Study Material Template

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Contents

The main tasks of Control Theory are

1. Stabilization
2. Tracking

1 Maths

2 Introduction

Dynamical systems

Let's consider the nth-order ordinary differential equation (ODE):

$$\mathbf{x}^{(n)} = \mathbf{f}(\mathbf{x}^{(n-1)}, \mathbf{x}^{(n-2)}, ..., \ddot{\mathbf{x}}, \dot{\mathbf{x}}, \mathbf{x}, t),$$

where $x(t)$ is a solutoin for the system, and t is an independent variable (usually - time). This equation is called **a dynamical system** and it represents a dynamics of the system.

State of

In canonical form, linear ODE is represented in the following way:

$$a_n z^{(n)} + a_{n-1} z^{(n-1)} + ... + a_2 \ddot{z} + a_1 \dot{z} + a_0 z = b_0$$

The set $\{\mathbf{x},\ \dot{\mathbf{x}}\ ...,\ \mathbf{x}^{(n-1)}\}$ is called the **state of the system**.

In case if relationships between state, output and control is **linear**, we can formulate the model of system in following form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases} \tag{1}$$

where * $\mathbf{x} \in \mathbb{R}^n$ states of the system * $\mathbf{y} \in \mathbb{R}^l$ output vector * $\mathbf{u} \in \mathbb{R}^m$ control inputs * $\mathbf{A} \in \mathbb{R}^{n \times n}$ state matrix * $\mathbf{B} \in \mathbb{R}^{n \times m}$ input matrix * $\mathbf{C} \in \mathbb{R}^{l \times n}$ output matrix * $\mathbf{D} \in \mathbb{R}^{l \times m}$ feedforward matrix

If matrices A, B, C, D are time-independent, then we call such systems **time-invariant.**

More frequently we work with systems when the output does not depend on conrol.

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \tag{2}$$

State vector, Control vector

Input vector

3 Stability

3.1 Critical Point (Node)

Critical Point (Node)

Consider the following LTI:

$$\dot{x} = f(\mathbf{x}, t)$$

x_0 is called a **Node**, or **Critical Point**, if $f(x_0) = 0$.

3.2 Stability

A system is stable if:

$$\|x(0) - x_0\| < \delta \quad \|x(t) - x_0\| < \epsilon$$

We can think of it as: if the starting point is in the δ-neighborhood of the node x_0, the rest of the trajectory $x(t)$ is in the ϵ-neighborhood of the node.

Or, the solutions starting from the δ-sized ball do not diverge.

Asymptotic Stability

A system is asymptotically stable if:

$$\|x(0) - x_0\| < \delta \to \lim_{t\to\infty} x(t) = x_0$$

For any initial point that lies in the δ-sized ball, the trajectory will asymptotically approach the node (point x_0). Or, the solutions starting from the δ-sized ball, converge to the node.

3.3 Stability of autonomous LTI

Autonomous Systems

A system is considered **autonomous** if its evolution depends only on time.

Example

$$\dot{x} = Ax$$

3.3.1 Diagonal matrices

Let's introduce a trick for autonomous Linear Time-Invariant (LTI) systems.

First of all, recall the properties of a diagonal matrix and eigen-decomposition.

- **Diagonal matrix**

$$\dot{z} = Dz$$

$$\begin{cases} \dot{x}_1 = d_1 x_1 \\ \dots \\ \dot{x}_n = d_n x_n \end{cases}$$

  The solution of each of the equations is: $x_i = C_i e^{d_i t}$. So, the system is asymptotically stable when for all i, $d_i < 0$. The system is stable when $d_i \leq 0$.

- **Eigen-decomposition**
  We can represent the matrix as $A = VDV^{-1}$, where D is a diagonal matrix.

Given an autonomous Linear Time-Invariant (LTI) system, let's switch to the system with a diagonal matrix:

$$\dot{x} = Ax$$

$$\dot{x} = VDV^{-1}x$$

$$V^{-1}\dot{x} = V^{-1}VDV^{-1}x = DV^{-1}x$$

Change of variables: $z = V^{-1}x$, $\dot{z} = Dz$,

$$V^{-1}\dot{x} = Dz$$

The system is asymptotically stable when for all the elements of D are < 0. The system is stable when the elements of D are ≤ 0.

