EXISTENCE OF $\mathcal{R}$-BOUNDED SOLUTION OPERATOR FAMILIES FOR A COMPRESSIBLE FLUID MODEL OF KORTEWEG TYPE ON THE HALF-SPACE

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ABSTRACT. The aim of this paper is to show the existence of $\mathcal{R}$-bounded solution operator families for a generalized resolvent problem on the half-space arising from a compressible fluid model of Korteweg type. Such a compressible fluid model was derived by Dunn and Serrin (1985) and studied by Kotschote (2008) as an initial-boundary value problem.

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. Let $\mathbf{R}^N_+$ and $\mathbf{R}^N_0$ be respectively the half-space and the boundary of $\mathbf{R}^N_+$ for $N \geq 2$, i.e.

$$\mathbf{R}^N_+ = \{x = (x', x_N) \mid x' = (x_1, \dots, x_{N-1}) \in \mathbf{R}^{N-1}, x_N > 0\},$$
$$\mathbf{R}^N_0 = \{x = (x', x_N) \mid x' = (x_1, \dots, x_{N-1}) \in \mathbf{R}^{N-1}, x_N = 0\}.$$

This paper is concerned with the existence of $\mathcal{R}$-bounded solution operator families for a generalized resolvent problem on $\mathbf{R}^N_+$ arising from a compressible fluid model of Korteweg type as follows:

$$\begin{cases} \lambda\rho + \operatorname{div} \mathbf{u} = d & \text{in } \mathbf{R}^N_+, \\ \lambda\mathbf{u} - \mu_*\Delta\mathbf{u} - \nu_*\nabla \operatorname{div} \mathbf{u} - \kappa_*\nabla\Delta\rho = \mathbf{f} & \text{in } \mathbf{R}^N_+, \\ \mathbf{n}\cdot\nabla\rho = g, \quad \mathbf{u} = 0 & \text{on } \mathbf{R}^N_0. \end{cases} \tag{1.1}$$

Here λ is the resolvent parameter varying in

$$\Sigma_\varepsilon = \{\lambda \in \mathbf{C} \setminus \{0\} \mid |\arg\lambda| < \pi - \varepsilon\} \tag{1.2}$$

for $\varepsilon \in (0, \pi/2)$; $\rho = \rho(x)$ and $\mathbf{u} = \mathbf{u}(x) = (u_1(x), \dots, u_N(x))^\mathsf{T}$ are respectively the fluid density and the fluid velocity that are unknown functions; $d = d(x)$, $\mathbf{f} = \mathbf{f}(x) = (f_1(x), \dots, f_N(x))^\mathsf{T}$, and $g = g(x)$ are given functions; $\mathbf{n} = (0, \dots, 0, -1)^\mathsf{T}$ is the outward unit normal vector on $\mathbf{R}^N_0$; $\mathbf{a}\cdot\mathbf{b} = \sum_{j=1}^N a_j b_j$ for any N-vectors $a = (a_1, \dots, a_N)^\mathsf{T}$ and $\mathbf{b} = (b_1, \dots, b_N)^\mathsf{T}$. Here and subsequently, one uses the following notation for differentiations: Let $u = u(x)$, $\mathbf{v} = (v_1(x), \dots, v_N(x))^\mathsf{T}$, and $\mathbf{M} = (M_{ij}(x))$ be a scalar-, a vector-, and an $N \times N$ matrix-valued function defined

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on a domain of $\mathbf{R}^N$, and then for $\partial_j = \partial/\partial x_j$

$$\nabla u = (\partial_1 u, \dots, \partial_N u)^\mathsf{T}, \quad \Delta u = \sum_{j=1}^N \partial_j^2 u, \quad \Delta \mathbf{v} = (\Delta v_1, \dots, \Delta v_N)^\mathsf{T},$$

$$\operatorname{div} \mathbf{v} = \sum_{j=1}^N \partial_j v_j, \quad \nabla \mathbf{v} = \{\partial_j v_k \mid j, k = 1, \dots, N\},$$

$$\nabla^2 \mathbf{v} = \{\partial_j \partial_k v_l \mid j, k, l = 1, \dots, N\}, \quad \operatorname{Div} \mathbf{M} = \left(\sum_{j=1}^N \partial_j M_{1j}, \dots, \sum_{j=1}^N \partial_j M_{Nj} \right)^\mathsf{T}.$$

The μ_* and ν_* are positive constants describing the viscosity coefficients, while κ_* is a positive constant describing the capillarity coefficient (cf. e.g. [8] for more detail). In [8], the author assumes

$$\eta_* := \left(\frac{\mu_* + \nu_*}{2\kappa_*} \right)^2 - \frac{1}{\kappa_*} \neq 0 \quad \text{and} \quad \kappa_* \neq \mu_* \nu_* \tag{1.3}$$

in order to show the existence of $\mathcal{R}$-bounded solution operator families for a generalized problem with free boundary condition. On the other hand, this paper shows the existence of $\mathcal{R}$-bounded solutions operator families associated with (1.1) for arbitrary positive constants μ_*, ν_*, and κ_*. To this end, we divide in the present paper the proof into five cases as follows:

Case I: $\eta_* < 0$;
Case II: $\eta_* > 0$ and $\kappa_* \neq \mu_* \nu_*$;
Case III: $\eta_* > 0$ and $\kappa_* = \mu_* \nu_*$;
Case IV: $\eta_* = 0$ and $\kappa_* \neq \mu_* \nu_*$;
Case V: $\eta_* = 0$ and $\kappa_* = \mu_* \nu_*$.

Throughout this paper, Cases I, II, III, IV, and V mean the above five cases.

The motion of barotropic compressible fluids is governed by

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) &= 0 && \text{(mass conservation)}, \\ \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) &= \operatorname{Div}(\mathbf{T} - P(\rho)\mathbf{I}) && \text{(momentum conservation)}, \end{aligned}$$

subject to the initial condition and suitable boundary conditions, where $\mathbf{I}$ is the $N \times N$ identity matrix and $P : [0, \infty) \to \mathbf{R}$ is a given smooth function describing the pressure. Dunn and Serrin have shown in [2] that for a special material of Korteweg type the stress tensor $\mathbf{T}$ is given by

$$\begin{aligned} &\mathbf{T} = \mathbf{S}(\mathbf{u}) + \mathbf{K}(\rho), \quad \mathbf{S}(\mathbf{u}) = \mu \mathbf{D}(\mathbf{u}) + (\nu - \mu) \operatorname{div} \mathbf{u} \mathbf{I}, \\ &\mathbf{K}(\rho) = \frac{\kappa}{2}(\Delta^2 \rho - |\nabla \rho|^2)\mathbf{I} - \kappa \nabla \rho \otimes \nabla \rho, \end{aligned}$$

where $\mathbf{S}(\mathbf{u})$ is the usual viscous stress tensor with viscosity coefficients μ, ν and $\mathbf{D}(\mathbf{u})$ is the doubled strain tensor whose (i, j)-component is given by $\partial_i u_j + \partial_j u_i$; $\mathbf{K}(\rho)$ is called the Korteweg tensor with a capillarity coefficient κ. Such a Korteweg-type model was mathematically studied by Kotschote [3, 4, 5, 6] as initial-boundary value problems. In a forthcoming paper [9], we prove a maximal regularity for some linearized system of an initial-boundary value problem of the Korteweg-type model on general domains by using results obtained in this paper. We also refer to [9] in order to see a brief history of mathematical studies of Korteweg-type models.

Throughout this paper, the letter C denotes generic constants and $C_{a,b,c,\dots}$ means that the constant depends on the quantities $a, b, c, \dots$. The values of constants C and $C_{a,b,c,\dots}$ may change from line to line.

1.2. **Main results.** To state our main results, we first introduce the notation and the definition of the $\mathcal{R}$-boundedness of operator families.

The set of all natural numbers is denoted by $\mathbf{N}$ and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, while the set of all complex numbers is denoted by $\mathbf{C}$ and $\mathbf{C}_+ = \{z \in \mathbf{C} \mid \Re z > 0\}$. Let $q \in [1, \infty]$ and G be a domain of $\mathbf{R}^N$. Then $L_q(G)$ and $H_q^m(G)$, $m \in \mathbf{N}$, denote respectively the usual Lebesgue spaces on G and the usual Sobolev spaces on G. One sets $H_q^0(G) = L_q(G)$ and denotes the norm of $H_q^n(G)$, $n \in \mathbf{N}_0$, by $\|\cdot\|_{H_q^n(G)}$.

Let X and Y be Banach spaces. Then X^m, $m \in \mathbf{N}$, denotes the m-product space of X, while the norm of X^m is usually denoted by $\|\cdot\|_X$ for short. The symbol $\mathcal{L}(X, Y)$ stands for the Banach space of all bounded linear operators from X to Y, and $\mathcal{L}(X)$ is the abbreviation of $\mathcal{L}(X, X)$. For a domain U of $\mathbf{C}$, $\mathrm{Hol}(U, \mathcal{L}(X, Y))$ is the set of all $\mathcal{L}(X, Y)$-valued holomorphic functions defined on U.

For the right member $(d, \mathbf{f}, g)$ of (1.1), we set

$$\mathcal{X}_q^1(G) = H_q^1(G) \times L_q(G)^N, \quad \mathcal{X}_q^2(G) = H_q^1(G) \times L_q(G)^N \times H_q^2(G).$$

Let $\mathbf{F}^1 = (d, \mathbf{f}) \in \mathcal{X}_q^1(G)$ and $\mathbf{F}^2 = (d, \mathbf{f}, g) \in \mathcal{X}_q^2(G)$. Symbols $\mathfrak{X}_q^j(G)$ and $\mathcal{F}_\lambda^j$ $(j = 1, 2)$ are then defined as follows:

$$\begin{aligned}
&\mathfrak{X}_q^1(G) = L_q(G)^{\mathsf{N}_1}, \quad \mathsf{N}_1 = N + 1 + N, \quad \mathcal{F}_\lambda^1 \mathbf{F}^1 = (\nabla d, \lambda^{1/2} d, \mathbf{f}) \in \mathfrak{X}_q^1(G);\\
&\mathfrak{X}_q^2(G) = L_q(G)^{\mathsf{N}_2}, \quad \mathsf{N}_2 = N + 1 + N + N^2 + N + 1,\\
&\mathcal{F}_\lambda^2 \mathbf{F}^2 = (\nabla d, \lambda^{1/2} d, \mathbf{f}, \nabla^2 g, \lambda^{1/2}\nabla g, \lambda g) \in \mathfrak{X}_q^2(G).
\end{aligned}$$

One also sets for solutions of (1.1)

$$\begin{aligned}
&\mathfrak{A}_q^0(G) = L_q(G)^{N^3+N^2+N+1}, \quad \mathcal{S}_\lambda^0 \rho = (\nabla^3 \rho, \lambda^{1/2}\nabla^2\rho, \lambda\nabla\rho, \lambda^{3/2}\rho);\\
&\mathfrak{B}_q(G) = L_q(G)^{N^3+N^2+N}, \qquad \mathcal{T}_\lambda \mathbf{u} = (\nabla^2 \mathbf{u}, \lambda^{1/2}\nabla\mathbf{u}, \lambda\mathbf{u}).
\end{aligned} \tag{1.4}$$

Remark 1.1. The above symbols $\mathcal{X}_q^1(G)$, $\mathcal{X}_q^2(G)$, $\mathfrak{X}_q^1(G)$, $\mathcal{F}_\lambda^1$, $\mathfrak{X}_q^2(G)$, $\mathcal{F}_\lambda^2$, $\mathfrak{A}_q^0(G)$, $\mathcal{S}_\lambda^0$, $\mathfrak{B}_q(G)$, and $\mathcal{T}_\lambda$ are defined in the same manner as [9].

The definition of the $\mathcal{R}$-boundedness of operator families is as follows:

Definition 1.2. *Let X and Y be Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X, Y)$, if there exist constants $p \in [1, \infty)$ and $C > 0$ such that the following assertion holds true: For each $m \in \mathbf{N}$, $\{T_j\}_{j=1}^m \subset \mathcal{T}$, $\{f_j\}_{j=1}^m \subset X$ and for all sequences $\{r_j(u)\}_{j=1}^m$ of independent, symmetric, $\{-1, 1\}$-valued random variables on $[0, 1]$, there holds*

$$\left(\int_0^1 \Big\|\sum_{j=1}^m r_j(u) T_j f_j\Big\|_Y^p \, du\right)^{1/p} \le C\left(\int_0^1 \Big\|\sum_{j=1}^m r_j(u) f_j\Big\|_X^p \, du\right)^{1/p}.$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X, Y)$ and denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

Remark 1.3. (1) The constant C in Definition 1.2 may depend on p.

(2) It is known that $\mathcal{T}$ is $\mathcal{R}$-bounded for any $p \in [1, \infty)$, provided that $\mathcal{T}$ is $\mathcal{R}$-bounded for some $p \in [1, \infty)$. This fact follows from Kahane's inequality (cf. e.g. [7, Theorem 2.4]).

One now states the main result of this paper.

Theorem 1.4. *Let $q \in (1,\infty)$ and assume that μ_*, ν_*, and κ_* are positive constants. Then, for any $\lambda \in \mathbf{C}_+$, there exist operators $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$, with*

$$\mathcal{A}(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(\mathfrak{X}_q^2(\mathbf{R}_+^N), H_q^3(\mathbf{R}_+^N)),$$
$$\mathcal{B}(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(\mathfrak{X}_q^2(\mathbf{R}_+^N), H_q^2(\mathbf{R}_+^N)^N)),$$

such that, for any $\mathbf{F} = (d, \mathbf{f}, g) \in \mathcal{X}_q^2(\mathbf{R}_+^N)$, $(\rho, \mathbf{u}) = (\mathcal{A}(\lambda)\mathcal{F}_\lambda^2\mathbf{F}, \mathcal{B}(\lambda)\mathcal{F}_\lambda^2\mathbf{F})$ is a unique solution to the system (1.1)*. In addition, for $n = 0, 1$,*

$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q^2(\mathbf{R}_+^N), \mathfrak{A}_q^0(\mathbf{R}_+^N))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n (\mathcal{S}_\lambda^0\mathcal{A}(\lambda)) \,\middle|\, \lambda \in \mathbf{C}_+\right\}\right) \leq C_{N,q,\mu_*,\nu_*,\kappa_*},$$
$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q^2(\mathbf{R}_+^N), \mathfrak{B}_q(\mathbf{R}_+^N))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n (\mathcal{T}_\lambda\mathcal{B}(\lambda)) \,\middle|\, \lambda \in \mathbf{C}_+\right\}\right) \leq C_{N,q,\mu_*,\nu_*,\kappa_*},$$

where $C_{N,q,\mu_*,\nu_*,\kappa_*}$ is a positive constant depending on at most N, q, μ_*, ν_*, and κ_*.

This paper is organized as follows: The next section first introduces some classes of symbols and their fundamental properties. Secondly, one computes characteristic roots that appear in analysis of a system of ordinary differential equations in the Fourier space in Sections 3, 4, 5, and 6. Thirdly, we introduce technical lemmas that play an important role in proving the existence of $\mathcal{R}$-bounded solution operator families. Fourthly, the system (1.1) is reduced to the case where $(d, \mathbf{f}) = (0, 0)$, and the main theorem (i.e. the existence of $\mathcal{R}$-bounded solution operator families) is stated for the reduced system. Furthermore, one proves Theorem 1.4, assuming the main theorem of the reduced system holds. Section 3 proves the main theorem of the reduced system for Cases I and II. Section 4, 5, and 6 treat respectively Case III, IV, and V, and prove the matin theorem of the reduced system.

2. Preliminaries

2.1. **Classes of symbols.** Recall that Σ_ε is given in (1.2) for $\varepsilon \in (0, \pi/2)$. Let $\Lambda = \Sigma_\varepsilon$ or $\Lambda = \mathbf{C}_+$, and let $m(\xi', \lambda)$ be a function, defined on $(\mathbf{R}^{N-1} \setminus \{0\}) \times \Lambda$, that is infinitely many times differentiable with respect to $\xi' = (\xi_1, \dots, \xi_{N-1})$ and holomorphic with respect to λ. For any multi-index $\alpha' = (\alpha_1, \dots, \alpha_{N-1}) \in \mathbf{N}_0^{N-1}$, let us define $\partial_{\xi'}^{\alpha'}$ by

$$\partial_{\xi'}^{\alpha'} = \frac{\partial^{|\alpha'|}}{\partial\xi_1^{\alpha_1} \dots \partial\xi_{N-1}^{\alpha_{N-1}}}, \quad |\alpha'| = \alpha_1 + \dots + \alpha_{N-1}.$$

If there exists a real number r such that for any multi-index $\alpha' = (\alpha_1, \dots, \alpha_{N-1}) \in \mathbf{N}_0^{N-1}$ and $(\xi', \lambda) \in (\mathbf{R}^{N-1} \setminus \{0\}) \times \Lambda$

$$\left|\partial_{\xi'}^{\alpha'}\left(\left(\lambda\frac{d}{d\lambda}\right)^n m(\xi', \lambda)\right)\right| \leq C(|\lambda|^{1/2} + |\xi'|)^{r-|\alpha'|} \quad (n = 0, 1)$$

with some positive constant C depending solely on N, r, α', and ε, then $m(\xi', \lambda)$ is called a multiplier of order r with type 1. If there exists a real number r such that for any multi-index $\alpha' = (\alpha_1, \dots, \alpha_{N-1}) \in \mathbf{N}_0^{N-1}$ and $(\xi', \lambda) \in (\mathbf{R}^{N-1} \setminus \{0\}) \times \Lambda$

$$\left|\partial_{\xi'}^{\alpha'}\left(\left(\lambda\frac{d}{d\lambda}\right)^n m(\xi', \lambda)\right)\right| \leq C(|\lambda|^{1/2} + |\xi'|)^r |\xi'|^{-|\alpha'|} \quad (n = 0, 1)$$

with some positive constant C depending solely on N, r, α', and ε, then $m(\xi',\lambda)$ is called a multiplier of order r with type 2.

Here and subsequently, we denote the set of all symbols of order r with type j on $(\mathbf{R}^{N-1}\setminus\{0\})\times\Lambda$ by $\mathbb{M}_{r,j}(\Lambda)$. For instance,

$$\xi_k/|\xi'|\in\mathbb{M}_{0,2}(\Lambda),\quad \xi_k,\lambda^{1/2}\in\mathbb{M}_{1,1}(\Lambda)\quad (k=1,\dots,N-1),$$

and also $|\xi'|^2,\lambda\in\mathbb{M}_{2,1}(\Lambda)$. One notes for $r\in\mathbf{R}$ and $j=1,2$ that $\mathbb{M}_{r,j}(\Lambda)$ are vector spaces on $\mathbf{C}$ and that $\mathbb{M}_{r,j}(\Sigma_\varepsilon)\subset\mathbb{M}_{r,j}(\mathbf{C}_+)$ for any $\varepsilon\in(0,\pi/2)$. In addition, we know the following fundamental properties of $\mathbb{M}_{r,j}(\Lambda)$ (cf. [11, Lemma 5.1]).

Lemma 2.1. *Let $r_1,r_2\in\mathbf{R}$, and let $\Lambda=\Sigma_\varepsilon$ for some $\varepsilon\in(0,\pi/2)$ or $\Lambda=\mathbf{C}_+$. Then the following assertions hold true:*

(1) *Given $l_j\in\mathbb{M}_{r_j,1}(\Lambda)$ $(j=1,2)$, we have $l_1l_2\in\mathbb{M}_{r_1+r_2,1}(\Lambda)$.*
(2) *Given $m_j\in\mathbb{M}_{r_j,j}(\Lambda)$ $(j=1,2)$, we have $m_1m_2\in\mathbb{M}_{r_1+r_2,2}(\Lambda)$.*
(3) *Given $n_j\in\mathbb{M}_{r_j,2}(\Lambda)$ $(j=1,2)$, we have $n_1n_2\in\mathbb{M}_{r_1+r_2,2}(\Lambda)$.*

2.2. **Characteristic roots.** Let μ_*, ν_*, and κ_* be positive constants, and let us define a polynomial $\mathcal{P}(s)$ by

$$\mathcal{P}(s)=s^2-\frac{\mu_*+\nu_*}{\kappa_*}s+\frac{1}{\kappa_*}.$$

The roofs $s_\pm$ of $\mathcal{P}(s)$ are then given by

$$s_\pm=\begin{cases}\dfrac{\mu_*+\nu_*}{2\kappa_*}\pm\sqrt{\eta_*} & (\eta_*\geq 0),\\[2mm] \dfrac{\mu_*+\nu_*}{2\kappa_*}\pm i\sqrt{|\eta_*|} & (\eta_*<0),\end{cases}$$

where $i=\sqrt{-1}$ and η_* is given in (1.3). Setting

$$s_1=s_-,\quad s_2=s_+,$$

we have

Lemma 2.2. *Let μ_*, ν_*, and κ_* be positive constants. Then the following assertions hold true:*

(1) *If μ_*, ν_*, and κ_* satisfy the condition of Case I, then s_1 and s_2 are imaginary numbers with $\Re s_j>0$ $(j=1,2)$. Especially, in Case I,*
$$s_1\neq s_2,\quad s_1\neq\mu_*^{-1},\quad s_2\neq\mu_*^{-1}.$$
(2) *If μ_*, ν_*, and κ_* satisfy the condition of Case II, then s_1 and s_2 are real numbers with $s_j>0$ $(j=1,2)$. Especially, in Case II,*
$$s_1\neq s_2,\quad s_1\neq\mu_*^{-1},\quad s_2\neq\mu_*^{-1}.$$
(3) *If μ_*, ν_*, and κ_* satisfy the condition of Case III, then $\mu_*\neq\nu_*$. Especially, in Case III,*
$$(s_1,s_2)=(\mu_*^{-1},\nu_*^{-1})\quad \textit{when } \mu_*>\nu_*;$$
$$(s_1,s_2)=(\nu_*^{-1},\mu_*^{-1})\quad \textit{when } \mu_*<\nu_*.$$
(4) *If μ_*, ν_*, and κ_* satisfy the condition of Case IV, then $\mu_*\neq\nu_*$. Especially, in Case IV,*
$$s_1=s_2=\frac{\mu_*+\nu_*}{2\kappa_*},\quad s_2\neq\mu_*^{-1}.$$

(5) *If μ_*, ν_*, and κ_* satisfy the condition of Case V, then $\mu_* = \nu_*$. Especially, in Case V,*

$$s_1 = s_2 = \frac{\mu_* + \nu_*}{2\kappa_*} = \mu_*^{-1}.$$

Let us define a positive number ε_* by

$$\varepsilon_* = \arg s_2 = \arg s_+ \in [0, \pi/2).$$

One then sets

$$t_j = \sqrt{|\xi'|^2 + s_j\lambda} \quad \text{for } j = 1, 2, \tag{2.1}$$

where $(\xi', \lambda) \in \mathbf{R}^{N-1} \times \Sigma_\varepsilon$ for $\varepsilon \in (\varepsilon_*, \pi/2)$. Here we have chosen a branch cut along the negative real axis and a branch of the square root so that $\Re\sqrt{z} > 0$ for $z \in \mathbf{C} \setminus (-\infty, 0]$. In addition, we set

$$\omega_\lambda = \sqrt{|\xi'|^2 + \mu_*^{-1}\lambda}, \tag{2.2}$$

where $(\xi', \lambda) \in \mathbf{R}^{N-1} \times \Sigma_\varepsilon$ for $\varepsilon \in (0, \pi/2)$. By Lemma 2.2, we have

Lemma 2.3. *Let μ_*, ν_*, and κ_* be positive constants, and let $\xi' \in \mathbf{R}^{N-1}$ and $\lambda \in \Sigma_\varepsilon$ for $\varepsilon \in (\varepsilon_*, \pi/2)$. Then the following assertions hold true:*

(1) *If μ_*, ν_*, and κ_* satisfy the conditions of Case I or Case II, then*

$$t_1 \neq t_2, \quad t_1 \neq \omega_\lambda, \quad t_2 \neq \omega_\lambda.$$

(2) *If μ_*, ν_*, and κ_* satisfy the condition of Case III, then*

$$t_1 = \omega_\lambda, \quad t_2 = \sqrt{|\xi'|^2 + \nu_*^{-1}\lambda} \quad \textit{when } \mu_* > \nu_*;$$
$$t_2 = \omega_\lambda, \quad t_1 = \sqrt{|\xi'|^2 + \nu_*^{-1}\lambda} \quad \textit{when } \mu_* < \nu_*.$$

(3) *If μ_*, ν_*, and κ_* satisfy the condition of Case IV, then*

$$t_1 = t_2 = \sqrt{|\xi'|^2 + \left(\frac{\mu_* + \nu_*}{2\kappa_*}\right)\lambda}, \quad t_2 \neq \omega_\lambda.$$

(4) *If μ_*, ν_*, and κ_* satisfy the condition of Case V, then*

$$t_1 = t_2 = \omega_\lambda.$$

Let us define a polynomial $P_\lambda(t)$ by

$$P_\lambda(t) = \lambda^2 - \lambda(\mu_* + \nu_*)(t^2 - |\xi'|^2) + \kappa_*(t^2 - |\xi'|^2)^2, \tag{2.3}$$

where $(\xi', \lambda) \in \times\mathbf{R}^{N-1} \times \Sigma_\varepsilon$ for $\varepsilon \in (\varepsilon_*, \pi/2)$. Since

$$\begin{aligned} P_\lambda(t) &= \kappa_*\lambda^2 \left\{ \frac{1}{\kappa_*} - \left(\frac{\mu_* + \nu_*}{\kappa_*}\right)\left(\frac{t^2 - |\xi'|^2}{\lambda}\right) + \left(\frac{t^2 - |\xi'|^2}{\lambda}\right)^2 \right\} \\ &= \kappa_*\lambda^2 \mathcal{P}\left(\frac{t^2 - |\xi'|^2}{\lambda}\right), \end{aligned}$$

the four roots of $P_\lambda(t)$ are given by $\pm t_1$ and $\pm t_2$. Hence, one has

Lemma 2.4. *Let μ_*, ν_*, and κ_* be positive constants, and let $\xi' \in \mathbf{R}^{N-1}$ and $\lambda \in \Sigma_\varepsilon$ for $\varepsilon \in (\varepsilon_*, \pi/2)$. Then the four roots of $P_\lambda(t)$ are given by $\pm t_1$ and $\pm t_2$. Especially, $\Re t_1 > 0$ and $\Re t_2 > 0$.*

Similarly to [8], we can prove

Lemma 2.5. *Assume that μ_*, ν_*, and κ_* are positive constants. Let $\varepsilon_1 \in (\varepsilon_*, \pi/2)$ and $\varepsilon_2 \in (0, \pi/2)$. Then the following assertions holds true:*

(1) *There exists a positive constant $C_{\varepsilon_1,\mu_*,\nu_*,\kappa_*}$ such that*
$$\Re t_j \geq C_{\varepsilon_1,\mu_*,\nu_*,\kappa_*}(|\lambda|^{1/2} + |\xi'|) \quad (j = 1, 2)$$
for any $(\xi', \lambda) \in \mathbf{R}^{N-1} \times \Sigma_{\varepsilon_1}$.

