Compendium

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Sommario

Il seguente documento contiene alcuni appunti di matematica e fisica del corso delle scienze applicate. Il materiale potrebbe essere incompleto o incoretto in alcune parti, esso non è da intendersi come materiale didattico. L'utilizzo è a discrezione del lettore. Si prega di contattare il curatore del documento nel caso si riscontrino errori o incorrettezze nel testo.

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Matematica

Capitolo 1

Analisi

1.1 Premesse all'analisi

1.1.1 Distanze e Intorni

Definizione 1.1. Si dice che un insieme $I \subseteq \mathbb{R}$ è un **intervallo** se per ogni terna di numeri $x_1 < x_2 < x_3$ si ha che $x_1, x_3 \in I \Rightarrow x_2 \in I$

I seguenti sono esempi di diversi tipi di intervalli:

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$
$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$
$$[a, +\infty) = \{x \in \mathbb{R} \mid a \leq x\}$$
$$(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$$

L'intorno di un punto è un insieme all'interno del quale ci si può avvicinare a piacere al punto

Definizione 1.2. $U(x_0)$ intorno di x_0 è un qualsiasi intervallo aperto che contenga x_0

Esempio: $(3, 7)$ è $U(4)$, ovvero un intorno di 4
Per andare a studiare gli intorni anche all'infinito introduciamo un'estensione dell'insieme $\mathbb{R}$

$$\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$$
$$U(+\infty) = (M, +\infty)$$
$$U(-\infty) = (-\infty, M)$$

Un intorno particolare è l'intorno circolare, ovvero l'insieme dei numeri che hanno distanza da un punto minore di una certa quantità. Tali intorni sono utili e vastamente utilizzati nello studio dei limiti

Definizione 1.3. Dato un punto $x_0 \in \mathbb{R}$ si dice **intorno circolare** di x_0 di **raggio** δ ogni intervallo del tipo $(x_0 - \delta, x_0 + \delta)$, con δ un qualsiasi numero positivo

$$I = \{x \in \mathbb{R} \mid x_0 - \delta < x < x_0 + \delta\} = (x_0 - \delta, x_0 + \delta)$$
$$x \in I \Leftrightarrow |x - x_0| < \delta$$

Si può parlare anche di intorno destro e sinistro

$$U^+ = \{x \in \mathbb{R} \mid x_0 < x < x_0 + \delta\} = (x_0, x_0 + \delta)$$
$$U^- = \{x \in \mathbb{R} \mid x_0 - \delta < x < x_0\} = (x_0 - \delta, x_0)$$

Definizione 1.4. Sia $A \subset \mathbb{R}$

- A è **superiormente limitato** se $\exists k \in \mathbb{R} \mid x \leq k \ \forall x \in A$
- A è **inferiormente limitato** se $\exists k \in \mathbb{R} \mid k \leq x \ \forall x \in A$
- A è **limitato** se è superiormente e inferiormente limitato
- b è **maggiorante** per A se $x \leq b \ \forall x \in A$
- b è **minorante** per A se $b \leq x \ \forall x \in A$
- dei maggioranti il più significativo è il più piccolo detto estremo superiore (**SUP**)
- dei minoranti il più significativo è il più grande detto estremo inferiore(**INF**)
- se il $SUP \in A$ è detto **massimo**
- se il $INF \in A$ è detto **minimo**

Il fatto che un numero sia SUP (o INF) di A significa che l'insieme A si può avvicinare a picare al SUP

Teorema 1.1. *Sia A un intervallo che ammetta SUP*

$$\forall \epsilon > 0 \ \exists a \in A \mid SUP - \epsilon < a \leq SUP$$

Dimostrazione. Per assurdo supponiamo che

$$\exists \bar{\epsilon} > 0 | \nexists a \in A \ , \ SUP - \bar{\epsilon} < a \leq SUP$$

ovvero

$$SUP - \bar{\epsilon} > a \forall a \in A$$

Quindi secondo la definizione 1.4 di maggiorante, $SUP - \bar{\epsilon}$ è maggiorante di A. Ma allora $SUP - \bar{\epsilon} < SUP$, che però contraddice il fatto che SUP sia il più piccolo dei maggioranti, secondo la definizione 1.4 di SUP. Dunque la supposizione fatta per assurdo è falsa. QED

