COMP9020 20T1
Week 8
Counting

- Textbook (R & W) - Ch. 5, Sec. 5.1, 5.3
- Problem set week 8 + quiz

Fact

Combinatorics and probability arise in many areas of Computer Science, e.g.

- *Complexity of algorithms, data management*
- *Reliability, quality assurance*
- *Computer security*
- *Data mining, machine learning, robotics*

Counting Techniques

General idea: find methods, algorithms or precise formulae to count the number of elements in various sets or collections derived, in a structured way, from some basic sets.

Examples

Single base set $S = \{s_1, \ldots, s_n\}$, $|S| = n$; find the number of

- all subsets of S
- ordered selections of r different elements of S
- unordered selections of r different elements of S
- selections of r elements from S s.t. ...
- functions $S \longrightarrow S$ (onto, 1-1)
- partitions of S into k equivalence classes
- graphs/trees with elements of S as labelled vertices/leaves
- ...and many more

Basic Counting Rules (1)

Union rule — S and T *disjoint*

$$|S \cup T| = |S| + |T|$$

$S_1, S_2, \ldots, S_n$ pairwise disjoint ($S_i \cap S_j = \emptyset$ for $i \neq j$)

$$|S_1 \cup \ldots \cup S_n| = \sum |S_i|$$

Exercise

How many numbers in $A = [1, 2, \ldots, 999]$ are divisible by 31 or 41?

$\lfloor 999/31 \rfloor = 32$ divisible by 31.
$\lfloor 999/41 \rfloor = 24$ divisible by 41.
No number in A divisible by both.
Hence, $32 + 24 = 56$ divisible by 31 or 41.

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Basic Counting Rules (2)

Product rule

$$|S_1 \times \ldots \times S_k| = |S_1| \cdot |S_2| \cdots |S_k| = \prod_{i=1}^{k} |S_i|$$

If all $S_i = S$ (the same set) and $|S| = m$ then $|S^k| = m^k$

Example

Let $\Sigma = \{a, b, c, d, e, f, g\}$.
How many 5-letter words? How many with no letter repeated?

$$|\Sigma^5| = |\Sigma|^5 = 7^5 = 16,807$$

$$\prod_{i=0}^{4}(|\Sigma| - i) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2,520$$

Exercise

S, T finite. How many functions $S \longrightarrow T$ are there?

$$|T|^{|S|}$$

Exercise

$\boxed{5.1.19}$ Consider a *complete* graph on n vertices.

(a) No. of paths of length 3

Take any vertex to start, then every next vertex different from the preceding one. Hence $n \cdot (n-1)^3$

(b) paths of length 3 with all vertices distinct

$n(n-1)(n-2)(n-3)$

(c) paths of length 3 with all edges distinct

$n(n-1)(n-2)^2$

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Inclusion-Exclusion

This is one of the most universal counting procedures. It allows you to compute the size of

$$A_1 \cup \ldots \cup A_n$$

from the sizes of all possible intersections

$$A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}, \;\; a_{i_1} < a_{i_2} < \ldots < a_{i_k}$$

Two sets $|A \cup B| = |A| + |B| - |A \cap B|$

Three sets $$\begin{aligned} |A \cup B \cup C| = & |A| + |B| + |C| \\ & -|A \cap B| - |A \cap C| - |B \cap C| \\ & +|A \cap B \cap C| \end{aligned}$$

NB

Inclusion-exclusion is often applied informally without making clear or explicit *why* certain quantities are subtracted or put back in.