3.3.2 Upper triangular matrices

Eigenvalues of upper triangular matrices are the diagonal elements.

3.3.3 General case

Consider the LTI:

$$\dot{x} = Ax$$

The system is called **stable** iff real parts of eigenvalues of A are non-positive.

The system is called **asymptotically stable** iff real parts of eigenvalues of A are strictly negative.

4 Control

The main tasks of control theory include **control design**, **trajectory tracking**, and **point-to-point control**.

4.1 Control Design

Dynamical Systems

The task of **stabilizing control** is defining the control law that makes a certain solution of a dynamical system stable.

This is true for both linear and nonlinear systems.
Consider a Linear Time-Invariant (LTI) system:

$$\dot{x} = Ax + Bu,$$

and choose the control as a linear function of the state:

$$u = -Kx,$$

where K is the **control gain**.

Stability Condition

Then, the closed-loop system can be represented as:

$$\dot{x} = (A - BK)x.$$

The system is asymptotically stable if the eigenvalues of the matrix $A - BK$ have strictly negative real parts.

4.2 Trajectory Tracking

The task is to stabilize the system around a reference trajectory.
Let $x^*(t)$ and $u^*(t)$ be solutions for the system $\dot{x} = Ax + Bu$. This means that:

$$\dot{x}^* = Ax^* + Bu^*.$$

Define the error as $e = x^* - x$ and $v = u^* - u$.
Then, the error dynamics become:

$$\dot{e} = Ae + Bv.$$

To stabilize the system, suggest $v = -Ke$, then:

$$\dot{e} = (A - BK)e.$$

Therefore, the control law becomes:

$$u = u^* + K(x^* - x).$$

4.3 Point-to-Point Control

Point-to-point control differs from trajectory tracking in that the reference input is constant, $x^* = \text{const}$, and feed-forward control is also constant, $u^* = \text{const}$.
Since the error dynamics and the stabilizing control are the same as in trajectory tracking, the control law becomes:

$$u = K(x^* - x) + u^*.$$

The dynamics of the system become:

$$\dot{x} = (A - BK)x + BKx^* + Bu^*.$$

5 Laplace Transform

5.1 Laplace Transform

The Laplace transform is an integral transform that converts a function of a real variable t (time domain) to a function of a complex variable s (frequency domain). The Laplace transform is a tool for solving differential equations by transforming them into algebraic equations.

The Laplace transform of a function $f(t)$ is given as:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt \tag{3}$$

where $F(s)$ is called an **image** of the function and $s = \alpha + \beta i$ is a complex frequency.

Laplace transform is defined as transformation from the time domain t to the frequency domain s.

5.1.1 Some Useful Properties

Linear properties:

$$\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} \tag{4}$$

$$\mathcal{L}\{af(t)\} = a\mathcal{L}\{f(t)\} \tag{5}$$

Final value theorem:

$$f(\infty) = \lim_{s\to 0} sF(s) \tag{6}$$

The final value theorem is useful because it gives the long-term behavior for a particular function.

5.1.2 Inverse Laplace Transform

The inverse Laplace transform transforms the image of your function $F(s)$ from the frequency domain to the time domain $x(t)$:

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)\,ds \tag{7}$$

However, in practice, we mostly use precalculated Laplace transforms and then try to decompose the image $X(s)$ into known transforms of functions obtained from a table, and construct the inverse by inspection, or just use some symbolic routines.

5.2 Laplace Transform of a Function's Derivative

For us, one of the most useful properties of Laplace transform is that if we apply it to the derivative of a given variable, it will result in the following:

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = s\mathcal{L}\left(x\right) = sX(s) \tag{8}$$

which is true for $x(0) = 0$.

