Fundamentals of Robotics: Assignment 2

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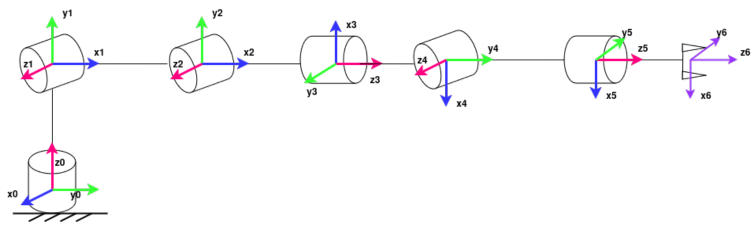
1 Link

Colab

2 Configuration of the chosen robot

Manipulator with antropomorphic elbow and a spherical wrist.



Forward kinematics:

$$^0T_1 = R_z(\theta_1^*)T_z(l_1)R_z(\frac{\pi}{2})R_x(\frac{\pi}{2})$$

$$^1T_2 = R_z(\theta_2^*)T_x(l_2)$$

$$^2T_3 = R_z(\theta_3^*)T_x(l_3)$$

$$^3T_4 = R_z(\theta_4^*)T_z(l_4)R_x(-\frac{\pi}{2})R_z(-\pi)$$

$$^4T_5 = R_z(\theta_5^*)T_y(l_5)R_x(-\frac{\pi}{2})$$

$$^5T_6 = R_z(\theta_6^*)T_z(l_6)$$

$$^0T_6 = {}^0T_1{}^1T_2{}^2T_3{}^3T_4{}^4T_5{}^5T_6$$

3 Task 1-3

Task 1. Derive inverse kinematics for your robot model.
Task 2. Solve inverse kinematics for multiple positions.
Task 3. Track the number of solutions along the way and choose the correct one and closest to the previous (current) configuration.

3.1 q_1

Given: End-effector position and orientation ${}^0T_6$.

First, let's define the configuration of the wrist center by solving forward kinematics. The end-effector position corresponds to the position of the wrist center; the 6th joint defines the end-effector orientation.

$$\begin{aligned}
T_{01} &= R_z(\theta_1) \cdot T_z(l_1) \cdot R_z(\frac{\pi}{2}) \cdot R_x(\frac{\pi}{2}) \\
T_{12} &= R_z(\theta_2) \cdot T_x(l_2) \\
T_{23} &= R_z(\theta_3) \cdot T_x(l_3 + l_4) \cdot R_y(\frac{\pi}{2}) \cdot R_z(\frac{\pi}{2}) \\
T_{\text{wrist_center}} &= T_0 \cdot T_{01} \cdot T_{12} \cdot T_{23}
\end{aligned}$$

Let's solve the forward kinematics problem for the wrist center symbolically. Notice the translation column of the transformation matrix. From it we can retrieve the value of $q_1 = \theta_1$.

$$\begin{matrix} x \\ y \\ z \end{matrix} = \begin{matrix} 0.8\sin(\theta_1)\sin(\theta_2)\sin(\theta_3)-0.8\sin(\theta_1)\cos(\theta_2)\cos(\theta_3)-1.0\sin(\theta_1)\cos(\theta_2) \\ -0.8\sin(\theta_2)\sin(\theta_3)\cos(\theta_1)+0.8\cos(\theta_1)\cos(\theta_2)\cos(\theta_3)+1.0\cos(\theta_1)\cos(\theta_2) \\ 0.8\sin(\theta_2)\cos(\theta_3)+1.0\sin(\theta_2)+0.8\sin(\theta_3)\cos(\theta_2)+0.3 \end{matrix}$$

We can see that:

$$\frac{x}{y} = -\frac{\sin(\theta_1)}{\cos(\theta_1)}$$

$$\tan(\theta_1) = -\frac{x}{y}$$

$$\theta_1 = -\arctan(\frac{x}{y})$$

3.2 q_3

We can find the value of $q_3 = \theta_3$ by the cosine theorem.

$$q3 = \arccos\left(\frac{(z - l1)^2 + (x^2 + y^2) - l2^2 - (l3 + l4)^2}{2 \cdot l2 \cdot (l3 + l4)}\right)$$

3.3 q_2

$$r = \sqrt{x^2 + y^2}$$

$$s = z - l1$$

$$q_2 = \arcsin\left(\frac{(l2 + (l3 + l4)\cos(q_3)) \cdot s - (l3 + l4)\sin(q_3) \cdot r}{r^2 + s^2}\right)$$

3.4 q_4

We know that

$$^0R_6 = {}^0R_3 \cdot {}^3R_6$$

$$^3R_6 = ({}^0R_3^T)^0R_6$$

We have already calculated the angles q_1, q_2, and q_3, so we can calculate $^0R_3^T$. Additionally, we know $^0R_6$ since the end-effector position is provided to us. Thus, with $^3R_6$, we can determine all remaining angles.

