Module 11

Relations

Ordered Pair (2-Tuple): given two sets A, B, an ordered pair is a tuple (a, b) where $a \in A$ and $b \in B$.

* order matters $(a, b) \neq (b, a)$.

Cartesian Product Set: of the two sets A and B, the Cartesian product set is the set of all ordered pairs that can be created from the two sets.

$$A \times B = \{ (a, b) \mid \forall a \in A, \forall b \in B \}$$

Ex 1: If $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, find $A \times B$

$A \times B \neq B \times A$

$(a, b) \neq (b, a)$

$$A \times B = \{\underbrace{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c), (3,a), (3,b), (3,c)}_{9 \text{ Ordered pairs.}}\}$$

$$\begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} (1,a) & (1,b) & (1,c) \\ (2,a) & (2,b) & (2,c) \\ (3,a) & (3,b) & (3,c) \end{bmatrix} \end{array}$$

Cardinality of the Cartesian Product Set The number of ordered pairs that make the cartesian Product Set.

$$|A \times B| = |A|\,|B|$$

$|A| = 3 = |B|$ $\quad |A \times B| = 3 \times 3 = 9.$

<u>Relation R</u>: A relation on two sets A and B is any subset of the cartesian product.

$$\underbrace{{}_A R_B}_{} \subseteq A \times B.$$

"Relation R from A to B"

① ${}_A R_B$

② $x R y$ where $x \in A$ and $y \in B$

③ $R(x, y)$ where $x \in A$ and $y \in B$.

The number of possible relations from sets $\boldsymbol{A}$ and $\boldsymbol{B} =$ $2^{|A||B|}$

Relation Examples $x R y$: x is less than y. $(x, y) \in \mathbb{Z}$.

$(4, 5)$

$(-8, 3)$

$(0, 2.5)$ X

$A \times B = \{(1, a), (1, b)\}$

<u>Ex 2</u>: If $A = \{1\}$ and $B = \{a, b\}$, find ALL possible relations R $\quad 2^{|A|\,|B|} = 2^{1 \cdot 2} = 2^2 = 4$

$\{\ \}, \{(1,a)\}, \{(1,b)\}, \{(1,a), (1,b)\}$

$\{(1,a)\} \subseteq \{(1,a), (1,b)\}$

The number of possible relations from sets A and B = $2^{|A||B|} = 2^2 = 4$.

<u>Note: R is a set</u>

If $A_1 \subseteq A_2$ then $R(A_1) \subseteq R(A_2)$

$R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$

$R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$.

Let R and S be relations from A and B

If $R(a) = S(a)$ $\forall a \in A$ then $R = S$.

Domain of a Relation R the set of all values that show up as first element or object or term of an ordered pair in the relation.

Range of a Relation R the set of all values that show up as second element/term of an ordered pair in the relation.

$R = \{(1, a), (1, b)\}$

$D = \{1\}$

Range $= \{a, b\}$.

Relation Representations $A = \{1, 2, 3\}$ $\quad {}_A R_A$

1. **Enumeration (List)**

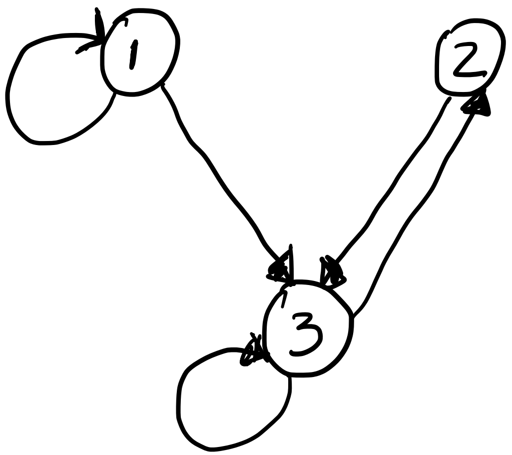
$${}_A R_A = \{(1,1), (1,3), (2,3), (3,2), (3,3)\}$$

~~2. Matrix~~

(2) Matrix.