Thus, we can define a **derivative operator**:

$$\frac{dx}{dt} \xrightarrow{\mathcal{L}} sX(s) \tag{9}$$

The proof is as follows, using the definition of Laplace transform:

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} = \int_0^\infty \frac{dx}{dt}e^{-st}dt \tag{10}$$

Then using integration by parts:

$$\int_0^\infty \frac{dx}{dt}e^{-st}dt = \left[xe^{-st}\right]_0^\infty - \int_0^\infty -se^{-st}xdt \tag{11}$$

which yields:

$$\left[xe^{-st}\right]_0^\infty + s\int_0^\infty e^{-st}xdt = x(0) + s\mathcal{L}\{x(t)\} = x(0) + sX(s) \tag{12}$$

By induction, it can be shown that:

$$\mathcal{L}\left\{\frac{d^n x}{dt^n}(t)\right\} = s^n \cdot \mathcal{L}\{x(t)\} + s^{n-1}x(0) + \cdots + x^{(n-1)}(0) \tag{13}$$

5.2.1 Applications to Linear ODEs

Let us consider the following ODE:

$$a_n x^{(n)} + a_{n-1}x^{(n-1)} + ... + a_2\ddot{x} + a_1\dot{x} + a_0 x = u_m b^{(m)} + b_{m-1}u^{(m-1)} + ... + b_2\ddot{u} + b_1\dot{u} + b_0 u \tag{14}$$

Notice that we introduce a new variable that we call the input u (control).

Applying the inverse Laplace transform with zero initial conditions yields:

$$a_n s^{(n)}X(s)+a_{n-1}s^{(n-1)}X(s)+...+a_2 s^2 X(s)+a_1 sX(s)+a_0 X(s) = b_m s^{(m)}U(s)+b_{m-1}s^{(m-1)}U(s)+...+b_2 s^2 U(s)+b_1 sU(s)+ \tag{15}$$

Example 1

$$\ddot{y} + a\dot{y} + by = u \tag{16}$$

$$S^2 Y(S) + ASY(S) + BY(S) = U(S) \tag{17}$$

$$Y(S) = \frac{1}{(S^2 + AS + B)}U(S) \tag{18}$$

This form is called a **transfer function**.

Example 2

$$2\ddot{y} + 5\dot{y} - 40y = 10u \tag{19}$$

$$2SY(S) - 4Y(S) = U(S) \tag{20}$$

$$Y(S)(2S - 4) = U(S) \tag{21}$$

$$Y(S) = \frac{1}{(2S - 4)}U(S) \tag{22}$$

Consider the ODE with the input u.

5.3 State-Space to Transfer Function

Let's study the relation between the input and the output of the dynamical system.

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Let's consider the 1st equation and rewrite it in the form:

$$SIX(S) = AX(S) + BU(S)$$

$$(SI - A)X(S) = BU(S)$$

$$X(S) = (SI - A)^{-1}BU$$

Then the initial system can be rewritten in the following form:

$$\begin{cases} X = (SI - A)^{-1}BU \\ Y = C((SI - A)^{-1}B + D)U \end{cases}$$

6 Discrete dynamics

Discrete case for dynamics:

$$x_{i+1} = Ax_i + Bu_i$$

Such an equation is easy to work with: no derivatives are used here, it is easy to simulate.
For this system, we can propose the control:

$$u_i = -Kx_i + u_i*$$

6.1 Stability

Let's consider the system:

$$x_{i+1} = Ax_i$$

$$x_{i+1} = V^{-1}DVx_i$$
$$Vx_{i+1} = VV^{-1}DVx_i$$

The change of variables: $z_i = Vx_i$, $z_{i+1} = Vx_{i+1}$

$$z_{i+1} = DVz_i$$

Let's consider the following example:

$$\begin{bmatrix} \vec{x}_{1,i+1} \\ x_{2,i+1} \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}$$

Let's find the norms of x_{i+1} and x_i.

$$\| \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} \|^2 = x_{1,i}^2 + x_{2,i}^2$$

$$\| \begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} \|^2 = \left| \begin{bmatrix} \alpha x_{1,i} - \beta x_{2,i} \\ \beta x_{1,i} + \alpha x_{2,i} \end{bmatrix} \right|^2 = (\alpha x_{1,i} - \beta x_{2,i})^2 + (\beta x_{1,i} + \alpha x_{2,i})^2 = \beta^2(x_{2,i}^2 + x_{1,i}^2) + \alpha^2(x_{2,i}^2 + x_{1,i}^2) = (\alpha^2 + \beta^2)(x_{1,i}^2 + x_{2,i}^2)$$

For stability we introduce:
Continuous case: Hurwitz matrix. **Discrete case:** Schur matrix.

