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INTELLECTUAL CONTROL SYSTEMS, DATA ANALYSIS

Meixner Nonorthogonal Filters

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Abstract—Consideration was given to a new representation of the Meixner filters which, in distinction to the previously proposed filters, have a rational form of representation of any integer values of the additional parameter α, can be used to describe the dynamic systems with fractional order for the noninteger α, and are obtained directly from the continuous generalized Laguerre filters through a modified bilinear transformation. The paper described a design of the proposed nonorthogonal Meixner filter, numerically stable algorithm to optimize the filter parameters, as well as the results of computer experiments corroborating efficiency of the nonorthogonal Meixner filters for solution of practical problems.

Keywords: Meixner filter, optimization of the filter poles, bilinear transformation, delay system

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1. INTRODUCTION

The discrete Laguerre filters enjoy broad use in diverse problems of identification, filtration, and analysis of the dynamic systems because they enable one to represent a long time series as a shorter spectrum of expansion [1, 3]. For description of the nonstationary systems and those with delay, this result can be improved by generalizing the discrete Laguerre filters and introducing the additional parameter α. Such generalized models are called the Meixner filters.

The paper of Nurges [4] considers application of the normalized and nonnormalized Meixner polynomials [5] to description of the discrete linear dynamic systems. It is noted at that that the normalized Meixner models generalize the Laguerre models [2], whereas the nonnormalized one generalizes the exponential power polynomials described by Perov [6]. Marking later utility of the Meixner functions as compared with the Laguerre ones in describing the delay systems, Brinker [7] made an emphasis on the fact that the Meixner functions do not have a rational z-transformation and, therefore, are nonrealizable. Two alternative methods to determine the Meixner filters in order to solve this problem were proposed in [7]. The first method lies in translating the continuous generalized Laguerre filters to the z-plane using the given modified bilinear transformation. However, the resulting filters are rational only for the even values of the additional parameter α. Another approach lies in generating a functional system akin to the system of Meixner functions with the use of the corresponding transformation matrix which is then used to determine the Meixner filters based on the Laguerre filters for the integer values of a parameter similar to α. Such filters were called z-transformations of the functions similar to the Meixner functions. They are widespread in practice [8–10]. In particular, [8] describes one of the most interesting applications of the z-transformation suggested by Brinker and related with analysis of the nonlinear delay systems at considering of the Volterra kernels [11, 12].

The paper [13] devoted to the theoretical substantiation of the uniqueness of the optimality condition of the generalized continuous Laguerre functions proposed new Meixner filters that are representable rationally for any integer α and can be used to describe the dynamic systems of fractional order [17, 18] for the noninteger α and are obtained directly from the continuous generalized

Laguerre filter with the use of the proposed bilinear transformation. The issues of optimization of the parameters of such Meixner filters were considered in [14, 15]. However, since the proposed filters are nonorthogonal and, as the result, their design can turn out to be specific, it is suggested here to consider in more detail the determination of the parameters of nonorthogonal Meixner filters at solving the two-parameter optimization problem, as well as the vectorized version of the optimization algorithm described in [15] for determination of a more stable solution and reduction of the computational burden.

2. REPRESENTATION OF THE MEIXNER FILTERS

2.1. Existing Representations of the Meixner Filters

There are two representations of the Meixner filters discussed in [7]. Consider them in more detail in order to bring to light their possible drawbacks. The first representation was obtained by transforming the generalized orthogonal Laguerre polynomials $\langle L_k(\tau,\alpha)\rangle_{k=0}^{\infty}$ for $\tau > 0$ and $\alpha > 1$ with the weight function $\omega(\tau,\alpha) = \tau^{\alpha}\exp(-\tau)$ and norm $\|L_k(\alpha)\|^2 = \Gamma(k+\alpha+1)/k!$ from the s-plane to the z-plane. According to [7], the Laplace transform of such system can be defined as

$$\Lambda_k^*(s,\alpha) = \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1+\alpha)}\sqrt{\frac{\Gamma(\alpha+m+1)}{m!}}\left(\frac{1}{s+\frac{1}{2}}\right)^{1+\frac{\alpha}{2}} F\left(-m, 1+\frac{\alpha}{2}; 1+\alpha; \frac{1}{s+\frac{1}{2}}\right), \tag{1}$$

where $F(\,)$ is a hypergeometric function. We notice that for the even values of α the transformation $\Lambda_k^*(s,\alpha)$ is a rational function. Then, by application of the modified transformation

$$G_k^*(z,\xi,\alpha) \mapsto \frac{2z}{\sqrt{D(z+1)}}\,\Lambda_k^*\left(\frac{2}{D}\,\frac{1-z^{-1}}{1+z^{-1}},\alpha\right)$$

(1) can be rearranged in the following z-transformation

$$\begin{aligned} G_k^*(z,\xi,\alpha) &= \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1+\alpha)}\sqrt{\frac{(1+\xi)\Gamma(\alpha+m+1)}{(1-\xi)m!}}(1-\xi)^{1+\alpha/2} \\ &\times\frac{z(z+1)^{\alpha/2}}{(z-\xi)^{1+\alpha/2}}\,F\left(-m,1+\frac{\alpha}{2};1+\alpha;\frac{(1-\xi)(z+1)}{z-\xi}\right), \end{aligned} \tag{2}$$

where $\xi = \frac{4-D}{4+D}$ and D is the parameter of the modified bilinear transformation ($D > 0$) [7]. The resulting z-transformation also represents the rational functions only for the even values of α.

An alternative z-transformation of the functions similar to the Meixner functions which gained a wider practical acceptance [8–10] was proposed in [7]. Its structure is depicted in Fig. 1, where $l_k^{\xi,\alpha}(n)$ and $g_k^{\xi,\alpha}(n)$ are, respectively, the inverse z-transformations of the Laguerre and Meixner functions. As can be seen from the figure, the gist of this approach lies in generating a matrix $A_m^{(q)}$ transforming the system of Laguerre functions into that of Meixner functions. The shortcoming of this representation lies in the presence of an additional transformation unit in the structure and as the result in an increase in the computational burden.

2.2. New Representation of the Meixner Filters

Let for each fixed $\gamma \in \Gamma$ with $\Gamma = \{\gamma \in \mathbb{R} : \gamma > 0\}$, $\alpha \in A$, where $A = \{\alpha \in \mathbb{R} : \alpha > -1\}$ and $k \in \mathbb{N}_0$ in the Hilbertian space $L_2(\mathbb{R}^+)$ the generalized Laguerre functions $L_k(\tau,\gamma,\alpha)$ defined by the system $\langle \exp(-\gamma\tau/2)L_k(\gamma\tau,\alpha)\rangle_{k=0}^{\infty}$ be orthogonal with nonnegative weight function $\omega(\tau,\alpha) = \tau^{\alpha}$

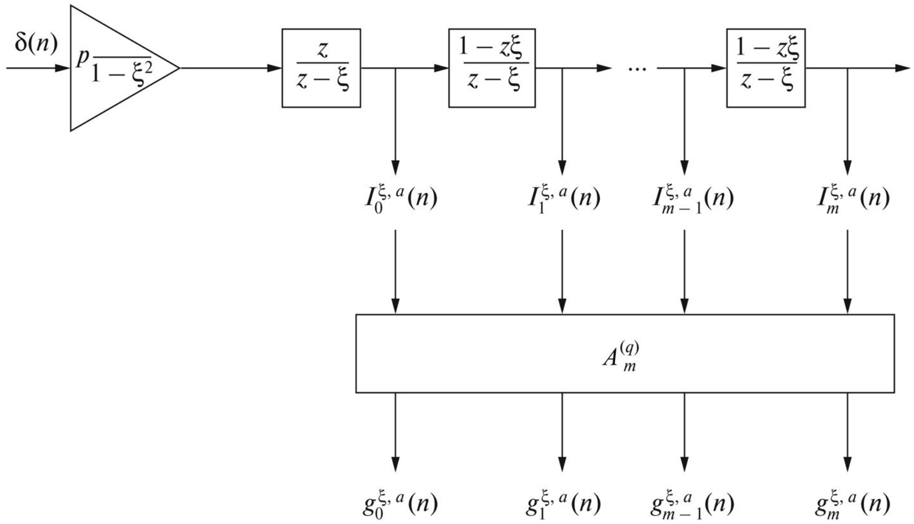

Fig. 1. Structure of the z-transformation of functions similar to the Meixner functions.

