Lecture 1: Set Theory

Set: A collection of elements/objects whose content can be clearly determined.

Elements of a Set: objects in a set

* unordered list

* no repetation of elements

$A = \{a, b, c\}$ $B = \{1,2\}, \{2,1\}, \{1,2,2,1\}$

$B = \{x \mid x$ is an integer $1 \le x \le 2\}$

Ex 1: True or False.

a. $8 \in \{1, 2, 3, 4, \ldots, 10\}$ T

b. $r \notin \{a, b, c, z\}$ T

Empty Set: contains no elements

$\{ \}$, ϕ

Notation	
Set	A, B, X, etc.
Element	a, b, #s 1, 2
$\in$	is an element of
$\notin$	not an element
{ } or Ø	empty set
$\mathbb{U}$ (or $\mathcal{U}$),	universal set
$\{x \mid \ldots\}$	such that

Ex 2: Determine which sets are the empty set.

$\cup, \wedge$

a. $A = \{x | x < 3 \text{ or } x > 5\}$ NOT empty set $A \neq \phi$

b. $B = \{x | x < 3 \text{ and } x > 5\}$ $B = \phi$

$\underbrace{x<3}_{1}$ $\cap$ $\vee$ $\underbrace{x>5}_{2}$

Two Characteristics of Sets:

- No repeated elements
- No order of elements

Universal Set: A set that contains all elements being considered in a given disscussion

U $\mathbb{U}$

Finite Set: Can explicitly list out all elements

$\{1,2,3,4\}$ $\{-2,-1,0,1,2,3\}$ $\{1,2,\ldots\ldots,10{,}000\}$

Notation	
$\lvert A\rvert$	cardinality.

Infinite Set: unlimited # of elements

$\mathbb{N}=\{1,2,3,4,\ldots\}$ $\lvert\mathbb{N}\rvert=\infty$ $\lvert\mathbb{Z}\rvert=\infty$

Cardinality (size): No. of elements in a set

$\lvert A\rvert$, $n(A)$

Ex 3: Find the cardinality of each set.

a. $A=\{6,10,14,15,16\}$

$\lvert A\rvert=4$

b. $B=\{872\}$

$\lvert B\rvert=1$

c. $C=\{9,10,11,\ldots,15,16\}$

$\lvert C\rvert=8$

d. $D=\{\ \}$?

$\lvert D\rvert=0$

<u>Ex 4</u>: Fill in the blank. Let $\mathbb{U} = \{1, 2, 3, \dots\}$

Roster   Set Builder Notation

Enumeration	Formal Rule
$A = \{1, 4, 9, \dots, 64, 81\}$	$A = \{K^2 \mid K \in \mathbb{U}, K \le 9\}$
$B = \{2, 4, 6, 8, \dots\}$	$B = \{2k \mid k \in \mathbb{U}\}$

$|A| = 9$

$\le$ less than or $=$ to

$\ge$ greater than or $=$ to

Recursive Rule

Ex 5: Represent the Fibonacci Sequence using:

a. Enumeration:

$F = \{0, 1, 1, 2, 3, 5, \ldots\}$

b. Recursive Rule:

$$F = \begin{cases} f_1 = 0 \\ f_2 = 1 \\ f_n = f_{n-2} + f_{n-1},\ n \in \mathbb{N},\ \boxed{n > 2} \end{cases}$$

Ex 6: Represent the set of positive odd numbers O using:

$O^+, Z^+, R^+, O^-, Z^-, R^-$

a. Enumeration

$O = \{1, 3, 5, 7, 9, 11, \ldots\}$

b. Formal Rule

$O = \{o \mid o, k \in \mathbb{W},\ o = 2k+1\}$

$O = \{o \mid o, k \in \mathbb{N},\ o = 2k-1\}$

c. Recursive Rule

$$\boxed{O = \begin{cases} o_1 = 1 \\ o_n = o_{n-1} + 2,\ n > 1,\ n \in \mathbb{N} \end{cases}}$$

