Study Material Template

Your Name

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4.1 Ituition behind poles

In control theory, the system state-space equation

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

has the transfer function

$$G(s) = C(sI - A)^{-1}B + D.$$

Since $(sI - A)^{-1} = \text{adj}(sI - A)\det(sI - A)$, where $\text{adj}(sI - A)$ is the adjugate of $sI - A$, the poles of $G(s)$ are the numbers that satisfy $\det(sI - A) = 0$. This is exactly the characteristic equation of matrix A, whose solutions are the eigenvalues of A.

4.2 LQR

In pole-placement method, we want to place the poles in the specific spots (or, we choose specific eigenvalues). But it is not intuitive where to place them, especially for complex systems, systems with numerous actuators. So, the new method is proposed. The key concept of the method lies in optimization of choosing K.

In LQR we find an optimal K by choosing parameters that are important to us, specifically how well the system performs and how much effort it takes to reach this performance.

If $Q >> R$, then we are turning the problem of Let J be an additive cost function:

$$J(x_0, p(x,t)) = \int_0^\infty g(x,u)$$

Q - how bad if x is not where it is supposed to be. Q - nonnegative, positive semidefinite.

if the system is a positions, velocity, and $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix}$ we penalize for

Suppose there is the best control law:

$$u = -kx$$

that minimizes the quadratic cost function.

$$J = \int x^T Q x + u^T R u$$

Hamiltonian-Jacobi-Bellman (HJB)

$$\min_u [g(x,u) + \frac{dJ}{dx f(x,u)}] = 0$$

Cost on effectiveness and energy to reach this effectiveness.

4.3 Subtopic 2.1

4.4 Subtopic 2.2

5 Controllability. Observability

Controllability

A system is controllable on $t_0 \leq t \leq t_f$ if it is possible to find a control input $u(t)$ that would drive the system to a desired state $x(t_f)$ from any initial state x_0.

Observability

A system is observable on $t_0 \leq t \leq t_f$ if it is possible to exactly estimate the state of the system $x(t_f)$, given any initial estimation error.

Observability (Alternative)

A system is observable on $t_0 \leq t \leq t_f$ if any initial state x_0 is uniquely determined by the output $y(t)$ on that interval.

Observability criterion
Consider discrete LTI.
And a Luenberger observer:

Error dynamics for the observer.
If the system is controllable, we would be able to find such controller to the system, whcih would make the system stable, which means that the error goes to zero.

$$O^T = [C^T (A^T) C^T]$$

Observability matrix
error dynamics need to be stabilized, which means thta it needs t be controllable.
Observability criterion.
Needs to be full column rank.

6 PBH controllability criterion

7 Kalman Filter

The way we pick our initial state estimate does not have a bias.

Assume you could pick your initial state estimate $\hat{x}_0$ such that the initial state estimation error behaves as a random variable sampled from a Gaussian distribution $x_0\ N(0, P_0)$

Knowing mean

$E[w_i] = 0$

All subsequent means will be $E[]$

Let's compute the autocovariance P_{i+1} knowing P_i

$$P_{i+1} = E[] = E[(Ax)]$$

We can assume thta the random process is uncorrelated with x, so

Since the two processes are uncorreleted then their covariance is 0.

$$P_{i+1} = E[Ax_i x_i^T A^T + w_i w_i^T]$$

Note that $x_i^T A^T = P_i - acovarianceofthepreviousstep.P_{i+1} = AP_iA^T + Q$

v_i - the random noise sampled from thhe Gaussian distribution, which represents the sensor error.

$$v_i\ N(0, R)$$

$\hat{x}_{i+1}^-$ is the estimation before measurements - apriori estimate.

$$x_i + 1$$

L_i is a control gain.

Apriori error

$$x_{i+1}^- = x_{i+1} - hatx_{i+1}^-$$

We claculate aposteriori covariance knowing our apriori covariance.

The question is how to minimize those covariances.

f we have all of the measurements up to and including time k available for use in our estimate of X k , then we can form an a posteriori estimate, which we denote as 2:. The "+" superscript denotes that the estimate is a posteriori. One way to form the a posteriori state estimate is to compute the expected value of x k conditioned on all of the measurements up to and including time k: 2; = E [X k / y l ,y 2 , . . ., Y k] = a posteriori estimate

