Module 3
Probability

S = set of all possible outcomes

Theoretical Probability

$$p(e) = \frac{\text{\# of outcomes of event}}{\text{total possible outcome}} = \frac{|e|}{|S|}$$

Ex: tossing a coin $S = \{H, T\}$ $|S| = 2$

e = getting a head $\Rightarrow e = \{H\}$ $|e| = 1$

$P(H) = 1/2$

$S = \{1, 2, 3, 4, 5, 6\}$

e = getting a even number

$= \{2, 4, 6\}$ $|e| = 3$

$P(\text{even no}) = 3/6 = 1/2 = 0.5 = 50\%$

experimental

Empirical Probability

$$p(e) = \frac{\text{\# of outcomes of event}}{\text{total observed outcomes / trials}}$$

Ex: 1000 times $H = 500$ $\boxed{P(H)} = \frac{500}{1000} = 0.5 = 50\%$

- **Law of Large Numbers** no. of trials / sample size $\longrightarrow \infty$

empirical probability $\longrightarrow$ theoritical probab.

TABLE 11.4 Empirical Probabilities of Heads as the Number of Tosses Increases		
Number of Tosses	Number of Heads Observed	Empirical Probability of Heads, or $P(H)$
10	4	$P(H) = \frac{4}{10} = 0.4$
50	27	$P(H) = \frac{27}{50} = 0.54$
100	44	$P(H) = \frac{44}{100} = 0.44$
1000	530	$P(H) = \frac{530}{1000} = 0.53$
10,000	4851	$P(H) = \frac{4851}{10,000} = 0.4851$
100,000	49,880	$P(H) = \frac{49,880}{100,000} = 0.4988$

Axioms of Probability

① $P(e) \geq 0$ $\boxed{1 \geq P(e) \geq 0}$

② S = sample space

$P(S) = 1 = \frac{|S|}{|S|} = 1$

③ e_1 and e_2 are mutually exclusive $P(e_1 \cup e_2) = P(e_1) + P(e_2)$

Ex 1: Three men and three women line up at a checkout counter in a store.

a. In how many ways can they line up?

b. In how many ways can they line up if the first person in line is a woman, and then the line alternates by gender—that is a woman, a man, a woman, a man, and so on?

c. Find the probability that the first person in line is a woman and the line alternates by gender.

Permutation = Order matters

Combination = Order does not matter

$$\boxed{{}_nP_K = \frac{n!}{(n-K)!}}$$

(a) $_6P_6 = \frac{6!}{(6-6)!} = \frac{6!}{0!} = 6! = 720$ ways.

(b) $\underset{W}{3} \times \underset{M}{3} \times \underset{W}{2} \times \underset{M}{2} \times \underset{W}{1} \times \underset{M}{1} = 9 \times 4 = \boxed{36 \text{ ways}}$

$E =$ W M W M W M

(c) $P(\text{line up alternate genders}) = \frac{|E|}{|S|} = \frac{36}{720} = \frac{1}{20} \approx 0.05 = 5\%$

Ex 2:

Powerball is a multi-state lottery played in most U.S. states. It is the first lottery game to randomly draw numbers from two drums. The game is set up so that each player chooses five different numbers from 1 to 59 and one Powerball number from 1 to 35. Twice per week 5 white balls are drawn randomly from a drum with 59 white balls, numbered 1 to 59, and then one red Powerball is drawn randomly from a drum with 35 red balls, numbered 1 to 35. A player wins the jackpot by matching all five numbers drawn from the white balls in any order and matching the number on the red Powerball. With one $2 Powerball ticket, what is the probability of winning the jackpot?

$${}_nC_k = \frac{n!}{(n-k)!\,k!}$$

$$P(\text{winning Jackpot}) = \frac{1}{\binom{59}{5} \text{ and } \binom{35}{1}}$$

or (+)

$$= \frac{1}{\frac{59!}{(59-5)!\,5!} \times \frac{35!}{(35-1)! \times !}} \Rightarrow \frac{1}{\frac{59!}{54! \times 5!} \times \frac{35!}{34!}} = \boxed{\frac{1}{175{,}223{,}510}}$$

Ex 3:

CHECK POINT 3 A club consists of six men and four women. Three members are selected at random to attend a conference. Find the probability that the selected group consists of

a. three men.

b. two men and one woman.