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

3. Digraph (Directional graph)

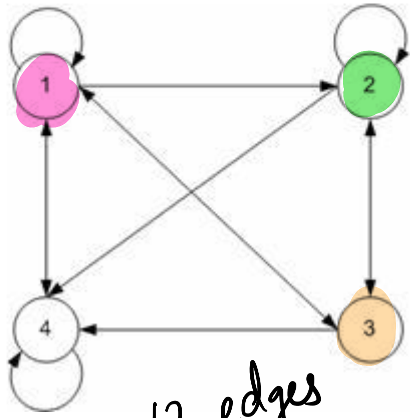


$$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

$${}_A R_A \subseteq A \times A.$$

$$2^{|A||A|} = 2^{3\times 3} = 2^9$$

Ex 3: Write the list and matrix representation for R on $A=1, 2, 3, 4$:



12 edges

① Enumeration /list :

$$A R_A = \{(1,1), (1,2), (1,4), (1,3), (2,2), (2,3), (2,4), \cancel{(2,1)}, (3,2), (3,1), (3,4), (4,1), (4,4)\}$$

12 ordered pairs

② Matrix

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array}$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Properties of Relations

$A = \{a, b, c\}$

$A \times A$ = cross product or Cartesian product

$R \subseteq A \times A$.

- **Reflexive** Every element in set A is related to itself.

\# $(a, a) \in R$

\# diagonal entries must be all 1.

$A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3), (1,2)\}$ ✓

$$\begin{array}{c|c} & \begin{matrix}1 & 2 & 3\end{matrix} \\ \begin{matrix}1\\2\\3\end{matrix} & \begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\end{array}$$

- **Irreflexive** No elements in A is related to itself.

\# $(a, a) \notin R$ or $a \not R a \ \forall a \in A$.

\# diagonal entries must be all 0.

$A = \{1, 2, 3\}$

$R = \{(1,2), (2,3), (3,1), (3,2)\}$

$$\begin{array}{c|c} & \begin{matrix}1 & 2 & 3\end{matrix} \\ \begin{matrix}1\\2\\3\end{matrix} & \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 1 & 0\end{bmatrix}\end{array}$$

- **Symmetric** If $(a, b) \in R$, then $(b, a) \in R$

\# matrix and traspose must be equal

$A = \{1, 2, 3\}$

$R = \{(1,2), (2,1), (2,2), (3,3)\}$

$$T(R) = \begin{bmatrix}0 & 1 & 0\\1 & 1 & 0\\0 & 0 & 0\end{bmatrix} \longleftrightarrow R = \begin{array}{c|c} & \begin{matrix}1 & 2 & 3\end{matrix} \\ \begin{matrix}1\\2\\3\end{matrix} & \begin{bmatrix}0 & 1 & 0\\1 & 1 & 0\\0 & 0 & 1\end{bmatrix}\end{array} \begin{matrix}\text{ROW1}\\\text{ROW2}\\\text{ROW3}\end{matrix}$$

=

Properties of Relations

- **Asymmetric** If $(a,b) \in R$, then $(b,a) \notin R$ and all 0's on the main diagonal.

\# every 1 is paired with 0 opposite to it across main diagonal.

$A = \{1, 2, 3\}$

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right] & \\ 3 & & & \end{array}$$

$R = \{(1,2), (3,1), (3,2)\}$

- **Antisymmetric** If $(a,b) \in R$ and $(b,a) \notin R$ then 1 may exist on the main diagonal

\# every 1 is paired with a zero opposite to it across main diagonal.

$A = \{1, 2, 3\}$

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right] & \\ 3 & & & \end{array}$$

$R = \{(1,1), (1,2), (3,1), (3,2)\}$

- **Transitive** If $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$ (In the diagonal 1's or 0's)

\# (If $a_{ij} = 1$ and $a_{jk} = 1$, then $a_{ik} = 1$)

$A = \{1, 2, 3\}$ $\quad R = \{(1,1), (1,2), (2,3), (1,3), (3,3)\}$

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right] & \\ 3 & & & \end{array}$$

Property	Definition	Identification with a matrix	Identification with a digraph
Reflexive	A relation R on a set A is reflexive if $(a, a) \in R$ for all $a \in A$.	All 1's on the main diagonal	Every vertex has a cycle of length 1
Irreflexive	A relation R on a set A is irreflexive if $(a, a) \notin R$ for all $a \in A$.	All 0's on the main diagonal	No vertex has a cycle of length 1
Symmetric	A relation R on a set A is symmetric if whenever $(a, b) \in R$, then $(b, a) \in R$.	The matrix is symmetric across the main diagonal	Every edge is undirected, i.e., it goes both ways.
Asymmetric	A relation R on a set A is asymmetric if whenever $(a, b) \in R$, then $(b, a) \notin R$.	All 0's on the main diagonal and every 1 in the matrix is paired with a 0 opposite it across the main diagonal.	No cycles of length 1 and every edge is directed, i.e., no edge can be paired with an equal edge in the opposite direction.
Antisymmetric	A relation R on a set A is antisymmetric if whenever $(a, b) \in R$ and $(b, a) \in R$, then $a = b$.	1's may exist on the main diagonal, but every 1 in the matrix is paired with a 0 opposite it across the main diagonal.	Cycles of length 1 are allowed, but every edge is directed, i.e., no edge can be paired with an equal edge in the opposite direction.
Transitive	A relation R on a set A is transitive if when $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.	Not obvious: If $a_{ij} = 1$ and $a_{jk} = 1$, then a_{ik} must equal 1.	Every path of length two must be "matched" with a path of length one.

