Module 4

Propositional Logic

Boolean Values: T, F

Variables: p, q, r, s

P: means Prove:

Operators: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$

"NOT p"

p	$\neg p$
T	F
F	T

"P AND q"

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

"P OR q"

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

"If P then q"

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

"p If and only If q"

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

If you got an A, then you Pass
p q

Proposition: A single logical statement composed of logical values, variables and operators.

Tautology: A proposition that is ALWAYS TRUE

Contradiction: A proposition that is ALWAYS FALSE

Truth Table: lists the value of a proposition resulting from all possible assignments of T and F to each variable.

Logical Equivalence: '$\equiv$' eg: $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$

they reduce to the same thruth table for all possible arguments of True and false for every variable.

Ex 1: **Construct a truth table for the given statement.**

a. $\neg(p \lor q)$

p	q	p∨q	¬(p∨q)
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

( ) ↔ ( )

( ) → ( )

∧ ∨

¬

b. $(\neg p) \land (\neg q)$

p	q	¬p	¬q	(¬p)∧(¬q)
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

same, $\neg(p \lor q) \equiv \neg p \land \neg q$

c. $(p \land \neg q) \lor \neg p$

p	q	¬q	p∧¬q	¬p	(p∧¬q)∨¬p
T	T	F	F	F	F
T	F	T	T	F	T
F	T	F	F	T	T
F	F	T	F	T	T

d. $p \lor \neg p$

p	¬p	p∨¬p
T	F	T
F	T	T

(Tautology)

e. $p \land \neg p$

p	¬p	p∧¬p
T	F	F
F	T	F

(Contradition)

Ex 2: I study hard, and I ace the final or fail the course.

a. Write the statement in symbolic form. Assign letters to simple statements that are not negated.

p: I study hard

q: I ace the final

r: I fail course

$\boxed{p \wedge (q \vee r)}$

b. Construct a truth table for the symbolic statement in part a.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

c. Determine the truth value for the statement when p is false, q is true, and r is true.

F T T

$\boxed{\text{False}}$

$\Leftrightarrow$ $\equiv$ logically equivalent

The Laws of Logic

For any primitive statements p, q, r, and any tautology T_o, and any contradiction F_o.

$\neg\neg p \Leftrightarrow p$	Law of Double Negation
$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	DeMorgan's Laws
$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	Commutative Laws
$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$	Associative Laws
$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	Distributive Laws
$p \vee p \Leftrightarrow \mathrm{p}$ $p \wedge p \Leftrightarrow \mathrm{p}$	Idempotent Laws
$p \vee F_o \Leftrightarrow \mathrm{p}$ $p \wedge T_o \Leftrightarrow \mathrm{p}$	Identity Laws
$p \vee \neg p \Leftrightarrow T_o$ $p \wedge \neg p \Leftrightarrow F_o$	Inverse Laws
$p \vee T_o \Leftrightarrow T_o$ $p \wedge F_o \Leftrightarrow F_o$	Domination Laws
$p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$	Absorption Laws

Other Laws

$p \rightarrow q \Leftrightarrow \neg p \vee q$ (conditional disjunction)

$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$ (contrapositive law)

Inference Operator

If then
conditional

TABLE 3.5 Common English Expressions for		
Symbolic Statement	**English Statement**	**Example** ***p*: A person is a father.** ***q*: A person is a male.**
$p \to q$	If p then q.	If a person is a father, then that person is a male.
$p \to q$	q if p.	A person is a male if that person is a father.
$p \to q$	p is sufficient for q.	Being a father is sufficient for being a male.
$p \to q$	q is necessary for p.	Being a male is necessary for being a father.
$p \to q$	p only if q.	A person is a father only if that person is a male.
$p \to q$	Only if q, p.	Only if a person is a male is that person a father.

Ex 3: Construct a truth table and show it is a tautology.

$$[(p \to q) \wedge \neg q] \to \neg p$$

p	q	$p \to q$	$\neg q$	$(p\to q)\wedge\neg q$	$\neg p$	$[(p\to q)\wedge \neg q] \to \neg p$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

$[(p\to q)\wedge\neg q] \to \neg p$ is a tautology.

<u>Ex 4</u>: An advertisement makes the following claim:

$p \qquad \wedge \qquad q \qquad \rightarrow$

If you use Hair Grow and apply it daily, then you will not go bald.

p : You use Hair grow
q : You apply it daily
r : You do not go bald

a. Construct a truth table for the claim.

$(p \wedge q) \rightarrow \neg r$

p	q	r	$p \wedge q$	$\neg r$	$(p \wedge q) \rightarrow \neg r$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	F	F	T
T	F	F	F	T	T
F	T	T	F	F	T
F	T	F	F	T	T
F	F	T	F	F	T
F	F	F	F	T	T

b. Suppose you use Hair Grow, forget to apply it every day, and you go bald. Under these conditions, is the ad false?

p	q	r
T	F	T

$\rightarrow$ T

No, ad is true!

Ex 5: a) P: $p \vee q \equiv \neg q \rightarrow p$

p	q	$p \vee q$	$\neg q$	p	$\neg q \rightarrow p$
T	T	T	F	T	T
T	F	T	T	T	T
F	T	T	F	F	T
F	F	F	T	F	F

Same

Therefore, $p \vee q \equiv \neg q \rightarrow p$.

b) Use the result from part (a) to write a statement that is equivalent to:

I attend classes or I lose my scholarship.

$\underbrace{p \quad \vee \quad q}$ $\equiv \neg q \rightarrow p$

I do not loose my scholarship then I attend classes.

