

Strongly correlated systems
in atomic and condensed matter physics

Lecture notes
by Eugene Demler
ETH Zurich

December 6, 2023

Chapter 12

Collective modes in interacting Fermi systems

12.1 Collective modes and response functions. RPA approach in the normal state.

12.1.1 From response functions to collective modes

Consider response function $\chi_A(q, \omega)$ for some quantity $A(q, \omega)$. It is defined as the expectation value of the operator \hat{A} induced by the external field $h_A(q, \omega)$, which is thermodynamically conjugate to it. Here thermodynamically conjugate means that the field $h_A(q, \omega)$ contributes a term to the Hamiltonian

$$\mathcal{H}_{\text{external}} = h_{Aq\omega} e^{i\omega t} \hat{A}_{-q} + \text{c.c.} \quad (12.1)$$

Hence the response function is defined by

$$\langle A(q, \omega) \rangle = \chi_A(q, \omega) h_A(q, \omega) \quad (12.2)$$

Generally to get a finite expectation value of $\langle A(q, \omega) \rangle$ one needs to have a nonzero external field $h_A(q, \omega)$. One important exception is when the response function $\chi_A(q, \omega)$ is infinite. Then it may be possible to have a finite $\langle A(q, \omega) \rangle$ with $h_A(q, \omega) = 0$. A more accurate argument shows that collective excitations in the system correspond to poles of the response function $\chi_A(q, \omega)$ [1, 7].

12.1.2 Equations of motion method

With few exceptions calculating response functions for interacting many-body systems can not be done exactly. The approach that we discuss in this chapter is based on writing equations of motion for density and spin operators $\hat{O} = \frac{1}{i}[\mathcal{H}, O]$, and then doing certain approximations to bring them to a closed form. When O is a two particle operator of the type $c_{k\sigma}^\dagger c_{k'\sigma'}$, its commutator with

the interaction term produces a four fermion term, so equations of motion do not close. RPA approximation replaces the true interacting Hamiltonian with an effective non-interacting Hamiltonian, where interactions are replaced by some effective field[7, 2]. For example, when the system is perturbed by the potential at wavevector q and frequency ω , we take the effective potential to be $V_{\text{eff}}(q\omega) = V(q)\langle\rho(q,\omega)\rangle e^{i\omega t}$, where $V(q)$ is the interaction strength and $\langle\rho(q,\omega)\rangle$ is the induced density. This potential can be understood as an effective time- and space- dependent Hartree potential created by other fermions. So fermions are scattered by the potential, which is the sum of the external and polarization potentials.

12.1.3 Density sector. Zero sound mode

We consider an interacting Fermi system in the normal state

$$\mathcal{H} = \sum_{k\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} + V \sum_k c_{k+q\uparrow}^\dagger c_{-k\downarrow}^\dagger c_{-k'\downarrow} c_{-k'+q\uparrow} \quad (12.3)$$

To calculate the density-density response function we assume that the system is perturbed by the external potential at wavevector q and frequency ω

$$\mathcal{H}_{\text{probe}} = \frac{\hbar q\omega}{2} e^{i\omega t} \sum_{p\sigma} c_{p-q\sigma}^\dagger c_{p\sigma} + \text{c.c.} \quad (12.4)$$

We define operators

$$\begin{aligned} \rho_{kq\sigma} &= c_{k+q\sigma}^\dagger c_{k\sigma} \\ \rho_{kq} &= \rho_{kq\uparrow} + \rho_{kq\downarrow} \end{aligned} \quad (12.5)$$

and write their equations of motion. In writing equations of motion we assume that the system develops expectation values of the density at wavevector q , which we will need to determine self-consistently. We take

$$\mathcal{H}_{\text{eff}} = \sum_{k\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} + \left(\frac{\hbar q\omega}{2} + V \frac{\langle\rho_{q\omega}\rangle}{2} \right) e^{i\omega t} \sum_k c_{k-q\sigma}^\dagger c_{k\sigma} \quad (12.6)$$

and find

$$\frac{1}{i} \dot{\rho}_{kq\sigma} = [\mathcal{H}_{\text{eff}}, \rho_{kq\sigma}] = (\xi_{k+q} - \xi_k) \rho_{kq\sigma} + \left(\frac{\hbar q\omega}{2} + V \frac{\langle\rho_{q\omega}\rangle}{2} \right) e^{i\omega t} (n_k - n_{k+q}) \quad (12.7)$$

