Exercises

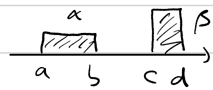
Exo 1 : Show that for $c(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x\neq y \end{cases}$ then

$$W_c(\alpha,\beta) \triangleq \inf_{\substack{\pi_1=\alpha \\ \pi_2=\beta}} \int c\, d\pi = \frac{1}{2}\|\alpha-\beta\|_{TV} = \frac{1}{2}|\alpha-\beta|(\mathcal{X})$$

Exo 2 : Show that $\mathcal{N}(m,\sigma^2) \xrightarrow{*} \mathcal{N}(m',\sigma'^2) \iff (m,\sigma) \to (m',\sigma')$

Exo 3 : Show that $\alpha \in \mathcal{M}(\mathcal{X}) \longmapsto \int f\, d\alpha \in \mathbb{R}$ if weak* continuous if f is continuous.

Exo 4 : For $\alpha = \mathcal{U}_{[a,b]}$, $\beta = \mathcal{U}_{[c,d]}$

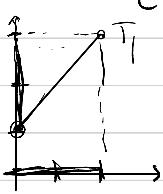


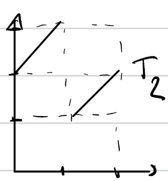
for $c(x,y) = |x-y|^2$, what are : $\begin{cases} \text{Monge sol}^{\circ}, \text{Kanto sol}^{\circ}, \\ W_2(\alpha,\beta), \text{ interpol}^{\circ} \end{cases}$?

Exo 5 : For $\begin{cases} \alpha = \mathcal{U}_{[0,2)} \\ \beta = \mathcal{U}_{[1,3)} \end{cases}$ show that $\begin{cases} T_1(x) = x+1 \\ T_2(x) = \begin{cases} x+2 & \text{for } x \leq 1 \\ x & \text{for } x > 1 \end{cases} \end{cases}$

are sol° for $c(x,y) = |x-y|$.

Display other sol° of Kanto.





Exo 6 : Show that Monge & Kanto. sol° for $c(x,y) = \|x-y\|^2$
are equiv. sol° for $c(x,y) = -\langle x, y\rangle$

Exo 7 : Let $T_u : x \longmapsto x+u$ and $\begin{cases} \alpha = T_{a\#}\alpha_0 & \int x\, d\alpha_0(y) = 0 \\ \beta = T_{b\#}\beta_0 & \int y\, d\beta_0(y) = 0 \end{cases}$

$$W_2^2(\alpha,\beta) = \|a - b\|^2 + W_2^2(\alpha_0, \beta_0)$$

$\hookrightarrow a = \int x\, d\alpha(x)$

Correction Exo 1 : For simplicity, we consider the discrete case

Denote : $\gamma(x) \triangleq \min(\alpha(x), \beta(x))$.

By conserv° of mass, for $\pi \in \mathcal{U}(\alpha,\beta)$, $\pi(x,x) \leq \gamma(x)$

thus $$W_c(\alpha,\beta) = \inf_{\pi_1 = \alpha,\, \pi_2 = \beta} \langle C, \pi\rangle = \sum_{x \neq y} \pi(x,y) = 1 - \sum_x \pi(x,x) \geq 1 - \sum_x \gamma(x)$$

$\leadsto$ Need to construct $\hat{\pi} \in \mathcal{U}(\alpha,\beta)$ s.t. $\mathrm{diag}(\hat{\pi}) = \gamma$
so that it will be optimal

let $\begin{cases} \bar{\alpha} \triangleq \alpha - \gamma = (\alpha - \beta)_+ \geq 0 \\ \bar{\beta} \triangleq \beta - \gamma = (\beta - \alpha)_+ \geq 0 \end{cases}$

One has $\dfrac{\bar{\alpha} \otimes \bar{\beta}}{\int \bar{\alpha}} \in \mathcal{U}(\bar{\alpha}, \bar{\beta})$ $\quad$ Rmq: $\int \bar{\alpha} = \int \bar{\beta} = 1 - \int \gamma$

$\int \bar{\alpha} = \langle \bar{\alpha}, 1\rangle$

One has $\hat{\pi} \triangleq \mathrm{diag}(\gamma) + \underbrace{\frac{\bar{\alpha} \otimes \bar{\beta}}{\langle \bar{\alpha}, 1 \rangle}}_{0 \text{ on the diagonal}} \geqslant 0$

and $\begin{cases} \hat{\pi}_1 = \gamma + \bar{\alpha} = \alpha \\ \hat{\pi}_2 = \gamma + \bar{\beta} = \beta \end{cases}$

$\Rightarrow \hat{\pi} \in U(\alpha, \beta)$

$\Rightarrow \hat{\pi}$ is optimal

Thus $W_C(\alpha, \beta) = \langle C, \hat{\pi} \rangle = \sum_{xy} \frac{\bar{\alpha}(x)\bar{\beta}(y)}{\langle \bar{\alpha}, 1 \rangle} = \frac{\overbrace{\sum \bar{\alpha}(x)}}{\sum_y \bar{\beta}(y)}$