(2) *There exists a positive constant C_{ε_2,μ_*} such that*
$$\Re\omega_\lambda \geq C_{\varepsilon_2,\mu_*}(|\lambda|^{1/2} + |\xi'|)$$
for any $(\xi', \lambda) \in \mathbf{R}^{N-1} \times \Sigma_{\varepsilon_2}$.

(3) *Let $r \in \mathbf{R}$ and $a > 0$. Then*
$$\begin{aligned} t_1^r, t_2^r, (t_1 + \omega_\lambda)^r, (t_2 + \omega_\lambda)^r &\in \mathbb{M}_{r,1}(\Sigma_{\varepsilon_1}), \\ \omega_\lambda^r &\in \mathbb{M}_{r,1}(\Sigma_{\varepsilon_2}), \\ (|\xi'|^2 + a\lambda)^r &\in \mathbb{M}_{2r,1}(\Sigma_{\varepsilon_2}). \end{aligned}$$

2.3. **Technical lemmas.** Let us introduce the following notation: for $x_N > 0$,

$$\mathcal{M}_0(x_N) = \frac{e^{-t_2 x_N} - e^{-t_1 x_N}}{t_2 - t_1}, \quad \mathcal{M}_j(x_N) = \frac{e^{-t_j x_N} - e^{-\omega_\lambda x_N}}{t_2 - t_1} \quad (j = 1, 2) \tag{2.4}$$

when $t_2 \neq t_1$;

$$\mathcal{M}(x_N) = \frac{e^{-t_2 x_N} - e^{-\omega_\lambda x_N}}{t_2 - \omega_\lambda} \tag{2.5}$$

when $t_2 \neq \omega_\lambda$. In addition, we define the partial Fourier transform with respect to $x' = (x_1, \ldots, x_{N-1})$ and its inverse transform by

$$\widehat{u} = \widehat{u}(x_N) = \widehat{u}(\xi', x_N) = \int_{\mathbf{R}^{N-1}} e^{-ix'\cdot\xi'} u(x', x_N)\, dx', \tag{2.6}$$

$$\mathcal{F}_{\xi'}^{-1}[v(\xi', x_N)](x') = \frac{1}{(2\pi)^{N-1}} \int_{\mathbf{R}^{N-1}} e^{ix'\cdot\xi'} v(\xi', x_N)\, d\xi', \tag{2.7}$$

respectively. Similarly to [8], one then has the following three lemmas:

Lemma 2.6. *Let $q \in (1, \infty)$, and let t_j $(j = 1, 2)$ and ω_λ be respectively given by (2.1) and (2.2) for positive constants μ_*, ν_*, and κ_*. Assume*
$$k(\xi', \lambda) \in \mathbb{M}_{-1,1}(\mathbf{C}_+), \quad l(\xi', \lambda) \in \mathbb{M}_{0,1}(\mathbf{C}_+),$$
and set for $x = (x', x_N) \in \mathbf{R}^N_+$
$$\begin{aligned} [K_0(\lambda)f](x) &= \mathcal{F}_{\xi'}^{-1}\left[k(\xi', \lambda)e^{-\omega_\lambda x_N}\widehat{f}(\xi', 0)\right](x'), \\ [L_0(\lambda)f](x) &= \mathcal{F}_{\xi'}^{-1}\left[l(\xi', \lambda)e^{-\omega_\lambda x_N}\widehat{f}(\xi', 0)\right](x'), \\ [K_j(\lambda)f](x) &= \mathcal{F}_{\xi'}^{-1}\left[k(\xi', \lambda)e^{-t_j x_N}\widehat{f}(\xi', 0)\right](x') \quad (j = 1, 2), \\ [L_j(\lambda)f](x) &= \mathcal{F}_{\xi'}^{-1}\left[l(\xi', \lambda)e^{-t_j x_N}\widehat{f}(\xi', 0)\right](x') \quad (j = 1, 2), \end{aligned}$$
with $\lambda \in \mathbf{C}_+$ and $f \in H^2_q(\mathbf{R}^N_+)$. Then the following assertions hold true:

(1) *For $j=0,1,2$ and $\lambda\in\mathbf{C}_+$, there are operators $\widetilde{K}_j(\lambda)$, with*
$$\widetilde{K}_j(\lambda)\in\mathrm{Hol}(\mathbf{C}_+,\mathcal{L}(L_q(\mathbf{R}^N_+)^{N^2+N+1},H^3_q(\mathbf{R}^N_+))),$$
such that for any $f\in H^2_q(\mathbf{R}^N_+)$
$$K_j(\lambda)f=\widetilde{K}_j(\lambda)(\nabla^2 f,\lambda^{1/2}\nabla f,\lambda f).$$
In addition, for $j=0,1,2$ and $n=0,1$,
$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+)^{N^2+N+1},\mathfrak{A}^0_q(\mathbf{R}^N_+))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\mathcal{S}^0_\lambda\widetilde{K}_j(\lambda)\right)\Big|\,\lambda\in\mathbf{C}_+\right\}\right)\leq C,$$
with some positive constant C depending solely on N, q, μ_*, ν_*, and κ_*. Here, $\mathfrak{A}^0_q(\mathbf{R}^N_+)$ and $\mathcal{S}^0_\lambda$ are given in (1.4) for $G=\mathbf{R}^N_+$.

(2) *For $j=0,1,2$ and $\lambda\in\mathbf{C}_+$, there are operators $\widetilde{L}_j(\lambda)$, with*
$$\widetilde{L}_j(\lambda)\in\mathrm{Hol}(\mathbf{C}_+,\mathcal{L}(L_q(\mathbf{R}^N_+)^{N^2+N+1},H^2_q(\mathbf{R}^N_+))),$$
such that for any $f\in H^2_q(\mathbf{R}^N_+)$
$$L_j(\lambda)f=\widetilde{L}_j(\lambda)(\nabla^2 f,\lambda^{1/2}\nabla f,\lambda f).$$
In addition, for $j=0,1,2$ and $n=0,1$,
$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+)^{N^2+N+1})}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\mathcal{T}_\lambda\widetilde{L}_j(\lambda)\right)\Big|\,\lambda\in\mathbf{C}_+\right\}\right)\leq C,$$
with some positive constant C depending solely on N, q, μ_*, ν_*, and κ_*. Here, $\mathcal{T}_\lambda$ are given in (1.4).

Lemma 2.7. *Let $q\in(1,\infty)$, and let t_j $(j=1,2)$ and ω_λ be respectively given by (2.1) and (2.2) for positive constants μ_*, ν_*, and κ_* satisfying the conditions of Case I or Case II. Assume*
$$m_0(\xi',\lambda)\in\mathbb{M}_{0,1}(\mathbf{C}_+),\quad m_1(\xi',\lambda),m_2(\xi',\lambda)\in\mathbb{M}_{1,1}(\mathbf{C}_+),$$
and set for $x=(x',x_N)\in\mathbf{R}^N_+$
$$[M_k(\lambda)f](x)=\mathcal{F}^{-1}_{\xi'}\left[m_k(\xi',\lambda)\mathcal{M}_k(x_N)\widehat{f}(\xi',0)\right](x'),$$
with $\lambda\in\mathbf{C}_+$ and $f\in H^2_q(\mathbf{R}^N_+)$. Then the following assertions hold true:

(1) *For $\lambda\in\mathbf{C}_+$, there is an operator $\widetilde{M}_0(\lambda)$, with*
$$\widetilde{M}_0(\lambda)\in\mathrm{Hol}(\mathbf{C}_+,L_q(\mathbf{R}^N_+)^{N^2+N+1},H^3_q(\mathbf{R}^N_+)),$$
such that for any $f\in H^2_q(\mathbf{R}^N_+)$
$$M_0(\lambda)f=\widetilde{M}_0(\lambda)(\nabla^2 f,\lambda^{1/2}\nabla f,\lambda f).$$
In addition, for $n=0,1$,
$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+)^{N^2+N+1},\mathfrak{A}^0_q(\mathbf{R}^N_+))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\mathcal{S}^0_\lambda\widetilde{M}_0(\lambda)\right)\Big|\,\lambda\in\mathbf{C}_+\right\}\right)\leq C,$$
with some positive constant C depending solely on N, q, μ_*, ν_*, and κ_*.

(2) *For $j = 1, 2$ and $\lambda \in \mathbf{C}_+$, there are operators $\widetilde{M}_j(\lambda)$, with*

$$\widetilde{M}_j(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(L_q(\mathbf{R}^N_+)^{N^2+N+1}, H^2_q(\mathbf{R}^N_+))),$$

such that for any $f \in H^2_q(\mathbf{R}^N_+)$

$$M_j(\lambda)f = \widetilde{M}_j(\lambda)(\nabla^2 f, \lambda^{1/2}\nabla f, \lambda f).$$

In addition, for $j = 1, 2$ and $n = 0, 1$,

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+)^{N^2+N+1})}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n \left(\mathcal{T}_\lambda \widetilde{M}_j(\lambda)\right)\middle| \lambda \in \mathbf{C}_+\right\}\right) \le C,$$

with some positive constant C depending solely on N, q, μ_*, ν_*, and κ_*.

Lemma 2.8. *Let $q \in (1, \infty)$, and let t_j $(j = 1, 2)$ and ω_λ be respectively given by (2.1) and (2.2) for positive constants μ_*, ν_*, and κ_* satisfying one of the following conditions:*

- *the condition of Case III and $\mu_* > \nu_*$;*
- *the condition of Case IV.*

Assume

$$u(\xi', \lambda) \in \mathbb{M}_{0,1}(\mathbf{C}_+), \quad v(\xi', \lambda) \in \mathbb{M}_{1,1}(\mathbf{C}_+),$$

and set for $x = (x', x_N) \in \mathbf{R}^N_+$

$$[U(\lambda)f](x) = \mathcal{F}^{-1}_{\xi'}\left[u(\xi', \lambda)\mathcal{M}(x_N)\widehat{f}(\xi', 0)\right](x'),$$

$$[V(\lambda)f](x) = \mathcal{F}^{-1}_{\xi'}\left[v(\xi', \lambda)\mathcal{M}(x_N)\widehat{f}(\xi', 0)\right](x'),$$

with $\lambda \in \mathbf{C}_+$ and $f \in H^2_q(\mathbf{R}^N_+)$. Then the following assertions hold true:

(1) *For $\lambda \in \mathbf{C}_+$, there is an operator $\widetilde{U}(\lambda)$, with*

$$\widetilde{U}(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, L_q(\mathbf{R}^N_+)^{N^2+N+1}, H^3_q(\mathbf{R}^N_+)),$$

such that for any $f \in H^2_q(\mathbf{R}^N_+)$

$$U(\lambda)f = \widetilde{U}(\lambda)(\nabla^2 f, \lambda^{1/2}\nabla f, \lambda f).$$

In addition, for $n = 0, 1$,

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+)^{N^2+N+1}, \mathfrak{A}^0_q(\mathbf{R}^N_+))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n \left(\mathcal{S}^0_\lambda \widetilde{U}(\lambda)\right)\middle| \lambda \in \mathbf{C}_+\right\}\right) \le C,$$

with some positive constant C depending solely on N, q, μ_*, ν_*, and κ_*.

(2) *For $\lambda \in \mathbf{C}_+$, there is an operator $\widetilde{V}(\lambda)$, with*

$$\widetilde{V}(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(L_q(\mathbf{R}^N_+)^{N^2+N+1}, H^2_q(\mathbf{R}^N_+))),$$

such that for any $f \in H^2_q(\mathbf{R}^N_+)$

$$V(\lambda)f = \widetilde{V}(\lambda)(\nabla^2 f, \lambda^{1/2}\nabla f, \lambda f).$$

In addition, for $n = 0, 1$,

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+)^{N^2+N+1})}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n \left(\mathcal{T}_\lambda \widetilde{V}(\lambda)\right)\middle| \lambda \in \mathbf{C}_+\right\}\right) \le C,$$

with some positive constant C depending solely on N, q, μ_*, ν_*, and κ_*.

In the last part of this subsection, we introduce fundamental properties of the $\mathcal{R}$-boundedness as follows (cf. e.g. [1, Proposition 3.4]):

Lemma 2.9. *Let X, Y, and Z be Banach spaces. Then the following assertions hold true:*

(1) *Let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families on $\mathcal{L}(X,Y)$. Then $\mathcal{T}+\mathcal{S}=\{T+S \mid T\in \mathcal{T}, S\in\mathcal{S}\}$ is also $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$, and*
$$\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}+\mathcal{S})\leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})+\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

(2) *Let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families on $\mathcal{L}(X,Y)$ and on $\mathcal{L}(Y,Z)$, respectively. Then $\mathcal{S}\mathcal{T}=\{ST \mid S\in\mathcal{S}, T\in\mathcal{T}\}$ is also $\mathcal{R}$-bounded on $\mathcal{L}(X,Z)$, and*
$$\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{S}\mathcal{T})\leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(Y,Z)}(\mathcal{S}).$$

2.4. **A reduced system of** (1.1)**.** This subsection reduces the system (1.1) to $(d,\mathbf{f})=(0,0)$. To this end, we start with the following problem on the whole space:
$$\begin{cases} \lambda\rho+\operatorname{div}\mathbf{u}=d & \text{in } \mathbf{R}^N, \\ \lambda\mathbf{u}-\mu_*\Delta\mathbf{u}-\nu_*\nabla\operatorname{div}\mathbf{u}-\kappa_*\nabla\Delta\rho=\mathbf{f} & \text{in } \mathbf{R}^N. \end{cases} \tag{2.8}$$
Recall that $\mathcal{X}_q^1(\mathbf{R}^N)$, $\mathfrak{X}_q^1(\mathbf{R}^N)$, $\mathcal{F}_\lambda^1$, $\mathfrak{A}_q^0(\mathbf{R}^N)$, $\mathcal{S}_\lambda^0$, $\mathfrak{B}_q(\mathbf{R}^N)$, and $\mathcal{T}_\lambda$ are symbols defined in Subsection 1.2 for $G=\mathbf{R}^N$. Concerning the system (2.8), one has

Theorem 2.10. *Let $q\in(1,\infty)$ and assume that μ_*, ν_*, and κ_* are positive constants. Then, for any $\lambda\in\mathbf{C}_+$, there exist operators $\mathcal{A}^1(\lambda)$ and $\mathcal{B}^1(\lambda)$, with*
$$\mathcal{A}^1(\lambda)\in\operatorname{Hol}(\mathbf{C}_+,\mathcal{L}(\mathfrak{X}_q^1(\mathbf{R}^N),H_q^3(\mathbf{R}^N))),$$
$$\mathcal{B}^1(\lambda)\in\operatorname{Hol}(\mathbf{C}_+,\mathcal{L}(\mathfrak{X}_q^2(\mathbf{R}^N),H_q^2(\mathbf{R}^N)^N)),$$
such that, for any $\mathbf{F}^1=(d,\mathbf{f})\in\mathcal{X}_q^1(\mathbf{R}^N)$, $(\rho,\mathbf{u})=(\mathcal{A}^1(\lambda)\mathcal{F}_\lambda^1\mathbf{F}^1,\mathcal{B}^1(\lambda)\mathcal{F}_\lambda^1\mathbf{F}^1)$ is a unique solution to the system (2.8)*. In addition, for $n=0,1$,*
$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q^1(\mathbf{R}^N),\mathfrak{A}_q^0(\mathbf{R}^N))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\mathcal{S}_\lambda^0\mathcal{A}^1(\lambda)\right)\,\middle|\,\lambda\in\mathbf{C}_+\right\}\right)\leq C_{N,q,\mu_*,\nu_*,\kappa_*},$$
$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q^1(\mathbf{R}^N),\mathfrak{B}_q(\mathbf{R}^N))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\mathcal{T}_\lambda\mathcal{B}^1(\lambda)\right)\,\middle|\,\lambda\in\mathbf{C}_+\right\}\right)\leq C_{N,q,\mu_*,\nu_*,\kappa_*},$$
with a positive constant $C_{N,q,\mu_*,\nu_*,\kappa_*}$.

Proof. The proof is similar to [8], so that the detailed proof may be omitted. □

One now reduces the system (1.1) to $(d,\mathbf{f})=(0,0)$ by using Theorem 2.10. For functions $f=f(x)$ with $x=(x',x_N)\in\mathbf{R}_+^N$, let E^ef and E^of be respectively the even extension of f and the odd extension of f, i.e.
$$E^ef=(E^ef)(x)=\begin{cases} f(x',x_N) & (x_N>0), \\ f(x',-x_N) & (x_N<0), \end{cases}$$
$$E^of=(E^of)(x)=\begin{cases} f(x',x_N) & (x_N>0), \\ -f(x',-x_N) & (x_N<0). \end{cases}$$
In addition, we set for $\mathbf{f}=(f_1,\dots,f_N)^\mathsf{T}$ defined on $\mathbf{R}_+^N$
$$\mathbf{E}\mathbf{f}=(E^ef_1,\dots,E^ef_{N-1},E^of_N)^\mathsf{T}.$$
Note that $E^e\in\mathcal{L}(H_q^1(\mathbf{R}_+^N),H_q^1(\mathbf{R}^N))$ and $\mathbf{E}\in\mathcal{L}(L_q(\mathbf{R}_+^N)^N,L_q(\mathbf{R}^N)^N)$.

Let $\mathcal{A}^1(\lambda)$ and $\mathcal{B}^1(\lambda)$ be the operators constructed in Theorem 2.10, and set for $(d, \mathbf{f}) \in H^1_q(\mathbf{R}^N_+) \times L_q(\mathbf{R}^N_+)^N$

$$R = \mathcal{A}^1(\lambda)\mathcal{F}^1_\lambda(E^e d, \mathbf{E}\mathbf{f}), \quad \mathbf{U} = \mathcal{B}^1(\lambda)\mathcal{F}^1_\lambda(E^e d, \mathbf{E}\mathbf{f}).$$

Furthermore, let us define $S = S(x', x_N)$ and $\mathbf{V} = \mathbf{V}(x', x_N)$ as

$$S = R(x', -x_N), \quad \mathbf{V} = (U_1(x', -x_N), \dots, U_{N-1}(x', -x_N), -U_N(x', -x_N))^{\mathsf{T}}.$$

Here and subsequently, U_J and V_J denote for $J = 1, \dots, N$ the Jth component of $\mathbf{U}$ and the Jth component of $\mathbf{V}$, respectively. It then holds that

$$\begin{aligned}
&(\lambda S + \operatorname{div} \mathbf{V})(x', x_N) \\
&= (\lambda R + \operatorname{div} \mathbf{U})(x', -x_N) = (E^e d)(x', -x_N) = (E^e d)(x', x_N).
\end{aligned}$$

Analogously, for $j = 1, \dots, N-1$,

$$\begin{aligned}
(\lambda V_j - \mu_* \Delta V_j - \nu_* \partial_j \operatorname{div} \mathbf{V} - \kappa_* \partial_j \Delta S)(x', x_N) &= (E^e f_j)(x', x_N), \\
(\lambda V_N - \mu_* \Delta V_N - \nu_* \partial_N \operatorname{div} \mathbf{V} - \kappa_* \partial_N \Delta S)(x', x_N) &= (E^o f_N)(x', x_N).
\end{aligned}$$

The uniqueness of solutions of (2.8) then implies $\mathbf{U}(x', x_N) = \mathbf{V}(x', x_N)$. Setting $x_N = 0$ in this equality yields $U_N(x', 0) = 0$.

Let $\rho = R + \widetilde{\rho}$ and $\mathbf{u} = \mathbf{U} + \widetilde{\mathbf{u}}$ in (1.1). One then achieves, by $U_N = 0$ on $\mathbf{R}^N_0$ mentioned above, the following reduced system:

$$\begin{cases}
\lambda \widetilde{\rho} + \operatorname{div} \widetilde{\mathbf{u}} = 0 & \text{in } \mathbf{R}^N_+, \\
\lambda \widetilde{\mathbf{u}} - \mu_* \Delta \widetilde{\mathbf{u}} - \nu_* \nabla \operatorname{div} \widetilde{\mathbf{u}} - \kappa_* \nabla \Delta \widetilde{\rho} = 0 & \text{in } \mathbf{R}^N_+, \\
\mathbf{n} \cdot \nabla \widetilde{\rho} = \widetilde{g}, \quad \widetilde{u}_j = \widetilde{h}_j, \quad \widetilde{u}_N = 0 & \text{on } \mathbf{R}^N_0,
\end{cases} \tag{2.9}$$

for $j = 1, \dots, N-1$ and

$$\begin{aligned}
\widetilde{g} &= g - \mathbf{n} \cdot \nabla \mathcal{A}^1(\lambda)\mathcal{F}^1_\lambda(E^e d, \mathbf{E}\mathbf{f}) = g + \partial_N \mathcal{A}^1(\lambda)\mathcal{F}^1_\lambda(E^e d, \mathbf{E}\mathbf{f}), \\
\widetilde{h}_j &= -(\mathcal{B}^1(\lambda)\mathcal{F}^1_\lambda(E^e d, \mathbf{E}\mathbf{f}))_j,
\end{aligned} \tag{2.10}$$

where $(\mathbf{v})_j$ denotes the jth component of $\mathbf{v}$.

