Introduction to Control Theory

The material is based on the Control Theory course, Innopolis, Spring 2024,
Steve Brunton's lectures

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Contents

1 Introduction

Dynamical systems

Let's consider the n-th order ordinary differential equation (ODE):

$$\mathbf{x}^{(n)} = \mathbf{f}(\mathbf{x}^{(n-1)}, \mathbf{x}^{(n-2)}, ..., \ddot{\mathbf{x}}, \dot{\mathbf{x}}, \mathbf{x}, t),$$

where $x(t)$ is a solution for the system, and t is an independent variable (usually time). This equation represents the dynamics of the system and it is called a **dynamical system**.
x is called the **state** of the dynamical system.

In canonical form, linear ODE is represented in the following way:

$$a_n z^{(n)} + a_{n-1} z^{(n-1)} + ... + a_2 \ddot{z} + a_1 \dot{z} + a_0 z = b_0$$

The set $\{\mathbf{x},\ \dot{\mathbf{x}}\ ...,\ \mathbf{x}^{(n-1)}\}$ is called the **state of the system**.
State of the system is a minimal set of variables that describe the system. Based on the current state and future inputs, we can predict the behaviour of the system.

A few examples of variables in a dynamical system include position, velocity, acceleration, electric charge, magnetic field, etc.

1.1 Input

General form of an n-th order linear ODE with an input can be presented as follows:

$$a_n y^{(n)} + ... + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = u(t) \tag{1}$$

State-space representation of a linear system with an input is:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{2}$$

A is called a **state matrix** and x is a **state vector**,
B is called a **control matrix** and u is a **control vector**.
u might be either a scalar or a vector.
<u>*Note:*</u> The transition from ODE to state-space representation is described further.

1.2 Output

Equations might also have an output, which can have plenty of physical meanings and interpretations. Let's list some of them: what we measure (position and orientation of a motor), what we want to control (the height of the quadrotor).
Output is usually defined as y.
Example of system wth input and output:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \tag{3}$$

If u and y are scalars, the system is called *single-input single-output (SISO)*, if they are vectors - *multi-input multi-output (MIMO)*.

Linear systems

In case if relationships between state, output and control are linear, we can model system in the following form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases} \tag{4}$$

Where

- $\mathbf{x} \in \mathbb{R}^n$: states of the system
- $\mathbf{y} \in \mathbb{R}^l$: output vector
- $\mathbf{u} \in \mathbb{R}^m$: control inputs
- $\mathbf{A} \in \mathbb{R}^{n\times n}$: state matrix
- $\mathbf{B} \in \mathbb{R}^{n\times m}$: input matrix
- $\mathbf{C} \in \mathbb{R}^{l\times n}$: output matrix
- $\mathbf{D} \in \mathbb{R}^{l\times m}$: feedforward matrix

If matrices A, B, C, D are time-independent, then we call such systems **time-invariant.**
More frequently we work with systems when the output does not depend on the control.

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \tag{5}$$

1.3 ODE to State-Space conversion

$$\dddot{y} + a_2\ddot{y} + a_0 y = u$$

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \\ \dddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix}$$

TO DO: explain it
Define the state-space variables:

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= \frac{dy(t)}{dt} \\ x_3(t) &= \frac{d^2y(t)}{dt^2} \\ &\vdots \\ x_n(t) &= \frac{d^{n-1}y(t)}{dt^{n-1}} \end{aligned}$$

They will comprise the state vector $\mathbf{x}$. The vector $\dot{\mathbf{x}}$ is essentially the derivative of the state vector. $\dot{\mathbf{x}}$ includes the variable with the highest degree (n), while the variable with the highest degree in a state vector $\mathbf{x}$ is of the order (n-1).

Example:

$$\ddot{x} + 3\dot{x} - x = 10u$$

Solution

$$\begin{cases} \dot{x} = \dot{x} \\ \ddot{x} = 10u + x - 3\dot{x} \end{cases}$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 10u \end{bmatrix}$$

1.4 State-Space to ODE conversion

$$\begin{cases} \dot{x} = Ax \\ y = Cx \end{cases}$$

We need to represent this system as an ODE in the form:

$$y^{(n)} = d_{n-1}y^{(n-1)} + d_{n-2}y^{(n-2)} + \cdots + d_1\dot{y} + d_0$$

Let's take the derivative of y:

$$\dot{y} = C\dot{x} = CAx$$

$$\dots$$

$$y^{(n)} = CA^{(n)}x$$

$$y = \begin{bmatrix} C \\ CA \\ \dots \\ CA^{(n-1)} \end{bmatrix} x = Ox,$$

O is called the **observability matrix**.

$$x = O^{-1}y$$

Then,

$$y^{(n)} = CA^{(n)}x = CA^{(n)}O^{-1}y = CA^{(n)}O^{-1} \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix}$$

2 Stability

2.1 Critical Point (Node)

Critical Point (Node)

Consider the following LTI:

$$\dot{x} = f(\mathbf{x}, t)$$

x_0 is called a **Node**, or **Critical Point**, if $f(x_0) = 0$.

2.2 Stability

A system is **stable** if:

$$\|x(0) - x_0\| < \delta \quad \|x(t) - x_0\| < \epsilon$$

We can think of it as: if the starting point is in the δ-neighborhood of the node x_0, the rest of the trajectory $x(t)$ is in the ϵ-neighborhood of the node.

Or, the solutions starting from the δ-sized ball do not diverge.