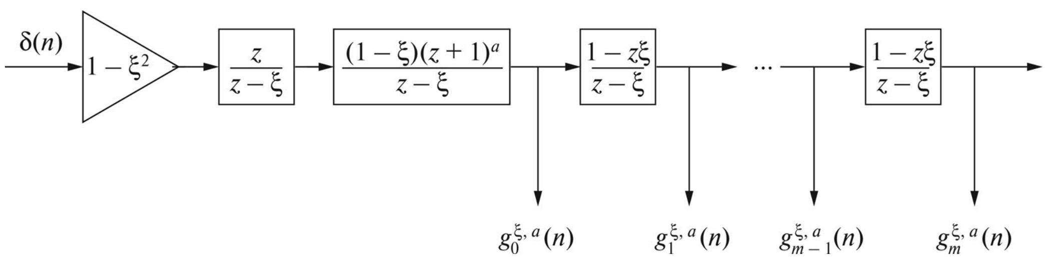

Fig. 2. Structure of the nonorthogonal Meixner filters.

over the interval $\tau \in \mathbb{R}^+$ and with the norm $\|L_k(\gamma,\alpha)\|^2 = \Gamma(k+\alpha+1)/(k!\gamma^{\alpha+1})$. Then, the Laplace transform $\Lambda_k(s,\gamma,\alpha)$ of the system of functions $L_k(\tau,\gamma,\alpha)$ can be defined as [13]

$$\Lambda_k(s,\gamma,\alpha) = \left(\frac{\gamma}{s+\gamma/2}\right)^{\alpha+1}\left(\frac{s-\gamma/2}{s+\gamma/2}\right)^k. \tag{3}$$

The Laplace transforms (3) are continuous generalized Laguerre filters representable in the z-plane as $G_k(z,\xi,\alpha)$ with the real field $|\xi| < 1$ by the aid of the following pair of bilinear transformations:

$$\Lambda_k(s,\gamma,\alpha) \mapsto \frac{z}{z+1}\Lambda_k\left(a\frac{z-1}{z+1},\gamma,\alpha\right) = \frac{(-1)^k 2a}{a+\gamma/2}G_k\left(z,\frac{a-\gamma/2}{a+\gamma/2},\alpha\right); \tag{4}$$
$$G_k(z,\xi,\alpha) \mapsto \frac{2a}{a+s}G_k\left(\frac{a+s}{a-s},\xi,\alpha\right) = \frac{(-1)^k}{1+\xi}\Lambda_k\left(s,2a\frac{1-\xi}{1+\xi},\alpha\right).$$

By applying (4) to (3), introduce a new representation of the Meixner filters [13–15] given by

$$G_k(z,\xi,\alpha) = \frac{(1-\xi^2)z}{z-\xi}\left(\frac{(1-\xi)(z+1)}{z-\xi}\right)^\alpha\left(\frac{1-\xi z}{z-\xi}\right)^k. \tag{5}$$

The intermediate mathematical transformations for determination of (5) on the basis of (4) are given in the Appendix. Figure 2 depicts the structure of the proposed Meixner filters (5).

As compared with its counterpart suggested in [7] (see Fig. 1), the given structure is free of the additional unit $A_m^{(q)}$ rearranging the system of discrete Laguerre filters in the z-transformation of functions similar to the Meixner ones. The structure of the proposed Meixner filters is distinguished only by the cells of infinite band, their number being dependent of the given parameter α. These filters have a rational form of representation and, therefore, are readily realizable for any integer values of the parameter α and not only for the even ones as was assumed previously (2). If the values of the additional parameter α are noninteger, then the proposed system of filters can be used to analyze the dynamic systems with fractional order [17, 18]. Detailed consideration of this problem is beyond the scope of the present study.

Nonorthogonality is the essential distinction of the given system of filters which as the result implies additional transformation of the coefficients. At representing the filter coefficients as solution of the normal equation, this drawback, however, may be leveled. The issues of calculation of the coefficients and optimization of the parameters of the nonorthogonal Meixner filters are considered in the next section in more detail.

We list below the basic advantages of the proposed nonorthogonal Meixner filters:

—simpler structure of the filters having a rational form of representation under any integer values of the parameter α;

—possibility of considering the dynamic systems with fractional order at noninteger α;

—direct dependence with the continuous generalized Laguerre filters through a pair of modified bilinear transformations.

3. OPTIMIZATION OF THE FILTER PARAMETERS

For the complete system $\langle G_k(z,\xi,\alpha)\rangle_{k=0}^{\infty}$, on the circumference of a radius smaller than 1 and greater than $1/|\xi|$ we define some transfer function $H(z)$ representable by a bounded orthogonal series with the approximation error [14, 15]

$$\Delta_m(\xi,\alpha) = \left\| H - \sum_{k=0}^{m} \beta_k(\xi,\alpha) G_k(\xi,\alpha) \right\|^2, \tag{6}$$

where the coefficients $\beta_k(\xi,\alpha)$ obey

$$\tilde{\beta}_k(\xi,\alpha) = \langle H, G_k(\xi,\alpha)\rangle, \tag{7}$$

with the use of the transformation [16]

$$\langle H, G_n(\xi,\alpha)\rangle = \sum_{k=0}^{m} \langle G_k(\xi,\alpha), G_n(\xi,\alpha)\rangle \beta_k(\xi,\alpha),$$

whence with regard for (7) we have

$$\beta_k(\xi,\alpha) = \sum_{k=0}^{m} M_{k,n}^{-1}(\xi,\alpha)\tilde{\beta}_k(\xi,\alpha), \quad M_{k,n}(\xi,\alpha) = \langle G_k(\xi,\alpha), G_n(\xi,\alpha)\rangle.$$

Formulate the following problem.

Problem. For each fixed $m \in \mathbb{N}_0$, optimization of the parameters of the system of nonorthogonal Meixner filters $\langle G_k(z,\xi,\alpha)\rangle_{k=0}^{\infty}$ can be formulated as the two-parameter problem given by

$$(\xi,\alpha) = \arg \min_{\substack{|\xi|<1 \\ \alpha>-1}} \Delta_m(\xi,\alpha). \tag{8}$$

A theoretical substantiation of the uniqueness of the optimality condition

$$\beta^{\alpha}_{m+1}(\xi) = 0$$

for the Meixner filters at solving the one-parameter problem

$$\xi = \arg\min_{|\xi|<1} \Delta^{\alpha}_{m}(\xi)$$

is considered in [14]. The present substantiation extends the theorem given in [13] for the case of generalized continuous Laguerre filters and discrete Laguerre filters. Henceforth, the extended theorem and the totality of lemmas preceeding it were presented in the [15] devoted to the two-parameter problem (8). Within the framework of the given publication, it is proposed to consider the extension of the algorithm described in [15] for optimization of calculations.

We proceed to the matrix representation of (8).

