<중간고사 해답>

1. (1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $\quad r(A) = 2 \quad$ A^{-1} 존재 안함.

(2) $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

소거법 사용

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 3 & 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

(-1)×(1행) + (2행)
(-1)×(1행) + (3행)
(-1)×(1행) + (4행)

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right]$$

(-1)×(2행) + (3행)
(-1)×(2행) + (4행)

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 & 0 & 1 \end{array}\right]$$

(-1)×(3행) + (4행)

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right]$$

(-1)×(2행) + (1행)

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right]$$

$(-1)\times$(3행) $+$ (2행)

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right]$$

$(-1)\times$(4행) $+$ (3행)

$$\Rightarrow B^{-1} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

(3) $C = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ $\quad r(C) = 2 \quad C^{-1}$ 존재 안함.

2. $\underset{\sim}{y} = \begin{bmatrix} K \\ 4 \\ 2 \\ 2 \end{bmatrix} \quad \underline{x}_1 = \begin{bmatrix} 1 \\ K \\ 1 \\ 1 \end{bmatrix} \quad \underline{x}_2 = \begin{bmatrix} 1 \\ 1 \\ K \\ 1 \end{bmatrix} \quad \underline{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ K \end{bmatrix}$

(a) $\underset{\sim}{y} = c_1 \underline{x}_1 + \underline{c_2} x_2 + c_3 x_3$

$$K = c_1 + c_2 + c_3 \quad \text{— ①}$$
$$4 = Kc_1 + c_2 + c_3 \quad \text{— ②}$$
$$2 = c_1 + Kc_2 + c_3 \quad \text{— ③}$$
$$2 = c_1 + c_2 + Kc_3 \quad \text{— ④}$$

②+③+④ 는

$$8 = (K+2)(c_1 + c_2 + c_3)$$

①과 같이 결합하면

$$8 = (K+2)K$$

$$\Rightarrow K^2 + 2K - 8 = 0$$

$$\Rightarrow (K+4)(K-2) = 0 \qquad \therefore K = -4, 2$$

(b) 위에서 $K = -4, 2$ 이면 선형독립이 아님.

또한 $K=1$ 인 경우 $x_1\ x_2\ x_3$ 가 선형독립이 아님

즉 $K = -4, 1, 2$ 를 제외한 모든 실수인 경우

y, x_1, x_2, x_3 는 선형독립.

2(b)

<u>다른 해법!</u>

4개의 벡터 $\underline{u}, \underline{x}_1, \underline{x}_2, \underline{x}_3$ 가 이루는 4×4 행렬의 행렬식이 0 이면 4 벡터는 선형독립이 아니다

$$\begin{vmatrix} 1 & 1 & 1 & K \\ K & 1 & 1 & 4 \\ 1 & K & 1 & 2 \\ 1 & 1 & K & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & k \\ 0 & 1-K & 1-K & 4-K^2 \\ 0 & K-1 & 0 & 2-K \\ 0 & 0 & K-1 & 2-K \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & K \\ 0 & 1-K & 1-K & 4-K^2 \\ 0 & 0 & 1-K & -(k-2)(k+3) \\ 0 & 0 & K-1 & 2-K \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & K \\ 0 & 1-K & 1-K & 4-K^2 \\ 0 & 0 & 1-K & -(k-2)(k+3) \\ 0 & 0 & 0 & -(k-2)(k+4) \end{vmatrix}$$

$$= -(K-1)^2(K-2)(K+4)$$

즉 $K=1, 2, -4$ 가 아니면 네 벡터들은 선형독립이다.

3. $X^2-2X+I=0$

$(X-I)^2=0$

let $X-I=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$
$a^2+bc=0$ — ①
$ab+bd=0$ — ② $(a+b)d=0$
$ac+cd=0$ — ③ $(a+d)c=0$
$bc+d^2=0$ — ④

$\Rightarrow$
$a^2+bc=0$ — ①
$b(a+d)=0$ — ②
$c(a+d)=0$ — ③
$(a+d)(a-d)=0$ — ① + ④

$(a+d)(a-d)=0$ 을 만족해야 하므로

case ① if $a+d=0$, then $a^2+bc=0$

case ② if $a-d=0$ 이면 $b(2a)=0$, $c(2a)=0$

여기서 $a\neq 0$ 이면 $b=0$, $c=0$ 하지만 $a^2+bc\neq 0$

$a=0$ $bc=0$ 즉 $b=0$ or c는 0

case ① 에서 $X - I = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ with $a^2+bc=0$

$$X = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a+1 & b \\ c & -a+1 \end{bmatrix}$$

