Example 3 Solve

(15) $$(1 + e^x y + xe^x y)\,dx + (xe^x + 2)\,dy = 0.$$

M N

$N = \frac{\partial f}{\partial y} = xe^x + 2$

$\frac{\partial M}{\partial y} = e^x + xe^x$

$\frac{\partial N}{\partial x} = e^x + xe^x$ (=)

$f^* = \int N\,dy = xye^x + 2y + h(x)$

Compare $\frac{\partial f^*}{\partial x}$ with $M = \frac{\partial f}{\partial x}$

$\frac{\partial f^*}{\partial x} = ye^x + xye^x + h'(x)$

$M = \frac{\partial f}{\partial x} = 1 + e^x y + xe^x y$

$h'(x) = 1$

$h(x) = x$

$f(x,y) = xye^x + 2y + x$

$$xye^x + 2y + x = C$$

$$y' - \frac{1}{x}y = 9x\,e^x \qquad y(1) = 9e + 2$$

$P(x)$

$$g(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\ln|x|} = \frac{1}{x}$$

$$\int \left(\frac{1}{x}y\right)' dx = \int 9\frac{x}{x}e^x\,dx$$

$$\frac{1}{x}y = 9e^x + C$$

$$y = 9xe^x + Cx \quad \leftarrow \text{General Solution}$$

IC: $y(1) = 9e + 2$

$$9e + 2 = 9e + C$$

$$C = 2$$

$$y = 9xe^x + 2x$$

$$e^t(9y - 7t)\,dt + (8 + 9e^t)\,dy = 0$$

$M = \frac{\partial f}{\partial t}$ $\qquad$ $N = \frac{\partial f}{\partial y}$

$\frac{\partial M}{\partial y} = 9e^t$ (=)

$\frac{\partial N}{\partial t} = 9e^t$

$$\int M\,dt = \int 9ye^t\,dt + 7\int te^t\,dt =$$

$u = t \quad dv = e^t dt$

$du = dt \quad v = e^t$

$$\int N\,dy = 8y + 9ye^t + h(t)$$

Compare

$$\int M\,dt = 9ye^t - 7(te^t - e^t) + g(y)$$

$$8y + 9ye^t - 7te^t + 7e^t = C$$

$$e^t(9y - 7t)\,dt + (8 + 9e^t)\,dy = 0$$

$M = \frac{\partial f}{\partial t}$ $\qquad$ $N = \frac{\partial f}{\partial y}$

$$f^* = \int N\,dy = 8y + 9ye^t + h(t)$$

compare $\frac{\partial f^*}{\partial t}$ with actual $M = \frac{\partial f}{\partial t}$

$$\frac{\partial f^*}{\partial t} = 9ye^t + h'(t)$$

compare

$$M = \frac{\partial f}{\partial t} = 9ye^t - 7te^t$$

$$h'(t) = -7te^t$$

$$h(t) = \int h'(t)\,dt = \int -7te^t\,dt$$

$u = t \quad dv = e^t dt$

$du = dt \quad v = e^t$

$$= -7(te^t - e^t)$$

$$8y + 9ye^t - 7te^t + 7e^t = C$$

$$y' + 8y = 2e^{-5x}$$

$$g(x) = e^{\int 8\,dx} = e^{8x}$$

$$\int \left(ye^{8x}\right)' dx = \int 2e^{3x}\,dx$$

$$ye^{8x} = \frac{2}{3}e^{3x} + c$$

$$y = \frac{2}{3}e^{-5x} + ce^{-8x}$$