$= \frac{1}{2} \sum_x \bar{\beta}(x) + \bar{\alpha}(x) = \frac{1}{2} \sum_x |\alpha(x) - \beta(x)| = \frac{1}{2} \|\alpha - \beta\|_{TV}$

Correct° exo 2: One has $W_2^2(\mathcal{N}(m, \sigma^2), \mathcal{N}(m', \sigma'^2)) = \left\| \begin{pmatrix} m \\ \sigma \end{pmatrix} - \begin{pmatrix} m' \\ \sigma' \end{pmatrix} \right\|^2$

and $\alpha \to \beta \iff W(\alpha, \beta) \to 0$

Correct° exo 3: Denote $\phi_f(\alpha) = \int f \, d\alpha \quad f \in \mathcal{C}^0(\mathcal{X})$

$\alpha_k \xrightarrow{\mathcal{L}} \alpha \underset{f \text{ cont}}{\Rightarrow} \int f \, d\alpha_k \to \int f \, d\alpha$ ie $\phi_f(\alpha_k) \xrightarrow{k} \phi_f(\alpha)$

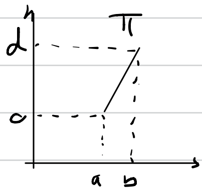
Correct° exo 4: Denote: $T(x) \triangleq c + \frac{x-a}{b-a}(d-c) = rx + t$

Since $\lambda > 0$, T is gradient of cvx f^{c}

Need to show $T_{\#} \dfrac{\mathbb{1}_{[a,b]}}{(b-a)} = \dfrac{\mathbb{1}_{[c,d]}}{(d-c)}$

i.e. $\forall f \in \mathcal{C}^0(\mathbb{R})$, $\displaystyle\int f \, d\left(T_{\#} \frac{\mathbb{1}_{[a,b]}}{b-a}\right) = \int_a^b f(\underbrace{\lambda x + t}_{=y}) \frac{dx}{b-a} = \int_c^d f(y) \frac{dy}{d-c}$

$$= \int f \, d\left(\frac{\mathbb{1}_{[c,d]}}{d-c}\right)$$

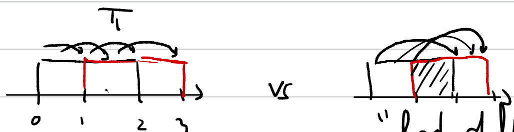


Interpol^c: $\alpha_t = \left((1-t)\,\mathrm{Id} + tT\right)_{\#} \dfrac{\mathbb{1}_{[a,b]}}{b-a}$

$$= \lambda_t \times \mathbb{1}_{[(1-t)a+tc,\,(1-t)b+td]}$$

$$\lambda_t = t(d-c) + (1-t)(b-a)$$

<u>Counter^c ex 5</u>:



"Book shifting"

$\text{cost} = \frac{1}{2}\int_0^2 \underbrace{|T(x)-x|}_{=1} dx = 1 \equiv \text{cost} = \frac{1}{2}\int_0^1 \underbrace{|T(x)-x|}_{=2} dx + \frac{1}{2}\int_1^2 \underbrace{|T(x)-x|}_{=0} dx$

T_1 is increasing $\longrightarrow$ it is optimal

T_2 has same cost

Correct° exo 6: For $\pi \in \mathcal{U}(\alpha,\beta)$,

$$\int |x-y|^2 d\pi(x,y) = \underbrace{\int |x|^2 d\pi(x,y)}_{\substack{=\int |x|^2 d\alpha(x) \\ = cst}} + \underbrace{\int |y|^2 d\pi(x,y)}_{\substack{=\int |y|^2 d\beta(y) \\ = cst}} - 2\int \langle x,y\rangle d\pi(x,y)$$

Correct° exo 7: Define $T:(x,y) \to (x-a, y-b)$

For $\pi \in \mathcal{U}(\alpha,\beta)$, let $\pi_0 = T_\# \pi$, one has $\begin{cases} (\pi_0)_1 = \alpha_0 \\ (\pi_0)_2 = \beta_0 \end{cases}$

ie. $\pi_0 \in \mathcal{U}(\alpha_0, \beta_0)$

$$\inf_{\pi \in \mathcal{U}(\alpha,\beta)} \int |x-y|^2 d\pi = \inf_{\pi_0 \in \mathcal{U}(\alpha_0,\beta_0)} \underbrace{\int |(x-a)-(y-b)|^2 d\pi_0}$$

$$= \inf_{\pi_0 \in \mathcal{U}(\alpha,\beta)} \int |x-y|^2 d\pi_0 + \underbrace{\int \|a-b\|^2 d\pi_0}_{=\|a-b\|^2} + \langle a-b, \int (x-y) d\pi_0 \rangle$$

$$\int (x-y)d\pi_0 = \underbrace{\int x \, d\pi_0}_{=\int x \, d\alpha_0} - \underbrace{\int y \, d\pi_0}_{=\int y \, d\beta_0} = 0 - 0$$