The exponential Distribution and Poisson Process

노트 제목 2009-04-14

(1) The Exponential Distribution

① Definition : $X \sim Exp(\lambda)$

The density of the exponential distribution with parameter $\lambda > 0$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

And the distribution function F is given by

$$F(x) = P(X \leq x) = \int_0^{\infty} \lambda e^{-\lambda x} dx = 1 - e^{-\lambda x}$$

<u>Note</u> $P(X \geq x) = 1 - F(x) = e^{-\lambda x} \equiv \bar{F}(x)$

② moments

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$ let $u = x$, $dv = \lambda e^{-\lambda x}$

$$= \int_0^{\infty} \lambda x e^{-\lambda x} dx$$ $\int u dv = uv - \int v du$

$$= -xe^{-\lambda x}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x}$$

$$= -\frac{1}{\lambda} e^{-\lambda x}\Big|_0^{\infty}$$

$$= 1/\lambda$$

$$E(X^2) = 2/\lambda^2$$

$$V(X) = EX^2 - (EX)^2 = 1/\lambda$$

The moment generating function $\phi(t)$ is

$$\phi(t) = E(e^{tX})$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} e^{(t-\lambda)x} dx$$

$$= \lambda \frac{1}{t-\lambda} e^{(t-\lambda)x}\Big|_0^{\infty}$$

$$= \frac{\lambda}{t-\lambda} \quad \text{when } t < \lambda$$

③ Properties

(a) memoryless (unique)

$$P(X > s+t \mid X > t) = P(X > s) \qquad s, t \geq 0$$

Note

$$P(X > s+t \mid X > t)$$
$$= P(X > s+t) / P(X > t)$$
$$= e^{-\lambda(s+t)} / e^{-\lambda t}$$
$$= e^{-\lambda s}$$
$$= P(X > s) = \bar{F}(s)$$

Ex: lifetime of some instrument

The probability that the instrument lives for at the least s+t hours given that it has survived t hours is the same as the initial probability $P(X > s)$

(b) failure rate

The failure rate $r(t)$ (hazard, risk, ...)

$$r(t) = f(t) / 1 - F(t)$$

where $f(t)$ is a density and $F(t)$ is a distribution function

Note

$$P\{ X \in (t, t+dt) \mid X > t \}$$
$$= \frac{P(X \in (t, t+dt) \text{ and } X > t \}}{P(X > t)}$$
$$= P(X \in (t, t+dt)) / P(X > t)$$
$$\simeq \frac{f(t)\, dt}{1 - F(t)} = r(t)\, dt$$

$\Rightarrow$ conditional probability that t-years old item fail.

If $X \sim Exp(\lambda)$,

$$r(t) = f(t) / \bar{F}(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

$\Rightarrow$ failure rate is "constant"

(2) Poisson Process

① Counting process

A stochastic process $\{N(t), t \geq 0\}$ is counting process if $N(t)$ represents the total number of events up to time t

$\Rightarrow$ (i) $N(t) \geq 0$

(ii) $N(t)$ is integer valued

(iii) If $s < t$, then $N(s) \leq N(t)$

(iv) For $s < t$, $N(t) - N(s)$ is the number of events that have occured in the interval (s, t)

② Independent Increment

$N(t)$ is independent increment process if the numbers of events which occur in disjoint time intervals are independent

$\Rightarrow$ for $s < t$, $N(t) - N(s)$ and $N(s)$ are independent

③ Stationary Increments

$N(t)$ is stationary increment process if the distribution of the number of events which occur in any interval of time depends only on the length of interval

For $t_1 < t_2$, $s > 0$,

Distribution of $N(t_2) - N(t_1)$ $\equiv$

Distribution of $N(t_2 + s) - N(t_1 + s)$

④ First Definition of Poisson Process

The counting process $\{N(t), t \geq 0\}$ is said to be Poisson Process with rate λ, $\lambda > 0$ if

(i) $N(0) = 0$

(ii) The process has independent increment.