Ex 6: Given P: $(p \to q) \wedge (p \vee q) \equiv q$

a) Prove P by truth table

p	q	$p \to q$	$p \vee q$	$(p \to q) \wedge (p \vee q)$
T	T	T	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	F

same

$\therefore (p \to q) \wedge (p \vee q) \equiv q$

b) Prove P by using the Laws of Logic

$(p \to q) \wedge (p \vee q)$

(i) $(\neg p \vee q) \wedge (p \vee q)$ Conditional disjunction.

(ii) $q \vee (\neg p \wedge p)$ distribution law.

(iii) $q \vee F$ Inverse law

(iv) q Identity law.

using steps (i) - (iv) we have prooved $(p \to q) \wedge (p \vee q) \equiv q$.

Ex 7: Write the negation of the conditional statement.

If you do not have a fever, then you do not have the flu.

$\neg(\neg p \to \neg q)$

p	q	$\neg p$	$\neg q$	$\neg p \to \neg q$	$\neg(\neg p \to \neg q)$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	F

p : You have fever

q : You have flue

Rules of Inference

Premises

Conclusion

Rule of Inference	Name of Rule
p $p \to q$ $\therefore\ q$	Rule of Detachment (Modus Ponens)
$p \to q$ $q \to r$ $\therefore\ p \to r$	Law of the Syllogism
$p \to q$ $\neg q$ $\therefore\ \neg p$	Modus Tollens
p q $\therefore\ p \wedge q$	Rule of Conjunction
$p \vee q$ $\neg p$ $\therefore\ q$	Rule of Disjunctive Syllogism
$\neg p \to F_o$ $\therefore\ p$	Rule of Contradiction
$p \wedge q$ $\therefore\ p$	Rule of Conjunctive Simplification
p $\therefore\ p \vee q$	Rule of Disjunctive Amplification
$p \wedge q$ $p \to (q \to r)$ $\therefore\ r$	Rule of Conditional Proof
$p \to r$ $q \to r$ $\therefore\ (p \vee q) \to r$	Rule for Proof by Cases
$p \to q$ $r \to s$ $p \vee r$ $\therefore\ q \vee s$	Rule of the Constructive Dilemma
$p \to q$ $r \to s$ $\neg q \vee \neg s$ $\therefore\ \neg p \vee \neg r$	Rule of the Destructive Dilemma

Arguments

TESTING THE VALIDITY OF AN ARGUMENT WITH A TRUTH TABLE

1. Use a letter to represent each simple statement in the argument.
2. Express the premises and the conclusion symbolically.
3. Write a symbolic conditional statement of the form

$$[\text{(premise 1)} \wedge \text{(premise 2)} \wedge \cdots \wedge \text{(premise } n)] \rightarrow \text{conclusion},$$

   where n is the number of premises.
4. Construct a truth table for the conditional statement in step 3.
5. If the final column of the truth table has all trues, the conditional statement is a tautology and the argument is valid. If the final column does not have all trues, the conditional statement is not a tautology and the argument is invalid.

Ex 8: Use a truth table to demonstrate the validity of the following Rules of Inference:

a. **Modus Ponens**

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore \quad q \end{array}$$

$[p \wedge (p \rightarrow q)] \rightarrow q$

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

tautology.

$\therefore$ the statement is valid.

b. Rule of Conditional Proof

$p \wedge q$

$p \rightarrow (q \rightarrow r)$

$\therefore \quad r$

$[(p \wedge q) \wedge (p \rightarrow (q \rightarrow r))] \longrightarrow r$

p	q	r	$p \wedge q$	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$(p \wedge q) \wedge (p \rightarrow (q \rightarrow r))$	$[(p \wedge q) \wedge (p \rightarrow (q \rightarrow r))] \rightarrow r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	T	F	T
F	T	T	F	T	T	F	T
F	T	F	F	F	T	F	T
F	F	T	F	T	T	F	T
F	F	F	F	F	T	F	T

tautology.

Rule of Conditional Proof is valid.

Logical Quantifiers

$\forall$: for all $\exists$: there exists

TABLE 3.1 Equivalent Ways of Expressing Quantified Statements

Statement	An Equivalent Way to Express the Statement	Example (Two Equivalent Quantified Statements)
All *A* are *B*.	There are no *A* that are not *B*.	All poets are writers. There are no poets that are not writers.
Some *A* are *B*.	There exists at least one *A* that is a *B*.	Some people are bigots. At least one person is a bigot.
No *A* are *B*.	All *A* are not *B*.	No common colds are fatal. All common colds are not fatal.
Some *A* are not *B*.	Not all *A* are *B*.	Some students do not work hard. Not all students work hard.

TABLE 3.2 Negations of Quantified Statements

Statement	Negation	Example (A Quantified Statement and Its Negation)
All *A* are *B*.	Some *A* are not *B*.	All people take exams honestly. Negation: Some people do not take exams honestly.
Some *A* are *B*.	No *A* are *B*.	Some roads are open. Negation: No roads are open.

TABLE 3.3 Examples of Negations of Quantified Statements

Statement	Negation
All humans are mortal.	Some humans are not mortal.
Some students do not come to class prepared.	All students come to class prepared.
Some psychotherapists are in therapy.	No psychotherapists are in therapy.
No well-behaved dogs shred couches.	Some well-behaved dogs shred couches.

Ex 9:

CHECK POINT 4 The board of supervisors told us, "All new tax dollars will be used to improve education." I later learned that the board of supervisors never tells the truth. What can I conclude? Express the conclusion in two equivalent ways.

① Some new tax dollars will not be used to improve education.

② Not all tax dollars will be used to improve education

③ There exists at least one new tax dollar that will not be used to improve education.