MA345 Differential Equations & Matrix Method

Lecture: Special cases

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Example 4 Show that

$$\overset{M}{(x + 3x^3 \sin y)}\, dx + \overset{N}{(x^4 \cos y)}\, dy = 0 \qquad (16)$$

$$\frac{\partial M}{\partial y} = 3x^3 \cos y \neq \frac{\partial N}{\partial x} = 4x^3 \cos y$$

is *not* exact but that multiplying this equation by the factor x^{-1} yields an exact equation. Use this fact to solve (16).

$$\underbrace{(1 + 3x^2 \sin y)}_{M^*} dx + \underbrace{(x^3 \cos y)}_{N^*} dy = 0 \qquad x \neq 0$$

$$\frac{\partial M^*}{\partial y} = 3x^2 \cos y$$

$$=$$

$$\frac{\partial N^*}{\partial x} = 3x^2 \cos y$$

f^*

$$\int M^* dx = x + x^3 \sin y + g(y)$$

$$\int N^* dy = x^3 \sin y + h(x) + 0$$

$$\boxed{x + x^3 \sin y = C} \qquad x \neq 0$$

Integrating Factor

Definition 3. If the equation

$$(1) \qquad M(x, y)\, dx + N(x, y)\, dy = 0$$

is not exact, but the equation

$$(2) \qquad \overbrace{\mu(x, y)M(x, y)}^{M^*}\, dx + \overbrace{\mu(x, y)N(x, y)}^{N^*}\, dy = 0\,,$$

which results from multiplying equation (1) by the function $\mu(x, y)$, *is* exact, then $\mu(x, y)$ is called an **integrating factor**[†] of the equation (1).

$$M^*_y = \frac{\partial}{\partial y}\left[\mu(x,y) M(x,y)\right] = \frac{\partial}{\partial x}\left[\mu(x,y) N(x,y)\right] = N^*_x$$

$$\mu_y \cdot M + \mu \cdot M_y = \mu_x N + \mu \cdot N_x$$

$$M \frac{\partial \mu}{\partial y} - N \frac{\partial \mu}{\partial x} = (N_x - M_y)\mu$$

if $\mu = \mu(x)$ only

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu$$

$$\int \frac{d\mu}{\mu} = \int \left(\frac{M_y - N_x}{N}\right) dx$$

$$\mu(x) = e^{\int \left(\frac{M_y - N_x}{N}\right) dx}$$

if $\mu = \mu(y)$

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu$$

$$\int \frac{d\mu}{\mu} = \int \left(\frac{N_x - M_y}{N}\right) dy$$

Method for Finding Special Integrating Factors

If $M\,dx + N\,dy = 0$ is neither separable nor linear, compute $\partial M/\partial y$ and $\partial N/\partial x$. If $\partial M/\partial y = \partial N/\partial x$, then the equation is exact. If it is not exact, consider

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} . \tag{10}$$

If (10) is a function of just x, then an integrating factor is given by formula (8). If not, consider

$$\frac{\partial N/\partial x - \partial M/\partial y}{M} . \tag{11}$$

If (11) is a function of just y, then an integrating factor is given by formula (9).

Special Integrating Factors

Theorem 3. If $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x, then

$$\mu(x) = \exp\left[\int\left(\frac{\partial M/\partial y - \partial N/\partial x}{N}\right)dx\right] \tag{8}$$

is an integrating factor for equation (1).

If $(\partial N/\partial x - \partial M/\partial y)/M$ is continuous and depends only on y, then

$$\mu(y) = \exp\left[\int\left(\frac{\partial N/\partial x - \partial M/\partial y}{M}\right)dy\right] \tag{9}$$

is an integrating factor for equation (1).

Example 2 Solve

$$\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = 2xy - 1$$

(12) $$(\underbrace{2x^2 + y}_{M})\, dx + (\underbrace{x^2y - x}_{N})\, dy = 0\,.$$

multiply

$$\frac{M_y - N_x}{N} = \frac{1-(2xy-1)}{x^2y - x} = \frac{2\cancel{(1-xy)}}{-x\cancel{(1-xy)}} = -\frac{2}{x}$$

in terms of "x" only.

$$\mu(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|} = x^{-2}$$

$$(\overbrace{2 + yx^{-2}}^{M^*})\,dx + (\overbrace{y - x^{-1}}^{N^*})\,dy = 0$$

$$\frac{\partial M^*}{\partial y} = x^{-2} = \frac{\partial N^*}{\partial x} = x^{-2}$$

$$f = \begin{cases} \int M^* dx = 2x - yx^{-1} + g(y) \\ \int N^* dy = \frac{y^2}{2} - yx^{-1} + h(x) \end{cases}$$

compare

$$\boxed{2x - yx^{-1} + \frac{y^2}{2} = c}$$