Definizione 1.5. Sia $A \subset \mathbb{R}$

- $x_0 \in A$ è un **punto isolato** se

$$\exists U = U(x_0) | U \cap A = \{x_0\}$$

- $x_0 \in A$ è un **punto di accumulazione** se

$$\forall U = U(x_0) \quad U \cap (A \backslash \{x_0\}) \neq \varnothing$$

1.2 Limiti

1.2.1 Definizione e concetti di base

Definizione 1.6 (Limite)**.** Sia $f : D \to \mathbb{R}$ e $x_0, L \in \bar{\mathbb{R}}$

$$\lim_{x\to x_0} f(x) = L \qquad \Longleftrightarrow \qquad \forall V = V(L) \quad \exists U = U(x_0) \mid \forall x \in U \cap D \setminus \{x_0\} \to f(x) \in V$$

Nota. Affinchè si possa andare a determinare un limite il punto deve essere un punto di accumulazione

Alcuni esempi di definizioni di limiti:

- x_0, L finiti

$$V = (L-\epsilon, L+\epsilon) \quad U = (x_0-\delta, x_0+\delta)$$

$$\lim_{x\to x_0} f(x) = L \qquad \Longleftrightarrow \qquad \forall\epsilon > 0 \quad \exists\delta > 0,\ \delta = \delta(\epsilon) \mid \forall x, 0 < |x-x_0| < \delta \to |f(x)-L| < \epsilon$$

- x_0, L infiniti

$$V = (N, +\infty) \quad U = (M, +\infty)$$

$$\lim_{x\to\pm\infty} f(x) = \pm\infty \qquad \Longleftrightarrow \qquad \forall N > 0 \quad \exists M > 0,\ M = M(\epsilon) \mid \forall x, x > M \to f(x) > N$$

Si può parlare anche di limite **destro** o **sinistro**

$$\lim_{x\to x_0^\pm} f(x) = L \qquad \Longleftrightarrow \qquad \forall V = V(L) \quad \exists U = U^\pm(x_0) \mid \forall x \in U \cap D \backslash \{x_0\} \to f(x) \in V$$

e di limite che tende per **eccesso** o **difetto**

$$\lim_{x\to x_0} f(x) = L^\pm \qquad \Longleftrightarrow \qquad \forall V = V^\pm(L) \quad \exists U = U(x_0) \mid \forall x \in U \cap D \backslash \{x_0\} \to f(x) \in V$$

Dalla definizione di limite destro e sinistro si può arrivare a una ridefinizione di limite:

$$f : D \to \mathbb{R} \qquad L, x_0 \in \bar{\mathbb{R}}$$

$$\lim_{x\to x_0} f(x) = L \Leftrightarrow \begin{cases} \lim_{x\to x_0^+} f(x) = L_1 \\ \lim_{x\to x_0^-} f(x) = L_2 \\ \qquad\qquad L_1 = L_2 \end{cases}$$

Definizione 1.7. Sia $f : D \to \mathbb{R}$
$r: \ y = k \quad k \in \mathbb{R}$ è **asintoto orizzontale** a $\pm\infty$ se

$$\lim_{x\to\pm\infty} f(x) = k$$

$r: \ x = x_0 \quad x_0 \in \mathbb{R}$ è **asintoto verticale** se vale una delle seguenti

$$\lim_{x\to x_0^+} f(x) = \pm\infty$$

$$\lim_{x\to x_0^-} f(x) = \pm\infty$$

Teorema 1.2 (Unicità del limite). *Sia* $f : D \to \mathbb{R} \qquad x_0 \in \bar{\mathbb{R}}$