3.1. Determination of the Coefficients as the Solution of the Normal Equation

Represent the approximation error (6) as

$$\Delta^{\beta,\xi,\alpha} = \left(\Sigma^{\beta,\xi,\alpha}\right)^{\mathrm{T}} \Sigma^{\beta,\xi,\alpha}; \quad \Delta^{\beta,\xi,\alpha} \in \mathbb{R}, \quad \Sigma^{\xi,\alpha} \in \mathbb{R}^{n\times 1}, \tag{9}$$

where

$$\Sigma^{\beta,\xi,\alpha} = H - G^{\xi,\alpha}\beta^{\xi,\alpha}; \quad H \in \mathbb{R}, \quad G^{\xi,\alpha} \in \mathbb{R}^{n\times m}, \quad \beta^{\xi,\alpha} \in \mathbb{R}^{m\times 1}.$$

Then for the vector of the coefficients $\beta^{\xi,\alpha}$ one can put down (9)

$$\frac{\partial \Delta^{\beta,\xi,\alpha}}{\partial \beta} = -2\left(G^{\xi,\alpha}\right)^{\mathrm{T}}\left(H - G^{\xi,\alpha}\beta^{\xi,\alpha}\right) = 0. \tag{10}$$

The normal equation

$$\left(G^{\xi,\alpha}\right)^{\mathrm{T}} G^{\xi,\alpha}\beta^{\xi,\alpha} = \left(G^{\xi,\alpha}\right)^{\mathrm{T}} H$$

follows from Eq. (10) according to which the vector of coefficients can be defined as

$$\beta^{\xi,\alpha} = \left(\left(G^{\xi,\alpha}\right)^{\mathrm{T}} G^{\xi,\alpha}\right)^{-1}\left(G^{\xi,\alpha}\right)^{\mathrm{T}} H. \tag{11}$$

3.2. Optimization of the Parameters ξ and α

Set down the necessary optimality condition ξ in compliance with the formulation of problem (8) as

$$\frac{\partial \Delta^{\beta,\xi,\alpha}}{\partial \xi} = -\left(\Sigma^{\beta,\xi,\alpha}\right)^{\mathrm{T}} \frac{\partial G^{\xi,\alpha}}{\partial \xi}\beta^{\xi,\alpha} - \left(\Sigma^{\beta,\xi,\alpha}\right)^{\mathrm{T}} G^{\xi,\alpha}\frac{\partial \beta^{\xi,\alpha}}{\partial \xi} = 0. \tag{12}$$

According to (10),

$$\left(G^{\xi,\alpha}\right)^{\mathrm{T}}\left(H - G^{\xi,\alpha}\beta^{\xi,\alpha}\right) = 0. \tag{13}$$

Transposition of (13) provides

$$\left(H - G^{\xi,\alpha}\beta^{\xi,\alpha}\right)^{\mathrm{T}} G^{\xi,\alpha} = 0 \quad \text{or} \quad \left(\Sigma^{\beta,\xi,\alpha}\right)^{\mathrm{T}} G^{\xi,\alpha} = 0. \tag{14}$$

Then, with regard for (14), (12) goes over to

$$\frac{\partial \Delta^{\beta,\xi,\alpha}}{\partial \xi} = -\left(\Sigma^{\beta,\xi,\alpha}\right)^{\mathrm{T}} \frac{\partial G^{\xi,\alpha}}{\partial \xi}\beta^{\xi,\alpha} = 0. \tag{15}$$

Now, we represent $\frac{\partial G^{\xi,\alpha}}{\partial \xi}$ as

$$\frac{\partial G^{\xi,\alpha}}{\partial \xi} = G^{\xi,\alpha}_{M+1}\left(\Phi^{\xi,\alpha}_{M+1}\right)^{\mathrm{T}}, \tag{16}$$

where the matrix of transformation $\Phi^{\xi,\alpha}_{M+1} \in \mathbb{R}^{m\times(m+1)}$ carries out transition from the space of vectors $\operatorname{span}\left\{G^{\xi,\alpha}_0, G^{\xi,\alpha}_1, \ldots, G^{\xi,\alpha}_{m+1}\right\}$ to the space $\operatorname{span}\left\{\frac{\partial G^{\xi,\alpha}_0}{\partial \xi}, \frac{\partial G^{\xi,\alpha}_1}{\partial \xi}, \ldots, \frac{\partial G^{\xi,\alpha}_m}{\partial \xi}\right\}$ according to the operation of transformation

$$G^{\xi,\alpha}\left(\frac{\partial G^{\xi,\alpha}}{\partial \xi}\right)^{\mathrm{T}} = G^{\xi,\alpha}\Phi^{\xi,\alpha}_{M+1}\left(G^{\xi,\alpha}_{M+1}\right)^{\mathrm{T}}.$$

By differentiating (9) with respect to the parameter α, we put down the necessary optimality condition for the parameter α as

$$\frac{\partial \Delta^{\beta,\xi,\alpha}}{\partial \alpha} = -\left(\Sigma^{\beta,\xi,\alpha}\right)^{\mathrm{T}} \frac{\partial G^{\xi,\alpha}}{\partial \alpha}\beta^{\xi,\alpha} = 0, \tag{17}$$

for which

$$\frac{\partial G^{\xi,\alpha}}{\partial \alpha} = G^{\xi,\alpha}_{M+1}\left(\Psi^{\xi,\alpha}_{M+1}\right)^{\mathrm{T}}, \tag{18}$$

where $\Psi^{\xi,\alpha}_{M+1} \in \mathbb{R}^{m\times(m+1)}$ is the matrix of transformation from the space of vectors $\operatorname{span}\left\{G^{\xi,\alpha}_0, G^{\xi,\alpha}_1, \ldots, G^{\xi,\alpha}_{m+1}\right\}$ to the space $\operatorname{span}\left\{\frac{\partial G^{\xi,\alpha}_0}{\partial \alpha}, \frac{\partial G^{\xi,\alpha}_1}{\partial \alpha}, \ldots, \frac{\partial G^{\xi,\alpha}_m}{\partial \alpha}\right\}$ according to the operation of transformation

$$G^{\xi,\alpha}\left(\frac{\partial G^{\xi,\alpha}}{\partial \alpha}\right)^{\mathrm{T}} = G^{\xi,\alpha}\Psi^{\xi,\alpha}_{M+1}\left(G^{\xi,\alpha}_{M+1}\right)^{\mathrm{T}}.$$

By differentiating (15) with respect to the parameter ξ, we represent the first sufficient optimality condition

$$\begin{aligned}\frac{\partial^2 \Delta^{\beta,\xi,\alpha}}{\partial \xi^2} = &-\left[\left(\beta^{\xi,\alpha}\right)^{\mathrm{T}}\left(\frac{\partial G^{\xi,\alpha}}{\partial \xi}\right)^{\mathrm{T}} + \left(\frac{\partial \beta^{\xi,\alpha}}{\partial \xi}\right)^{\mathrm{T}}\left(G^{\xi,\alpha}\right)^{\mathrm{T}}\right]\frac{\partial G^{\xi,\alpha}}{\partial \xi}\beta^{\xi,\alpha} \\ &-\left(\Sigma^{\beta,\xi,\alpha}\right)^{\mathrm{T}}\left(\frac{\partial^2 G^{\xi,\alpha}}{\partial \xi^2}\beta^{\xi,\alpha} + \frac{\partial G^{\xi,\alpha}}{\partial \xi}\frac{\partial \beta^{\xi,\alpha}}{\partial \xi}\right) > 0,\end{aligned} \tag{19}$$

where

$$\frac{\partial \beta^{\xi,\alpha}}{\partial \xi} = \left(\left(G^{\xi,\alpha}\right)^{\mathrm{T}} G^{\xi,\alpha}\right)^{-1}\left[\left(G^{\xi,\alpha}\right)^{\mathrm{T}} H - \left[\left(\frac{\partial G^{\xi,\alpha}}{\partial \xi}\right)^{\mathrm{T}} G^{\xi,\alpha} + \left(G^{\xi,\alpha}\right)^{\mathrm{T}}\frac{\partial G^{\xi,\alpha}}{\partial \xi}\right]\beta^{\xi,\alpha}\right]. \tag{20}$$

