MAE 5730 : Final Project

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Contents

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MAE 5730
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1 Introduction

This report describes the simulation (in Matlab) of a triple pendulum and a four-bar linkage using three different methods for obtaining the necessary equations. Also included are a quadruple pendulum, a five-bar linkage, and a system that I have named "String of Lights", which is a combination of the five-bar linkage and single pendulums hinged at the moving hinges of the five-bar linkage. In all simulations, friction forces are ignored, and the rods are infinitely thin, with the mass distributed evenly throughout their lengths. In some of the simulations, you can add external, constant, nonconservative forces in the horizontal direction.

2 Triple Pendulum

A triple pendulum is a single pendulum with another pendulum hinged at its end, which in turn has another pendulum hinged at its end. See Appendix 2.01 for a drawing of our triple pendulum setup. In order to find the motion of this system given initial positions and velocities, we need to know the accelerations of at least three variables: θ_1, θ_2, and θ_3. In simpler systems (such as systems with point masses,) we can often use linear momentum balance to find equations of motion. In this case, however, doing so would leave us with unknown values for the tension forces at the hinges. We can, however, use angular momentum balance, which eliminates these terms.

2.1 Solving Using Angular Momentum Balance

We know that the total angular momentum of a system is:

$$\vec{L} = \sum \vec{r} \times \vec{p} \tag{1}$$

By differentiating both sides of this equation, we get the expression:

$$\dot{\vec{L}} = \sum \vec{r} \times \vec{F} \tag{2}$$

We can use this expression to get our equations of motion. To ensure that we get all the information we need about the system, we must find the total angular momentum about each hinge. See Appendix 2.02 for free-body diagrams and their corresponding angular momentum balance equations. The tension forces are eliminated because the force at the hinge about which we are taking angular moment balance has a position vector of $\vec{0}$, and the forces at the other hinges come in equal and opposite pairs, canceling each other out. The result is six equations that can be put into matrix form to solve for our three desired accelerations: $\ddot{\theta}_1$, $\ddot{\theta}_2$, and $\ddot{\theta}_3$. Note that when writing the time-derivative of the angular momentum, we always write it with respect to the fixed point O, regardless of the hinge about which we are taking angular momentum balance.

* To run this simulation, open the file "runner.m" and uncomment line 15. You can add horizontal forces using the F_i variables on line 15.

2.2 Solving Using Lagrange Equations

Another valid approach for finding the motion of a triple pendulum is Lagrange equations. We need only find the total kinetic and total potential energies of the system. The total kinetic energy of the system is:

$$KE_{tot} = \sum_{1}^{3} \frac{1}{2} m_i v_{G_i/O} + \frac{1}{2} I_{G_i} \dot{\theta}_i^{\,2} \tag{3}$$

The total potential energy of the system is:

$$PE_{tot} = \sum_{1}^{3} \frac{1}{2} m_i g y_{G_i/O} \tag{4}$$

where $y_{G_i/O}$ is the height of the ith mass. We can plug these into our Lagrangian:

$$\mathcal{L} = KE_{tot} - PE_{tot} \tag{5}$$

Which we use to get three Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} \quad , \quad \frac{\partial \mathcal{L}}{\partial \theta_2} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} \quad , \quad \frac{\partial \mathcal{L}}{\partial \theta_3} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_3} \tag{6}$$

More detail about the velocities and heights can be found in Appendix 2.03. As with angular momentum balance, we can put these equations into a matrix form that allows us to solve for $\ddot{\theta}_1$, $\ddot{\theta}_2$, and $\ddot{\theta}_3$.

* To run this simulation, open the file "runner.m" and uncomment line 18.

2.3 Solving Using Differential Algebraic Equations

A third way to approach the triple pendulum is with differential algebraic equations. This method involves using linear momentum balance, angular momentum balance, and constraint equations. Although it may seem overly complex compared to angular momentum balance alone or Lagrange equations, it becomes useful when modeling more complex systems where eliminating tension forces is not as straightforward.