(a) $P(3 \text{ men})$

event = 3 men $= {}_6C_3 = \dfrac{6!}{(6-3)!\,3!} = \dfrac{\cancel{6}^{2} \times 5 \times 4 \times \cancel{3!}}{\cancel{3!} \times \cancel{3} \times \cancel{2} \times 1}$

$= 20$ ways

$${}_{10}C_3 = \frac{10!}{7! \times 3!} = \frac{10 \times \cancel{9}^{3} \times \cancel{8}^{4} \times \cancel{7!}}{\cancel{7!} \times \cancel{3} \times \cancel{2} \times 1} = 120.$$

$$P(3 \text{ men}) = \frac{2\cancel{0}}{12\cancel{0}} = \boxed{\frac{1}{6}}$$

(b) $P(2M \text{ and } 1W) = \dfrac{{}_6C_2 \times {}_4C_1}{{}_{10}C_3}$

$$= \frac{\dfrac{6!}{4! \times 2!} \times \dfrac{4!}{3! \times 1!}}{120} = \frac{\dfrac{6 \times 5 \times \cancel{4}^{2} \times 3!}{\cancel{2} \times 1 \times 3!}}{120} = \frac{\cancel{6} \times \cancel{5} \times \cancel{2}}{\cancel{120}} = \boxed{1/2}$$

$4\ 2\cancel{40}\ 2$

Complement Rule for Events

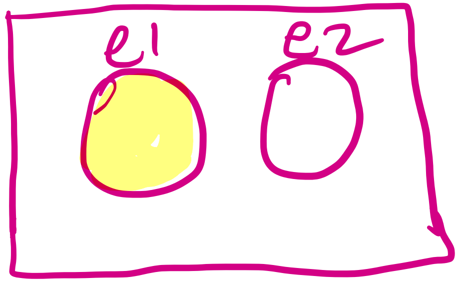
e   e' ← compliment of event e.

¬e

$$\boxed{P(e') = 1 - P(e)}$$

Mutually Exclusive: e_1 and e_2 are mutually exclusive / DISJOINT

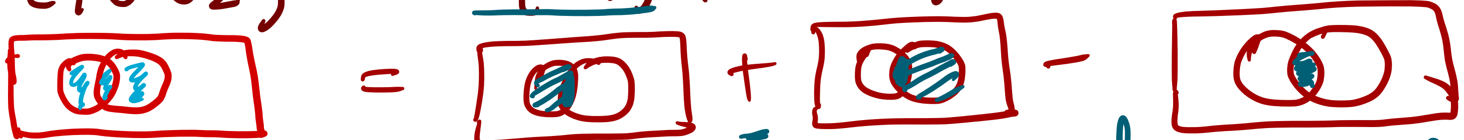
$$\boxed{P(e_1 \cap e_2) = 0}$$



Addition Rules

- **General Addition Rule:**

$$P(e_1 \cup e_2) = P(e_1) + P(e_2) - P(e_1 \cap e_2)$$



- **Simplified Addition Rule:** if event e_1 and e_2 are disjoint

$$\boxed{P(e_1 \cup e_2) = P(e_1) + P(e_2)}$$

Ex 4: The table below shows the soft drinks preferences of people in three age groups.

	cola	root beer	lemon-lime	
under 21 years of age	40	25	20	85
between 21 and 40	35	20	30	85
over 40 years of age	20	30	35	85
	95	75	85	255

a. Find the probability that a randomly selected person is over 40 and drinks root beer.

$$P(>40 \cap RB) = \frac{30}{255} \approx 0.12.$$

b. Find the probability that a randomly selected person is over 40 or drinks root beer.

$$P(>40 \text{ or } RB) = \frac{85}{255} + \frac{75}{255} - \frac{30}{255} = \frac{130}{255} \approx 0.51$$

