substack{\text{highest order}\\ \downarrow}}{\frac{\partial^4 u}{\partial x^4}} + \frac{\partial^2 u}{\partial t^2} = 0$$

are examples of a second-order ordinary differential equation and a fourth-order partial differential equation, respectively.

$$y'' + 2y' + 3y + e^x = 0$$

General form

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

$$y'' = -2y' - 3y - e^x$$

Normal form

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots y^{(n-1)})$$

$$y'' + 2y' + 3y = -e^x$$

Standard form

In symbols, we can express an nth-order ordinary differential equation in one dependent variable by the **general form**

$$F(x, y, y', \ldots, y^{(n)}) = 0, \tag{4}$$

where F is a real-valued function of $n + 2$ variables: $x, y, y', \ldots, y^{(n)}$. For both practical and theoretical reasons, we shall also make the assumption hereafter that it is possible to solve an ordinary differential equation in the form (4) uniquely for the highest derivative $y^{(n)}$ in terms of the remaining $n + 1$ variables. The differential equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}), \tag{5}$$

where f is a real-valued continuous function, is referred to as the **normal form** of (4). Thus, when it suits our purposes, we shall use the normal forms

$$\frac{dy}{dx} = f(x, y) \quad \text{and} \quad \frac{d^2y}{dx^2} = f(x, y, y')$$

differential form : $M(x,y)\,dx + N(x,y)\,dy = 0$

$$(y-x)\,dx + 4x\,dy = 0$$

$$(y-x) + 4x\frac{dy}{dx} = 0$$ General form

$$4x\,y' = x-y$$

$$y' = \frac{x-y}{4x}$$ → Normal form

$$y' + \frac{y}{4x} = \frac{1}{4}$$ → Standard form

Definition 2.3.1 Linear Equation

A first-order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1}$$

is said to be a **linear equation** in the dependent variable y.

When $g(x) = 0$, the linear equation (1) is said to be **homogeneous**; otherwise, it is **nonhomogeneous**.

A second-order ODE is called **linear** if it can be written

$$y'' + p(x)y' + q(x)y = r(x) \tag{1}$$

and **nonlinear** if it cannot be written in this form.

The distinctive feature of this equation is that it is *linear in y and its derivatives,* whereas the functions p, q, and r on the right may be any given functions of x. If the equation begins with, say, $f(x)y''$, then divide by $f(x)$ to have the **standard form** (1) with y'' as the first term.

(b) The equations

nonlinear term: coefficient depends on y $\downarrow$

$$(1-y)y' + 2y = e^x,$$

nonlinear term: nonlinear function of y $\downarrow$

$$\frac{d^2y}{dx^2} + \sin y = 0,$$

nonlinear term: power not 1 $\downarrow$

$$\frac{d^4y}{dx^4} + y^2 = 0,$$

$$\frac{d^2y}{dx^2} + y^3 = 0$$

ODE
2nd order
non-linear
homogeneous

$$t^3\frac{dx}{dt} = t^3 + x$$

ODE
1st order
linear

$$\frac{d^2y}{dx^2} - y\frac{dy}{dx} = \cos x$$

ODE
2nd order
non linear
non homogeneous

$$y' + 2y = e^x$$

ODE
1st order
linear
$y = f(x)$

$$y''' + \cos x\, y' + xy = 0$$

ODE
2nd
linear
homogeneous

Explicit Solution

Definition 1. A function $\phi(x)$ that when substituted for y in equation (1) [or (2)] satisfies the equation for all x in the interval I is called an **explicit solution** to the equation on I.

Definition 1.1.2 Solution of an ODE

Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth-order ordinary differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.