DERIVATIONOF THE DISCRETE-TIME KALMAN FILTER 125 the estimate is a posteriori. One way to form the a posteriori state estimate is to compute the expected value of x k conditioned on all of the measurements up to and including time k: 2; = E [X k / y l ,y 2 , . . ., Y k] = a posteriori estimate (5.3) If we have all of the measurements before (but not including) time k available for use in our estimate of X k , then we can form an a praori estimate, which we denote as 2 ; . The "-"superscript denotes that the estimate is a priori. One way to form the a priori state estimate is to compute the expected value of 51, conditioned on all of the measurements before (but not including) time k: 2; = E [X k l y l , y 2 , . ., Y k - l] = a priori estimate (5.4) It is important to note that 2; and 2; are both estimates of the same quantity; they are both estimates of X k . However, 2 i is our estimate of Xk before the measurement Y k is taken into account, and 2: is our estimate of 21, after the measurement y k is taken into account. We naturally expect 2; to be a better estimate than 2 i , because we use more information to compute 2;: 2 i = 2' k = estimate of Xk after we process the measurement at time k (5.5) If we have measurements after time k available for use in our estimate of X k , then we can form a smoothed estimate. One way to form the smoothed state estimate is to compute the expected value of x k conditioned on all of the measurements that are available: estimate of Xk before we process the measurement at time k ? k l k + N = E [x k l Y l i Y 2 , ' . . , Y k , " ' , Y k + N] = smoothed estimate (5.6) where N is some positive integer whose value depends on the specific problem that is being solved. If we want to find the best prediction of x k more than one time step ahead of the available measurements, then we can form a predicted estimate. One way to form the predicted state estimate is to compute the expected value of Xk conditioned on all of the measurements that are available: 2 k l k - M = E [x k j y 1 , y 2 , - . .,y k -]= predicted estimate

8 Reference materials

(Controllability. Observability)

9 Linearization

Linear models occur only in specific cases: DC motors. mass-spring damper, 3D-printer. Most real-world systems are non-linear.

Taylor expansion around node.

Let's consider a non-linear dynamical system $\dot{x} = f(x, u)$ and a trajectory $x_0 = f(x_0, u_0)$

Node - any position at which robot remains static.

Physical meaning: node is any position where the robot remains static.

The main "magic" aboout the node happens with u_0, which is the control that enables the robot to be static.

$$A = \frac{\partial f}{\partial x}$$

Taylor expansion along a trajectory.

Consider a non-linear dynamical system: $\dot{x} = f(x, u)$ and a trajectory

$$f(x, u) \approx f(x_0, u_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial f}(u - u_0) + \text{H.O.T}$$

Since $\dot{e} = \dot{\xi} - \dot{x}_0$, we re-werite:

$$\dot{e} = Ae + Bv + \text{H.O.T}$$

Let's drop the high-order term and obtain linearization.

$$\dot{e} = Ae + Bv$$

Expansion around the Node and expansion around trajectory stay the same, nothing actually changes.

If we drop the higher-order terms from the Taylor expansion, we obtain linearization of the system dynamics.

$$\dot{e} = Ae + Bv$$

Nothing changes between expansion around a node and expansion along a trajectory. The original function and the local approximation behave in the same way in the region. However, the change of variables is different

So, can we linearaze around each and any point?

1) If it is a node - yes. 2) If we want to linearize around some other point - yes, but the chnage of variables would entail slightly different results.

While we had:

$$e = x - x_0, \dot{e} = \dot{x}$$

, now $\dot{e}$ is the differrence between the derivative of the state of our non-linear system and derivative of the state of our trajectory distance from the trajectory linearization.

The meaning of the variables changes slightly, but the mathematical expression is the exact same expression.

Affine expansion

The other way to obtain linearization, without change of varibales is as follows.

$$f(x, u) \approx f(x_0, u_0) + A(x - x_0) + B(u - u_0)$$

Denoting

$$f(x_0, u_0) - Ax_0 - Bu_0 = c$$

$$\dot{x} = Ax + Bu + c$$

c makes it not a linear model, but affine, which means it has a constant term.

It can be a constant in the case of a node, a function of time in the case of expansion along the trajectory.

Often we choose u to compensate c, so u will also be affine in this case.

Manipulator equations (describe ???)

Cars, underwater robots are desccribed by the other equations - Euler or Lagrangian equaitons.

Consider Manipulator equation - quadratic, symmetric, another form of kinetic energy.

H - generalized inertia matrix

$$x = q \vec{-} q_0$$

We can express $\ddot{q}$ as long as H is invertible. For any mechanical system, H happens to be always invertible.

Let's introduce

$$\phi() = H^{-}1(\tau - C\dot{q} - g)$$

mechanical systems are second-order systems, while the state space is a 1st order system. d() = Let's replace $d()$ with $\dot{x}$.

State matrices are: A = []

$$\frac{\partial \phi}{\partial q} = H{-}1(\frac{\partial C\dot{q}}{\partial q} - \bar{})$$

$$dfrac\partial\phi\partial\dot{q} = \frac{\partial}{\partial \dot{q}}$$

9.1 Sylvester equation

10 Lyapunov theory

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10.1 Sylvester equation