In view of (2.9), let us consider

$$\begin{cases}
\lambda \rho + \operatorname{div} \mathbf{u} = 0 & \text{in } \mathbf{R}^N_+, \\
\lambda u_J - \mu_* \Delta u_J - \nu_* \partial_J \operatorname{div} \mathbf{u} - \kappa_* \partial_J \Delta \rho = 0 & \text{in } \mathbf{R}^N_+, \\
\mathbf{n} \cdot \nabla \rho = g, \quad u_j = h_j, \quad u_N = 0 & \text{on } \mathbf{R}^N_0,
\end{cases} \tag{2.11}$$

where $J = 1, \dots, N$ and $j = 1, \dots, N-1$, for given $g \in H^2_q(\mathbf{R}^N_+)$ and $h_j \in H^2_q(\mathbf{R}^N_+)$. The main part of the proof of Theorem 1.4 is to show

Theorem 2.11. *Let $q \in (1, \infty)$ and set*

$$\begin{aligned}
&\mathcal{Y}_q(\mathbf{R}^N_+) = H^2_q(\mathbf{R}^N_+)^N, \quad \mathfrak{Y}_q(\mathbf{R}^N_+) = L_q(\mathbf{R}^N_+)^{N(N^2+N+1)}, \\
&\mathcal{G}_\lambda(g, h_1, \dots, h_{N-1}) = (\mathcal{T}_\lambda g, \mathcal{T}_\lambda h_1, \dots, \mathcal{T}_\lambda h_{N-1}),
\end{aligned}$$

for $(g, h_1, \dots, h_{N-1}) \in \mathcal{Y}_q(\mathbf{R}^N_+)$. Assume that μ_*, ν_*, and κ_* are positive constants. Then, for any $\lambda \in \mathbf{C}_+$, there are operators $\mathcal{A}^2(\lambda)$ and $\mathcal{B}^2(\lambda)$, with

$$\begin{aligned}
&\mathcal{A}^2(\lambda) \in \operatorname{Hol}(\mathbf{C}_+, \mathcal{L}(\mathfrak{Y}_q(\mathbf{R}^N_+), H^3_q(\mathbf{R}^N_+))), \\
&\mathcal{B}^2(\lambda) \in \operatorname{Hol}(\mathbf{C}_+, \mathcal{L}(\mathfrak{Y}_q(\mathbf{R}^N_+), H^2_q(\mathbf{R}^N_+)^N)),
\end{aligned}$$

such that, for any $\mathbf{G} = (g, h_1, \dots, h_{N-1}) \in \mathcal{Y}_q(\mathbf{R}^N_+)$,

$$(\rho, \mathbf{u}) = (\mathcal{A}^2(\lambda)\mathcal{F}^2_\lambda \mathbf{G}, \mathcal{B}^2(\lambda)\mathcal{F}^2_\lambda \mathbf{G})$$

is a unique solution to the system (2.11). *In addition, for* $n = 0, 1$,

$$\mathcal{R}_{\mathcal{L}(\mathfrak{Y}_q(\mathbf{R}^N_+), \mathfrak{A}^0_q(\mathbf{R}^N_+))}\left(\left\{\left(\lambda \frac{d}{d\lambda}\right)^n \left(\mathcal{S}^0_\lambda \mathcal{A}^2(\lambda)\right) \middle| \lambda \in \mathbf{C}_+\right\}\right) \leq C_{N,q,\mu_*,\nu_*,\kappa_*},$$
$$\mathcal{R}_{\mathcal{L}(\mathfrak{Y}_q(\mathbf{R}^N_+), \mathfrak{B}_q(\mathbf{R}^N_+))}\left(\left\{\left(\lambda \frac{d}{d\lambda}\right)^n \left(\mathcal{T}_\lambda \mathcal{B}^2(\lambda)\right) \middle| \lambda \in \mathbf{C}_+\right\}\right) \leq C_{N,q,\mu_*,\nu_*,\kappa_*},$$

with a positive constant $C_{N,q,\mu_*,\nu_*,\kappa_*}$.

In the remaining part of this subsection, we prove Theorem 1.4, assuming that Theorem 2.11 holds.

Let $\mathbf{F} = (d, \mathbf{f}, g) \in \mathcal{X}_q(\mathbf{R}^N_+)$. Noting $\nabla E^e d = \mathbf{E}\nabla d$, we see $\mathcal{F}^1_\lambda(E^e d, \mathbf{E}\mathbf{f}) = (\mathbf{E}\nabla d, E^e(\lambda^{1/2} d), \mathbf{E}\mathbf{f})$. In view of this relation and (2.10), one sets, for[1] $\mathbf{H} = (H_1, \dots, H_6) \in \mathfrak{X}^2_q(\mathbf{R}^N_+)$ and $\mathcal{Z} \in \{\mathcal{A}, \mathcal{B}\}$,

$$\begin{aligned}
\mathcal{Z}(\lambda)\mathbf{H} = {} & \mathcal{Z}^1(\lambda)(\mathbf{E}H_1, E^e H_2, \mathbf{E}H_3) \\
& + \mathcal{Z}^2(\lambda)\Big(H_4 + \nabla^2 \partial_N \mathcal{A}^1(\lambda)(\mathbf{E}H_1, E^e H_2, \mathbf{E}H_3), \\
& H_5 + \lambda^{1/2}\nabla\partial_N \mathcal{A}^1(\lambda)(\mathbf{E}H_1, E^e H_2, \mathbf{E}H_3), \\
& H_6 + \lambda\partial_N \mathcal{A}^1(\lambda)(\mathbf{E}H_1, E^e H_2, \mathbf{E}H_3), \\
& - \mathcal{T}_\lambda(\mathcal{B}^1(\lambda)(\mathbf{E}H_1, E^e H_2, \mathbf{E}H_3))_1, \dots, \\
& - \mathcal{T}_\lambda(\mathcal{B}^1(\lambda)(\mathbf{E}H_1, E^e H_2, \mathbf{E}H_3))_{N-1}\Big).
\end{aligned}$$

It is then clear that $(\rho, \mathbf{u}) = (\mathcal{A}(\lambda)\mathcal{F}^2_\lambda \mathbf{F}, \mathcal{B}(\lambda)\mathcal{F}^2_\lambda \mathbf{F})$ is a solution to (1.1), and also $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ satisfy the estimates required in Theorem 1.4 by Lemma 2.9 and Theorems 2.10 and 2.11. The uniqueness of solutions of (1.1) can be proved similarly to [8]. This completes the proof of Theorem 1.4.

Remark 2.12. The following sections are devoted to the proof of Theorem 2.11, and the proof is divided into Cases I, II, III, IV, and V.

3. Proof of Theorem 2.11 for Cases I and II

This section proves Theorem 2.11 for Cases I and II. Throughout this section, we assume that μ_*, ν_*, and κ_* are positive constants satisfying the conditions of Case I or Case II. One then recalls Lemma 2.3 (1), i.e.

$$t_1 \neq t_2, \quad t_1 \neq \omega_\lambda, \quad t_2 \neq \omega_\lambda,$$

which are often used in the following computations. Let $J = 1, \dots, N$ and $j = 1, \dots, N-1$ in this section.

---

[1] H_1, H_2, H_3, H_4, H_5, and H_6 are variables corresponding to ∇d, $\lambda^{1/2} d$, $\mathbf{f}$, $\nabla^2 g$, $\lambda^{1/2}\nabla g$, and λg, respectively.

3.1. **Solution formulas.** Set $\varphi = \operatorname{div} \mathbf{u}$. Applying the partial Fourier transform given by (2.6) to the system (2.11) yields the ordinary differential equations:

$$\lambda\widehat{\rho} + \widehat{\varphi} = 0, \ x_N > 0, \tag{3.1}$$

$$\lambda\widehat{u}_j - \mu_*(\partial_N^2 - |\xi'|^2)\widehat{u}_j - \nu_* i\xi_j\widehat{\varphi} - \kappa_* i\xi_j(\partial_N^2 - |\xi'|^2)\widehat{\rho} = 0, \ x_N > 0, \tag{3.2}$$

$$\lambda\widehat{u}_N - \mu_*(\partial_N^2 - |\xi'|^2)\widehat{u}_N - \nu_*\partial_N\widehat{\varphi} - \kappa_*\partial_N(\partial_N^2 - |\xi'|^2)\widehat{\rho} = 0, \ x_N > 0, \tag{3.3}$$

with the boundary conditions:

$$\partial_N\widehat{\rho}(0) = -\widehat{g}(0), \tag{3.4}$$

$$\widehat{u}_j(0) = \widehat{h}_j(0), \quad \widehat{u}_N(0) = 0. \tag{3.5}$$

One inserts (3.1) into (3.2), (3.3), and (3.4), and then

$$\lambda^2\widehat{u}_j - \lambda\mu_*(\partial_N^2 - |\xi'|^2)\widehat{u}_j - i\xi_j\{\lambda\nu_* - \kappa_*(\partial_N^2 - |\xi'|^2)\}\widehat{\varphi} = 0, \ x_N > 0, \tag{3.6}$$

$$\lambda^2\widehat{u}_N - \lambda\mu_*(\partial_N^2 - |\xi'|^2)\widehat{u}_N - \partial_N\{\lambda\nu_* - \kappa_*(\partial_N^2 - |\xi'|^2)\}\widehat{\varphi} = 0, \ x_N > 0, \tag{3.7}$$

$$\partial_N\widehat{\varphi}(0) = \lambda\widehat{g}(0). \tag{3.8}$$

Multiplying (3.6) by $i\xi_j$ and applying ∂_N to (3.7), we sum the resultant equations in order to obtain

$$\lambda^2\widehat{\varphi} - \lambda(\mu_* + \nu_*)(\partial_N^2 - |\xi'|^2)\widehat{\varphi} + \kappa_*(\partial_N^2 - |\xi'|^2)^2\widehat{\varphi} = 0, \ x_N > 0,$$

where we have used the fact that $\widehat{\varphi} = \sum_{j=1}^{N-1} i\xi_j\widehat{u}_j + \partial_N\widehat{u}_N$. By $P_\lambda(t)$ given in (2.3), the last equation is written as

$$P_\lambda(\partial_N)\widehat{\varphi} = 0. \tag{3.9}$$

On the other hand, (3.6) and (3.7) are respectively equivalent to

$$\mu_*\lambda(\partial_N^2 - \omega_\lambda^2)\widehat{u}_j + i\xi_j\{\nu_*\lambda - \kappa_*(\partial_N^2 - |\xi'|^2)\}\widehat{\varphi} = 0, \ x_N > 0, \tag{3.10}$$

$$\mu_*\lambda(\partial_N^2 - \omega_\lambda^2)\widehat{u}_N + \partial_N\{\nu_*\lambda - \kappa_*(\partial_N^2 - |\xi'|^2)\}\widehat{\varphi} = 0, \ x_N > 0. \tag{3.11}$$

Applying $P_\lambda(\partial_N)$ to (3.10) and (3.11) then furnishes by (3.9)

$$(\partial_N^2 - \omega_\lambda^2)P_\lambda(\partial_N)\widehat{u}_J = 0. \tag{3.12}$$

Remark 3.1. In what follows, we solve equations (3.9)-(3.12) with boundary conditions (3.5) and (3.8) with respect to $\widehat{u}_J$ and $\widehat{\varphi}$ under the following constraint:

$$\widehat{\varphi} = \sum_{j=1}^{N-1} i\xi_j\widehat{u}_j + \partial_N\widehat{u}_N, \quad x_N > 0. \tag{3.13}$$

In view of (3.9), (3.12), and Lemma 2.4, we look for solutions $\widehat{u}_J$ and $\widehat{\varphi}$ of the forms:

$$\widehat{u}_J = \alpha_J e^{-\omega_\lambda x_N} + \beta_J(e^{-t_1 x_N} - e^{-\omega_\lambda x_N}) + \gamma_J(e^{-t_2 x_N} - e^{-\omega_\lambda x_N}), \tag{3.14}$$

$$\widehat{\varphi} = \sigma e^{-t_1 x_N} + \tau e^{-t_2 x_N}. \tag{3.15}$$

It then holds by (3.13) that

$$i\xi' \cdot \alpha' - i\xi' \cdot \beta' - i\xi' \cdot \gamma' - \omega_\lambda\alpha_N + \omega_\lambda\beta_N + \omega_\lambda\gamma_N = 0, \tag{3.16}$$

$$\sigma = i\xi' \cdot \beta' - t_1\beta_N, \quad \tau = i\xi' \cdot \gamma' - t_2\gamma_N, \tag{3.17}$$

where $i\xi'\cdot a'=\sum_{j=1}^{N-1} i\xi_j a_j$ for $a\in\{\alpha,\beta,\gamma\}$. On the other hand, inserting (3.14) and (3.15) into (3.10) and (3.11) yields

$$\begin{aligned}
\mu_*\lambda\beta_j(t_1^2-\omega_\lambda^2)+i\xi_j\sigma\{\nu_*\lambda-\kappa_*(t_1^2-|\xi'|^2)\}&=0,\\
\mu_*\lambda\gamma_j(t_2^2-\omega_\lambda^2)+i\xi_j\tau\{\nu_*\lambda-\kappa_*(t_2^2-|\xi'|^2)\}&=0,\\
\mu_*\lambda\beta_N(t_1^2-\omega_\lambda^2)-t_1\sigma\{\nu_*\lambda-\kappa_*(t_1^2-|\xi'|^2)\}&=0,\\
\mu_*\lambda\gamma_N(t_2^2-\omega_\lambda^2)-t_2\tau\{\nu_*\lambda-\kappa_*(t_2^2-|\xi'|^2)\}&=0.
\end{aligned}$$

By these relations, we have

$$\mu_*\lambda(t_1^2-\omega_\lambda^2)\left(\beta_j+\frac{i\xi_j}{t_1}\beta_N\right)=0,\quad \mu_*\lambda(t_2^2-\omega_\lambda^2)\left(\gamma_j+\frac{i\xi_j}{t_2}\gamma_N\right)=0.$$

Since $t_1\neq\omega_\lambda$ and $t_2\neq\omega_\lambda$, the last two equations imply

$$\beta_j=-\frac{i\xi_j}{t_1}\beta_N,\quad \gamma_j=-\frac{i\xi_j}{t_2}\gamma_N. \tag{3.18}$$

These relations yield

$$i\xi'\cdot\beta'=\frac{|\xi'|^2}{t_1}\beta_N,\quad i\xi'\cdot\gamma'=\frac{|\xi'|^2}{t_2}\gamma_N, \tag{3.19}$$

and thus

$$i\xi'\cdot\beta'-t_1\beta_N=-\left(\frac{t_1^2-|\xi'|^2}{t_1}\right)\beta_N,\quad i\xi'\cdot\gamma'-t_2\gamma_N=-\left(\frac{t_2^2-|\xi'|^2}{t_2}\right)\gamma_N. \tag{3.20}$$

Next, we consider the boundary conditions. By (3.5) and (3.14),

$$\alpha_j=\widehat{h}_j(0),\quad \alpha_N=0. \tag{3.21}$$

It then holds by the first relation of (3.21) that

$$i\xi'\cdot\alpha'=i\xi'\cdot\widehat{\mathbf{h}}'(0),\quad \widehat{\mathbf{h}}'(0)=(\widehat{h}_1(0),\dots,\widehat{h}_{N-1}(0))^{\mathsf{T}}. \tag{3.22}$$

On the other hand, by (3.8) and (3.15),

$$-t_1\sigma-t_2\tau=\lambda\widehat{g}(0),$$

which, combined with (3.17) and (3.20), furnishes

$$(t_1^2-|\xi'|^2)\beta_N+(t_2^2-|\xi'|^2)\gamma_N=\lambda\widehat{g}(0). \tag{3.23}$$

One now derives simultaneous equations with respect to β_N and γ_N. Inserting (3.19), (3.22), and $\alpha_N=0$ of (3.21) into (3.16) furnishes

$$i\xi'\cdot\widehat{\mathbf{h}}'(0)-t_1^{-1}|\xi'|^2\beta_N-t_2^{-1}|\xi'|^2\gamma_N+\omega_\lambda\beta_N+\omega_\lambda\gamma_N=0.$$

Hence,

$$-t_2(t_1\omega_\lambda-|\xi'|^2)\beta_N-t_1(t_2\omega_\lambda-|\xi'|^2)\gamma_N=t_1t_2 i\xi'\cdot\widehat{\mathbf{h}}'(0),$$

which, combined with (3.23), yields

$$\begin{aligned}
&\mathbf{L}\begin{pmatrix}\beta_N\\ \gamma_N\end{pmatrix}=\begin{pmatrix}\lambda\widehat{g}(0)\\ t_1t_2i\xi'\cdot\widehat{\mathbf{h}}'(0)\end{pmatrix},\\
&\mathbf{L}=\begin{pmatrix}t_1^2-|\xi'|^2 & t_2^2-|\xi'|^2\\ -t_2(t_1\omega_\lambda-|\xi'|^2) & -t_1(t_2\omega_\lambda-|\xi'|^2)\end{pmatrix}.
\end{aligned} \tag{3.24}$$

Let us solve (3.24). By direct calculations,

$$\det\mathbf{L}=t_2(t_2^2-|\xi'|^2)(t_1\omega_\lambda-|\xi'|^2)-t_1(t_1^2-|\xi'|^2)(t_2\omega_\lambda-|\xi'|^2) \tag{3.25}$$

$$= (t_2 - t_1)\{t_1 t_2 \omega_\lambda (t_2 + t_1) - |\xi'|^2 (t_2^2 + t_1 t_2 + t_1^2 - |\xi'|^2)\}.$$

One here proves

Lemma 3.2. *There holds* $\det \mathbf{L} \neq 0$ *for any* $(\xi', \lambda) \in \mathbf{R}^{N-1} \times (\overline{\mathbf{C}_+} \setminus \{0\})$*, where* $\overline{\mathbf{C}_+} = \{z \in \mathbf{C} \mid \Re z \geq 0\}$.

Proof. The lemma is proved by contradiction. Suppose that $\det \mathbf{L} = 0$ for some $(\xi', \lambda) \in \mathbf{R}^{N-1} \times (\overline{\mathbf{C}_+} \setminus \{0\})$. Then there is $(\beta_N, \gamma_N) \neq (0, 0)$ satisfying (3.24) with $\widehat{g}(0) = 0$ and $\widehat{\mathbf{h}}'(0) = 0$. This implies that the equations (3.10) and (3.11), with $\widehat{\varphi} = \sum_{j=1}^{N-1} i\xi_j \widehat{u}_j + \partial_N \widehat{u}_N$ and homogeneous boundary conditions $\widehat{u}_J(0) = 0$ and $\partial_N \widehat{\varphi}(0) = 0$, admits a non-trivial solution $(\widehat{u}_1, \dots, \widehat{u}_N, \widehat{\varphi})$ sufficiently smooth and decaying exponentially as $x_N \to \infty$. Let us denote the non-trivial solution by $(u_1, \dots, u_N, \varphi)$ for notational simplicity.

Recall that (3.10) and (3.11) are respectively equivalent to (3.6) and (3.7). One multiplies (3.6) and (3.7) by λ^{-1}, and then

$$\lambda u_j - \mu_*(\partial_N^2 - |\xi'|^2) u_j - \nu_* i\xi_j \varphi + \kappa_* \lambda^{-1} i\xi_j (\partial_N^2 - |\xi'|^2)\varphi = 0, \ x_N > 0, \tag{3.26}$$

$$\lambda u_N - \mu_*(\partial_N^2 - |\xi'|^2) u_N - \nu_* \partial_N \varphi + \kappa_* \lambda^{-1} \partial_N (\partial_N^2 - |\xi'|^2)\varphi = 0, \ x_N > 0. \tag{3.27}$$

In this proof, we set $(a, b) = \int_0^\infty a(x_N)\,\overline{b(x_N)}\,dx_N$ and $\|a\| = \sqrt{(a, a)}$ for functions $a = a(x_N)$ and $b = b(x_N)$ on $\mathbf{R}_+$.

Step 1. Multiplying (3.26) by $\overline{u_j(x_N)}$ and integrating the resultant formula with respect to $x_N \in (0, \infty)$ yield

$$\begin{aligned} \lambda \|u_j\|^2 - \mu_*((\partial_N^2 u_j, u_j) - |\xi'|^2 \|u_j\|^2) - \nu_*(i\xi_j \varphi, u_j) \\ + \kappa_* \lambda^{-1} (i\xi_j (\partial_N^2 - |\xi'|^2)\varphi, u_j) = 0, \end{aligned}$$

which, combined with $(\partial_N^2 u_j, u_j) = -\|\partial_N u_j\|^2$ following from integration by parts with $u_j(0) = 0$ and combined with the properties:

$$(i\xi_j \varphi, u_j) = -(\varphi, i\xi_j u_j), \quad (i\xi_j (\partial_N^2 - |\xi'|^2)\varphi, u_j) = -((\partial_N^2 - |\xi'|^2)\varphi, i\xi_j u_j),$$

furnishes that

$$\begin{aligned} \lambda \|u_j\|^2 + \mu_*(\|\partial_N u_j\|^2 + |\xi'|^2 \|u_j\|^2) + \nu_*(\varphi, i\xi_j u_j) \\ - \kappa_* \lambda^{-1} ((\partial_N^2 - |\xi'|^2)\varphi, i\xi_j u_j) = 0. \end{aligned} \tag{3.28}$$

It similarly follows from (3.27) that

$$\begin{aligned} \lambda \|u_N\|^2 + \mu_*(\|\partial_N u_N\|^2 + |\xi'|^2 \|u_N\|^2) + \nu_*(\varphi, \partial_N u_N) \\ - \kappa_* \lambda^{-1} ((\partial_N^2 - |\xi'|^2)\varphi, \partial_N u_N) = 0. \end{aligned} \tag{3.29}$$

Step 2. Summing (3.28) with respect to $j = 1, \dots, N-1$ and (3.29), we have by $\varphi = \sum_{j=1}^{N-1} i\xi_j u_j + \partial_N u_N$

$$\begin{aligned} \lambda \sum_{J=1}^{N} \|u_J\|^2 + \mu_* \sum_{J=1}^{N} \left(\|\partial_N u_J\|^2 + |\xi'|^2 \|u_J\|^2\right) \\ + \nu_* \|\varphi\|^2 - \kappa_* \lambda^{-1} ((\partial_N^2 - |\xi'|^2)\varphi, \varphi) = 0. \end{aligned} \tag{3.30}$$

On the other hand, by integration by parts with $\partial_N \varphi(0) = 0$,

$$((\partial_N^2 - |\xi'|^2)\varphi, \varphi) = -(\|\partial_N \varphi\|^2 + |\xi'|^2 \|\varphi\|^2).$$

Inserting this relation into (3.30) and noting $\lambda^{-1}=\overline{\lambda}|\lambda|^{-2}$ furnish

$$\begin{aligned}\lambda\sum_{J=1}^N\|u_J\|^2+\mu_*\sum_{J=1}^N\left(\|\partial_N u_J\|^2+|\xi'|^2\|u_J\|^2\right)&\\ +\nu_*\|\varphi\|^2+\kappa_*\overline{\lambda}|\lambda|^{-2}(\|\partial_N\varphi\|^2+|\xi'|^2\|\varphi\|^2)&=0.\end{aligned}\tag{3.31}$$

Step 3. One takes the real part of (3.31) and the imaginary part of (3.31) in order to obtain

$$\begin{aligned}(\Re\lambda)\left\{\sum_{J=1}^N\|u_J\|^2+\kappa_*|\lambda|^{-2}(\|\partial_N\varphi\|^2+|\xi'|^2\|\varphi\|^2)\right\}&\\ +\mu_*\sum_{J=1}^N\left(\|\partial_N u_J\|^2+|\xi'|^2\|u_J\|^2\right)+\nu_*\|\varphi\|^2&=0,\end{aligned}\tag{3.32}$$

$$(\Im\lambda)\left\{\sum_{J=1}^N\|u_J\|^2-\kappa_*|\lambda|^{-2}(\|\partial_N\varphi\|^2+|\xi'|^2\|\varphi\|^2)\right\}=0.\tag{3.33}$$

It now holds by (3.32) that $\varphi=0$. One then sees that $(u_1,\dots,u_N)=(0,\dots,0)$ by (3.32) when $\Re\lambda>0$ and by (3.33) when $\Re\lambda=0$. Hence, $(u_1,\dots,u_N,\varphi)=(0,0,\dots,0)$, which contradicts the fact that $(u_1,\dots,u_N,\varphi)$ is a non-trivial solution. This completes the proof of the lemma. □

Let us write the inverse matrix $\mathbf{L}^{-1}$ of $\mathbf{L}$ as follows:

$$\mathbf{L}^{-1}=\frac{1}{\det\mathbf{L}}\begin{pmatrix}L_{11}&L_{12}\\L_{21}&L_{22}\end{pmatrix},$$

where

$$\begin{aligned}L_{11}&=-t_1(t_2\omega_\lambda-|\xi'|^2),\quad L_{12}=-(t_2^2-|\xi'|^2),\\ L_{21}&=t_2(t_1\omega_\lambda-|\xi'|^2),\qquad L_{22}=t_1^2-|\xi'|^2.\end{aligned}\tag{3.34}$$

One then sees that, by solving (3.24),

$$\begin{aligned}\beta_N&=\frac{\lambda L_{11}}{\det\mathbf{L}}\widehat{g}(0)+\frac{t_1t_2L_{12}}{\det\mathbf{L}}i\xi'\cdot\widehat{\mathbf{h}}'(0),\\ \gamma_N&=\frac{\lambda L_{21}}{\det\mathbf{L}}\widehat{g}(0)+\frac{t_1t_2L_{22}}{\det\mathbf{L}}i\xi'\cdot\widehat{\mathbf{h}}'(0).\end{aligned}\tag{3.35}$$

On the other hand, one has, by (3.14), (3.15), (3.17), (3.18), (3.20), and (3.21),

$$\begin{aligned}\widehat{u}_j(x_N)&=\widehat{h}_j(0)e^{-\omega_\lambda x_N}-\frac{i\xi_j}{t_1}\beta_N(e^{-t_1x_N}-e^{-\omega_\lambda x_N})\\ &\quad-\frac{i\xi_j}{t_2}\gamma_N(e^{-t_2x_N}-e^{-\omega_\lambda x_N}),\\ \widehat{u}_N(x_N)&=\beta_N\left(e^{-t_1x_N}-e^{-\omega_\lambda x_N}\right)+\gamma_N\left(e^{-t_2x_N}-e^{-\omega_\lambda x_N}\right),\\ \widehat{\varphi}(x_N)&=-\left(\frac{t_1^2-|\xi'|^2}{t_1}\right)\beta_Ne^{-t_1x_N}-\left(\frac{t_2^2-|\xi'|^2}{t_2}\right)\gamma_Ne^{-t_2x_N},\end{aligned}$$

and sets $\widehat{\rho}(x_N)=-\lambda^{-1}\widehat{\varphi}(x_N)$ in view of (3.1). Recall the inverse partial Fourier transform given in (2.7) and set $\rho=\mathcal{F}_{\xi'}^{-1}[\widehat{\rho}(x_N)](x')$ and $u_J=\mathcal{F}_{\xi'}^{-1}[\widehat{u}_J(x_N)](x')$. Then ρ and $\mathbf{u}=(u_1,\dots,u_N)^{\mathsf{T}}$ solve the system (2.11).

