과제 2 1번~6번 해답

1. $$E(y_{ij}) = E(\mu + \tau_{(i)} + A_i + e_{ij})$$
$$= \mu + \tau_{(i)} + E(A_i) + E(e_{ij})$$
$$= \mu + \tau_{(i)}$$

2. $$V(y_{ij}) = V(\mu + \tau_{(i)} + A_i + e_{ij})$$
$$= V(A_i + e_{ij})$$
$$= V(A_i) + V(e_{ij})$$
$$= \sigma_a^2 + \sigma_e^2$$

3. 다른 학교의 두 학생 성적은 $y_{i_1 j}$, $y_{i_2 k}$ 라고 하면
$(i_1 \neq i_2)$

$$Cov(y_{i_1 j}, y_{i_2 k})$$

$$= Cov(\mu + \tau_{(i_1)} + A_{i_1} + e_{i_1 j}, \mu + \tau_{(i_2)} + A_{i_2} + e_{i_2 k})$$

$$= Cov(A_{i_1} + e_{i_1 j}, A_{i_2} + e_{i_2 k})$$
$$= Cov(A_{i_1}, A_{i_2}) + Cov(A_{i_1}, e_{i_2 k}) + Cov(e_{i_1 j}, A_{i_2}) + Cov(e_{i_1 j}, e_{i_2 k})$$

$$= 0 + 0 + 0 + 0 = 0$$

왜냐하면 $A_1 \cdots A_4$ 는 독립이고 $e_{11} \cdots e_{2,10}$ 도 독립이며 A_i 와 e_{ij} 도 독립이다.

따라서 상관계도 0 이다

4. $Cov(y_{ij}, y_{ik})$

$= Cov(A_i, A_i)$

$= \sigma_a^2$

$$Corr(y_{ij}, y_{ik}) = \frac{Cov(y_{ij}, y_{ik})}{\sqrt{V(y_{ij})V(y_{ik})}}$$

$$= \frac{\sigma_a^2}{\sigma_a^2 + \sigma_e^2}$$

5 모형을 행렬 형식 $\underline{y} = X\underline{\beta} + Z\underline{b} + \underline{\epsilon}$ 로 나타내면

$$\underset{40\times 2}{X} = \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ \hline 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix} \qquad \underset{2\times 1}{\beta} = \begin{bmatrix} 0 \\ \tau \end{bmatrix}$$

$$\underset{10\times 1}{Z_1} = \underset{10\times 1}{Z_2} = \underset{10\times 1}{Z_3} = \underset{10\times 1}{Z_4} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \underset{\sim}{1}_{10}$$

$$\underset{40\times 4}{Z} = \begin{bmatrix} \underset{\sim}{1}_{10} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1}_{10} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1}_{10} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1}_{10} \end{bmatrix}$$

여기서 $\underset{10\times 1}{\underset{\sim}{0}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$$\underset{4\times 1}{\underset{\sim}{b}} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

6 $V(\underset{\sim}{y}) = V(X\beta + Z\underset{\sim}{b} + \underset{\sim}{\varepsilon})$

$= V(Z\underset{\sim}{b}) + V(\underset{\sim}{\varepsilon})$

$= Z V(\underset{\sim}{b}) Z^t + \sigma_e^2 I_{40}$

여기서 $V(\underset{\sim}{b}) = \sigma_a^2 I_4$

따라서

$Z V(\underset{\sim}{b}) Z^t = Z \sigma_a^2 I_4 Z^t$

$= \sigma_a^2 Z Z^t$

$$= \sigma_a^2 \begin{bmatrix} \underset{\sim}{1}\underset{\sim}{1}^t & 0 & 0 & 0 \\ 0 & \underset{\sim}{1}\underset{\sim}{1}^t & 0 & 0 \\ 0 & 0 & \underset{\sim}{1}\underset{\sim}{1}^t & 0 \\ 0 & 0 & 0 & \underset{\sim}{1}\underset{\sim}{1}^t \end{bmatrix} -$$

위에서 $\underset{\sim}{1}\underset{\sim}{1}^t = \begin{bmatrix} 1 & \cdots & 1 \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$ = 모든 원소가 1 인 10×10 행렬

$\mathbb{0} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix}$ = 모든 원소가 0 인 10×10 행렬

여기서

$$\underset{10\times10}{\Sigma} = \sigma_a^2 \underset{\sim}{1}\underset{\sim}{1}^t + \sigma_e^2 I$$

$$= \begin{bmatrix} \sigma_a^2+\sigma_e^2 & & \sigma_a^2 \\ & \ddots & \\ \sigma_a^2 & & \sigma_a^2+\sigma_e^2 \end{bmatrix}$$

따라서

$$\underset{40\times40}{V(\underset{\sim}{y})} = \begin{bmatrix} \Sigma & 0 & 0 & 0 \\ 0 & \Sigma & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ 0 & 0 & 0 & \Sigma \end{bmatrix}$$