There is more than one way to set up the equations for DAEs. The first thing to do is to identify how many unknowns you have. This is how many equations you will need to find. The goal is to get a square matrix containing the coefficients of our unknowns, and a column vector of the leftover terms in our equations that do not belong to any of our unknowns. If we call our matrix M and our vector $\vec{b}$, then our equations in matrix form will be:

$$M\vec{q} = \vec{b} \tag{7}$$

Where $\vec{q}$ is the vector containing all our unknowns for which we want to solve. We need simply solve for $\vec{q}$ (using the backslash command in Matlab) to get our accelerations. In this case, we have 15 unknowns:

$$\ddot{x_1}, \ddot{y_1}, \ddot{x_2}, \ddot{y_2}, \ddot{x_3}, \ddot{y_3}, \ddot{\theta_1}, \ddot{\theta_2}, \ddot{\theta_3}, T_{Ox}, T_{Oy}, T_{1x}, T_{1y}, T_{2x}, T_{2y}$$

We need, therefore, 15 equations. Our first six will come from doing linear momentum balance on each individual bar. Our next three will come from doing angular momentum balance of each bar about its center of mass. Our last 6 equations will come from the constraints at each hinge. See Appendix 2.04 for the free-body diagrams and equations.

* To run this simulation, open the file "runner.m" and uncomment line 22. You can add horizontal forces using the F_i variables on line 22.

2.4 Comparing Methods

The three previously described methods produce very close results for the first several seconds of the simulation, then diverge wildly from each other. Depending on the initial conditions, the time to this divergence varies. Run the file called "lagrange_vs_AMB_vs_DAEs.m" to see plots of the difference between the three methods. There are three plots, each showing the difference between two of the methods. Below is the plot for differential algebraic equations vs. angular momentum balance equations. The plot shows that the solutions are equal until around t = 42, at which point they start to diverge.

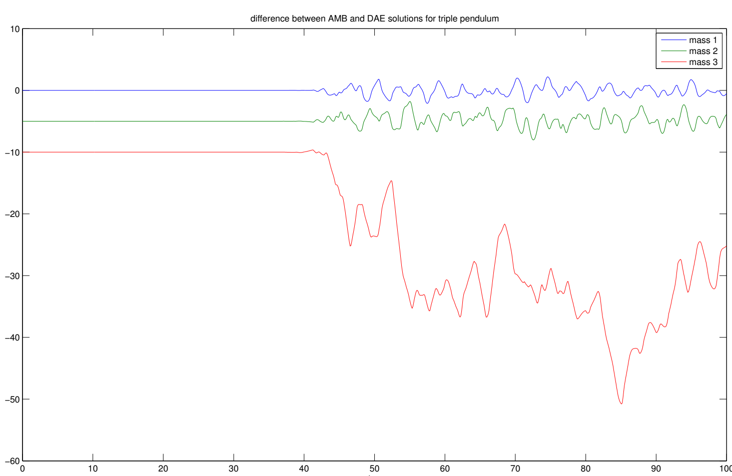

Figure 1: Difference over time between AMB and DAE solutions for triple pendulum

2.5 Checking for Accuracy

There are a few ways that we can verify our solutions. One way is to reduce our system to one whose motion we already know - such as the single pendulum. Another is to check for conservation of total energy. Note that total energy is only conserved when we have no external, nonconservative forces.

2.5.1 Reducing System to Simpler Systems

With a triple pendulum, there are a couple different ways to reduce our system to a single pendulum:

- Make the first two masses much larger than the third, and start them at $\theta = 0$, $\dot{\theta} = 0$.
- Make two of the three rods (it doesn't matter which two) very short.

Below in Figure 2 is a plot of rod angles over time of a system whose first two masses are 1000 times larger than the third, and whose initial angles and angular velocities are zero.

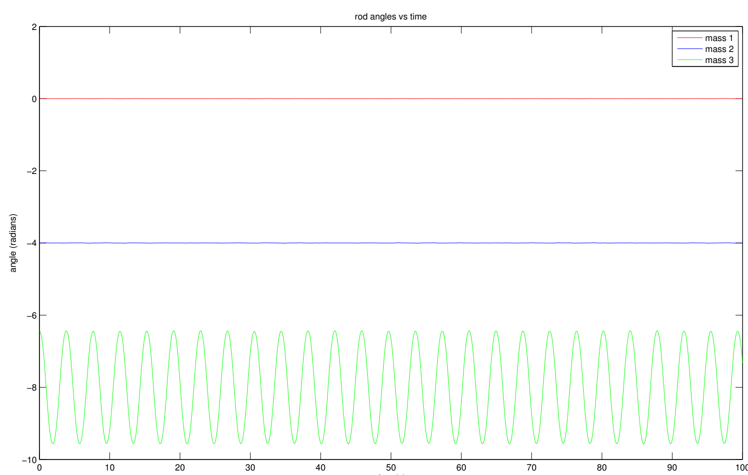

Figure 2: Rod angles over time of a triple pendulum reduced to a single pendulum

Another simplification is to set gravity equal to zero, start all three rods at the same angle, and give them the same initial angular velocity. This should result in the three rods moving together at constant angular velocity. Figure 3 below is a plot of this system's angles over time.