Notation: Number Sets

Natural Numbers

$\mathbb{N} = \{1, 2, 3, \ldots\}$

$\mathbb{N}$ $\mathbb{N}$

Whole Numbers

$\mathbb{W} = \{0, 1, 2, 3, \ldots\}$

$\mathbb{W}$ $\mathbb{W}$

Integers

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Z}$ $\mathbb{Z}$

$\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$

p/q, $5/0$ $\mathbb{Q}$

Real Numbers

$\mathbb{R}$

$\mathbb{R}$

Complex Numbers

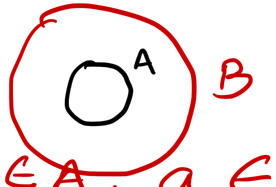
$\mathbb{C}$ $a+ib$

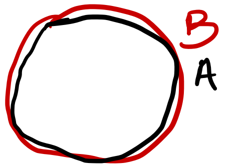
$\mathbb{C}$ $\mathbb{C}$

Subset If A is a subset of B, then every element of A must be element of B.

$A \subseteq B$

if $A \subseteq B$, then $\forall a \in A$, $a \in B$





Proper Subset $A \subset B$ every element of A is an element of B but $A \neq B$, if $A \subset B$ $\forall a \in A, a \in B \wedge \exists\, b \in B, b \notin A$

Ex 7: Determine whether $\subseteq$, $\subset$, both or neither can be placed in each blank to form a true statement.

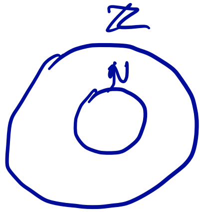
a. $\{2, 4, 6, 8\} \underset{\subset}{\overset{\subseteq}{\_\_\_}} \{2, 8, 4, 6, 10\}$ BOTH

b. $\{1, 5, 7, 9\} \overset{\subseteq}{\_\_\_} \{9, 7, 1, 5\}$

c. $\{\ \} \underset{\subset}{\overset{\subseteq}{\_\_\_}} \{1, 2, 3\}$ BOTH

d. $\mathbb{Z}$___$\mathbb{N}$ NEITHER $\mathbb{Z} = \{\dots -2, -1, 0, 1, 2, 3 \dots\}$

$\mathbb{N} = \{1, 2, 3, 4, \dots\}$



e. $\{2, 4, 8, 16, \dots\}$___$\{x \mid x, k \in \mathbb{N}, x = 2^k\}$

$\subseteq$

Notation

$\forall$ for all

$\exists$ there exists

$\wedge$ and

$\vee$ (inclusive or) and/OR

Notation: Subsets

$\subseteq$ is a subset of

$\nsubseteq$ is NOT a subset of

$\subset$ proper subset of

$\not\subset$ NOT a proper subset

$=$ $A = B$

Ex 8: Calculate the number of distinct subsets and the number of distinct proper subsets for the set given.

$$A = \{a \mid a \in \mathbb{N}, 3 < a < 8\}$$

$A = \{4, 5, 6, 7\} \Rightarrow A$

- $\{\ \}$
- $\{4\}, \{5\}, \{6\}, \{7\}$ ~ single elements
- $\{4,5\}, \{4,6\}, \{4,7\}, \{5,6\}, \{5,7\}, \{6,7\}$
- $\{4,5,6\}, \{4,5,7\}, \{4,6,7\}, \{5,6,7\}$
- $\cancel{\{4,5,6,7\}}$

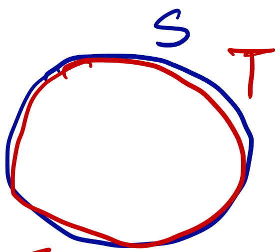
\# No of subsets = 16

\# No of proper subsets = 15

$$\boxed{\begin{array}{l} \text{No. of subsets} = 2^{|A|} \\ \text{No. of Proper subsets} = 2^{|A|} - 1 \end{array}}$$

Set Equality

$S = T$

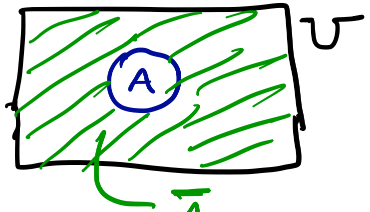


$\forall s \in S, s \in T$

$$\boxed{S \subseteq T \text{ and } T \subseteq S}$$

Complement: $\bar{A}$ the set of all element in U that are not in A.

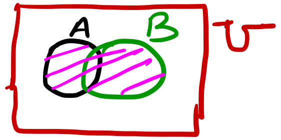
$\bar{A} = \{x \mid x \in U \wedge x \notin A\}$



Union:

$A \cup B$ set of elements that are members of A or B or Both.

