Assignment 2

Due: Wednesday, September 4

MATH 486 - Mathematical Modeling
Prof. Tier

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Problem 1: A rocket of mass m is tracked in the (x, y) plane where y is the vertical direction. The rocket is launched on a flat surface in the positive x direction from location $(0, 0)$ with an initial velocity vector $\vec{v} = v_x\vec{i} + v_y\vec{j}$.

(a) Use dimensional analysis to find a formula for the range R of the rocket assuming that the range is a function of v_x, v_y, m, and g.
(b) Derive an exact formula for the range using Newton's 2nd law.

Solution:

(a) Under the given assumption, we perform the following analysis:

$$\begin{aligned} R &= f(v_x, v_y, m, g) \\ [R] &= [v_x]^a[v_y]^b[m]^c[g]^d \\ L &= (LT^{-1})^a(LT^{-1})^b M^c (LT^{-2})^d \\ L &= L^{a+b+d}T^{-a-b-2d}M^c \end{aligned}$$

We can then form the following system of equations:

$$\begin{aligned} a + b + d &= 1 \\ -a - b - 2d &= 0 \\ c &= 0 \end{aligned}$$

Which we can then use to form an augmented matrix and solve accordingly:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ -1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{[1]} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{[2],[3]} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

Thus, we get $d = -1 \implies c = 0 \implies b = t \implies a = 1 - b - d = 2 - t$. We can then write the solution as:

$$\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 2 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Therefore, we can write the formula for R as follows:

$$R = \frac{v_x^2}{g} F\left(\frac{v_y}{v_x}\right). \tag{0.1}$$

[1]: $r_1 + r_2 \rightarrow r_2$

[2]: $r_2 \leftrightarrow r_3$

[3]: $\frac{r_3}{-1} \rightarrow r_3$

(b) To derive an exact formula, we consider the kinematic equation

$$v_y - v_{y_o} = a_t t \tag{0.2}$$

Note that at the highest point of the trajectory, $v_y = 0$ and $a_y = -g$. Thus, plugging these back into (0.2) and solving for t we get

$$t_h = \frac{-\,v_{y_0}}{-g} = \frac{v_o \sin\theta}{g}.$$

Observe that the time it takes to reach the maximum height is only *half* the time of the total trajectory. Thus, to find range R, we let

$$t_r = 2t_h = \frac{2v_o \sin\theta}{g}.$$

Substituting this back into the kinematic equation for displacement, we get:

$$\begin{aligned}
R &= (v_o \cos\theta)t_r + \frac{1}{2}gt_r^2 \\
&= (v_o \cos\theta)\frac{2v_o \sin\theta}{g} + 0 \\
&= \frac{2v_x v_y}{g} \\
&= \frac{v_x^2}{g}\frac{2v_y}{v_x}.
\end{aligned}$$

This implies that the function F from (0.1) is equal to $F\left(\frac{v_y}{v_x}\right) = 2\frac{v_y}{v_x}$. In any case, we have found the exact formula for the range R as:

$$R = \frac{2v_x v_y}{g} = \frac{v_o^2 \sin 2\theta}{g}.$$

Problem 2: (adapted from Holmes 1.4) The luminosity of certain giant and super-giant stars varies in a periodic manner. It is hypothesized that the period p depends on the star's average radius r, its mass m and the gravitational constant G.

(a) Newton's law of gravitation states that the attractive force between two bodies is proportional to the product of their masses divided by the square of the distance between them, i.e.

$$F = \frac{Gm_1m_2}{d^2}$$

where G is the gravitational constant. Find the dimensions of G.

(b) Use dimensional analysis to determine the functional dependence of p on m, r, and G. Identify any dimensionless groups.

(c) Eddington devised a theory which gave the period as

$$p = \sqrt{\frac{3\pi}{2\gamma G\rho}}$$

where ρ is the average density of the star and γ is the ratio of the specific heats for stellar material. Assuming the star is of mass m and spherical, use Eddington's formula to determine an unknown constants in your formula in (b).

Solution:

(a) Using the following equivalency, we deduce the dimensions of G as follows:

$$\begin{aligned} [F] &= \frac{[G][m_1][m_2]}{[d]^2} \\ MLT^{-2} &= [G]M^2L^{-2} \\ \implies [G] &= L^3T^{-2}M^{-1} \end{aligned}$$

(b) We begin with the assumption that $p = f(m, r, G)$. Thus, we perform the following analysis:

$$\begin{aligned} [p] &= [m]^a[r]^b[G]^c \\ T &= M^aL^b(L^3T^{-2}M^{-1})^c = M^{a-c}L^{b+3c}T^{-2c} \end{aligned}$$

We can then form the following system of equations:

$$\begin{aligned} a - c &= 0 \\ b + 3c &= 0 \\ -2c &= 1 \end{aligned}$$

We can then form an augmented matrix and solve as such:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right] \xrightarrow{[1]} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & \frac{-1}{2} \end{array}\right]$$

This implies that $c = \frac{-1}{2} \implies b = -3c = \frac{3}{2} \implies a = c = \frac{-1}{2}$.