In the last term $n_{k\sigma} = c_{k\sigma}^\dagger c_{k\sigma}$ and $n_{k+q\sigma} = c_{k+q\sigma}^\dagger c_{k+q\sigma}$ are, in principle, operators, but we replace them by their expectation values in the ground state. We take $\langle\rho_{kq}(t)\rangle = \langle\rho_{kq\omega}\rangle e^{i\omega t}$ and find

$$(\omega - (\xi_{k+q} - \xi_k)) \rho_{kq\omega} = (V \langle\rho_{q\omega}\rangle + \hbar q\omega)(n_k - n_{k+q}) \quad (12.8)$$

We have $\langle \rho_{q\omega} \rangle = \sum_k \langle \rho_{kq(\omega)} \rangle$ and find

$$\langle \rho_{q\omega} \rangle = \chi_0(q, \omega) (V \langle \rho_{q\omega} \rangle + h_{q\omega}) \quad (12.9)$$

where

$$\chi_0(q, \omega) = \sum_k \frac{n_k - n_{k+q}}{\omega - (\xi_{k+q} - \xi_k)} \quad (12.10)$$

Thus we find RPA expression for the response function

$$\chi_{\text{RPA}}(q, \omega) = \frac{\chi_0(q, \omega)}{1 - V \chi_0(q, \omega)} \quad (12.11)$$

Poles of the density-density response function describe collective excitations in the system. So we look for the solution of equation

$$1 - V \chi_0(q, \omega) = 0 \quad (12.12)$$

In the longwavelength limit ($q \rightarrow 0$) and at $T = 0$ we can take

$$\chi_0(q, \omega) = \sum_k \frac{(-\frac{\partial n}{\partial \epsilon})(\vec{q}\vec{v}_p)}{(\omega - \vec{q}\vec{v}_p)} = \frac{N(0)}{2} \int \frac{d\Omega_p}{4\pi} \frac{\vec{q}\vec{v}_p}{(\omega - \vec{q}\vec{v}_p)} = N(0) \left[-1 + \frac{\lambda}{2} \log \frac{\lambda + 1}{\lambda - 1} \right] \quad (12.13)$$

where $\lambda = \omega/qv_f$. In writing equation (12.13) we used $\frac{\partial n}{\partial \epsilon} = -\delta(\epsilon - \epsilon_f)$. So we have

$$\frac{1}{VN(0)} = -1 + \frac{\lambda}{2} \log \frac{\lambda + 1}{\lambda - 1} \quad (12.14)$$

For repulsive interactions this equation describes zero sound excitations[7, 1]. Note that zero sound mode is above the particle-hole continuum.

12.1.4 Spin sector. Stoner instability

We now consider probing the system in the spin sector

$$\mathcal{H}_{\text{probe}} = \frac{h_{q\omega}^Z}{2} e^{i\omega t} \sum_{p\sigma} \sigma c_{p-q\sigma}^\dagger c_{p\sigma} + \text{c.c.} \quad (12.15)$$

Analysis similar to the one we discussed in the previous subsection gives

$$\chi_{\text{RPA}}^Z(q, \omega) = \frac{\chi_0(q, \omega)}{1 + V \chi_0(q, \omega)} \quad (12.16)$$

Spin wave modes can be found from the equation

$$-\frac{1}{VN(0)} = -1 + \frac{\lambda^z}{2} \log \frac{\lambda^z + 1}{\lambda^z - 1} \quad (12.17)$$

For repulsive interactions with $VN(0) > 1$ we find imaginary frequency of the modes. This corresponds to exponentially growing magnetization and signals instability of the system to spontaneous polarization[7, 1]. This is the so-called Stoner instability[13].