3.2. **Analysis of symbols.** This subsection estimates several symbols arising from the representation formulas of solutions obtained in Subsection 3.1.

Let us define the following symbols:

$$
\begin{aligned}
\mathfrak{m}_k(\xi',\lambda) &= \frac{t_k(t_k+\omega_\lambda)\det\mathbf{L}}{\lambda(t_2-t_1)} \quad (k=1,2),\\
\mathfrak{n}_1(\xi',\lambda) &= \frac{(t_2+\omega_\lambda)L_{11}}{\lambda}, \quad \mathfrak{n}_2(\xi',\lambda)=\frac{(t_1+\omega_\lambda)L_{21}}{\lambda},\\
\mathfrak{p}_1(\xi',\lambda) &= \frac{t_1+\omega_\lambda}{t_2+\omega_\lambda}, \quad \mathfrak{p}_2(\xi',\lambda)=\frac{t_2+\omega_\lambda}{t_1+\omega_\lambda}.
\end{aligned}
\tag{3.36}
$$

Subsequently, $k=1$ or $k=2$, and also one often denotes $\mathfrak{m}_k(\xi,\lambda)$, $\mathfrak{n}_k(\xi',\lambda)$, and $\mathfrak{p}_k(\xi',\lambda)$ by $\mathfrak{m}_k$, $\mathfrak{n}_k$, and $\mathfrak{p}_k$, respectively, for short. Recall by (2.1) and (2.2) that

$$
t_k^2-|\xi'|^2=s_k\lambda, \quad t_k^2-\omega_\lambda^2=(s_k-\mu_*^{-1})\lambda, \quad \omega_\lambda^2-|\xi'|^2=\mu_*^{-1}\lambda. \tag{3.37}
$$

Since $\det\mathbf{L}$ is written as

$$
\begin{aligned}
\det\mathbf{L} &= (t_2-t_1)\{t_2\omega_\lambda(t_2+t_1)(t_1-\omega_\lambda)-\omega_\lambda^2(t_1^2-|\xi'|^2)\\
&\quad+(\omega_\lambda^2-|\xi'|^2)(t_2^2+t_1t_2+t_1^2-|\xi'|^2)\}\\
&= (t_2-t_1)\{t_1\omega_\lambda(t_2+t_1)(t_2-\omega_\lambda)-\omega_\lambda^2(t_2^2-|\xi'|^2)\\
&\quad+(\omega_\lambda^2-|\xi'|^2)(t_2^2+t_1t_2+t_1^2-|\xi'|^2)\},
\end{aligned}
$$

one has by (3.37)

$$
\begin{aligned}
\mathfrak{m}_k(\xi',\lambda) &= (s_k-\mu_*^{-1})t_1t_2\omega_\lambda(t_2+t_1)-s_kt_k\omega_\lambda^2(t_k+\omega_\lambda)\\
&\quad+\mu_*^{-1}t_k(t_k+\omega_\lambda)(t_2^2+t_1t_2+t_1^2-|\xi'|^2).
\end{aligned}
\tag{3.38}
$$

In addition, by (3.37),

$$
\begin{aligned}
(t_k+\omega_\lambda)(t_k\omega_\lambda-|\xi'|^2) &= (t_k+\omega_\lambda)\{(t_k-\omega_\lambda)\omega_\lambda+\omega_\lambda^2-|\xi'|^2\}\\
&= (s_k-\mu_*^{-1})\lambda\omega_\lambda+\mu_*^{-1}\lambda(t_k+\omega_\lambda),
\end{aligned}
$$

which furnishes

$$
\begin{aligned}
\mathfrak{n}_1(\xi',\lambda) &= -t_1\{(s_2-\mu_*^{-1})\omega_\lambda+\mu_*^{-1}(t_2+\omega_\lambda)\},\\
\mathfrak{n}_2(\xi',\lambda) &= t_2\{(s_1-\mu_*^{-1})\omega_\lambda+\mu_*^{-1}(t_1+\omega_\lambda)\}.
\end{aligned}
\tag{3.39}
$$

By (3.34), (3.38), (3.39), and Lemmas 2.1 and 2.5, we immediately obtain

Lemma 3.3. *Let $k=1,2$. It then holds that*

$$
L_{k1}\in\mathbb{M}_{3,1}(\mathbf{C}_+), \quad \mathfrak{m}_k\in\mathbb{M}_{4,1}(\mathbf{C}_+), \quad \mathfrak{n}_k\in\mathbb{M}_{2,1}(\mathbf{C}_+), \quad \mathfrak{p}_k\in\mathbb{M}_{0,1}(\mathbf{C}_+).
$$

To treat $\mathfrak{m}_k(\xi',\lambda)^{-1}$, we prove

Lemma 3.4. *There exists a positive constant C_{μ_*,κ_*,ν_*} such that, for any $(\xi',\lambda)\in\mathbf{R}^{N-1}\times(\overline{\mathbf{C}_+}\setminus\{0\})$, there holds the estimate:*

$$
|\mathfrak{m}_k(\xi',\lambda)|\geq C_{\mu_*,\kappa_*,\nu_*}(|\lambda|^{1/2}+|\xi'|)^4 \quad (k=1,2). \tag{3.40}
$$

Proof. **Case 1**. One considers the case $|\xi'|^2/|\lambda|\leq R_1$, where $\xi'\in\mathbf{R}^{N-1}$, $\lambda\in\overline{\mathbf{C}_+}\setminus\{0\}$, and some number $R_1\in(0,1)$ determined below.

Let $z=|\xi'|^2/\lambda$. Then,

$$
t_k=\sqrt{s_k\lambda}(1+O(z)), \quad \omega_\lambda=\sqrt{\mu_*^{-1}\lambda}(1+O(z)) \quad \text{as } |z|\to 0,
$$

which yields that

$$t_k + \omega_\lambda = \left(\sqrt{s_k} + \sqrt{\mu_*^{-1}}\right)\sqrt{\lambda}(1 + O(z)) \quad \text{as } |z| \to 0$$

and that by the second equality of (3.25)

$$\frac{\det \mathbf{L}}{t_2 - t_1} = \sqrt{s_2}\sqrt{s_1}\sqrt{\mu_*^{-1}}\left(\sqrt{s_2} + \sqrt{s_1}\right)\lambda^2(1 + O(z)) \quad \text{as } |z| \to 0.$$

One thus has by (3.36)

$$\mathfrak{m}_k(\xi', \lambda) = \sqrt{s_k}\left(\sqrt{s_k} + \sqrt{\mu_*^{-1}}\right)\sqrt{s_2}\sqrt{s_1}\sqrt{\mu_*^{-1}}\left(\sqrt{s_2} + \sqrt{s_1}\right)\lambda^2(1 + O(z))$$

as $|z| \to 0$. Since there is a positive constant M, depending on at most μ_*, ν_*, and κ_*, such that $\Re\sqrt{s_k} \geq M$, one observes that

$$\begin{aligned} |\sqrt{s_k}| &\geq \Re\sqrt{s_k} \geq M, \\ \left|\sqrt{s_k} + \sqrt{\mu_*^{-1}}\right| &\geq \Re\left(\sqrt{s_k} + \sqrt{\mu_*^{-1}}\right) \geq M + \sqrt{\mu_*^{-1}}, \\ |\sqrt{s_2} + \sqrt{s_1}| &\geq \Re(\sqrt{s_1} + \sqrt{s_2}) \geq 2M. \end{aligned}$$

Combining these inequalities with the last equality yields that there is a constant $R_1 \in (0, 1)$ such that, for any $(\xi', \lambda) \in \mathbf{R}^{N-1} \times (\overline{\mathbf{C}_+} \setminus \{0\})$ with $|\xi'|^2/|\lambda| \leq R_1$,

$$|\mathfrak{m}_\ell(\xi', \lambda)| \geq C_{\mu_*,\nu_*,\kappa_*}|\lambda|^2$$

for some positive constant C_{μ_*,ν_*,κ_*}. On the other hand, it holds by $(a + b)^2/2 \leq (a^2 + b^2)$ with $a, b \geq 0$ that

$$\begin{aligned} |\lambda|^2 &= \frac{1}{2}(|\lambda|^2 + |\lambda|^2) \geq \frac{1}{2}\left\{|\lambda|^2 + \left(\frac{|\xi'|^2}{R_1}\right)^2\right\} \geq \frac{1}{4}\left(|\lambda| + \frac{|\xi'|^2}{R_1}\right)^2 \\ &\geq \frac{1}{16}\left(|\lambda|^{1/2} + \frac{|\xi'|}{\sqrt{R_1}}\right)^4. \end{aligned}$$

By the last two inequalities, we have (3.40) for $(\xi', \lambda) \in \mathbf{R}^{N-1} \times (\overline{\mathbf{C}_+} \setminus \{0\})$ with $|\xi'|^2/|\lambda| \leq R_1$.

Case 2. One considers the case $|\lambda|/|\xi'|^2 \leq R_2$, where $\xi' \in \mathbf{R}^{N-1} \setminus \{0\}$, $\lambda \in \overline{\mathbf{C}_+} \setminus \{0\}$, and some number $R_2 \in (0, 1)$ determined below.

Let $y = \lambda/|\xi'|^2$. Then,

$$t_k = |\xi'|(1 + O(y)), \quad \omega_\lambda = |\xi'|(1 + O(y)) \quad \text{as } |y| \to 0,$$

which, combined with (3.38), furnishes that

$$\mathfrak{m}_k(\xi', \lambda) = 2\mu_*^{-1}|\xi'|^4(1 + O(y)) \quad \text{as } |y| \to 0.$$

Thus there is a constant $R_2 \in (0, 1)$ such that, for any $(\xi', \lambda) \in (\mathbf{R}^{N-1} \setminus \{0\}) \times (\overline{\mathbf{C}_+} \setminus \{0\})$ with $|\lambda|/|\xi'|^2 \leq R_2$,

$$|\mathfrak{m}_\ell(\xi', \lambda)| \geq \mu_*^{-1}|\xi'|^4.$$

Similarly to Step 1, this inequality yields (3.40) for $(\xi', \lambda) \in (\mathbf{R}^{N-1}\setminus\{0\})\times(\overline{\mathbf{C}_+}\setminus\{0\})$ with $|\lambda|/|\xi'|^2 \leq R_2$.

Case 3. One considers the case $|\xi'|^2/|\lambda| \geq R_1/2$ and $|\lambda|/|\xi'|^2 \geq R_2/2$, where $\xi' \in \mathbf{R}^{N-1} \setminus \{0\}$, $\lambda \in \overline{\mathbf{C}_+} \setminus \{0\}$, and R_1, R_2 are positive constants introduced in the above two steps. The condition of this case is equivalent to

$$\frac{R_2}{2}|\xi'|^2 \leq |\lambda| \leq \frac{2}{R_1}|\xi'|^2. \tag{3.41}$$

Let $\widetilde{\xi}'$, $\widetilde{\lambda}$, $\widetilde{t}_k$, and $\widetilde{\omega}_\lambda$ be given by

$$\widetilde{\xi}' = (|\lambda|^{1/2} + |\xi'|)^{-1}\xi', \qquad \widetilde{\lambda} = (|\lambda|^{1/2} + |\xi'|)^{-2}\lambda,$$
$$\widetilde{t}_k = \sqrt{|\widetilde{\xi}'|^2 + s_k\widetilde{\lambda}}, \qquad \widetilde{\omega}_\lambda = \sqrt{|\widetilde{\xi}'|^2 + \mu_*^{-1}\widetilde{\lambda}}.$$

One then observes that

$$\mathfrak{m}_k(\xi', \lambda) = (|\lambda|^{1/2} + |\xi'|)^4 \mathfrak{m}_k(\widetilde{\xi}', \widetilde{\lambda})$$

and that (3.41) implies $r_1 \leq |\widetilde{\xi}'| \leq r_2$ and $r_3 \leq |\widetilde{\lambda}| \leq r_4$, where

$$r_1 = \left(\sqrt{\frac{2}{R_1}} + 1\right)^{-1}, \quad r_2 = \left(\sqrt{\frac{R_2}{2}} + 1\right)^{-1},$$
$$r_3 = \left(\sqrt{\frac{2}{R_2}} + 1\right)^{-2}, \quad r_4 = \left(\sqrt{\frac{R_1}{2}} + 1\right)^{-2}.$$

We here define a compact set K as follows:

$$K = \{(\widetilde{\xi}', \widetilde{\lambda}) \in \mathbf{R}^{N-1} \times \overline{\mathbf{C}_+} \mid r_1 \leq |\widetilde{\xi}'| \leq r_2, r_3 \leq |\widetilde{\lambda}| \leq r_4\}.$$

Since $\mathfrak{m}_k(\xi', \lambda)$ is continuous and $\mathfrak{m}_k(\xi', \lambda) \neq 0$ on $\mathbf{R}^{N-1} \times (\overline{\mathbf{C}_+} \setminus \{0\})$ by (3.36) and Lemma 3.2, there exists at least one minimum of $|\mathfrak{m}_\ell(\widetilde{\xi}', \widetilde{\lambda})|$ over K such that

$$m_k := \min_{(\widetilde{\xi}', \widetilde{\lambda}) \in K} |\mathfrak{m}_k(\widetilde{\xi}', \widetilde{\lambda})| > 0.$$

Thus, for any $(\xi', \lambda) \in (\mathbf{R}^{N-1} \setminus \{0\}) \times (\overline{\mathbf{C}_+} \setminus \{0\})$ with (3.41),

$$|\mathfrak{m}_k(\xi', \lambda)| = (|\lambda|^{1/2} + |\xi'|)^4 |\mathfrak{m}_k(\widetilde{\xi}', \widetilde{\lambda})| \geq m_k(|\lambda|^{1/2} + |\xi'|)^4,$$

which implies that (3.40) holds for Case 3.

Summing up the above estimates, we have completed the proof of Lemma 3.4. □

Corollary 3.5. *Let $k = 1, 2$. Then $\mathfrak{m}_k^{-1} \in \mathbb{M}_{-4,1}(\mathbf{C}_+)$.*

Proof. Recall Bell's formula for derivatives of the composite function of $f(t)$ and $g(\xi')$ as follows: for any multi-index $\alpha' \in \mathbf{N}_0^{N-1}$,

$$\partial_{\xi'}^{\alpha'} f(g(\xi')) = \sum_{k=1}^{|\alpha|} f^{(k)}(g(\xi')) \sum_{\substack{\alpha_1' + \cdots + \alpha_k' = \alpha', \\ |\alpha_j'| \geq 1}} \Gamma^{\alpha'}_{\alpha_1', \ldots, \alpha_k'} (\partial_{\xi'}^{\alpha_1'} g(\xi')) \ldots (\partial_{\xi'}^{\alpha_k'} g(\xi'))$$

with suitable coefficients $\Gamma^{\alpha'}_{\alpha_1', \ldots, \alpha_k'}$, where $f^{(k)}(t)$ is the kth derivative of $f(t)$.

One first proves, for any $r \in \mathbf{R}$ and multi-index $\alpha' \in \mathbf{N}_0^{N-1}$,

$$|\partial_{\xi'}^{\alpha'} \mathfrak{m}_k(\xi', \lambda)^r| \leq C_{r,\alpha',\mu_*,\nu_*,\kappa_*}(|\lambda|^{1/2} + |\xi'|)^{4r-|\alpha'|}, \tag{3.42}$$

where $(\xi', \lambda) \in (\mathbf{R}^{N-1} \setminus \{0\}) \times \mathbf{C}_+$. Using Bell's formula with $f(t) = t^r$ and $g(\xi') = \mathfrak{m}_k(\xi', \lambda)$, we have by Lemmas 3.3 and 3.4

$$\begin{aligned}
|\partial_{\xi'}^{\alpha'} \mathfrak{m}_k(\xi', \lambda)^r| &\leq C_{r,\alpha',\mu_*,\nu_*,\kappa_*} \sum_{k=1}^{|\alpha'|} |\mathfrak{m}_k(\xi', \lambda)|^{r-k} (|\lambda|^{1/2} + |\xi'|)^{4k-|\alpha'|} \\
&\leq C_{r,\alpha',\mu_*,\nu_*,\kappa_*} \sum_{k=1}^{|\alpha'|} (|\lambda|^{1/2} + |\xi'|)^{4(r-k)} (|\lambda|^{1/2} + |\xi'|)^{4k-|\alpha'|} \\
&\leq C_{r,\alpha',\mu_*,\nu_*,\kappa_*} (|\lambda|^{1/2} + |\xi'|)^{4r-|\alpha'|}
\end{aligned}$$

for any $(\xi', \lambda) \in (\mathbf{R}^{N-1} \setminus \{0\}) \times \mathbf{C}_+$. This implies that (3.42) holds true.

Now it holds that

$$\lambda \frac{d}{d\lambda} \mathfrak{m}_k(\xi', \lambda)^{-1} = -\mathfrak{m}_k(\xi', \lambda)^{-2} \left(\lambda \frac{d}{d\lambda} \mathfrak{m}_k(\xi', \lambda) \right).$$

Combining this relation with (3.42) for $r = -2$ and Lemma 3.3 furnishes, by Leibniz's rule, that for any multi-index $\alpha' \in \mathbf{N}_0^{N-1}$ and $(\xi', \lambda) \in (\mathbf{R}^{N-1} \setminus \{0\}) \times \mathbf{C}_+$

$$\begin{aligned}
&\left| \partial_{\xi'}^{\alpha'} \left(\lambda \frac{d}{d\lambda} \mathfrak{m}_k(\xi', \lambda)^{-1} \right) \right| \\
&\leq C_{\alpha'} \sum_{\beta'+\gamma'=\alpha'} \left| \partial_{\xi'}^{\beta'} \mathfrak{m}_k(\xi', \lambda)^{-2} \right| \left| \partial_{\xi'}^{\gamma'} \left(\lambda \frac{d}{d\lambda} \mathfrak{m}_k(\xi', \lambda) \right) \right| \\
&\leq C_{\alpha',\mu_*,\nu_*,\kappa_*} \sum_{\beta'+\gamma'=\alpha'} (|\lambda|^{1/2} + |\xi'|)^{-8-|\beta'|} (|\lambda|^{1/2} + |\xi'|)^{4-|\gamma'|} \\
&\leq C_{\alpha',\mu_*,\nu_*,\kappa_*} (|\lambda|^{1/2} + |\xi'|)^{-4-|\alpha'|}.
\end{aligned}$$

This inequality and (3.42) with $r = -1$ complete the proof of the corollary. $\square$

3.3. **Proof of Theorem 2.11.** This subsection proves Theorem 2.11 by means of results obtained in Subsections 3.1 and 3.2. Let $\mathcal{M}_0(x_N)$, $\mathcal{M}_1(x_N)$, and $\mathcal{M}_2(x_N)$ be symbols given in (2.4).