$$P(>40 \cup RB) = P(>40) + P(RB) - P(>40 \cap RB)$$

Conditional Probability

$P(A|B)$

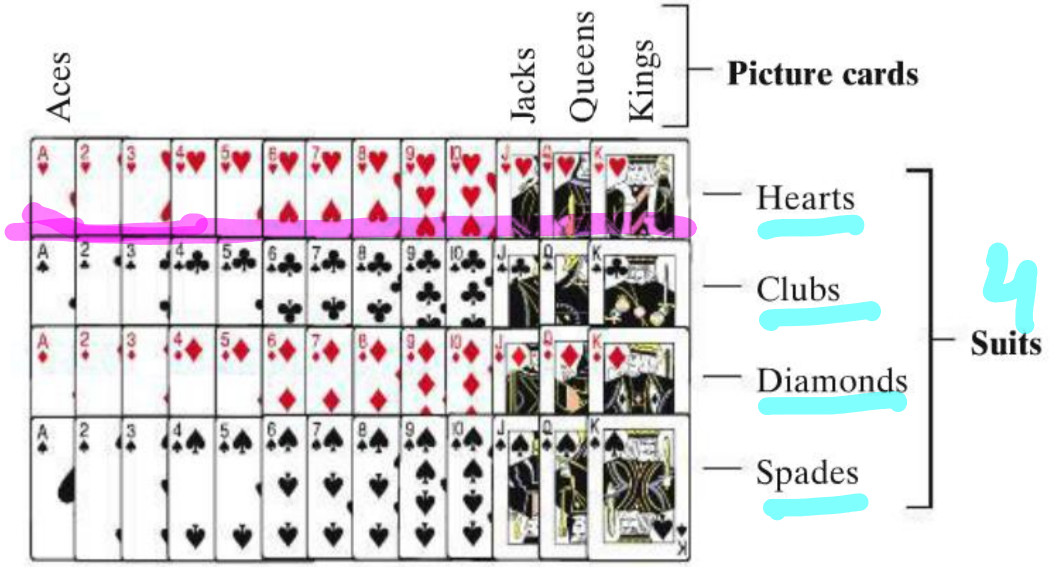


FIGURE 11.5 A standard 52-card bridge deck

w/o FORMULA

Ex 5: **Two cards are drawn from a standard deck of cards, without replacement.**

a. What is the probability of choosing a heart on the second draw, given that the first card is a club?

$P(\text{2nd } \heartsuit \mid \text{1st club}) = \frac{13}{51} \approx 0.25$

b. What is the probability of choosing a heart on the second draw, given that the first card is a heart?

$P(\text{2nd } \heartsuit \mid \text{1st } \heartsuit) = \frac{12}{51} \approx 0.24$

Conditional Probability

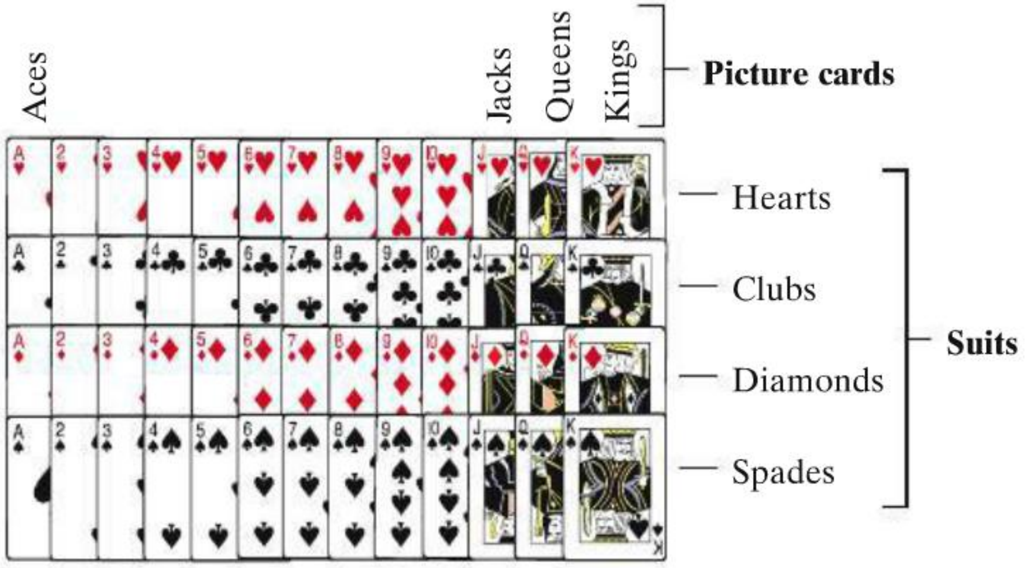


FIGURE 11.5 A standard 52-card bridge deck

WITH FORMULA

Ex 5: Two cards are drawn from a standard deck of cards, without replacement.

a. What is the probability of choosing a heart on the second draw, given that the first card is a club?

$$P(2^{nd}\,\heartsuit \mid 1^{st}\,\text{club}) = \frac{P(\heartsuit \cap \text{club})}{P(\text{club})} = \frac{P(\heartsuit)\;P(\text{club})}{P(\text{club})} = \frac{\frac{13}{51}\cdot\cancel{13/52}}{\cancel{13/52}} = \frac{13}{51} \approx 0.25$$

b. What is the probability of choosing a heart on the second draw, given that the first card is a heart?

