MA345 Differential Equations & Matrix Method

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COAS.301.12

MODULE I - 1ST ORDER ODE

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MODULE II - 2ND ORDER LINEAR ODE

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- 4.6 Variation of Parameters
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QUIZ 2

Method for Solving Linear Equations

(a) Write the equation in the standard form

$$\frac{dy}{dx} + P(x)y = Q(x) \; .$$

(b) Calculate the integrating factor $\mu(x)$ by the formula

$$\mu(x) = \exp\left[\int P(x)dx\right] .$$

(c) Multiply the equation in standard form by $\mu(x)$ and, recalling that the left-hand side is just $\frac{d}{dx}[\mu(x)y]$, obtain

$$\underbrace{\mu(x)\frac{dy}{dx} + P(x)\mu(x)y}_{\frac{d}{dx}[\mu(x)y]} = \mu(x)Q(x) \; ,$$

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x) \; .$$

(d) Integrate the last equation and solve for y by dividing by $\mu(x)$ to obtain (8).

Solve the given IVP

Standard form $y' = 3x^2 - \frac{y}{x}$; $y(1) = 5$

linear ✓
1st order ✓

$$\rightarrow \boxed{y' + \frac{1}{x} y = 3x^2}$$

$P(x)$

Integrating factor: $g(x) = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$ $\quad x>0$

$$\underbrace{xy' + \frac{x}{x}y}_{} = 3x^3$$

$$\int (xy)' dx = \int 3x^3 dx$$

$$xy = \frac{3}{4}x^4 + C$$

general Solution

$$\boxed{y = \frac{3}{4}x^3 + \frac{C}{x}}$$

Particular Solution:

$y(1) = 5$

$5 = \frac{3}{4} + C \quad C = 5 - \frac{3}{4} = \frac{17}{4}$

$$\boxed{y = \frac{3}{4}x^3 + \frac{17}{4}x^{-1}}$$

Example 1 Find the general solution to

$$\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos x\,, \qquad x > 0\,. \tag{9}$$

$a^{rs} = (a^r)^s$

Standard form

$$\boxed{y' - \frac{2}{x}y = x^2\cos x}$$

Integrating factor: $g(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|} = e^{\ln x^{-2}} = x^{-2}$