Asymptotic Stability

A system is **asymptotically stable** if:

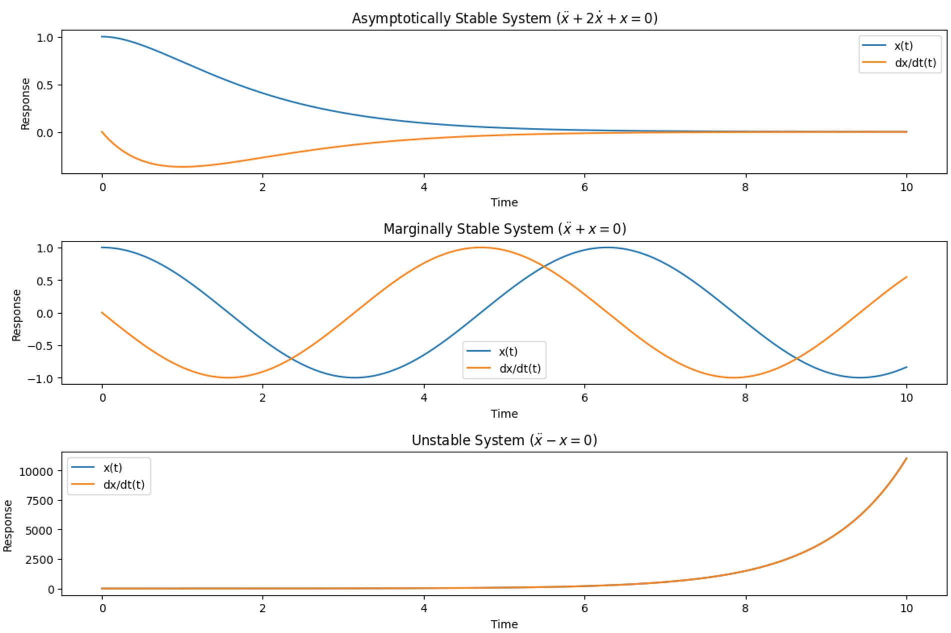
$$\|x(0) - x_0\| < \delta \to \lim_{t\to\infty} x(t) = x_0$$

For any initial point that lies in the δ-sized ball, the trajectory will asymptotically approach the node (point x_0). Or, the solutions starting from the δ-sized ball, converge to the node.

Lyapunov Stability: A more general concept applicable to all systems, ensuring that trajectories remain close to the equilibrium point if they start close enough. Assymptotically stable systems are stable in the sense of Lyapunov.

An equilibrium point which is Lyapunov stable but not asymptotically stable is called marginally stable point

Marginal Stability: Specific to LTI systems and defined by the eigenvalues' locations. It implies that the system's response neither grows unbounded nor decays, often resulting in sustained oscillations.



2.3 Stability of autonomous LTI

Autonomous Systems

A system is considered **autonomous** if its evolution depends only on time.

Example

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

2.3.1 Diagonal matrices

Let's introduce a trick for autonomous Linear Time-Invariant (LTI) systems.
First of all, recall the properties of a diagonal matrix and eigen-decomposition.

- **Diagonal matrix**

$$\dot{\mathbf{z}} = \mathbf{D}\mathbf{z}$$

$$\begin{cases} \dot{x}_1 = d_1 x_1 \\ \dots \\ \dot{x}_n = d_n x_n \end{cases}$$

The solution of each of the equations is: $x_i = C_i e^{d_i t}$. So, the system is asymptotically stable when for all i, $d_i < 0$. The system is stable when $d_i \leq 0$.

- **Eigen-decomposition**
We can represent the matrix as $A = VDV^{-1}$, where D is a diagonal matrix.

Given an autonomous Linear Time-Invariant (LTI) system, let's switch to the system with a diagonal matrix:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

$$\dot{\mathbf{x}} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x}$$

$$\mathbf{V}^{-1}\dot{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x} = \mathbf{D}\mathbf{V}^{-1}\mathbf{x}$$

Change of variables: $\mathbf{z} = \mathbf{V}^{-1}\mathbf{x}$, $\dot{\mathbf{z}} = \mathbf{D}\mathbf{z}$,

$$\mathbf{V}^{-1}\dot{\mathbf{x}} = \mathbf{D}\mathbf{z}$$

The system is asymptotically stable when for all the elements of D are < 0. The system is stable when the elements of D are ≤ 0.

2.3.2 Upper triangular matrices

Eigenvalues of upper triangular matrices are the diagonal elements.

2.3.3 General case

Consider the LTI:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

The system is called **stable** iff real parts of eigenvalues of A are non-positive.

The system is called **asymptotically stable** iff real parts of eigenvalues of A are strictly negative.

2.4 Pole placement method

Pole placement is a fundamental and useful method for achieving stabilization in control systems. With this method, we can manually choose the locations of the system's poles, which correspond to the points where the system would ideally converge to.

Example:

Check stability of the system and in case the system is unstable, stabilize it using the Pole Placement method:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Let's check the eigenvalues:

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{bmatrix}\right) = \lambda(1+\lambda) - 2 = \lambda^2 + \lambda - 2$$

Eigenvalues $\lambda_1, \lambda_2 = -2, 1$. Since one of the roots is positive, the system is unstable. Let $U = -K$, Then, $\dot{x} = Ax - BKx$.

Consider the matrix $A - BK$:

$$A - BK = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -k_1 - \lambda & 1 - k_2 \\ 2 & -1-\lambda \end{bmatrix}$$

Let's choose negative eigenvalues -1 and -2. Solving the system of two equations, we get $k_1 = 2$, $k_2 = 1$. And the control law will be:

$$u = -Kx = \begin{bmatrix} 2 & 1 \end{bmatrix} x$$

3 Control

The main tasks of control theory include **control design**, **trajectory tracking**, and **point-to-point control**.

3.1 Control Design

Stabilizing control

The task of **stabilizing control** is defining the control law that makes a certain solution of a dynamical system stable.

This is true for both linear and nonlinear systems.
Consider a Linear Time-Invariant (LTI) system:

$$\dot{x} = Ax + Bu,$$

and choose the control as a linear function of the state:

$$u = -Kx,$$

where K is the **control gain**.