By analogy with (16), represent $\frac{\partial^2 G^{\xi,\alpha}}{\partial \xi^2}$ as

$$\frac{\partial^2 G^{\xi,\alpha}}{\partial \xi^2} = \frac{\partial G^{\xi,\alpha}_{M+1}}{\partial \xi} \left(\Theta^{\xi,\alpha}_{M+1}\right)^{\mathrm{T}}, \tag{21}$$

$$\frac{\partial G^{\xi,\alpha}}{\partial \xi} \left(\frac{\partial^2 G^{\xi,\alpha}}{\partial \xi^2}\right)^{\mathrm{T}} = \frac{\partial G^{\xi,\alpha}}{\partial \xi} \Theta^{\xi,\alpha}_{M+1} \left(\frac{\partial G^{\xi,\alpha}_{M+1}}{\partial \xi}\right)^{\mathrm{T}} = G^{\xi,\alpha}_{M+1} \left(\Phi^{\xi,\alpha}_{M+1}\right)^{\mathrm{T}} \Theta^{\xi,\alpha}_{M+1} \Phi^{\xi,\alpha}_{M+2} \left(G^{\xi,\alpha}_{M+2}\right)^{\mathrm{T}},$$

where $\Theta^{\xi,\alpha}_{M+1} \in \mathbb{R}^{m\times(m+1)}$ is the matrix of transformation from the vector space span $\left\{\frac{\partial G^{\xi,\alpha}_0}{\partial \xi}, \frac{\partial G^{\xi,\alpha}_1}{\partial \xi}, \ldots, \frac{\partial G^{\xi,\alpha}_{m+1}}{\partial \xi}\right\}$ to the space span $\left\{\frac{\partial^2 G^{\xi,\alpha}_0}{\partial \xi^2}, \frac{\partial^2 G^{\xi,\alpha}_1}{\partial \xi^2}, \ldots, \frac{\partial^2 G^{\xi,\alpha}_m}{\partial \xi^2}\right\}$, and

$$\left(\Phi^{\xi,\alpha}_{M+1}\right)^{\mathrm{T}} \Theta^{\xi,\alpha}_{M+1} \Phi^{\xi,\alpha}_{M+2} \in \mathbb{R}^{(m+1)\times(m+2)}$$

carries out the last transformation from the source space span $\left\{G^{\xi,\alpha}_0, G^{\xi,\alpha}_2, \ldots, G^{\xi,\alpha}_{m+2}\right\}$.

Put down the second sufficient condition in the form of equations similar to (19), (20) and (21). Namely,

$$\frac{\partial^2 \Delta^{\beta,\xi,\alpha}}{\partial \alpha^2} = -\left[\left(\beta^{\xi,\alpha}\right)^{\mathrm{T}} \left(\frac{\partial G^{\xi,\alpha}}{\partial \alpha}\right)^{\mathrm{T}} + \left(\frac{\partial \beta^{\xi,\alpha}}{\partial \alpha}\right)^{\mathrm{T}} \left(G^{\xi,\alpha}\right)^{\mathrm{T}}\right] \frac{\partial G^{\xi,\alpha}}{\partial \alpha} \beta^{\xi,\alpha}$$
$$- \left(\Sigma^{\beta,\xi,\alpha}\right)^{\mathrm{T}} \left(\frac{\partial^2 G^{\xi,\alpha}}{\partial \alpha^2} \beta^{\xi,\alpha} + \frac{\partial G^{\xi,\alpha}}{\partial \alpha} \frac{\partial \beta^{\xi,\alpha}}{\partial \alpha}\right) > 0, \tag{22}$$

$$\frac{\partial \beta^{\xi,\alpha}}{\partial \alpha} = \left(\left(G^{\xi,\alpha}\right)^{\mathrm{T}} G^{\xi,\alpha}\right)^{-1} \left[\left(G^{\xi,\alpha}\right)^{\mathrm{T}} H - \left[\left(\frac{\partial G^{\xi,\alpha}}{\partial \alpha}\right)^{\mathrm{T}} G^{\xi,\alpha} + \left(G^{\xi,\alpha}\right)^{\mathrm{T}} \frac{\partial G^{\xi,\alpha}}{\partial \alpha}\right] \beta^{\xi,\alpha}\right].$$

Finally,

$$\frac{\partial^2 G^{\xi,\alpha}}{\partial \alpha^2} = \frac{\partial G^{\xi,\alpha}_{M+1}}{\partial \alpha} \left(\Omega^{\xi,\alpha}_{M+1}\right)^{\mathrm{T}}, \tag{23}$$

$$\frac{\partial G^{\xi,\alpha}}{\partial \alpha} \left(\frac{\partial^2 G^{\xi,\alpha}}{\partial \alpha^2}\right)^{\mathrm{T}} = \frac{\partial G^{\xi,\alpha}}{\partial \alpha} \Omega^{\xi,\alpha}_{M+1} \left(\frac{\partial G^{\xi,\alpha}_{M+1}}{\partial \alpha}\right)^{\mathrm{T}} = G^{\xi,\alpha}_{M+1} \left(\Psi^{\xi,\alpha}_{M+1}\right)^{\mathrm{T}} \Omega^{\xi,\alpha}_{M+1} \Psi^{\xi,\alpha}_{M+2} \left(G^{\xi,\alpha}_{M+2}\right)^{\mathrm{T}},$$

where $\Omega^{\xi,\alpha}_{M+1} \in \mathbb{R}^{m\times(m+1)}$ is the matrix of transformation from the space of vectors span$\left\{\frac{\partial G^{\xi,\alpha}_0}{\partial \alpha}, \frac{\partial G^{\xi,\alpha}_1}{\partial \alpha}, \ldots, \frac{\partial G^{\xi,\alpha}_{m+1}}{\partial \alpha}\right\}$ into the space span$\left\{\frac{\partial^2 G^{\xi,\alpha}_0}{\partial \alpha^2}, \frac{\partial^2 G^{\xi,\alpha}_1}{\partial \alpha^2}, \ldots, \frac{\partial^2 G^{\xi,\alpha}_m}{\partial \alpha^2}\right\}$, and consequently, $\left(\Psi^{\xi,\alpha}_{M+1}\right)^{\mathrm{T}} \Omega^{\xi,\alpha}_{M+1} \Psi^{\xi,\alpha}_{M+2} \in \mathbb{R}^{(m+1)\times(m+2)}$ executes the last transformation from the source span $\left\{G^{\xi,\alpha}_0, G^{\xi,\alpha}_2, \ldots, G^{\xi,\alpha}_{m+2}\right\}$.

It deserves noting that the expressions for $\frac{\partial^2 G^{\xi,\alpha}}{\partial \xi^2}$ and $\frac{\partial^2 G^{\xi,\alpha}}{\partial \alpha^2}$ that are similar to (21) and (23) can be established by direct differentiation, respectively, of (16) and (18).