(iii) The number of events in any interval of lenghth t is poisson distributed with mean λt

$\Leftrightarrow$ for all $s, t > 0$

$$P\{N(s+t) - N(s) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad n = 0, 1, 2 \cdots$$

Note 1

(iii) implies "stationary increment".

and $P(N(t) = n) = \dfrac{e^{-\lambda t}(\lambda t)^n}{n!}$ (let $s = 0$) so that

$E(N(t)) = \lambda t$

Note 2 Definition of $o(h)$ (small "o")

The function $f(t)$ is said to be $o(h)$ if

$$\lim_{h \to 0} \frac{f(h)}{h} \to 0$$

eg) $f(x) = x^2 \Rightarrow o(h)$ $\quad \lim_{h \to 0} \frac{h^2}{h} = 0$

$f(x) = x \Rightarrow$ not $o(h)$ $\quad \lim_{h \to 0} \frac{h}{h} = 1 \neq 0$

"f(h) goes to 0 faster than h goes to 0"

Also if $f(t)$ and $g(t)$ are both $o(h)$, then, for any c_1, c_2,

$c_1 f(t) + c_2 g(t)$ is $o(h)$

⑤ Second Definition of Poisson Process

(i) $N(t) = 0$

(ii) The process has stationary and independent increment

(iii) $P\{N(h)=1\} = \lambda h + o(h)$

(iv) $P\{N(h) \geq 2\} = o(h)$

<u>Note 1</u>

(iii) implies underlying rate is "λ"

(iv) implies "No" two points can occurs at same time.

<u>Note2</u> The first and second definitions are equivalent !

(proof)

Let $P_n(t) = P\{N(t)=n\}$

ⓐ $P_0(t+h) = P\{N(t+h)=0\}$

$$= P\{N(t)=0,\ N(t+h)-N(t)=0\}$$

$$= P\{N(t)=0\}\, P\{N(t+h)-N(t)=0\} \quad \text{independent increment}$$

$$= P_0(t)\cdot P[N(h)=0] \quad \text{stationary}$$

$$= P_0(t)\left[1 - P(N(h)=1) - P(N(h)\geq 2)\right]$$

$$= P_0(t)\left[1-\lambda h - o(h)\right]$$

Hence, $\dfrac{P_0(t+h) - P_0(h)}{h} = -\lambda P_0(t) + \dfrac{o(h)}{h}$

By letting $h \to 0$, we have

$$P_0'(t) = -\lambda P_0(t)$$

$$\Leftrightarrow \quad \frac{P_0'(t)}{P_0(t)} = -\lambda$$

$$\Leftrightarrow \quad \log P_0(t) = -\lambda t + c$$

$$\Leftrightarrow \quad P_0(t) = k \cdot e^{-\lambda t}$$

Since $P_0(0) = P(N(0)=0) = 1$, $P_0(t) = e^{-\lambda t}$

(b) for $n \geq 0$

$$P_n(t+h) = P(N(t+h) = n)$$

$$= P(N(t) = n,\ N(t+h) - N(t) = 0)$$

$$+ P(N(t) = n-1,\ N(t+h) - N(t) = 1)$$

$$+ \sum_{k=2}^{n} P(N(t) = n-k,\ N(t+h) - N(t) = k)$$

$$= P(N(t) = n) \cdot P(N(t+h) - N(t) = 0)$$

$$+ P(N(t) = n-1)\ P(N(t+h) - N(t) = 1)$$

$$+ o(h)$$

$$= P_n(t)\ P_0(h) + P_{n-1}(t)\ P_1(h) + o(h)$$

$$= [1 - \lambda h - o(h)]\ P_n(t) + \lambda h\ P_{n-1}(t) + o(h)$$

$$= (1 - \lambda h)\ P_n(t) + \lambda h\ P_{n-1}(t) + o(h)$$

Thus

$$\frac{P_n(t+h) - P_n(t)}{h} = -\lambda P_n(t) + \lambda P_{n-1}(t) + \frac{o(h)}{h}$$