2.5.2 Energy Conservation

Another way to verify your solutions is to check that the total kinetic energy plus the total potential energy stays constant over time. An easy way to do this is to plot the current total energy over time minus the initial total energy. In Figure 4 below, we can see that this error is on the order of 10^{-10}. This accuracy is improved by using smaller time-steps for the integration functions in Matlab, as well as by choosing smaller tolerances.

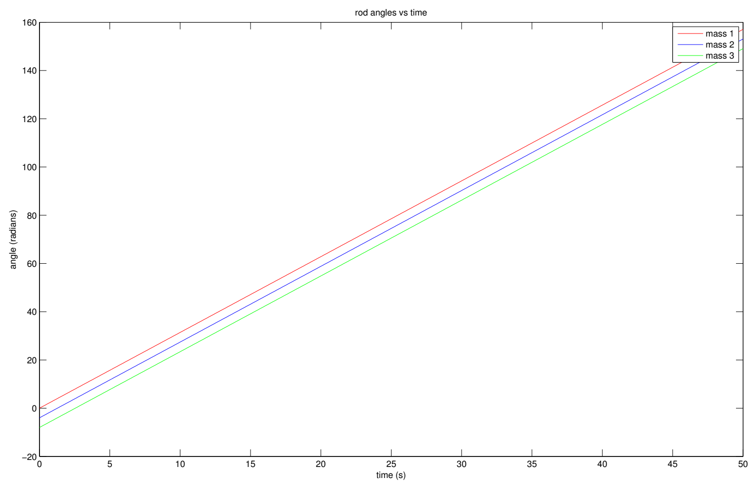

Figure 3: Rod angles over time of a triple pendulum with no gravity and equal initial positions and angular velocities

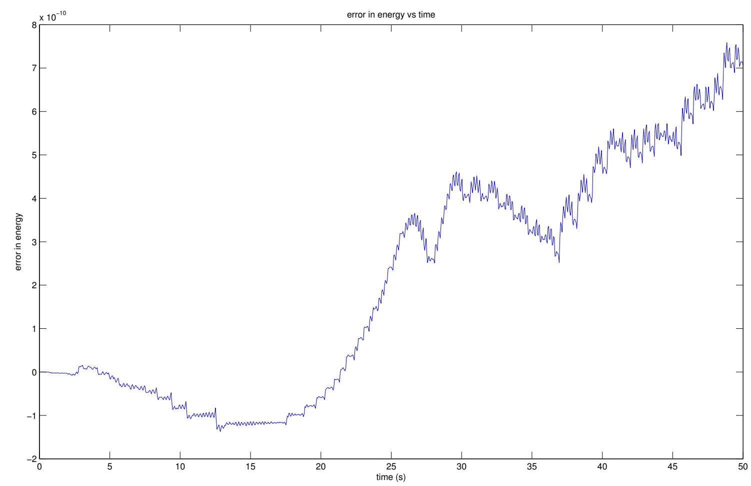

Figure 4: Difference in current total energy and initial total energy over time

To see these plots yourself, select any of the methods in "runner.m," uncomment it, and run. Three windows will appear. The first will contain a plot of rod angles vs. time, the second, a plot of error in energy vs time, and the third will be an animation of the system.

3 Quadruple Pendulum

Included in this simulation package is a quadruple pendulum set up using angular momentum balance. Its setup is analogous to the triple pendulum using angular momentum balance, so I have not included the derivations. To run this simulation, open "runner.m" and uncomment line 26. As with the triple pendulum, you will find plots of the angles vs. time, error in energy vs. time, and an animation of the pendulum. You can add horizontal forces using the F_i variables on line 26.