3.3. Algorithm of Two-parameter Optimization

We present an algorithm of two-parameter optimization based on the aforementioned matrix transformation [15, 16]. Equations (15) and (17) may be interpreted as a pair of scalar products $P^{\xi,\alpha}\left(\frac{\partial G^{\xi,\alpha}}{\partial \xi}\right)H$ and $P^{\xi,\alpha}\left(\frac{\partial G^{\xi,\alpha}}{\partial \alpha}\right)H$ where $P^{\xi,\alpha}$ is the projection of the vector on the finite-dimensional subspace generated by $G^{\xi,\alpha}$. Then,

$$Q^{\xi,\alpha}\left(\frac{\partial G^{\xi,\alpha}}{\partial \xi}\right)H = 0 \quad \text{and} \quad Q^{\xi,\alpha}\left(\frac{\partial G^{\xi,\alpha}}{\partial \alpha}\right)H = 0,$$

where $Q^{\xi,\alpha}$ is the vector projection on the orthogonal complement such that $P^{\xi,\alpha} + Q^{\xi,\alpha} = I$. Consequently, Problem is solved by varying the finite-dimensional subspace generated by $G^{\xi,\alpha}$ in the Hilbertian space under which the components of the vectors $\frac{\partial G^{\xi,\alpha}}{\partial \xi}$ and $\frac{\partial G^{\xi,\alpha}}{\partial \alpha}$ orthogonal to the space $G^{\xi,\alpha}$ are orthogonal to H.

Algorithm 1.

1. Calculate the vector of coefficients $\beta^{\xi,\alpha}$ with the condition of the normal Eq. (11).
2. Generate the spaces of vectors $\frac{\partial G^{\xi,\alpha}}{\partial \xi}$ and $\frac{\partial G^{\xi,\alpha}}{\partial \alpha}$ in compliance with (16) and (18).
3. Calculate the complements $\frac{\partial G^{\xi,\alpha}}{\partial \xi}$ and $\frac{\partial G^{\xi,\alpha}}{\partial \alpha}$ orthogonal to $G^{\xi,\alpha}$ and H.
4. Generate the vector spaces $\frac{\partial^2 G^{\xi,\alpha}}{\partial \xi^2}$ and $\frac{\partial^2 G^{\xi,\alpha}}{\partial \alpha^2}$ in compliance with (21) and (23) and verify (19) and (22) for satisfiability of the sufficient optimality conditions.

We notice that the proposed algorithm of the two-parameter optimization can be adapted to design in the s-domain the nonorthogonal Meiksner filters defined in compliance with (3).

4. COMPUTER EXPERIMENTS

Consider the effect of the above algorithm by way of three examples selected to demonstrate the degree of impact of the parameter α on the final result depending on the considered in [15] kind of the approximated function. The experimental studies used MacBook Air 11 OS X EI Captain with processor 1.3 GHz Intel Core i5 and memory of 4 GB 1600 MHz DDR3, as well as GNU Octave 3.8.2.

Example 1. Consider the transfer function $H(z)$ of start-up of the supersonic engine described in [3] given by

$$H(z) = z\frac{2.034z^6 - 4.9825z^5 + 6.57z^4 - 5.8189z^3 + 3.636z^2 - 1.4105z + 0.2997}{z^7 - 2.46z^6 + 3.433z^5 - 3.333z^4 + 2.546z^3 - 1.584z^2 + 0.7478z - 0.252}.$$

It is used in [13, 15] to verify the algorithms to optimize filter parameters. Figure 3a shows a surface describing the approximation error (9) as function of $\xi \in (-1; 1)$ and $\alpha \in [-0.5; 2]$ under $m = 6$. For interpretation convenience, Fig. 3b also presents an outline graph where the cross marks solution of the Problem. The established values of the optimal pair $(\xi; \alpha)$ for the orders $m = [0; 6]$ are given in the column "Example 1" of the table.

Numerical values of the optimal pair (ξ, α) for the range $m = [0; 6]$

Example 1			Example 2		
m	$(\xi; \alpha)$	$\Delta^{\beta,\xi,\alpha}$	m	$(\xi; \alpha)$	$\Delta^{\beta,\xi,\alpha}$
0	(−0.133; 0.208)	0.1287	0	(0.9; 1.979)	0.6916
1	(0.233; 0.104)	0.1176	1	(0.85; 1.979)	0.4352
2	(0.583; 0.021)	0.0647	2	(0.9; 1.958)	0.2529
3	(0.617; 0.021)	0.0645	3	(0.866; 1.854)	0.1166
4	(0.333; 0.271)	0.0422	4	(0.883; 1.979)	0.0505
5	(0.25; -0.083)	0.0235	5	(0.866; 1.854)	0.0197
6	(0.133; 0)	0.0194	6	(0.883; 1.979)	0.0072

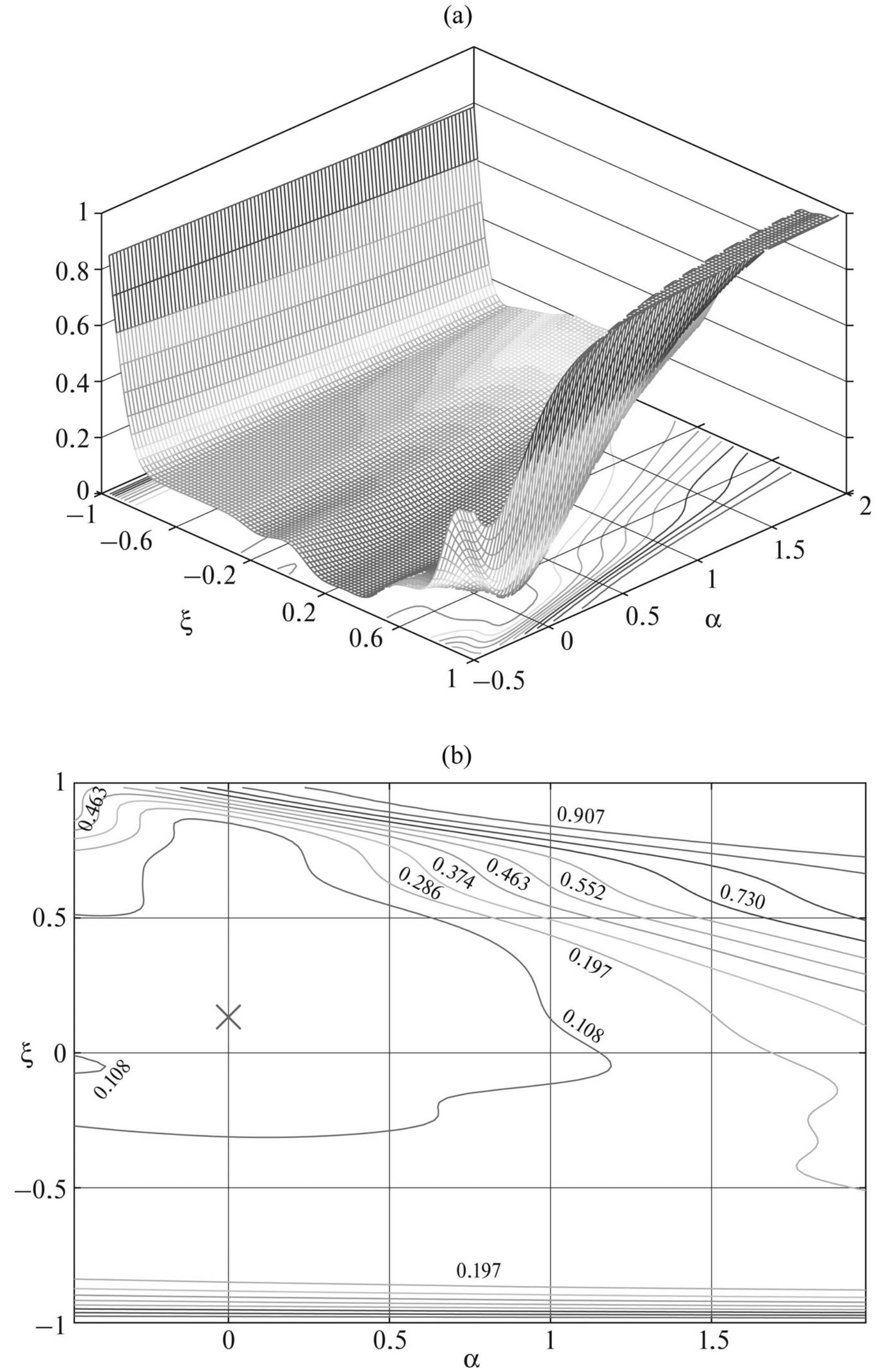

Fig. 3. Error $\Delta^{\beta,\xi,\alpha}$ vs. $\xi \in (-1; 1)$ and $\alpha \in [-0.5; 2]$ for $m = 6$ (Example 1).

