

# Notes on Bethe Ansatz

Mikhail Zvonarev

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Coordinate and Algebraic Bethe Ansatz techniques are reviewed; the former is applied to the Lieb-Liniger model, and the latter to the Heisenberg model. Suggestions for the further reading are given.

## 1 Coordinate Bethe Ansatz

Bethe Ansatz means “Bethe’s substitution”, the method is named after the work by Hans Bethe [1]. Bethe found eigenfunctions and spectrum of the one-dimensional spin-1/2 isotropic magnet (often called one-dimensional spin-1/2 isotropic Heisenberg model). I will demonstrate the usage of the Algebraic Bethe Ansatz on this model in the second part of my lecture.

Next model was solved by coordinate Bethe Ansatz more than 30 years later [2]. This is a model of one-dimensional gas of bosons interacting via the  $\delta$ -function potential. Since it was solved by Lieb and Liniger, it is often called the Lieb-Liniger model. I will demonstrate how coordinate Bethe Ansatz works using this model as an example.

### 1.1 Hamiltonian of the Lieb-Liniger model in the first and second-quantized form

The Hamiltonian of the one-dimensional Bose gas with zero-range interaction potential is

$$H = \int dx [-\Psi^\dagger(x) \partial_x^2 \Psi(x) + c \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x)], \quad (1.1)$$

where the units have been chosen such that  $\hbar = 2m = 1$ . The interaction constant  $c \geq 0$  has the dimension of inverse length. The dimensionless coupling strength  $\gamma$  is given by

$$\gamma = \frac{c}{\rho}. \quad (1.2)$$

The boson fields  $\Psi$  and  $\Psi^\dagger$  satisfy canonical equal-time commutation relations:

$$[\Psi(x), \Psi^\dagger(y)] = \delta(x - y), \quad [\Psi^\dagger(x), \Psi^\dagger(y)] = [\Psi(x), \Psi(y)] = 0. \quad (1.3)$$

The number of particles operator  $N$  is

$$N = \int dx \Psi^\dagger(x) \Psi(x), \quad (1.4)$$

and the momentum operator  $P$  is

$$P = -i \int dx \Psi^\dagger(x) \partial_x \Psi(x). \quad (1.5)$$

The number of particles and momentum are integrals of motion:

$$[H, N] = [H, P] = 0. \quad (1.6)$$

The equation of motion for the Hamiltonian (1.1)

$$i\partial_t \Psi = -\partial_x^2 \Psi + 2c \Psi^\dagger \Psi \Psi \quad (1.7)$$

is called the quantum nonlinear Schrödinger equation.

Since the Hamiltonian commutes with  $N$ , Eq. (1.6), the number of particles in the model is conserved and one can look for the eigenfunctions of the Hamiltonian (1.1) in a sector with the fixed number of particles. Write

$$|\Psi_N(k_1, \dots, k_N)\rangle = \frac{1}{\sqrt{N!}} \int d^N x \chi_N(x_1, \dots, x_N | k_1, \dots, k_N) \Psi^\dagger(x_1) \dots \Psi^\dagger(x_N) |0\rangle. \quad (1.8)$$

Here  $|\Psi_N(k_1, \dots, k_N)\rangle$  is an eigenfunction of the second-quantized Hamiltonian (1.1):

$$H |\Psi_N(k_1, \dots, k_N)\rangle = E_N(k_1, \dots, k_N) |\Psi_N(k_1, \dots, k_N)\rangle, \quad (1.9)$$

the symbol  $|0\rangle$  denotes the Fock vacuum state

$$\Psi(x) |0\rangle = 0, \quad \langle 0 | \Psi^\dagger(x) = 0, \quad \langle 0 | 0 \rangle = 1, \quad (1.10)$$

the symbol  $d^N x \equiv dx_1 \dots dx_N$ , and the parameters  $k_1, \dots, k_N$  are called quasi-momenta. The coordinate wave function  $\chi_N$  is an eigenfunction of the Hamiltonian (1.1) written in the first-quantized form:

$$\mathcal{H}_N = - \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + 2c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j), \quad (1.11)$$

$$\mathcal{H}_N \chi_N = E_N(k_1, \dots, k_N) \chi_N. \quad (1.12)$$

The Bose statistics of the model is encoded in the symmetry properties of  $\chi_N$ : it should be symmetric with respect to the arbitrary permutations of the coordinates  $x_1, \dots, x_N$ . Finally, write the momentum operator (1.5) in the first-quantized form:

$$\mathcal{P}_N = -i \sum_{j=1}^N \frac{\partial}{\partial x_j}. \quad (1.13)$$

No boundary conditions were imposed so far. Since it is our purpose to get the gas at a finite density in the limit  $N \rightarrow \infty$ , we should compactify the system in the space direction

when working at finite  $N$ . This can be done in several different ways: one can put the system on a ring of circumference  $L$  and impose the periodic boundary conditions, or put it in the hard-walls box of length  $L$ , or, more generally, impose twisted boundary conditions. The influence of the boundary conditions should disappear in the thermodynamic limit, but for finite  $N$  they are, of course, affect the results. The exact eigenfunctions of the model are known for the general case of the twisted boundary conditions. The particular case of the periodic boundary conditions is, however, enough for our purposes and will be assumed henceforth.

## 1.2 Lieb-Liniger model in the $c \rightarrow \infty$ limit: Tonks-Girardeau gas

The limit of infinite repulsion,  $c \rightarrow \infty$ , is often called the Tonks-Girardeau limit [3, 4]. We will consider this limit in full details since it will give us a lot of intuition about the behavior of the model at arbitrary  $c$ .

### 1.2.1 Eigenfunctions and spectrum of the Tonks-Girardeau gas

The eigenfunctions of the Hamiltonian (1.11) should go to zero when  $x_i \rightarrow x_j$  for arbitrary  $i \neq j$ :

$$\chi_N(x_1, \dots, x_i, \dots, x_j, \dots, x_N) \rightarrow 0, \quad x_i \rightarrow x_j, \quad i \neq j, \quad (1.14)$$

and they should solve the free Schrödinger equation when all  $x_j$  are different. A possible candidate is the determinant of an  $N \times N$  matrix with the entries  $\exp\{ik_j z_l\}$ :

$$\det[\exp\{ik_j x_l\}]. \quad (1.15)$$

This function should be symmetrized in coordinates:

$$\chi_N(x_1, \dots, x_N | k_1, \dots, k_N) = \frac{C}{\sqrt{N!}} \det[\exp\{ik_j x_l\}] \prod_{1 \leq l < j \leq N} \operatorname{sgn}(x_j - x_l), \quad (1.16)$$

where  $C$  is a normalization constant. It is easy to check that (1.16) is a common eigenfunction of the operators  $\mathcal{H}$ ,  $\mathcal{P}$  and  $N$  with the corresponding eigenvalues being equal to

$$E_N = \sum_{j=1}^N k_j^2, \quad P_N = \sum_{j=1}^N k_j. \quad (1.17)$$

In getting (1.17) we did not specify the boundary conditions. Let us now put a system on a ring of circumference  $L$  and impose the periodic boundary conditions. The wave function (1.16) should thus satisfy

$$\chi_N(x_1, \dots, x_j = 0, \dots, x_N | k_1, \dots, k_N) = \chi_N(x_1, \dots, x_j = L, \dots, x_N | k_1, \dots, k_N) \quad (1.18)$$

for all  $j = 1, \dots, N$ . This implies the following conditions on the quasimomenta  $k_j$ :

$$\exp\{ik_j L\} = (-1)^{N-1}, \quad j = 1, \dots, N. \quad (1.19)$$

These equations are called Bethe equations; in this particular case they can be solved easily:

$$k_j = \frac{2\pi}{L} n_j, \quad j = 1, \dots, N. \quad (1.20)$$

Here  $n_j$  are arbitrary odd numbers for  $N$  even and arbitrary even for  $N$  odd. Apart from this selection rule, the quantization condition for quasimomenta are the same as for free fermions: the allowed values of  $k_j$  are equidistant in the momentum space and multiple occupation of the same position in the momentum space is prohibited: the determinant in Eq. (1.19) becomes equal to zero. The fermion-like behavior of this boson system is very typical for the quantum systems in one spacial dimension; this phenomenon is called statistical transmutation. It should be stressed, however, that the Tonks-Girardeau gas is not a system of the non-interacting fermions: the wave function (1.19) is symmetric in coordinates, while the wave functions of the fermion system should be antisymmetric. Thus, correlation properties of the Tonks-Girardeau gas are very different in many aspects from those of the free fermion system. An example illustrating this difference, the one-particle density matrix, will be studied in section 1.5.

The norm of the wave function (1.19) with the periodic boundary conditions can be calculated easily:

$$\int_0^L d^N x |\chi(x_1, \dots, x_N | k_1, \dots, k_N)|^2 = C^2 L^N, \quad (1.21)$$

therefore

$$C = \frac{1}{\sqrt{L^N}}. \quad (1.22)$$

It is not difficult to prove that the wave functions (1.16) with the quasi-momenta quantized by the conditions (1.19) form a complete set of eigenfunctions of the Hamiltonian (1.11) in the limit  $c \rightarrow \infty$ .

### 1.2.2 The ground state and thermodynamics of the Tonks-Girardeau gas at zero temperature

The ground state of the Tonks-Girardeau gas can be found easily. To minimize  $E_N$ , Eq. (1.17), one should take the following subset of  $k_j$  from the set (1.20):

$$k_j = \frac{2\pi}{L} \left( j - \frac{N+1}{2} \right), \quad j = 1, \dots, N. \quad (1.23)$$

The momentum  $P_N$  of the ground state is thus equal to zero.

The thermodynamic limit is the limit of infinite number of particles at finite density  $\rho$ :

$$\rho = \frac{N}{L} = \text{const}, \quad N, L \rightarrow \infty. \quad (1.24)$$

One has in this limit

$$\sum_{j=1}^N \rightarrow \frac{L}{2\pi} \int_{-\pi\rho}^{\pi\rho} dk. \quad (1.25)$$

The ground state energy in the canonical ensemble is, therefore,

$$E_0 = \frac{L}{2\pi} \int_{-\pi\rho}^{\pi\rho} dk k^2 = \frac{L}{3}\pi^2\rho^3. \quad (1.26)$$

In the grand canonical ensemble the number of particles is not fixed and depends on the chemical potential; if the average number of particles in the system is  $N$  then

$$E_N = \sum_{j=1}^N k_j^2 - \mu. \quad (1.27)$$

In the thermodynamic limit the chemical potential is related to the density of particles as follows:

$$\mu = (\pi\rho)^2. \quad (1.28)$$

One can see that the thermodynamics of the Tonks-Girardeau gas at zero temperature is the same as that of the spinless free-fermion gas, with  $\pi\rho$  playing a role of the Fermi momentum  $k_F$ :

$$k_F = \pi\rho. \quad (1.29)$$

The pressure of the 1D gas is given by the formula

$$P = - \left( \frac{\partial E_0}{\partial L} \right)_{\sigma,N} = - \left( \frac{\partial F}{\partial L} \right)_{T,N}. \quad (1.30)$$

Here  $\sigma$  is the entropy of the gas, and  $F$  is the Helmholtz free energy,  $F = E_0 - T\sigma$ . The partial derivatives taken at a fixed  $T$  coincide with the partial derivatives taken at a fixed  $\sigma$  for the zero temperature case. Because of this fact we will omit the corresponding subscripts in the zero temperature case. For example, we will write  $P = -(\partial E_0 / \partial L)_N$ . The thermodynamic definition of the sound velocity is

$$v_S = \sqrt{-\frac{L}{m\rho} \left( \frac{\partial P}{\partial L} \right)_N}. \quad (1.31)$$

We set  $m = 1/2$  in the definition of the Lieb-Liniger model (1.1). One thus gets for the Tonks-Girardeau gas

$$P = \frac{2}{3}\pi^2\rho^3 \quad (1.32)$$

and

$$v_S = 2\pi\rho. \quad (1.33)$$

One can see from the expression for  $k_F$  that the Fermi velocity  $v_F$  is

$$v_F \equiv \frac{k_F}{m} = 2\pi\rho \quad (1.34)$$

(recall that we set  $m = 1/2$ ). Comparing Eqs. (1.33) and (1.34) one concludes that the Fermi velocity coincides with the sound velocity in the Tonks-Girardeau gas. This is the expected result since the spectrum of the Tonks-Girardeau gas is the same as that of the free-fermion gas. In our studies of the Lieb-Liniger model with finite coupling  $c$  we will use the notion of the Fermi velocity as given by Eq. (1.34); the sound velocity  $v_S$  at arbitrary  $c$  is always less than  $v_F$ , see Fig. 1.

### 1.3 Lieb-Liniger model at arbitrary repulsion strength

We will repeat, for arbitrary repulsion strength  $c$ , the steps performed in section 1.2: we construct the eigenfunctions and the spectrum of the Lieb-Liniger model (1.11) at arbitrary coupling strength  $c$ , identify the ground state of the system and take the thermodynamic limit. We then calculate the velocity of sound  $v_S$ , the compressibility  $\kappa$ , and the Luttinger parameter  $K$  from the thermodynamics of the Lieb-Liniger model. We discuss the weak (Bogoliubov) and strong (Tonks-Girardeau) coupling limits in the solution of the Lieb-Liniger model.

#### 1.3.1 Eigenfunctions and spectrum of the Lieb-Liniger model

The structure of the eigenfunctions at general  $c$  is much more complicated as compared to the eigenfunctions at  $c = \infty$ , considered in section 1.2.1. That particular case, however, shows a way one should proceed: The eigenfunctions  $\chi_N(x_1, \dots, x_N)$  in the  $N$ -particle sector are symmetric under any interchange of coordinates, therefore it is sufficient to consider the following domain  $T$  in the coordinate space:

$$T : \quad x_1 < x_2 < \dots < x_N. \quad (1.35)$$

In this domain  $\chi_N$  is an eigenfunction of the free Hamiltonian

$$\mathcal{H}_N^0 = - \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} \quad (1.36)$$

$$\mathcal{H}_N^0 \chi_N = E_N(k_1, \dots, k_N) \chi_N \quad (1.37)$$

The following boundary condition should be satisfied:

$$\left( \frac{\partial}{\partial x_{j+1}} - \frac{\partial}{\partial x_j} - c \right) \chi_N = 0, \quad x_{j+1} = x_j + 0. \quad (1.38)$$

This boundary condition obviously reduces to Eq. (1.14) in the Tonks-Girardeau limit. To prove Eq. (1.38) use the variables  $z = x_{j+1} - x_j$  and  $Z = (x_{j+1} + x_j)/2$ . One has

$$\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1} = 2 \frac{\partial}{\partial z}, \quad \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_1^2} = 2 \frac{\partial^2}{\partial z^2} + \frac{1}{2} \frac{\partial^2}{\partial Z^2}. \quad (1.39)$$

Integrating Eq. (1.12) over  $z$  in the small domain  $|z| < \epsilon \rightarrow 0$  one can see that the eigenfunction of the free-particle Hamiltonian (1.36) obeying the boundary condition (1.38) is indeed an eigenfunction of the Hamiltonian (1.11).

