$$(x^2+4)\,y' + xy = 2x$$

Standard form:

$$\boxed{y' + \frac{x}{x^2+4}\,y = \frac{2x}{x^2+4}}$$

Integrating factor

$$g(x) = e^{\int \frac{x}{x^2+4}\,dx} =$$

$u = x^2+4$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$$= e^{\frac{1}{2}\ln|x^2+4|} = (x^2+4)^{1/2}$$

$$\int \frac{\left(y(x^2+4)^{1/2}\right)'}{dx} = \int \frac{2x}{(x^2+4)^{1/2}}\,dx$$

$u = x^2+4$
$du = 2x\,dx$
$\int u^{-1/2}\,du$

$$y(x^2+4)^{1/2} = 2(x^2+4)^{1/2} + C$$

$$\boxed{y = 2 + \frac{C}{(x^2+4)^{1/2}}}$$

$G(ax+by)$

$$y' = \sqrt{8x+y} - 8$$

$$z = 8x+y$$

$$z' = 8 + y'$$

$$y' = z' - 8$$

$$z' - \cancel{8} = \sqrt{z} \cancel{-8}$$

$$\frac{dz}{dx} = z^{1/2}$$

$$\frac{dz}{z^{1/2}} = dx$$

$$2z^{1/2} = x + c$$

$$\boxed{2\sqrt{8x+y} = x + c}$$

$$y' + \frac{y}{x} = 4x^2 y^2$$

$$y^{-2} y' + \frac{1}{x} y^{-1} = 4x^2$$

$$z = y^{-1}$$

$$\frac{dz}{dx} = -y^{-2} y'$$

$$-z' + \frac{1}{x} z = 4x^2$$

$$z' - \frac{1}{x} z = -4x^2$$

$$g(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln|x|} = \frac{1}{x}$$

$$\left(\frac{1}{x} z\right)' = -4x$$

$$\frac{1}{x} z = -2x^2 + C$$

$$z = -2x^3 + Cx$$

$$\boxed{y^{-1} = -2x^3 + Cx}$$

Second-Order Linear ODEs

A second-order ODE is called **linear** if it can be written

$$y'' + p(x)y' + q(x)y = r(x) \tag{1}$$

and **nonlinear** if it cannot be written in this form.

The distinctive feature of this equation is that it is *linear in* y *and its derivatives,* whereas the functions p, q, and r on the right may be any given functions of x. If the equation begins with, say, $f(x)y''$, then divide by $f(x)$ to have the **standard form** (1) with y'' as the first term.

A second-order ODE is called **linear** if it can be written

$$y'' + p(x)y' + q(x)y = r(x) \tag{1}$$

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

and is called **homogeneous**. If $r(x) \not\equiv 0$, then (1) is called **nonhomogeneous**

This nomenclature is not related to the way we used the term for first-order equations

$$y'' + p(x)y' + q(x)y = r(x) \tag{1}$$

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

The functions p and q in (1) and (2) are called the **coefficients** of the ODEs.

Solutions are defined similarly as for first-order ODEs in Chap. 1. A function

$$y = h(x)$$

is called a *solution* of a (linear or nonlinear) second-order ODE on some open interval I if h is defined and twice differentiable throughout that interval and is such that the ODE

$\frac{dv}{dt} = \ldots \quad v = \frac{dx}{dt}$

Homogeneous Linear ODEs: Superposition Principle

homogeneous

Linear ODEs have a rich solution structure. For the homogeneous equation the backbone of this structure is the *superposition principle* or *linearity principle,* which says that we can obtain further solutions from given ones by adding them or by multiplying them with any constants. Of course, this is a great advantage of homogeneous linear ODEs. Let us first discuss an example.

The functions $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE

$$y'' + y = 0$$

$y_1 = \cos x$

$y_1' = -\sin x$

$y_1'' = -\cos x$

$-\cos x + \cos x = 0$

$0 = 0$

$y_2 = \sin x$

$y_2' = \cos x$

$y_2'' = -\sin x$

$-\sin x + \sin x = 0$

$0 = 0$

$$y = 4.7 \cos x - 2 \sin x$$

$$(4.7 \cos x - 2 \sin x)'' + (4.7 \cos x - 2 \sin x) = 0$$

$$(-4.7 \cos x + 2 \sin x) + (4.7 \cos x - 2 \sin x) = 0$$

$$0 = 0$$

$$\boxed{y = C_1 y_1 + C_2 y_2}$$

Fundamental Theorem for the Homogeneous Linear ODE (2)

For a homogeneous linear ODE (2), *any linear combination of two solutions on an open interval I is again a solution of* (2) *on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.*

CAUTION! Donît forget that this highly important theorem holds for *homogeneous linear* ODEs only but ***does not hold*** for nonhomogeneous linear or nonlinear ODEs,

2nd order linear differential equation without source (homogeneous linear differential equation) can be always written as y'' + P(x)y' + Q(x)y = 0 where ' means differentiation. Full solution is obtained by solving this equation with initial conditions (equation describes evolution of the system from initial conditions). If we know, y(0) and y'(0) then the equation tells y''(0). These initial conditions gives next time values y(t1), y'(t1) and substituting these into the equation tells y''(t1). Next time values are obtained in similar process so full solution is obtained.

As coefficients of independent solutions are rooms for initial conditions and we need only two conditions to fully describe system evolution thus 2nd order homogeneous differential equation only requires two independent solutions as constituents of general solutions.

Initial Value Problem. Basis. General Solution

For a second-order homogeneous linear ODE (2) an **initial value problem** consists of (2) and two **initial conditions**

(4)
$$y(x_0) = K_0, \qquad y'(x_0) = K_1.$$

These conditions prescribe given values K_0 and K_1 of the solution and its first derivative (the slope of its curve) at the same given $x = x_0$ in the open interval considered.

The conditions (4) are used to determine the two arbitrary constants c_1 and c_2 in a **general solution**

(5)
$$y = c_1 y_1 + c_2 y_2$$

$$y' = c_1 y_1' + c_2 y_2'$$

of the ODE; here, y_1 and y_2 are suitable solutions of the ODE, with ìsuitableñ to be explained after the next example. This results in a unique solution, passing through the point (x_0, K_0) with K_1 as the tangent direction (the slope) at that point. That solution is called a **particular solution** of the ODE (2).

$V(t)$ $\quad$ $x(t)$

$x = V'$

Solve the initial value problem

$$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5.$$

General solution: $y = A\cos x + B\sin x$

IC: $y(0) = 3$ $y'(0) = -\frac{1}{2}$

$3 = A\underbrace{\cos 0}_{=1} + B\underbrace{\sin 0}_{=0}$ $A = 3$

$y' = -A\sin x + B\cos x$

$-\frac{1}{2} = -A\underbrace{\sin 0}_{=0} + B\underbrace{\cos 0}_{=1}$ $B = -\frac{1}{2}$

Particular Solution $\boxed{y = 3\cos x - \frac{1}{2}\sin x}$