As can be seen from the figures and table, an increase in α does not improve the result. To corroborate the results obtained, Fig. 4a shows the results of approximation of the pulse transfer characteristic $h(n)$ for $m = 6$, optimal ξ, and various $\alpha = \{0; 0.5; 1; 1.5; 2\}$. The best result corresponds to $\alpha = 0$. It follows from the table that for other orders m various fractional nonzero α provide better results. Nevertheless, taking into account the need for handling the fractional filters and complication of the filter structure, it is advisable to make choice in favor of simpler discrete Laguerre filters ($\alpha = 0$).

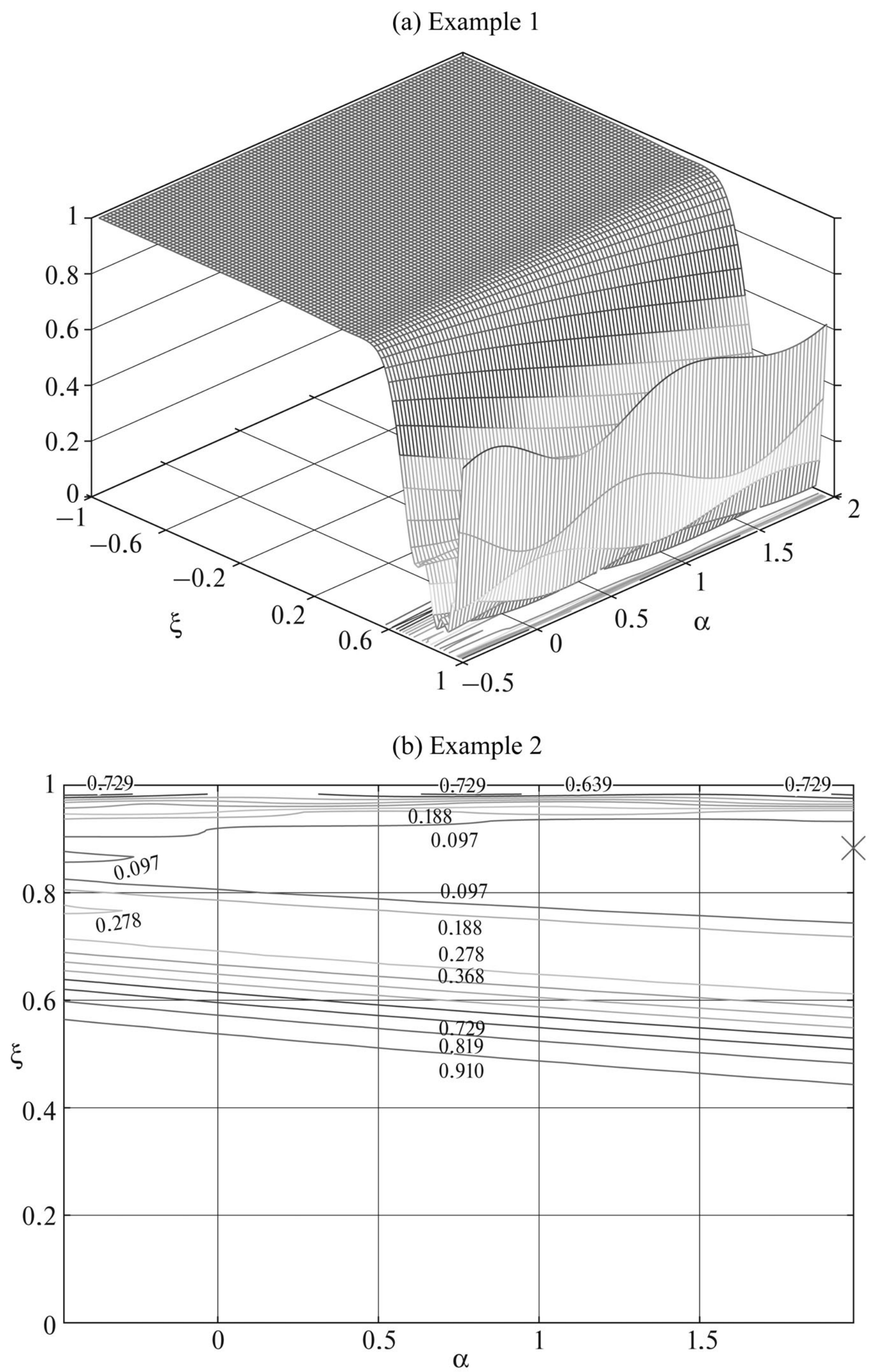

Fig. 4. Approximation of $h(n)$ for $\xi_{\rm opt}$ and $\alpha = \{0; 0.5; 1; 1.5; 2\}$, $m = 6$.

Example 2. Consider the transfer function

$$H(z) = \left(\frac{z}{z - (0.9 + 0.1\mathrm{j})}\right)^4 + \left(\frac{z}{z - (0.9 - 0.1\mathrm{j})}\right)^4$$

described in [7]. It deserves noting that the given example is not sufficiently good in the sense that the fields $H(z)$ are complex, although selected near the real axis. However, it allows one to demonstrate that the Meixner filters represent a constructive alternative to the discrete Laguerre