To get an explicit expression for the eigenfunctions (1.12) we start from the trial expression (1.15). We suggest the following form of the eigenfunctions:

$$\chi_N = \text{const} \left[ \prod_{N \geq j > l \geq 1} \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_l} + c \right) \right] \det[\exp\{ik_j x_l\}]. \quad (1.40)$$

One should show that this expression satisfies Eqs. (1.36) and (1.38). Check, for instance, that

$$\left( \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1} - c \right) \chi_N = 0, \quad x_2 = x_1 + 0. \quad (1.41)$$

Do do this, rewrite Eq. (1.40) as follows:

$$\chi_N = \left( \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1} + c \right) \tilde{\chi}_N, \quad (1.42)$$

where

$$\begin{aligned} \tilde{\chi}_N = \text{const} \prod_{j=3}^N & \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_1} + c \right) \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_2} + c \right) \\ & \times \prod_{N \geq j > l \geq 3} \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_l} + c \right) \det[\exp\{ik_j x_l\}]. \end{aligned} \quad (1.43)$$

The function  $\tilde{\chi}_N$  is antisymmetric with respect to the interchange of  $x_1$  and  $x_2$ ,

$$\tilde{\chi}_N(x_1, x_2) = -\tilde{\chi}_N(x_2, x_1). \quad (1.44)$$

Writing the left hand side of the Eq. (1.41) in the form

$$\left[ \left( \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1} \right)^2 - c^2 \right] \tilde{\chi}_N \quad (1.45)$$

one sees that it is antisymmetric with respect to the interchange of  $x_1$  and  $x_2$ , and, therefore, equal to zero when  $x_2 \rightarrow x_1$ . The boundary condition (1.38) for the other  $x_j$  can be checked similarly. We thus showed that the wave function (1.40) is indeed an eigenfunction of the Hamiltonian (1.11). Using the definition of the determinant

$$\det[\exp\{ik_j x_l\}] = \sum_P (-1)^{[P]} \exp \left\{ i \sum_{n=1}^N x_n k_{P_n} \right\} \quad (1.46)$$

one gets for (1.40)

$$\begin{aligned} \chi_N = \text{const} & \left\{ N! \prod_{j>l} [(k_j - k_l)^2 + c^2] \right\}^{-1/2} \\ & \times \sum_P (-1)^{[P]} \exp \left\{ i \sum_{n=1}^N x_n k_{P_n} \right\} \prod_{j>l} [k_{P_j} - k_{P_l} - ic \operatorname{sign}(x_j - x_l)]. \end{aligned} \quad (1.47)$$

This is the desired expression for the eigenfunctions of the Hamiltonian (1.11). We wrote Eq. (1.47) in such a form that it is valid for arbitrary values of  $x_1, \dots, x_N$ , thus removing the restriction (1.35). One can see that the wave functions given by Eq. (1.47) reduce to the wave functions given by Eq. (1.16) in the  $c \rightarrow \infty$  limit.

The eigenvalues of the Hamiltonian (1.11) and the momentum operator (1.13) on the eigenfunctions (1.47) are

$$E_N = \sum_{j=1}^N k_j^2, \quad P_N = \sum_{j=1}^N k_j, \quad (1.48)$$

respectively. These formulas look very similar to that of a free system and because of this similarity the parameters  $k_j$  are called quasi-momenta.

No boundary conditions were imposed so far. Let us do it now and use the periodic boundary conditions. The condition (1.18) imposed on the function (1.47) gives

$$\exp\{ik_jL\} = \prod_{l \neq j}^N \frac{k_j - k_l + ic}{k_j - k_l - ic}, \quad j = 1, \dots, N. \quad (1.49)$$

These coupled nonlinear equations are called Bethe equations. In the  $c \rightarrow \infty$  limit these equations reduce to Eqs. (1.19). It can be shown that all their solutions are real (see, for example, Ref. [5]). It is often more convenient to use the logarithmic form of Bethe equations:

$$Lk_j + \sum_{m=1}^N \theta(k_j - k_m) = 2\pi \left( n_j - \frac{N+1}{2} \right), \quad j = 1, \dots, N. \quad (1.50)$$

Here  $\{n_j\}$  is a set of integers, and the function  $\theta$  is

$$\theta(k) = i \ln \left( \frac{ic+k}{ic-k} \right), \quad \theta(\pm\infty) = \pm\pi. \quad (1.51)$$

An important property of the representation (1.50) is that the sets of  $n_j$  and the sets of quasi-momenta  $k_j$  are in one to one correspondence. It is often more convenient to parameterize a quantum state by the set  $n_j$  rather than by the set of quasi-momenta  $k_j$ .

Let us sum up Bethe equations (1.50) over  $j$ . The function  $\theta(k)$  defined by Eq. (1.51) is antisymmetric, therefore

$$\sum_{j,m=1}^N \theta(k_j - k_m) = 0 \quad (1.52)$$

and one gets

$$L \sum_{j=1}^N k_j = 2\pi \sum_{j=1}^N \left( n_j - \frac{N+1}{2} \right). \quad (1.53)$$

Comparing this equation with Eq. (1.48) one gets a relation between the momentum of an arbitrary state and the set of quantum numbers  $n_j$ :

$$P_N = \frac{2\pi}{L} \sum_{j=1}^N \left( n_j - \frac{N+1}{2} \right). \quad (1.54)$$

The calculation of the normalization constant in Eq. (1.47) is a nontrivial task (recall that in the Tonks-Girardeau limit this constant is given by Eqs. (1.21) and (1.22)). A closed expression for this constant was suggested by Gaudin (see [6] and references therein) and proved in Ref. [7]; a detailed discussion can be found in Ref. [5].

### 1.3.2 The ground state and thermodynamics at zero temperature

The ground state of the model in the  $N$ -particle sector corresponds to the following choice of the quantum numbers  $n_j$  in Eq. (1.50):

$$n_j = j, \quad j = 1, \dots, N, \quad (1.55)$$

that is the Bethe equations (1.50) take the form

$$Lk_j + \sum_{m=1}^N \theta(k_j - k_m) = 2\pi \left( j - \frac{N+1}{2} \right), \quad j = 1, \dots, N. \quad (1.56)$$

in the ground state of the model. The momentum of the ground state is equal to zero.

Consider now the thermodynamic limit of the model, defined by Eq. (1.24). Write

$$\sum_{j=1}^N = \sum_{j=1}^N \frac{k_{j+1} - k_j}{k_{j+1} - k_j} = L \sum_{j=1}^N \rho(k_j)(k_{j+1} - k_j) \quad (1.57)$$

where we have defined  $\rho(k_j)$  as follows

$$\rho(k_j) = \frac{1}{L(k_{j+1} - k_j)}. \quad (1.58)$$

In the thermodynamic limit one has for an arbitrary function  $f(k_j)$ :

$$\sum_{j=1}^N f(k_j) \rightarrow L \int_{-\Lambda}^{\Lambda} dk \rho(k) f(k). \quad (1.59)$$

The parameter  $\Lambda$  plays a role of the Fermi momentum: all the states with  $|k| < \Lambda$  are occupied, and all the states with  $|k| > \Lambda$  are empty. The value of  $\Lambda$  is given by the normalization condition

$$\int_{-\Lambda}^{\Lambda} dk \rho(k) = \frac{N}{L} = \rho. \quad (1.60)$$

To get the thermodynamic limit of the Bethe equations (1.56), take the difference of the equations with the indices  $j+1$  and  $j$ :

$$L(k_{j+1} - k_j) + \sum_{j=1}^N [\theta(k_{j+1} - k_m) - \theta(k_j - k_m)] = 2\pi. \quad (1.61)$$

The difference  $k_{j+1} - k_j$  is small in the thermodynamic limit, therefore Eq. (1.61) can be written as follows

$$L(k_{j+1} - k_j) + (k_{j+1} - k_j) \sum_{m=1}^N \theta'(k_j - k_m) = 2\pi, \quad N, L \rightarrow \infty, \quad (1.62)$$

where  $\theta'(k) \equiv \partial\theta(k)/\partial k$ . Using Eqs. (1.58) and (1.59) one gets after some algebra

$$\rho(k) - \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dq \rho(q) K(k, q) = \frac{1}{2\pi}, \quad -\Lambda \leq k \leq \Lambda, \quad (1.63)$$

where

$$K(k, q) \equiv \theta'(k - q) = \frac{2c}{c^2 + (k - q)^2}. \quad (1.64)$$

The linear integral equation (1.63) is called Lieb equation. Its solution, together with the normalization condition (1.60), gives the quasi-momentum distribution function  $\rho(k)$ . All

thermodynamic parameters can be calculated from  $\rho(k)$  and  $\Lambda$ . The ground state energy in the canonical ensemble is

$$E_0 = \sum_{j=1}^N k_j^2 \rightarrow L \int_{-\Lambda}^{\Lambda} dk \rho(k) k^2. \quad (1.65)$$

Following [2], change the variables as follows:

$$k = \Lambda z, \quad c = \Lambda \alpha, \quad \rho(\Lambda z) = g(z). \quad (1.66)$$

Written in these variables, Eqs. (1.60), (1.63), and (1.65) are, respectively,

$$\gamma \int_{-1}^1 dz g(z) = \alpha, \quad (1.67)$$

$$g(z) - \frac{1}{2\pi} \int_{-1}^1 dy \frac{2\alpha g(y)}{\alpha^2 + (y-z)^2} = \frac{1}{2\pi}, \quad (1.68)$$

and

$$e(\gamma) \equiv \frac{E_0}{N\rho^2} = \frac{\gamma^3}{\alpha^3} \int_{-1}^1 dy g(y) y^2. \quad (1.69)$$

Recall that  $\gamma$  is given by Eq. (1.2). The system of equations (1.67) and (1.68) is much more suitable for the numerical analysis than the system (1.60) and (1.63). To solve the system (1.67) and (1.68) the following two steps should be done:

- (i) Solve Eq. (1.68) for a given  $\alpha$ ;
- (ii) use Eq. (1.67) to determine  $\gamma$  as a function of  $\alpha$ ;

Having solved the system (1.67) and (1.68), one can easily go back to the equations in the form (1.60) and (1.63), if necessary. The Fermi momentum  $\Lambda$  can be found by combining Eqs. (1.2) and (1.66):

$$\Lambda = \rho \frac{\gamma}{\alpha} = \rho \left[ \int_{-1}^1 dz g(z) \right]^{-1}. \quad (1.70)$$

The quasi-momentum density  $\rho(k)$  is related to  $g(z)$  by Eq. (1.66).

### 1.3.3 Velocity of sound, compressibility, and Luttinger parameter

Discuss first the thermodynamic definition of the sound velocity, Eq. (1.31). Taking  $E_0$  from the Eq. (1.69), and using Eq. (1.30) one gets

$$v_S = 2\rho \left[ 3e(\gamma) - 2\gamma e'(\gamma) + \frac{1}{2}\gamma^2 e''(\gamma) \right]^{1/2} = 2 \left[ \mu(\gamma) - \frac{1}{2}\gamma\mu'(\gamma) \right]^{1/2}, \quad (1.71)$$

where “prime” denotes differentiation with respect to  $\gamma$  and  $\mu$  is the chemical potential given by

$$\mu = \left( \frac{\partial E_0}{\partial N} \right)_L = \rho^2 [3e(\gamma) - \gamma e'(\gamma)]. \quad (1.72)$$

Note an important technical point: since  $P$  and  $\mu$  depend on  $N$  and  $L$  only via  $\rho = N/L$ , the following identities are valid:

$$\left( \frac{\partial P}{\partial \rho} \right)_N = \left( \frac{\partial P}{\partial \rho} \right)_L, \quad \left( \frac{\partial \mu}{\partial \rho} \right)_N = \left( \frac{\partial \mu}{\partial \rho} \right)_L. \quad (1.73)$$

These identities are very useful in getting various thermodynamic relations.

Another, microscopic definition of the sound velocity is

$$v_S = \left. \frac{\partial \epsilon(k)}{\partial P(k)} \right|_{k=\Lambda}, \quad (1.74)$$

where  $\epsilon(k)$  is the energy change of the system when a particle with the quasi-momentum  $k$  is added;  $P(k)$  is the momentum change of the system. The equivalence of the definitions (1.31) and (1.74) in the Lieb-Liniger was proven in Ref. [2]; for more modern presentation one can consult Ref. [5]. We will omit this proof since it is quite long; we only present here several equivalent representations for  $v_S$  obtained in course of this proof:

$$v_S = \sqrt{2\rho \frac{\partial \mu}{\partial \rho}} = \frac{\partial \mu}{\partial \Lambda} = \frac{\rho}{2\pi\rho(\Lambda)^2}. \quad (1.75)$$

Using the compressibility

$$\kappa \equiv -\frac{1}{L} \left( \frac{\partial L}{\partial P} \right)_N = \frac{1}{\rho^2} \left( \frac{\partial \rho}{\partial \mu} \right)_L \quad (1.76)$$

we can write the velocity of sound as

$$v_S = \sqrt{\frac{2}{\kappa \rho}}. \quad (1.77)$$

The Luttinger Liquid Hamiltonian contains two parameters: the velocity of sound,  $v_S$ , and the Luttinger parameter,  $K$ . These parameters are related to the compressibility of the system [8]:

$$\frac{K}{v_S} = \pi \rho^2 \kappa \quad \text{any model} \quad (1.78)$$

In case the microscopic theory has some extra symmetry, there could exist some other relations. For example, if the system is Galilean-invariant, that is, the kinetic energy of any particle it consists of is  $p^2/2m$ , one can show [9] that

$$K v_S = 2\pi\rho \quad \text{Galilean-invariant model} \quad (1.79)$$

One can see that Eqs. (1.78) and (1.79) are consistent with Eq. (1.77), as it should be since the Lieb-Liniger model is Galilean-invariant. Note that  $2\pi\rho = v_F$ , where  $v_F$  is the sound velocity of the Tonks-Girardeau gas, Eqs. (1.33) and (1.34).