Interpretation

Each A_i defined as the set of objects that satisfy some property P_i

$$A_i = \{ x \in X : P_i(x) \}$$

Union $A_1 \cup \ldots \cup A_n$ is the set of objects that satisfy **at least one** property P_i

$$A_1 \cup \ldots \cup A_n = \{ x \in X : P_1(x) \vee P_2(x) \vee \ldots \vee P_n(x) \}$$

Intersection $A_{i_1} \cap \ldots \cap A_{i_r}$ is the set of objects that satisfy **all** properties $P_{i_1}, \ldots, P_{i_r}$

$$A_{i_1} \cap \ldots \cap A_{i_r} = \{ x \in X : P_{i_1}(x) \wedge P_{i_2}(x) \wedge \ldots \wedge P_{i_r}(x) \}$$

Special case $r = 1$: $A_{i_1} = \{x \in X : P_{i_1}(x)\}$

Examples

$\boxed{5.3.1}$ 200 people. 150 swim or jog, 85 swim and 60 do both. How many jog?

S – (set of) people who swim, J – people who jog
$|S \cup J| = |S| + |J| - |S \cap J|$; thus $150 = 85 + |J| - 60$ hence $|J| = 125$; answer *does not* depend on the number of people overall

$\boxed{\text{5.6.38 (supp)}}$ There are 100 problems, 75 of which are 'easy' and 40 'important'.
What's the smallest number of easy *and* important problems?

$|E \cap I| = |E| + |I| - |E \cup I| = 75 + 40 - |E \cup I| \geq 75 + 40 - 100 = 15$

Exercise

$\boxed{5.3.2}$ $S = [100 \ldots 999]$, thus $|S| = 900$.

(a) How many numbers have at least one digit that is a 3 or 7?

$A_3 = \{\text{at least one '3'}\}$
$A_7 = \{\text{at least one '7'}\}$

$$(A_3 \cup A_7)^c = \{\ n \in [100, 999] : n \text{ digits } \in \{0, 1, 2, 4, 5, 6, 8, 9\}\ \}$$

7 choices for the first digit and 8 choices for the later digits:

$$|(A_3 \cup A_7)^c| = |\{1, 2, 4, 5, 6, 8, 9\}| \cdot |\{0, 1, 2, 4, 5, 6, 8, 9\}|^2$$

Therefore $|A_3 \cup A_7| = 900 - 448 = 452$.

(b) How many numbers have a 3 *and* a 7?

$|A_3 \cap A_7| = |A_3| + |A_7| - |A_3 \cup A_7| =$
$(900 - 8 \cdot 9 \cdot 9) + (900 - 8 \cdot 9 \cdot 9) - 452 = 2 \cdot 252 - 452 = 52$

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Corollaries

- If $|S \cup T| = |S| + |T|$ then S and T are disjoint
- If $|\bigcup_{i=1}^{n} S_i| = \sum_{i=1}^{n} |S_i|$ then S_i are pairwise disjoint
- If $|T \setminus S| = |T| - |S|$ then $S \subseteq T$

These properties can serve to identify cases when sets are disjoint (resp. one is contained in the other).

Proof.

$|S| + |T| = |S \cup T|$ implies $|S \cap T| = |S| + |T| - |S \cup T| = 0$

$|T \setminus S| = |T| - |S|$ implies $|S \cap T| = |S|$, which implies $S \subseteq T$ $\square$

Basic Counting Rules (3)

permutations
Ordering of all objects from a set S; equivalently: Selecting all objects while *recognising* the order of selection.
The number of permutations of n elements is

$$n! = n \cdot (n-1) \cdots 1, \quad 0! = 1! = 1$$

r-**permutations**
Selecting any r objects from a set S of size n without repetition while *recognising* the order of selection.
Their number is

$$\Pi(n, r) = n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

r-**selections** (or: r-**combinations**)
Collecting any r distinct objects without repetition;
equivalently: selecting r objects from a set S of size n and *not* recognising the order of selection.
Their number is

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n \cdot (n-1) \cdots (n-r+1)}{1 \cdot 2 \cdots r}$$

NB

These numbers are usually called *binomial coefficients* due to

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + b^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

NB

Also defined for any $\alpha \in \mathbb{R}$ as $\binom{\alpha}{r} = \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!}$

Examples

- Number of edges in a complete graph K_n
- Number of diagonals in a convex polygon
- Number of poker hands
- Decisions in games, lotteries etc.