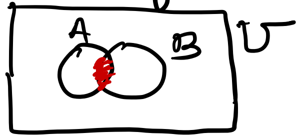
$A \cup B = \{x \mid x \in A \vee x \in B\}$



Intersection:

$A \cap B$ set of elements that are members of A & B

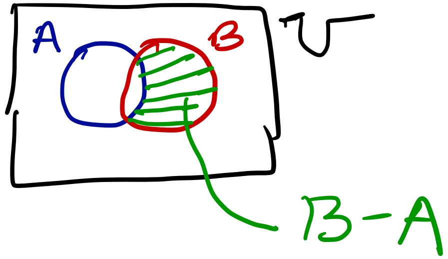
$A \cap B = \{x \mid x \in A \wedge x \in B\}$



Minus (Difference):

$B - A$ Set of elements of B that are NOT elements of A.

$B - A = \{x \mid x \in B \wedge x \notin A\}$



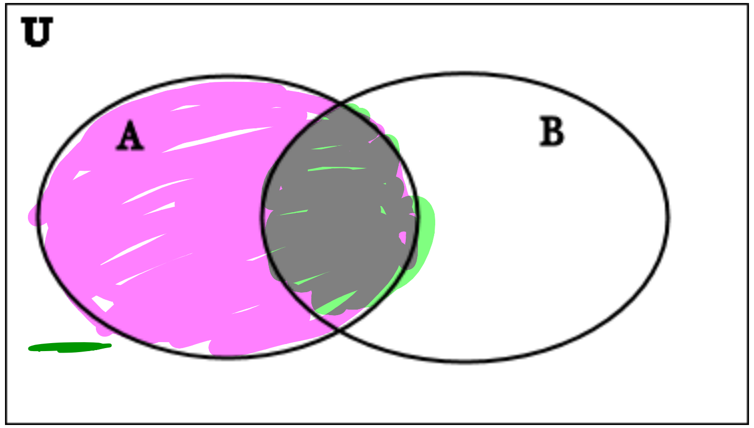
Notation: Operators

- $\bar{A}$ — $\bar{A}$, A' complement
- $\cup$ — union
- $\cap$ — intersection
- $-$ — Difference

Ex 9: Let $\mathbb{U} = \{1, \cancel{2}, 3, \cancel{4}, 5, \cancel{6}, 7\}$, $A = \{2, 4, 6\}$, $B = \{1, 2, 3, 4\}$

a. $\bar{A} = \{1, 3, 5, 7\}$

Venn Diagram



b. $A \cup B = \{1, 2, 3, 4, 6\}$

c. $A \cap B = \{2, 4\}$

d. $A - B = \{6\}$

$A \cap \bar{B}$

$B - A = \{1, 3\}$

$\overline{A \cup B} = \{5, 7\}$

Ex 10:

CHECK POINT 3 A survey of 250 memorabilia collectors showed the following results: 108 collected baseball cards. 92 collected comic books. 62 collected stamps. 29 collected baseball cards and comic books. 5 collected baseball cards and stamps. 2 collected comic books and stamps. 2 collected all three types of memorabilia. Use a Venn diagram to illustrate the survey's results.