It now holds that

$$\begin{aligned}
\rho &= \mathcal{F}_{\xi'}^{-1} \left[\left\{ \left(\frac{t_1^2 - |\xi'|^2}{\lambda t_1} \right) \beta_N + \left(\frac{t_2 - |\xi'|^2}{\lambda t_2} \right) \gamma_N \right\} e^{-t_1 x_N} \right] (x') \\
&\quad + \mathcal{F}_{\xi'}^{-1} \left[\left(\frac{t_2^2 - |\xi'|^2}{\lambda t_2} \right) \gamma_N \left(e^{-t_2 x_N} - e^{-t_1 x_N} \right) \right] (x') \\
&=: \rho_1 + \rho_2.
\end{aligned}$$

By (3.34), (3.35), (3.36), and (3.37), we see that

$$\begin{aligned}
\rho_1 &= \mathcal{F}_{\xi'}^{-1} \left[\left(\frac{s_1 \lambda L_{11}}{t_1 \det \mathbf{L}} + \frac{s_2 \lambda L_{21}}{t_2 \det \mathbf{L}} \right) \widehat{g}(0) e^{-t_1 x_N} \right] (x') \\
&\quad + \mathcal{F}_{\xi'}^{-1} \left[\left(\frac{s_1 t_2 L_{12} + s_2 t_1 L_{22}}{\det \mathbf{L}} \right) i\xi' \cdot \widehat{\mathbf{h}}'(0) e^{-t_1 x_N} \right] (x') \\
&= \sum_{k=1}^{2} \mathcal{F}_{\xi'}^{-1} \left[\frac{s_k (t_2 + t_1) \mathfrak{p}_k(\xi', \lambda) \mathfrak{n}_k(\xi', \lambda)}{(s_2 - s_1) \mathfrak{m}_k(\xi', \lambda)} e^{-t_1 x_N} \widehat{g}(0) \right] (x')
\end{aligned}$$

$$
-\sum_{l=1}^{N-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{s_1 s_2 i\xi_l t_1(t_1+\omega_\lambda)}{\mathfrak{m}_1(\xi',\lambda)}e^{-t_1 x_N}\widehat{h}_l(0)\right](x').
$$

On the other hand, β_N and γ_N are written as

$$
\begin{aligned}
\beta_N &= \frac{t_1(t_1+\omega_\lambda)L_{11}}{(t_2-t_1)\mathfrak{m}_1(\xi',\lambda)}\widehat{g}(0) - \frac{s_2 t_1^2 t_2(t_1+\omega_\lambda)}{(t_2-t_1)\mathfrak{m}_1(\xi',\lambda)}i\xi'\cdot\widehat{\mathbf{h}}'(0),\\
\gamma_N &= \frac{t_2(t_2+\omega_\lambda)L_{21}}{(t_2-t_1)\mathfrak{m}_2(\xi',\lambda)}\widehat{g}(0) + \frac{s_1 t_1 t_2^2(t_2+\omega_\lambda)}{(t_2-t_1)\mathfrak{m}_2(\xi',\lambda)}i\xi'\cdot\widehat{\mathbf{h}}'(0),
\end{aligned} \tag{3.43}
$$

which furnishes, together with (3.37), that

$$
\begin{aligned}
\rho_2 &= \mathcal{F}_{\xi'}^{-1}\left[\frac{s_2(t_2+\omega_\lambda)L_{21}}{\mathfrak{m}_2(\xi',\lambda)}\mathcal{M}_0(x_N)\widehat{g}(0)\right](x')\\
&\quad+\sum_{l=1}^{N-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{s_1 s_2 i\xi_l t_1 t_2(t_2+\omega_\lambda)}{\mathfrak{m}_2(\xi',\lambda)}\mathcal{M}_0(x_N)\widehat{h}_l(0)\right](x').
\end{aligned}
$$

Recalling $\rho=\rho_1+\rho_2$, one obtains

$$
\begin{aligned}
\rho &= \sum_{k=1}^{2}\mathcal{F}_{\xi'}^{-1}\left[\frac{s_k(t_2+t_1)\mathfrak{p}_k(\xi',\lambda)\mathfrak{n}_k(\xi',\lambda)}{(s_2-s_1)\mathfrak{m}_k(\xi',\lambda)}e^{-t_1 x_N}\widehat{g}(0)\right](x')\\
&\quad-\sum_{l=1}^{N-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{s_1 s_2 i\xi_l t_1(t_1+\omega_\lambda)}{\mathfrak{m}_1(\xi',\lambda)}e^{-t_1 x_N}\widehat{h}_l(0)\right](x')\\
&\quad+\mathcal{F}_{\xi'}^{-1}\left[\frac{s_2(t_2+\omega_\lambda)L_{21}}{\mathfrak{m}_2(\xi',\lambda)}\mathcal{M}_0(x_N)\widehat{g}(0)\right](x')\\
&\quad+\sum_{l=1}^{N-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{s_1 s_2 i\xi_l t_1 t_2(t_2+\omega_\lambda)}{\mathfrak{m}_2(\xi',\lambda)}\mathcal{M}_0(x_N)\widehat{h}_l(0)\right](x').
\end{aligned} \tag{3.44}
$$

Similarly, we observe by (3.43) that, for u_j ($j=1,\dots,N-1$) and u_N,

$$
\begin{aligned}
u_j &= \mathcal{F}_{\xi'}^{-1}\left[e^{-\omega_\lambda x_N}\widehat{l}_j(0)\right](x')\\
&\quad-\sum_{k=1}^{2}\mathcal{F}_{\xi'}^{-1}\left[\frac{i\xi_j(t_k+\omega_\lambda)L_{k1}}{\mathfrak{m}_k(\xi',\lambda)}\mathcal{M}_k(x_N)\widehat{g}(0)\right](x')\\
&\quad+\sum_{k=1}^{2}\sum_{l=1}^{N-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{(-1)^k s_1 s_2\xi_j\xi_l t_1 t_2(t_k+\omega_\lambda)}{s_k\mathfrak{m}_k(\xi',\lambda)}\mathcal{M}_k(x_N)\widehat{h}_l(0)\right](x'),\\
u_N &= \sum_{k=1}^{2}\mathcal{F}_{\xi'}^{-1}\left[\frac{t_k(t_k+\omega_\lambda)L_{k1}}{\mathfrak{m}_k(\xi',\lambda)}\mathcal{M}_k(x_N)\widehat{g}(0)\right](x')\\
&\quad+\sum_{k=1}^{2}\sum_{l=1}^{N-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{(-1)^k s_1 s_2 i\xi_l t_1 t_2 t_k(t_k+\omega_\lambda)}{s_k\mathfrak{m}_k(\xi',\lambda)}\mathcal{M}_k(x_N)\widehat{h}_l(0)\right](x').
\end{aligned} \tag{3.45}
$$

By Lemmas 2.1, 2.5, and 3.3 and by Corollary 3.5, the symbols of ρ, u_J ($J=1,\dots,N$) satisfy the following conditions: For ρ, there hold

$$
\frac{s_k(t_2+t_1)\mathfrak{p}_k(\xi',\lambda)\mathfrak{n}_k(\xi',\lambda)}{(s_2-s_1)\mathfrak{m}_k(\xi',\lambda)},\ \frac{s_1 s_2 i\xi_l t_1(t_1+\omega_\lambda)}{\mathfrak{m}_1(\xi',\lambda)}\in\mathbb{M}_{-1,1}(\mathbf{C}_+), \tag{3.46}
$$

$$\frac{s_2(t_2+\omega_\lambda)L_{21}}{\mathfrak{m}_2(\xi',\lambda)},\ \frac{s_1s_2i\xi_l t_1t_2(t_2+\omega_\lambda)}{\mathfrak{m}_2(\xi',\lambda)}\in\mathbb{M}_{0,1}(\mathbf{C}_+);$$

For u_J, there hold

$$\frac{i\xi_j(t_k+\omega_\lambda)L_{k1}}{\mathfrak{m}_k(\xi',\lambda)},\ \frac{(-1)^k s_1s_2\xi_j\xi_l t_1t_2(t_k+\omega_\lambda)}{s_k\mathfrak{m}_k(\xi',\lambda)}\in\mathbb{M}_{1,1}(\mathbf{C}_+),$$
$$\frac{t_k(t_k+\omega_\lambda)L_{k1}}{\mathfrak{m}_k(\xi',\lambda)},\ \frac{(-1)^k s_1s_2i\xi_l t_1t_2t_k(t_k+\omega_\lambda)}{s_k\mathfrak{m}_k(\xi',\lambda)}\in\mathbb{M}_{1,1}(\mathbf{C}_+). \tag{3.47}$$

Finally, combing (3.44)-(3.47) with Lemmas 2.6, 2.7, and 2.9 shows the existence of solution operators $\mathcal{A}^2(\lambda)$ and $\mathcal{B}^2(\lambda)$ stated in Theorem 2.11. This completes the proof of Theorem 2.11 for Cases I and II.

4. Proof of Theorem 2.11 for Case III

This section proves Theorem 2.11 for Case III. Throughout this section, we assume that μ_*, ν_*, and κ_* are positive constants satisfying the condition of Case III. One then recalls Lemmas 2.2 (3) and 2.3 (2), and considers the case $\mu_*>\nu_*$ only, i.e.

$$t_1=\omega_\lambda=\sqrt{|\xi'|^2+\mu_*^{-1}\lambda},\quad t_2=\sqrt{|\xi'|^2+\nu_*^{-1}\lambda},$$

which are often used in the following computations. Let $J=1,\dots,N$ and $j=1,\dots,N-1$ in this section.

4.1. **Solution formulas.** One first considers (3.9)-(3.12) with (3.5), (3.8), and (3.13) in order to derive solution formulas of (2.11). In view of (3.9), (3.12), and Lemma 2.4, we look for solutions $\widehat{u}_J$ and $\widehat{\varphi}$ of the forms:

$$\widehat{u}_J=\alpha_J e^{-\omega_\lambda x_N}+\beta_J x_N e^{-\omega_\lambda x_N}+\gamma_J(e^{-t_2x_N}-e^{-\omega_\lambda x_N}), \tag{4.1}$$
$$\widehat{\varphi}=\sigma e^{-\omega_\lambda x_N}+\tau e^{-t_2x_N}. \tag{4.2}$$

It then holds by (3.13) that

$$\sigma=i\xi'\cdot\alpha'-i\xi'\cdot\gamma'-\omega_\lambda\alpha_N+\beta_N+\omega_\lambda\gamma_N, \tag{4.3}$$
$$0=i\xi'\cdot\beta'-\omega_\lambda\beta_N, \tag{4.4}$$
$$\tau=i\xi'\cdot\gamma'-t_2\gamma_N. \tag{4.5}$$

On the other hand, by the assumption $\kappa_*=\mu_*\nu_*$,

$$\nu_*\lambda-\kappa_*(\partial_N^2-|\xi'|^2)=-\mu_*\nu_*(\partial_N^2-\omega_\lambda^2),$$

and thus (3.10) and (3.11) are respectively equivalent to

$$\lambda(\partial_N^2-\omega_\lambda^2)\widehat{u}_j-\nu_*i\xi_j(\partial_N^2-\omega_\lambda^2)\widehat{\varphi}=0,$$
$$\lambda(\partial_N^2-\omega_\lambda^2)\widehat{u}_N-\nu_*\partial_N(\partial_N^2-\omega_\lambda^2)\widehat{\varphi}=0. \tag{4.6}$$

Here note that

$$(\partial_N^2-\omega_\lambda^2)(x_Ne^{-\omega_\lambda x_N})=-2\omega_\lambda e^{-\omega_\lambda x_N}.$$

Inserting (4.1) and (4.2), together with the last relation, into (4.6) yields

$$-2\lambda\omega_\lambda\beta_j=0,\quad (t_2^2-\omega_\lambda^2)(\lambda\gamma_j-\nu_*i\xi_j\tau)=0,$$
$$-2\lambda\omega_\lambda\beta_N=0,\quad (t_2^2-\omega_\lambda^2)(\lambda\gamma_N+\nu_*t_2\tau)=0,$$

which, combined with $t_2\neq\omega_\lambda$, furnishes

$$\beta_J=0, \tag{4.7}$$

$$\lambda\gamma_j - \nu_* i\xi_j\tau = 0, \tag{4.8}$$

$$\lambda\gamma_N + \nu_* t_2\tau = 0. \tag{4.9}$$

Remark 4.1. The relation (4.7) implies (4.4), while the relations (4.8) and (4.9) imply (4.5).

One has by (4.1), (4.3), and (4.7),

$$\widehat{u}_J = \alpha_J e^{-\omega_\lambda x_N} + \gamma_J(e^{-t_2 x_N} - e^{-\omega_\lambda x_N}), \tag{4.10}$$

$$\sigma = i\xi'\cdot\alpha' - i\xi'\cdot\gamma' - \omega_\lambda\alpha_N + \omega_\lambda\gamma_N, \tag{4.11}$$

and also by (4.8) and (4.9)

$$\gamma_j = -\frac{i\xi_j}{t_2}\gamma_N. \tag{4.12}$$

Furthermore, (4.12) yields

$$i\xi'\cdot\gamma' = \frac{|\xi'|^2}{t_2}\gamma_N. \tag{4.13}$$

Next, we consider the boundary conditions. By (3.5) and (4.10),

$$\alpha_j = \widehat{h}_j(0), \quad \alpha_N = 0. \tag{4.14}$$

It then holds by the first relation of (4.14) that

$$i\xi'\cdot\alpha' = i\xi'\cdot\widehat{\mathbf{h}}'(0), \quad \widehat{\mathbf{h}}'(0) = (\widehat{h}_1(0),\dots,\widehat{h}_{N-1}(0))^{\mathsf{T}}.$$

Combining this relation with (4.11), (4.13), and $\alpha_N = 0$ of (4.14) furnishes

$$\sigma = i\xi'\cdot\widehat{\mathbf{h}}'(0) - \frac{|\xi'|^2}{t_2}\gamma_N + \omega_\lambda\gamma_N, \tag{4.15}$$

while (4.9) implies

$$\tau = -\frac{\nu_*^{-1}\lambda}{t_2}\gamma_N. \tag{4.16}$$

One the other hand, by (3.8) and (4.2),

$$-\omega_\lambda\sigma - t_2\tau = \lambda\widehat{g}(0),$$

which, combined with (4.15) and (4.16), furnishes

$$-\omega_\lambda\left(i\xi'\cdot\widehat{\mathbf{h}}'(0) - \frac{|\xi'|^2}{t_2}\gamma_N + \omega_\lambda\gamma_N\right) + \nu_*^{-1}\lambda\gamma_N = \lambda\widehat{g}(0).$$

Solving this equation with respect to γ_N, we have

$$\gamma_N = \frac{t_2(\lambda\widehat{g}(0) + \omega_\lambda i\xi'\cdot\widehat{\mathbf{h}}'(0))}{\omega_\lambda|\xi'|^2 - t_2\omega_\lambda^2 + t_2\nu_*^{-1}\lambda}.$$

Here note that by $t_2^2 = |\xi'|^2 + \nu_*^{-1}\lambda$

$$\begin{aligned}\omega_\lambda|\xi'|^2 - t_2\omega_\lambda^2 + t_2\nu_*^{-1}\lambda &= \omega_\lambda(t_2^2 - \nu_*^{-1}\lambda) - t_2\omega_\lambda^2 + t_2\nu_*^{-1}\lambda \\ &= (t_2 - \omega_\lambda)(t_2\omega_\lambda + \nu_*^{-1}\lambda).\end{aligned}$$

The above formula of γ_N is thus written as[2]

$$\gamma_N = \frac{t_2(\lambda\widehat{g}(0) + \omega_\lambda i\xi' \cdot \widehat{\mathbf{h}}'(0))}{(t_2 - \omega_\lambda)(t_2\omega_\lambda + \nu_*^{-1}\lambda)}. \tag{4.17}$$

Finally, one has, by (4.2), (4.10), (4.12), (4.14), (4.15), and (4.16),

$$\begin{aligned}
\widehat{u}_j(x_N) &= \widehat{h}_j(0)e^{-\omega_\lambda x_N} - \frac{i\xi_j}{t_2}\gamma_N(e^{-t_2x_N} - e^{-\omega_\lambda x_N}), \\
\widehat{u}_N(x_N) &= \gamma_N(e^{-t_2x_N} - e^{-\omega_\lambda x_N}), \\
\widehat{\varphi}(x_N) &= \left(i\xi' \cdot \widehat{\mathbf{h}}'(0) - \frac{|\xi'|^2}{t_2}\gamma_N + \omega_\lambda\gamma_N\right)e^{-\omega_\lambda x_N} - \frac{\nu_*^{-1}\lambda}{t_2}\gamma_N e^{-t_2x_N},
\end{aligned}$$

and sets $\widehat{\rho}(x_N) = -\lambda^{-1}\widehat{\varphi}(x_N)$ in view of (3.1). Recall the inverse partial Fourier transform given in (2.7) and set $\rho = \mathcal{F}_{\xi'}^{-1}[\widehat{\rho}(x_N)](x')$ and $u_J = \mathcal{F}_{\xi'}^{-1}[\widehat{u}_J(x_N)](x')$. Then ρ and $\mathbf{u} = (u_1, \dots, u_N)^\mathsf{T}$ solve the system (2.11).

The last part of this subsection is devoted to the proof of the following lemma.

Lemma 4.2. (1) $t_2\omega_\lambda + \nu_*^{-1}\lambda \in \mathbb{M}_{2,1}(\mathbf{C}_+)$.
(2) *There is a positive constant C_{μ_*,ν_*,κ_*} such that*

$$|t_2\omega_\lambda + \nu_*^{-1}\lambda| \geq C_{\mu_*,\nu_*,\kappa_*}(|\lambda|^{1/2} + |\xi'|)^2 \tag{4.18}$$

for any $(\xi', \lambda) \in \mathbf{R}^{N-1} \times \mathbf{C}_+$.
(3) $(t_2\omega_\lambda + \nu_*^{-1}\lambda)^{-1} \in \mathbb{M}_{-2,1}(\mathbf{C}_+)$.

Proof. (1) The required property follows from Lemmas 2.1 and 2.5 immediately.

(2). First, let us prove

$$\Re\left(\frac{\nu_*^{-1}\lambda}{t_2}\right) > 0. \tag{4.19}$$

By $t_2^2 = |\xi'|^2 + \nu_*^{-1}\lambda$,

$$\frac{\nu_*^{-1}\lambda}{t_2} = \frac{t_2^2 - |\xi'|^2}{t_2} = t_2 - \frac{|\xi'|^2}{|t_2|^2}\overline{t_2},$$

which implies

$$\Re\left(\frac{\nu_*^{-1}\lambda}{t_2}\right) = (\Re t_2)\left(1 - \frac{|\xi'|^2}{|t_2|^2}\right) = \frac{(\Re t_2)(|t_2|^4 - |\xi'|^4)}{|t_2|^2(|t_2|^2 + |\xi'|^2)}. \tag{4.20}$$

One here observes by $|t_2|^4 = ||\xi'|^2 + \nu_*^{-1}\lambda|^2$ that

$$\begin{aligned}
|t_2|^4 - |\xi'|^4 &= (|\xi'|^2 + \nu_*^{-1}\Re\lambda)^2 + (\nu_*^{-1}\Im\lambda)^2 - |\xi'|^4 \\
&= \nu_*^{-2}|\lambda|^2 + 2|\xi'|^2\nu_*^{-1}\Re\lambda,
\end{aligned}$$

which, combined with $\Re\lambda > 0$, furnishes

$$|t_2|^4 - |\xi'|^4 \geq \nu_*^{-2}|\lambda|^2.$$

Since $\Re t_2 > 0$ by Lemma 2.4, the last inequality and (4.20) imply (4.19).

Next, we prove (4.18). By (4.19),

$$\left|\omega_\lambda + \frac{\nu_*^{-1}\lambda}{t_2}\right| \geq \Re\left(\omega_\lambda + \frac{\nu_*^{-1}\lambda}{t_2}\right) \geq \Re\omega_\lambda,$$

[2] Estimates of $t_2\omega_\lambda + \nu_*^{-1}\lambda$ are given in Lemma 4.2 below.

which, combined with Lemma 2.5, furnishes

$$\begin{aligned}|t_2\omega_\lambda+\nu_*^{-1}\lambda| &= |t_2|\left|\omega_\lambda+\frac{\nu_*^{-1}\lambda}{t_2}\right| \\ &\geq (\Re t_2)(\Re\omega_\lambda) \\ &\geq C_{\mu_*,\nu_*,\kappa_*}(|\lambda|^{1/2}+|\xi'|)^2.\end{aligned}$$

This completes the proof of (4.18).

(3). The required property follows from (1) and (2) in the same manner as one has proved Corollary 3.5 from Lemmas 3.3 and 3.4, so that the detailed proof may be omitted. □

4.2. **Proof of Theorem 2.11.** This subsection proves Theorem 2.11 by means of results obtained in Subsection 4.1. Let $\mathcal{M}(x_N)$ be given in (2.5).

First, one considers the formula of ρ, which is written as

$$\begin{aligned}\rho &= \mathcal{F}_{\xi'}^{-1}\left[\left\{-\frac{1}{\lambda}\left(i\xi'\cdot\widehat{\mathbf{h}}'(0)-\frac{|\xi'|^2}{t_2}\gamma_N+\omega_\lambda\gamma_N\right)+\frac{\nu_*^{-1}}{t_2}\gamma_N\right\}e^{-\omega_\lambda x_N}\right](x') \\ &+\mathcal{F}_{\xi'}^{-1}\left[\frac{\nu_*^{-1}}{t_2}\gamma_N(e^{-t_2x_N}-e^{-\omega_\lambda x_N})\right](x') \\ &=: \rho_1+\rho_2.\end{aligned}$$

The relation $t_2^2=|\xi'|^2+\nu_*^{-1}\lambda$ yields

$$\rho_1=\mathcal{F}_{\xi'}^{-1}\left[\frac{1}{\lambda}\left\{-i\xi'\cdot\widehat{\mathbf{h}}'(0)+(t_2-\omega_\lambda)\gamma_N\right\}e^{-\omega_\lambda x_N}\right](x').$$

Since it holds by (4.17) that

$$-i\xi'\cdot\widehat{\mathbf{h}}'(0)+(t_2-\omega_\lambda)\gamma_N=\lambda\left(\frac{t_2}{t_2\omega_\lambda+\nu_*^{-1}\lambda}\widehat{g}(0)-\frac{\nu_*^{-1}}{t_2\omega_\lambda+\nu_*^{-1}\lambda}i\xi'\cdot\widehat{\mathbf{h}}'(0)\right)$$

the above formula of ρ_1 is reduced to

$$\begin{aligned}\rho_1 &= \mathcal{F}_{\xi'}^{-1}\left[\left(\frac{t_2}{t_2\omega_\lambda+\nu_*^{-1}\lambda}\right)e^{-\omega_\lambda x_N}\widehat{g}(0)\right](x') \\ &-\sum_{k=1}^{N-1}\left[\left(\frac{\nu_*^{-1}i\xi_k}{t_2\omega_\lambda+\nu_*^{-1}\lambda}\right)e^{-\omega_\lambda x_N}\widehat{h}_k(0)\right](x').\end{aligned}$$

On the other hand, by (4.17),

$$\begin{aligned}\rho_2 &= \mathcal{F}_{\xi'}^{-1}\left[\left(\frac{\nu_*^{-1}\lambda}{t_2\omega_\lambda+\nu_*^{-1}\lambda}\right)\mathcal{M}(x_N)\,\widehat{g}(0)\right](x') \\ &+\sum_{k=1}^{N-1}\mathcal{F}_{\xi'}^{-1}\left[\left(\frac{\nu_*^{-1}\omega_\lambda i\xi_k}{t_2\omega_\lambda+\nu_*^{-1}\lambda}\right)\mathcal{M}(x_N)\,\widehat{h}_k(0)\right](x').\end{aligned}$$

Recalling $\rho=\rho_1+\rho_2$, one obtains

$$\begin{aligned}\rho &= \mathcal{F}_{\xi'}^{-1}\left[\left(\frac{t_2}{t_2\omega_\lambda+\nu_*^{-1}\lambda}\right)e^{-\omega_\lambda x_N}\widehat{g}(0)\right](x') \\ &-\sum_{k=1}^{N-1}\left[\left(\frac{\nu_*^{-1}i\xi_k}{t_2\omega_\lambda+\nu_*^{-1}\lambda}\right)e^{-\omega_\lambda x_N}\widehat{h}_k(0)\right](x')\end{aligned}\tag{4.21}$$

$$
\begin{aligned}
&+ \mathcal{F}^{-1}_{\xi'}\left[\left(\frac{\nu_*^{-1}\lambda}{t_2\omega_\lambda+\nu_*^{-1}\lambda}\right)\mathcal{M}(x_N)\,\widehat{g}(0)\right](x')\\
&+ \sum_{k=1}^{N-1}\mathcal{F}^{-1}_{\xi'}\left[\left(\frac{\nu_*^{-1}\omega_\lambda i\xi_k}{t_2\omega_\lambda+\nu_*^{-1}\lambda}\right)\mathcal{M}(x_N)\,\widehat{h}_k(0)\right](x').
\end{aligned}
$$

Next, we consider the formulas of u_J. By (4.17),

$$
\begin{aligned}
u_j &= \mathcal{F}^{-1}_{\xi'}\left[e^{-\omega_\lambda x_N}\widehat{h}_j(0)\right](x')\\
&\quad - \mathcal{F}^{-1}_{\xi'}\left[\left(\frac{i\xi_j\lambda}{t_2\omega_\lambda+\nu_*^{-1}\lambda}\right)\mathcal{M}(x_N)\,\widehat{g}(0)\right](x')\\
&\quad + \sum_{k=1}^{N-1}\mathcal{F}^{-1}_{\xi'}\left[\left(\frac{\xi_j\xi_k\omega_\lambda}{t_2\omega_\lambda+\nu_*^{-1}\lambda}\right)\mathcal{M}(x_N)\,\widehat{h}_k(0)\right](x'),\\
u_N &= \mathcal{F}^{-1}_{\xi'}\left[\frac{t_2\lambda}{t_2\omega_\lambda+\nu_*^{-1}\lambda}\mathcal{M}(x_N)\,\widehat{g}(0)\right](x')\\
&\quad + \sum_{k=1}^{N-1}\mathcal{F}^{-1}_{\xi'}\left[\frac{t_2\omega_\lambda i\xi_k}{t_2\omega_\lambda+\nu_*^{-1}\lambda}\mathcal{M}(x_N)\,\widehat{h}_k(0)\right](x').
\end{aligned}
\tag{4.22}
$$

By Lemmas 2.1, 2.5, and 4.2, the symbols of ρ and u_J satisfy the following condition: For ρ, there hold

$$
\begin{aligned}
&\frac{t_2}{t_2\omega_\lambda+\nu_*^{-1}\lambda},\ \frac{\nu_*^{-1}i\xi_k}{t_2\omega_\lambda+\nu_*^{-1}\lambda} \in \mathbb{M}_{-1,1}(\mathbf{C}_+),\\
&\frac{\nu_*^{-1}\lambda}{t_2\omega_\lambda+\nu_*^{-1}\lambda},\ \frac{\nu_*^{-1}\omega_\lambda i\xi_k}{t_2\omega_\lambda+\nu_*^{-1}\lambda} \in \mathbb{M}_{0,1}(\mathbf{C}_+);
\end{aligned}
\tag{4.23}
$$

For u_J, there hold

$$
\begin{aligned}
1 &\in \mathbb{M}_{0,1}(\mathbf{C}_+),\\
\frac{i\xi_j\lambda}{t_2\omega_\lambda+\nu_*^{-1}\lambda},\ \frac{\xi_j\xi_k\omega_\lambda}{t_2\omega_\lambda+\nu_*^{-1}\lambda} &\in \mathbb{M}_{1,1}(\mathbf{C}_+),\\
\frac{t_2\lambda}{t_2\omega_\lambda+\nu_*^{-1}\lambda},\ \frac{t_2\omega_\lambda i\xi_k}{t_2\omega_\lambda+\nu_*^{-1}\lambda} &\in \mathbb{M}_{1,1}(\mathbf{C}_+).
\end{aligned}
\tag{4.24}
$$

Finally, combining (4.21)-(4.24) with Lemmas 2.6, 2.8, and 2.9 shows the existence of solution operators $\mathcal{A}^2(\lambda)$ and $\mathcal{B}^2(\lambda)$ stated in Theorem 2.11. This completes the proof of Theorem 2.11 for Case III.