12.2 Anderson-Bogoliubov mode in the paired state

12.2.1 RPA approach in the paired state.

In the presence of pairing we need to change the "bare" quadratic Hamiltonian to the BCS form

$$\mathcal{H} = \sum_{k\sigma} \left\{ \xi_k c_{k\sigma}^\dagger c_{k\sigma} + (\Delta c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + \text{c.c.}) \right\} \quad (12.18)$$

Moreover, external field that couples to the density operator induces not only the expectation value of the density $\rho_q = \sum_{k\sigma} c_{k+q\sigma}^\dagger c_{k\sigma}$, but also anomalous expectation values

$$\begin{aligned} \langle b_q^\dagger \rangle &= \sum_k \langle c_{k+q\uparrow}^\dagger c_{-k\downarrow}^\dagger \rangle \\ \langle b_{-q} \rangle &= \sum_k \langle c_{-k-q\downarrow} c_{k\uparrow} \rangle \end{aligned} \quad (12.19)$$

This reflects particle-hole mixing in the superfluid state arising from the condensate of Cooper pairs. Hence

$$\begin{aligned} \mathcal{H}_{\text{eff}} &= \sum_{k\sigma} \left\{ \xi_k c_{k\sigma}^\dagger c_{k\sigma} + (\Delta c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + \text{c.c.}) \right\} + \left(\frac{h_{q\omega}}{2} + V \frac{\langle \rho_{q\omega} \rangle}{2} \right) e^{i\omega t} \sum_k c_{k-q\sigma}^\dagger c_{k\sigma} \\ &+ V \langle b_q^\dagger \rangle \sum_k c_{-k\downarrow} c_{-k+q\uparrow} + V \langle b_{-q} \rangle \sum_k c_{k-q\uparrow}^\dagger c_{-k\downarrow}^\dagger \end{aligned} \quad (12.20)$$

Equations of motion couple particle-hole operators $\rho_{kq\sigma} = c_{k+q\sigma}^\dagger c_{k\sigma}$ with two-particle operators $b_{kq}^\dagger = c_{k+q\uparrow}^\dagger c_{-k\downarrow}^\dagger$ and two-hole operators $b_{k-q} = c_{-k-q\downarrow}^\dagger c_{k\uparrow}$. We have

$$\begin{aligned} [\mathcal{H}_{\text{eff}}, \rho_{kq\uparrow}] &= (\xi_{k+q} - \xi_k) \rho_{kq\uparrow} - \Delta b_{kq}^\dagger + \Delta^* b_{k-q} + \left(\frac{h_{q\omega}}{2} + V \frac{\langle \rho_{q\omega} \rangle}{2} \right) e^{i\omega t} (n_k - n_{k+q}) \\ &+ V \langle b_q^\dagger \rangle \langle c_{-k\downarrow} c_{k\uparrow} \rangle + V \langle b_{-q} \rangle \langle c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \rangle \end{aligned} \quad (12.21)$$

and similar equations for $[\mathcal{H}_{\text{eff}}, \rho_{kq\downarrow}]$, $[\mathcal{H}_{\text{eff}}, b_{kq}^\dagger]$ and $[\mathcal{H}_{\text{eff}}, b_{kq}]$. These equations of motion can be solved and in the BCS regime we obtain a collective mode with the long wavelength dispersion [2]

$$\omega_q = \frac{v_f}{\sqrt{3}} q \quad (12.22)$$

This is the celebrated Anderson-Bogoliubov mode [3, 2].