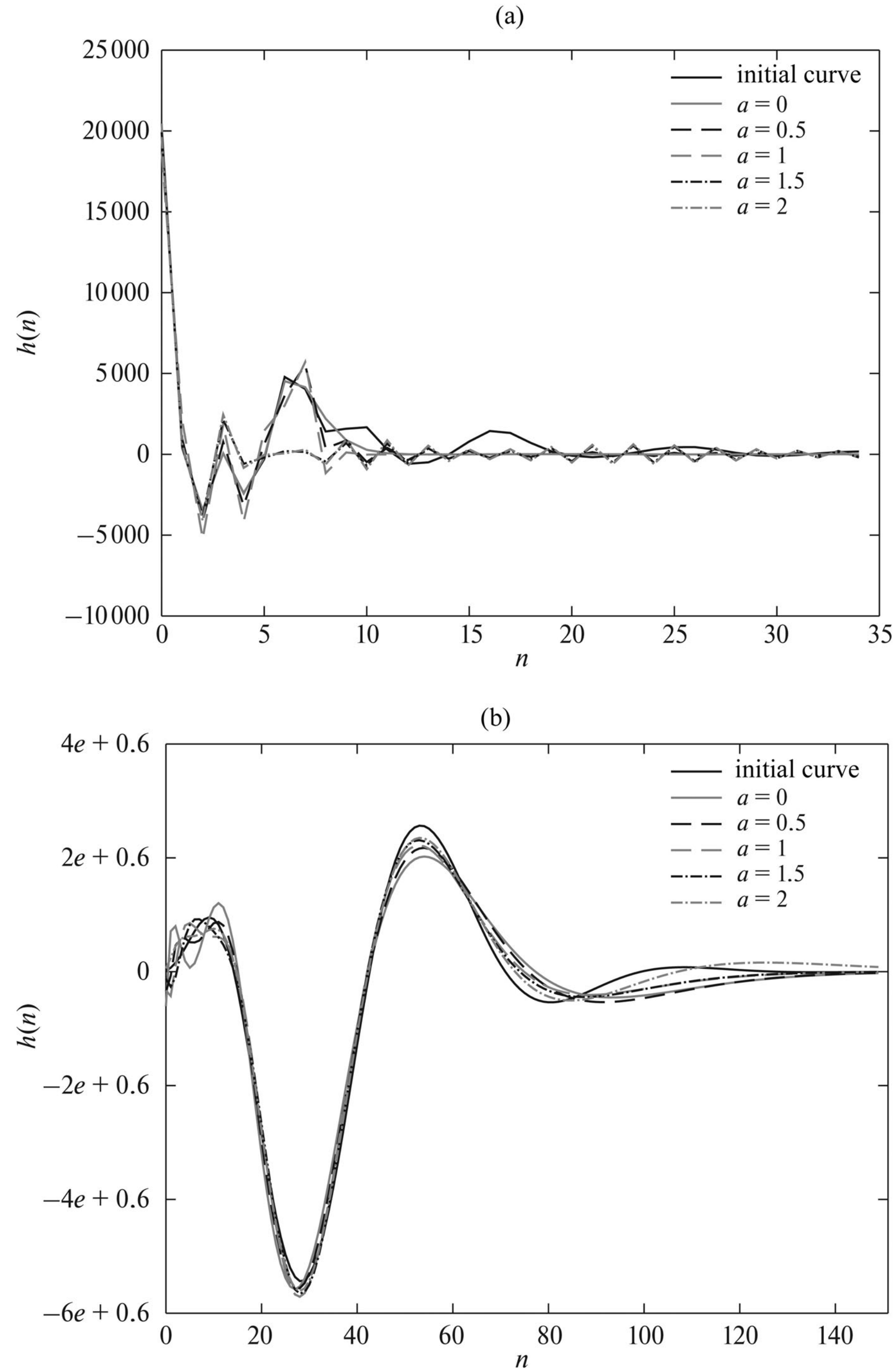

Fig. 5. Error $\Delta^{\beta,\xi,\alpha}$ vs. $\xi \in (-1;1)$ and $\alpha \in [-0.5;2]$ for $m = 6$ (Example 2).

filters. As was noticed in [7], in comparison with the Laguerre functions those of Meixner suit better for representation of the delay functions and also enable one to extract the low-frequency components at the expense of the high-frequency components.

For the given example, Fig. 5a shows the surface describing approximation error (9) vs. $\xi \in (-1;1)$ and $\alpha \in [-0.5;2]$ for $m = 6$. For convenience of interpretation, Fig. 5b shows an outline graph where the cross marks solution of Problem. We notice that for better scaling the outline graph has the range of values $\xi \in [0;1)$ because the values of all filter fields lie on the positive

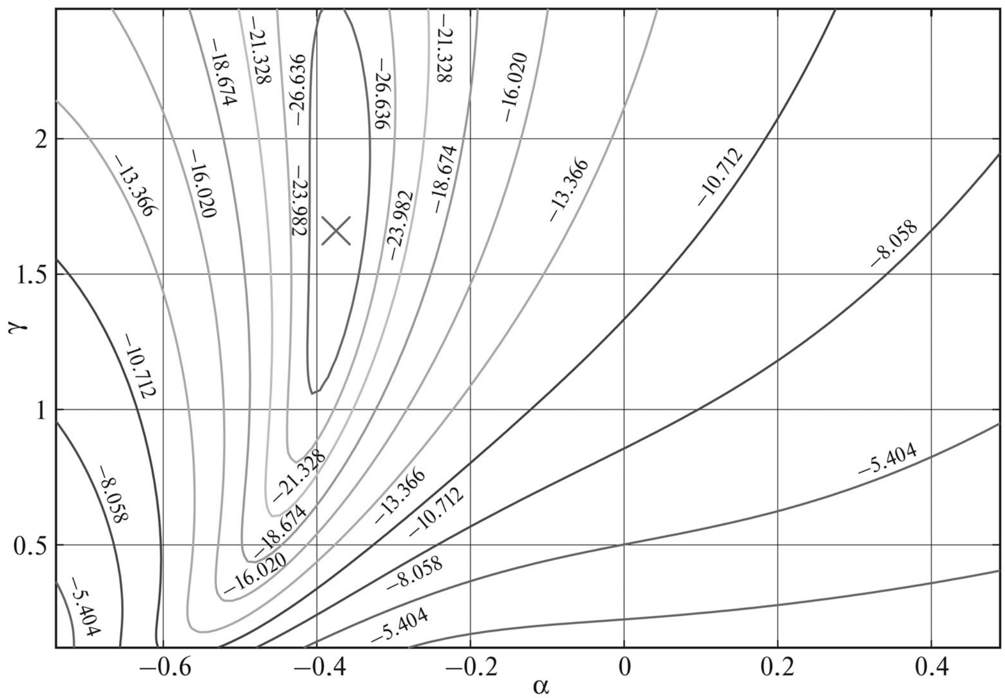

Fig. 6. Error $\Delta^{\beta,\gamma,\alpha}$ vs. $\gamma \in (0; 2.5]$, $\alpha \in [-0.75; 0.5]$ for $m = 1$ (Example 3).

half-axis of the range of definition of ξ. The corresponding column of the table contains the values of the optimal pair $(\xi; \alpha)$ for the orders $m = [0; 6]$.

One can see from this example that further increase in α beyond the considered interval might provide better results because all optimal values of the given parameter lie near the right boundary of the interval (see the table and Fig. 5b). The results of approximation of the pulse transfer characteristic are shown in Fig. 4b. As was expected, the best result corresponds to $\alpha = 2$. Moreover, as the fragment of $h(n)$ shows over the range $n \in [0; 20]$, the Meixner filters demonstrate better extraction of the low-frequency component with increased α. Whence it follows a conclusion of advisability of using the Meixner filters for the given example.

Consider an example where the optimal α that was established by solving Problem is not an integer. For completeness of the computer experiment, we estimate efficiency of the Meixner filters with regard for formulation of the problem of two-parameter optimization in the s-range.

Example 3. Given be the transfer function

$$H(s) = \frac{1}{s^{0.7} + 2} + \frac{1}{s^{0.8} + 2} + \frac{1}{s^{0.9} + 1}$$

[18]. Figure 6 depicts an outline graph where solution of Problem is marked for $\gamma \in (0; 2.5]$, $\alpha \in [-0.75; 0.5]$ and $m = 1$. Figure 7 shows the Bode diagrams:

(a) for $\gamma^* = 1.66$ and $\alpha = 0$;

(b) for $\gamma^* = 1.66$ and $\alpha^* = -0.375$

under increased order $m = \{0; 1; 2\}$.

One can see from Fig. 7 that under the integer $\alpha = 0$ which is the nearest to the optimal one $\alpha = -0.375$ increased filter order m does not improve the result in distinction to the choice of α^*.