Stability Condition

Then, the closed-loop system can be represented as:

$$\dot{x} = (A - BK)x.$$

The system is asymptotically stable if the eigenvalues of the matrix $A - BK$ have strictly negative real parts.
Or, matrix $(A - BK) \in \mathcal{H}$ should be Hurwitz.

3.2 Trajectory Tracking

The task is to stabilize the system around a reference trajectory.
Let $x^*(t)$ and $u^*(t)$ be solutions for the system $\dot{x} = Ax + Bu$. This means that:

$$\dot{x}^* = Ax^* + Bu^*.$$

Define the error as $e = x^* - x$ and $v = u^* - u$.
Then, the error dynamics become:

$$\dot{e} = Ae + Bv.$$

To stabilize the system, suggest $v = -Ke$, then:

$$\dot{e} = (A - BK)e.$$

We choose such control gains that will make $(A - BK) \in \mathcal{H}$ (and therefore make the system stable).
And then, the control law becomes:

$$u = u^* + K(x^* - x).$$

3.3 Point-to-Point Control

Point-to-point control differs from trajectory tracking in that the reference input is constant, $x^* = \text{const}$, and feed-forward control is also constant, $u^* = \text{const}$.
Since the error dynamics and the stabilizing control are the same as in trajectory tracking, the control law becomes:

$$u = K(x^* - x) + u^*.$$

The dynamics of the system become:

$$\dot{x} = (A - BK)x + BKx^* + Bu^*.$$

4 Laplace Transform

4.1 Laplace Transform

The Laplace transform is an integral transformation that converts a function of a real variable t (time domain) to a function of a complex variable s (frequency domain).

We can think of Laplace transform as a general case of Fourier transform (Steve Brunton).

The Laplace transform is a tool for solving differential equations by transforming them into algebraic equations.

The Laplace transform of a function $f(t)$ is given as:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt \tag{6}$$

where $F(s)$ is called an **image** of the function and $s = \alpha + \beta i$ is a complex frequency.

4.1.1 Some Useful Properties

Linear properties:

$$\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} \tag{7}$$

$$\mathcal{L}\{af(t)\} = a\mathcal{L}\{f(t)\} \tag{8}$$

Final value theorem:

$$f(\infty) = \lim_{s\to 0} sF(s) \tag{9}$$

The final value theorem is useful because it gives the long-term behavior for a particular function.

4.1.2 Inverse Laplace Transform

The inverse Laplace transform transforms the image of your function $F(s)$ from the frequency domain to the time domain $x(t)$:

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} e^{st}F(s)\,ds \tag{10}$$

However, in practice, we mostly use precalculated Laplace transforms and then try to decompose the image $X(s)$ into known transforms of functions obtained from the table, and construct the inverse by inspection, or just use some symbolic routines.

4.2 Laplace Transform of a Function's Derivative

For us, one of the most useful properties of Laplace transform is that if we apply it to the derivative of a given variable, it will result in the following:

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = s\mathcal{L}\left(x\right) = sX(s) \tag{11}$$

which is true for $x(0) = 0$.

Thus, we can define a derivative operator:

$$\frac{dx}{dt} \xrightarrow{\mathcal{L}} sX(s) \tag{12}$$

The proof is as follows, using the definition of Laplace transform:

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} = \int_0^\infty \frac{dx}{dt}e^{-st}dt \tag{13}$$

Then using integration by parts:

$$\int_0^\infty \frac{dx}{dt}e^{-st}dt = \left[xe^{-st}\right]_0^\infty - \int_0^\infty -se^{-st}xdt \tag{14}$$

which yields:

$$\left[xe^{-st}\right]_0^\infty + s\int_0^\infty e^{-st}xdt = x(0) + s\mathcal{L}\{x(t)\} = x(0) + sX(s) \tag{15}$$

By induction, it can be shown that:

$$\mathcal{L}\left\{\frac{d^n x}{dt^n}(t)\right\} = s^n \cdot \mathcal{L}\{x(t)\} + s^{n-1}x(0) + \cdots + x^{(n-1)}(0) \tag{16}$$

4.2.1 Applications to Linear ODEs

Let us consider the following ODE:

$$a_n x^{(n)} + a_{n-1}x^{(n-1)} + ... + a_2\ddot{x} + a_1\dot{x} + a_0 x = u_m b^{(m)} + b_{m-1}u^{(m-1)} + ... + b_2\ddot{u} + b_1\dot{u} + b_0 u \tag{17}$$

Notice that we introduce a new variable that we call the input u (control).

Applying the inverse Laplace transform with zero initial conditions yields:

$$\begin{aligned} &a_n s^{(n)}X(s) + a_{n-1}s^{(n-1)}X(s) + \cdots + a_2 s^2 X(s) + a_1 sX(s) + a_0 X(s) \\ &= b_m s^{(m)}U(s) + b_{m-1}s^{(m-1)}U(s) + \cdots + b_2 s^2 U(s) + b_1 sU(s) + b_0 U(s) \end{aligned} \tag{18}$$

Example 1

$$\ddot{y} + a\dot{y} + by = u \tag{19}$$

$$S^2 Y(S) + ASY(S) + BY(S) = U(S) \tag{20}$$

$$Y(S) = \frac{1}{(S^2 + AS + B)}U(S) \tag{21}$$

This form is called a **transfer function**.

Example 2

Consider the ODE with the input u.

$$2\,\ddot{y} + 5\dot{y} - 40y = 10u \tag{22}$$

$$2SY(S) - 4Y(S) = U(S) \tag{23}$$

$$Y(S)(2S - 4) = U(S) \tag{24}$$

$$Y(S) = \frac{1}{(2S - 4)}U(S) \tag{25}$$

4.3 State-Space to Transfer Function

Let's study the relation between the input and the output of the dynamical system.