### 1.3.4 Weak and strong coupling limits and numerics

The large  $\gamma$  limit is easy to work out since Eq. (1.68) admits a regular perturbative expansion with  $1/\alpha$  being a small parameter. One has

$$g(z) = \frac{1}{2\pi} + \frac{1}{\pi^2 \alpha} + \frac{2}{\pi^3 \alpha^2} + \dots, \quad \alpha \rightarrow \infty, \quad (1.80)$$

Equation (1.67) then gives

$$\gamma = \pi\alpha - 2 + \frac{4}{3\alpha^2} - \frac{32}{15\alpha^4} - \frac{16}{45\pi\alpha^5} + \dots, \quad \alpha \rightarrow \infty. \quad (1.81)$$

Inverting this expression, one gets

$$\alpha = \frac{\gamma}{\pi} + \frac{2}{\pi} - \frac{4\pi}{3\gamma^2} + \frac{16\pi}{3\gamma^3} + \left( \frac{2\pi^2}{15} - 1 \right) \frac{16\pi}{\gamma^4} + \dots, \quad \gamma \rightarrow \infty. \quad (1.82)$$

For the ground state energy per particle defined by Eq. (1.69) one gets

$$e(\gamma) = \frac{\pi^2}{3} \left[ 1 - \frac{4}{\gamma} + \frac{12}{\gamma^2} + (\pi^2 - 15) \frac{32}{15\gamma^3} + \dots \right] \quad (1.83)$$

From the exact expression (1.71) for the sound velocity in the Lieb-Liniger model, one gets the following large  $\gamma$  expansion:

$$v_{large} = v_F \left( 1 - \frac{4}{\gamma} + \frac{12}{\gamma^2} + \dots \right), \quad \gamma \rightarrow \infty \quad (\text{error} < 1\% \text{ for } \gamma > 10). \quad (1.84)$$

The leading terms in all the expansions written in the present section correspond to the Tonks-Girardeau limit and were obtained in the section 1.2.2.

The opposite limit,  $\gamma \rightarrow 0$ , is much more difficult to analyze since the kernel in Eq. (1.63) becomes singular when  $\alpha \rightarrow 0$ . The leading term is obtained in Ref. [2]:

$$g(z) \simeq \frac{\sqrt{1-z^2}}{2\pi\alpha}, \quad \alpha \rightarrow 0. \quad (1.85)$$

For the ground state energy per particle defined by Eq. (1.69) one gets

$$e(\gamma) \simeq \gamma \left( 1 - \frac{4}{3\pi} \sqrt{\gamma} \right), \quad \gamma \rightarrow 0. \quad (1.86)$$

Let us substitute the expansion (1.86) into Eq. (1.71):

$$v_{small} = v_F \frac{1}{\pi} \left( \gamma - \frac{1}{2\pi} \gamma^{3/2} \right)^{1/2}, \quad \gamma \rightarrow 0 \quad (\text{error} < 1\% \text{ for } \gamma < 10). \quad (1.87)$$

Since only two terms are written explicitly in the expansion (1.86), one should expand Eq. (1.87) and drop all the terms except the leading and the first subleading. However, the numerical analysis shows that Eq. (1.87) gives much better approximation to the exact expression (1.71) if not expanded. We plot  $v_S/v_F$ ,  $v_{large}/v_F$  and  $v_{small}/v_F$  versus  $\gamma$  in Fig. 1. Recall that  $v_F$  is the velocity of sound in the Tonks-Girardeau gas, Eq. (1.34), and the ratio  $v_S/v_F$  is the inverse Luttinger parameter, Eq. (1.79),

$$K^{-1} = \frac{v_S}{v_F}. \quad (1.88)$$

## 1.4 Excitations in the Lieb-Liniger model

We construct “elementary” (single quasi-particle and single quasi-hole) excitations in the Lieb-Liniger model and construct more complex excitations out of them.

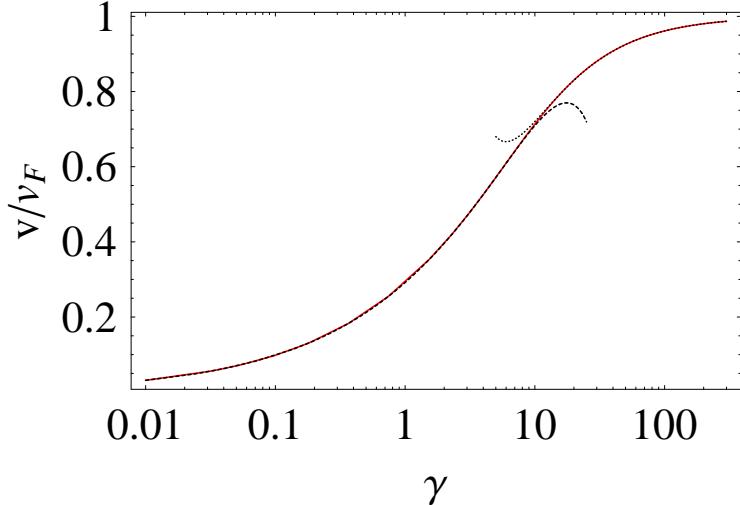


Figure 1: Shown is the log-linear plot of the inverse Luttinger parameter in the Lieb-Liniger model, Eq. (1.88), versus  $\gamma$ . Red curve corresponds to the exact (numerical) result for  $v_S$ , black dotted curve is obtained from Eq. (1.84), and black dashed curve from Eq. (1.87).

#### 1.4.1 One-particle excitations in the Tonks-Girardeau gas

The ground state quasimomentum distribution in the  $N$ -particle sector of the Tonks-Girardeau gas is given by Eq. (1.23):

$$Lk_j = 2\pi \left( j - \frac{N+1}{2} \right), \quad j = 1, \dots, N, \quad (1.89)$$

while in the  $N+1$ -particle sector it takes a form

$$L\tilde{k}_j = 2\pi \left( j - \frac{N+2}{2} \right), \quad j = 1, \dots, N+1. \quad (1.90)$$

Therefore, every  $\tilde{k}_j$  is shifted by  $\pi/L$  with respect to  $k_j$ . A way to avoid dealing with this shift is to impose anti-periodic boundary conditions in the  $N+1$ -particle sector. Then the ground state quasimomentum distribution is

$$L\tilde{k}_j = 2\pi \left( j - \frac{N+1}{2} \right), \quad j = 1, \dots, N+1. \quad (1.91)$$

Note that the ground state is 2-fold degenerate, with another configuration corresponding to  $j = 0, 1, \dots, N$ . While the above choice of the boundary conditions eliminates the  $\pi/L$  shift, its influence onto the ground state energy is negligible small (vanishes as  $1/N$ ) in the large  $N$  limit.

The structure of the one-particle excitations is now evident. They can be decomposed into the “topological” part resulting in the boundary conditions change and “truly one-particle” excitation. The corresponding Bethe equations are

$$L\tilde{k}_j = 2\pi \left( j - \frac{N+1}{2} \right), \quad j = 1, \dots, N. \quad (1.92)$$

and

$$L\tilde{k}_{N+1} = 2\pi M, \quad |M| > N/2. \quad (1.93)$$

One can see by comparing Eqs. (1.89) and (1.92) that  $\tilde{k}_j = k_j$  for  $j = 1, \dots, N$ . This result is specific to the Tonks-Girardeau gas and does not hold true for the finite repulsion case. However, the separation of the topological part of the one-particle excitation by using the above trick with the boundary conditions will prove useful in the finite repulsion case as well.

### 1.4.2 One-particle shift function

We consider the  $N$ -particle ground state, Eq. (1.56). We then add one article and impose anti-periodic boundary conditions, like in the Tonks-Girardeau limit. The resulting Bethe equations are

$$L\tilde{k}_j + \sum_{k=1}^N \theta(\tilde{k}_j - \tilde{k}_k) + \theta(\tilde{k}_j - \tilde{k}_{N+1}) = 2\pi \left( j - \frac{N+1}{2} \right), \quad j = 1, \dots, N, \quad (1.94)$$

and

$$L\tilde{k}_{N+1} + \sum_{k=1}^N \theta(\tilde{k}_{N+1} - \tilde{k}_k) = 2\pi M, \quad |M| > \frac{N}{2}. \quad (1.95)$$

The difference of Eqs. (1.56) and (1.94) is

$$L(k_j - \tilde{k}_j) + \sum_{k=1}^N [\theta(k_j - k_k) - \theta(\tilde{k}_j - \tilde{k}_k)] - \theta(\tilde{k}_j - \tilde{k}_{N+1}) = 0. \quad (1.96)$$

Using that  $k_{j+1} - k_j \sim 1/L$  in the thermodynamic limit one gets

$$\begin{aligned} & \sum_{k=1}^N [\theta(k_j - k_k) - \theta(\tilde{k}_j - \tilde{k}_k)] \\ &= (k_j - \tilde{k}_j) \sum_{k=1}^N \theta'(k_j - k_k) - \sum_{k=1}^N (k_k - \tilde{k}_k) \theta'(k_j - k_k), \quad N, L \rightarrow \infty \end{aligned} \quad (1.97)$$

and it follows from Eq. (1.56) that

$$(k_{j+1} - k_j) \left[ L + \sum_{k=1}^N \theta'(k_j - k_k) \right] = 2\pi, \quad N, L \rightarrow \infty. \quad (1.98)$$

Let us define the so-called shift function  $F(k_j|k_{N+1})$  as follows (we write  $k_p$  instead of  $k_{N+1}$  in order to lighten the notations)

$$F(k_j|k_p) = \frac{k_j - \tilde{k}_j}{k_{j+1} - k_j}. \quad (1.99)$$

Using Eqs. (1.97)–(1.99) one gets from Eq. (1.96) an integral equation (often called dressing equation) onto the shift function:

$$F(k|k_p) - \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dq K(k, q) F(q|k_p) = -\frac{1}{2\pi} \theta(k_p - k). \quad (1.100)$$

This dressing equation is of crucial importance for the analysis of one-particle excitations.

### 1.4.3 Momentum of the excited state

We introduce the so-called dressed momentum  $k(\lambda)$  by the formula

$$p(k) = k + \int_{-\Lambda}^{\Lambda} dq \rho(q) \theta(k - q). \quad (1.101)$$

An important property of  $k(k)$ , valid for any  $\gamma$  is

$$p(\pm\Lambda) = \pm k_F. \quad (1.102)$$

It can be proven by integrating Eq. (1.63) with respect to  $k$ . Note that in the infinite repulsion limit the integral in Eq. (1.101) vanishes:

$$p(k) = k \quad \text{as} \quad \gamma = \infty. \quad (1.103)$$

The importance of the definition (1.101) becomes evident when one calculates the momentum difference  $\Delta P(k_p)$  between the  $N$ -particle ground state and  $N + 1$ -particle excited state. One has

$$\begin{aligned} \Delta P(k_p) &= \sum_{k=1}^N (\tilde{k}_k - k_k) + k_p = - \sum_{j=1}^N F(k_j|k_p)(k_{j+1} - k_j) + k_p \\ &\rightarrow - \int_{-\Lambda}^{\Lambda} dk F(k|k_p) + k_p, \quad N \rightarrow \infty. \end{aligned} \quad (1.104)$$

Multiplying both parts of Eq. (1.100) by  $\rho(k)$ , integrating over  $k$ , and taking into account Eq. (1.63) one gets an identity

$$\int_{-\Lambda}^{\Lambda} dk F(k|k_p) = - \int_{-\Lambda}^{\Lambda} dk \rho(k) \theta(k_p - k) \quad (1.105)$$

which implies

$$\Delta P(k_p) = p(k_p), \quad |k_p| \geq \Lambda. \quad (1.106)$$

### 1.4.4 Energy of the excited state

Let us introduce the integral equation onto the so-called dressed energy  $\varepsilon(\lambda)$ :

$$\varepsilon(k) - \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dq \varepsilon(q) K(k, q) = k^2 - \mu \quad (1.107)$$

with an extra condition

$$\varepsilon(\pm\Lambda) = 0. \quad (1.108)$$

It can be shown [5] that  $\varepsilon(k)$  possesses the following properties

$$\varepsilon(k) = \varepsilon(-k) \quad (1.109)$$

$$\varepsilon'(k) > 0, \quad k > 0 \quad (1.110)$$

and

$$\varepsilon(k) < 0, \quad |k| < \Lambda \quad (1.111)$$

$$\varepsilon(k) > 0, \quad |k| > \Lambda \quad (1.112)$$

Differentiating Eq. (1.107) and taking the integral by parts one gets

$$\varepsilon'(k) - \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dq \varepsilon'(q) K(k, q) = 2k. \quad (1.113)$$

Let us now calculate the energy difference  $\Delta E(k_p)$  between the  $N$ -particle ground state and  $N + 1$ -particle excited state. One has

$$\begin{aligned} \Delta E(k_p) &= k_p^2 - \mu + \sum_{j=1}^N (\tilde{k}_j^2 - k_j^2) \\ &= k_p^2 - \mu - \sum_{j=1}^N 2k_j(k_j - \tilde{k}_j) \rightarrow k_p^2 - \mu - \int_{-\Lambda}^{\Lambda} dk 2k F(k|k_p). \end{aligned} \quad (1.114)$$

Multiplying Eq. (1.113) by  $F(k|k_p)$ , integrating over  $k$ , and using Eq. (1.100), one arrives at the identity

$$\int_{-\Lambda}^{\Lambda} dk 2k F(k|k_p) = -\frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dq \varepsilon(q) K(q, k_p). \quad (1.115)$$

Substituting this identity into Eq. (1.114) we finally get

$$\Delta E(k_p) = \varepsilon(k_p), \quad |k_p| \geq \Lambda. \quad (1.116)$$

This formula clarifies a physical meaning of  $\varepsilon(k)$  introduced earlier by Eq. (1.107): it can be interpreted as the energy of an excitation with the momentum  $p(k)$ .

#### 1.4.5 One-hole excitations

One-hole type of excitations can be generated by removing one particle from the  $N$ -particle ground state. An analysis similar to the one carried out for the one-particle excitation gives

$$\Delta P(k_h) = -p(k_h), \quad -\Lambda \leq k_h \leq \Lambda \quad (1.117)$$

for the dressed momentum of the excitation, and

$$\Delta E(k_h) = -\varepsilon(k_h), \quad -\Lambda \leq k_h \leq \Lambda \quad (1.118)$$

for the dressed energy.

