

# Study Material Template

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### 4.1 Intuition behind poles

In control theory, the system state-space equation

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

has the transfer function

$$G(s) = C(sI - A)^{-1}B + D.$$

Since  $(sI - A)^{-1} = \text{adj}(sI - A) \det(sI - A)^{-1}$ , where  $\text{adj}(sI - A)$  is the adjugate of  $sI - A$ , the poles of  $G(s)$  are the numbers that satisfy  $\det(sI - A) = 0$ . This is exactly the characteristic equation of matrix  $A$ , whose solutions are the eigenvalues of  $A$ .

### 4.2 LQR

In pole-placement method, we want to place the poles in the specific spots (or, we choose specific eigenvalues). But it is not intuitive where to place them, especially for complex systems, systems with numerous actuators. So, the new method is proposed. The key concept of the method lies in optimization of choosing  $K$ .

In LQR we find an optimal  $K$  by choosing parameters that are important to us, specifically how well the system performs and how much effort it takes to reach this performance.

If  $Q \gg R$ , then we are turning the problem of Let  $J$  be an additive cost function:

$$J(x_0, p(x, t)) = \int_0^\infty g(x, u)$$

$Q$  - how bad if  $x$  is not where it is supposed to be.  $Q$  - nonnegative, positive semidefinite.

if the system is a positions, velocity, and  $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix}$  we penalize for

Suppose there is the best control law:

$$u = -kx$$

that minimizes the quadratic cost function.

$$J = \int x^T Q x + u^T R u$$

### Hamiltonian-Jacobi-Bellman (HJB)

$$\min_u [g(x, u) + \frac{dJ}{dx} f(x, u)] = 0$$

Cost on effectiveness and energy to reach this effectiveness.

#### 4.3 Subtopic 2.1

#### 4.4 Subtopic 2.2

### 5 Controllability. Observability

**Controllability**

A system is controllable on  $t_0 \leq t \leq t_f$  if it is possible to find a control input  $u(t)$  that would drive the system to a desired state  $x(t_f)$  from any initial state  $x_0$ .

**Observability**

A system is observable on  $t_0 \leq t \leq t_f$  if it is possible to exactly estimate the state of the system  $x(t_f)$ , given any initial estimation error.

**Observability (Alternative)**

A system is observable on  $t_0 \leq t \leq t_f$  if any initial state  $x_0$  is uniquely determined by the output  $y(t)$  on that interval.

Observability criterion

Consider discrete LTI.

And a Luenberger observer:

Error dynamics for the observer.

If the system is controllable, we would be able to find such controller to the system, which would make the system stable, which means that the error goes to zero.

$$O^T = [C^T(A^T)C^T]$$

Observability matrix

error dynamics need to be stabilized, which means that it needs to be controllable.

Observability criterion.

Needs to be full column rank.

## 6 PBH controllability criterion

## 7 Kalman Filter

The way we pick our initial state estimate does not have a bias.

Assume you could pick your initial state estimate  $\hat{x}_0$  such that the initial state estimation error behaves as a random variable sampled from a Gaussian distribution  $x_0 \sim N(0, P_0)$

Knowing mean

$$E[w_i] = 0$$

All subsequent means will be  $E[\cdot]$

Let's compute the autocovariance  $P_{i+1}$  knowing  $P_i$

$$P_{i+1} = E[\cdot] = E[(Ax)]$$

We can assume that the random process is uncorrelated with  $x$ , so

Since the two processes are uncorrelated then their covariance is 0.

$$P_{i+1} = E[Ax_i x_i^T A^T + w_i w_i^T]$$

Note that  $x_i^T A^T = P_i - \text{covariance of the previous step}$ .  $P_{i+1} = AP_i A^T + Q$

$v_i$  - the random noise sampled from the Gaussian distribution, which represents the sensor error.

$$v_i \sim N(0, R)$$

$\hat{x}_{i+1}^-$  is the estimation before measurements - a priori estimate.

$$x_i + 1$$

$L_i$  is a control gain.

A priori error

$$x_{i+1}^- = x_{i+1} - \text{hat } x_{i+1}^-$$

We calculate a posteriori covariance knowing our a priori covariance.

The question is how to minimize those covariances.

If we have all of the measurements up to and including time  $k$  available for use in our estimate of  $X_k$ , then we can form an a posteriori estimate, which we denote as  $\hat{x}_k^+$ . The "+" superscript denotes that the estimate is a posteriori. One way to form the a posteriori state estimate is to compute the expected value of  $x_k$  conditioned on all of the measurements up to and including time  $k$ :  $\hat{x}_k^+ = E[X_k / y_1, y_2, \dots, Y_k] = \text{a posteriori estimate}$

**DERIVATION OF THE DISCRETE-TIME KALMAN FILTER** 125 the estimate is a posteriori. One way to form the a posteriori state estimate is to compute the expected value of  $x_k$  conditioned on all of the measurements up to and including time  $k$ :  $\hat{x}_k^+ = E[X_k / y_1, y_2, \dots, Y_k] = \text{a posteriori estimate}$  (5.3) If we have all of the measurements before (but not including) time  $k$  available for use in our estimate of  $X_k$ , then we can form an a priori estimate, which we denote as  $\hat{x}_k^-$ . The "-" superscript denotes that the estimate is a priori. One way to form the a priori state estimate is to compute the expected value of  $x_k$ , conditioned on all of the measurements before (but not including) time  $k$ :  $\hat{x}_k^- = E[X_k / y_1, y_2, \dots, Y_{k-1}] = \text{a priori estimate}$  (5.4) It is important to note that  $\hat{x}_k^-$  and  $\hat{x}_k^+$  are both estimates of the same quantity; they are both estimates of  $X_k$ . However,  $\hat{x}_k^-$  is our estimate of  $X_k$  before the measurement  $Y_k$  is taken into account, and  $\hat{x}_k^+$  is our estimate of  $x_k$  after the measurement  $y_k$  is taken into account. We naturally expect  $\hat{x}_k^+$  to be a better estimate than  $\hat{x}_k^-$ , because we use more information to compute  $\hat{x}_k^+$ :  $\hat{x}_k^+ = \hat{x}_k^- + \text{estimate of } X_k \text{ after we process the measurement at time } k$  (5.5) If we have measurements after time  $k$  available for use in our estimate of  $X_k$ , then we can form a smoothed estimate. One way to form the smoothed state estimate is to compute the expected value of  $x_k$  conditioned on all of the measurements that are available: estimate of  $X_k$  before we process the measurement at time  $k$ :  $\hat{x}_{k|k+N} = E[x_k / Y_1, Y_2, \dots, Y_k, \dots, Y_{k+N}] = \text{smoothed estimate}$  (5.6) where  $N$  is some positive integer whose value depends on the specific problem that is being solved. If we want to find the best prediction of  $x_k$  more than one time step ahead of the available measurements, then we can form a predicted estimate. One way to form the predicted state estimate is to compute the expected value of  $X_k$  conditioned on all of the measurements that are available:  $\hat{x}_{k|k-M} = E[x_k / y_1, y_2, \dots, y_{k-M}] = \text{predicted estimate}$

## 8 Reference materials

(Controllability. Observability)

## 9 Linearization

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Linear models occur only in specific cases: DC motors. mass-spring damper, 3D-printer. Most real-world systems are non-linear.