$|B \cap C \cap S| = 2$

$|C \cap S| = 2$

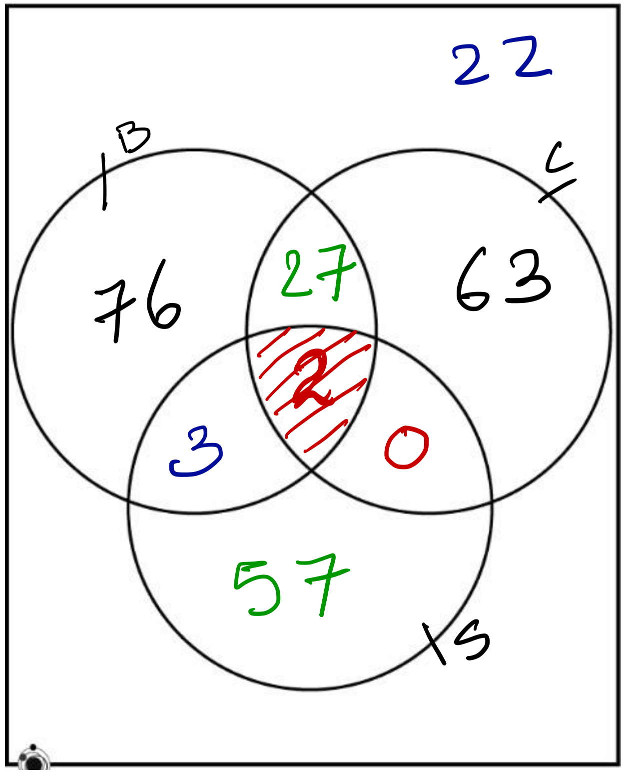
$|B \cap S| = 5$

$|B \cap C| = 29$

$|S| = 62$

$|B| = 108$

$|C| = 92$



$|U| = 250$

$U = 250 - 76 - 27 - 63 - 2 - 3 - 57 = 22$

$B = 108$

$C = 92 - 27 - 2 = 63$

$S = 62 - 5 = 57$

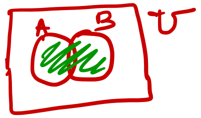
Laws of Set Theory

For any sets A, B, and C taken from a universe $\mathcal{U}$

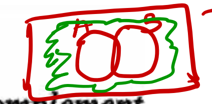
1)	$\overline{\overline{A}} = A$	Law of *Double Complement*
2)	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	*DeMorgan's* Laws
	$\overline{A \cap B} = \overline{A} \cup \overline{B}$	
3)	$A \cup B = B \cup A$	*Commutative* Laws
	$A \cap B = B \cap A$	
4)	$A \cup (B \cup C) = (A \cup B) \cup C$	*Associative* Laws
	$A \cap (B \cap C) = (A \cap B) \cap C$	
5)	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	*Distributive* Laws
	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	
6)	$A \cup A = A$	*Idempotent* Laws
	$A \cap A = A$	
7)	$A \cup \emptyset = A$	*Identity* Laws
	$A \cap \mathcal{U} = A$	
8)	$A \cup \overline{A} = \mathcal{U}$	*Inverse* Laws
	$A \cap \overline{A} = \emptyset$	
9)	$A \cup \mathcal{U} = \mathcal{U}$	*Domination* Laws
	$A \cap \emptyset = \emptyset$	
10)	$A \cup (A \cap B) = A$	*Absorption* Laws
	$A \cap (A \cup B) = A$	

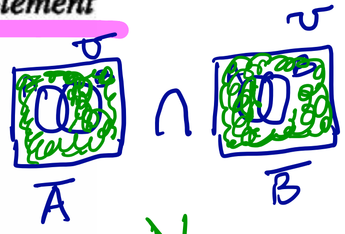
LHS = $\overline{A \cup B}$

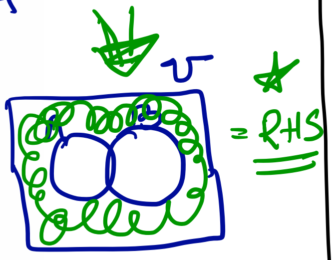
$A \cup B$



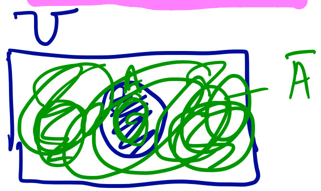
$\overline{A \cup B}$







$\overline{A} \cap \overline{B}$



Ex 11: Simplify: $\overline{\overline{(A \cup B) \cap C} \cup \overline{B}}$

X

$\cap$

$\overline{\overline{(A \cup B) \cap C}} \cap \overline{\overline{B}}$ → DeMorgans LAW

$((A \cup B) \cap C) \cap B$ ⇒ Double Complement

$(A \cup B) \cap (C \cap B)$ ⇒ ASSOCIATIVE LAW

$(A \cup B) \cap (B \cap C)$ ⇒ commutative

$((A \cup B) \cap B) \cap C$ ⇒ ASSOCIATIVE LAW

$B \cap C$ ⇒ Absorption law.