5. Proof of Theorem 2.11 for Case IV

This section proves Theorem 2.11 for Case IV. Throughout this section, we assume that μ_*, ν_*, and κ_* are positive constants satisfying the condition of Case IV. One then recalls Lemmas 2.2 (4) and 2.3 (3), i.e.

$$
\mu_*\neq\nu_*,\quad t_1=t_2=\sqrt{|\xi'|^2+\left(\frac{\mu_*+\nu_*}{2\kappa_*}\right)\lambda},\quad t_2\neq\omega_\lambda,
$$

which are often used in the following computations. Let $J=1,\dots,N$ and $j=1,\dots,N-1$ in this section.

5.1. **Solution formulas.** One first considers (3.9)-(3.12) with (3.5), (3.8), and (3.13) in order to derive solution formulas of (2.11). In view of (3.9), (3.12), and Lemma 2.4, we look for solutions $\widehat{u}_J$ and $\widehat{\varphi}$ of the forms:

$$\widehat{u}_J = \alpha_J e^{-\omega_\lambda x_N} + \beta_J(e^{-t_2 x_N} - e^{-\omega_\lambda x_N}) + \gamma_J x_N e^{-t_2 x_N}, \tag{5.1}$$

$$\widehat{\varphi} = \sigma e^{-t_2 x_N} + \tau x_N e^{-t_2 x_N}. \tag{5.2}$$

It then holds by (3.13) that

$$0 = i\xi' \cdot \alpha' - i\xi' \cdot \beta' - \omega_\lambda \alpha_N + \omega_\lambda \beta_N, \tag{5.3}$$

$$\sigma = i\xi' \cdot \beta' - t_2 \beta_N + \gamma_N, \tag{5.4}$$

$$\tau = i\xi' \cdot \gamma' - t_2 \gamma_N. \tag{5.5}$$

Here note that

$$\begin{aligned}
(\partial_N^2 - \omega_\lambda)^2 \widehat{u}_J &= \beta_J(t_2^2 - \omega_\lambda^2)e^{-t_2 x_N} + \gamma_J\{-2t_2 e^{-t_2 x_N} + (t_2^2 - \omega_\lambda^2)x_N e^{-t_2 x_N}\}, \\
(\partial_N^2 - |\xi'|^2)\widehat{\varphi} &= \sigma(t_2^2 - |\xi'|^2)e^{-t_2 x_N} + \tau\{-2t_2 e^{-t_2 x_N} + (t_2^2 - |\xi'|^2)x_N e^{-t_2 x_N}\}, \\
\partial_N(\partial_N^2 - |\xi'|^2)\widehat{\varphi} &= -\sigma t_2(t_2^2 - |\xi'|^2)e^{-t_2 x_N} \\
&\quad + \tau\{2t_2^2 e^{-t_2 x_N} + (t_2^2 - |\xi'|^2)e^{-t_2 x_N} - t_2(t_2^2 - |\xi'|^2)x_N e^{-t_2 x_N}\},
\end{aligned}$$

and also

$$\partial_N \widehat{\varphi} = (-\sigma t_2 + \tau)e^{-t_2 x_N} - \tau t_2 x_N e^{-t_2 x_N}.$$

Inserting (5.1) and (5.2), together with the last four relations, into (3.10) and (3.11) furnishes that

$$\mu_*\lambda\{(t_2^2 - \omega_\lambda^2)\beta_j - 2t_2\gamma_j\} + i\xi_j\sigma\{\nu_*\lambda - \kappa_*(t_2^2 - |\xi'|^2)\} + 2\kappa_* i\xi_j t_2 \tau = 0, \tag{5.6}$$

$$\mu_*\lambda(t_2^2 - \omega_\lambda^2)\gamma_j + i\xi_j\tau\{\nu_*\lambda - \kappa_*(t_2^2 - |\xi'|^2)\} = 0, \tag{5.7}$$

$$\mu_*\lambda\{(t_2^2 - \omega_\lambda^2)\beta_N - 2t_2\gamma_N\} + (-t_2\sigma + \tau)\{\nu_*\lambda - \kappa_*(t_2^2 - |\xi'|^2)\} - 2\kappa_* t_2^2 \tau = 0, \tag{5.8}$$

$$\mu_*\lambda(t_2^2 - \omega_\lambda^2)\gamma_N - t_2\tau\{\nu_*\lambda - \kappa_*(t_2^2 - |\xi'|^2)\} = 0. \tag{5.9}$$

It then holds by $t_2^2 = |\xi'|^2 + (\mu_* + \nu_*)\lambda/(2\kappa_*)$ that

$$\nu_*\lambda - \kappa_*(t_2^2 - |\xi'|^2) = \frac{(\nu_* - \mu_*)\lambda}{2},$$

which, inserted into (5.6)-(5.9), furnishes

$$2\mu_*\lambda\{(t_2^2 - \omega_\lambda^2)\beta_j - 2t_2\gamma_j\} + i\xi_j\sigma(\nu_* - \mu_*)\lambda + 4\kappa_* i\xi_j t_2\tau = 0, \tag{5.10}$$

$$2\mu_*(t_2^2 - \omega_\lambda^2)\gamma_j + i\xi_j\tau(\nu_* - \mu_*) = 0, \tag{5.11}$$

$$2\mu_*\lambda\{(t_2^2 - \omega_\lambda^2)\beta_N - 2t_2\gamma_N\} + (-t_2\sigma + \tau)(\nu_* - \mu_*)\lambda - 4\kappa_* t_2^2\tau = 0, \tag{5.12}$$

$$2\mu_*(t_2^2 - \omega_\lambda^2)\gamma_N - t_2\tau(\nu_* - \mu_*) = 0. \tag{5.13}$$

By (5.11) and (5.13),

$$2\mu_*(t_2^2 - \omega_\lambda^2)(t_2\gamma_j + i\xi_j\gamma_N) = 0,$$

which, combined with $t_2 \neq \omega_\lambda$, furnishes

$$\gamma_j = -\frac{i\xi_j}{t_2}\gamma_N. \tag{5.14}$$

This relation yields

$$i\xi' \cdot \gamma' = \frac{|\xi'|^2}{t_2}\gamma_N,$$

and thus by (5.5)

$$\tau = i\xi' \cdot \gamma' - t_2\gamma_N = -\left(\frac{t_2^2 - |\xi'|^2}{t_2}\right)\gamma_N = -\left(\frac{\mu_* + \nu_*}{2\kappa_*}\right)\frac{\lambda}{t_2}\gamma_N. \tag{5.15}$$

On the other hand, one multiplies (5.10) by t_2 and (5.12) by $i\xi_j$, and sum the resultant equations in order to obtain

$$2\mu_* t_2\{(t_2^2 - \omega_\lambda^2)\beta_j - 2t_2\gamma_j\} + 2\mu_* i\xi_j\{(t_2^2 - \omega_\lambda^2)\beta_N - 2t_2\gamma_N\} + i\xi_j\tau(\nu_* - \mu_*) = 0.$$

Inserting (5.14) into the last equation furnishes

$$2\mu_* t_2(t_2^2 - \omega_\lambda^2)\beta_j = -2\mu_* i\xi_j(t_2^2 - \omega_\lambda^2)\beta_N - i\xi_j\tau(\nu_* - \mu_*),$$

which, combined with $t_2 \neq \omega_\lambda$, yields

$$\beta_j = -\frac{i\xi}{t_2}\beta_N - \frac{i\xi_j}{t_2}\left(\frac{\tau}{t_2^2 - \omega_\lambda^2}\right)\left(\frac{\nu_* - \mu_*}{2\mu_*}\right). \tag{5.16}$$

One here notes by the assumption $\eta_* = 0$

$$\frac{\mu_* + \nu_*}{2\kappa_*} = \frac{2}{\mu_* + \nu_*}, \tag{5.17}$$

and thus one observes, by $t_2^2 = |\xi'|^2 + (\mu_* + \nu_*)\lambda/(2\kappa_*)$ and $\omega_\lambda^2 = |\xi'|^2 + \mu_*^{-1}\lambda$,

$$t_2^2 - \omega_\lambda^2 = \left(\frac{\mu_* + \nu_*}{2\kappa_*} - \frac{1}{\mu_*}\right)\lambda = -\frac{(\nu_* - \mu_*)}{\mu_*(\mu_* + \nu_*)}\lambda. \tag{5.18}$$

Then, by $\mu_* \neq \nu_*$, (5.15), (5.17), and (5.18),

$$\begin{aligned}\frac{\tau}{t_2^2 - \omega_\lambda^2} &= \left\{-\frac{(\nu_* - \mu_*)}{\mu_*(\mu_* + \nu_*)}\lambda\right\}^{-1}\left\{-\left(\frac{2}{\mu_* + \nu_*}\right)\frac{\lambda}{t_2}\gamma_N\right\} \\ &= \left(\frac{2\mu_*}{\nu_* - \mu_*}\right)\frac{1}{t_2}\gamma_N,\end{aligned}$$

which, combined with (5.16), furnishes

$$\beta_j = -\frac{i\xi}{t_2}\beta_N - \frac{i\xi_j}{t_2^2}\gamma_N. \tag{5.19}$$

This relation yields

$$i\xi' \cdot \beta' = \frac{|\xi'|^2}{t_2}\beta_N + \frac{|\xi'|^2}{t_2^2}\gamma_N, \tag{5.20}$$

and thus by (5.4)

$$\sigma = i\xi' \cdot \beta' - t_2\beta_N + \gamma_N = -\left(\frac{t_2^2 - |\xi'|^2}{t_2}\right)\beta_N + \left(\frac{t_2^2 + |\xi'|^2}{t_2^2}\right)\gamma_N. \tag{5.21}$$

Next, we consider the boundary conditions. By (3.5) and (5.1),

$$\alpha_j = \widehat{h}_j(0), \quad \alpha_N = 0. \tag{5.22}$$

It then holds by the first relation of (5.22) that

$$i\xi' \cdot \alpha' = i\xi' \cdot \widehat{\mathbf{h}}'(0), \quad \widehat{\mathbf{h}}'(0) = (\widehat{h}_1(0), \dots, \widehat{h}_{N-1}(0))^{\mathsf{T}}. \tag{5.23}$$

On the other hand, by (5.2),

$$\partial_N\widehat{\varphi}(0) = -t_2\sigma + \tau,$$

which, combined with (3.8), (5.15), (5.21), and $t_2^2 = |\xi'|^2 + (\mu_* + \nu_*)\lambda/(2\kappa_*)$, furnishes that

$$(t_2^2 - |\xi'|^2)\beta_N - 2t_2\gamma_N = \lambda\widehat{g}(0). \tag{5.24}$$

From now on, we derive simultaneous equations with respect to β_N and γ_N. To this end, we note by (5.17) and (5.18) that the following relation holds:

$$\begin{aligned}
&2\mu_*(t_2^2 - \omega_\lambda^2) - (\nu_* - \mu_*)|\xi'|^2 \\
&= -2\mu_*\frac{(\nu_* - \mu_*)}{\mu_*(\mu_* + \nu_*)}\lambda - (\nu_* - \mu_*)|\xi'|^2 \\
&= -(\nu_* - \mu_*)\left(\frac{2}{\mu_* + \nu_*}\lambda + |\xi'|^2\right) \\
&= -(\nu_* - \mu_*)t_2^2.
\end{aligned} \tag{5.25}$$

One multiplies (5.3) by $-(\nu_* - \mu_*)t_2^2$ in order to obtain

$$-(\nu_* - \mu_*)t_2^2(i\xi'\cdot\alpha' - i\xi'\cdot\beta' - \omega_\lambda\alpha_N + \omega_\lambda\beta_N) = 0,$$

which, combined with (5.20), (5.23), (5.25), and $\alpha_N = 0$ of (5.22) furnishes

$$\begin{aligned}
-(\nu_* - \mu_*)t_2^2 i\xi'\cdot\widehat{\mathbf{h}}'(0) + (\nu_* - \mu_*)(t_2|\xi'|^2\beta_N + |\xi'|^2\gamma_N) \\
+\{2\mu_*(t_2^2 - \omega_\lambda^2) - (\nu_* - \mu_*)|\xi'|^2\}\omega_\lambda\beta_N = 0.
\end{aligned}$$

It thus holds that

$$\begin{aligned}
&(t_2 - \omega_\lambda)\{2\mu_*\omega_\lambda(t_2 + \omega_\lambda) + (\nu_* - \mu_*)|\xi'|^2\}\beta_N + (\nu_* - \mu_*)|\xi'|^2\gamma_N \\
&= (\nu_* - \mu_*)t_2^2 i\xi'\cdot\widehat{\mathbf{h}}'(0).
\end{aligned} \tag{5.26}$$

On the other hand, (5.25) is written as

$$(\nu_* - \mu_*)(t_2^2 - |\xi'|^2) = -2\mu_*(t_2 - \omega_\lambda)(t_2 + \omega_\lambda). \tag{5.27}$$

One then multiplies (5.24) by $\nu_* - \mu_*$, and inserts the last relation into the resultant formula in order to obtain

$$-2\mu_*(t_2 - \omega_\lambda)(t_2 + \omega_\lambda)\beta_N - 2(\nu_* - \mu_*)t_2\gamma_N = (\nu_* - \mu_*)\lambda\widehat{g}(0).$$

Summing up this equation and (5.26), we have achieved

$$\mathbf{M}\begin{pmatrix}\beta_N \\ \gamma_N\end{pmatrix} = \begin{pmatrix}(\nu_* - \mu_*)\lambda\widehat{g}(0) \\ (\nu_* - \mu_*)t_2^2 i\xi'\cdot\widehat{\mathbf{h}}'(0)\end{pmatrix}, \tag{5.28}$$

where

$$\mathbf{M} = \begin{pmatrix} -2\mu_*(t_2 - \omega_\lambda)(t_2 + \omega_\lambda) & -2(\nu_* - \mu_*)t_2 \\ (t_2 - \omega_\lambda)\{2\mu_*\omega_\lambda(t_2 + \omega_\lambda) + (\nu_* - \mu_*)|\xi'|^2\} & (\nu_* - \mu_*)|\xi'|^2 \end{pmatrix}.$$

Let us solve (5.28). By direct calculations,

$$\det\mathbf{M} = (\nu_* - \mu_*)(t_2 - \omega_\lambda)\mathfrak{q}(\xi', \lambda),$$

where

$$\mathfrak{q}(\xi', \lambda) = 2\{(2\mu_*(t_2 + \omega_\lambda)\omega_\lambda + (\nu_* - \mu_*)|\xi'|^2)t_2 - \mu_*(t_2 + \omega_\lambda)|\xi'|^2\}.$$

One here has

Lemma 5.1. *There holds* $\det \mathbf{M} \neq 0$ *for any* $(\xi', \lambda) \in \mathbf{R}^{N-1} \times (\overline{\mathbf{C}_+} \setminus \{0\})$.

Proof. The proof is similar to the proof of Lemma 3.2, so that the detailed proof may be omitted. $\square$

Let us write the inverse matrix $\mathbf{M}^{-1}$ of $\mathbf{M}$ as follows:

$$\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where

$$\begin{aligned} M_{11} &= (\nu_* - \mu_*)|\xi'|^2, \quad M_{12} = 2(\nu_* - \mu_*)t_2, \\ M_{21} &= -(t_2 - \omega_\lambda)\{2\mu_*(t_2 + \omega_\lambda)\omega_\lambda + (\nu_* - \mu_*)|\xi'|^2\}, \\ M_{22} &= -2\mu_*(t_2 - \omega_\lambda)(t_2 + \omega_\lambda). \end{aligned} \tag{5.29}$$

One then sees that, by solving (5.28),

$$\begin{aligned} \beta_N &= \frac{(\nu_* - \mu_*)M_{11}}{\det \mathbf{M}} \lambda \widehat{g}(0) + \frac{(\nu_* - \mu_*)M_{12}}{\det \mathbf{M}} t_2^2 i\xi' \cdot \widehat{\mathbf{h}}'(0), \\ \gamma_N &= \frac{(\nu_* - \mu_*)M_{21}}{\det \mathbf{M}} \lambda \widehat{g}(0) + \frac{(\nu_* - \mu_*)M_{22}}{\det \mathbf{M}} t_2^2 i\xi' \cdot \widehat{\mathbf{h}}'(0). \end{aligned} \tag{5.30}$$

On the other hand, one has, by (5.1), (5.2), (5.14), (5.15), (5.19), (5.21), and (5.22),

$$\begin{aligned} \widehat{u}_j(x_N) &= \widehat{h}_j(0)e^{-\omega_\lambda x_N} \\ &\quad - \left(\frac{i\xi_j}{t_2}\beta_N + \frac{i\xi_j}{t_2^2}\gamma_N\right)(e^{-t_2 x_N} - e^{-\omega_\lambda x_N}) - \frac{i\xi_j}{t_2}\gamma_N x_N e^{-t_2 x_N}, \\ \widehat{u}_N(x_N) &= \beta_N(e^{-t_2 x_N} - e^{-\omega_\lambda x_N}) + \gamma_N x_N e^{-t_2 x_N}, \\ \widehat{\varphi}(x_N) &= \left\{-\left(\frac{t_2^2 - |\xi'|^2}{t_2}\right)\beta_N + \left(\frac{t_2^2 + |\xi'|^2}{t_2^2}\right)\gamma_N\right\} e^{-t_2 x_N} \\ &\quad - \left(\frac{\mu_* + \nu_*}{2\kappa_*}\right)\frac{\lambda}{t_2}\gamma_N x_N e^{-t_2 x_N}, \end{aligned}$$

and sets $\widehat{\rho}(x_N) = -\lambda^{-1}\widehat{\varphi}(x_N)$ in view of (3.1). Recall the inverse partial Fourier transform given in (2.7) and set $\rho = \mathcal{F}_{\xi'}^{-1}[\widehat{\rho}(x_N)](x')$ and $u_J = \mathcal{F}_{\xi'}^{-1}[\widehat{u}_J(x_N)](x')$. Then ρ and $\mathbf{u} = (u_1, \ldots, u_N)^{\mathsf{T}}$ solve the system (2.11).

5.2. **Analysis of symbols.** This subsection estimates several symbols arising from the representation formulas of solutions obtained in Subsection 5.1. We often denote $\mathfrak{q}(\xi', \lambda)$ by $\mathfrak{q}$ for short in what follows.

One starts with

Lemma 5.2. (1) $\mathfrak{q} \in \mathbb{M}_{3,1}(\mathbf{C}_+)$.
(2) $M_{11}, M_{22} \in \mathbb{M}_{2,1}(\mathbf{C}_+)$, $M_{12} \in \mathbb{M}_{1,1}(\mathbf{C}_+)$, *and* $M_{21} \in \mathbb{M}_{3,1}(\mathbf{C}_+)$.
(3) *It holds that*

$$\frac{t_2^2 - |\xi'|^2}{t_2 - \omega_\lambda} \in \mathbb{M}_{1,1}(\mathbf{C}_+), \quad \frac{M_{21}}{t_2 - \omega_\lambda} \in \mathbb{M}_{2,1}(\mathbf{C}_+), \quad \frac{M_{22}}{t_2 - \omega_\lambda} \in \mathbb{M}_{1,1}(\mathbf{C}_+).$$

Proof. (1), (2). The required properties follow from (5.29) and Lemmas 2.1 and 2.5 immediately.

(3). We here prove the first assertion only. Since it holds by (5.17) that

$$t_2^2 - |\xi'|^2 = \left(\frac{\mu_* + \nu_*}{2\kappa_*}\right)\lambda = \frac{2}{\mu_* + \nu_*}\lambda,$$

one has by (5.18)

$$\begin{aligned}\frac{t_2^2-|\xi'|^2}{t_2-\omega_\lambda}&=\frac{t_2^2-|\xi'|^2}{t_2^2-\omega_\lambda^2}(t_2+\omega_\lambda)\\&=\left\{-\frac{(\nu_*-\mu_*)}{\mu_*(\mu_*+\nu_*)}\lambda\right\}^{-1}\frac{2\lambda}{\mu_*+\nu_*}(t_2+\omega_\lambda)\\&=-\frac{2\mu_*}{\nu_*-\mu_*}(t_2+\omega_\lambda).\end{aligned}$$

Combining this relation with Lemmas 2.1 and 2.5 furnishes

$$\frac{t_2^2-|\xi'|^2}{t_2-\omega_\lambda}\in\mathbb{M}_{1,1}(\mathbf{C}_+).$$

This completes the proof of the lemma. $\square$

Similarly to the proof of Lemma 3.4 and Corollary 3.5, one can prove the following lemma by using Lemmas 5.1 and 5.2 (1).

Lemma 5.3. $\mathfrak{q}^{-1}\in\mathbb{M}_{-3,1}(\mathbf{C}_+)$.

5.3. **Proof of Theorem 2.11.** This subsection proves Theorem 2.11 by means of results obtained in Subsections 5.1 and 5.2. Let $\mathcal{M}(x_N)$ be given in (2.5).