$\hookrightarrow (e^{\ln|x|})^{-2} = x^{-2}$

$$\int (y x^{-2})'\,dx = \int \frac{x^2}{x^2}\cos x\,dx$$

$$y x^{-2} = \sin x + C$$

$$\boxed{y = x^2 \sin x + Cx^2}$$

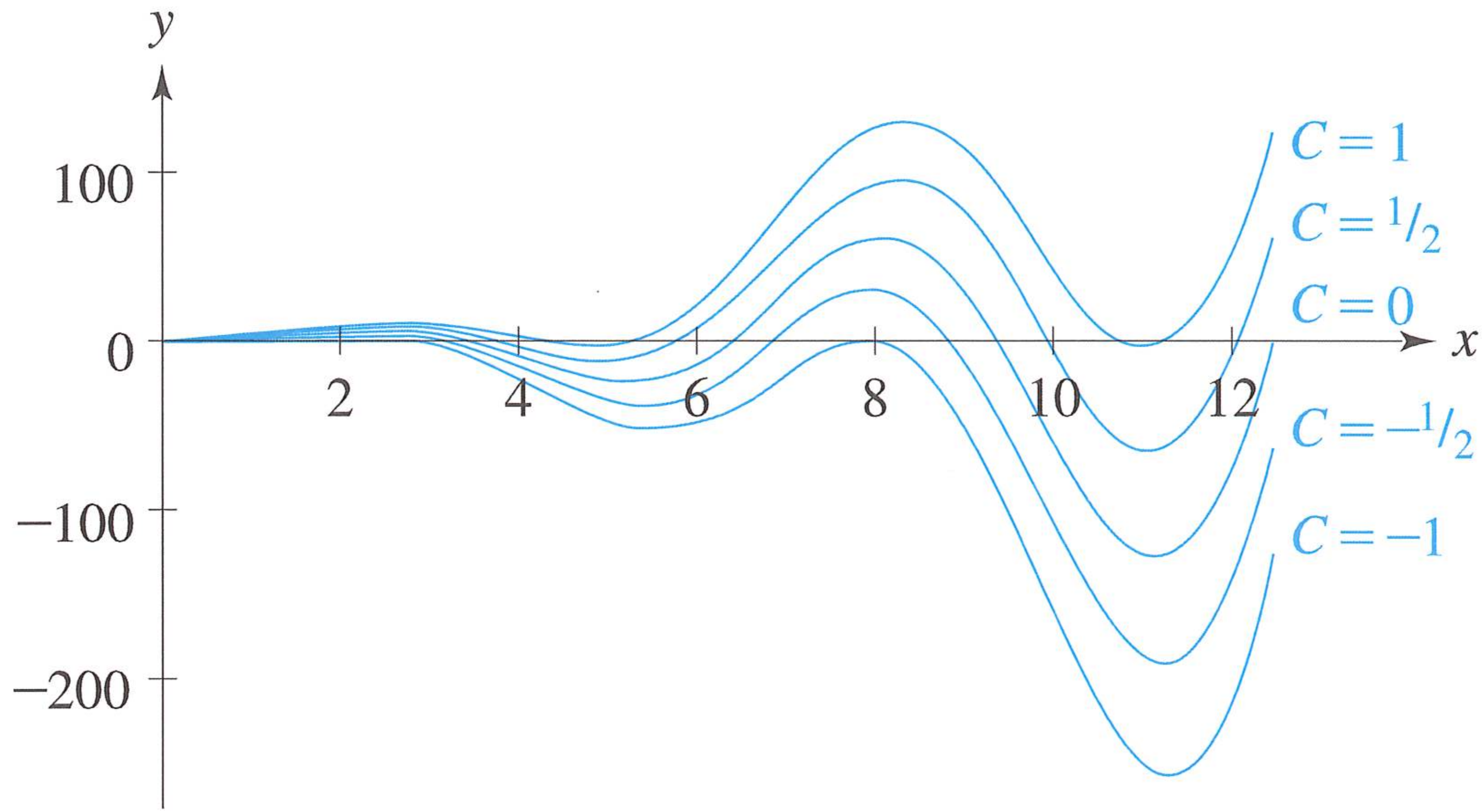

Figure 2.5 Graph of $y = x^2 \sin x + Cx^2$ for five values of the constant C

Solve $\dfrac{dy}{dx} - 3y = 0.$

linear: integrating factor

$$y' - 3 y = 0$$

$$g(x) = e^{\int -3dx} = e^{-3x}$$

$$\int (e^{-3x}y)' \, dx = \int 0 \, dx$$

$$e^{-3x} y = C$$

$$\boxed{y = Ce^{3x}}$$

Separation of variables:

$$\frac{dy}{dx} = 3y$$

$$\int \frac{dy}{y} = \int 3dx$$

$$\ln y = 3x + C$$

$$e^{\ln y} = e^{3x+c} = e^{3x} \cdot \underbrace{e^{c}}_{A} = Ae^{3x}$$

$$\boxed{y = Ae^{3x}}$$

Exact Equations

$$y\,dx + x\,dy = 0$$

$$d(xy) = 0$$

$$xy = C$$

implicit solution

Differential of a Function of Two Variables If $z = f(x, y)$ is a function of two variables with continuous first partial derivatives in a region R of the xy-plane, then its **differential** (also called the total differential) is

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{1}$$

Now if $f(x, y) = c$, it follows from (1) that

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0. \tag{2}$$

$$\overbrace{x^2 - 5xy + y^3}^{f(x,y)} = C$$

$$\boxed{\underbrace{(2x-5y)}_{f_x}dx + \underbrace{(-5x+3y^2)}_{f_y}dy = 0}$$

Definition 2.4.1 Exact Equation

A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$. A first-order differential equation of the form

$$\overset{f_x}{M(x, y)}\,dx + \overset{f_y}{N(x, y)}\,dy = 0$$

is said to be an **exact equation** if the expression on the left side is an exact differential.

$$f_{xy} = f_{yx}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Theorem 2.4.1 Criterion for an Exact Differential

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b$, $c < y < d$. Then a necessary and sufficient condition that $M(x, y)\,dx + N(x, y)\,dy$ be an exact differential is

$$f_{xy} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = f_{yx} \tag{4}$$

Solve $\underbrace{2xy}_{M}\,dx + \underbrace{(x^2 - 1)}_{N}\,dy = 0.$

Check exactness $2x = \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = 2x$ ✓

That means that there is some $f(x,y)$ such that $M = \dfrac{\partial f}{\partial x}$ and $N = \dfrac{\partial f}{\partial y}$

$M = \dfrac{\partial f}{\partial x} = 2xy$

$N = \dfrac{\partial f}{\partial y} = x^2 - 1$

$\boxed{f(x,y) = ?}$

function of "x" only

$f^* = \int N\,dy = x^2y - y + h(x)$

$M = \dfrac{\partial f}{\partial x} = 2xy + 0$

compare: $\dfrac{\partial f^*}{\partial x} = 2xy + h'(x)$

$f(x,y) = x^2y - y$

$$\boxed{x^2y - y = C}$$

Solve $(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0.$

$\underbrace{(e^{2y} - y\cos xy)}_{M}$ $\underbrace{(2xe^{2y} - x\cos xy + 2y)}_{N}$

$\frac{\partial M}{\partial y} = 2e^{2y} - \cos xy + yx\sin xy$

$=$

$\frac{\partial N}{\partial x} = 2e^{2y} - \cos xy + xy\sin xy$

$M = \frac{\partial f}{\partial x} = e^{2y} - y\cos xy$

$N = \frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$

$f^* \to \int M\,dx = xe^{2y} - y\frac{\sin xy}{y} + g(y) + 0$

$\to \int N\,dy = \frac{2x}{2}e^{2y} - x\frac{\sin xy}{x} + y^2 + h(x)$

$f^* = xe^{2y} - \sin xy + y^2$

$xe^{2y} - \sin xy + y = C$

$xe^{2y} - \sin xy + y^2 = C$