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Let's consider the 1st equation and rewrite it in the form:

$$SIX(S) = AX(S) + BU(S)$$
$$(SI - A)X(S) = BU(S)$$
$$X(S) = (SI - A)^{-1}BU$$

Then the initial system can be rewritten in the following form:

$$\begin{cases} X = (SI - A)^{-1}BU \\ Y = C((SI - A)^{-1}B + D)U \end{cases}$$

5 Bode

By steady-state we mean that initial conditoins have stopped playing a role. After some time passed we expect the output of the system to not depend on the initial conditions, but on the input.

Consider a system:

$$Y(s) = G(s)U(s),$$

$G(s)$ is a transfer function, $U(s)$ is a Laplace space input.

$u(t) = sin(\omega t)$ in time domain takes form $\frac{\omega}{\omega^2+s^2}$ in Laplace domain.

$$Y(s) = G(s)\frac{\omega}{\omega^2 + s^2}$$

,

If a transfer function is a rational fraction, it can be reprresented in the following way:

$$G(s) = \frac{n(s)}{(s+p_1)(s+p_2)+\cdots+(s+p_n)} = \frac{r_1}{s+p_1} + \cdots + \frac{r_n}{s+p_n}$$

p_i in this equations are the **poles**.

Previously, we introduced the stability analysis based on eigenvalues, but there is an equvivalent analysis based on poles.

A Bode plot usually consists of magnitude and phase response of a transfer function.

Transfer functions in s-domain quickly become cumbersome to analyse as the control system gets complicated. It's easy to understand the critical properties of the system by looking at the Bode plot.

6 Discrete dynamics

Discrete case for dynamics:

$$x_{i+1} = Ax_i + Bu_i$$

Such an equation is easy to work with: no derivatives are used here, it is easy to simulate.
For this system, we can propose the control:

$$u_i = -Kx_i + u_i*$$

6.1 Stability

Let's consider the system:

$$x_{i+1} = Ax_i$$

$$x_{i+1} = V^{-1}DVx_i$$

$$Vx_{i+1} = VV^{-1}DVx_i$$

The change of variables: $z_i = Vx_i$, $z_{i+1} = Vx_{i+1}$

$$z_{i+1} = DVz_i$$

Let's consider the following example:

$$\begin{bmatrix} \vec{x}_{1,i+1} \\ x_{2,i+1} \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}$$

Let's find the norms of x_{i+1} and x_i.

$$\| \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} \|^2 = x_{1,i}^2 + x_{2,i}^2$$

$$\| \begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} \|^2 = \left| \begin{bmatrix} \alpha x_{1,i} - \beta x_{2,i} \\ \beta x_{1,i} + \alpha x_{2,i} \end{bmatrix} \right|^2 = (\alpha x_{1,i} - \beta x_{2,i})^2 + (\beta x_{1,i} + \alpha x_{2,i})^2$$
$$= \beta^2(x_{2,i}^2 + x_{1,i}^2) + \alpha^2(x_{2,i}^2 + x_{1,i}^2) = (\alpha^2 + \beta^2)(x_{1,i}^2 + x_{2,i}^2)$$

For stability we introduce:
Continuous case: Hurwitz matrix. **Discrete case:** Schur matrix.

Dynamical Systems

- Discrete system $x_{i+1} = Ax_i$ is **stable** if and only if the absolute values of A's eigenvalues are less than or equal to 1: $|\lambda_i(A)| \leq 1$.
- Discrete system $x_{i+1} = Ax_i$ is **asymptotically stable** if and only if the absolute values of A's eigenvalues are less than 1: $|\lambda_i(A)| < 1$.

6.2 Analytical solution of Continuous LTI

Case: $\dot{x} = ax(t)$

Recall that we can describe exponent as a series:

$$e^a = 1 + a + \frac{a^2}{2} + \frac{a^3}{6} + \dots$$

$$e^A = I + A + \frac{AA}{2} + \frac{AAA}{6} + \dots$$

Suppose that the solution of the dynamical system is:

$$x(t) = e^{at}x(0)$$

$$x(t) = e^{At}x(0)$$

Let's rewrite the exponent:

$$x(t) = (I + At + \frac{AAt^2}{2} + \frac{AAAt^3}{6} + \dots)x(0)$$

$$\dot{x}(t) = (A + AAt + \frac{AAAt^2}{2} + \dots)x(0)$$

$$\dot{x}(t) = A(I + At + \frac{AAt^2}{2} + \dots)x(0)$$

$$\dot{x}(t) = Ae^{At}x(0)$$

$$\dot{x}(t) = Ax(t)$$

<u>**Case:** $\dot{x} = ax(t) + bu(t)$</u>

$$\dot{x} = ax(t) + bu(t)$$

$$\dot{x}e^{-at} - ae^{-at}x(t) = be^{-at}u(t)$$

$$\frac{d}{dt}(xe^{-at}) = be^{-at}u(t)$$

$$\int_0^t \frac{d}{d\tau}(e^{-a(t-\tau)}x(\tau))d\tau = \int_0^t \frac{d}{d\tau}(e^{-a(t-\tau)}bu(\tau))d\tau$$

$$e^{-at}x(t) - x(0) = \int_0^t e^{-a\tau}bu(\tau)d\tau$$

$$x(t) = e^{at}x(0) + e^{at}\int_0^t e^{-a\tau}bu(\tau)d\tau$$

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau$$

$$\boxed{x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}$$

6.3 Discretization

$$\begin{cases} x_0 = x(0) \\ x_1 = x(\Delta t) \\ x_2 = x(2\Delta t) \\ \dots \\ x_n = x(n\Delta t) \end{cases}$$

$$\dot{x} = \frac{1}{\Delta t}(x_{i+1} - x_i)$$

$$Ax_i = \frac{1}{\Delta t}(x_{i+1} - x_i)$$

$$x_{i+1} = Ax_i\Delta t + x_i$$

$$\boxed{x_{i+1} = (A\Delta t + I)x_i}$$

or

$$\boxed{x_{i+1} = (I - A\Delta t)^{-1}x_i}$$

Thus, the discrete state matrix is: $A_d = A\Delta t + I$ The control matrix is: $B\Delta t$.