4 Four-Bar Linkage

The four-bar linkage is very similar to the triple pendulum, with the addition of a new constraint on the end of the third bar. See Appendix 4.01 for a drawing of the setup. It is an example of a system where using angular momentum balance alone would not be straightforward, due to unknown tension forces. We will therefore use DAEs.

4.1 Solving Using Differential Algebraic Equations

Since this system is very similar to the triple pendulum, we can use most of the equations from it. We have, however, two new unknowns:

$$T_{3x}, T_{3y}$$

which are the horizontal and vertical components of the tension force from the fourth hinge. This means that we need two new equations for our DAE matrix. We also need to redo our AMB and LMB equations for our third bar, which now has this new tension force at its end, $\vec{T_3}$. See Appendix 4.01 for the new and modified free-body diagram and equations. To run this simulation, open the file "runner.m" and uncomment line 30. You can add horizontal forces using the F_i variables on line 30.

4.2 Checking for Accuracy

As with the triple pendulum, we can simplify this system to one that will behave like a single pendulum by making the first and third bars the same length, and hinged at the same height. Figure 5 below shows a plot of angles vs. time of such a system.

We can also check for error in total energy as was described for the triple pendulum.

5 Five-Bar Linkage

Included in this simulation package is a five-bar linkage set up with DAEs. Its setup is analogous to the four-bar linkage, so I have not included the derivations. To run this simulation, open "runner.m" and uncomment line 34. As with the four-bar linkage, you will find plots of the angles vs. time, error in energy vs. time, and an animation of the system. You can add horizontal forces using the F_i variables on line 34.

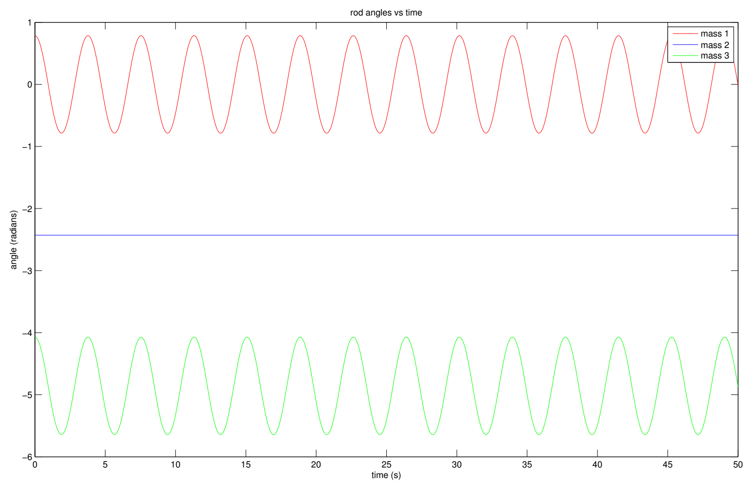

Figure 5: Rod angles vs. time of simplified four-bar-linkage

6 Branched Double Pendulum

The last section of this report includes a simulation of what I have called a "String of Lights." It includes a five-bar linkage with short, single pendulums attached at each moving hinge, in a manner that resembles string lights. In preparation for setting up that system, I have simulated a simpler system - a double pendulum with an extra single pendulum hinged at the one moving hinge. A drawing of this system can be found in Appendix 6.01. To simulate this system using Lagrange equations, open "runner.m" and uncomment line 37. To simulate it with DAEs, uncomment line 41. You can add horizontal forcing to the DAE version using the F_i variables on line 41. Derivations for these methods can be found in Appendix 6.02 and 6.03, respectively.

7 String of Lights

The last system simulated in this report is what I have called a "String of Lights". It includes a five-bar linkage with three short, single pendulums attached at each moving hinge, in a manner that resembles string lights. A diagram of this system can be found in Appendix 7.01.

7.1 Solving using DAEs

I solved this system using differential algebraic equations. Free-body diagrams and derivations can be found in Appendix 7.01. The system has a total of 37 unknowns. To run this simulation, open the file "runner_lights.m" and run.

7.2 Checking for Accuracy

Energy is conserved for this system when no external nonconservative forces are applied. A file similar to that for the previous systems checks for energy conservation. There are also a couple interesting initial conditions we can use to check the validity of this simulation.