Let's solve $^3R_6$ symbolically,

$$\begin{matrix} \sin(\theta_4)\sin(\theta_6) - \cos(\theta_4)\cos(\theta_5)\cos(\theta_6) & \sin(\theta_4)\cos(\theta_6) + \sin(\theta_6)\cos(\theta_4)\cos(\theta_5) & \sin(\theta_5)\cos(\theta_4) \\ -\sin(\theta_4)\cos(\theta_5)\cos(\theta_6) - \sin(\theta_6)\cos(\theta_4) & \sin(\theta_4)\sin(\theta_6)\cos(\theta_5) - \cos(\theta_4)\cos(\theta_6) & \sin(\theta_4)\sin(\theta_5) \\ \sin(\theta_5)\cos(\theta_6) & -\sin(\theta_5)\sin(\theta_6) & \cos(\theta_5) \end{matrix}$$

In order to find q_4 we can solve $\frac{sin(\theta_4)\sin(\theta_5)}{\sin(\theta_5)\cos(\theta_4)}$.

If $\sin(\theta_5) \neq 0$, then $q_4 = \arctan(\frac{\sin(\theta_4)}{\cos(\theta_4)})$.

3.5 q_5

$$(\sin(\theta_5)\cos(\theta_4))^2 + (\sin(\theta_4)\sin(\theta_5))^2 = \sin(\theta_5)^2$$

$$\theta_5 = \arctan 2(\sqrt{(\sin(\theta_5)\cos(\theta_4))^2 + (\sin(\theta_4)\sin(\theta_5))^2}, \cos(\theta_5))$$

3.6 q_6

Analogically to q_4, we can find the value of q_6 from $^3R_6$:

$$\frac{-\sin(\theta_5)\sin(\theta_6)}{\sin(\theta_5)\cos(\theta_6)}$$

If $\sin(\theta_5) \neq 0$, then $q_6 = \arctan(\frac{-\sin(\theta_6)}{\cos(\theta_6)})$.

3.7 Singularity analysis

Singularity occurs when:

1. $\theta_3 = 0$ or $\theta_3 = \pi$.
2. $\theta_5 = 0$ or $\theta_5 = \pi$.

When a singularity situation arises in the inverse kinematics solution, we can revert back to the previous non-singular solution for this joint.

For specific examples of inverse kinematics solutions for different configurations, please refer to the code in the Colab link.

4 Task 4. Derive the jacobian matrix for your robot model.

$$J = [J_1 \quad J_2 \quad \dots \quad J_6]$$

In screw theory, the Jacobian is defined as:

$$J = \begin{bmatrix} U_{i-1} \times (O_n - O_{i-1}) \\ U_{i-1} \end{bmatrix}$$

U_{i-1} is the column vector corresponding to the axis of rotation matrix (in our case, z). O_{i-1} is the translation vector of transformation matrix for T_{i-1}.

Singularity

To check the singularity of the Jacobian, we need to check whether its determinant is not equal to 0.

Alternatively, we can check the rank of the matrix. A non-singular matrix is full-rank, whereas a singular matrix is not full-rank.

5 Task 5-6.

Task 5. Plan a synchronized trajectory for all 6 joints between two poses. (consider 20Hz controller frequency).

Task 6. Use the Jacobian matrix to check for singularities along the planned trajectory.

5.1 Trajectory Time

Based on the q parameters $[q_0, q_f, \dot{q}_{\max}, \ddot{q}_{\max}]$, we can determine the velocity profile and calculate the time for the trajectory.

Triangular profile:

$$\text{If } \sqrt{\delta_q \times \ddot{q}_{\max}} \leq \dot{q}_{\max}, \text{ then:}$$

$$\tau = \frac{\delta_q}{\dot{q}_{\max}}, \quad T = \tau, \quad tf = 2\tau$$

Trapezoidal profile:

Otherwise, we have:

$$T = \frac{\delta_q}{\dot{q}_{\max}}, \quad \tau = \frac{\dot{q}_{\max}}{\ddot{q}_{\max}}$$

Based on the time parameters obtained, we can determine the trajectory for the trapezoidal profile in the following way:

$$\begin{cases} q = q_0 + \frac{1}{2}\ddot{q}_{\max} \cdot (t_i - t_0)^2, & 0 < t_i \leq \tau \\ v = \min(\ddot{q}_{\max} \cdot t, \dot{q}_{\max}) \\ a = \ddot{q}_{\max}, & 0 < t_i \leq \tau \end{cases}$$

$$\begin{cases} q = q_0 + dq_0 \cdot (t_i - \tau), & \tau < t_i \leq T \\ v = \dot{q}_{\max} \\ a = 0 \end{cases}$$

$$\begin{cases} q = q_{\mathrm{T}} + \dot{q}_{\mathrm{T}} \cdot (t_i - T) - \frac{1}{2}\ddot{q}_{\max} \cdot (t_i - T)^2, & T < t_i \leq t_f \\ v = \min(\ddot{q}_{\max} \cdot (t_f - t_i), \dot{q}_{\max}) \\ a = -\ddot{q}_{\max} \end{cases}$$

5.2 Synchronization

In the case of multiple joints, trajectory planning needs to be synchronized based on time. To incorporate control with a frequency of 20 Hz, we can discretize time parameters. Additionally, along the trajectory, we track the Jacobian to ensure a non-zero determinant.

For more details, please, refer to the code in colab link.