7 LQR

We can consider stability as a necessary requirement, but just because we are stable doesn't necessarily mean that we are performing at our best. How can we do the best control? This is the question of optimality.

There is also another way that we can think of LQR. In pole-placement method, we want to place the poles in the specific spots (or, we choose specific eigenvalues). But it is not intuitive where to place them, especially for complex systems, systems with numerous actuators. So, the new method is proposed. The key concept of the method lies in optimization of choosing K.

The optimal control aims to stabilize the system while minimizing the time spent in the process.

In LQR we find an optimal K by choosing parameters that are important to us, specifically how well the system performs and how much effort it takes to reach this performance.

7.1 Cost. Optimal cost

Let's introduce $u = \pi(x)$ - a **control policy**. It is like a recipe for obtaining u when we know x and t. Whereas control policy is a function, control is its output.

How to find the best control policy?

Let's introduce an **additive cost function** J:

$$J(x_0, \pi(x,t)) = \int_0^\infty g(x,u),$$

It basically means that at each time stamp, we always add to the cost.

$x_0 = x(0)$ in the equation is the initial conditions.

$g(x,u)$, **instantaneous cost**, is a subject of our choice.

Function $g(x,u) \geq 0$ can be interpreted as a rate of change of cost.

Note that the additive cost function depends on the initial state x_0 rather than x since the trajectory x itself depends on initial conditions and control law. Consider an example with a robot arm that is supposed to grip an object. If in the initial position this robot arm is far away from the object, then the control policy is to be high.

Applying the optimal control policy $\pi^*(x)$ to the system $\dot{x} = f(x,u)$, we obtain optimal dynamics $\dot{x} = f^*(x,u) = f(x, \pi^*(x))$.

Given initial condition $z = x_0$, we can obtain the optimal trajectory $x^* = x^*(t,z)$.

Knowing the optimal trajectory and the optimal control policy, we can calculate the optimal cost: $J^*(z) = J(z, \pi^*(x))$. And the optimal instantaneous cost is: $g^*(x) = g(x, \pi^*(x))$.

Let J^* be the optimal (lowest possible) cost.

$$J^*(x_0) = \inf_\pi J(x_0, \pi(x,t))$$

We obtain this optimal cost via the optimal control policy $\pi = \pi^*(x,t)$.

This is a key subject in our solution, since if we can find the optimal control policy, we can define the optimal trajectory (recall that it depends on the policy and initial conditions).

7.2 Optimality conditions

When is our policy the best policy? When is our cost the best cost?

Hamiltonian-Jacobi-Bellman (HJB)

$$\min_u [g(x,u) + \frac{dJ}{dx} f(x,u)] = 0$$

We can see that $\frac{dJ}{dx} f(x,u)$ is a derivative of J w.r.t t (by the chain rule).

We can think of $g(x,u)$ as $\frac{\partial J}{\partial t}$. Thus, the HJB equation seems as an attempt to get the full derivative of J

$$u^* = \arg\min_u \left[g(x,u) + \frac{dJ}{dx} f(x,u) \right]$$

7.3 Algebraic Riccati

For LTI, dynamics is:

$$\dot{x} = Ax + Bu$$

We can choose quadratic cost:

$$g(x, u) = x^\top Qx + u^\top Ru$$

R and Q are symmetric and positive-definite (CHECK!!! which is semi-definite) since we do not want the cost to be negative.

Then HJB becomes:

$$J^* = x^T Sx$$

$$\frac{\partial J^*}{\partial t} = \dot{x}^T Sx + x^T S\dot{x}$$

$$\min_u [x^T Qx + u^T Ru + x^T S(Ax + Bu) + (Ax + Bu)^T Sx] = 0$$

After simplification:

$$\min_u [x^T (Q + SA + A^T S)x + u^T Ru + x^T SBu + u^T B^T Sx] = 0$$

Let's take its derivative w.r.t u and set it to 0:

$$2u^T R + 2x^T SB = 0$$

$$u^T R + x^T SB = 0$$

Let's transpose this equation and reconfigure it:

$$R^T u = -B^T Sx$$
$$u = -(R^T)^{-1} B^T Sx$$

Recall that R is symmetric

$$u = -R^{-1} B^T Sx$$

The control law:

$$u = -R^{-1} B^T Sx$$

is called **LQR (linear quadratic regulator)**

u is proportional to S, which is a **cost-to-go**, R is a cost matrix on the control actions.

How to define S?

Substituting the found optimal control back into the HJB, we get:

$$\min_{\mathbf{u}} [\mathbf{x}^\top (\mathbf{Q} + \mathbf{SA} + \mathbf{A}^\top \mathbf{S})\mathbf{x} + \mathbf{x}^\top \mathbf{SBR}^{-1}\mathbf{RR}^{-1}\mathbf{B}^\top \mathbf{Sx} - \\ -\mathbf{x}^\top \mathbf{SBR}^{-1}\mathbf{B}^\top \mathbf{Sx} - \mathbf{x}^\top \mathbf{SBR}^{-1}\mathbf{B}^\top \mathbf{Sx}] = 0 \tag{26}$$

Simplifying, we get:

$$\mathbf{x}^\top (\mathbf{Q} + \mathbf{SA} + \mathbf{A}^\top \mathbf{S} - \mathbf{SBR}^{-1}\mathbf{B}^\top \mathbf{S})\mathbf{x} = 0 \tag{27}$$

which would hold for all $\mathbf{x}$ iff:

$$\mathbf{Q} - \mathbf{SBR}^{-1}\mathbf{B}^\top \mathbf{S} + \mathbf{SA} + \mathbf{A}^\top \mathbf{S} = 0 \tag{28}$$

This is the ***Algebraic Riccati equation***.