12.2.2 Hydrodynamic approach

We now present an alternative approach to calculating the sound mode (12.22). We write hydrodynamic equations of motion[11, 6]

$$\begin{aligned}\frac{\partial n}{\partial t} + \vec{\nabla}(n\vec{v}) &= 0 \\ m\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}\left(\frac{1}{2}mv^2 + \mu(n)\right) &= 0\end{aligned}\quad (12.23)$$

In the linearized regime, the second equation can be written as

$$m\frac{\partial \vec{v}}{\partial t} + \left(\frac{\partial n}{\partial \mu}\right) \vec{\nabla} n = 0 \quad (12.24)$$

Hence

$$\frac{\partial^2 n}{\partial t^2} - \frac{n}{m} \left(\frac{\partial n}{\partial \mu}\right) \nabla^2 n = 0 \quad (12.25)$$

In the BCS regime $\mu = E_F = \frac{k_f^2}{2m} \sim n^{2/3}$ and $\frac{\partial \mu}{\partial n} = \frac{2}{3} \frac{\mu}{n}$. This gives

$$\omega_q^2 = \frac{k_f^2}{3m} q^2 = \left(\frac{v_f}{\sqrt{3}} q\right)^2 \quad (12.26)$$

We can apply hydrodynamic analysis in the BEC regime using

$$\mu = g_M n_M \left(1 + \frac{32}{3\sqrt{\pi}} \sqrt{n_M a_M^3} + \dots\right) \quad (12.27)$$

Here n_M is the density of molecules and g_M is the interaction strength between molecules, which can be expressed using the mass of molecules M and their scattering length a_M as $g_M = \frac{2\hbar^2 a_M}{M}$. Analysis in ref[10] shows that $a_M = 0.6a$. So we find

$$v = \frac{\hbar}{M} \sqrt{2a_M n_M} \quad (12.28)$$

At unitarity we expect to find

$$\mu = (1 + \beta) \frac{\hbar^2}{2m} (6\pi^2)^{2/3} n^{2/3} \quad (12.29)$$

where β is known from theory. In the mean-field analysis $\beta = -0.41$, Quantum Monte Carlo gives $\beta = -0.58$.

Hydrodynamic approach can be extended to include effects of the confining potential[14, 15]. Equations of motion in this case can be written as

$$\frac{\partial n}{\partial t} + \vec{\nabla}(n\vec{v}) = 0 \quad (12.30)$$

$$m\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}\left(\frac{mv^2}{2} + \mu(n) + V_{\text{ext}}\right) = 0 \quad (12.31)$$

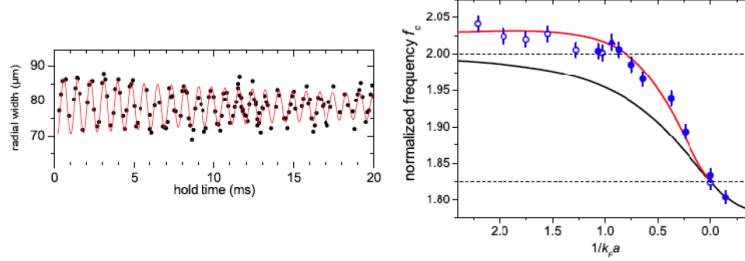


Figure 12.1: Compression oscillations in superfluid Fermi gas. Second figure shows good agreement of theoretical calculations with experiments. Figures taken from [4].

To obtain linearized equations we can take the time derivative of (12.30) and substitute (12.31) for $\partial \vec{v} / \partial t$. In the second term in (12.30), the time derivative should only be applied to \vec{v} , since both $\partial n / \partial t$ and \vec{v} are first order in the deviation from equilibrium. However we can not take n outside $\vec{\nabla}$ since the equilibrium density is not uniform. We find the following linearized equation

$$\frac{\partial^2 \delta n}{\partial t^2} - \frac{1}{m} \vec{\nabla} \left(n \frac{\partial \mu}{\partial n} \vec{\nabla} \delta n \right) = 0 \quad (12.32)$$

Far on the BEC side these equations can be simplified further. In this case the system is similar to a gas of weakly interacting bosons, so we can take $\frac{\partial \mu}{\partial n} = g$, where g is a constant. Then we have

$$\frac{\partial^2 \delta n}{\partial t^2} - \frac{g}{m} \vec{\nabla} (n \vec{\nabla} \delta n) = 0 \quad (12.33)$$

In equilibrium $gn(r) = \mu - V_{\text{ext}}(r)$. Hence we find

$$\frac{\partial^2 \delta n}{\partial t^2} + \frac{1}{m} [\vec{\nabla} V_{\text{ext}} \vec{\nabla} \delta n - (\mu - V(r)) \nabla^2 \delta n] = 0 \quad (12.34)$$

This equation needs to be solved subject to the boundary conditions that the solution is nonsingular at the origin and vanishes at the Thomas-Fermi radius, where the equilibrium density goes to zero.