Example

$\boxed{5.1.2}$ Give an example of a counting problem whose answer is

(a) $\Pi(26, 10)$

(b) $\binom{26}{10}$

Draw 10 cards from a half deck (eg. black cards only)
(a) the cards are recorded in the order of appearance
(b) only the complete draw is recorded

Exercise

$\boxed{5.1.6}$ From a group of 12 men and 16 women, how many committees can be chosen consisting of

(a) 7 members? $\binom{12+16}{7}$

(b) 3 men and 4 women? $\binom{12}{3}\binom{16}{4}$

(c) 7 women or 7 men? $\binom{12}{7}+\binom{16}{7}$

$\boxed{5.1.7}$ As above, but any 4 people (male or female) out of 9 and two, Alice and Bob, unwilling to serve on the same committee.

$\{\text{all committees}\} - \{\text{committees with both } A \text{ and } B\}$
$= \binom{9}{4} - \binom{7}{2} = 126 - 21 = 105$

equivalently, $\{\text{A in, B out}\} + \{\text{A out, B in}\} + \{\text{none in}\}$
$= \binom{7}{3} + \binom{7}{3} + \binom{7}{4} = 35 + 35 + 35 = 105$

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Counting Poker Hands

Exercise

$\boxed{5.1.15}$ A poker hand consists of 5 cards drawn without replacement from a standard deck of 52 cards

$$\{A, 2\text{-}10, J, Q, K\} \times \{\text{club, spade, heart, diamond}\}$$

(a) Number of "4 of a kind" hands (e.g. 4 Jacks)

$|\text{rank of the 4-of-a-kind}| \cdot |\text{any other card}| = 13 \cdot (52 - 4)$

(b) Number of non-straight flushes, i.e. all cards of same suit but *not* consecutive (e.g. 8,9,10,J,K)

$|\text{all flush}| - |\text{straight flush}|$

$= |\text{suit}| \cdot |\text{5-hand in a given suit}| -$

$\quad |\text{suit}| \cdot |\text{rank of a straight flush in a given suit}|$

$= 4 \cdot \binom{13}{5} - 4 \cdot 10$

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Difficult Counting Problems

Example (Ramsay numbers)

An example of a *Ramsay number* is $R(3,3) = 6$, meaning that

> *"K_6 is the smallest complete graph s.t. if all edges are painted using two colours, then there must be at least one monochromatic triangle"*

This serves as the basis of a game called S-I-M (invented by Simmons), where two adversaries connect six dots, respectively using blue and red lines. The objective is to *avoid* closing a triangle of one's own colour. The second player has a winning strategy, but the full analysis requires a computer program.

Using Programs to Count

Two dice, a red die and a black die, are rolled.
(Note: one *die*, two or more *dice*)

Write a program to list all the pairs $\{(R, B) : R > B\}$

Similarly, for three dice, list all triples $R > B > G$

Generally, for n dice, all of which are m-sided ($n \leq m$), list all *decreasing* n-tuples

NB

In order to just find the number of such n-tuples, it is not necessary to list them all. One can write a recurrence relation for these numbers and compute (or try to solve) it.

Approximate Counting

NB

A *Count* may be a precise value or an **estimate**.

The latter should be asymptotically correct or at least give a good asymptotic bound, whether upper or lower. If S is the base set, $|S| = n$ its size, and we denote by $c(S)$ some collection of objects from S we are interested in, then the estimate $\text{est}(c(S))$ is asymptotically correct if

$$\lim_{n\to\infty} \frac{\text{est}(|c(S)|)}{|c(S)|} = 1$$

Summary

- Union rule
- Product rule
- $n!$
- $\Pi(n, r)$
- $\binom{n}{r}$

Coming up . . .

- Ch. 5, Sec. 5.2 (Probability)
- Ch. 9, Sec. 9.1 (Conditional probability)