Study Material Template

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4.1 Intuition behind poles

In control theory, the system state-space equation

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

has the transfer function

$$G(s) = C(sI - A)^{-1}B + D.$$

Since $(sI - A)^{-1} = \text{adj}(sI - A)\det(sI - A)$, where $\text{adj}(sI - A)$ is the adjugate of $sI - A$, the poles of $G(s)$ are the numbers that satisfy $\det(sI - A) = 0$. This is exactly the characteristic equation of matrix A, whose solutions are the eigenvalues of A.

4.2 LQR

In pole-placement method, we want to place the poles in the specific spots (or, we choose specific eigenvalues). But it is not intuitive where to place them, especially for complex systems, systems with numerous actuators. So, the new method is proposed. The key concept of the method lies in optimization of choosing K.

In LQR we find an optimal K by choosing parameters that are important to us, specifically how well the system performs and how much effort it takes to reach this performance.

If $Q >> R$, then we are turning the problem of Let J be an additive cost function:

$$J(x_0, p(x,t)) = \int_0^{\infty} g(x,u)$$

Q - how bad if x is not where it is supposed to be. Q - nonnegative, positive semidefinite.

if the system is a positions, velocity, and $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix}$ we penalize for

Suppose there is the best control law:

$$u = -kx$$

that minimizes the quadratic cost function.

$$J = \int x^T Q x + u^T R u$$

Hamiltonian-Jacobi-Bellman (HJB)

$$\min_u [g(x,u) + \frac{dJ}{dx f(x,u)}] = 0$$

Cost on effectiveness and energy to reach this effectiveness.

4.3 Subtopic 2.1

4.4 Subtopic 2.2

5 Observer

When the full state feedback is unavailablle, we introducce an observer to estimate the state x.

We created a system with a controller, but how to check the current state of the system.

We can try to estimate it with the measurements, for example form sensors. But in real life the task is not that trivial due to some problems:

1. Lack of sensors. For a quadrotor we can not measure the height straight forwardly
2. Measurements can be imprecise or biased
3. Measurements can be only made in discrete time

The key problem arises when the output of the system y is not the whole state x, but $y = Cx$, which means that we are able to get the state partially.

These are just several problems that create the difficulties for us to measure the state of the system.

Let's introduce a new idea of how to introduce an observer in our system.

We estimate x $\hat{x}$ based on the history of y values.

x and y are the state and outpuut (actual or true)

Estimation error

State estimation error:

$$\epsilon = \hat{x} - x$$

But since we do not know x, we cannot compute this error ϵ.

However, we can always find $y = C\hat{x} - y$.

Let's suggest that the dynamics should also hold for the observed state:

$$\dot{\hat{x}} = A\hat{x} + Bu$$

Let's introduce in our equation a linear correction law $-L\ y$. Since $y = C\hat{x} - y$, we get:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{1}$$

This is called Luenberger observer.

But how to find suitable observer gain L?

Let's subtract $\dot{x} = Ax + Bu$ from (?). The equation we got is the observer error dynamics:

With no knowledge of x nad $\hat{x}$, we can define the stability of the system.

The observer $\dot{\epsilon} = (A - LC)\epsilon$ is stable.

5.1 Luenberger Observer

Based on the output y, we wish to estimate input x.

6 Filters

In literature, the system is frequently called a Plant, which in our case is a robot.

Static system

Static systemns are linear systems for whhich the relation between input and output is constant

Dynamic system

Dynamic systemns are linear systems for whhich the relation between input and output depends on time.

In Laplace domain, plant $Y(S) =$

Time does not appear in this equaiton, but if we rewrite it into ode form,

$$\ddot{y} + 5\dot{y} + y = u$$

For these systems, the output depends not only on the current (time-wise) value of input, but on the entire history of input values.

Static systems form algebraic linear equaitons. Dynamical systems create linear differential equaitons

For the system we can create the controller and Luenbberger observer:

7 Controllability. Observability

Controllability

A system is controllable on $t_0 \leq t \leq t_f$ if it is possible to find a control input $u(t)$ that would drive the system to a desired state $x(t_f)$ from any initial state x_0.

Observability

A system is observable on $t_0 \leq t \leq t_f$ if it is possible to exactly estimate the state of the system $x(t_f)$, given any initial estimation error.

Observability (Alternative)

A system is observable on $t_0 \leq t \leq t_f$ if any initial state x_0 is uniquely determined by the output $y(t)$ on that interval.

Observability criterion

Consider discrete LTI.

And a Luenberger observer:

Error dynamics for the observer.

If the system is controllable, we would be able to find such controller to the system, whcih would make the system stable, which means that the error goes to zero.

$$O^T = [C^T (A^T) C^T]$$

Observability matrix

error dynamics need to be stabilized, which means thta it needs t be controllable.

Observability criterion.

Needs to be full column rank.

8 PBH controllability criterion

9 Kalman Filter

10 Reference materials

(Controllability. Observability)

11 Linearization

12 Lyapunov theory

Stability is not defined only for linear systems, but for nonlinear systems as well.

There exist a small neighbourhood. Every solution that starts from this neighbourhood will asymptotically approach the same solution.

Asymptotic stability criteria:

Autonomous dynamic system $\dot{x} = f(x)$ is asymptotically stable if there exists a scalar function $V = V(x) > 0$.

Note that computing $\dot{V}(x)$ comes along with computing $\dot{x}$. We can think of $V(x)$ as energy (pseudo-energy).

Asymptotic stability means that the system converges, and marginal stability means that the system does not diverge.

Asymptotic stability

The system is called asymptotically stable if there exists a function $V > 0$ such that $\dot{V} < 0$, except for $V = 0, \dot{V} = 0$.

Marginal stability

The system $\dot{x} = f(x)$ is stable in the sense of Lyapunov if there exists $V > 0$ such that $\dot{V} \leq 0$.

Lyapunov function

A function $V > 0$ in this case is called a Lyapunov function.

Example 1 Consider the following system:

$$\dot{x} = -x$$

Let's introduce a function $V = x^2 > 0$ for all $x \neq 0$. $\dot{V} = 2x \cdot \dot{x} = 2x(-x) = -2x^2 < 0$. V satisfies the Lyapunov criteria, so the system is (asymptotically) stable.

Example 2 Consider the equation of the pendulum:

$$\ddot{q} = -\dot{q} - \sin(q)$$

12.1 LaSalle's Principle

The system is called stable if

LaSalle's principle allows us to prove asymptotic stability even for $\dot{V}(x) \leq 0$ only for the trivial solution.

12.2 Linear case

Lyapunov theory applies for both nonlinear and linear systems. For linear systems the following feature exist:

For a system $\dot{x} = Ax$, we can always choose a Lyapunov candidate function in the form:

$$V = x^T S x,$$

where S is positive-definite

$$\dot{V} = \dot{x}^T S x + x^T S \dot{x} = (Ax)^T S x + x^T S A x = x^T A^T S x + x^T S A x = x^T (A^T S + SA) x$$