Dynamical Systems

- Discrete system $x_{i+1} = Ax_i$ is **stable** if and only if the absolute values of A's eigenvalues are less than or equal to 1: $|\lambda_i(A)| \leq 1$.
- Discrete system $x_{i+1} = Ax_i$ is **asymptotically stable** if and only if the absolute values of A's eigenvalues are less than 1: $|\lambda_i(A)| < 1$.

6.2 Analytical solution of Continuous LTI

Case: $\dot{x} = ax(t)$

Recall that we can describe exponent as a series:

$$e^a = 1 + a + \frac{a^2}{2} + \frac{a^3}{6} + \dots$$

$$e^A = I + A + \frac{AA}{2} + \frac{AAA}{6} + \dots$$

Suppose that the solution of the dynamical system is:

$$x(t) = e^{at}x(0)$$

$$x(t) = e^{At}x(0)$$

Let's rewrite the exponent:

$$x(t) = (I + At + \frac{AAt^2}{2} + \frac{AAAt^3}{6} + \dots)x(0)$$

$$\dot{x}(t) = (A + AAt + \frac{AAAt^2}{2} + \dots)x(0)$$

$$\dot{x}(t) = A(I + At + \frac{AAt^2}{2} + \dots)x(0)$$

$$\dot{x}(t) = Ae^{At}x(0)$$

$$\dot{x}(t) = Ax(t)$$

Case: $\dot{x} = ax(t) + bu(t)$

$$\dot{x} = ax(t) + bu(t)$$

$$\dot{x}e^{-at} - ae^{-at}x(t) = be^{-at}u(t)$$

$$\frac{d}{dt}(xe^{-at}) = be^{-at}u(t)$$

$$\int_0^t \frac{d}{d\tau}(e^{-a(t-\tau)}x(\tau))d\tau = \int_0^t \frac{d}{d\tau}(e^{-a(t-\tau)}bu(\tau))d\tau$$

$$e^{-at}x(t) - x(0) = \int_0^t e^{-a\tau}bu(\tau)d\tau$$

$$x(t) = e^{at}x(0) + e^{at}\int_0^t e^{-a\tau}bu(\tau)d\tau$$

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau$$

$$\boxed{x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}$$

6.3 Discretization

$$\begin{cases} x_0 = x(0) \\ x_1 = x(\Delta t) \\ x_2 = x(2\Delta t) \\ \dots \\ x_n = x(n\Delta t) \end{cases}$$

$$\dot{x} = \frac{1}{\Delta t}(x_{i+1} - x_i)$$

$$Ax_i = \frac{1}{\Delta t}(x_{i+1} - x_i)$$

$$x_{i+1} = Ax_i\Delta t + x_i$$

$$\boxed{x_{i+1} = (A\Delta t + I)x_i}$$

or

$$\boxed{x_{i+1} = (I - A\Delta t)^{-1}x_i}$$

Thus, the discrete state matrix is: $A_d = A\Delta t + I$ The control matrix is: $B\Delta t$.

7 LQR

7.1 Intuition behind poles

In control theory, the system state-space equation

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

has the transfer function

$$G(s) = C(sI - A)^{-1}B + D.$$

Since $(sI - A)^{-1} = \text{adj}(sI - A)\det(sI - A)$, where $\text{adj}(sI - A)$ is the adjugate of $sI - A$, the poles of $G(s)$ are the numbers that satisfy $\det(sI - A) = 0$. This is exactly the characteristic equation of matrix A, whose solutions are the eigenvalues of A.

7.2 LQR

In pole-placement method, we want to place the poles in the specific spots (or, we choose specific eigenvalues). But it is not intuitive where to place them, especially for complex systems, systems with numerous actuators. So, the new method is proposed. The key concept of the method lies in optimization of choosing K.

In LQR we find an optimal K by choosing parameters that are important to us, specifically how well the system performs and how much effort it takes to reach this performance.

If $Q >> R$, then we are turning the problem of Let J be an additive cost function:

$$J(x_0, p(x,t)) = \int_0^{\infty} g(x,u)$$

Q - how bad if x is not where it is supposed to be. Q - nonnegative, positive semidefinite.

if the system is a positions, velocity, and $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix}$ we penalize for

Suppose there is the best control law:

$$u = -kx$$

that minimizes the quadratic cost function.

$$J = \int x^T Q x + u^T R u$$

Hamiltonian-Jacobi-Bellman (HJB)

$$\min_u [g(x,u) + \frac{dJ}{dx f(x,u)}] = 0$$

Cost on effectiveness and energy to reach this effectiveness.