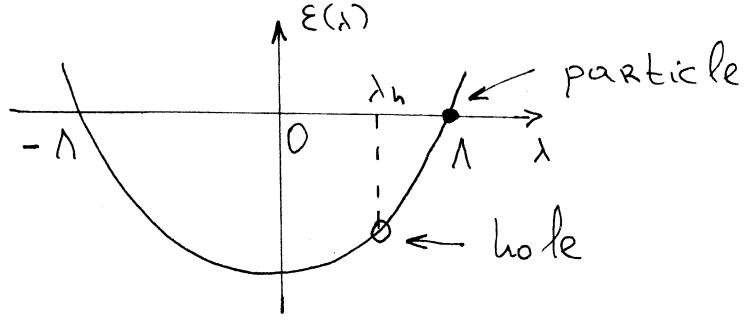


Figure 2: Particle-hole excitation corresponding to  $\min \Delta E$  for a given momentum  $0 \leq \Delta P \leq 2k_F$ .

#### 1.4.6 Multiple particle-hole excitations

An arbitrary excitation in the Lieb-Liniger model can be created from one-particle and one-hole excitations. The momentum difference between the excited state and the ground state is

$$\Delta P = \sum_{\text{particles}} p(k_p) - \sum_{\text{holes}} p(k_h) \quad (1.119)$$

and the energy difference is

$$\Delta E = \sum_{\text{particles}} \varepsilon(k_p) - \sum_{\text{holes}} \varepsilon(k_h). \quad (1.120)$$

#### 1.4.7 Dispersion curves and their exact lower bound

By the dispersion curve we call the curve of  $\Delta E$  plotted as a function of  $\Delta P$ . What is the minimal value of  $\Delta E$  for a given  $\Delta P$  lying in the region  $0 \leq \Delta P \leq 2k_F$ ? The answer to this question (first given in Ref. [10]) is very elegant and intuitively clear: the corresponding excitation is shown in Fig. 2. It consists of a single particle-hole pair, with the particle lying at the right Fermi point and

$$-k_F \leq -p(k_h) = \Delta P - k_F \leq k_F. \quad (1.121)$$

We plot  $\Delta E$  as a function of  $\Delta P$  for several values of  $\gamma$ , see Fig. 3. Note that the explicit analytic expression for  $\min \Delta E$  can be found easily for  $\gamma = \infty$ :

$$\frac{\Delta E}{k_F^2} = \frac{\Delta P}{k_F} \left( 2 - \frac{\Delta P}{k_F} \right), \quad 0 \leq \Delta P \leq 2k_F \quad \text{as} \quad \gamma = \infty. \quad (1.122)$$

### 1.5 Correlation functions of the Tonks-Girardeau gas

The knowledge of the exact eigenfunctions and spectrum for the Lieb-Liniger model makes it possible to develop the microscopic approach to the calculation of its correlation functions.

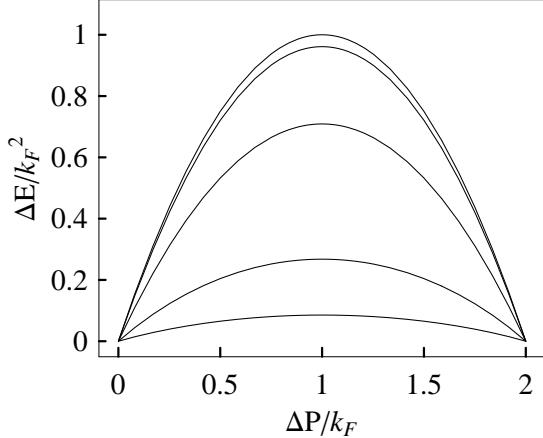


Figure 3: Shown are the plots of  $\min \Delta E$  as a function of  $\Delta P$  for  $\gamma = \infty, 100, 10, 1, 0.1$  (from upper to lower).

The procedure is straightforward: calculate the form-factors of local field operators in the basis formed by the eigenfunctions of the Hamiltonian (1.1) in the sector with given number of particles  $N$ , then sum up over all intermediate states, and, finally, take the thermodynamic limit. It is very easy to formulate such a program, and very difficult to implement it because of the complicated structure of the Bethe wave function (1.47) and Bethe equations (1.49). The book [5] is mostly devoted to the development of this program and is, probably, the most comprehensive monograph in this field. Among the results not included in the cited book one should mention the papers [11, 12, 13], where the local field operators were represented via the fundamental operators of the Quantum Inverse Scattering Problem (this approach work for spin chains, where the quantum space is finite-dimensional, but not for the Lieb-Liniger model; there is, however a trick (unpublished) solving the inverse scattering problem for the Lieb-Liniger model as well).

Unfortunately, the final expressions for the correlation functions obtained within this straightforward approach are, in general, too complicated to be successfully analyzed either by analytic methods or numerically. There are, however, several important exceptions. One of them, the Tonks-Girardeau gas, is discussed here.

The relatively simple form of the eigenfunctions of the Tonks-Girardeau gas make it possible to calculate the correlation functions explicitly for arbitrary number of particles  $N$ . The first non-trivial correlation function which was calculated is the one-particle density matrix. The calculations were done by Lenard [14] and the result is now known as the Lenard's formula. We present this formula in section 1.5.1. Lenard's formula contains the determinant of an  $N \times N$  matrix. The  $N \rightarrow \infty$  limit of the determinant of an  $N \times N$  matrix is called the Fredholm determinant. It is discussed in section 1.5.2. The thermodynamic limit of the Lenard's formula is studied in sections 1.5.3 and 1.5.4. The momentum distribution function for the Tonks-Girardeau gas is calculated in section 1.5.5.

### 1.5.1 One-particle density matrix: Lenard's formula

The one-particle density matrix in the  $N$ -particle sector is defined as follows

$$\rho_N(x) = \langle \Psi^\dagger(x) \Psi(0) \rangle, \quad (1.123)$$

where the average is taken over the ground state of the system in the  $N$ -particle sector. The expression for the  $\rho_N(x)$  in the first-quantized representation can be obtained easily (remember the commutation relations (1.3) and the representation (1.8)):

$$\begin{aligned} \rho_N(x) &= \langle \Psi_N(k_1, \dots, k_N) | \Psi^\dagger(x) \Psi(0) | \Psi_N(k_1, \dots, k_N) \rangle = \frac{1}{N!} \int_0^L d^N z d^N z' \\ &\times \bar{\chi}_N(z_1, \dots, z_N) \chi_N(z'_1, \dots, z'_N) \langle 0 | \Psi(z_1) \dots \Psi(z_N) | \Psi^\dagger(x) \Psi(0) | \Psi^\dagger(z'_1) \dots \Psi^\dagger(z'_N) | 0 \rangle \\ &= N \int_0^L dz_2 \dots dz_N \bar{\chi}_N(x, z_2, \dots, z_N) \chi_N(0, z_2, \dots, z_N) \quad (1.124) \end{aligned}$$

In the Tonks-Girardeau case  $\chi_N$  is given by Eqs. (1.16), the normalization constant  $C$  for the periodic boundary conditions is given by Eq. (1.22), and the ground state quasi-momentum distribution is given by Eq. (1.23). Therefore

$$\begin{aligned} \rho_N(x) &= \frac{1}{(N-1)! L^N} \int_0^L dz_2 \dots dz_N \\ &\times \sum_P \sum_{\tilde{P}} (-1)^{[P]+[\tilde{P}]} \exp\{-ixk_{P_1}\} \prod_{a=2}^N \exp\{iz_a(k_{\tilde{P}_a} - k_{P_a})\} \text{sgn}(z_a - x). \quad (1.125) \end{aligned}$$

All the integrals in this expression can be taken explicitly and give

$$\frac{1}{L} \int_0^L dz_a \exp\{iz_a(k_{\tilde{P}_a} - k_{P_a})\} \text{sgn}(z_a - x) = f(k_{\tilde{P}_a} - k_{P_a}), \quad a = 2, 3, \dots, N, \quad (1.126)$$

where

$$f(k_a - k_b) = \delta_{k_a k_b} - \frac{2}{L} \frac{\exp\{ix(k_a - k_b)\} - 1}{i(k_a - k_b)}. \quad (1.127)$$

One gets, therefore,

$$\rho_N(x) = \frac{1}{(N-1)! L} \sum_P \sum_{\tilde{P}} (-1)^{[P]+[\tilde{P}]} \frac{\exp\{-ixk_{P_1}\}}{f(k_{\tilde{P}_1} - k_{P_1})} \prod_{a=1}^N f(k_{\tilde{P}_a} - k_{P_a}). \quad (1.128)$$

Represent the permutation  $\tilde{P}$  in the internal sum as a sequence of two permutations

$$\tilde{P} = P' P \quad (1.129)$$

where  $P$  is the running index of the external sum. Further, introduce the notation

$$k_{P_a} \equiv P[k_a]. \quad (1.130)$$

Written in these notations, Eq. (1.128) reads

$$\rho_N(x) = \frac{1}{(N-1)!L} \sum_P \sum_{P'} (-1)^{[P']} \frac{\exp\{-ixP[k_1]\}}{f(P'[P[k_1]] - P[k_1])} \prod_{a=1}^N f(P'[P[k_a]] - P[k_a]). \quad (1.131)$$

One can easily notice the following identity:

$$\prod_{a=1}^N f(P'[P[k_a]] - P[k_a]) = \prod_{a=1}^N f(P'[k_a] - k_a), \quad (1.132)$$

therefore

$$\begin{aligned} \rho_N(x) &= \frac{1}{(N-1)!L} \sum_{P'} (-1)^{[P']} \prod_{a=1}^N f(P'[k_a] - k_a) \sum_P \frac{\exp\{-ixP[k_1]\}}{f(P'[P[k_1]] - P[k_1])} \\ &= \frac{1}{L} \sum_{P'} (-1)^{[P']} \prod_{a=1}^N f(P'[k_a] - k_a) \sum_{j=1}^N \frac{\exp\{-ixk_j\}}{f(P'[k_j] - k_j)} \\ &= \frac{1}{L} \sum_{j=1}^N \exp\{-ixk_j\} \sum_P (-1)^{[P]} \prod_{a \neq j} f(P[k_a] - k_a). \end{aligned} \quad (1.133)$$

Consider the determinant of an  $N \times N$  matrix with the entries  $f(k_j - k_l)$ :

$$\begin{vmatrix} f(k_1 - k_1) & \cdots & f(k_1 - k_N) \\ \vdots & \ddots & \vdots \\ f(k_N - k_1) & \cdots & f(k_N - k_N) \end{vmatrix} = \sum_P (-1)^{[P]} \prod_{a=1}^N f(P[k_a] - k_a). \quad (1.134)$$

To reproduce the last line of the Eq. (1.133) one should replace the  $l$ -th column of the matrix  $f(k_j - k_l)$  with the expression  $\exp\{-iz_1 k_l\}/L$ :

$$\rho_N(x) = \frac{\partial}{\partial \lambda} \det_N(I + \tilde{V}_1 + \lambda \tilde{V}_2) \Big|_{\lambda=0} = \det_N(I + \tilde{V}_1 + \tilde{V}_2) - \det_N(I + \tilde{V}_1), \quad (1.135)$$

where

$$(\tilde{V}_1)_{ab} = -\frac{2}{L} \frac{\exp\{ix(k_a - k_b)\} - 1}{i(k_a - k_b)}, \quad (\tilde{V}_2)_{ab} = \frac{1}{L} \exp\{-ixk_b\}, \quad a, b = 1, \dots, N. \quad (1.136)$$

Equation (1.135) can be rewritten in the following form

$$\rho_N(x) = \frac{\partial}{\partial \lambda} \det_N(I + V_1 + \lambda V_2) \Big|_{\lambda=0} = \det_N(I + V_1 + V_2) - \det_N(I + V_1), \quad (1.137)$$

where

$$(V_1)_{ab} = -2 \frac{2\pi}{L} \frac{\sin[\frac{x}{2}(k_a - k_b)]}{\pi(k_a - k_b)}, \quad (V_2)_{ab} = \frac{1}{L} \exp\left\{-i\frac{x}{2}(k_a + k_b)\right\}, \quad a, b = 1, \dots, N. \quad (1.138)$$

Equations (1.137) and (1.138) are the desired Lenard's formula for the one-particle density matrix of the Tonks-Girardeau gas at zero temperature. It can be easily generalized to a finite temperature case.

It is proper place to stress again that the Tonks-Girardeau gas is *not* equivalent to the free-fermion gas: though the spectrum of the Tonks-Girardeau gas is the same as that of the free fermion gas, the wave functions have different symmetry. This changes some observables drastically: for example, the one-particle density matrix for the free fermion gas

$$\rho_N^{ff}(x) = \frac{1}{L} \sum_{j=1}^N \exp\{-ixk_j\} \quad (1.139)$$

is totally different from that of the Tonks-Girardeau gas, Eqs. (1.137) and (1.138).

### 1.5.2 The Fredholm determinant

We have seen that in the  $N$ -particle sector the density matrix (1.123) is expressed via the determinants of  $N \times N$  matrices, Eqs. (1.137) and (1.138). In the thermodynamic limit (1.24) these matrices becomes infinite-dimensional. Thus, an important object comes into play: the determinant of an infinite-dimensional matrix (it is called Fredholm determinant). This object appears regularly in the calculations of the correlation functions of the integrable models.

We give in this section two alternative definitions of the Fredholm determinant. The first one is: Let  $V$  be an  $N \times N$  matrix with the entries  $V_{ab} = V(k_a, k_b)$ . Let

$$k_a = \left( \frac{2a}{N-1} - 1 \right), \quad a = 0, 1, \dots, N-1. \quad (1.140)$$

Then the Fredholm determinant  $\det(\hat{I} + \hat{V})$  is defined as follows:

$$\det(\hat{I} + \hat{V}) = \lim_{N \rightarrow \infty} \det_N \left( I + \frac{2}{N-1} V \right). \quad (1.141)$$

The matrix  $I$  on the right hand side of Eq. (1.141) is the  $N \times N$  identity matrix. We see that with increasing number of divisions,  $N$ , the possible values of  $k_a$ , Eq. (1.140), fill the interval  $-1 \leq k_a \leq 1$  densely. The objects  $\hat{I}$  and  $\hat{V}$  on the left hand side of the Eq. (1.141) should thus be some integral operators with the kernels defined on  $[-1, 1] \times [-1, 1]$ . This can be readily seen from an alternative definition of the Fredholm determinant:

$$\det(\hat{I} + \hat{V}) = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{-1}^1 dk_1 \dots \int_{-1}^1 dk_N \det \begin{bmatrix} V(k_1, k_1) & \dots & V(k_1, k_N) \\ \vdots & \ddots & \vdots \\ V(k_N, k_1) & \dots & V(k_N, k_N) \end{bmatrix}, \quad (1.142)$$

where  $\hat{V}$  is a linear integral operator with the kernel  $V(k, p)$  defined on  $[-1, 1] \times [-1, 1]$ . The equivalence of the definitions (1.141) and (1.142) is proven, for example, in Ref. [15]. These definitions are somehow complementary: the former one is more suitable for a numerical analysis, while the latter one reveals better the analytic structure of the problem.