$$P(2^{nd}\,\heartsuit \mid 1^{st}\,\heartsuit) = \frac{P(2^{nd}\,\heartsuit \cap 1^{st}\,\heartsuit)}{P(1^{st}\,\heartsuit)} = \frac{12/51\cdot\cancel{13/52}}{\cancel{13/52}} = \frac{12}{51} \approx 0.24.$$

Conditional Probability Formula

$$P(A|B) = \boxed{\frac{P(A \cap B)}{P(B)}}$$

given | conditioned on

$P(B|A) = P(B \cap A) / P(A)$

Ex 6: A university cafeteria surveyed the students for their coffee preferences. The findings are summarized as follows:

	Do not Drink Coffee	Prefer Regular Coffee	Prefer Decaf Coffee	Total
Female	23	145	69	237 F
Male	18	196	46	260 M
Total	41	341	115	497

What is the probability that a randomly selected student:

a) Does not drink coffee, given that he is male?

b) Is female given she prefers regular coffee?

a) $P(\text{No Coffee} \mid M) = \frac{18}{260} \approx 0.069$

$P(\text{No Coffee} \mid M) = \frac{P(\text{no coffee} \cap M)}{P(M)} = \frac{18/497}{260/497} = 18/260$

b) $P(F \mid \text{Prefers RC}) = 145/341 \approx 0.43$

$\frac{P(F \cap \text{Preference RC})}{P(\text{Prefer RC})} = \frac{145/497}{341/497} = 145/341$

D A

<u>Ex 7</u>: At a certain airport, 89% of flights depart on time, and 87% arrive on time. The probability that a flight both departs and arrives on time is 83%.

$P(D) = 89\% = 0.89$ $P(D \cap A) = 83\% = 0.83$

$P(A) = 87\% = 0.87$

Find the probability that a flight arrives on time, <u>given</u> that it departed on time.

$$P(A \mid D) = \frac{P(A \cap D)}{P(D)} = \frac{0.83}{0.89} \approx 0.93.$$

Discussion on Bayes' Theorem The Probability of a hypothesis H given a new piece of evidence E.

$$\boxed{P(H|E) = \frac{P(E|H)\,P(H)}{P(E)}}$$

Bays Rate fallacy: $P(H|E) \neq P(E|H)$

H: You have a very rare disease

E: Symptoms of this rare disease

$P(H) = 0.0001$

$P(E|H) = 0.95$

$P(E) = 0.01$

given

$$P(H|E) = \frac{P(E|H)\,P(H)}{P(E)}$$

$$= \frac{0.95 \times 0.0001}{0.01}$$

$$= 0.00095$$

Independent Events: The Probability that one event occurs in no way affects the probability of another event occuring.

Dependent Events: The probability that one event occurs affects the probability of another event occuring.

Multiplication Rules $P(A) \times P(B|A) = \frac{P(A \cap B)}{\cancel{P(A)}} \cancel{\times P(A)}$

General Multiplication Rule	Simplified Multiplication Rule when event A and B are independant
$\boxed{P(A \cap B) = P(A) \times P(B \mid A)}$	$P(A \cap B) = P(A) \cdot P(B)$ A & B are independant $P(A \mid B) = P(A)$ $P(B \mid A) = P(B)$

Ex 8: The probability that a person in the U.S. has type A^+ blood is 31%. Suppose that three (unrelated) people are selected at random.

a. What is the probability that all three have type A^+ blood?

$$P(A^+ \cap A^+ \cap A^+) = P(A^+) \cdot P(A^+) \cdot P(A^+) = 0.31 \times 0.31 \times 0.31 = \boxed{0.0298}$$

b. What is the probability that none of the three have type A^+ blood?

$$P(\overline{A^+} \cap \overline{A^+} \cap \overline{A^+}) = P(\overline{A^+}) \cdot P(\overline{A^+}) \cdot P(\overline{A^+})$$
$$= (1 - P(A^+))^3 = (1 - 0.31)^3 \approx \boxed{0.3285}$$

c. What is the probability that at least one of the individuals has type A^+ blood?