7.2.1 Reducing System to Simpler Systems

One way to test this simulation is to make the center short mass (mass #6 in the diagram,) very heavy, and to give the long rods initial angles that cause them to form a "V." Given initial thetas not equal to zero, and no external, nonconservative forces, the outer two lights should oscillate like single pendulums. Figure 6 below shows a plot of the angles over time for such a system.

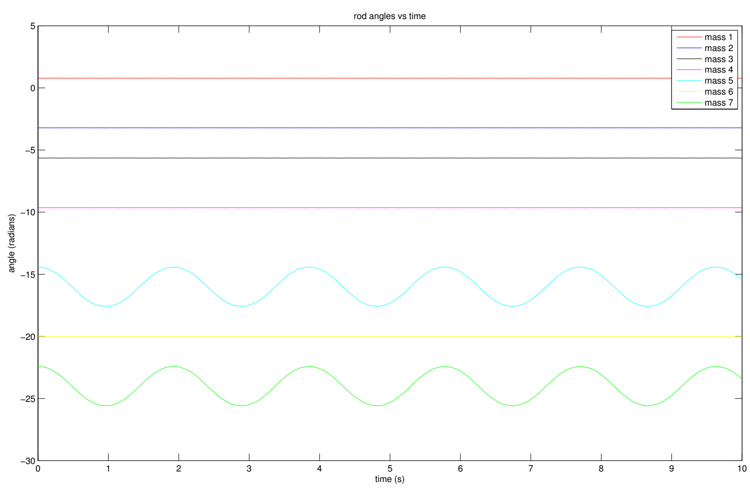

Figure 6: Rod angles vs. time of simplified four-bar-linkage

8 Appendix

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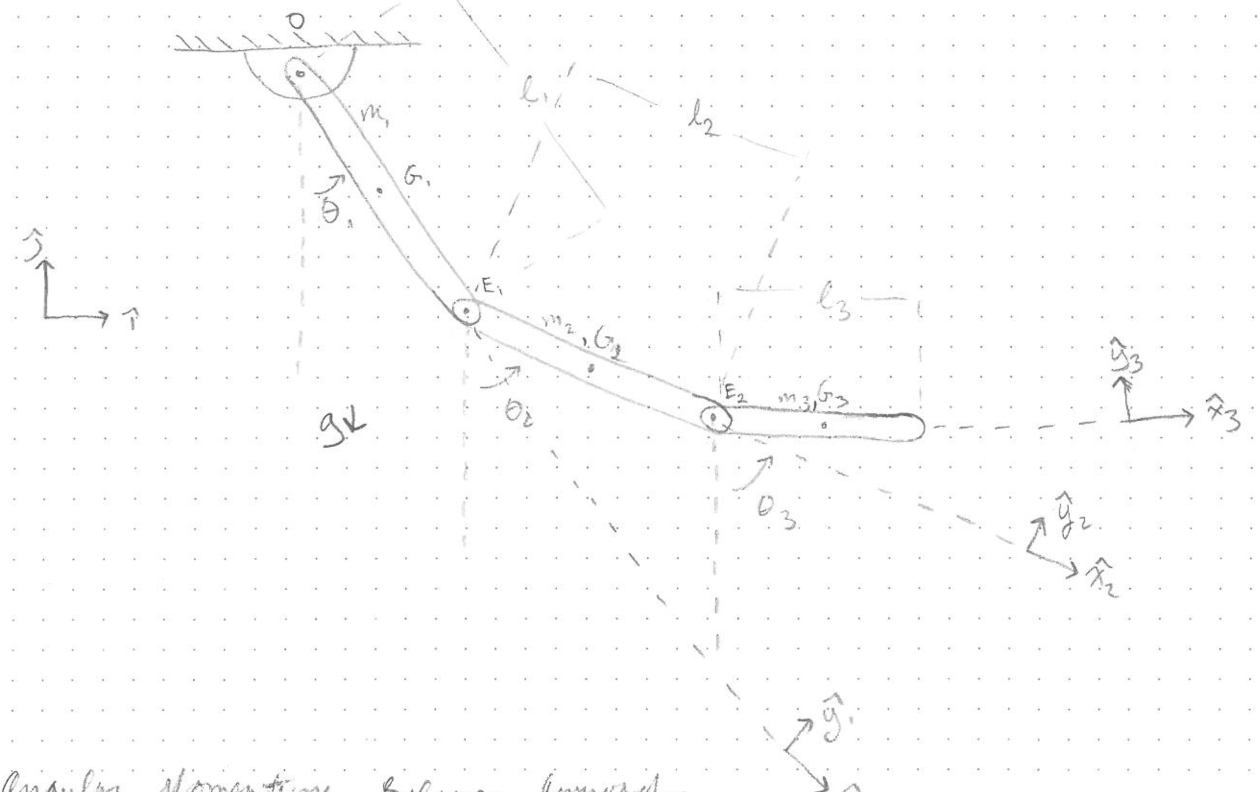
Triple Pendulum