### 1.5.3 One-particle density matrix: thermodynamic limit

Let us set up a convention first. The one-dimensional density  $\rho$  is controlled by the chemical potential  $\mu$  in the infinite  $\gamma$  limit as given by Eq. (1.28):

$$\sqrt{\mu} = \pi\rho. \quad (1.143)$$

It follows from this equation that  $\sqrt{\mu}$  has a dimension of inverse length. In the rest of section 1.5 we measure distances in units of  $1/\sqrt{\mu}$ . This convention implies that the momentum is measured in units of the Fermi momentum  $k_F$ , Eq. (1.29).

The ground state and zero temperature thermodynamics of the Tonks-Girardeau gas are described in section 1.2.2. Using formulas from that section, write the thermodynamic limit of the Lenard's formula (1.137) in the form prescribed by the definition (1.141):

$$\frac{\rho(x)}{\rho(0)} = \det(\hat{I} + \hat{V}_1 + \hat{V}_2) - \det(\hat{I} + \hat{V}_1). \quad (1.144)$$

The kernels of the operators  $\hat{V}_1$  and  $\hat{V}_2$  in Eq. (1.144) are given by the thermodynamic limit of Eq. (1.138):

$$V_1(k, p) = \alpha \frac{\sin \frac{x}{2}(k - p)}{\pi(k - p)} \quad (1.145)$$

and

$$V_2(k, p) = \frac{1}{2} \exp\left\{-i\frac{x}{2}(k + p)\right\}. \quad (1.146)$$

These kernels are defined on  $[-1, 1] \times [-1, 1]$ , in accordance with our convention formulated below Eq. (1.143). The parameter  $\alpha$  is

$$\alpha = -2. \quad (1.147)$$

Note that  $\rho(0) = \rho$ ; one has  $\rho = 1/\pi$  if measured in units of  $\sqrt{\mu}$ .

### 1.5.4 One-particle density matrix: asymptotics and numerics

Since  $\rho(x) = \rho(-x)$ , we assume  $x \geq 0$  in this section. For short distances ( $x \ll 1$ ) one can easily find from Eqs. (1.144) and (1.142) that

$$\frac{\rho(x)}{\rho(0)} = 1 - \frac{x^2}{6} - \frac{\alpha x^3}{\pi 18} + \frac{x^4}{120} + \frac{\alpha 11x^5}{\pi 2700} + \dots, \quad x \rightarrow 0. \quad (1.148)$$

For  $\alpha = 0$  this is simply the expansion of  $\sin x/x$ . Notice that higher order terms that are not exhibited on the r.h.s. of Eq. (1.148), depend on  $\alpha$  nonlinearly.

Next, consider the long-distance expansion of Eq. (1.144). Compared to the short-distance analysis, this is a more sophisticated task. The analysis carried out in Refs. [16, 17, 18] shows the power-law decay of the density matrix:

$$\frac{\rho(x)}{\rho(0)} = \frac{C}{\sqrt{x}} \left[ 1 + \frac{1}{8x^2} \left( \cos 2x - \frac{1}{4} \right) + \frac{3}{16x^3} \sin 2x \right], \quad x \rightarrow \infty \quad (1.149)$$

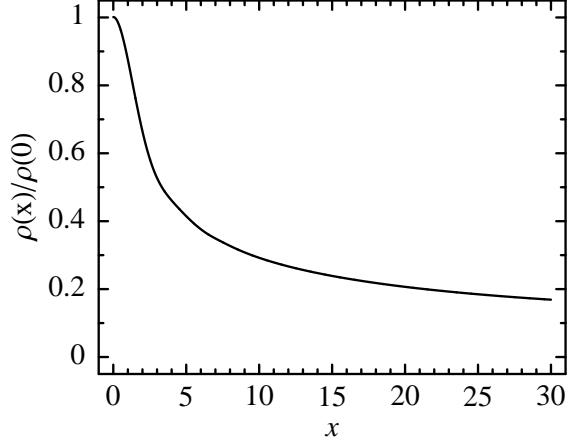


Figure 4: Shown is the function  $\rho(x)/\rho(0)$  versus  $x$ . The density matrix  $\rho(x)$  is given by Eq. (1.144); in getting the plot the Fredholm determinants entering Eq. (1.144) were approximated by finite-dimensional matrices, Eq. (1.141), with  $N = 800$ .

with relative corrections of the order of  $x^{-1}$ . The constant  $C$  is given by

$$C = \pi e^{1/2} 2^{-1/3} A^{-6} \approx 0.924, \quad (1.150)$$

with  $A = 1.2824271\dots$  being Glaisher's constant. The result (1.149) confirms the Luttinger liquid theory predictions.

For the curve in Fig. 4 we took  $N = 800$  in using the representation (1.141) for the Fredholm determinants entering Eq. (1.144). For  $x$  large than 30 it is safe to use the asymptotic expansion (1.149) instead of the exact expression (1.144): the relative difference  $\Delta(x)$  between the exact expression and the leading term of the large  $x$  expansion

$$\Delta(x) = \frac{\rho(x) - C/\sqrt{x}}{\rho(x)} \quad (1.151)$$

is very small for  $x = 30$ :

$$\Delta(30) \approx 6 \cdot 10^{-4}. \quad (1.152)$$

so the subleading terms in expansion (1.149) are not relevant for most applications. On the other hand, the function  $1/\sqrt{x}$  converges to zero very slowly and the knowledge of constant  $C$  entering the expansion (1.149) is required with a good precision. Thus, the exact expression (1.150) is very much appreciated.

### 1.5.5 The momentum distribution

The momentum distribution function  $n(k)$  is given by

$$n(k) = \int dx e^{-ikx} \rho(x), \quad (1.153)$$

where  $\rho(x)$  is the one-particle density matrix (1.123). Recall that  $x$  is measured in units of  $1/\sqrt{\mu}$  as defined below Eq. (1.143), and that  $\rho(x) = \rho(-x)$ . The momentum distribution is

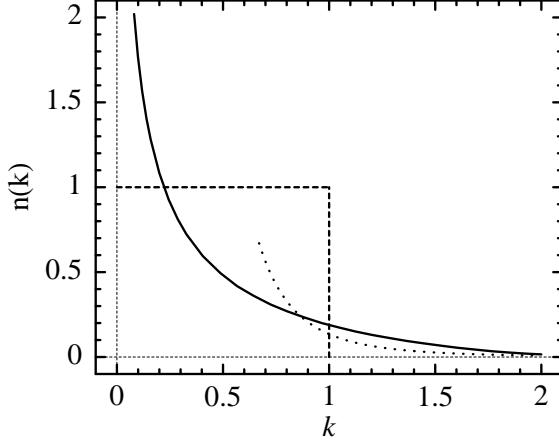


Figure 5: Shown is the momentum distribution for the Tonks-Girardeau gas (solid curve) and for the free fermion gas (dashed curve). Both distributions are normalized by the condition (1.154). Dotted curve represents the large  $k$  asymptotics of the momentum distribution, Eq. (1.155).

also symmetric,  $n(k) = n(-k)$ . It is normalized as follows

$$\int_0^\infty dk n(k) = 1. \quad (1.154)$$

We plot  $n(k)$  in Fig. 5. The density matrix  $\rho(x)$  for  $x \leq 30$  was calculated from the Fredholm determinant representation; for  $x \geq 30$  we have used the asymptotic expansion (1.149). This momentum distribution is compared with the momentum distribution of the free fermion gas.

For the large momentum tail of  $n(k)$  one gets

$$n(k) \simeq \frac{4}{3\pi^2} \frac{1}{k^4}, \quad k \rightarrow \infty. \quad (1.155)$$

The  $1/k^4$  decay originates from the cusp of  $\rho(x)$  at  $x = 0$  seen in the third term on the right-hand side of Eq. (1.148). The coefficient  $3/4\pi^2$  in Eq. (1.155) is taken from the Ref. [19]. It is useful to note that Ref. [19] contains more general result: the large momentum tail of  $n(k)$  decays as  $1/k^4$  at any coupling strength; the coefficient stands at  $1/k^4$  term was also found in the cited reference.

It follows from the large  $x$  expansion (1.149) that the momentum distribution is divergent at small momenta,

$$n(k) \simeq C \sqrt{\frac{2}{\pi k}}, \quad k \rightarrow 0. \quad (1.156)$$

where the coefficient  $C$  is given by Eq. (1.150). This singularity is predicted by the Luttinger liquid theory for any coupling strength  $\gamma$ , and is the manifestation of quasi-long-range order, that is, a quasicondensate. The singularity is integrable for any  $\gamma \neq 0$ , therefore zero momentum state is not macroscopically occupied. Note, however, that a significant fraction of particles lies in low-momentum states: for the Tonks-Girardeau gas about 15% of particles have a momentum lower than 0.01 (in units of  $k_F$ , as discussed below Eq. (1.143)). When  $\gamma$  lowers this fraction is increased.

## 2 Algebraic Bethe Ansatz

The Quantum Inverse Scattering Method, and its descendant, Algebraic Bethe Ansatz, were proposed in late 70's (several crucial works appeared in the years 1978 and 1979). One can consult chapters VI and VII of the book [5] for the list of references. Various relativistic and non-relativistic quantum field theory models in  $1 + 1$  (one space one time) dimension were analyzed. Given the time limitations of a single lecture, we will learn the method by using it to solve the spin-1/2 Heisenberg chain. This is, probably, the simplest model which can be solved by the Algebraic Bethe Ansatz: it is non-relativistic (remember how careful one should treat infinite Dirac sea in the Luttinger model) and has finite-dimensional quantum space.

### 2.1 Coordinate Bethe Ansatz for the spin-1/2 XXZ Heisenberg chain

The coordinate Bethe Ansatz for the spin-1/2 XXZ Heisenberg spin chain is reviewed. It is similar to the coordinate Bethe Ansatz for the Lieb-Liniger model presented in section 1.

#### 2.1.1 Hamiltonian, eigenfunctions and spectrum

Recall some basic notations. The Pauli spin matrices are

$$\sigma^{(x)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^{(y)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^{(z)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.1)$$

It is often convenient to use  $\sigma^\pm$  instead of  $\sigma^{(x)}$  and  $\sigma^{(y)}$ :

$$\sigma^{(x)} = \sigma^{(+)} + \sigma^{(-)}, \quad \sigma^{(y)} = -i(\sigma^{(+)} - \sigma^{(-)}) \quad (2.2)$$

Useful multiplication formula

$$\sigma^{(p)}\sigma^{(q)} = i\epsilon_{pqr}\sigma^{(r)} + \delta_{pq}I \quad (2.3)$$

The tensor product for  $2 \times 2$  matrices  $a$  and  $b$  is defined as follows<sup>1</sup>:

$$a \otimes b = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}. \quad (2.4)$$

We will work with the operators living in the tensor product of  $M$   $2 \times 2$  matrices. Periodic boundary conditions are imposed. The Hamiltonian of the  $XXZ$  chain is

$$H = H_0 + M\Delta - 2hS_z, \quad (2.5)$$

---

<sup>1</sup>When the type of the operator's product, ordinary or tensorial, is clear from the context, the symbol “ $\otimes$ ” is usually omitted to lighten the formulas. Expression (2.6) is an example, there  $\sigma_m\sigma_{m+1}$  means  $\sigma_m \otimes \sigma_{m+1}$ .

$$H_0 = - \sum_{m=1}^M \left[ \sigma_m^{(x)} \sigma_{m+1}^{(x)} + \sigma_m^{(y)} \sigma_{m+1}^{(y)} + \Delta \sigma_m^{(z)} \sigma_{m+1}^{(z)} \right], \quad (2.6)$$

$$S_z = \frac{1}{2} \sum_{m=1}^M \sigma_m^{(z)}. \quad (2.7)$$

Eigenfunctions of this Hamiltonian are

$$|\psi_N(\lambda_1, \dots, \lambda_N)\rangle = \frac{1}{N!} \sum_{m_1, \dots, m_N=1}^M \chi_N(m_1, \dots, m_N | \lambda_1, \dots, \lambda_N) \sigma_{m_1}^{(-)} \cdots \sigma_{m_N}^{(-)} |0\rangle \quad (2.8)$$

where  $|0\rangle$  is the ferromagnetic vacuum (all spins up),  $|0\rangle = \otimes_{j=1}^M |\uparrow\rangle$ . There exists an operator  $\mathcal{V}$  performing an important unitary transformation

$$\mathcal{V}H(h)\mathcal{V}^{-1} = H(-h), \quad \mathcal{V} = \prod_{m=1}^M \sigma_m^{(z)}. \quad (2.9)$$

Since  $\sigma^{(x)} |\uparrow\rangle = |\downarrow\rangle$  and  $\sigma^{(x)} \sigma^{(-)} |\uparrow\rangle = \sigma^{(+)} |\downarrow\rangle$ , the wave function  $\mathcal{V}|\psi_N(\lambda_1, \dots, \lambda_N)\rangle$  is

$$\mathcal{V}|\psi_N(\lambda_1, \dots, \lambda_N)\rangle = \frac{1}{N!} \sum_{m_1, \dots, m_N=1}^M \chi_N(m_1, \dots, m_N | \lambda_1, \dots, \lambda_N) \sigma_{m_1}^{(+)} \cdots \sigma_{m_N}^{(+)} |0'\rangle, \quad (2.10)$$

where  $|0'\rangle$  is the dual ferromagnetic vacuum (all spins down),  $|0'\rangle = \otimes_{j=1}^M |\downarrow\rangle$ . Comparing Eqs. (2.8) and (2.10) we see that the first-quantized wave function is the same in these expressions. Therefore, it is sufficient to consider  $h > 0$ , since all the observables will have a definite parity with respect to  $h$  in our problem. In particular, the ground state energy is an even function of  $h$ , and the magnetization is an odd one. Another useful similarity transformation, which exists for even number of sites only, is

$$\mathcal{U}H(\Delta, h=0)\mathcal{U}^{-1} = -H(-\Delta, h=0), \quad \mathcal{U} = \prod_{m=1}^{M/2} \sigma_{2m}^{(z)}. \quad (2.11)$$

Due to the existence of this transformation, some authors choose the “+” sign in front of the Hamiltonian (2.5). We, however, will keep the “-” sign throughout this text. In our further calculations  $M$  is assumed to be even.