Experimentally one often studies lowest energy collective modes. Figure (12.1) shows analysis of the quadrupolar (compressional) mode across the BCS/BEC crossover. It may seem surprising that the mode frequency does not change dramatically with interactions. The sound velocity should depend very strongly on the scattering length on the BEC side and at unitarity. A hand-waving argument is that for stronger interactions the sound wave velocity goes up dramatically, but so does the size of the cloud (think of the Thomas-Fermi approximation). And in the oscillation frequency, which we can estimate as the ratio of the two quantities, the net change ends up being small [9].

12.3. COMPETITION OF STONER INSTABILITY AND MOLECULE FORMATION NEAR FESHBACH RESONANCE

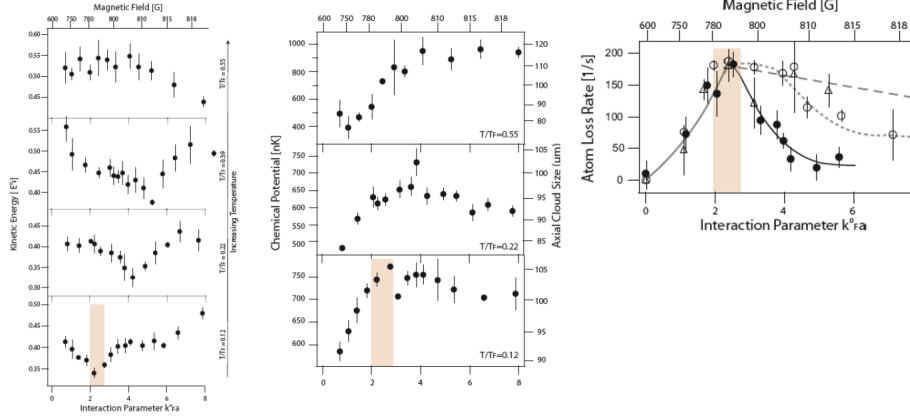


Figure 12.2: Signatures of Stoner instability. Figures taken from [5].

12.3 Competition of Stoner instability and molecule formation near Feshbach resonance

12.3.1 Ferromagnetism in itinerant electron systems

Question of magnetism in itinerant systems of fermions is one of the longest standing problems in physics. Mean-field argument by Stoner suggests that such transition should take place when interactions become strong enough. Using analysis of collective modes we derived earlier that such transition occurs when $N(0)V = 1$. Several counter-arguments have been suggested to Stoner's original mean-field reasoning. For example, Kanamori argued that one should take screened value of the interaction, which may never exceed the Fermi energy. Since the density of states $N(0)$ is inversely proportional to the Fermi energy, Kanamori argued that $N(0)V$ can not be made arbitrarily large, as Stoner's simple argument would suggest.

The idea of studying ferromagnetism with ultracold fermions has been proposed by several groups[?].

12.3.2 MIT experiments

Recent experiments on the BEC side of the Feshbach resonance [5] raised questions of the competition of molecule formation and the Stoner instability [16, 8]. Repulsive scattering length suggests ferromagnetism, whereas the existence of molecular bound states suggests dynamical pair formation. Main features observed in experiments are summarized in figure 12.2. There is an abrupt increase in the kinetic energy as the system approaches the resonance, a maximum in the system size, and suppression of the molecule formation rate.

Subsequent experiments demonstrated that pair formation was the dominant process[12].

