Module 9

Methods of Proof: Induction

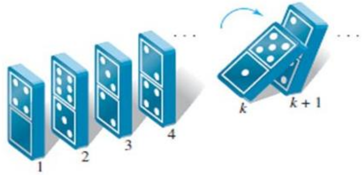
Proof by Induction

Recall:

Law of syllogism.

$$\begin{array}{l} P_1 \to P_2 \\ P_2 \to P_3 \\ \hline \therefore P_1 \to P_3 \end{array}$$

$$\begin{array}{l} P_1 \to P_2 \\ P_2 \to P_3 \\ P_3 \to P_4 \\ \vdots \\ P_{k-1} \to P_k \\ \hline \therefore P_1 \to P_k \end{array}$$



If the kth domino falls backward, it pushes the $(k + 1)$st domino backward also.

Give P_1 and infinite # of inferences how can we prove $P_n \to P_{n+1}$ exists?

Prove an arbitary form of $P_n \to P_{n+1}$ then use law of universal Generalization to obtain all other inferences.

If you can prove P_1, you prove them all.

Step 1: **Basis** / Basis case

Choose the Smallest value of n and perform a direct proof. (proving P_1)

Conclusion: steps ( ) show that P is true for n = smallest val.

Step 2: **Induction Hypothesis**

Let $n = k$ and state the arbitary case : P_k

DO NOT PROVE ANYTHING.

Step 3: **Induction Step** Prove that $P_k \longrightarrow P_{k+1}$

Let $n = k+1$ and use direct Proof that relies on Induction hypothesis (P_k)

Proof for $n = k+1$ case must rely on $n = k$ case.

Conclusion: steps ( ) - ( ) show P is true for $n = k+1$

Step 4: **Conclusion**

Because the Base case is true, and the Induction step is true then the Property P is true for all $n \geq$ the n of the base case.

Ex 1: $\mathbb{P}: \forall n \in \mathbb{N}, (-1)^{2n} = 1$

$n=1$

STEP 1: Base Case

Let $n=1$

$\mathbb{P}: (-1)^{2n} = 1$ for $n=1$

(i) $(-1)^{2n}$ Premise

(ii) $(-1)^{2\cdot 1}$ Substitute $n=1$

(iii) $(-1)\times(-1) = 1$ math

Conclusion: steps (i)-(iii) show $\mathbb{P}$ is true for $n=1$

STEP 2: Induction Hypothesis

Let $n=k$, assume $\boxed{(-1)^{2k} = 1}$

STEP 3: Induction step

Let $n=k+1$

$\mathbb{P}: (-1)^{2n} = 1$ for $n=k+1$ given $(-1)^{2k} = 1$

(i) $(-1)^{2n}$ Premise

(ii) $(-1)^{2\cdot(k+1)}$ Substitute $n=k+1$

(iii) $(-1)^{2k}\,(-1)^{2}$ algebra

(iv) 1 math

Conclusion: steps (i)-(iv) show $\mathbb{P}$ is true for $n=k+1$

STEP 4: Induction Proof Conclusions

Because the base case is true, and the Induction step is true, $\mathbb{P}$ is true for all $n \in \mathbb{N}$.

Ex 2: $\mathbb{P}$: $\forall n \in \mathbb{N}$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$1+2+3+4+\cdots+n = \frac{n(n+1)}{2}$.

Step 1: Base Case

Let $n = 1$

$\mathbb{P}: \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for $n = 1$

(i)	$\sum_{i=1}^{n} i$	Premise
(ii)	$\sum_{i=1}^{1} i$	Substitute $n=1$
(iii)	1	def^n of Σ
(iv)	$\frac{1(1+1)}{2}$	math
(v)	$n(n+1)/2$	Substitute $1=n$

Conclusion: step (i)-(V) prove $\mathbb{P}$ is valid for $n=1$

Step 2: Induction Hypothesis

Let $n=k$, assume $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

Step 3: Induction step

Let $n=k+1$

$\boxed{\mathbb{P}: \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \text{ for } n=k+1 \text{ using } \sum_{i=1}^{k} i = \frac{k(k+1)}{2}}$

(i)	$\sum_{i=1}^{n} i$	Premise
(ii)	$\sum_{i=1}^{k+1} i$	Substitute $n=k+1$
(iii)	$\sum_{i=1}^{k} i + (k+1)$	algebra of Σ.
(iv)	$\frac{k(k+1)}{2} + (k+1)$	Induction hypothesis.
(v)	$\frac{k(k+1)+2(k+1)}{2}$	math
(vi)	$(k+1)(k+1+1)/2$	simplification.
(vii)	$n(n+1)/2$	$k+1 = n$.

Conclusion: steps (i)-(vi) Prove $\mathbb{P}$ is valid for $n=k+1$

Step 4: Induction Proof Conclusions

Because the base case is true and the Induction step is true, then $\mathbb{P}$ is true for all $n \in \mathbb{N}$

Ex 3: $\mathbb{P}$: The sum of the first n odd natural numbers is n^2

$$\mathbb{P}: \forall n \in \mathbb{N}, \sum_{i=1}^{n} (2i-1) = n^2$$

Step 1: Basis Case : Let $n = 1$

$$\boxed{\mathbb{P}: \sum_{i=1}^{n} (2i-1) = n^2 \text{ for } n = 1}$$

(i)	$\sum_{i=1}^{n} (2i-1)$	Premise
(ii)	$\sum_{i=1}^{1} (2i-1)$	Substituting $n=1$
(iii)	1	def^n of Σ
(iv)	1^2	math
(v)	n^2	Substitute $1 = n$

Conclusion : steps (i) – (v) show that $\mathbb{P}$ is true for $n = 1$

Ex 3: $\mathbb{P}$: **The sum of the first n odd natural numbers is n^2**

Step 2: Induction hypothesis

Let $n = k$, assume $\boxed{\sum_{i=1}^{k}(2i-1) = k^2}$

Step 3: Induction Step.

$\mathbb{P}$: $\sum_{i=1}^{n}(2i-1) = n^2$ for $n = k+1$ using $\sum_{i=1}^{k}(2i-1) = k^2$

(i) $\sum_{i=1}^{n}(2i-1)$ Premise

(ii) $\sum_{i=1}^{k+1}(2i-1)$ Substituting $n = k+1$

(iii) $\sum_{i=1}^{k}(2i-1) + 2k+1$ def^n of Σ

(iv) $k^2 + 2k + 1$ Induction hypothesis.

(v) $(k+1)^2$ algebra

(vi) n^2 Substitute $k+1 = n$

Conclusion: steps (i)-(vi) show that $\mathbb{P}$ is valid for $n = k+1$

Step 4: Induction Proof Conclusions

Because the base case is true, and Induction step is true, then $\mathbb{P}$ is true for all $n \in \mathbb{N}$.