The function  $\chi_N$  is the Bethe ansatz wave function:

$$\begin{aligned} \chi_N = & \left[ \prod_{N \geq b > a \geq 1} \text{sgn}(m_b - m_a) \right] \sum_{\mathcal{Q}} (-1)^{\mathcal{Q}} \exp \left[ -i \sum_{a=1}^N m_a p_0(\lambda_{Q_a}) \right] \\ & \times \exp \left[ -\frac{i}{2} \sum_{N \geq b > a \geq 1} \theta(\lambda_{Q_b} - \lambda_{Q_a}) \text{sgn}(m_b - m_a) \right] \end{aligned} \quad (2.12)$$

The summation runs through all the permutations  $\mathcal{Q}$  of the integers  $1, 2, \dots, N$ . The anisotropy parameter  $\Delta$  is convenient to represent as follows

$$\Delta = \cos(2\eta). \quad (2.13)$$

Since the model is defined on a lattice, the momentum operator should be defined via the shift operator. The eigenfunctions of the shift operator  $Q$  in the  $N$ -particle sector have the form

$$Q_N = \sum_{\alpha=1}^N p_0(\lambda_\alpha) \quad (2.14)$$

Thus, one calls sometimes  $p_0$  by a momentum density. For the parameters  $\lambda_\alpha$  the word rapidities us used. Unfortunately, the word “quasi-momentum distribution” is used for  $\lambda_\alpha$  as well, so one should be careful. Momentum density  $p_0$  and scattering phase  $\theta$  are expressed via rapidities as follows:

$$p_0(\lambda) = i \ln \frac{\cosh(\lambda - i\eta)}{\cosh(\lambda + i\eta)}, \quad p_0(0) = 0, \quad (2.15)$$

and

$$\theta(\lambda_1 - \lambda_2) = i \ln \frac{\sinh(2i\eta + \lambda_1 - \lambda_2)}{\sinh(2i\eta - \lambda_1 + \lambda_2)}, \quad \theta(0) = 0. \quad (2.16)$$

Note that  $p_0(\lambda)$  and  $\theta(\lambda)$  are antisymmetric, and

$$\theta(\lambda) \rightarrow \pi - 4\eta \quad \text{as} \quad \lambda \rightarrow \infty \quad (2.17)$$

Periodic boundary conditions imposed onto the eigenfunctions of the Hamiltonian (2.5) imply a set of nonlinear equations onto rapidities. These equations are called Bethe equations:

$$\left[ \frac{\cosh(\lambda_\alpha - i\eta)}{\cosh(\lambda_\alpha + i\eta)} \right]^M = (-1)^{N-1} \prod_{\beta \neq \alpha}^N \frac{\sinh(2i\eta - \lambda_\alpha + \lambda_\beta)}{\sinh(2i\eta + \lambda_\alpha - \lambda_\beta)}. \quad (2.18)$$

They can be written in the logarithmic form

$$Mp_0(\lambda_\alpha) + \sum_{\beta=1}^N \theta(\lambda_\alpha - \lambda_\beta) = 2\pi n_\alpha, \quad \alpha = 1, \dots, N. \quad (2.19)$$

Here  $n_\alpha$  are integers (half-integers) when  $N$  is odd (even). We stress that these equations do not contain magnetic field. The eigenvalues of the Hamiltonian (2.5) are, of course, field-dependent:

$$E_N = \sum_{\alpha=1}^N \varepsilon_0(\lambda_\alpha) - hM, \quad (2.20)$$

where the one-particle bare dispersion  $\varepsilon_0$  is the followin function of the quasi-momentum  $p_0$ :

$$\begin{aligned} \varepsilon_0(\lambda) &= -2 \sin(2\eta) \frac{dp_0(\lambda)}{d\lambda} + 2h = -\frac{2 \sin^2(2\eta)}{\cosh(\lambda + i\eta) \cosh(\lambda - i\eta)} + 2h \\ &= -\frac{4 \sin^2(2\eta)}{\cosh(2\lambda) + \cos(2\eta)} + 2h. \end{aligned} \quad (2.21)$$

Here

$$\frac{dp_0(\lambda)}{d\lambda} = \frac{2 \sin(2\eta)}{\cosh(2\lambda) + \cos(2\eta)}. \quad (2.22)$$

We see that the state  $\psi_N$  contains  $N$  particles with quasi-momenta  $p_0(\lambda_\alpha)$  ( $\alpha = 1, \dots, N$ ) and energies  $\varepsilon_0(\lambda_\alpha)$ . These excitations are excitations over the ferromagnetic vacuum  $|0\rangle$  and they are usually called spin waves.

### 2.1.2 Particular case: XXX model

The limiting case  $\Delta = 1$  corresponds to the isotropic ferromagnetic. The ground state is the ferromagnetic vacuum, so the magnetization takes maximal possible value. For  $\Delta > 1$  the ground state remains to be ferromagnetic. The point  $\Delta = -1$  corresponds to the isotropic antiferromagnetic.

(I) Consider  $\Delta = -1$  (antiferromagnetic). To take this limit we perform the rescaling procedure

$$\lambda \rightarrow \delta\lambda, \quad \eta \rightarrow \frac{\pi}{2} - \frac{\delta}{2}, \quad \delta \rightarrow 0. \quad (2.23)$$

The Bethe equations (2.18) become (recall that  $M$  is even)

$$\left( \frac{\lambda_\alpha + i/2}{\lambda_\alpha - i/2} \right)^M = \prod_{\beta \neq \alpha}^N \frac{\lambda_\alpha - \lambda_\beta + i}{\lambda_\alpha - \lambda_\beta - i}, \quad \alpha = 1, \dots, N. \quad (2.24)$$

The bare dispersion (2.21) becomes

$$\varepsilon_0(\lambda) = -\frac{2}{\lambda^2 + \frac{1}{4}} + 2h. \quad (2.25)$$

(II) Consider  $\Delta = 1$  (ferromagnetic). To take this limit we perform the rescaling procedure

$$\lambda \rightarrow \delta\lambda + i\frac{\pi}{2}, \quad \eta \rightarrow \frac{\delta}{2}, \quad \delta \rightarrow 0. \quad (2.26)$$

The Bethe equations (2.18) become the same as for the antiferromagnetic case, while the bare dispersion is

$$\varepsilon_0(\lambda) = \frac{2}{\lambda^2 + \frac{1}{4}} + 2h. \quad (2.27)$$

## 2.2 Thermodynamics of the XXZ model (optional reading)

The thermodynamic limit of the Bethe equations is taken. The procedure is similar to the one carried out for the Lieb-Liniger model, section 1.

### 2.2.1 Lieb equation

To begin with, note some important properties of the ground state. It can be proven that the parameters  $n_\alpha$  fill the symmetric interval around zero and the rapidities  $\lambda_\alpha$  are real in the ground state. It will be convenient to work with the density of particles in the rapidity space (we will sometimes call it the distribution function of  $\lambda_\alpha$ 's):

$$\rho(\lambda_\alpha) = \frac{1}{M(\lambda_{\alpha+1} - \lambda_\alpha)}, \quad (2.28)$$

that is

$$\sum_{\alpha=1}^N f(\lambda_\alpha) = M \sum_{\alpha=1}^N (\lambda_{\alpha+1} - \lambda_\alpha) \rho(\lambda_\alpha) f(\lambda_\alpha), \quad (2.29)$$

where  $f(\lambda_\alpha)$  is an arbitrary function.

We consider now the limit of large number of sites  $M$  and large number of particles  $N$ :

$$D = \frac{N}{M} \rightarrow \text{const}, \quad N, M \rightarrow \infty. \quad (2.30)$$

This limit will be called thermodynamic limit. One has in this limit

$$\sum_{\alpha=1}^N (\lambda_{\alpha+1} - \lambda_\alpha) \rho(\lambda_\alpha) f(\lambda_\alpha) \rightarrow \int_{-\Lambda}^{\Lambda} d\lambda \rho(\lambda) f(\lambda). \quad (2.31)$$

The integration limits are defined by the normalization condition

$$D = \int_{-\Lambda}^{\Lambda} d\lambda \rho(\lambda). \quad (2.32)$$

The Bethe equations (2.19) become in the thermodynamic limit

$$2\pi\rho(\lambda) - \int_{-\Lambda}^{\Lambda} d\mu K(\lambda, \mu) \rho(\mu) = \frac{dp_0(\lambda)}{d\lambda}. \quad (2.33)$$

where

$$\begin{aligned} K(\lambda, \mu) &= \frac{\partial \theta(\lambda - \mu)}{\partial \lambda} = \frac{\sin(4\eta)}{\sinh(\lambda - \mu + 2i\eta) \sinh(\lambda - \mu - 2i\eta)} \\ &= \frac{2 \sin(4\eta)}{\cosh[2(\lambda - \mu)] - \cos(4\eta)}. \end{aligned} \quad (2.34)$$

The magnetization  $\sigma$  is

$$\sigma = \langle \sigma_n^{(z)} \rangle = 1 - 2D. \quad (2.35)$$

Having calculated  $\rho(\lambda)$  one gets all the thermodynamics of the model. The ground state energy density  $\epsilon$

$$\epsilon = \frac{E}{M} \quad (2.36)$$

can be found from Eq. (2.20):

$$\epsilon = -h + \int_{-\Lambda}^{\Lambda} d\lambda \rho(\lambda) \varepsilon_0(\lambda) = -h(1 - 2D) - \int_{-\Lambda}^{\Lambda} d\lambda \rho(\lambda) \frac{4 \sin^2(2\eta)}{\cosh(2\lambda) + \cos(2\eta)}. \quad (2.37)$$

### 2.2.2 Interaction with magnetic field

We need to know how to calculate the ground state density at a given value of a chemical potential. We write

$$\varepsilon(\lambda) - \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} d\mu K(\lambda, \mu) \varepsilon(\mu) = \varepsilon_0(\lambda), \quad (2.38)$$

with the condition

$$\varepsilon(\pm\Lambda) = 0. \quad (2.39)$$

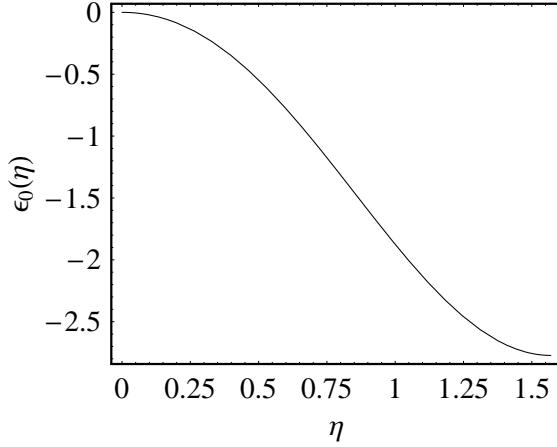


Figure 6: Shown is the ground state energy density  $\varepsilon(\eta)$  as a function of anisotropy  $\eta$  for  $h = 0$ .

When  $h = 0$  one has  $\sigma = 0$  and  $D = 1/2$ . The system of equations (2.32) and (2.33) can be solved analytically and gives

$$\rho(\lambda) = \left[ 2(\pi - 2\eta) \cosh \left( \frac{\pi\lambda}{\pi - 2\eta} \right) \right]^{-1}. \quad (2.40)$$

The ground state at zero field is antiferromagnetic. The ground state has nontrivial structure in the region  $0 \leq h < h_c$ , where

$$h_c = 4 \sin^2 \eta = 2(1 - \Delta). \quad (2.41)$$

Above the critical field  $h_c$  the ground state is ferromagnetic. The parameter  $\Lambda$  decays from infinity to zero when  $h$  is changing from 0 to  $h_c$ . At  $h = 0$  one can calculate an energy per site,  $\epsilon$ , from Eqs. (2.37) and (2.40). For several values of  $\Delta$  it is given in the table.

$\eta$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\Delta = \cos 2\eta$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1
$\epsilon$	0	$2 - \frac{3\sqrt{3}}{2}$	$-\frac{4}{\pi}$	-2	-2.77

A result for an arbitrary value of  $\Delta$  is shown in Fig. 6

An important characteristics of the system is the magnetic susceptibility

$$\chi = \left( \frac{\partial \sigma}{\partial h} \right)_D. \quad (2.43)$$

We obtain the curves of  $\sigma$  and  $\chi$  at  $\Delta = 1/2$  solving the system of Eqs. (2.32) and (2.33) numerically. The technique and the results are discussed in a separate section.

### 2.2.3 Zero temperature thermodynamics at $\Delta = 0$

Bethe equations can be solved explicitly in this case. One has from Eq. (2.33)

$$\rho(\lambda) = \frac{1}{\pi} \frac{1}{\cosh(2\lambda)}. \quad (2.44)$$

The dressing equation for  $\varepsilon(\lambda)$ , Eq. (2.38) becomes

$$\varepsilon(\lambda) = -\frac{4}{\cosh(2\lambda)} + 2h \quad (2.45)$$

and the condition (2.39) gives us an explicit relation between  $\Lambda$  and  $h$  :

$$\cosh(2\Lambda) = \frac{2}{h}. \quad (2.46)$$

Equation (2.32) gives us, in turn, the relation between  $\Lambda$  and  $D$  :

$$\tan\left(\frac{\pi}{2}D\right) = \tanh(\Lambda). \quad (2.47)$$

Using these relations and the identities

$$\tanh^2(\Lambda) = \frac{\cosh(2\Lambda) - 1}{\cosh(2\Lambda) + 1} = \frac{2-h}{2+h}, \quad \tanh(2\Lambda) = \frac{2\tanh\Lambda}{1+\tanh^2\Lambda} = \frac{1}{2}\sqrt{4-h^2} \quad (2.48)$$

one gets the relation between  $D$  and  $h$  :

$$h = 2 \cos(\pi D), \quad 0 \leq D \leq \frac{1}{2}. \quad (2.49)$$

For the energy per site, Eq. (2.37), one has

$$\begin{aligned} \epsilon &= -h(1-2D) - \frac{4}{\pi} \tanh(2\Lambda) \\ &= -h \left( 1 - \frac{4}{\pi} \arctan \sqrt{\frac{2-h}{2+h}} \right) - \frac{4}{\pi} \sqrt{1 - \frac{h^2}{4}}, \quad 0 \leq h \leq 2 \\ &= -2 \cos(\pi D)(1-2D) - \frac{4}{\pi} \sin(\pi D), \quad 0 \leq D \leq \frac{1}{2}. \end{aligned} \quad (2.50)$$

For the magnetic susceptibility, Eq. (2.43), one has

$$\chi = \frac{1}{\pi} \left( 1 - \frac{h^2}{4} \right)^{-\frac{1}{2}}, \quad 0 \leq h \leq 2. \quad (2.51)$$

The magnetization is plotted in Fig. 7. The behavior of susceptibility is obvious from this plot.