이면서 $a^2+bc=0$ 을 만족하는 모든 X —— ①

case ② 에서

$$X - I = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \quad \text{with} \quad b \neq 0$$

or

$$X - I = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \quad \text{with} \quad c \neq 0$$

따라서

$$X = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \quad \text{with} \quad b \neq 0$$

or

$$X = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \quad \text{with} \quad c \neq 0$$

②

경우 ② 는 ① 의 경우에서 $a=0$ 인 경우

답: $X = \begin{bmatrix} a+1 & b \\ c & -a+1 \end{bmatrix}$ with $a^2+bc=0$

⇒ 무수히 많은 X 가 존재한다.

4. 다음과 같은 사실을 이용하자

① 하나의 행이나 열에 c배를 하면 행렬식이 c배

② 열이나 행을 바꿀때는 행렬식이 (−1)배

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & xz & xy \end{vmatrix}$$

$$= \frac{1}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & xyz & xyz \end{vmatrix}$$

각 열에 x, y, z를 각각 곱한다.

$$= \frac{xyz}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix}$$

3행에서 xyz를 밖으로 빼아냄.

$$= 1\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & y^3 \end{vmatrix}$$

1행과 3행을 바꾸고 다시 2행과 3행을 바꾼다

5. $A = \begin{bmatrix} 1 & 2 & 1 & 2 & -1 \\ 1 & 2 & 1 & 2 & -1 \\ 2 & 4 & 2 & 4 & -1 \\ 1 & 1 & 2 & 1 & 2 \end{bmatrix}$

2행과 4행을 교환

$$E_{24}A = \begin{bmatrix} 1 & 2 & 1 & 2 & -1 \\ 1 & 1 & 2 & 1 & 2 \\ 2 & 4 & 2 & 4 & -1 \\ 1 & 2 & 1 & 2 & -1 \end{bmatrix}$$

3열과 5열을 교환

$$E_{24}AE_{35} = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 2 \\ 2 & 4 & -1 & 4 & 2 \\ 1 & 2 & -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$$

$$Z = [1 \ 2 \ -1] = [1 \ 0 \ 0]\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = [1 \ 0 \ 0]\, X = FX$$

$$Y = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \\ 2 & 4 & -1 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} = X\begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} = XH$$

$$W = [2 \ 1] = FXH = [1 \ 2 \ -1]\begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$E_{24}AE_{35} = \begin{bmatrix} I \\ F \end{bmatrix} \times [I \ H]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \\ 2 & 4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\underset{4\times 5}{E_{24} A E_{35}} = \begin{bmatrix} I \\ F \end{bmatrix} \times \begin{bmatrix} I & H \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

따라서

$$A = E_{24}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} E_{35}^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 2 & 4 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 2 & -1 \\ 1 & 1 & 2 & 1 & 2 \\ 2 & 4 & 2 & 4 & -1 \end{bmatrix}$$

6 (a)

$$(I - xy^t)\left(I - \frac{xy^t}{y^tx-1}\right)$$

$$= I - xy^t - \frac{xy^t}{y^tx-1} + \frac{1}{y^tx-1}(xy^t xy^t)$$

$$= I - \left(1 + \frac{1}{y^tx-1}\right)xy^t + \left(\frac{xy^t}{y^tx-1}\right)$$

$$= I + \underbrace{\left[\frac{-y^tx+1-1+y^tx}{y^tx-1}\right]}_{0} xy^t$$

$$(xy^t)(xy^t) = x(y^tx)y^t = (y^tx)\,x\,y^t$$

$$= I$$

$$\Rightarrow \quad (I - xy^t)^{-1} = I - \frac{xy^t}{y^tx-1}$$

6 (b)

$$(A^{-1} + B^{-1})^{-1}$$

$$= (A^{-1} + B^{-1}AA^{-1})^{-1}$$

$$= [(I + B^{-1}A)(A^{-1})]^{-1}$$

$$= A(I + B^{-1}A)^{-1}$$

$$= A(B^{-1}B + B^{-1}A)^{-1}$$

$$= A[B^{-1}(B+A)]^{-1}$$

$$= A(A+B)^{-1}B$$