7.3 Subtopic 2.1

7.4 Subtopic 2.2

8 Observer

When the full state feedback is unavailablle, we introducce an observer to estimate the state x.

We created a system with a controller, but how to check the current state of the system.

We can try to estimate it with the measurements, for example form sensors. But in real life the task is not that trivial due to some problems:

1. Lack of sensors. For a quadrotor we can not measure the height straight forwardly
2. Measurements can be imprecise or biased
3. Measurements can be only made in discrete time

The key problem arises when the output of the system y is not the whole state x, but $y = Cx$, which means that we are able to get the state partially.

These are just several problems that create the difficulties for us to measure the state of the system.

Let's introduce a new idea of how to introduce an observer in our system.

We estimate x $\hat{x}$ based on the history of y values.

x and y are the state and outpuut (actual or true)

Estimation error

State estimation error:

$$\epsilon = \hat{x} - x$$

But since we do not know x, we cannot compute this error ϵ.

However, we can always find $y = C\hat{x} - y$.

Let's suggest that the dynamics should also hold for the observed state:

$$\dot{\hat{x}} = A\hat{x} + Bu$$

Let's introduce in our equation a linear correction law $-L\ y$. Since $y = C\hat{x} - y$, we get:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{23}$$

This is called Luenberger observer.

But how to find suitable observer gain L?

Let's subtract $\dot{x} = Ax + Bu$ from (?). The equation we got is the observer error dynamics:

With no knowledge of x nad $\hat{x}$, we can define the stability of the system.

The observer $\dot{\epsilon} = (A - LC)\epsilon$ is stable.

8.1 Luenberger Observer

Based on the output y, we wish to estimate input x.

9 Filters

In literature, the system is frequently called a Plant, which in our case is a robot.

Static system

Static systemns are linear systems for whhich the relation between input and output is constant

Dynamic system

Dynamic systemns are linear systems for whhich the relation between input and output depends on time.

In Laplace domain, plant $Y(S) =$

Time does not appear in this equaiton, but if we rewrite it into ode form,

$$\ddot{y} + 5\dot{y} + y = u$$

For these systems, the output depends not only on the current (time-wise) value of input, but on the entire history of input values.

Static systems form algebraic linear equaitons. Dynamical systems create linear differential equaitons

For the system we can create the controller and Luenbberger observer:

10 Controllability. Observability

Not always can we control the system. The same applies to an observer. Sometimes, we cannot obtain the full information that we desire from the system.

Let's suggest the tools which can inform us whether we can succeed in designing a controller or observer or not.

Controllability

A system is controllable on $t_0 \leq t \leq t_f$ if it is possible to find a control input $u(t)$ that would drive the system to a desired state $x(t_f)$ from any initial state x_0.

Observability

A system is observable on $t_0 \leq t \leq t_f$ if it is possible to exactly estimate the state of the system $x(t_f)$, given any initial estimation error.

Observability (Alternative)

A system is observable on $t_0 \leq t \leq t_f$ if any initial state x_0 is uniquely determined by the output $y(t)$ on that interval.

Cayley-Hamilton Theorem: A matrix M satisfies its own characteristic equation.

A characteristic equation can be written like:

$$\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ldots + a_0 = 0$$

So,

$$M^n + a_{n-1}M^{n-1} + a_{n-2}M^{n-2} + \ldots + a_0 I = 0$$

Meaning that M^n is a linear combination of $M^{n-1} + M^{n-2} + \ldots + I$. Remember that if the system is controllable, we would be able to find such controller to the system, whcih would make the system stable, which means that the error goes to zero.

10.1 Controllability of Discrete LTI

Consider discrete LTI.

$$x_{i+1} = Ax_i + Bu_i$$

Let's expand it numerically:

$$x_2 = Ax_1 + Bu_1$$

$$x_3 = Ax_2 + Bu_2 = A(Ax_1 + Bu_1) + Bu_2 = A^2x_1 + ABu_1 + Bu_2$$

$$\ldots$$

$$x_{n+1} = A^n x_1 + A^{n-1}Bu_1 + ... + A^{n-k}Bu_k + ... + Bu_n$$

$$x_{n+1} - A^n x_1 = \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} u_n \\ u_{n-1} \\ \vdots \\ u_1 \end{bmatrix}$$

Thus, we showed that $x_{n+1} - A^n x_1$ must be in the column space of $C = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$

Now, note that since x_{n+1} can be any real number and x_1 can take a zero value, $\mathbb{R}$ must be in the column space of C. So, C must be a full-rank matrix.