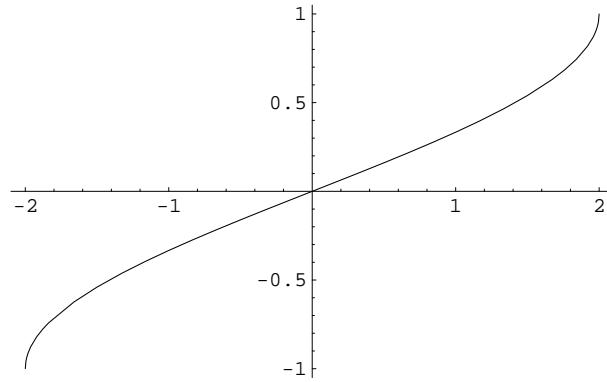


Figure 7: Shown is the magnetization  $\sigma$  as a function of  $h$  for  $\Delta = 0$  and  $T = 0$ .

#### 2.2.4 Finite-temperature thermodynamics at $\Delta = 0$

We will follow the notations of Ref. [20] in this section. Namely,

$$\begin{aligned} H &= \sum_{j=1}^M (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) - 2h \sum_{j=1}^M S_j^z \\ &= \frac{1}{2} \sum_{j=1}^M (\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+) - h \sum_{j=1}^M (2\sigma_j^+ \sigma_j^- - 1). \end{aligned} \quad (2.52)$$

Perform the Jordan-Wigner transformation

$$c_m = e^{\pi i \sum_{j=1}^{m-1} \sigma_j^+ \sigma_j^-} \sigma_m^-, \quad c_m^\dagger = \sigma_m^+ e^{-\pi i \sum_{j=1}^{m-1} \sigma_j^+ \sigma_j^-}. \quad (2.53)$$

The operators  $c^\dagger$  and  $c$  are Fermi operators

$$\{c_m, c_n^\dagger\} = \delta_{mn} \quad (2.54)$$

Since  $c_m^\dagger c_m = \sigma_m^+ \sigma_m^-$ , the inverse transform is

$$\sigma_m^- = e^{-\pi i \sum_{j=1}^{m-1} c_j^\dagger c_j} c_m, \quad \sigma_m^+ = c_m^\dagger e^{\pi i \sum_{j=1}^{m-1} c_j^\dagger c_j}. \quad (2.55)$$

The Hamiltonian takes the form (we have omitted  $c$ -number term)

$$H = \frac{1}{2} \sum_{j=1}^M (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - 2h \sum_{j=1}^M c_j^\dagger c_j. \quad (2.56)$$

Take the Fourier transform

$$c_k^\dagger = \frac{1}{M} \sum_{j=1}^M e^{-ikj} c_j^\dagger, \quad c_j^\dagger = \sum_k e^{ikj} c_k^\dagger, \quad k = \frac{2\pi}{M} q, \quad q = 1, 2, \dots, M. \quad (2.57)$$

Therefore

$$H = \sum_k (\cos k - 2h) c_k^\dagger c_k. \quad (2.58)$$

We calculate now the partition function of the model. It seems that the closed expression for it does not exist on a finite lattice. However, in the thermodynamic limit the closed expression can be obtained. Indeed,

$$\mathcal{Z} = \text{tr } e^{-\beta H} = \text{tr } e^{-\beta \sum_k \epsilon_k c_k^\dagger c_k} = \prod_k (1 + e^{-\beta \epsilon_k}). \quad (2.59)$$

Introducing the free energy,

$$F = -T \log \mathcal{Z} \quad (2.60)$$

and taking the limit of large system size,

$$\sum_k \rightarrow \frac{M}{2\pi} \int dk, \quad (2.61)$$

one gets

$$\frac{F}{M} = -\frac{T}{2\pi} \int dk \log (1 + e^{-\beta \epsilon_k}). \quad (2.62)$$

Finally, taking into account the definition of the Fermi distribution function,

$$f_k = \frac{1}{e^{\beta \epsilon_k} + 1}, \quad 1 - f_k = \frac{1}{1 + e^{-\beta \epsilon_k}} \quad (2.63)$$

one represents  $F$  as follows

$$\frac{F}{M} = \frac{T}{2\pi} \int dk \log (1 - f_k). \quad (2.64)$$

It is useful to calculate the entropy from this expression. Indeed,

$$\frac{S}{M} = -\frac{1}{M} \left( \frac{\partial F}{\partial T} \right)_M = -\frac{1}{2\pi} \int dk \log(1 - f_k) - \frac{T}{2\pi} \int dk \frac{\partial}{\partial T} \log(1 - f_k). \quad (2.65)$$

We get then the internal energy  $U$  of the fermion gas,

$$\frac{U}{M} = \frac{F}{M} + T \frac{S}{M} = -\frac{T^2}{2\pi} \int dk \frac{\partial}{\partial T} \log(1 - f_k) = \frac{1}{2\pi} \int dk \epsilon_k f_k. \quad (2.66)$$

The density of the particles,  $D = N/M$ , is

$$D = \frac{1}{2\pi} \int dk f_k. \quad (2.67)$$

We consider the dispersion  $\epsilon_k$  of the form

$$\epsilon_k = -\cos k - 2h, \quad -\pi \leq k \leq \pi. \quad (2.68)$$

Note that the sign in front of  $\cos k$  is immaterial, it depends on the range of  $k$ . At  $T = 0$  one has  $D = 1$  when  $h = 1/2$  and  $D = 0$  when  $h = -1/2$ . So,  $h = \pm 1/2$  are the critical values of the magnetic field, and we define the magnetization  $\sigma$  as follows

$$\sigma = 2D - 1. \quad (2.69)$$

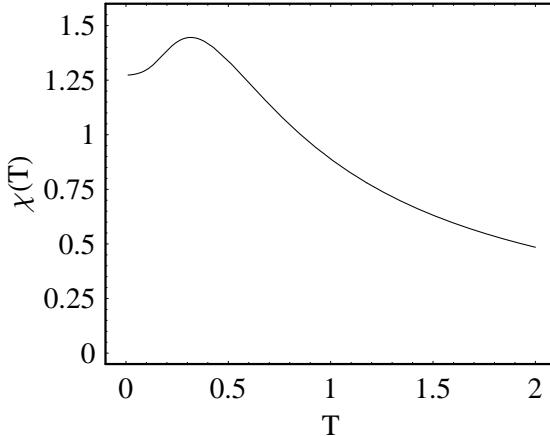


Figure 8: Shown is the zero field susceptibility  $\chi(T, h = 0)$  as a function of  $T$  for  $\Delta = 0$ .

For  $T = 0$  it is shown in Fig. 7. Note that the sign of the above equation is different from that of Eq. (2.35).

Let us calculate the magnetic susceptibility of the system, Eq. (2.43), focusing on the case of zero field. One can easily show that

$$\chi(T, h = 0) = \frac{1}{\pi} \int_{-\pi}^{\pi} dk \frac{\beta/2}{\cosh^2(\frac{\beta}{2} \cos k)}, \quad \chi(T = 0, h = 0) = \frac{4}{\pi}. \quad (2.70)$$

For arbitrary temperature this function is plotted in Fig. 8. Note non-monotonous behavior of this function.

### 2.3 Algebraic Bethe Ansatz for the Heisenberg chain

The eigenfunctions (2.8) are given in the form of non-trivial combinations of the local spin operators acting onto the vacuum (sometimes called “pseudovacuum”) state. At present, there exist an alternative formulation of the Bethe-ansatz. In this formulation the operators acting onto the vacuum state are not local spin operators: each of them acts non-trivially at *all* lattice sites. This is the so-called Algebraic Bethe Ansatz, while the constructions along the line of Bethe’s work are often referred to as Coordinate Bethe Ansatz. The ABA makes explicit the fact that the model (2.5) is integrable, that is it have infinitely many mutually commuting integrals of motion. All these integrals of motion can be calculated explicitly using ABA.

#### 2.3.1 Generating functional for the integrals of motion

We will stick to the notations used in the book [5]. We consider a lattice with  $L$  sites and impose periodic boundary conditions. The ABA solution of the model (2.5) starts with defining the so-called  $\mathcal{L}$ -operator

$$\mathcal{L}(n|\lambda) = -i \begin{pmatrix} \cosh(\lambda - i\eta\sigma_n^z) & \sigma_n^- \sin 2\eta \\ \sigma_n^+ \sin 2\eta & \cosh(\lambda + i\eta\sigma_n^z) \end{pmatrix}, \quad (2.71)$$

where  $\sigma$  are Pauli matrices, Eq. (2.1), and  $\lambda$  is a free (complex-valued) parameter, customarily called a spectral parameter. The entries of  $\mathcal{L}(n|\lambda)$  are quantum operators acting nontrivially on the  $n$ -th lattice site. The monodromy matrix  $T(\lambda)$  is defined as a product of the  $\mathcal{L}$ -operators taken over all lattice sites:

$$T(\lambda) \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = \mathcal{L}(L|\lambda) \cdots \mathcal{L}(2|\lambda) \mathcal{L}(1|\lambda). \quad (2.72)$$

The entries of  $T(\lambda)$  are quantum operators acting nontrivially on the whole lattice. They are polynomials of the local spin operators, but the complicated structure of these polynomials makes them hard to be used for calculations of any observable. In the contrast, local spin operator  $\sigma_m^j$  can be expressed through  $A$ ,  $B$ ,  $C$ , and  $D$  in a closed and relatively simple form [11, 12, 13, 21]. This representation (called the solution of the quantum inverse scattering problem), combined with numerics, was used quite successfully for the analysis of the dynamical properties (dynamical correlation functions) of the Heisenberg chain [22, 23]. However, how to extract the analytic (and numerical) values of the dynamical exponents near the threshold is an open problem within this approach; its discussion leave aside the scope of the present lecture.

The commutation relations between the entries of the monodromy matrix (2.72) are nontrivial. To get them we introduce an object which plays a crucial role in the ABA formalism. It is the so-called  $R$ -matrix

$$R(\lambda, \mu) = \begin{pmatrix} f(\mu, \lambda) & 0 & 0 & 0 \\ 0 & g(\mu, \lambda) & 1 & 0 \\ 0 & 1 & g(\mu, \lambda) & 0 \\ 0 & 0 & 0 & f(\mu, \lambda) \end{pmatrix}, \quad (2.73)$$

where

$$f(\mu, \lambda) = \frac{\sinh(\mu - \lambda + 2i\eta)}{\sinh(\mu - \lambda)}, \quad g(\mu, \lambda) = i \frac{\sin 2\eta}{\sinh(\mu - \lambda)} \quad (2.74)$$

One can check that the  $R$ -matrix (2.73) and the  $\mathcal{L}$ -operator (2.71) obey the so-called intertwining relation

$$R(\lambda, \mu) (\mathcal{L}(n|\lambda) \otimes \mathcal{L}(n|\mu)) = (\mathcal{L}(n|\mu) \otimes \mathcal{L}(n|\lambda)) R(\lambda, \mu). \quad (2.75)$$

We recall that the tensor product for  $2 \times 2$  matrices  $a$  and  $b$  is defined by Eq. (2.4). Note that

$$(a \otimes b)(c \otimes d) = (ac) \otimes (bd). \quad (2.76)$$

Equation (2.75) implies the intertwining relation for the monodromy matrix:

$$R(\lambda, \mu) (T(\lambda) \otimes T(\mu)) = (T(\mu) \otimes T(\lambda)) R(\lambda, \mu). \quad (2.77)$$

The proof of Eq. (2.77) is by induction. Equation (2.75) is the basis of the induction. By forming the product  $\mathcal{L}(m|\lambda) \cdots \mathcal{L}(2|\lambda) \mathcal{L}(1|\lambda)$  and increasing  $m$  from 1 to  $L$  we get Eq. (2.77).

Since Eq. (2.77) is the relation between  $4 \times 4$  matrices, it generates 16 relations between their entries. In particular, the operators of the same sort commute:

$$[A(\lambda), A(\mu)] = [B(\lambda), B(\mu)] = [C(\lambda), C(\mu)] = [D(\lambda), D(\mu)] = 0. \quad (2.78)$$

However, the commutation relations between different operators are nontrivial:

$$[A(\lambda), D(\mu)] = g(\lambda, \mu) (C(\lambda)B(\mu) - C(\mu)B(\lambda)) \quad (2.79)$$

$$[A(\lambda), D(\mu)] = g(\mu, \lambda) \{B(\lambda)C(\mu) - B(\mu)C(\lambda)\} \quad (2.80)$$

$$[B(\lambda), C(\mu)] = g(\lambda, \mu) (D(\lambda)A(\mu) - D(\mu)A(\lambda)) \quad (2.81)$$

$$[B(\lambda), C(\mu)] = g(\mu, \lambda) \{A(\lambda)D(\mu) - A(\mu)D(\lambda)\} \quad (2.82)$$

$$A(\mu)B(\lambda) = f(\mu, \lambda)B(\lambda)A(\mu) + g(\lambda, \mu)B(\mu)A(\lambda) \quad (2.83)$$

$$B(\lambda)A(\mu) = f(\lambda, \mu)A(\mu)B(\lambda) + g(\mu, \lambda)A(\lambda)B(\mu) \quad (2.84)$$

$$A(\lambda)C(\mu) = f(\mu, \lambda)C(\mu)A(\lambda) + g(\lambda, \mu)C(\lambda)A(\mu) \quad (2.85)$$

$$C(\mu)A(\lambda) = f(\lambda, \mu)A(\lambda)C(\mu) + g(\mu, \lambda)A(\mu)C(\lambda) \quad (2.86)$$

$$B(\lambda)D(\mu) = f(\mu, \lambda)D(\mu)B(\lambda) + g(\lambda, \mu)D(\lambda)B(\mu) \quad (2.87)$$

$$D(\mu)B(\lambda) = f(\lambda, \mu)B(\lambda)D(\mu) + g(\mu, \lambda)B(\mu)D(\lambda) \quad (2.88)$$

$$C(\lambda)D(\mu) = f(\lambda, \mu)D(\mu)C(\lambda) + g(\mu, \lambda)D(\lambda)C(\mu) \quad (2.89)$$

$$D(\mu)C(\lambda) = f(\mu, \lambda)C(\lambda)D(\mu) + g(\lambda, \mu)C(\mu)D(\lambda) \quad (2.90)$$

Next important object of the Algebraic Bethe Ansatz is the transfer matrix  $\tau(\lambda)$ , which is defined as the trace of the monodromy matrix

$$\tau(\lambda) = \text{Tr } T(\lambda) = A(\lambda) + D(\lambda). \quad (2.91)$$

It follows from Eq. (2.77) and the fact that the trace of the tensor product of matrices is equal to the product of their traces that

$$[\tau(\lambda), \tau(\mu)] = 0. \quad (2.92)$$

Equation (2.92) implies that  $\tau$  is a generating function of the integrals of motion of the problem: expanding  $\tau(\lambda)$  in  $\lambda$  one gets a set of commuting integrals of motion  $I_m$ :

$$[I_m, I_n] = 0, \quad n, m = 1, 2, 3, \dots \quad (2.93)$$

This set can be chosen in many different ways since any analytic function of  $\tau(\lambda)$  can play the role of the generating functional. In practice, one works with local integrals of motion. Locality means that  $I_m$  should be written in the following form:

$$I_m = \sum_{j=1}^M \mathcal{I}_m(j), \quad (2.94)$$

where the operators  $\mathcal{I}_m(j)$  act non-trivially in  $m$  neighboring lattice sites only. In the model (2.6) a set of  $I_m$  satisfying locality condition is generated as follows<sup>2</sup>

$$I_{m+1} = i \left. \frac{d^m}{d\lambda^m} \ln \tau(\lambda) \right|_{\lambda=\lambda_0}, \quad \lambda_0 = \frac{i\pi}{2} - i\eta, \quad m = 1, 2, 3, \dots \quad (2.95)$$

---

<sup>2</sup>Note the analogy with the generating functional of the irreducible diagrams in Quantum Field Theory.