First, we consider the solution formula of ρ. It holds by (5.30) that

$$\begin{aligned}&-\left(\frac{t_2^2-|\xi'|^2}{t_2}\right)\beta_N+\left(\frac{t_2^2+|\xi'|^2}{t_2^2}\right)\gamma_N\\&=\left\{-\left(\frac{t_2^2-|\xi'|^2}{t_2-\omega_\lambda}\right)t_2M_{11}+(t_2^2+|\xi'|^2)\left(\frac{M_{21}}{t_2-\omega_\lambda}\right)\right\}\frac{\lambda\widehat{g}(0)}{t_2^2\mathfrak{q}(\xi',\lambda)}\\&+\left\{-t_2(t_2^2-|\xi'|^2)M_{12}+(t_2^2+|\xi'|^2)M_{22}\right\}\frac{i\xi'\cdot\widehat{\mathbf{h}}'(0)}{(t_2-\omega_\lambda)\mathfrak{q}(\xi',\lambda)}.\end{aligned}$$

On the other hand, since

$$\begin{aligned}&-t_2(t_2^2-|\xi'|^2)M_{12}+(t_2^2+|\xi'|^2)M_{22}\\&=-2t_2^2(\nu_*-\mu_*)(t_2^2-|\xi'|^2)-2\mu_*(t_2^2+|\xi'|^2)(t_2-\omega_\lambda)(t_2+\omega_\lambda),\end{aligned}$$

one has by (5.17) and (5.27)

$$\begin{aligned}&-t_2(t_2^2-|\xi'|^2)M_{12}+(t_2^2+|\xi'|^2)M_{22}\\&=2\mu_*(t_2-\omega_\lambda)(t_2+\omega_\lambda)(t_2^2-|\xi'|^2)\\&=\left(\frac{4\mu_*}{\mu_*+\nu_*}\right)\lambda(t_2-\omega_\lambda)(t_2+\omega_\lambda).\end{aligned}$$

Thus we have

$$\begin{aligned}\rho=&-\mathcal{F}_{\xi'}^{-1}\left[\left(-\frac{t_2^2-|\xi'|^2}{t_2-\omega_\lambda}\frac{M_{11}}{t_2\mathfrak{q}(\xi',\lambda)}+\frac{t_2^2+|\xi'|^2}{t_2^2\mathfrak{q}(\xi',\lambda)}\frac{M_{21}}{t_2-\omega_\lambda}\right)e^{-t_2x_N}\widehat{g}(0)\right](x')\\&-\left(\frac{4\mu_*}{\mu_*+\nu_*}\right)\sum_{k=1}^{N-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{i\xi_k(t_2+\omega_\lambda)}{\mathfrak{q}(\xi',\lambda)}e^{-t_2x_N}\widehat{h}_k(0)\right](x')\\&+\left(\frac{\mu_*+\nu_*}{2\kappa_*}\right)\mathcal{F}_{\xi'}^{-1}\left[\frac{\lambda}{t_2\mathfrak{q}(\xi',\lambda)}\frac{M_{21}}{t_2-\omega_\lambda}x_Ne^{-t_2x_N}\widehat{g}(0)\right](x')\end{aligned}\tag{5.31}$$

$$+\left(\frac{\mu_*+\nu_*}{2\kappa_*}\right)\sum_{k=1}^{N-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{i\xi_k t_2}{\mathfrak{q}(\xi',\lambda)}\frac{M_{22}}{t_2-\omega_\lambda}x_N e^{-t_2x_N}\widehat{h}_k(0)\right](x').$$

Next, we consider the formulas of u_J. By (5.30),

$$\begin{aligned}
u_j &= \mathcal{F}_{\xi'}^{-1}\left[e^{-\omega_\lambda x_N}\widehat{h}_j(0)\right](x') \\
&\quad - \mathcal{F}_{\xi'}^{-1}\left[\left(\frac{\lambda i\xi_j M_{11}}{t_2\mathfrak{q}(\xi',\lambda)}+\frac{\lambda i\xi_j M_{21}}{t_2^2\mathfrak{q}(\xi',\lambda)}\right)\mathcal{M}(x_N)\widehat{g}(0)\right](x') \\
&\quad + \sum_{k=1}^{N-1}\mathcal{F}_{\xi'}^{-1}\left[\left(\frac{\xi_j\xi_k t_2 M_{12}}{\mathfrak{q}(\xi',\lambda)}+\frac{\xi_j\xi_k M_{22}}{\mathfrak{q}(\xi',\lambda)}\right)\mathcal{M}(x_N)\widehat{h}_k(0)\right](x') \\
&\quad - \mathcal{F}_{\xi'}^{-1}\left[\frac{\lambda i\xi_j}{t_2\mathfrak{q}(\xi',\lambda)}\frac{M_{21}}{t_2-\omega_\lambda}x_N e^{-t_2x_N}\widehat{g}(0)\right](x') \\
&\quad + \sum_{k=1}^{N-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{\xi_j\xi_k t_2}{\mathfrak{q}(\xi',\lambda)}\frac{M_{22}}{t_2-\omega_\lambda}x_N e^{-t_2x_N}\widehat{h}_k(0)\right](x'), \\
u_N &= \mathcal{F}_{\xi'}^{-1}\left[\frac{\lambda M_{11}}{\mathfrak{q}(\xi',\lambda)}\mathcal{M}(x_N)\widehat{g}(0)\right](x') \\
&\quad + \sum_{k=1}^{N-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{i\xi_k t_2^2 M_{12}}{\mathfrak{q}(\xi',\lambda)}\mathcal{M}(x_N)\widehat{h}_k(0)\right](x') \\
&\quad + \mathcal{F}_{\xi'}^{-1}\left[\frac{\lambda}{\mathfrak{q}(\xi',\lambda)}\frac{M_{21}}{t_2-\omega_\lambda}x_N e^{-t_2x_N}\widehat{g}(0)\right](x') \\
&\quad + \sum_{k=1}^{N-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{i\xi_k t_2^2}{\mathfrak{q}(\xi',\lambda)}\frac{M_{22}}{t_2-\omega_\lambda}x_N e^{-t_2x_N}\widehat{h}_k(0)\right](x').
\end{aligned}\tag{5.32}$$

By Lemmas 2.1, 2.5, 5.2, and 5.3, the symbols of ρ and u_J satisfy the following conditions: For ρ, there hold

$$\begin{aligned}
&\frac{t_2^2-|\xi'|^2}{t_2-\omega_\lambda}\frac{M_{11}}{t_2\mathfrak{q}(\xi',\lambda)},\ \frac{t_2^2+|\xi'|^2}{t_2^2\mathfrak{q}(\xi',\lambda)}\frac{M_{21}}{t_2-\omega_\lambda},\ \frac{i\xi_k(t_2+\omega_\lambda)}{\mathfrak{q}(\xi',\lambda)}\in\mathbb{M}_{-1,1}(\mathbf{C}_+), \\
&\qquad \frac{\lambda}{t_2\mathfrak{q}(\xi',\lambda)}\frac{M_{21}}{t_2-\omega_\lambda},\ \frac{i\xi_k t_2}{\mathfrak{q}(\xi',\lambda)}\frac{M_{22}}{t_2-\omega_\lambda}\in\mathbb{M}_{0,1}(\mathbf{C}_+);
\end{aligned}\tag{5.33}$$

For u_J, there hold

$$\begin{aligned}
&1\in\mathbb{M}_{0,1}(\mathbf{C}_+), \\
&\frac{\lambda i\xi_j M_{11}}{t_2\mathfrak{q}(\xi',\lambda)},\ \frac{\lambda i\xi_j M_{21}}{t_2^2\mathfrak{q}(\xi',\lambda)},\ \frac{\xi_j\xi_k t_2 M_{12}}{\mathfrak{q}(\xi',\lambda)},\ \frac{\xi_j\xi_k M_{22}}{\mathfrak{q}(\xi',\lambda)}\in\mathbb{M}_{1,1}(\mathbf{C}_+), \\
&\frac{\lambda i\xi_j}{t_2\mathfrak{q}(\xi',\lambda)}\frac{M_{21}}{t_2-\omega_\lambda},\ \frac{\xi_j\xi_k t_2}{\mathfrak{q}(\xi',\lambda)}\frac{M_{22}}{t_2-\omega_\lambda}\in\mathbb{M}_{1,1}(\mathbf{C}_+), \\
&\frac{\lambda M_{11}}{\mathfrak{q}(\xi',\lambda)},\ \frac{i\xi_k t_2^2 M_{12}}{\mathfrak{q}(\xi',\lambda)},\ \frac{\lambda}{\mathfrak{q}(\xi',\lambda)}\frac{M_{21}}{t_2-\omega_\lambda},\ \frac{i\xi_k t_2^2}{\mathfrak{q}(\xi',\lambda)}\frac{M_{22}}{t_2-\omega_\lambda}\in\mathbb{M}_{1,1}(\mathbf{C}_+).
\end{aligned}\tag{5.34}$$

At this point, we introduce the following lemma in order to show the existence of $\mathcal{R}$-bounded solutions operator families associated with ρ and u_J (cf. the appendix below for the proof).

Lemma 5.4. *Let $q \in (1,\infty)$ and $a \in (0,\infty)$. Assume*

$$w_l(\xi',\lambda) \in \mathbb{M}_{l-1,1}(\mathbf{C}_+) \quad (l=1,2),$$

and set for $x=(x',x_N) \in \mathbf{R}^N_+$

$$[W_l(\lambda)f](x) = \mathcal{F}^{-1}_{\xi'}\left[w_l(\xi',\lambda)x_N e^{-\sqrt{|\xi'|^2+a\lambda}x_N}\widehat{f}(\xi',0)\right](x') \quad (l=1,2),$$

with $\lambda \in \mathbf{C}_+$ and $f \in H^2_q(\mathbf{R}^N_+)$. Then the following assertions hold true:

(1) *For $\lambda \in \mathbf{C}_+$, there is an operator $\widetilde{W}_1(\lambda)$, with*

$$\widetilde{W}_1(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(L_q(\mathbf{R}^N_+)^{N^2+N+1}, H^3_q(\mathbf{R}^N_+))),$$

such that for any $f \in H^2_q(\mathbf{R}^N_+)$

$$W_1(\lambda)f = \widetilde{W}_1(\lambda)(\nabla^2 f, \lambda^{1/2}\nabla f, \lambda f).$$

In addition, for $n=0,1$,

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+)^{N^2+N+1},\mathfrak{A}^0_q(\mathbf{R}^N_+))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\mathcal{S}^0_\lambda\widetilde{W}_1(\lambda)\right)\middle|\lambda \in \mathbf{C}_+\right\}\right) \leq C,$$

with some positive constant C depending solely on N, q, and a. Here, $\mathfrak{A}^0_q(\mathbf{R}^N_+)$ and $\mathcal{S}^0_\lambda$ are given in (1.4) *for $G=\mathbf{R}^N_+$.*

(2) *For $\lambda \in \mathbf{C}_+$, there is an operator $\widetilde{W}_2(\lambda)$, with*

$$\widetilde{W}_2(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(L_q(\mathbf{R}^N_+)^{N^2+N+1}, H^2_q(\mathbf{R}^N_+))),$$

such that for any $f \in H^2_q(\mathbf{R}^N_+)$

$$W_2(\lambda)f = \widetilde{W}_2(\lambda)(\nabla^2 f, \lambda^{1/2}\nabla f, \lambda f).$$

In addition, for $n=0,1$,

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+)^{N^2+N+1})}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\mathcal{T}_\lambda\widetilde{W}_2(\lambda)\right)\middle|\lambda \in \mathbf{C}_+\right\}\right) \leq C,$$

with some positive constant C depending solely on N, q, and a. Here, $\mathcal{T}_\lambda$ is given in (1.4).

Finally, combining (5.31)-(5.34) with Lemmas 2.6, 2.8, 2.9, and 5.4 shows the existence of solution operators $\mathcal{A}^2(\lambda)$ and $\mathcal{B}^2(\lambda)$ stated in Theorem 2.11. This completes the proof of Theorem 2.11 for Case IV.

6. Proof of Theorem 2.11 for Case V

This section proves Theorem 2.11 for Case V. Throughout this section, we assume that μ_*, ν_*, and κ_* are positive constants satisfying the condition of Case V. One then recalls Lemmas 2.2 (5) and 2.3 (4), i.e.

$$\mu_* = \nu_*, \quad t_1 = t_2 = \omega_\lambda = \sqrt{|\xi'|^2 + \mu_*^{-1}\lambda},$$

which are often used in the following computations. Let $J = 1,\dots,N$ and $j = 1,\dots,N-1$ in this section.

6.1. **Solution formulas.** One first considers (3.9)-(3.12) with (3.5), (3.8), and (3.13) in order to derive solution formulas of (2.11). In view of (3.9), (3.12), and Lemma 2.4, we look for solutions $\widehat{u}_J$ and $\widehat{\varphi}$ of the forms:

$$\widehat{u}_J = \alpha_J e^{-\omega_\lambda x_N} + \beta_J x_N e^{-\omega_\lambda x_N} + \gamma_J x_N^2 e^{-\omega_\lambda x_N}, \tag{6.1}$$

$$\widehat{\varphi} = \sigma e^{-\omega_\lambda x_N} + \tau x_N e^{-\omega_\lambda x_N}. \tag{6.2}$$

It then holds by (3.13) that

$$\sigma = i\xi' \cdot \alpha' - \omega_\lambda \alpha_N + \beta_N, \tag{6.3}$$

$$\tau = i\xi' \cdot \beta' - \omega_\lambda \beta_N + 2\gamma_N, \tag{6.4}$$

$$0 = i\xi' \cdot \gamma' - \omega_\lambda \gamma_N. \tag{6.5}$$

On the other hand, by the assumption $\kappa_* = \mu_* \nu_*$,

$$\nu_* \lambda - \kappa_*(\partial_N^2 - |\xi'|^2) = -\mu_* \nu_*(\partial_N^2 - \omega_\lambda^2),$$

which, combined with $\mu_* = \nu_*$, yields

$$\nu_* \lambda - \kappa_*(\partial_N^2 - |\xi'|^2) = -\mu_*^2(\partial_N^2 - \omega_\lambda^2).$$

Therefore, (3.10) and (3.11) are respectively equivalent to

$$\begin{aligned} \lambda(\partial_N^2 - \omega_\lambda^2)\widehat{u}_j - \mu_* i\xi_j(\partial_N^2 - \omega_\lambda^2)\widehat{\varphi} &= 0, \\ \lambda(\partial_N^2 - \omega_\lambda^2)\widehat{u}_N - \mu_* \partial_N(\partial_N^2 - \omega_\lambda^2)\widehat{\varphi} &= 0. \end{aligned} \tag{6.6}$$

Here note that

$$\begin{aligned} (\partial_N^2 - \omega_\lambda^2)(x_N e^{-\omega_\lambda x_N}) &= -2\omega_\lambda e^{-\omega_\lambda x_N}, \\ (\partial_N^2 - \omega_\lambda^2)(x_N^2 e^{-\omega_\lambda x_N}) &= 2e^{-\omega_\lambda x_N} - 4\omega_\lambda x_N e^{-\omega_\lambda x_N}. \end{aligned}$$

Inserting (6.1) and (6.2), together with the last two relations, into (6.6) furnishes

$$\begin{aligned} -4\lambda\omega_\lambda\gamma_j = 0, &\quad \lambda(-2\omega_\lambda\beta_j + 2\gamma_j) + 2\mu_*\omega_\lambda i\xi_j\tau = 0, \\ -4\lambda\omega_\lambda\gamma_N = 0, &\quad \lambda(-2\omega_\lambda + 2\gamma_N) - 2\mu_*\omega_\lambda^2\tau = 0, \end{aligned}$$

which yields

$$\gamma_J = 0, \tag{6.7}$$

$$-\lambda\beta_j + \mu_* i\xi_j\tau = 0, \tag{6.8}$$

$$-\lambda\beta_N - \mu_*\omega_\lambda\tau = 0. \tag{6.9}$$

Remark 6.1. The relations (6.7)-(6.9) imply (6.4) and (6.5).

One has by (6.1) and (6.7)

$$\widehat{u}_J = \alpha_J e^{-\omega_\lambda x_N} + \beta_J x_N e^{-\omega_\lambda x_N}, \tag{6.10}$$

and also by (6.8) and (6.9)

$$\beta_j = -\frac{i\xi_j}{\omega_\lambda}\beta_N. \tag{6.11}$$

Next, we consider the boundary conditions. By (3.5) and (6.10),

$$\alpha_j = \widehat{h}_j(0), \quad \alpha_N = 0. \tag{6.12}$$

It then holds by the first relation of (6.12) that

$$i\xi' \cdot \alpha' = i\xi' \cdot \widehat{\mathbf{h}}'(0), \quad \widehat{\mathbf{h}}'(0) = (\widehat{h}_1(0), \dots, \widehat{h}_{N-1}(0))^\mathsf{T}.$$

Combining this relation with (6.3) and $\alpha_N = 0$ of (6.12) furnishes

$$\sigma = i\xi' \cdot \widehat{\mathbf{h}}'(0) + \beta_N, \tag{6.13}$$

while (6.9) implies

$$\tau = -\frac{\mu_*^{-1}\lambda}{\omega_\lambda}\beta_N. \tag{6.14}$$

On the other hand, by (3.8) and (6.2),

$$-\omega_\lambda \sigma + \tau = \lambda \widehat{g}(0),$$

which, combined with (6.13) and (6.14), furnishes

$$-\omega_\lambda(i\xi' \cdot \widehat{\mathbf{h}}'(0) + \beta_N) - \frac{\mu_*^{-1}\lambda}{\omega_\lambda}\beta_N = \lambda\widehat{g}(0).$$

Solving this equation with respect to β_N, we have

$$\beta_N = -\frac{\omega_\lambda}{\omega_\lambda^2 + \mu_*^{-1}\lambda}\left(\lambda\widehat{g}(0) + \omega_\lambda i\xi' \cdot \widehat{\mathbf{h}}'(0)\right). \tag{6.15}$$

Finally, one has, by (6.2), (6.10), (6.11), (6.12), (6.13), and (6.14),

$$\begin{aligned}
\widehat{u}_j(x_N) &= \widehat{h}_j(0)e^{-\omega_\lambda x_N} - \frac{i\xi_j}{\omega_\lambda}\beta_N x_N e^{-\omega_\lambda x_N}, \\
\widehat{u}_N(x_N) &= \beta_N x_N e^{-\omega_\lambda x_N}, \\
\widehat{\varphi}(x_N) &= (i\xi' \cdot \widehat{\mathbf{h}}'(0) + \beta_N)e^{-\omega_\lambda x_N} - \frac{\mu_*^{-1}\lambda}{\omega_\lambda}\beta_N x_N e^{-\omega_\lambda x_N},
\end{aligned}$$

and sets $\widehat{\rho}(x_N) = -\lambda^{-1}\widehat{\varphi}(x_N)$ in view of (3.1). Recall the inverse partial Fourier transform given in (2.7) and set $\rho = \mathcal{F}_{\xi'}^{-1}[\widehat{\rho}(x_N)](x')$ and $u_J = \mathcal{F}_{\xi'}^{-1}[\widehat{u}_J(x_N)](x')$. Then ρ and $\mathbf{u} = (u_1, \dots, u_N)^\mathsf{T}$ solve the system (2.11).

In the last part of this section, we prove the following lemma.

Lemma 6.2. $(\omega_\lambda^2 + \mu_*^{-1}\lambda)^{-1} \in \mathbb{M}_{-2,1}(\mathbf{C}_+)$.

Proof. Since $\omega_\lambda^2 = |\xi'|^2 + \mu_*^{-1}\lambda$, it holds that $\omega_\lambda^2 + \mu_*^{-1}\lambda = |\xi'|^2 + 2\mu_*^{-1}\lambda$. One thus sees by Lemma 2.5 that

$$(\omega_\lambda^2 + \mu_*^{-1}\lambda)^{-1} = (|\xi'|^2 + 2\mu_*^{-1}\lambda)^{-1} \in \mathbb{M}_{-2,1}(\mathbf{C}_+).$$

This completes the proof of the lemma. $\square$

6.2. **Proof of Theorem 2.11.** This subsection proves Theorem 2.11 by means of solution formulas obtained in Subsection 6.1.

First, we consider the formula of ρ. Note that by (6.15)

$$i\xi' \cdot \widehat{\mathbf{h}}'(0) + \beta_N = -\frac{\omega_\lambda}{\omega_\lambda^2 + \mu_*^{-1}\lambda}\lambda\widehat{g}(0) + \frac{\mu_*^{-1}\lambda}{\omega_\lambda^2 + \mu_*^{-1}\lambda}i\xi' \cdot \widehat{\mathbf{h}}'(0).$$

It thus holds that

$$\begin{aligned}
\rho &= \mathcal{F}_{\xi'}^{-1}\left[\frac{\omega_\lambda}{\omega_\lambda^2 + \mu_*^{-1}\lambda}e^{-\omega_\lambda x_N}\widehat{g}(0)\right](x') \\
&\quad - \sum_{k=1}^{N-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{\mu_*^{-1}i\xi_k}{\omega_\lambda^2 + \mu_*^{-1}\lambda}e^{-\omega_\lambda x_N}\widehat{h}_k(0)\right](x')
\end{aligned} \tag{6.16}$$

$$
\begin{aligned}
&- \mathcal{F}_{\xi'}^{-1}\left[\frac{\mu_*^{-1}\lambda}{\omega_\lambda^2+\mu_*^{-1}\lambda} x_N e^{-\omega_\lambda x_N}\widehat{g}(0)\right](x') \\
&- \sum_{k=1}^{N-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{\mu_*^{-1}i\xi_k\omega_\lambda}{\omega_\lambda^2+\mu_*^{-1}\lambda} x_N e^{-\omega_\lambda x_N}\widehat{h}_k(0)\right](x').
\end{aligned}
$$

Next, we consider the formulas of u_J. By (6.15),

$$
\begin{aligned}
u_j &= \mathcal{F}_{\xi'}^{-1}\left[e^{-\omega_\lambda x_N}\widehat{h}_j(0)\right](x') \\
&\quad + \mathcal{F}_{\xi'}^{-1}\left[\frac{i\xi_j\lambda}{\omega_\lambda^2+\mu_*^{-1}\lambda} x_N e^{-\omega_\lambda x_N}\widehat{g}(0)\right](x') \\
&\quad - \sum_{k=1}^{N-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{\xi_j\xi_k\omega_\lambda}{\omega_\lambda^2+\mu_*^{-1}\lambda} x_N e^{-\omega_\lambda x_N}\widehat{h}_k(0)\right](x'), \\
u_N &= -\mathcal{F}_{\xi'}^{-1}\left[\frac{\lambda\omega_\lambda}{\omega_\lambda^2+\mu_*^{-1}\lambda} x_N e^{-\omega_\lambda x_N}\widehat{g}(0)\right](x') \\
&\quad - \sum_{k=1}^{N-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{i\xi_k\omega_\lambda^2}{\omega_\lambda^2+\mu_*^{-1}\lambda} x_N e^{-\omega_\lambda x_N}\widehat{h}_k(0)\right](x').
\end{aligned}
\tag{6.17}
$$

By Lemmas 2.1, 2.5, and 6.2, the symbols of ρ and u_J satisfy the following conditions: For ρ, there hold

$$
\begin{aligned}
&\frac{\omega_\lambda}{\omega_\lambda^2+\mu_*^{-1}\lambda},\ \frac{\mu_*^{-1}i\xi_k}{\omega_\lambda^2+\mu_*^{-1}\lambda} \in \mathbb{M}_{-1,1}(\mathbf{C}_+), \\
&\frac{\mu_*^{-1}\lambda}{\omega_\lambda^2+\mu_*^{-1}\lambda},\ \frac{\mu_*^{-1}i\xi_k\omega_\lambda}{\omega_\lambda^2+\mu_*^{-1}\lambda} \in \mathbb{M}_{0,1}(\mathbf{C}_+);
\end{aligned}
\tag{6.18}
$$

For u_J, there hold

$$
\begin{aligned}
&1 \in \mathbb{M}_{0,1}(\mathbf{C}_+), \\
&\frac{i\xi_j\lambda}{\omega_\lambda^2+\mu_*^{-1}\lambda},\ \frac{\xi_j\xi_k\omega_\lambda}{\omega_\lambda^2+\mu_*^{-1}\lambda} \in \mathbb{M}_{1,1}(\mathbf{C}_+), \\
&\frac{\lambda\omega_\lambda}{\omega_\lambda^2+\mu_*^{-1}\lambda},\ \frac{i\xi_k\omega_\lambda^2}{\omega_\lambda^2+\mu_*^{-1}\lambda} \in \mathbb{M}_{1,1}(\mathbf{C}_+).
\end{aligned}
\tag{6.19}
$$

Finally, combining (6.16)-(6.19) with Lemmas 2.6 and 5.4 shows the existence of solution operators $\mathcal{A}^2(\lambda)$ and $\mathcal{B}^2(\lambda)$ stated in Theorem 2.11. This completes the proof of Theorem 2.11 for Case V.

A.

This appendix proves Lemma 5.4 for[3] $a = 1$. Set

$$
B = \sqrt{|\xi'|^2+\lambda} \quad \text{for } (\xi',\lambda) \in \mathbf{R}^{N-1}\times\Sigma_\varepsilon.
$$

One then has

Lemma A.1. *Let $\varepsilon \in (0,\pi/2)$.*
(1) *For $r \in \mathbf{R}$, there holds $B^r \in \mathbb{M}_{r,1}(\Sigma_\varepsilon)$.*

[3] Lemma 5.4 for $a > 0$ follows from the result for $a = 1$ and the definition of the $\mathcal{R}$-boundedness (cf. Definition 1.2).

(2) *For $x_N > 0$, $n = 0, 1$, and multi-index $\alpha' \in \mathbf{N}_0^{N-1}$, there holds*

$$\left|\partial_{\xi'}^{\alpha'} \left(\lambda \frac{d}{d\lambda}\right)^n e^{-Bx_N}\right| \leq C_{\alpha',\varepsilon}(|\lambda|^{1/2} + |\xi'|)^{-|\alpha'|} e^{-b_\varepsilon(|\lambda|^{1/2}+|\xi'|)x_N},$$

where $(\xi', \lambda) \in \mathbf{R}^{N-1} \times \Sigma_\varepsilon$, with a positive constant $C_{\alpha',\varepsilon}$ independent of x_N and a positive constant b_ε depending only on ε.