12.3.3 Molecule formation

Our goal is to analyze the rate of molecule formation taking into account the many-body character of the system. We start with interacting fermions in the normal state and analyze the rate of molecule formation in the same approach as we used earlier. We analyze collective modes that correspond to the Cooper channel. If we find imaginary value for the frequency, it will suggest that the system is unstable toward developing pairing with the rate given by the magnitude of the mode frequency[8].

We use RPA analysis

$$\mathcal{H} = \sum_k \xi_k c_{k\sigma}^\dagger c_{k\sigma} + (Vb_q^* + h_{\Delta q}^*) \sum_k c_{-k\downarrow} c_{k+q\uparrow} + (Vb_q + h_{\Delta q}) \sum_k c_{k+q\uparrow}^\dagger c_{-k\downarrow}^\dagger \quad (12.35)$$

We take the operator $b_{pq}^\dagger = c_{p+q\uparrow}^\dagger c_{-p\downarrow}^\dagger$. Equations of motion for b_{pq}^\dagger are given by

$$[\mathcal{H}, b_{pq}^\dagger] = (\xi_{p+q} + \xi_{-p})b_{pq}^\dagger + (Vb_q^* + h_{\Delta q}^*)(1 - n_{p+q} - n_{-p}) \quad (12.36)$$

We find for the Cooper pair response functions (Cooperon)

$$\begin{aligned} \chi_{RPA}^C &= \frac{\chi_0^C}{1 - V\chi_0^C} \\ \chi_0^C &= \sum_p \frac{1 - n_{p+q} - n_{-p}}{\omega - (\xi_{p+q} - \xi_{-p})} \end{aligned} \quad (12.37)$$

To find collective modes we need to solve

$$\frac{1}{V} - \sum_p \frac{1 - n_{p+q} - n_{-p}}{\omega - (\xi_{p+q} - \xi_{-p})} = 0 \quad (12.38)$$

We can again trade the bare interaction for the scattering length using

$$\frac{m}{4\pi a_s} = \frac{1}{V} + \sum_k \frac{m}{k^2} \quad (12.39)$$

For the most unstable mode at $q = 0$ we obtain

$$\frac{m}{4\pi a_s} - \sum_p \left[\frac{1 - 2n_p}{\omega - 2\xi_p} + \frac{m}{p^2} \right] = 0 \quad (12.40)$$

The rate of molecule formation as a function of the scattering length is shown in fig. 12.3. We can also use this analysis to calculate the rate of change of the kinetic energy of fermions that do not become converted into molecules. Essentially the argument is that the imaginary part of the pole of the Cooperon

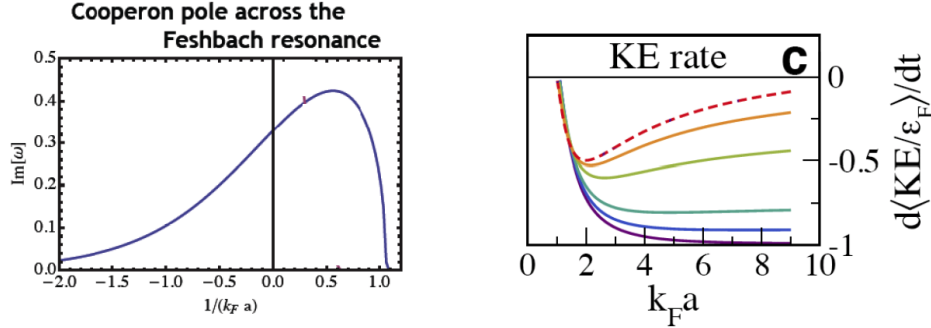


Figure 12.3: The left plot shows the rate of molecule formation. The right plot shows the rate of change of the kinetic energy of fermions due to molecule formation [8].

gives the rate of molecule formation and the real part gives the energy of the molecular state, i.e. the energy of fermions being "extracted" from the sea as a result of the molecule formation. Then the product of the two gives the rate of change of the fermion kinetic energy. Both quantities behave qualitatively similar to the experimental results shown in fig. 12.2.

We observe that the physics of molecule formation can explain two features observed in experiments [5].