By letting $h \to 0$,

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t)$$

$$\Leftrightarrow \quad e^{\lambda t}\left[P_n'(t) + \lambda P_n(t)\right] = \lambda e^{\lambda t} P_{n-1}(t)$$

$$\Leftrightarrow \quad \frac{\partial}{\partial t} e^{\lambda t} P_n(t) = \lambda e^{\lambda t} P_{n-1}(t)$$

Now $n=1$

$$\frac{\partial}{\partial t} e^{\lambda t} P_1(t) = \lambda e^{\lambda t} P_0(t) = \lambda e^{\lambda t} e^{-\lambda t} = \lambda$$

$$\Rightarrow \quad P_1(t) = (\lambda t + c)e^{-\lambda t}$$

Since $P_1(0) = 0 \Rightarrow P_1(0) = c = 0$,

$$P_1(t) = \lambda t e^{-\lambda t}$$

To show $P_n(t) = e^{-\lambda t}(\lambda t)^n / n!$, use mathematical induction

Assume first for $n-1$, them

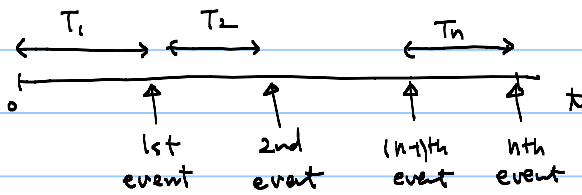
$$\frac{\partial}{\partial t} e^{\lambda t} P_n(t) = \frac{\lambda^n t^{n-1}}{(n-1)!}$$

$$\Rightarrow \quad e^{\lambda t} P_n(t) = \frac{(\lambda t)^n}{n!} + c$$

use $P_n(0) = 0$

⑥ Inter-arrival and Waiting time distribution

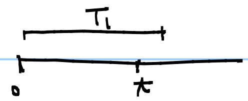
ⓐ T_n : inter-arrival time



T_n : time between (n-1)th event and nth event

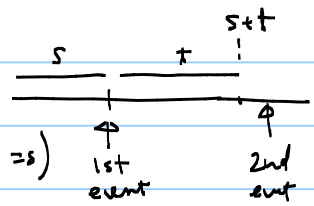
= inter-arrival time

(a) $P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t}$



$\Rightarrow$ T_1 has exponential distribution with λ

(b) $P(T_2 > t) = E[P(T_2 > t \mid T_1)]$



But $P(T_2 > t \mid T_1 = s)$

$= P(\text{0 event on } [s, s+t] \mid T_1 = s)$

$= P(\text{0 event on } (s, s+t))$

$= P(N(t) = 0) = e^{-\lambda t}$

Hence, $P(T_2 > t) = E[P(T_2 > t \mid T_1)]$

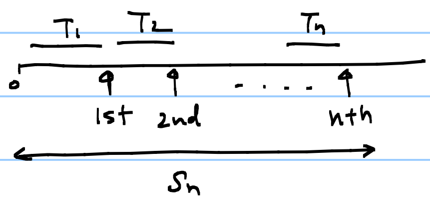
$= E e^{-\lambda t}$

$= e^{-\lambda t}$

$\Rightarrow$ T_2 has exponential distribution with λ

$\Rightarrow$ $T_1, T_2, T_3, \cdots T_n \cdots$ are independent identically distributed exponential random variable with λ.

ⓑ waiting time $S_n = T_1 + T_2 + \cdots + T_n$



<u>Proposition</u> If $T_1, T_2, \dots T_n$ are independent exponential random variable with λ,

$S = T_1 + T_2 + \dots + T_n$ has a gamma distribution with n and λ

$$\Leftrightarrow \quad f_{S_n}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}$$

<u>Note</u> $\{N(t) \geq n\} \Leftrightarrow \{S_n \leq t\}$