$$\exists!\ L \in \bar{\mathbb{R}} \mid \lim_{x \to x_0} f(x) = L$$

Dimostrazione. Per assurdo supponiamo che

$$\exists L \neq L' \in \bar{\mathbb{R}} \mid \lim_{x \to x_0} f(x) = L \quad \text{e} \quad \lim_{x \to x_0} f(x) = L'$$

allora

$$\exists V_1 = V(L), V_2 = V(L') \mid V_1 \cap V_2 = \varnothing$$

Per definizione 1.6 di limite

$$\lim_{x \to x_0} f(x) = L \quad \Longrightarrow \quad \exists U_1 = U(x_0) \mid \forall x \in U_1 \cap D \backslash \{x_0\} \to f(x) \in V_1$$

$$\lim_{x \to x_0} f(x) = L' \quad \Longrightarrow \quad \exists U_2 = U(x_0) \mid \forall x \in U_2 \cap D \backslash \{x_0\} \to f(x) \in V_2$$

Ma allora si ottiene $U_1 \cap U_2 = U(x_0)$ e x_0 punto di accumulazione, il che implica

$$\exists x^* \in (U_1 \cap U_2) \cap D \backslash \{x_0\} \Rightarrow f(x^*) \in V_1,\ f(x^*) \in V_2$$

Questo contraddice il fatto che $V_1 \cap V_2 = \varnothing$, dunque la supposizione fatta per assurdo è falsa. QED

1.2.2 Operazioni

Teorema 1.3 (Somma di limiti). *Siano* $f : D \to \mathbb{R}$ *e* $g : D' \to \mathbb{R}$ *due funzioni tali che* $\exists U = U(x_0) | U \subset D \cap D'$. *Se* $\lim_{x \to x_0} f(x) = L$ *e* $\lim_{x \to x_0} g(x) = L'$ *e* L,L' *sono finiti, allora*

$$\lim_{x \to x_0} (f(x) + g(x)) = L + L'$$

Dimostrazione. Per la definizione 1.6 di limite si ha

- $\lim_{x \to x_0} f(x) = L \quad \Leftrightarrow \quad \forall \epsilon > 0 \quad \exists \delta_f > 0,\ \delta_f = \delta_f(\epsilon) \mid \forall x, 0 < |x - x_0| < \delta_f \to$
$$|f(x) - L| < \epsilon \tag{1.1}$$
- $\lim_{x \to x_0} g(x) = L' \quad \Leftrightarrow \quad \forall \epsilon > 0 \quad \exists \delta_g > 0,\ \delta_g = \delta_g(\epsilon) \mid \forall x, 0 < |x - x_0| < \delta_g \to$
$$|g(x) - L'| < \epsilon \tag{1.2}$$

Valutiamo quindi $|f(x) + g(x) - (L + L')|$, per la diugualianza triangolare vale

$$|f(x) + g(x) - (L + L')| \leq |f(x) - L| + |g(x) - L'|$$

Prendendo $\delta = \min(\delta_f, \delta g)$ e considerando valori $x \neq x_0 \mid |x - x_0| < \delta$, sfruttando le disugualianze (1.1), (1.2); si ottiene

$$|f(x) + g(x) - (L + L')| \leq |f(x) - L| + |g(x) - L'| < \epsilon + \epsilon = 2\epsilon$$

Data l'arbitrarietà di ϵ il limite è dimostrato QED

Discutiamo ora i casi con limiti infiniti

- Un limite finito e l'altro infinito
  $\lim_{x\to x_0} f(x) = L$ e $\lim_{x\to x_0} g(x) = +\infty$
  $$\lim_{x\to x_0} (f(x) + g(x)) = +\infty$$
- Entrambi i limiti infiniti **concordi**
  $\lim_{x\to x_0} f(x) = +\infty$ e $\lim_{x\to x_0} g(x) = +\infty$
  $$\lim_{x\to x_0} (f(x) + g(x)) = +\infty$$
- Se i due limiti infiniti **discordi**
  $$\lim_{x\to x_0} (f(x) + g(x)) = -\infty$$
  non è possibile dire nulla a priori della somma dei limiti. Si dice che $[+\infty - \infty]$ è una **forma indeterminata**