Controllability

A system $x_{i+1} = Ax_i + Bu_i$ is controllable (or any state can be reached) if $C = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$ is full row rank.

10.2 Observability of Discrete LTI

Observability criterion shows whether it is in principle possible to estimate the system or not.

Consider a discrete LTI:

$$\begin{cases} x_{i+1} = Ax_i + Bu_i \\ y_i = Cx_i \end{cases}$$

And a Luenberger observer:

$$\hat{x}_{i+1} = A\hat{x}_i + Bu_i + L(y_i - C\hat{x}_i)$$

Change of variables: $e_i = \hat{x}_i - x_i$ will provide us with the following equation:

$$e_{i+1} = Ae_i + L(y_i - C\hat{x}_i)$$
$$e_{i+1} = Ae_i - LCe_i$$

Introduce the dual system (which is stable iff the original system is stable):

$$\epsilon_{i+1} = A^T\epsilon_i - C^TL^T\epsilon_i$$

$$\begin{cases} \epsilon_{i+1} = A^T\epsilon_i - C^Tv_i \\ v_i = -L^T\epsilon_i \end{cases}$$

Controllability matrix for the system:

$$O^T = \begin{bmatrix} C^T & A^TC^T ... & A^{T^{n-1}}C^T \end{bmatrix}$$

$$O = \begin{bmatrix} C \\ CA \\ ... \\ CA^{n-1} \end{bmatrix}$$

Observability

A system $x_{i+1} = Ax_i + Bu_i$ is **observable** (or observation error can go to 0 from any initial position) if $O = \begin{bmatrix} C \\ ... \\ CA \\ CA^{n-1} \end{bmatrix}$ is full column rank.

11 Kalman Filter

12 Reference materials

(Controllability. Observability)

13 Linearization

While linear models are present in the real world (examples include DC motors, 3D printers, and mass-spring-damper systems), the majority of real-world systems are nonlinear (such as cars, airplanes, robot arms, underwater robots, and almost any mechanical system with linkages).

However, there is a method to describe these systems using our toolkits - through linearization.

Linearization is the technique that enables us to create a linear model for a nonlinear system.

So, we can introduce linear models for nonlinear systems that could describe the system pretty well locally. We can do control for this local approximation.

13.1 Taylor expansion

Taylor expansion around node.

Let's consider a non-linear dynamical system $\dot{x} = f(x, u)$ and a Taylor expansion around the node (x_0, u_0), i.e., $f(x_0, u_0) = 0$, $x_0 = \text{const}$, $u_0 = \text{const}$.

Recall the physical meaning of a **node**: it is any position at which the robot remains static.

$$f(x, u) \approx f(x_0, u_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial u}(u - u_0) + \text{HOT}$$

HOT = high-order terms.

(x_0, u_0) are an expansion point.

Now let's introduce new variables e and v that represent the distances from the expansion point:

$$e = x - x_0$$

$$\dot{e} = \dot{x}$$

$$v = u - u_0$$

Then we can rewrite the Taylor expansion:

$$\dot{e} \approx \frac{\partial f}{\partial x} e + \frac{\partial f}{\partial u} v + \text{HOT}$$

$$\dot{e} \approx Ae + Bv + \text{HOT}$$

Let's drop the high-order term and get an approximation. And this approximation is called linearization.

$$\dot{e} = Ae + Bv$$

In this context, (x_0, u_0) is a linearization point.

This type of linearization is proved to provide the best possible linear model for the given system.

Taylor expansion along a trajectory.

Now let's consider a non-linear dynamical system:

$$\dot{x} = f(x, u)$$

and a trajectory $x_0 = f(x_0, u_0)$. Note that now x_0 is not a point - it is a whole trajectory

$$f(x, u) \approx f(x_0, u_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial u}(u - u_0) + \text{HOT}$$

We can denote $\dot{e} = \dot{x} - \dot{x}_0 = f(x, u) - f(x_0, u_0)$.