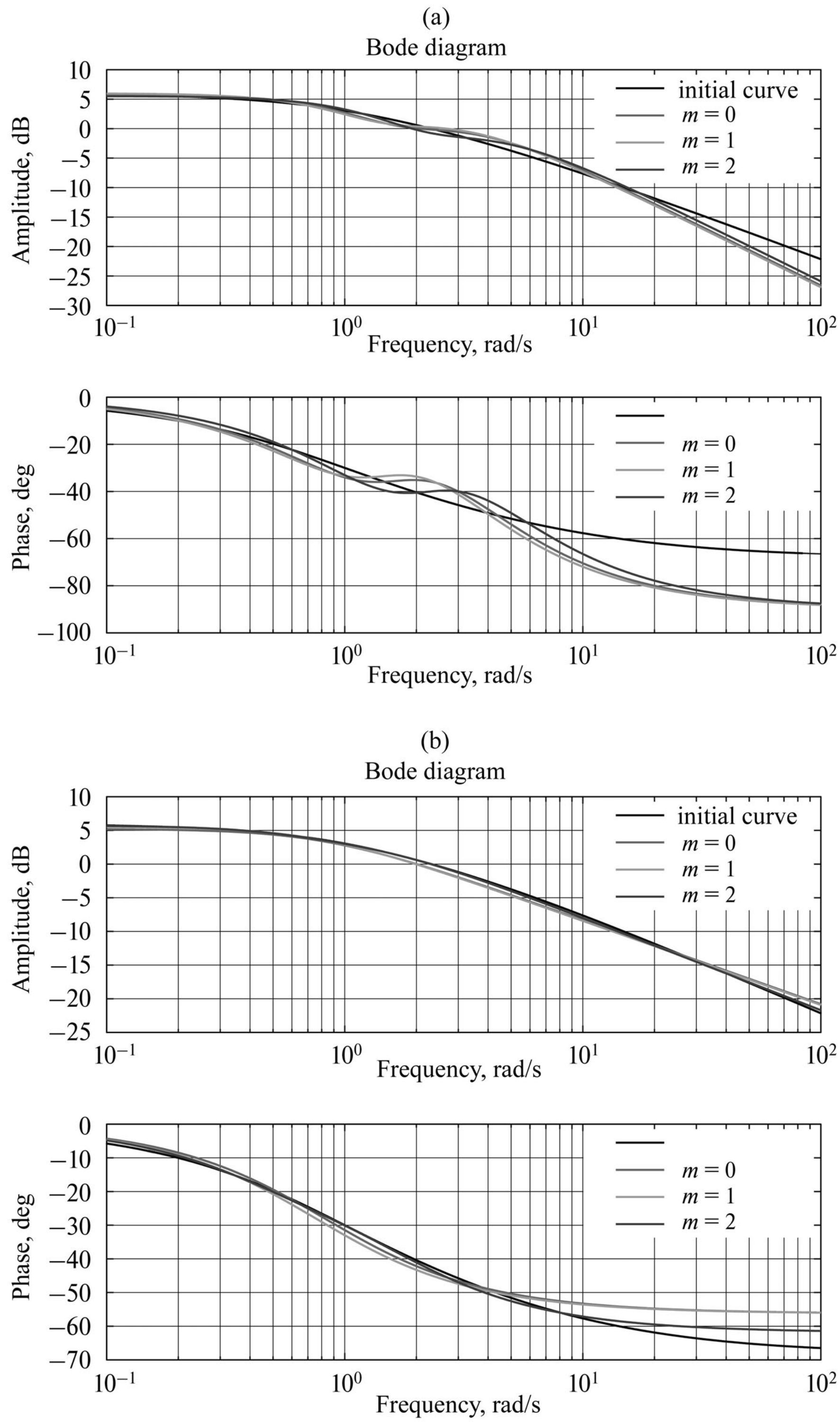

Fig. 7. Bode diagrams (Example 3).

5. CONCLUSIONS

The present paper considers a new representation of the Meixner filters featuring the following advantages as compared with those proposed previously: simpler structure, rational form of representation under any integer values of the additional parameter α, and possibility of describing the dynamic systems with fractional order under noninteger values of α. Moreover, these filters are directly related with continuous generalized Laguerre filters by a modified bilinear transformation. The distinctive feature of the proposed filters is their nonorthogonality. To design such filters, it was suggested to define the coefficients as the solution of the normal equation and solve the problem of optimization of the pair (ξ, α) of parameters in the matrix form. As the result of such formulation, proposed was an efficient algorithm to solve the two-parameter optimization problem providing a more stable result with smaller computational burden. At that, by more stable result is meant increased convergence of the algorithm for estimation of the model parameters as the result of

1) passage from numerical method of estimation of the parameters of filter model to the normal equation;

2) introduction of the procedure of implicit regularization of the filter model in the form of optimization problem of the pair (ξ, α).

Lower computer burden is attained owing to the simpler structure of the nonorthogonal Meixner filters as compared with the previous structure including the additional unit transforming the system of Laguerre filters into the z-transformation of functions similar to the Meixner functions. Expedience of the proposed filters was corroborated by the computer experiments. The basic purpose of the new nonorthogonal Meixner filters, as their predecessors lies in describing the nonstationary and delay systems where an identical approximation can be obtained by a smaller number of series terms with the use of the additional parameter α enabling one to extract the low-frequency components to the best advantage. The proposed algorithm to estimate the filter model parameters allows one in turn to establish a better result at solving the problem of real-time identification and verification requiring efficient realization of the models such as ARX, ARMAX, and their modifications.

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APPENDIX

Determine nonorthogonal Meixner filters

$$G_k(z,\xi,\alpha) = \frac{(1-\xi^2)z}{z-\xi}\left(\frac{(1-\xi)(z+1)}{z-\xi}\right)^{\alpha}\left(\frac{1-\xi z}{z-\xi}\right)^{k} \tag{A.1}$$

using the bilinear transformation

$$\Lambda_k(s,\gamma,\alpha) \mapsto \frac{z}{z+1}\Lambda_k\left(a\frac{z-1}{z+1},\gamma,\alpha\right) = \frac{(-1)^k 2a}{a+\gamma/2}G_k\left(z,\frac{a-\gamma/2}{a+\gamma/2},\alpha\right). \tag{A.2}$$

Assume that the Laplace transform $\Lambda_k(s,\gamma,\alpha)$ of the function system $L_k(\tau,\gamma,\alpha)$ is given by

$$\Lambda_k(s,\gamma,\alpha) = \left(\frac{\gamma}{s+\gamma/2}\right)^{\alpha+1}\left(\frac{s-\gamma/2}{s+\gamma/2}\right)^{k}. \tag{A.3}$$

Then, by replacing s by $a\frac{z-1}{z+1}$ in compliance with (A.2), we can put down

$$s+\frac{\gamma}{2} = \frac{\left(a+\frac{\gamma}{2}\right)\left(z-\frac{a-\gamma/2}{a+\gamma/2}\right)}{z+1}, \quad s-\frac{\gamma}{2} = \frac{\left(a+\frac{\gamma}{2}\right)\left(z\frac{a-\gamma/2}{a+\gamma/2}-1\right)}{z+1}. \tag{A.4}$$

In turn, replacement of $\frac{a-\gamma/2}{a+\gamma/2}$ by ξ according to (A.2) provides

$$\frac{s-\gamma/2}{s+\gamma/2}=\frac{z\xi-1}{z-\xi}. \tag{A.5}$$

Go to the transformation $\left(\frac{\gamma}{s+\gamma/2}\right)^{\alpha+1}$ in (A.3). Taking in consideration (A.4), one can put down

$$\left(\frac{\gamma}{s+\gamma/2}\right)^{\alpha+1}=\left(\frac{\gamma}{(a+\gamma/2)}\frac{(z+1)}{(z-\xi)}\right)^{\alpha+1}. \tag{A.6}$$

Obviously,

$$1-\xi=\frac{\gamma}{a+\gamma/2};\quad 1+\xi=\frac{2a}{a+\gamma/2}. \tag{A.7}$$

Use of (A.5) and (A.6) with allowance for (A.7), multiplication by $\frac{z}{z+1}$ according to (A.2), and division by $\frac{(-1)^k 2a}{a+\gamma/2}$ equal to $(-1)^k(1+\xi)$ according to (A.7) provide (A.1).

Similarly, the Laplace transform (A.3) can be obtained from the z-transformation (A.1) in compliance with

$$G_k(z,\xi,\alpha)\mapsto\frac{2a}{a+s}G_k\left(\frac{a+s}{a-s},\xi,\alpha\right)=\frac{(-1)^k}{1+\xi}\Lambda_k\left(s,2a\frac{1-\xi}{1+\xi},\alpha\right).$$

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