In particular, the Hamiltonian (2.6) can be written as

$$H = 2I_2 \sin 2\eta + L \cos 2\eta. \quad (2.96)$$

To understand why  $\lambda_0$  chosen as indicated in Eq. (2.95) implies the locality of  $I_m$ , and to prove Eq. (2.96), let us analyze the  $\mathcal{L}$ -operator (2.71):

$$\mathcal{L}(n|\lambda_0) = -\frac{i}{2}(1 + \vec{\sigma} \otimes \vec{\sigma}_n) \sin 2\eta. \quad (2.97)$$

Given two lattice sites,  $i$  and  $j$ , the operator

$$\Pi_{ij} = \frac{1}{2}(1 + \vec{\sigma}_i \otimes \vec{\sigma}_j) \quad (2.98)$$

is the permutation operator:

$$\Pi_{ij}\Pi_{ij} = 1, \quad \Pi_{ij}(\sigma_i^a \otimes 1_j)\Pi_{ij} = 1_i \otimes \sigma_j^a, \quad a = x, y, z. \quad (2.99)$$

Here by  $1_j$  we denote the identity matrix acting at the site  $j$ . Therefore, the operator (2.97) can be written as

$$\mathcal{L}(n|\lambda_0) = -i \sin 2\eta \Pi_{0n}. \quad (2.100)$$

The subscript “0” indicate the auxiliary (“matrix”) space, in which the operators  $\vec{\sigma}$  act, and the subscript “ $n$ ” indicate the quantum space, in which the operators  $\vec{\sigma}_n$  act in Eq. (2.97). Substituting the  $\mathcal{L}$ -operator (2.100) into Eq. (2.72) we get for the transfer matrix (2.91)

$$(-i \sin 2\eta)^{-L} \tau(\lambda_0) = \text{Tr}_0 (\Pi_{L0} \Pi_{L-10} \cdots \Pi_{20} \Pi_{10}). \quad (2.101)$$

The trace on the right hand side of this expression is taken over the auxiliary space. An alternative representation for the right hand side of Eq. (2.101) is

$$U \equiv (-i \sin 2\eta)^{-L} \tau(\lambda_0) = \Pi_{LL-1} \Pi_{L-1L-2} \cdots \Pi_{32} \Pi_{21}. \quad (2.102)$$

Bearing in mind Eq. (2.99), we recognize in Eq. (2.102) the shift operator:

$$U \sigma_j^a U^{-1} = \sigma_{j-1}^a, \quad a = x, y, z, \quad j = 1, 2, \dots, L, \quad (2.103)$$

where  $\sigma_0 \equiv \sigma_L$ . In the continuum models the momentum operator,  $P$ , generates translations:  $e^{-iaP} \psi(x) e^{iaP} = \psi(x + a)$ . In the case of the lattice model, Eq. (2.103) indicates that the operator  $P$  defined by

$$U = e^{iP} \quad (2.104)$$

is a generator of the lattice translations, that is, Bloch quasi-momenta.

For the first derivative of  $\tau(\lambda)$  at  $\lambda = \lambda_0$  the calculations similar to those used to get Eq. (2.103) give

$$(-i \sin 2\eta)^{-L+1} \partial_\lambda \tau(\lambda)|_{\lambda=\lambda_0} = \sum_{m=1}^L \Pi_{LL-1} \cdots \Pi_{m+1m} \mathcal{L}'_{m-1}(m|\lambda_0) \Pi_{m-1m-2} \cdots \Pi_{21}. \quad (2.105)$$

Here the notation  $\mathcal{L}'_{m-1}(m|\lambda_0)$  is used for  $\partial_\lambda \mathcal{L}_{m-1}(m|\lambda)|_{\lambda=\lambda_0}$ ; the  $\mathcal{L}$ -operator is given by Eq. (2.71) and the subscript “ $m-1$ ” indicates that the operator acts in the tensor product of

the *quantum* spaces. Combining Eqs. (2.102) and (2.105) and taking into account Eq. (2.99) we arrive at Eq. (2.96). Higher  $I_m$  can be obtained similarly. The locality of higher  $I_m$  follows from the structure of the derivatives of the transfer matrix and can be seen rather easy. Note that  $I_1$  is exceptionally non-local since it is proportional to the Bloch quasimomentum,

$$I_1 = iL \ln[\sinh(-2i\eta)] - P, \quad (2.106)$$

and the latter is the logarithm of the shift operator, acting in all lattice sites, Eqs. (2.102) and (2.104).

### 2.3.2 Eigenfunctions and spectrum of the integrals of motion

We are ready to describe the eigenfunctions of the transfer matrix (2.91). We start with introducing the pseudovacuum

$$|0\rangle = \prod_{j=1}^L |\uparrow\rangle_j, \quad (2.107)$$

where  $|\uparrow\rangle_j$  is the spin-up state in the  $j$ -th lattice site. The down-left entry of the  $\mathcal{L}$ -operator  $\mathcal{L}(n|\lambda)$  vanishes when acting onto the  $|\uparrow\rangle_n$ . This implies for the monodromy matrix (2.72)

$$C(\lambda)|0\rangle = 0, \quad A(\lambda)|0\rangle = a(\lambda)|0\rangle, \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle, \quad (2.108)$$

where

$$a(\lambda) = [-i \cosh(\lambda - i\eta)]^L, \quad d(\lambda) = [-i \cosh(\lambda + i\eta)]^L. \quad (2.109)$$

One can check that the commutation relations between the  $z$ -component of the total spin and the entries of the monodromy matrix are

$$[S_z, B(\lambda)] = -B(\lambda), \quad [S_z, C(\lambda)] = C(\lambda), \quad [S_z, A(\lambda)] = [S_z, D(\lambda)] = 0. \quad (2.110)$$

Thus, the operators  $B$  and  $C$  resemble the creation and annihilation operators of the harmonic oscillator, with  $L/2 - S_z$  counting the number of excitations. A difference with the harmonic oscillator is that  $B$  and  $C$  commute non-canonically, Eq. (2.81). Nonetheless, we take the states

$$|\Psi_N(\{\lambda_j\})\rangle = \prod_{j=1}^N B(\lambda_j)|0\rangle \quad (2.111)$$

as a candidate for the eigenfunctions of the transfer matrix. To move  $A(\mu)$  and  $D(\mu)$  past  $B(\lambda)$  one can use Eqs. (2.83) and (2.88), respectively. Such an operation generates “unwanted” terms  $g(\lambda, \mu)B(\mu)A(\lambda)$  and  $g(\mu, \lambda)B(\mu)D(\lambda)$ . It can be shown (see, for example, section VII.1 of the book [5]) that all unwanted terms can be canceled out by the proper choice of the parameters  $\lambda_j$  in Eq. (2.111). These parameters (called rapidities) should satisfy the Bethe equations:

$$\frac{a(\lambda_n)}{d(\lambda_n)} \prod_{\substack{j=1 \\ j \neq n}}^N \frac{f(\lambda_n, \lambda_j)}{f(\lambda_j, \lambda_n)} = 1, \quad n = 1, \dots, N, \quad (2.112)$$

where  $a$  and  $d$  are given by Eq. (2.109). They are often written in the “logarithmic” form:

$$Lp_0(\lambda_j) + \sum_{k=1}^N \theta(\lambda_j - \lambda_k) = 2\pi \left( n_j - \frac{N+1}{2} \right), \quad j = 1, \dots, N, \quad (2.113)$$

where

$$p_0(\lambda) = i \ln \frac{\cosh(\lambda - i\eta)}{\cosh(\lambda + i\eta)}, \quad p_0(-\lambda) = -p_0(\lambda), \quad p_0(+\infty) = 2\eta \quad (2.114)$$

and

$$\theta(\lambda) = i \ln \frac{\sinh(\lambda + 2i\eta)}{\sinh(-\lambda + 2i\eta)}, \quad \theta(\lambda) = -\theta(-\lambda), \quad \theta(+\infty) = \pi - 4\eta. \quad (2.115)$$

The parameters  $n_j$  take arbitrary integer values. Equation (2.113) is the Bethe equation (2.19) obtained by the Coordinate Bethe Ansatz.

We can now exploit the full power of the Algebraic Bethe Ansatz. For the transfer matrix (2.91) acting onto the state (2.111) with  $\lambda_j$  satisfying Eq. (2.112) we get

$$\tau(\lambda)|\Psi_N(\{\lambda_j\})\rangle = \theta(\lambda, \{\lambda_j\})|\Psi_N(\{\lambda_j\})\rangle, \quad (2.116)$$

where

$$\theta(\lambda, \{\lambda_j\}) = a(\lambda) \prod_{j=1}^N f(\lambda, \lambda_j) + d(\lambda) \prod_{j=1}^N f(\lambda_j, \lambda). \quad (2.117)$$

This expression, being combined with Eq. (2.95) gives the spectrum of  $I_m$ :

$$I_{m+1}(\{\lambda_j\}) = \sum_{j=1}^N v_m(\lambda_j) + L\alpha_m, \quad m = 0, 1, \dots, L-1, \quad (2.118)$$

where

$$v_m(\lambda_j) = \left. \frac{\partial^m p_0(\lambda_j - \lambda)}{\partial \lambda^m} \right|_{\lambda=0}, \quad \alpha_m = i \left. \frac{\partial^m}{\partial \lambda^m} \ln[\sinh(\lambda - 2i\eta)] \right|_{\lambda=0}, \quad (2.119)$$

and  $p_0$  is defined by Eq. (2.114). In getting Eq. (2.118) we used that  $\left. \frac{\partial^m}{\partial \lambda^m} d(\lambda) \right|_{\lambda=\lambda_0} = 0$  for  $m = 0, 1, \dots, L-1$ .

The spectrum of the Hamiltonian (2.6) follows from Eqs. (2.96) and (2.118). It follows from Eqs. (2.107), (2.110), and (2.111) that

$$S_z |\Psi_N(\{\lambda_j\})\rangle = \left( \frac{L}{2} - N \right) |\Psi_N(\{\lambda_j\})\rangle. \quad (2.120)$$

Therefore, the spectrum of the Hamiltonian (2.5) is

$$E_h = \sum_{j=1}^N \epsilon_0(\lambda_j) - hL, \quad (2.121)$$

where

$$\epsilon_0(\lambda) = 2h - \frac{2 \sin^2 2\eta}{\cosh(\lambda + i\eta) \cosh(\lambda - i\eta)}. \quad (2.122)$$

We thus recovered results for the spectrum obtained by the Coordinate Bethe Ansatz.

## 3 Discussion and open horizons

### 3.1 Normalization of wave functions

We know how to calculate norms of the wave functions written in the form (2.111) [7]. No solution available through the Coordinate Bethe Ansatz wave functions, Eqs. (1.47) and (2.12).

### 3.2 How to go from Coordinate to Algebraic BA: a solution of the Inverse Scattering Problem

The state (2.111) resembles the state of the harmonic oscillator. The Hamiltonian of the XXZ chain, Eq. (2.6), written through the operators  $A$ ,  $B$ ,  $C$ , and  $D$  is a non-local function of  $A$  and  $D$  operators taken at some point

$$H \sim \frac{\partial}{\partial \lambda} \ln(A + D). \quad (3.1)$$

The problem of representation of the local fields through the operators  $A$ ,  $B$ ,  $C$ , and  $D$ , often referred to as the Quantum Inverse Scattering Problem, is solved for the XXZ Heisenberg chain [11, 12, 13, 21]:

$$\sigma_j^{(-)} = \prod_{i=1}^{j-1} (A + D)(\lambda_j) B(\lambda_i) \prod_{i=j+1}^L (A + D)(\lambda_j). \quad (3.2)$$

The discussion of the solution to Quantum Inverse Scattering Problem for other models lies outside the scope of the present lecture.

### 3.3 Correlation functions

When the spectrum of a model is the same as of free fermions (which is the case for the Tonks-Girardeau gas and for the XXZ chain at  $\Delta = 0$ ) one can write the correlation functions as determinants of infinite-dimensional matrices with relatively simple entries. Numerics and analytical methods works quite well for such determinants, as one can see for the Tonks-Girardeau gas in section 1.5.

Away from the points of free-fermion spectra, the analytic expression for the correlation function are tremendously complex, if exist. The book [5] provides a very decent account of approaches and results on the subject until year 1993. The solution of the Quantum Inverse Scattering Problem boosted the efforts in the direction of getting closed-form analytic expressions for the correlation functions. However, their analysis still requires extensive numerics at some stage - so extensive that the required computational time is compatible with the computational time required for purely numerical analysis. One can check Refs. [22, 23] for further details.

### 3.4 Limitations of Algebraic Bethe Ansatz

The ABA scheme does not work if there is no “pseudovacuum”  $|0\rangle$  above which creation and annihilation can be defined. This is the case for the totally anisotropic (XYZ) Heisenberg spin chain:

$$H = - \sum_{m=1}^M \left[ J_x \sigma_m^{(x)} \sigma_{m+1}^{(x)} + J_y \sigma_m^{(y)} \sigma_{m+1}^{(y)} + J_z \sigma_m^{(z)} \sigma_{m+1}^{(z)} \right]. \quad (3.3)$$

This model is exactly solvable [24] but Functional Bethe Ansatz should be used.

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