We can rewrite this as

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1/2 \\ 3/2 \\ -1/2 \end{bmatrix}$$

Therefore, we can write p as

$$p \propto \sqrt{\frac{r^3}{mG}} \text{ or } p = \alpha\sqrt{\frac{r^3}{mG}} \text{, where } [\alpha] = 1$$

(c) To find the unknown constant α in our formula, we will use Eddington's formula.

$$p = \alpha\sqrt{\frac{r^3}{mG}} = \sqrt{\frac{3\pi}{2\gamma G\rho}}$$

$$\alpha\sqrt{\frac{r^3}{m}} = \sqrt{\frac{3\pi}{2\gamma\rho}}$$

$$\alpha = \sqrt{\frac{3\pi}{2\gamma\rho}}\sqrt{\frac{m}{r^3}}$$

$$\alpha = \sqrt{\frac{3m\pi}{2\gamma\rho r^3}}$$

Thus, we can write our final formula as follows:

$$p = \sqrt{\frac{3m\pi}{2\gamma\rho r^3}}\sqrt{\frac{r^3}{mG}}$$

$$= \sqrt{\frac{3\pi}{2\gamma\rho G}}$$

Problem 3: (Logan) In an indentation experiment, a slab of metal of thickness h is subjected to a constant pressure P on its upper surface by a cylinder of radius a. A vertical displacement of U is observed of the indentation. The displacement depends on two material properties: the Poisson ratio v, which is dimensionless, and assume the Lame' constant μ, which has dimensions M/L^3T^2. Find the functional form of U.

Solution: We begin by assuming U depends on all given parameters, and as such can be written as

$$U = f(P, a, h, v, \mu).$$

Thus we get the following:

$$\begin{aligned} [U] &= [P]^a[a]^b[h]^c[v]^d[\mu]^e \\ L &= (ML^{-1}T^{-2})^a L^b L^c \theta^d (ML^{-3}T^{-2})^c \\ L &= L^{-a+b+c-3e} M^{a+e} T^{-2a-2e} \theta^d \end{aligned}$$

This can be rewritten as the following system of equations:

$$\begin{aligned} -a + b + c - 3e &= 1 \\ a + e &= 0 \\ -2a - 2e &= 0 \implies a + e = 0 \textit{ (duplicate)} \\ d &= 0 \end{aligned}$$

We form the following augmented matrix with the three equations and solve as such:

$$\left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & -3 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{[1]} \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{[2]} \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

Therefore, we get the following solutions:

$$\begin{aligned} a &= b + c - 1 - 3e = -e \\ b &= 1 + 2e - c \\ c &= c \textit{ (free)} \\ d &= 0 \\ e &= e \textit{ (free)} \end{aligned}$$

Which can be rewritten as

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} c + \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} e + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} d$$

Therefore, we can write U as such

$$U = aF\left(\frac{P}{\mu a^2}, \frac{h}{a}, v\right)$$

[1]: $r_1 + r_2 \to r_2$

[2]: $\frac{r_1}{-1} \to r_1$

Problem 4: (Nonlinear Pendulum) The differential equation for the angular displacement $\psi(t)$ of a pendulum (see class notes) is:

$$\psi''(t) + \frac{g}{r}\sin\psi = 0 \tag{0.3}$$

with initial conditions $\psi(0) = \theta$, $\psi'(0) = 0$.

(a) Repeat the (in-class) derivation of the calculation of t_p of a nonlinear pendulum in the absence of damping.
(b) Generate a graph of t_p/T_o as a function of $\theta \in (0, 2)$ with $r = 2$ and g in MKS units. Note: Fix the range in the vertical axis to be $[0, 4]$.
(c) Does your graph illustrate that the linear approximation gives a reasonably accurate result for the period if θ is small?

Solution:

(a) Let t_p be the period of the pendulum. Using the relationship $t_p = f(m, g, r, \theta)$, we perform some dimensional analysis as follows:

$$\begin{aligned}[t_p] &= [m]^a[g]^b[r]^c[\theta]^d \\ T &= M^a L^{b+c} T^{-2b} \\ &\Downarrow \\ a &= 0 \\ b + c &= 0 \\ -2b &= 1 \\ &\Downarrow \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{[1],[2]} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/2 \end{array}\right]$$

[1]:$r_1 + r_2 \to r_2$

[2]:$r_2 \leftrightarrow r_3$

For which we write the solution as

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ -1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} t$$

$$\begin{aligned} \therefore t_p &\propto \sqrt{\frac{r}{g}}\theta^t \\ &= \alpha\sqrt{\frac{r}{g}}\theta^t \text{ , where } [\alpha] = 1 \\ &= \sqrt{\frac{r}{g}}\left[\alpha_1\theta^{t_1} + \alpha_2\theta^{t_2} + \dots\right] \\ &= \sqrt{\frac{r}{g}}h(\theta) \end{aligned}$$

