통계학 특강 과제 1 해답

1. (1) $L(\lambda : y) = f(y ; \lambda) = \dfrac{\lambda^y e^{-\lambda}}{y!}$

(2) $\ell(\lambda ; y) = \log L(\lambda ; y) = y \log\lambda - \lambda - \log(y!)$

(3) $S(\lambda) = \dfrac{\partial}{\partial\lambda} \ell(\lambda ; y) = \dfrac{y}{\lambda} - 1$

(4) $J(\lambda) = -\dfrac{\partial^2}{\partial\lambda^2} \ell(\lambda ; y) = y/\lambda^2$

(5) $I(\lambda) = E[J(\lambda)] = \dfrac{1}{\lambda^2} E(y) = \dfrac{1}{\lambda}$

(6) $S(\lambda) = 0 \Rightarrow \hat{\lambda} = y$

2 (1) $L_n(\lambda : \underset{\sim}{y}) = \prod_{i=1}^{n} f(y_i ; \lambda) = \prod_{i=1}^{n} \dfrac{\lambda^{y_i} e^{-\lambda}}{y_i!}$

$= \lambda^{\Sigma y_i} e^{-n\lambda} \prod_{i=1}^{n} \dfrac{1}{y_i!}$

(2) $\ell_n(\lambda ; \underset{\sim}{y}) = \Sigma y_i \log\lambda - n\lambda + \log\left[\prod_{i=1}^{n} \dfrac{1}{y_i!}\right]$

(3) $S_n(\lambda) = \dfrac{\Sigma y_i}{\lambda} - n$

(4) $J_n(\lambda) = \dfrac{\Sigma y_i}{\lambda^2}$

(5) $I_n(\lambda) = \dfrac{n}{\lambda}$

(b) $\hat{\lambda} = \frac{\sum y_i}{n}$

3. $y_i \overset{ind}{\sim} N(\mu, \sigma^2)$

편의를 위하여 $\sigma^2 = \tau$ 로 놓고 모수벡터 $\underset{\sim}{\theta} = (\mu, \tau)^t$ 라고 하자

$$L_n(\underset{\sim}{\theta} ; \underset{\sim}{y}) = \prod_{i=1}^{n} (2\pi\tau)^{-1/2} \exp\left\{-\frac{(y_i-\mu)^2}{2\tau}\right\}$$

$$= [2\pi\tau]^{-n/2} \exp\left\{-\frac{1}{2\tau}\sum_{i=1}^{n}(y_i-\mu)^2\right\}$$

$$l_n(\underset{\sim}{\theta} : \underset{\sim}{y}) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\tau - \frac{1}{2\tau}\sum_{i=1}^{n}(y_i-\mu)^2$$

MLE는 다음의 방정식을 만족하는 $\hat{\underset{\sim}{\theta}} = (\hat{\mu}, \hat{\tau})^t$ 이다.

$$S_n(\underset{\sim}{\theta} : \underset{\sim}{y}) = \begin{bmatrix} \frac{\partial}{\partial\mu} l_n(\underset{\sim}{\theta} ; \underset{\sim}{y}) \\ \frac{\partial}{\partial\tau} l_n(\underset{\sim}{\theta} : \underset{\sim}{y}) \end{bmatrix} = 0$$

① $\frac{\partial}{\partial\mu} l_n(\underset{\sim}{\theta} ; \underset{\sim}{y}) = \frac{1}{\tau}\sum(y_i-\mu) = 0$

② $\frac{\partial}{\partial\tau} l_n(\underset{\sim}{\theta} : \underset{\sim}{y}) = -\frac{n}{2\tau} + \frac{1}{2\tau^2}\sum(y_i-\mu)^2 = 0$

따라서 위의 두 방정식을 만족하는 MLE는

$$\hat{\mu} = \frac{\sum y_i}{n} = \bar{y} \qquad \hat{\tau} = \hat{\sigma}^2 = \frac{\sum(y_i-\bar{y})^2}{n}$$

$$J_n(\underset{\sim}{\theta}) = -\begin{bmatrix} \frac{\partial^2}{\partial\mu^2} l_n(\underline{\theta}) & \frac{\partial^2}{\partial\mu\partial\tau} l_n(\underline{\theta}) \\ \frac{\partial^2}{\partial\tau\partial\mu} l_n(\underline{\theta}) & \frac{\partial^2}{\partial\tau^2} l_n(\underline{\theta}) \end{bmatrix}$$