7.4 LQR vs Pole-Placement

Pole-Placement: may require unreasonably high control gains. LQR: easy to produce reasonable control gains.

Let's introduce the **cost-to-go** V_i which represents a difference between the optimal cost and the incurred cost.

$$V_i = J^*(z) - S_i$$

The function V_i is monotonously decreasing with the rate of change $-g^*(x)$.

Let J be an additive cost function:

$$J(x_0, p(x,t)) = \int_0^\infty g(x,u),$$

Intuition behind cost function: Q - how bad if x is not where it is supposed to be. Q - nonnegative, positive semidefinite.

if the system is a positions, velocity, and $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix}$ we penalize for

Suppose there is the best control law:

$$u = -kx$$

that minimizes the quadratic cost function.

$$J = \int x^T Q x + u^T R u$$

Cost on effectiveness and energy to reach this effectiveness. The optimal control is supposed to make the system stable and spend on it as less time as possible.

8 Observer

When it comes to applying control in real practice, how do we do it? Quite frequently, the full real state is unavailable to us. The only option is to try *estimating* it. We can do this using sensors, cameras, for example.

But in real life, the task is not that trivial due to some problems:

1. Lack of sensors. For a quadrotor, we cannot measure the height straightforwardly.
2. Measurements can be imprecise or biased.
3. Measurements can only be made in discrete time.

The key problem arises when the output of the system y is not the whole state x, but $y = Cx$, which means that we are only able to get a partial state.

These are just several problems that create difficulties for us to measure the state of the system.

This brings us to the motivation of introducing an observer to estimate the state x. With the observer, we will be able to check the current state of the system.

We already know how to create a system with a controller, but how do we check the current state of the system?

$$\begin{cases} \dot{x} = Ax + Bu \\ u = -Kx \end{cases}$$

8.1 Measurement and estimation

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \\ \hat{x}(t) = \text{estimate}(y(t)) \\ u = -K\hat{x} \end{cases}$$

x and y are the state and output (actual or true). When we do not know the exact state x, we can only estimate it. We estimate x ($\hat{x}$) based on the history of y values. The control law is now governed by the estimated state $\hat{x}$.

Estimation error

State estimation error is the following:

$$\epsilon = \hat{x} - x$$

8.2 Dynamics estimation

We can always find $\tilde{y} = C\hat{x} - y$.

$$\tilde{y} = C\hat{x} - y = C\hat{x} - Cx = C\epsilon$$

Let's suggest that the dynamics should also hold for the observed state:

$$\dot{\hat{x}} = A\hat{x} + Bu$$

Let's introduce in our equation a linear correction law $-L\,\tilde{y}$. Since $\tilde{y} = C\hat{x} - y$, we get:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{29}$$

This is called the Luenberger observer.

But how to find a suitable observer gain L?

Let's subtract $\dot{x} = Ax + Bu$ from Equation (29) . The equation we got is the observer error dynamics:

$$\dot{\hat{x}} - \dot{x} = A\hat{x} - Ax + L(y - C\hat{x}) \tag{30}$$

$$\dot{\hat{x}} - \dot{x} = A(\hat{x} - x) + L(Cx - C\hat{x}) = A\hat{x} - Ax + LC(x - \hat{x}) \tag{31}$$

$$\dot{\hat{x}} - \dot{x} = A\hat{x} - Ax + LC(\hat{x} - x) \tag{32}$$

$$\dot{\epsilon} = A\epsilon - LC\epsilon = (A - LC)\epsilon \tag{33}$$

With no knowledge of x nad $\hat{x}$, we can define the stability of the system.

What we want is the error converging to 0. To obtain this, the observer $\dot{\epsilon} = (A - LC)\epsilon$ needs to be stable. $(A - LC) \in \mathcal{H}$.

Recall:

- Controller design: find such K that $A - BK \in \mathcal{H}$.
- Observer design: find such L that: $A - LC \in \mathcal{H}$

But now the gain L is in the left side for the observer (unlike K for the controller), so we cannot use any stabilization methods (LQR, pole placement) right away.

We need to introduce the following change (or, we can solve the *dual problem*), find such L that:

$$A^T - C^T L^T \in \mathcal{H}$$

And now, for this equation we can use LQR or pole placement.

8.3 Observer + Controller

$$\begin{cases} \dot{x} = Ax - BK\hat{x} \\ \dot{\hat{x}} = A\hat{x} - BK\hat{x} + LC(x - \hat{x}) \end{cases}$$

$$\begin{cases} \dot{x} = Ax - BK\hat{x} \\ \dot{\hat{x}} = A\hat{x} - BK\hat{x} - LC(\hat{x} - x) \end{cases} \tag{34}$$

Let's do a change of variables so that it would be easier to analyze the eigenvalues: $e = x - \hat{x}$, $\dot{e} = \dot{x} - \dot{\hat{x}}$. Let's subtract (1) of 34 from (2) of 34:

$$\dot{e} = A\hat{x} - BK\hat{x} - LC(\hat{x} - x) - Ax + BK\hat{x}$$
$$\dot{e} = Ae - LCe$$

$$\dot{e} = (A - LC)e \tag{35}$$

Now let's turn back to the equation 1 of the system and rewrite it.

$$\dot{x} = Ax - BK\hat{x} = Ax - BK(x - e)$$

$$\dot{x} = (A - BK)x + BKe \tag{36}$$

$$\begin{cases} \dot{e} = (A - LC)e \\ \dot{x} = (A - BK)x + BKe \end{cases}$$

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

The eigenvalues of the system are the union of eigenvalues of $(A - BK)$ and $(A - LC)$.

Separation Principle

As long as the observer and the controller are stable independently, the overall system is stable too.

9 Filters

In literature, the system is frequently called a Plant, which in our case is actually a robot.

Static system

Linear systems for which the relation between input and output is constant are called **static systems**.