$(a, b) \wedge (b, a) \rightarrow \underline{(a, a)}$

Ex 4: **Given the following matrix representations of relations, identify if each of them are reflexive, irreflexive, symmetric, asymmetric, antisymmetric, and/or transitive:**

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \mathbf{M}_{R_1}$$

(a)

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \mathbf{M}_{R_2}$$

(b)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{M}_{R_3}$$

(c)

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \mathbf{M}_{R_4}$$

(d)

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{M}_{R_5}$$

(e)

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{M}_{R_6}$$

(f) HW.

Ex	Reflexive	Irreflexive	Symmetric	Asymmetric	Antisymmetric	Transitive
a	N	N	Y	N	N	N
b	N	N	Y	N	N	N
c	N	N	N	N	Y	N
d	N	Y	N	N	N	N
e	Y	N	N	N	Y	T
f	N	Y	N	Y	Y	N

<u>Ex 5</u>: Given the following digraph representations of relations, identify if each of them are reflexive, irreflexive, symmetric, asymmetric, antisymmetric, and/or transitive:

Fig 1

$A = \{1,2,3,4,5\}$

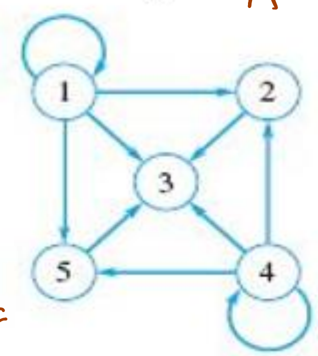
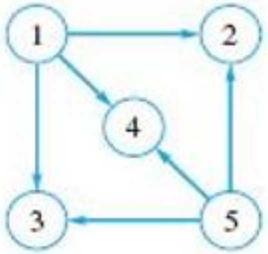


Fig 2



(1)

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 & 1 & 1 \\ 5 & 0 & 0 & 1 & 0 & 0 \end{array}$$

Not reflexive
Not Irreflexive.
Not symmetric
Not Asymmetric
Anti-symmetric
Not transitive

$(1,2) \wedge (2,3) \rightarrow (1,3)$ ✓

$(1,3) \wedge (3,..) \rightarrow (1,..)$ ✗

(2)

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 1 & 1 & 1 & 0 \end{array}$$

not reflexive
Irreflective
Not symmetric
Asymmetric
Antisymmetric
transitive

** If asymmetric then anti-symmetric **

Closure The closure of a relation is a relation derived from the original relation where ordered pairs are added until the relation exhibits a desired property.

	a	b	c
a	1	1	0
b	0	1	1
c	0	0	0

1

Ex 6: Let set $A = \{a, b, c\}$ and Relation R =

R = $\{(a,a), (b,b), (a,b), (b,c)\}$. **Find the following closures:**

a. **Reflexive Closure**

$\{(a,a), (b,b), (a,b), (b,c), (c,c)\}$ ← reflexive closure of R.

	a	b	c
a	1	1	0
b	0	1	1
c	0	0	0

1 1

b. **Symmetric Closure**

$\{(a,a), (b,b), (a,b), (b,c), (b,a), (c,b)\}$

c. **Transitive Closure**

$(a,a) \wedge (a,b) \rightarrow (a,b)$ ✓

$(a,b) \wedge (b,c) \rightarrow (a,c)$

	a	b	c
a	1	1	0
b	0	1	1
c	0	0	0

(a,c)

$(a,b) \wedge (b,c) \rightarrow (a,c)$

add (a,c)

R = $\{(a,a), (b,b), (a,b), (b,c), (a,c)\}$

Equivalence Relation A relation R on a set A that is reflexive, symmetric and transitive.

Examples Let $A = \{1,2,3,4\}$

$(1,2) \wedge (2,1) \rightarrow (1,1)$ ✓

$(1,2) \wedge (2,2) \rightarrow (1,2)$ ✓

$(3,3) \wedge (3,4) \rightarrow (3,4)$ ✓

$(3,4) \wedge (4,3) \rightarrow (3,3)$ ✓

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 1 \\ 4 & 0 & 0 & 1 & 1 \end{array}$$

$R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$