Contents

§ 1 基本积分公式

1.
$$\int x^k \,\mathrm{d}x = \frac{1}{k+1}x^{k+1} + C, k \neq -1$$

2.
$$\int e^x \,\mathrm{d}x = e^x + C, \int a^x \,\mathrm{d}x = \frac{a^x}{\ln a} + C, a > 0, a \neq 1$$

3.
$$\int \frac{1}{x} \,\mathrm{d}x = \ln x + C$$

4.
$$\int \sin x \,\mathrm{d}x = -\cos x + C; \int \cos x \,\mathrm{d}x = \sin x + C$$

$$\int \tan x \,\mathrm{d}x = \ln|\cos x| + C; \int \cot x \,\mathrm{d}x = \ln|\sin x| + C$$

$$\int \sec^2 x \,\mathrm{d}x = \tan x + C; \int \csc^2 x \,\mathrm{d}x = -\cot x + C$$

$$\int \sec x \tan x \,\mathrm{d}x = \sec x + C; \int \csc x \cot x \,\mathrm{d}x = -\csc x + C$$

$$\int \frac{\mathrm{d}x}{\sin x} = \int \csc x \,\mathrm{d}x = \ln(|\csc x - \cot x|) + C$$

$$\int \frac{\mathrm{d}x}{\cos x} = \int \sec x \,\mathrm{d}x = \ln(|\sec x + \tan x|) + C$$

5.
$$\int \frac{1}{a^2 + x^2} \,\mathrm{d}x = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C, (a > 0)$$

6.
$$\int \frac{1}{\sqrt{a^2 - x^2}} \,\mathrm{d}x = \arcsin \frac{x}{a} + C, (a > 0)$$

7.
$$\int \frac{1}{\sqrt{x^2 + a^2}} \,\mathrm{d}x = \ln\left|x + \sqrt{x^2 + a^2}\right| + C$$

8.
$$\int \frac{1}{x^2 - a^2} \,\mathrm{d}x = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C, (a > 0)$$

9.
$$\int \sqrt{a^2 - x^2} \,\mathrm{d}x = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C$$

10.
$$\int \sin^2 x \,\mathrm{d}x = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

$$\int \cos^2 x \,\mathrm{d}x = \frac{x}{2} + \frac{\sin 2x}{4} + C$$

$$\int \tan^2 x \,\mathrm{d}x = \tan x - x + C$$

$$\int \cot^2 x \,\mathrm{d}x = -\cot x - x + C$$

§2 不定积分积分法

凑微分

$$\int f[g(x)]g'(x)\,\mathrm{d}x = \int f[g(x)]\,\mathrm{d}g(x) = \int f(u)\,\mathrm{d}u$$

换元法

1. 三角换元法：

$$\sqrt{a^2-x^2} \to x = a\sin t, \sqrt{a^2+x^2} \to x = a\tan t, \sqrt{x^2-a^2} \to x = a\sec t$$

2. 恒等变形后做三角代换：
被积函数含有根式$\sqrt{ax^2+bx+c}$时，化为

$$\sqrt{\varphi^2(x)+k^2}, \sqrt{\varphi^2(x)-k^2}, \sqrt{k^2-\varphi^2(x)}$$

再做三角代换

3. 根式代换法：
被积函数含有根式 $\sqrt[n]{ax+b}, \sqrt{\frac{ax+b}{cx+d}}, \sqrt{ae^{bx}+c}$时，一般令$\sqrt{*}=t$, 若同时含有 $\sqrt[n]{ax+b}$, $\sqrt[m]{ax+b}$, 取m,n最小公倍数l, 令$\sqrt[l]{ax+b}=t$, 再做根式代换法
    1. 倒代换法：

    被积函数分母次数比分子高两次及以上时，令$u=\frac{1}{x}$

分部积分法

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

推广有：

$$\int uv^{(n+1)}\,\mathrm{d}x = uv^{(n)} - u'v^{(n-1)} + u''v^{(n-2)} - ... + (-1)^n u^{(n)}v + (-1)^{n+1}\int u^{(n+1)}v\,\mathrm{d}x$$

有理函数积分

将有理函数拆成若干项最简有理分式之和 方法：

1. 分母一次单因式产生一项$\frac{A}{ax+b}$
2. 分母 k 重单因式产生 k 项，分别为$\frac{A_1}{ax+b}, \frac{A_2}{(ax+b)^2}, \dots \frac{A_k}{(ax+b)^k}(k>0, k\neq 1)$
3. 分母二次单因式产生一项$\frac{A}{ax^2+bx+c}$
4. 分母 k 重二次单因式产生 k 项，分别为$\frac{A_1x+B_1}{ax^2+bx+c}, \frac{A_2x+B_2}{(ax^2+bx+c)^2}, \dots \frac{A_kx+B_k}{(ax^2+bx+c)^k}(k>0, k\neq 1)$

三角有理式可用万能代换法：

$$t = \tan\frac{x}{2}, \sin x = \frac{2t}{1+t^2}, \cos t = \frac{1-t^2}{1+t^2}, \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}x$$

§3 定积分计算

设函数$F(x)$是连续函数$f(x)$在区间$[a,b]$上的一个原函数，则：

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

1. 定积分的分部积分法：

$$\int_a^b u\,\mathrm{d}v = uv|_a^b - \int_a^b v(x)u'(x)\,\mathrm{d}x$$

2. wallias 公式：

$$\int_0^{\frac{\pi}{2}} \sin^n x\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x\,\mathrm{d}x = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\ldots\cdot\frac{2}{3}\cdot 1, n\text{为大于1的奇数} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\ldots\cdot\frac{1}{2}\cdot\frac{\pi}{2}, n\text{为正偶数} \end{cases}$$

$$\int_0^{\pi} \sin^n x\,\mathrm{d}x = \begin{cases} 2\cdot\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\ldots\cdot\frac{2}{3}\cdot 1, n\text{为大于1的奇数} \\ 2\cdot\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\ldots\cdot\frac{1}{2}\cdot\frac{\pi}{2}, n\text{为正偶数} \end{cases}$$

$$\int_0^{\pi} \cos^n x\,\mathrm{d}x = \begin{cases} 0, n\text{为正奇数} \\ 2\cdot\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\ldots\cdot\frac{1}{2}\cdot\frac{\pi}{2}, n\text{为正偶数} \end{cases}$$

$$\int_0^{2\pi} \sin^n x\,\mathrm{d}x = \int_0^{2\pi} \cos^n x\,\mathrm{d}x = \begin{cases} 0, n\text{为正奇数} \\ 4\cdot\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\ldots\cdot\frac{1}{2}\cdot\frac{\pi}{2}, n\text{为正偶数} \end{cases}$$

3. Γ函数
   definition：

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}\,\mathrm{d}x, n > 0$$

递推式:

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

$\Gamma(1) = 1, \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$ 故$\Gamma(n+1) = n!$