Example
(Laplace domain) $Y(s) = 10X(s)$.
Note that when we are talking about independency on time not the signal, but the relation has to be time-independent.

Dynamic system

Dynamic systemns are linear systems for whhich the relation between input and output depends on time.

Dynamic system Example 1 State-space plant:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Apparently, the output y does not depend on the input u.
Example 2
Laplace plant: $Y(S) = \frac{1}{s^2+2s+7}X(s)$ Time does not appear in this equaiton, but if we rewrite it into ode form:

$$\ddot{y} + 2\dot{y} + 7y = x$$

Example 3 ODE plant:

$$\ddot{y} + 5\dot{y} + y = u$$

For these systems, the output depends not only on the current (time-wise) value of input, but on the entire history of input values.
Static systems form algebraic linear equaitons. Dynamical systems create linear differential equaitons
Dynamic systems have **state**, which changes over the time.
For the system we can create the controller and Luenbberger observer:

10 Controllability. Observability

Not always can we control the system. The same applies to an observer. Sometimes, we cannot obtain the full information that we desire from the system.

Let's suggest the tools which can inform us whether we can succeed in designing a controller or observer or not.

Controllability

A system is controllable on $t_0 \leq t \leq t_f$ if it is possible to find a control input $u(t)$ that would drive the system to a desired state $x(t_f)$ from any initial state x_0.

Observability

A system is observable on $t_0 \leq t \leq t_f$ if it is possible to exactly estimate the state of the system $x(t_f)$, given any initial estimation error.

Observability (Alternative)

A system is observable on $t_0 \leq t \leq t_f$ if any initial state x_0 is uniquely determined by the output $y(t)$ on that interval.

Cayley-Hamilton Theorem: A matrix M satisfies its own characteristic equation.

A characteristic equation can be written like:

$$\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ldots + a_0 = 0$$

So,

$$M^n + a_{n-1}M^{n-1} + a_{n-2}M^{n-2} + \ldots + a_0 I = 0$$

Meaning that M^n is a linear combination of $M^{n-1} + M^{n-2} + \ldots + I$. Remember that if the system is controllable, we would be able to find such controller to the system, whcih would make the system stable, which means that the error goes to zero.

10.1 Controllability of Discrete LTI

Consider discrete LTI.

$$x_{i+1} = Ax_i + Bu_i$$

Let's expand it numerically:

$$x_2 = Ax_1 + Bu_1$$

$$x_3 = Ax_2 + Bu_2 = A(Ax_1 + Bu_1) + Bu_2 = A^2x_1 + ABu_1 + Bu_2$$

$$\ldots$$

$$x_{n+1} = A^n x_1 + A^{n-1}Bu_1 + \ldots + A^{n-k}Bu_k + \ldots + Bu_n$$

$$x_{n+1} - A^n x_1 = \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} u_n \\ u_{n-1} \\ \vdots \\ u_1 \end{bmatrix}$$

Thus, we showed that $x_{n+1} - A^n x_1$ must be in the column space of $C = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$

Now, note that since x_{n+1} can be any real number and x_1 can take a zero value, $\mathbb{R}$ must be in the column space of C. So, C must be a full-rank matrix.

Controllability

A system $x_{i+1} = Ax_i + Bu_i$ is controllable (or any state can be reached) if $C = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$ is full row rank.

10.2 Observability of Discrete LTI

Observability criterion shows whether it is in principle possible to estimate the system or not.

Consider a discrete LTI:

$$\begin{cases} x_{i+1} = Ax_i + Bu_i \\ y_i = Cx_i \end{cases}$$

And a Luenberger observer:

$$\hat{x}_{i+1} = A\hat{x}_i + Bu_i + L(y_i - C\hat{x}_i)$$

Change of variables: $e_i = \hat{x}_i - x_i$ will provide us with the following equation:

$$e_{i+1} = Ae_i + L(y_i - C\hat{x}_i)$$
$$e_{i+1} = Ae_i - LCe_i$$

Introduce the dual system (which is stable iff the original system is stable):

$$\epsilon_{i+1} = A^T\epsilon_i - C^TL^T\epsilon_i$$

$$\begin{cases} \epsilon_{i+1} = A^T\epsilon_i - C^Tv_i \\ v_i = -L^T\epsilon_i \end{cases}$$

Controllability matrix for the system:

$$O^T = \begin{bmatrix} C^T & A^TC^T ... & A^{T^{n-1}}C^T \end{bmatrix}$$

$$O = \begin{bmatrix} C \\ CA \\ ... \\ CA^{n-1} \end{bmatrix}$$

Observability

A system $x_{i+1} = Ax_i + Bu_i$ is **observable** (or observation error can go to 0 from any initial position) if $O = \begin{bmatrix} C \\ ... \\ CA \\ CA^{n-1} \end{bmatrix}$ is full column rank.

10.3 Controllability of Continuous-time systems

10.4 PBH controllability criterion

It is an alternative way to test whether the pair (A, B) is controllable.

PBH criterion

The task of **stabilizing control** is defining the control law that makes a certain solution of a dynamical system stable.

11 Kalman Filter

CHECK!

Without a Kalman filter, if your system is not observable, you won't be able to retrieve all the state variables xx from the output yy alone. By using a Kalman filter, you can estimate all the state variables xx even if they are not all directly observable in yy. The filter uses the model of the system and the measurements to infer the values of the unobserved states.

12 Reference materials

(Controllability. Observability)

13 Linearization

While linear models are present in the real world (examples include DC motors, 3D printers, and mass-spring-damper systems), the majority of real-world systems are nonlinear (such as cars, airplanes, robot arms, underwater robots, and almost any mechanical system with linkages).

However, there is a method to describe these systems using our toolkits - through linearization.

Linearization is the technique that enables us to create a linear model for a nonlinear system.

So, we can introduce linear models for nonlinear systems that could describe the system pretty well locally. We can do control for this local approximation.

