MA345 Differential Equations & Matrix Method

Professor Berezovski
COAS.301.12

MODULE I - 1ST ORDER ODE

Week 1: 1st order ODE: Assignment: Homework 1

1.1 Background

- 1.2 Solutions and Initial Value Problems
- 2.2 Separable Equations

Week 2: 1st order ODE: Assignment: Homework 2

- 2.3 Linear Equations
- 2.4 Exact Equations

Week 3: 1st order ODE: Substitutions / summary Assignment: Homework 3

- 2.5 Special Integrating Factors
- 2.6 Substitutions and Transformations

MODULE II - 2ND ORDER LINEAR ODE

Week 5: 2nd order linear ODE: Characteristic equation Assignment: Homework 4

- 4.2 Homogeneous Linear Equations: The General Solution
- 4.3 Characteristic Equations with Complex Roots
- 6.2 Higher Order Homogeneous Linear Equations with Constant Coefficients

Week 6: 2nd order linear ODE: Undetermined Coefficients Assignment: Homework 5

- 4.4 Nonhomogeneous Equations: The Method of Undetermined Coefficients
- 4.5 The Superposition Principle and Undetermined Coefficients Revisited
- 6.3 Undetermined Coefficients and the Annihilator Method

Week 7: 2nd order linear ODE: Variation of Parameters Assignment: Homework 6

- 4.6 Variation of Parameters
- 6.4 Method of Variation of Parameters

QUIZ 2

Method for Solving Separable Equations

To solve the equation

$$\frac{dy}{dx} = g(x)p(y) \tag{2}$$

multiply by dx and by $h(y) := 1/p(y)$ to obtain

$$h(y)\,dy = g(x)\,dx\,.$$

Then integrate both sides:

$$\int h(y)\,dy = \int g(x)\,dx\,,$$

$$H(y) = G(x) + C\,, \tag{3}$$

where we have merged the two constants of integration into a single symbol C. The last equation gives an implicit solution to the differential equation.

Solve the initial-value problem $\frac{dy}{dx} = -\frac{x}{y}$, $y(4) = -3$.

$$\int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$y^2 = -x^2 + 2C$$

general solution

explicit

$$y = \pm\sqrt{2C - x^2}$$

implicit

$$x^2 + y^2 = 2C$$

IC: $y(4) = -3$

$x = 4 \quad y = -3$

$$16 + 9 = 2C$$

$$25 = 2C$$

Particular Solution: $x^2 + y^2 = 25$

$$y = \pm\sqrt{25 - x^2}$$

EXAMPLE 3 **Initial Value Problem (IVP). Bell-Shaped Curve**

Solve $y' = -2xy$, $y(0) = 1.8$.

$$\frac{dy}{dx} = -2xy$$

$$e^{\ln y} = y$$

$$\int \frac{dy}{y} = \int -2x\,dx$$

$$\ln|y| = -x^2 + C$$

$$e^{\ln|y|} = e^{-x^2+C} = e^{-x^2} \cdot e^{C} = Ae^{-x^2}$$

$$y = Ae^{-x^2}$$ → General Solution

IC = $y(0) = 1.8$ $\quad 1{,}8 = A \cdot e^{0}$ $\quad A = 1{,}8$

Particular Solution: $\boxed{y = 1{,}8\, e^{-x^2}}$

Example 3 Solve the nonlinear equation

$$\frac{dy}{dx} = \frac{6x^5 - 2x + 1}{\cos y + e^y}. \qquad (9)$$

$$\int (\cos y + e^y)\,dy = \int (6x^5 - 2x + 1)\,dx$$

$$\sin y + e^y = x^6 - x^2 + x + C$$

$$\frac{dy}{dx} = y^2 e^{-x}$$

$y \neq 0$

$$\int \frac{dy}{y^2} = \int e^{-x} dx$$

$y = 0$

Singular solution

$$-\frac{1}{y} = -e^{-x} + c$$

$$y = \frac{1}{e^{-x} + c}$$

Linear Equations

A type of first-order differential equation that occurs frequently in applications is the linear equation. Recall from Section 1.1 that a **linear first-order equation** is an equation that can be expressed in the form

(1) $$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)\ ,$$

where $a_1(x)$, $a_0(x)$, and $b(x)$ depend only on the independent variable x, not on y.

Linear Equations

A type of first-order differential equation that occurs frequently in applications is the linear equation. Recall from Section 1.1 that a **linear first-order equation** is an equation that can be expressed in the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)\,, \tag{1}$$

where $a_1(x)$, $a_0(x)$, and $b(x)$ depend only on the independent variable x, not on y.

One can seldom rewrite a linear differential equation so that it reduces to a form as simple as (2). However, the form (3) can be achieved through multiplication of the original equation (1) by a well-chosen function $\mu(x)$. Such a function $\mu(x)$ is then called an "integrating fae tor" for equation (1). The easiest way to see this is first to divide the original equation (1) by $a_1(x)$ and put it into **standard form**

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)\,,} \tag{4}$$

where $P(x) = a_0(x)/a_1(x)$ and $Q(x) = b(x)/a_1(x)$.

$$y' + y = x$$

Integrating factor

$$g(x) = e^x$$

$$e^x y' + e^x y = xe^x$$

$$(uv)' = u'v + uv'$$

$$(e^x y)' = xe^x$$

Integration by parts

$$\int (e^x y)'\,dx = \int x e^x\,dx$$

$$\int u\,dv = uv - \int v\,du$$

$u = x$ $\quad dv = e^x\,dx$

$du = dx$ $\quad v = e^x$

$$e^x y = xe^x - e^x + C$$

$$\boxed{y = x - 1 + \frac{C}{e^x}}$$

Standard form:

$$y' + p(x)y = q(x)$$

Integrating Factor

$$\boxed{g(x) = e^{\int p(x)dx}}$$

$$g'(x) = p(x)e^{\int p(x)dx} = p(x)g(x)$$

$$g(x)y' + p(x)\cdot g(x)\cdot y = g(x)\cdot q(x)$$

$$\underbrace{g(x)y' + g'(x)y}_{} = g(x)\cdot q(x)$$

$$\int \big(g(x)\,y\big)'dx = \int g(x)q(x)dx$$

$$g(x)\,y = \int g(x)q(x)dx$$

$$\boxed{y = \frac{1}{g(x)}\int g(x)q(x)dx}$$