Proof. (1), The required property follows from Lemma 2.5 with $\mu_* = 1$.
(2). See [11, Lemma 5.3]. □

At this point, we introduce two lemmas. The first one is essentially proved in [11, Lemma 5.4], while the second one can be proved similarly to [11, Lemmas 5.4 and 5.6] (cf. also [8]).

Lemma A.2. *Let $q \in (1, \infty)$ and $\mathcal{K}_\lambda(x)$ $(\lambda \in \mathbf{C}_+)$ be a function on $\mathbf{R}_+^N$. Assume*

$$\left|\left(\lambda \frac{d}{d\lambda}\right)^n \mathcal{K}_\lambda(x)\right| \leq \frac{M}{|x|^N} \quad (x \in \mathbf{R}_+^N, \lambda \in \mathbf{C}_+)$$

for a positive constant M and $n = 0, 1$, and set for $x = (x', x_N) \in \mathbf{R}_+^N$ and $\lambda \in \mathbf{C}_+$

$$[K(\lambda)f](x) = \int_{\mathbf{R}_+^N} \mathcal{K}_\lambda(x' - y', x_N + y_N) f(y)\, dy.$$

Then $K(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(L_q(\mathbf{R}_+^N)))$, and also for $n = 0, 1$

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}_+^N))}\left(\left\{\left(\lambda \frac{d}{d\lambda}\right)^n K(\lambda) \,\middle|\, \lambda \in \mathbf{C}_+\right\}\right) \leq C_{N,q} M,$$

with some positive constant $C_{N,q}$.

Lemma A.3. *Let $q \in (1, \infty)$. Assume*

$$k(\xi', \lambda) \in \mathbb{M}_{1,1}(\mathbf{C}_+), \quad l_m(\xi', \lambda) \in \mathbb{M}_{m-3}(\mathbf{C}_+) \quad (m = 1, 2),$$

and set for $x = (x', x_N) \in \mathbf{R}_+^N$ and $\lambda \in \mathbf{C}_+$

$$[K(\lambda)f](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[k(\xi', \lambda) e^{-B(x_N+y_N)} \widehat{f}(\xi', y_N)\right](x')\, dy_N,$$

$$[L_m(\lambda)f](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[l_m(\xi', \lambda) e^{-B(x_N+y_N)} \widehat{f}(\xi', y_N)\right](x')\, dy_N \quad (m = 1, 2).$$

Then the following assertions hold true:

(1) *There holds $K(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(L_q(\mathbf{R}_+^N)))$, and also for $n = 0, 1$*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}_+^N))}\left(\left\{\left(\lambda \frac{d}{d\lambda}\right)^n K(\lambda) \,\middle|\, \lambda \in \mathbf{C}_+\right\}\right) \leq C_{N,q},$$

with some positive constant $C_{N,q}$.

(2) *There holds $L_1(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(L_q(\mathbf{R}_+^N), H_q^3(\mathbf{R}_+^N)))$, and also for $n = 0, 1$*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}_+^N), \mathfrak{A}_q^0(\mathbf{R}_+^N))}\left(\left\{\left(\lambda \frac{d}{d\lambda}\right)^n \left(\mathcal{S}_\lambda^0 L_1(\lambda)\right) \,\middle|\, \lambda \in \mathbf{C}_+\right\}\right) \leq C_{N,q},$$

with some positive constant $C_{N,q}$.

(3) *There holds* $L_2(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(L_q(\mathbf{R}^N_+), H^2_q(\mathbf{R}^N_+)))$, *and also for* $n = 0, 1$

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+), L_q(\mathbf{R}^N_+)^{N^2+N+1})}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n (\mathcal{T}_\lambda L_2(\lambda))\,\middle|\, \lambda \in \mathbf{C}_+\right\}\right) \leq C_{N,q},$$

with some positive constant $C_{N,q}$.

One now proves

Lemma A.4. *Let* $q \in (1, \infty)$. *Assume*

$$k(\xi', \lambda) \in \mathbb{M}_{1,1}(\mathbf{C}_+), \quad l(\xi', \lambda) \in \mathbb{M}_{1,2}(\mathbf{C}_+), \quad m(\xi', \lambda) \in \mathbb{M}_{2,1}(\mathbf{C}_+),$$

and set for $x = (x', x_N) \in \mathbf{R}^N_+$ *and* $\lambda \in \mathbf{C}_+$

$$[K(\lambda)f](x) = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\lambda^{1/2}k(\xi', \lambda)(x_N + y_N)e^{-B(x_N+y_N)}\widehat{f}(\xi', y_N)\right](x')\,dy_N,$$

$$[L(\lambda)f](x) = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[|\xi'|l(\xi', \lambda)(x_N + y_N)e^{-B(x_N+y_N)}\widehat{f}(\xi', y_N)\right](x')\,dy_N,$$

$$[M(\lambda)f](x) = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m(\xi', \lambda)(x_N + y_N)e^{-B(x_N+y_N)}\widehat{f}(\xi', y_N)\right](x')\,dy_N.$$

Then $K(\lambda), L(\lambda), M(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(L_q(\mathbf{R}^N_+)))$, *and also for* $n = 0, 1$

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n K(\lambda)\,\middle|\, \lambda \in \mathbf{C}_+\right\}\right) \leq C_{N,q},$$

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n L(\lambda)\,\middle|\, \lambda \in \mathbf{C}_+\right\}\right) \leq C_{N,q},$$

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n M(\lambda)\,\middle|\, \lambda \in \mathbf{C}_+\right\}\right) \leq C_{N,q},$$

with some positive constant $C_{N,q}$

Proof. Let $n = 0, 1$ and $\lambda \in \mathbf{C}_+$ in this proof.

Case 1. This case considers $K(\lambda)$. Let $\mathcal{K}_\lambda(x)$ be given by

$$\mathcal{K}_\lambda(x) = \mathcal{K}_\lambda(x', x_N) = \mathcal{F}^{-1}_{\xi'}\left[\lambda^{1/2}k(\xi', \lambda)x_N e^{-Bx_N}\right](x').$$

Then $K(\lambda)f$ is written as

$$[K(\lambda)f](x) = \int_{\mathbf{R}^N_+} \mathcal{K}_\lambda(x' - y', x_N + y_N)f(y)\,dy.$$

Since $k(\xi', \lambda) \in \mathbb{M}_{1,1}(\mathbf{C}_+)$, it holds by the Leibniz formula and Lemma A.1 that

$$\begin{aligned}\text{(A.1)}\qquad &\left|\partial^{\alpha'}_{\xi'}\left(\lambda\frac{d}{d\lambda}\right)^n\left(\lambda^{1/2}k(\xi', \lambda)x_N e^{-Bx_N}\right)\right| \\ &\leq C_{\alpha'}|\lambda|^{1/2}(|\lambda|^{1/2} + |\xi'|)^{1-|\alpha'|}x_N e^{-b(|\lambda|^{1/2}+|\xi'|)x_N} \\ &\leq C_{\alpha'}|\lambda|^{1/2}(|\lambda|^{1/2} + |\xi'|)^{-|\alpha'|}e^{-(b/2)(|\lambda|^{1/2}+|\xi'|)x_N},\end{aligned}$$

where $b = b_\epsilon$ of Lemma A.1. On the other hand, using the identity:

$$e^{ix'\cdot\xi'} = \sum_{|\alpha'|=j}\left(\frac{-ix'}{|x'|^2}\right)^{\alpha'}\partial^{\alpha'}_{\xi'}e^{ix'\cdot\xi'} \quad (j \in \mathbf{N}_0),$$

we have, by integration by parts,

$$\begin{aligned}\left(\lambda\frac{d}{d\lambda}\right)^n\mathcal{K}_\lambda(x) &= \frac{1}{(2\pi)^{N-1}}\int_{\mathbf{R}^{N-1}} e^{ix'\cdot\xi'}\left(\lambda\frac{d}{d\lambda}\right)^n\left(\lambda^{1/2}k(\xi',\lambda)x_Ne^{-Bx_N}\right)d\xi' \\ &= \frac{1}{(2\pi)^{N-1}}\sum_{|\alpha'|=N}\left(\frac{ix'}{|x'|^2}\right)^{\alpha'} \\ &\quad\cdot\int_{\mathbf{R}^{N-1}} e^{ix'\cdot\xi'}\partial_{\xi'}^{\alpha'}\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\lambda^{1/2}k(\xi',\lambda)x_Ne^{-Bx_N}\right)\right\}d\xi'.\end{aligned}$$

Combining this relation with (A.1) yields

$$\left|\left(\lambda\frac{d}{d\lambda}\right)^n\mathcal{K}_\lambda(x)\right| \le \frac{C}{|x'|^N}|\lambda|^{1/2}\int_{\mathbf{R}^{N-1}}(|\lambda|^{1/2}+|\xi'|)^{-N}\,d\xi',$$

which, combined with

$$\int_{\mathbf{R}^{N-1}}(|\lambda|^{1/2}+|\xi'|)^{-N}\,d\xi' = |\lambda|^{-1/2}\int_{\mathbf{R}^{N-1}}(1+|\eta'|)^{-N}\,d\eta',$$

furnishes

$$\left|\left(\lambda\frac{d}{d\lambda}\right)^n\mathcal{K}_\lambda(x)\right| \le \frac{C}{|x'|^N} \quad (x\in\mathbf{R}^N_+,\lambda\in\mathbf{C}_+). \tag{A.2}$$

By direct calculations, we have by (A.1) with $|\alpha'|=0$

$$\begin{aligned}\left|\left(\lambda\frac{d}{d\lambda}\right)^n\mathcal{K}_\lambda(x)\right| &\le C|\lambda|^{1/2}\int_{\mathbf{R}^{N-1}} e^{-(b/2)(|\lambda|^{1/2}+|\xi'|)x_N}\,d\xi' \\ &\le C(x_N)^{-1}\int_{\mathbf{R}^{N-1}} e^{-(b/2)|\xi'|x_N}\,d\xi' \\ &\le C(x_N)^{-N}.\end{aligned}$$

Combining this inequality with (A.2) yields

$$\left|\left(\lambda\frac{d}{d\lambda}\right)^n\mathcal{K}_\lambda(x)\right| \le \frac{C}{|x|^N} \quad (x\in\mathbf{R}^N_+,\lambda\in\mathbf{C}_+),$$

which, combined with Lemma A.2, proves that $K(\lambda)$ satisfies the required properties of Lemma A.4.

Case 2. This case considers $L(\lambda)$. Let $\mathcal{L}_\lambda(x)$ be given by

$$\mathcal{L}_\lambda(x) = \mathcal{L}_\lambda(x',x_N) = \mathcal{F}^{-1}_{\xi'}\left[|\xi'|l(\xi',\lambda)x_Ne^{-Bx_N}\right](x').$$

Then $L(\lambda)f$ is written as

$$[L(\lambda)f](x) = \int_{\mathbf{R}^N_+}\mathcal{L}_\lambda(x'-y',x_N+y_N)f(y)\,dy.$$

Since $l(\xi',\lambda)\in\mathbb{M}_{1,2}(\mathbf{C}_+)$, it holds by the Leibniz formula and Lemma A.1 that

$$\begin{aligned}&\left|\partial_{\xi'}^{\alpha'}\left(\lambda\frac{d}{d\lambda}\right)^n\left(|\xi'|l(\xi',\lambda)x_Ne^{-Bx_N}\right)\right| \\ &\le C_{\alpha'}|\xi'|^{1-|\alpha'|}(|\lambda|^{1/2}+|\xi'|)x_Ne^{-b(|\lambda|^{1/2}+|\xi'|)x_N} \\ &\le C_{\alpha'}|\xi'|^{1-|\alpha'|}e^{-(b/2)(|\lambda|^{1/2}+|\xi'|)x_N}.\end{aligned} \tag{A.3}$$

Applying [10, Theorem 2.3] to $\lambda^n(d/d\lambda)^n\mathcal{L}_\lambda(x)$ thus furnishes

$$\left|\left(\lambda\frac{d}{d\lambda}\right)^n\mathcal{L}_\lambda(x)\right|\leq\frac{C}{|x'|^N}\quad(x\in\mathbf{R}^N_+,\lambda\in\mathbf{C}_+).\tag{A.4}$$

Similarly to Case 1, we also have by (A.3) with $|\alpha'|=0$

$$\left|\left(\lambda\frac{d}{d\lambda}\right)^n\mathcal{L}_\lambda(x)\right|\leq C(x_N)^{-N}\quad(x\in\mathbf{R}^N_+,\lambda\in\mathbf{C}_+).$$

Combining this inequality with (A.4) yields

$$\left|\left(\lambda\frac{d}{d\lambda}\right)^n\mathcal{L}_\lambda(x)\right|\leq\frac{C}{|x|^N}\quad(x\in\mathbf{R}^N_+,\lambda\in\mathbf{C}_+),$$

which, combined with Lemma A.2, proves that $L(\lambda)$ satisfies the required properties of Lemma A.4.

Case 3. This case considers $M(\lambda)$. Since

$$m(\xi',\lambda)=\frac{B^2m(\xi',\lambda)}{B^2}=\lambda^{1/2}\frac{\lambda^{1/2}m(\xi',\lambda)}{B^2}+\sum_{j=1}^{N-1}|\xi'|\frac{\xi_j}{|\xi'|}\frac{\xi_jm(\xi',\lambda)}{B^2},$$

one observes that

$$\begin{aligned}&[M(\lambda)f](x)\\&=\int_0^\infty\mathcal{F}^{-1}_{\xi'}\left[\lambda^{1/2}\frac{\lambda^{1/2}m(\xi',\lambda)}{B^2}(x_N+y_N)e^{-B(x_N+y_N)}\widehat{f}(\xi',y_N)\right](x')\,dy_N\\&+\sum_{j=1}^{N-1}\int_0^\infty\mathcal{F}^{-1}_{\xi'}\left[|\xi'|\frac{\xi_j}{|\xi'|}\frac{\xi_jm(\xi',\lambda)}{B^2}(x_N+y_N)e^{-B(x_N+y_N)}\widehat{f}(\xi',y_N)\right](x')\,dy_N.\end{aligned}$$

It then holds by Lemmas 2.1 and A.1 that

$$\frac{\lambda^{1/2}m(\xi',\lambda)}{B^2}\in\mathbb{M}_{1,1}(\mathbf{C}_+),\qquad\frac{\xi_j}{|\xi'|}\frac{\xi_jm(\xi',\lambda)}{B^2}\in\mathbb{M}_{1,2}(\mathbf{C}_+),$$

which, combined with Lemma 2.9 and the results obtained in Cases 1 and 2, proves that $M(\lambda)$ satisfies the required properties of Lemma A.4. This completes the proof of the lemma. □

From now on, we prove Lemma 5.4[4]. For functions $g(s)$ and $h(s)$ $(s\geq0)$ with $\lim_{y_N\to\infty}g(x_N+y_N)h(y_N)=0$, one notes that

$$\begin{aligned}g(x_N)h(0)&=-\int_0^\infty\frac{\partial}{\partial y_N}\left(g(x_N+y_N)h(y_N)\right)\,dy_N\\&=-\int_0^\infty g'(x_N+y_N)h(y_N)\,dy_N\\&\quad-\int_0^\infty g(x_N+y_N)h'(y_N)\,dy_N,\end{aligned}$$

where $g'(s)=(dg/ds)(s)$ and $h'(s)=(dh/ds)(s)$. By this relation,

$$[W_2(\lambda)f](x)=-\int_0^\infty\mathcal{F}^{-1}_{\xi'}\left[w_2(\xi',\lambda)e^{-B(x_N+y_N)}\widehat{f}(\xi',y_N)\right](x')$$

[4] We here prove Lemma 5.4 (2) only. The proof of (1) is similar to one of (2).

$$+\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[w_2(\xi',\lambda)B(x_N+y_N)e^{-B(x_N+y_N)}\widehat{f}(\xi',y_N)\right](x')$$
$$-\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[w_2(\xi',\lambda)(x_N+y_N)e^{-B(x_N+y_N)}\widehat{\partial_N f}(\xi',y_N)\right](x'),$$

which, combined with

$$1=\frac{B^2}{B^2}=\frac{\lambda}{B^2}-\sum_{j=1}^{N-1}\frac{(i\xi_j)^2}{B^2},\quad B=\frac{B^2}{B}=\frac{\lambda}{B}-\sum_{j=1}^{N-1}\frac{(i\xi_j)^2}{B},$$

furnishes that

$$\begin{aligned}
[W_2(\lambda)f](x) &= -\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{w_2(\xi',\lambda)}{B^2}e^{-B(x_N+y_N)}\widehat{\lambda f}(\xi',y_N)\right](x')\\
&+\sum_{j=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{w_2(\xi',\lambda)}{B^2}e^{-B(x_N+y_N)}\widehat{\partial_j^2 f}(\xi',y_N)\right](x')\\
&+\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{w_2(\xi',\lambda)}{B}(x_N+y_N)e^{-B(x_N+y_N)}\widehat{\lambda f}(\xi',y_N)\right](x')\\
&-\sum_{j=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{w_2(\xi',\lambda)}{B}(x_N+y_N)e^{-B(x_N+y_N)}\widehat{\partial_j^2 f}(\xi',y_N)\right](x')\\
&-\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{\lambda^{1/2}w_2(\xi',\lambda)}{B^2}(x_N+y_N)e^{-B(x_N+y_N)}\widehat{\lambda^{\frac12}\partial_N f}(\xi',y_N)\right](x')\\
&+\sum_{j=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{i\xi_j w_2(\xi',\lambda)}{B^2}(x_N+y_N)e^{-B(x_N+y_N)}\widehat{\partial_j\partial_N f}(\xi',y_N)\right](x').
\end{aligned}$$

From this viewpoint, for

$$\mathbf{F}=(\{F_{jk}\}_{j,k=1}^N,\{G_j\}_{j=1}^N,H)\in L_q(\mathbf{R}^N_+)^{N^2}\times L_q(\mathbf{R}^N_+)^N\times L_q(\mathbf{R}^N_+),$$

let us define

$$\begin{aligned}
Z_1(\lambda)\mathbf{F} &= -\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{w_2(\xi',\lambda)}{B^2}e^{-B(x_N+y_N)}\widehat{H}(\xi',y_N)\right](x'),\\
Z_2(\lambda)\mathbf{F} &= \sum_{j=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{w_2(\xi',\lambda)}{B^2}e^{-B(x_N+y_N)}\widehat{F_{jj}}(\xi',y_N)\right](x'),\\
Z_3(\lambda)\mathbf{F} &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{w_2(\xi',\lambda)}{B}(x_N+y_N)e^{-B(x_N+y_N)}\widehat{H}(\xi',y_N)\right](x'),\\
Z_4(\lambda)\mathbf{F} &= -\sum_{j=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{w_2(\xi',\lambda)}{B}(x_N+y_N)e^{-B(x_N+y_N)}\widehat{F_{jj}}(\xi',y_N)\right](x'),\\
Z_5(\lambda)\mathbf{F} &= -\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{\lambda^{1/2}w_2(\xi',\lambda)}{B^2}(x_N+y_N)e^{-B(x_N+y_N)}\widehat{G_N}(\xi',y_N)\right](x'),\\
Z_6(\lambda)\mathbf{F} &= \sum_{j=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{i\xi_j w_2(\xi',\lambda)}{B^2}(x_N+y_N)e^{-B(x_N+y_N)}\widehat{F_{jN}}(\xi',y_N)\right](x').
\end{aligned}$$

It is then clear that

$$W_2(\lambda)f = \sum_{k=1}^{6} Z_k(\lambda)(\nabla^2 f, \lambda^{1/2} f, \lambda f).$$

In addition, one has by $w_2(\xi', \lambda) \in \mathbb{M}_{1,1}(\mathbf{C}_+)$ and Lemmas 2.1 and A.1

$$\frac{w_2(\xi', \lambda)}{B^2} \in \mathbb{M}_{-1,1}(\mathbf{C}_+),$$

which, combined with Lemma A.3, furnishes that for $k = 1, 2$ and $n = 0, 1$

$$Z_k(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(L_q(\mathbf{R}_+^N)^{N^2+N+1}, H_q^2(\mathbf{R}_+^N))),$$
$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}_+^N)^{N^2+N+1})}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n (\mathcal{T}_\lambda Z_k(\lambda)) \,\middle|\, \lambda \in \mathbf{C}_+\right\}\right) \le C_{N,q},$$

with some positive constant $C_{N,q}$.

Next, we consider $Z_3(\lambda)$. By direct calculations, we see for $j, k = 1, \dots, N-1$

$$\begin{aligned}
\partial_N^2 Z_3(\lambda)\mathbf{F} &= -2\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[w_2(\xi', \lambda)e^{-B(x_N+y_N)}\widehat{H}(\xi', y_N)\right](x') \\
&\quad + \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[w_2(\xi', \lambda)B(x_N+y_N)e^{-B(x_N+y_N)}\widehat{H}(\xi', y_N)\right](x'), \\
\partial_j\partial_N Z_3(\lambda)\mathbf{F} &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{i\xi_j w_2(\xi', \lambda)}{B}e^{-B(x_N+y_N)}\widehat{H}(\xi', y_N)\right](x') \\
&\quad - \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[i\xi_j w_2(\xi', \lambda)(x_N+y_N)e^{-B(x_N+y_N)}\widehat{H}(\xi', y_N)\right](x'), \\
\partial_j\partial_k Z_3(\lambda)\mathbf{F} &= -\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\xi_j\xi_k w_2(\xi', \lambda)}{B}(x_N+y_N)e^{-B(x_N+y_N)}\widehat{H}(\xi', y_N)\right](x'), \\
\lambda^{1/2}\partial_N Z_3(\lambda)\mathbf{F} &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\lambda^{1/2} w_2(\xi', \lambda)}{B}e^{-B(x_N+y_N)}\widehat{H}(\xi', y_N)\right](x') \\
&\quad - \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\lambda^{1/2} w_2(\xi', \lambda)(x_N+y_N)e^{-B(x_N+y_N)}\widehat{H}(\xi', y_N)\right](x'), \\
\lambda^{1/2}\partial_j Z_3(\lambda)\mathbf{F} &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\lambda^{1/2} i\xi_j w_2(\xi', \lambda)}{B}(x_N+y_N)e^{-B(x_N+y_N)}\widehat{H}(\xi', y_N)\right](x'), \\
\lambda Z_3(\lambda)\mathbf{F} &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\lambda w_2(\xi', \lambda)}{B}(x_N+y_N)e^{-B(x_N+y_N)}\widehat{H}(\xi', y_N)\right](x').
\end{aligned}$$

Note that by $w_2(\xi', \lambda) \in \mathbb{M}_{1,1}(\mathbf{C}_+)$ and Lemmas 2.1 and A.1

$$\begin{aligned}
&\frac{i\xi_j w_2(\xi', \lambda)}{B},\ \frac{\lambda^{1/2} w_2(\xi', \lambda)}{B} \in \mathbb{M}_{1,1}(\mathbf{C}_+); \\
&w_2(\xi', \lambda)B,\ i\xi_j w_2(\xi', \lambda),\ \frac{\xi_j\xi_k w_2(\xi', \lambda)}{B},\ \lambda^{1/2} w_2(\xi', \lambda), \\
&\frac{\lambda^{1/2} i\xi_j w_2(\xi', \lambda)}{B},\ \frac{\lambda w_2(\xi', \lambda)}{B} \in \mathbb{M}_{2,1}(\mathbf{C}_+).
\end{aligned}$$

One thus observes by Lemmas A.3 and A.4 that for $n = 0, 1$

$$Z_3(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(L_q(\mathbf{R}_+^N)^{N^2+N+1}, H_q^2(\mathbf{R}_+^N))),$$
$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}_+^N)^{N^2+N+1})}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n (\mathcal{T}_\lambda Z_3(\lambda)) \,\middle|\, \lambda \in \mathbf{C}_+\right\}\right) \le C_{N,q}.$$

Similarly to $Z_3(\lambda)$, it holds that for $k = 4, 5, 6$ and $n = 0, 1$

$$Z_k(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(L_q(\mathbf{R}^N_+)^{N^2+N+1}, H^2_q(\mathbf{R}^N_+))),$$

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+)^{N^2+N+1})}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n (\mathcal{T}_\lambda Z_k(\lambda))\,\middle|\,\lambda\in\mathbf{C}_+\right\}\right) \leq C_{N,q}.$$

This completes the proof of Lemma 5.4.

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