τ^{-2}

-2τ

$$= \begin{bmatrix} \frac{n}{\tau} & \frac{1}{\tau^2}\Sigma(y_i-\mu) \\ \frac{1}{\tau^2}\Sigma(y_i-\mu) & -\frac{n}{2\tau^2} + \frac{1}{\tau^3}\Sigma(y_i-\mu)^2 \end{bmatrix}$$

$$I_n(\underline{\theta}) = E[J_n(\underline{\theta})]$$

$$= \begin{bmatrix} \frac{n}{\tau} & 0 \\ 0 & -\frac{n}{2\tau^2} + \frac{n}{\tau^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{n}{\tau} & 0 \\ 0 & \frac{n}{2\tau^2} \end{bmatrix} \Rightarrow$$

$$I_n^{-1}(\underset{\sim}{\theta}) = \begin{bmatrix} \tau/n & 0 \\ 0 & 2\tau^2/n \end{bmatrix}$$

$\underline{\theta}_0$는 $\underline{\theta}$의 참값이라고 하면

$$\hat{\theta} = \begin{bmatrix} \hat{\mu} \\ \hat{\tau} \end{bmatrix} \sim N(\underline{\theta_0}, I_n^{-1}(\underline{\theta_0}))$$

4. $y_i \overset{iid}{\sim} exp(\lambda)$ $\quad f(y;\lambda) = \frac{1}{\lambda} e^{-y/\lambda}$, $\lambda > 0$

$$L_n(\lambda) = \lambda^{-n} e^{-\Sigma y_i/\lambda}$$

$$l_n(\lambda) = -n \log \lambda - \frac{\Sigma y_i}{\lambda}$$

$$S_n(\lambda) = -\frac{n}{\lambda} + \frac{\Sigma y_i}{\lambda^2}$$

$S_n(\lambda) = 0$ 을 만족하는 MLE 는

$$\hat{\lambda} = \frac{\Sigma y_i}{n}$$

$$J_n(\lambda) = -\frac{n}{\lambda^2} + \frac{2\Sigma y_i}{\lambda^3}$$

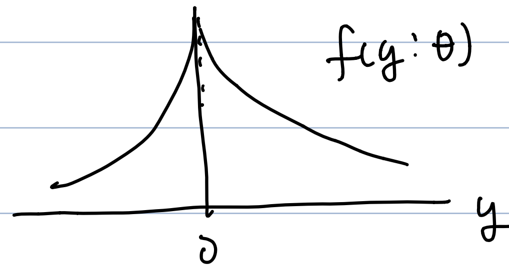
$$I_n(\lambda) = -\frac{n}{\lambda^2} + \frac{2n}{\lambda^2} = \frac{n}{\lambda^2}$$

λ_0는 λ의 참값이라고 하면

$$\hat{\lambda} \sim N\left(\lambda_0, \frac{\lambda_0^2}{n}\right)$$

5 $y_i \overset{iid}{\sim} f(y|\theta) = \frac{1}{2\theta} \exp\left(-\frac{|y_i|}{\theta}\right)$

$f(y|\theta)$ 의 확률밀도함수는 다음과 같다.



하지만 $f(y|\theta)$를 θ의 함수로 보면 미분가능한 함수이다.

$$L_n(\theta;\underset{\sim}{y}) = (2\theta)^{-n} \exp\left(-\frac{1}{\theta}\sum_{i=1}^{n}|y_i|\right)$$

$$l_n(\theta;\underline{y}) = -n\log 2 - n\log\theta - \frac{1}{\theta}\sum_{i=1}^{n}|y_i|$$

$$S_n(\theta:\underline{y}) = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n}|y_i|$$

$S_n(\theta;y)$를 만족하는 MLE는

$$\hat{\theta} = \frac{\sum|y_i|}{n}$$

$$J_n(\theta;\underline{y}) = -\frac{n}{\theta^2} + \frac{2}{\theta^3}\sum|y_i|$$

$$I_n(\theta;\underline{y}) = -\frac{n}{\theta^2} + \frac{4n}{\theta^2} = \frac{3n}{\theta^2}$$

여기서

$$E[|Y|] = \int_{-\infty}^{\infty} |y| f(y;\theta)\, dy$$

$$= \int_{-\infty}^{\infty} |y| \cdot \frac{1}{\theta} e^{-|y|/\theta} dy$$ 대칭

$$= 2 \cdot \int_{0}^{\infty} y \frac{1}{\theta} e^{-y/\theta} dy$$

Y가 평균이 θ인 지수분포를 따르는 경우 $E(Y) = \theta$

$$= 2\theta$$

따라서

$$\hat{\theta} \sim N\left(\theta_0, \frac{\theta_0^2}{3n}\right)$$