The rest stays the same: $e = x - x_0, v = u - u_0$

$$\dot{e} \approx Ae + Bv + \text{HOT}$$

Let's drop HOT and obtain linearization:

$$\dot{e} = Ae + Bv$$

Basically, nothing changes between expansion around a node and expansion along a trajectory as long as we introduce the change of variables described above. The change of variables is slightly different, but original function and the local approximation behave in the same way in the region of the linearization point.

So, can we linearize around each and any point?

1) If it is a node - yes.

2) If we want to linearize around some other point - yes, but the chnage of variables would entail slightly different results.

While we had:

$$e = x - x_0, \dot{e} = \dot{x},$$

now $\dot{e}$ is the differrence between the derivative of the state of our non-linear system and derivative of the state of our trajectory distance from the trajectory linearization.

The meaning of the variables changes slightly, but the mathematical expression is the exact same expression.

Affine expansion

The other way to obtain linearization, without change of varibales is as follows.

$$f(x, u) \approx f(x_0, u_0) + A(x - x_0) + B(u - u_0)$$

Denoting

$$f(x_0, u_0) - Ax_0 - Bu_0 = c$$

$$\dot{x} = Ax + Bu + c$$

c makes it not a linear model, but affine, which means it has a constant term.

It can be a constant in the case of a node, and a function of time in the case of expansion along the trajectory.

Often we choose u to compensate c, so u will also be affine in this case.

13.2 Manipulator equations

Frequently, in robotics we deal with the following equation when describing a system:

$$H\ddot{q} + C\dot{q} + g = \tau$$

which is called a **manipulator equation**

However, cars, underwater robots are usually described by the other equations - Euler or Lagrangian equaitons.

Let's derive linearization for the manipulator equation.

Begin by proposing new variables:

$$x = q \vec{-} q_0, \dot{q} \vec{-} \dot{q}_0$$

$$u = \tau - \tau_0$$

Points around which we linearize are denoted from equation:

Let's introduce

$$\phi(\dot{q}, q, t) = H^{-1}(\tau - C\dot{q} - g)$$

14 Lyapunov theory

Stability is not defined only for linear systems, but for nonlinear systems as well.

There exist a small neighbourhood. Every solution that starts from this neighbourhood will asymptotically approach the same solution.

Asymptotic stability criteria:

Autonomous dynamic system $\dot{x} = f(x)$ is asymptotically stable if there exists a scalar function $V = V(x) > 0$.

Note that computing $\dot{V}(x)$ comes along with computing $\dot{x}$. We can think of $V(x)$ as energy (pseudo-energy).

Asymptotic stability means that the system converges, and marginal stability means that the system does not diverge.

Asymptotic stability

The system is called asymptotically stable if there exists a function $V > 0$ such that $\dot{V} < 0$, except for $V = 0, \dot{V} = 0$.

Marginal stability

The system $\dot{x} = f(x)$ is stable in the sense of Lyapunov if there exists $V > 0$ such that $\dot{V} \leq 0$.

Lyapunov function

A function $V > 0$ in this case is called a Lyapunov function.

Example 1 Consider the following system:

$$\dot{x} = -x$$

Let's introduce a function $V = x^2 > 0$ for all $x \neq 0$. $\dot{V} = 2x \cdot \dot{x} = 2x(-x) = -2x^2 < 0$. V satisfies the Lyapunov criteria, so the system is (asymptotically) stable.

Example 2 Consider the equation of the pendulum:

$$\ddot{q} = -\dot{q} - \sin(q)$$

14.1 LaSalle's Principle

The system is called stable if

LaSalle's principle allows us to prove asymptotic stability even for $\dot{V}(x) \leq 0$ only for the trivial solution.

14.2 Linear case

Lyapunov theory applies for both nonlinear and linear systems. For linear systems the following feature exist:

For a system $\dot{x} = Ax$, we can always choose a Lyapunov candidate function in the form:

$$V = x^T S x,$$

where S is positive-definite

$$\dot{V} = \dot{x}^T S x + x^T S \dot{x} = (Ax)^T S x + x^T S A x = x^T A^T S x + x^T S A x = x^T (A^T S + S A) x$$