$\boxed{B \cap C}$

Membership Tables

1: $x \in$ given set

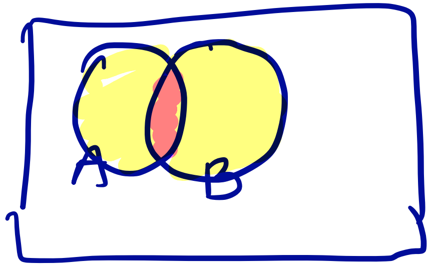
0: $x \notin$ given set

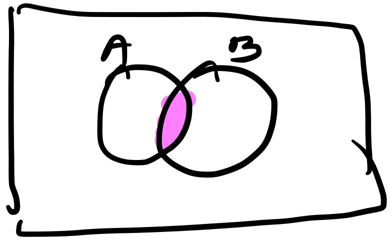
opposite

A	$\bar{A}$
1	0
0	1

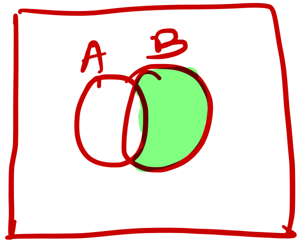
A	B	$A \cup B$
0	0	0
0	1	1
1	0	1
1	1	1

A	B	$A \cap B$
0	0	0
0	1	0
1	0	0
1	1	1





A	B	$B-A$
0	0	0
0	1	1
1	0	0
1	1	0



Ex 12: Use membership tables to establish:

a. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

A	B	$A \cap B$	$\overline{A \cap B}$	$\bar{A}$	$\bar{B}$	$\bar{A} \cup \bar{B}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

LHS = RHS

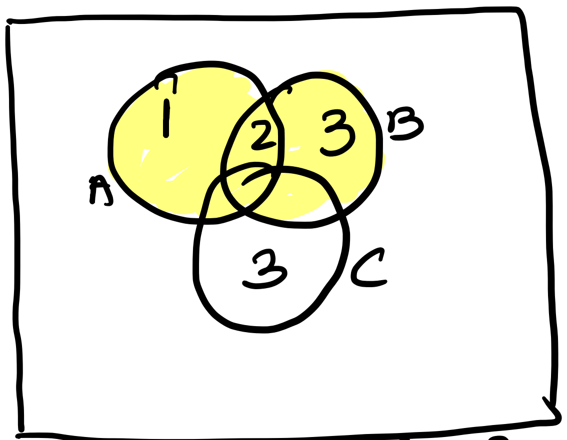
b. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

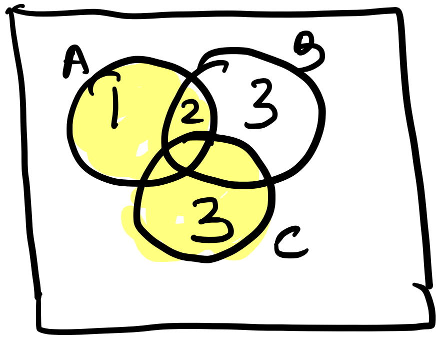
A	B	C	$B\cap C$	$A\cup(B\cap C)$	$(A\cup B)$	$(A\cup C)$	$(A\cup B)\cap(A\cup C)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

LHS = RHS

given find

<u>Ex 13</u>: If $A \cup B = A \cup C$, must $B = C$?





$A = \{1,2\}$ $B = \{2,3\}$ $C = \{3\}$

$A \cup B = \{1,2,3\}$ $A \cup C = \{1,2,3\}$

$A \cup B = A \cup C$

However, $B \neq C$

Prove by Contradiction

Assume true $A \cup B = A \cup C \Rightarrow B = C$

Counter example, Let $A = \{1,2\}$

$B = \{2,3\}$, $C = \{3\}$

$A \cup B = \{1,2,3\} = A \cup C$

However $B \neq C$

$\therefore$ $B = C$ does not follow from $A \cup B = A \cup C$

<u>Ex 14</u>: If $A \cap B = A \cap C$, must $B = C$?

$\boxed{\text{NO}}$, Prove by Contradiction

Assume true $A \cap B = A \cap C \Rightarrow B = C$

Counterexample: $A = \{1,2,3\}$, $B = \{3,4,5\}$ $C = \{3, 10, 11, 12\}$

$A \cap B = \{3\} = A \cap C$, $B \neq C$ $\therefore$ $B = C$ does not follow from $A \cap B = A \cap C$.