Teorema 1.4 (Prodotto di limiti)**.** *Siano $f : D \to \mathbb{R}$ e $g : D' \to \mathbb{R}$ due funzioni tali che $\exists U = U(x_0)|U \subset D \cap D'$. Se $\lim_{x\to x_0} f(x) = L$ e $\lim_{x\to x_0} g(x) = L'$ e L,L' sono finiti, allora*

$$\lim_{x\to x_0} (f(x) \cdot g(x)) = L \cdot L'$$

Si ha la forma indeterminata $[0 \cdot \infty]$

Teorema 1.5 (Reciproco di limite)**.** *Siano $f : D \to \mathbb{R}$ una funzione tale che $\exists U = U(x_0)|U \subset D \cap D'$. Se $\lim_{x\to x_0} f(x) = L$ e $L \neq 0$ finito, allora*

$$\lim_{x\to x_0} \frac{1}{f(x)} = \frac{1}{L}$$

Discutiamo ora i casi particolari

- Se $L = 0$ non è possibile calcolare il reciproco perchè la divisione non è definita, ma è dimostrabile che
  $$\lim_{x\to x_0} f(x) = 0^{\pm} \;\Rightarrow\; \lim_{x\to x_0} \frac{1}{f(x)} = \pm\infty$$
  Il che **non significa che** $\frac{1}{0} = \infty$
- Se $L = \infty$ non è possibile calcolare il reciproco perchè la divisione non è definita, ma è dimostrabile che
  $$\lim_{x\to x_0} f(x) = \infty \;\Rightarrow\; \lim_{x\to x_0} \frac{1}{f(x)} = 0$$
  Il che **non significa che** $\frac{1}{\infty} = 0$

Teorema 1.6 (Rapporto di limiti)**.** *Siano $f : D \to \mathbb{R}$ e $g : D' \to \mathbb{R}$ due funzioni tali che $\exists U = U(x_0)|U \subset D \cap D'$. Se $\lim_{x\to x_0} f(x) = L$ e $\lim_{x\to x_0} g(x) = L'$ e $L, L' \neq 0$ sono finiti, allora*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{L'}$$

Si hanno le forme indeterminate $\left[\frac{0}{0}\right]$ e $\left[\frac{\infty}{\infty}\right]$

Teorema 1.7 (Potenze di limiti)**.** *Siano* $f : D \to \mathbb{R}$ *e* $g : D' \to \mathbb{R}$ *due funzioni tali che* $\exists U = U(x_0) | U \subset D \cap D'$*. Se* $\lim_{x \to x_0} f(x) = L > 0$ *e* $\lim_{x \to x_0} g(x) = L'$ *e* L*,* L' *sono finiti, allora*

$$\lim_{x \to x_0} f(x)^{g(x)} = L^{L'}$$

Si hanno le forme indeterminate $\left[0^0\right]$, $\left[\infty^0\right]$ e $[1^\infty]$

1.2.3 Teoremi fondamentali

Teorema 1.8 (Permanenza del segno)**.** *Sia* $f(x)$ *una funzione definita su un intervallo* I *e* x_0 *un punto appartenente a* I *o un suo estremo.*

$$\lim_{x \to x_0} f(x) > 0 \ (< 0) \ \Rightarrow \quad \exists U = U(x_0) \mid \forall x \in U \cap I \backslash \{x_0\} \quad f(x) > 0 \ (< 0)$$