12.4 Problems to Chapter 12

Problem 1. Stoner instability

a) Derive eq. (12.16) for the spin susceptibility of a model with contact interactions between fermions using the RPA approximation discussed in this chapter. Show that it leads to equation (12.17) for the spin collective mode.

b) Show that for $VN(0) > 1$ the system has unstable modes with imaginary frequencies.

Problem 2. Scaling solution

In the limit when $\partial\mu/\partial n = g$ is constant and the trap is parabolic, one can solve nonequilibrium hydrodynamic equations beyond the linearized approximation discussed in section 12.2.2. For example, let us consider exciting the system by changing the strength of the periodic potential. Equation (12.31) becomes

$$m \frac{\partial \vec{v}}{\partial t} + \vec{\nabla} \left(\frac{mv^2}{2} + gn + \frac{1}{2} m \omega^2(t) r^2 \right) = 0 \quad (12.41)$$

a) Show that equations (12.30) and (12.41) can be solved by the scaling

ansatz

$$\begin{aligned}\vec{v}(r, t) &= \vec{r} \frac{\dot{b}(t)}{b(t)} \\ n(r, t) &= \frac{1}{b^3(t)} n_0\left(\frac{r}{b(t)}\right)\end{aligned}\tag{12.42}$$

Here $n_0(r)$ is the equilibrium density when the confining potential strength is ω_0 , and $b(t)$ is the scaling function that satisfies

$$\frac{\ddot{b}}{b} = -\omega^2(t) + \frac{\omega_0^2}{b^4}\tag{12.43}$$

b) Solve the problem of sudden release of the condensate from the trap. This corresponds to taking $\omega(t) = \theta(-t)\omega_0$ and initial conditions $b(t=0) = \dot{b}(t=0) = 0$.

Bibliography

- [1] A. Abrikosov, L. Gorkov, and I. Dzyaloshinski. *Methods of Quantum Field Theory in Statistical Physics*. 1963.
- [2] P.W. Anderson. *Phys. B*, 112:1900, 1958.
- [3] N.N. Bogoliubov. *Sov. Phys. JETP*, 34(7):698, 1958.
- [4] A. Altmeyer et al. *Phys. Rev. Lett.*, 98:40401, 2007.
- [5] G.B. Jo et al. *Science*, 325:1521, 2009.
- [6] Stefano Giorgini, Lev P. Pitaevskii, and Sandro Stringari. Theory of ultra-cold atomic fermi gases. *Rev. Mod. Phys.*, 80(4):1215–1274, Oct 2008.
- [7] P. Nozieres and D. Pines. *The theory of quantum liquids*. 1966.
- [8] David Pekker, Mehrtash Babadi, Rajdeep Sensarma, Nikolaj Zinner, Lode Pollet, Martin Zwierlein, and Eugene Demler. Competition between Pairing and Ferromagnetic Instabilities in Ultracold Fermi Gases near Feshbach Resonances. *Physical Review Letters*, 106(5):1–4, February 2011.
- [9] C.J. Pethick and H. Smith. *Bose-Einstein Condensation in Dilute Gases*. Cambridge University Press, 2002.
- [10] D. Petrov, C. Salomon, and G. Shlyapnikov. *Phys. Rev. A*, 71:12708, 2005.
- [11] L. Pitaevskii and S. Stringari. *Bose-Einstein Condensation*. Oxford Science Publications, 2002.
- [12] Christian Sanner, Edward J Su, Wujie Huang, Aviv Keshet, Jonathon Gillen, and Wolfgang Ketterle. Correlations and Pair Formation in a Repulsively Interacting Fermi Gas. 12(1):1–5.
- [13] Stoner. *Philos. Mag*, 15:1018, 1933.
- [14] S. Stringari. *Phys. Rev. Lett.*, 77:2360, 1996.
- [15] S. Stringari. *Phys. Rev. A*, 58:2385, 1998.
- [16] Shizhong Zhang and Tin-Lun Ho. Atom loss maximum in ultra-cold Fermi gases. *New Journal of Physics*, 13(5):055003, May 2011.