1 or 2 or 3

$$P(\text{at least one } A^+) = 1 - P(\text{none } A^+)$$
$$= 1 - 0.3285 = \boxed{0.6715}$$

<u>Ex 9</u>: A student goes to the library. Define events:

B **= the student checks out a book, and**

D **= the student checks out a DVD.**

Suppose that P(*B*) = 0.40, P(*D*) = 0.30, and P(*B* and *D*) = 0.20. Find:

a. P(*B* or *D*) $= P(B) + P(D) - P(B \cap D) = 0.4 + 0.3 - 0.2 = \boxed{0.5}$

a. Find P(*B* | *D*). $\dfrac{P(B \cap D)}{P(D)} = 0.2/0.3 = \boxed{2/3 \approx 0.67}$

c. Find P(*D* | *B*). $\dfrac{P(D \cap B)}{P(B)} = \dfrac{0.2}{0.4} = 1/2 = \boxed{0.5}$

d. Are *B* and *D* independent? Explain.

$P(B \cap D) \stackrel{?}{=} P(B) \cdot P(D)$

$0.2 \stackrel{?}{=} 0.4 \times 0.3$

$0.2 \neq 0.12$ $\boxed{\text{NOT INDEPENDENT}}$

e. Are *B* and *D* mutually exclusive? Explain.

$P(B \cap D) \neq 0$ not mutually exclusive

Odds

$$\text{Odds in favor} = \frac{\#\text{ of ways } e \text{ can occur}}{\#\text{ of ways } e \text{ cannot occur}}.$$

$$\text{odds againt} = \frac{\#\text{ of ways } e \text{ cannot occur}}{\#\text{ of ways } e \text{ can occur}}.$$

Ex 10:

CHECK POINT 7 You are dealt one card from a 52-card deck.

a. Find the odds in favor of getting a red queen.

b. Find the odds against getting a red queen.

(a) odds in favor $= \dfrac{2}{50}$

(b) odds against $= \dfrac{50}{2}$

Ex 11: CHECK POINT 9 The odds against a particular horse winning a race are 15 to 1. Find the odds in favor of the horse winning the race and the probability of the horse winning the race.

odds against $= \frac{15}{1}$

e can occur

odds in favor $= 1/15$

e cannot occur

total $= 1+15$

$P(\text{horse winning}) = \frac{1}{16} = 0.0625$

Discrete Random Variable

A variable that can take only countable # of values. * Probability of each value is between 0 and 1
* sum of all probabilities is 1.

Expected Value

(mean) Sum of all possible values weighted by their probabilities.

$$\boxed{E(x) = \Sigma\,(x \cdot p(x))}$$

Ex 12: Suppose that a fair coin is tossed three times in succession. List the outcomes in the sample space as sequences of *H*'s and *T*'s. Let *X* = the number of heads in three tosses. Find the expected number of heads in three tosses.

HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

X	P(x)	X·P(x)
0	0.125	0
1	0.375	0.375
2	0.375	0.75
3	0.125	0.375

$$E(x) = \sum x \cdot P(x) = 0 + 0.375 + 0.75 + 0.375$$

$$\boxed{E(X) = 1.5}$$

<u>Ex 13</u>: **You are playing a game of chance in which four cards are drawn from a standard deck of 52 cards. You guess the suit of each card before it is drawn. The cards are replaced in the deck on each draw. You pay $1 to play. If you guess the right suit very time, you get your money back and $256. What is your expected profit of playing the game over the long term?**

	x	$P(x)$	$x\,P(x)$
Win	$256	$\left(\frac{1}{4}\right)^4 = \frac{1}{256}$	1
lose	-$1	$1 - \frac{1}{256} = \frac{255}{256}$	$-\frac{255}{256}$

$$E(\text{Profit}) = \sum x\,P(x) = 1 + \left(-\frac{255}{256}\right) = \frac{1}{256} = \boxed{\$0.0039}$$

Ex 14:

An automobile insurance company has determined the probabilities for various claim amounts for drivers ages 16 through 21, shown in **Table 11.10**.

a. Calculate the expected value and describe what this means in practical terms.

$E(x) = \$1100$

b. How much should the company charge as an average premium so that it does not lose or gain money on its claim costs?

$\boxed{\$1100}$

TABLE 11.10 Probabilities for Auto Claims

Amount of Claim (to the nearest $2000)	Probability	$x \cdot P(x)$
$0	0.70	= 0
$2000	0.15	= 300
$4000	0.08	= 320
$6000	0.05	= 300
$8000	0.01	= 80
$10,000	0.01	= 100
		1100