Now that we have:

$$t_p = \sqrt{\frac{r}{g}} h(\theta) \tag{0.4}$$

We will now look at (0.3). Observe that $[\psi''(t)] = T^{-2}$, $[\frac{g}{r}\sin\psi] = [\frac{g}{r}] = T^{-2}$. Supposing that θ is small $\implies \psi(t)$ is small. Using the first-order approximation of $\sin\psi = \psi$, we get:

$$\psi''(t) + \frac{g}{r}\psi = 0 \tag{0.5}$$

with $\psi(0) = \theta$, $\psi'(0) = 0$. Assuming $\psi = e^{\lambda t} \implies \lambda^2 + \frac{g}{r} = 0 \implies \lambda = \pm i\sqrt{\frac{g}{r}} \implies$ the fundamental set of solutions is:

$$\left\{\cos\frac{g}{r}t, \sin\frac{g}{r}t\right\}.$$

Thus, the general solution is $\psi = c_1\cos\frac{g}{r}t + c_2\sin\frac{g}{r}t$, with $\psi(0) = \theta \implies c_1 = \theta$, $\psi'(0) = 0 \implies c_2 = 0$. Since $c_2 = 0$, our solution becomes $\psi = \theta\cos\frac{g}{r}t$. Thus the period t_p is:

$$t_p = \frac{2\pi}{\sqrt{\frac{g}{r}}} = 2\pi\sqrt{\frac{r}{g}} \xrightarrow{(0.4)} h(\theta) = 2\pi.$$

Thus,

$$t_p = 2\pi\sqrt{\frac{r}{g}} \tag{0.6}$$

To find t_p directly, we multiply (0.5) across by $\dfrac{d\psi}{dt}$. We then get:

$$\frac{d\psi}{dt}\frac{d^2\psi}{dt^2} + \frac{g}{r}\psi\frac{d\psi}{dt} = 0 \implies \frac{1}{2}\left(\frac{d\psi}{dt}\right)^2 + \frac{g}{r}\frac{1}{2}\psi^2 = c \xrightarrow{t=0} \frac{g}{r}\frac{1}{2}\theta^2 = c$$

This then implies:

$$\begin{aligned}
\left(\frac{d\psi}{dt}\right)^2 &= \frac{g}{r}(\theta^2 - \psi^2) \\
\frac{d\psi}{dt} &= \sqrt{\frac{g}{r}}\sqrt{\theta^2 - \psi^2} \\
\frac{dt}{d\psi} &= \sqrt{\frac{r}{g}}\frac{1}{\sqrt{\theta^2 - \psi^2}} \\
\int dt &= \sqrt{\frac{r}{g}}\int_0^\theta \frac{d\psi}{\sqrt{\theta^2 - \psi^2}} \\
t_p &= 4\sqrt{\frac{r}{g}}\int_0^\theta \frac{d\psi}{\sqrt{\theta^2 - \psi^2}}.
\end{aligned}$$

(b) Entering the following expressions into Mathematica produces the following graph from $\theta = (0, 2)$.

$$f(\theta\_) := 4\sqrt{\frac{2}{9.81}} \int_0^1 \frac{1}{\sqrt{(1-t^2)\left(1-t^2\sin^2\left(\frac{\theta}{2}\right)\right)}}\,dt$$

$$\text{Plot}\left[\left\{f(\theta), 2\pi\sqrt{\frac{2}{9.81}}\right\}, \{\theta, 0, 2\}, \text{PlotRange} \to \{0, 4\}\right]$$

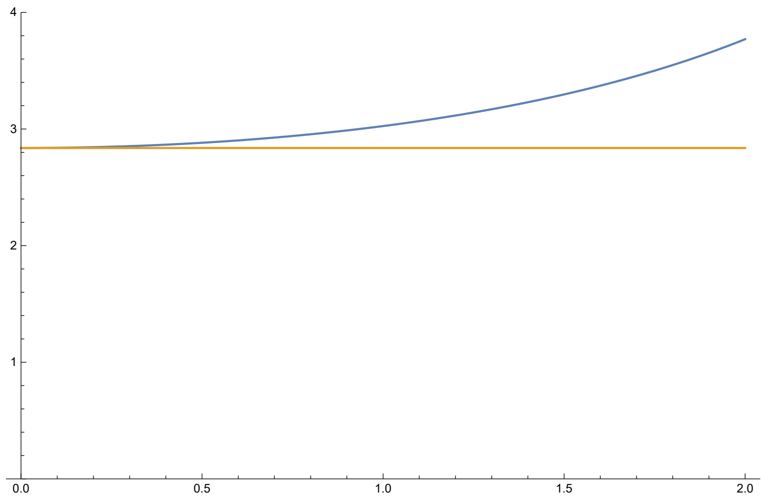

FIGURE 0.1. t_p/T_o

(c) Yes, for a reasonably small θ, the linear approximation T_o gives an equally reasonable result.