13.1 Taylor expansion

Taylor expansion around node.

Let's consider a non-linear dynamical system $\dot{x} = f(x, u)$ and a Taylor expansion around the node (x_0, u_0), i.e., $f(x_0, u_0) = 0$, $x_0 = \text{const}$, $u_0 = \text{const}$.

Recall the physical meaning of a **node**: it is any position at which the robot remains static.

$$f(x, u) \approx f(x_0, u_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial u}(u - u_0) + \text{HOT}$$

HOT = high-order terms.

(x_0, u_0) are an expansion point.

Now let's introduce new variables e and v that represent the distances from the expansion point:

$$e = x - x_0$$
$$\dot{e} = \dot{x}$$
$$v = u - u_0$$

Then we can rewrite the Taylor expansion:

$$\dot{e} \approx \frac{\partial f}{\partial x} e + \frac{\partial f}{\partial u} v + \text{HOT}$$

$$\dot{e} \approx Ae + Bv + \text{HOT}$$

Let's drop the high-order term and get an approximation. And this approximation is called linearization.

$$\dot{e} = Ae + Bv$$

In this context, (x_0, u_0) is a linearization point.

This type of linearization is proved to provide the best possible linear model for the given system.

Taylor expansion along a trajectory.

Now let's consider a non-linear dynamical system:

$$\dot{x} = f(x, u)$$

and a trajectory $x_0 = f(x_0, u_0)$. Note that now x_0 is not a point - it is a whole trajectory

$$f(x, u) \approx f(x_0, u_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial u}(u - u_0) + \text{HOT}$$

We can denote $\dot{e} = \dot{x} - \dot{x}_0 = f(x, u) - f(x_0, u_0)$.

The rest stays the same: $e = x - x_0, v = u - u_0$

$$\dot{e} \approx Ae + Bv + \text{HOT}$$

Let's drop HOT and obtain linearization:

$$\dot{e} = Ae + Bv$$

Basically, nothing changes between expansion around a node and expansion along a trajectory as long as we introduce the change of variables described above. The change of variables is slightly different, but original function and the local approximation behave in the same way in the region of the linearization point.

So, can we linearize around each and any point?

1) If it is a node - yes.

2) If we want to linearize around some other point - yes, but the chnage of variables would entail slightly different results.

While we had:

$$e = x - x_0, \dot{e} = \dot{x},$$

now $\dot{e}$ is the differrence between the derivative of the state of our non-linear system and derivative of the state of our trajectory distance from the trajectory linearization.

The meaning of the variables changes slightly, but the mathematical expression is the exact same expression.

Affine expansion

The other way to obtain linearization, without change of varibales is as follows.

$$f(x, u) \approx f(x_0, u_0) + A(x - x_0) + B(u - u_0)$$

Denoting

$$f(x_0, u_0) - Ax_0 - Bu_0 = c$$

$$\dot{x} = Ax + Bu + c$$

c makes it not a linear model, but affine, which means it has a constant term.

It can be a constant in the case of a node, and a function of time in the case of expansion along the trajectory.

Often we choose u to compensate c, so u will also be affine in this case.

13.2 Manipulator equations

Frequently, in robotics we deal with the following equation when describing a system:

$$H\ddot{q} + C\dot{q} + g = \tau$$

which is called a **manipulator equation**

However, cars, underwater robots are usually described by the other equations - Euler or Lagrangian equaitons.

Let's derive linearization for the manipulator equation.

Begin by proposing new variables:

$$x = q \overset{\rightarrow}{-} q_0, \dot{q} \overset{\rightarrow}{-} \dot{q}_0$$

$$u = \tau - \tau_0$$

Points around which we linearize are denoted from equation:

Let's introduce

$$\phi(\dot{q}, q, t) = H^{-1}(\tau - C\dot{q} - g)$$

14 Lyapunov theory

Stability is not defined only for linear systems, but for nonlinear systems as well.

There exist a small neighbourhood. Every solution that starts from this neighbourhood will asymptotically approach the same solution.

Asymptotic stability criteria:
Autonomous dynamic system $\dot{x} = f(x)$ is asymptotically stable if there exists a scalar function $V = V(x) > 0$.

Note that computing $\dot{V}(x)$ comes along with computing $\dot{x}$. We can think of $V(x)$ as energy (pseudo-energy).

Asymptotic stability means that the system converges, and marginal stability means that the system does not diverge.

Asymptotic stability

The system is called asymptotically stable if there exists a function $V > 0$ such that $\dot{V} < 0$, except for $V = 0, \dot{V} = 0$.

Marginal stability

The system $\dot{x} = f(x)$ is stable in the sense of Lyapunov if there exists $V > 0$ such that $\dot{V} \leq 0$.

Lyapunov function

A function $V > 0$ in this case is called a Lyapunov function.

Example 1 Consider the following system:

$$\dot{x} = -x$$

Let's introduce a function $V = x^2 > 0$ for all $x \neq 0$. $\dot{V} = 2x \cdot \dot{x} = 2x(-x) = -2x^2 < 0$. V satisfies the Lyapunov criteria, so the system is (asymptotically) stable.

Example 2 Consider the equation of the pendulum:

$$\ddot{q} = -\dot{q} - \sin(q)$$

14.1 LaSalle's Principle

The system is called stable if

LaSalle's principle allows us to prove asymptotic stability even for $\dot{V}(x) \leq 0$ only for the trivial solution.

14.2 Linear case

Lyapunov theory applies for both nonlinear and linear systems. For linear systems the following feature exist:

For a system $\dot{x} = Ax$, we can always choose a Lyapunov candidate function in the form:

$$V = x^T S x,$$

where S is positive-definite

$$\dot{V} = \dot{x}^T S x + x^T S \dot{x} = (Ax)^T S x + x^T S A x = x^T A^T S x + x^T S A x = x^T (A^T S + SA) x$$