Dimostrazione. Supponiamo $\lim_{x \to x_0} f(x) = L > 0$. Per la definizione 1.6 di limite si ha

$$\lim_{x \to x_0} f(x) = L \quad \Leftrightarrow \quad \forall \epsilon > 0 \quad \exists \delta > 0, \ \delta = \delta(\epsilon) \mid \forall x, 0 < |x - x_0| < \delta \to$$

$$L - \epsilon < f(x) < L + \epsilon \tag{1.3}$$

Essendo $L > 0$ è possibile porre $\epsilon = L$; allora la disequazione (1.3) assume la forma $0 < f(x) < 2L$. In particolare $0 < |x - x_0| < \delta \Rightarrow \quad f(x) > 0$ QED

Nota. Intorni più larghi di $f(x)$ possono assumere segno opposto

Teorema 1.9 (Opposto della permanenza del segno)**.** *Sia* $f(x)$ *una funzione definita su un intervallo* I *e* x_0 *un punto appartenente a* I *o un suo estremo.*

$$\exists U = U(x_0) \mid \forall x \in U \cap I \backslash \{x_0\} \quad f(x) \geq 0 \ (\leq 0) \ \Rightarrow \quad \lim_{x \to x_0} f(x) \geq 0 \ (\leq 0)$$

Dimostrazione. Per assurdo supponiamo che $\lim_{x \to x_0} f(x) < 0$. Allora per il teorema 1.8 della permanenza del segno si ha che $\exists U' = U(x_0) \mid \forall x \in U' \quad f(x) < 0$. Posto $U'' = U \cap U'$, si ha che

$$x \in U'' \subset U \Rightarrow f(x) \geq 0$$

$$x \in U'' \subset U' \Rightarrow f(x) < 0$$

Si giunge quindi a una contraddizione, il che implica che la supposizione fatta per assurdo è falsa. QED

Teorema 1.10 (Teorema del confronto)**.** *Siano* $f(x), g(x), h(x)$ *funzioni definite ciascuna su un intervallo e l'intersezione dei tre intervalli contenga* $U = U(x_0) \mid \forall x \in U \quad f(x) \leq h(x) \leq g(x)$*. Allora*

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} f(x) = L \quad \Longrightarrow \quad \lim_{x \to x_0} h(x) = L$$

Dimostrazione. Per la definizione 1.6 di limite si ha

- $\lim_{x \to x_0} f(x) = L \quad \Leftrightarrow \quad \forall \epsilon > 0 \quad \exists \delta_f > 0,\ \delta_f = \delta_f(\epsilon) \mid \forall x, 0 < |x - x_0| < \delta_f \rightarrow$
$$L - \epsilon < f(x) < L + \epsilon \tag{1.4}$$
- $\lim_{x \to x_0} g(x) = L \quad \Leftrightarrow \quad \forall \epsilon > 0 \quad \exists \delta_g > 0,\ \delta_g = \delta_g(\epsilon) \mid \forall x, 0 < |x - x_0| < \delta_g \rightarrow$
$$L - \epsilon < g(x) < L + \epsilon \tag{1.5}$$

Prendendo $\delta = \min(\delta_f, \delta g)$ e considerando valori $x \neq x_0 \mid |x - x_0| < \delta$, sfruttando le disugualianze (1.4), (1.5); si ottiene
$$L - \epsilon < f(x) \quad \leq h(x) \leq \quad g(x) < L + \epsilon$$
In particolare $L - \epsilon < h(x) < L + \epsilon\ \forall \epsilon > 0$, che corrisponde alla definizione 1.6 di limite; dunque si può affermare che $\lim_{x \to x_0} h(x) = L$ QED

1.2.4 Limiti notevoli

- $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x} = 1$
- $\lim\limits_{x \to 0} \dfrac{1 - \cos(x)}{x^2} = \dfrac{1}{2}$
- $\lim\limits_{x \to 0} (1 + x)^{\frac{1}{x}} = e$
- $\lim\limits_{x \to 0} \dfrac{\log_a(1 + x)}{x} = \log_a e$
- $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \ln a$

Parte II

Fisica

Capitolo 2

Relatività

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