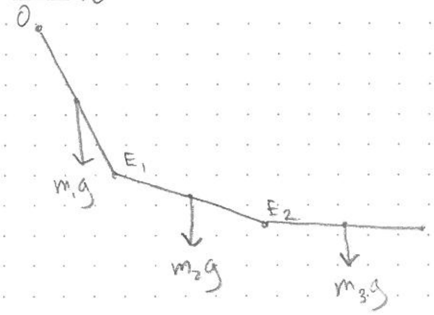
2.01



Angular Momentum Balance Approach

2.02

FBD/O

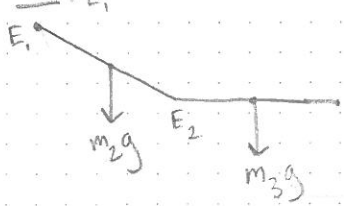


AMB/O

$$\sum \vec{M}_{/O} = \dot{\vec{H}}_{/O}$$

$$\sum_{i}^{3} \vec{r}_{G_i/O} \times -m_i g \hat{j} = \sum_{i}^{3} \vec{r}_{G_i/O} \times m_i \ddot{\vec{r}}_{G_i/O} + I_{G_i} \ddot{\theta}_i \hat{k}$$

FBD/E_1

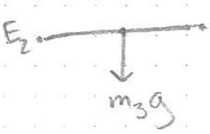


AMB/E_1

$$\sum \vec{M}_{/E_1} = \dot{\vec{H}}_{/E_1}$$

$$\sum_{2}^{3} \vec{r}_{G_i/E_1} \times -m_i g \hat{j} = \sum_{2}^{3} \vec{r}_{G_i/E_1} \times m_i \ddot{\vec{r}}_{G_i/O} + I_{G_i} \ddot{\theta}_i \hat{k}$$

FBD/E_2



AMB/E_2

$$\vec{r}_{G3/E_2} \times -m_3 g \hat{j} = \vec{r}_{G3/E_2} \times m_3 \ddot{\vec{r}}_{G3/O} + I_{G3} \ddot{\theta}_3 \hat{k}$$

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Triple Pendulum (cont.)

12.03

$$KE_{TOT} = \sum_1^3 \frac{1}{2} m_i v_{G_i/O}^2 + \frac{1}{2} I_{G_i} \dot{\theta}_i^2$$

$$\begin{aligned} PE_{TOT} = \; & m_1 g \cdot \frac{1}{2} \ell_1 (1 - \cos\theta_1) \\ & + m_2 g \cdot \left[\ell_1 (1-\cos\theta_1) + \frac{1}{2}\ell_2 (1-\cos\theta_2) \right] \\ & + m_3 g \cdot \left[\ell_1 (1-\cos\theta_1) + \ell_2 (1-\cos\theta_2) + \frac{1}{2}\ell_3 (1-\cos\theta_3) \right] \end{aligned}$$

$$\mathcal{L} = KE_{TOT} - PE_{TOT}$$

Lagrange Equations:

$$\frac{\partial \mathcal{L}}{\partial \theta_1} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \theta_3} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_3} = 0$$

useful information about the velocities:

$$\vec{v}_{G2/O} = 2\vec{v}_{G1/O} + \vec{v}_{G2/E1}$$

$$\vec{v}_{G3/O} = 2\vec{v}_{G1/O} + 2v_{G2/E1} + \vec{v}_{G3/E2}$$

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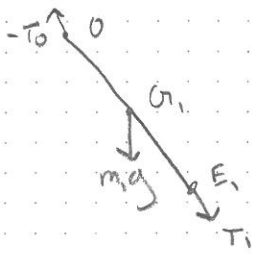
Triple Pendulum (cont.)

Differential Algebraic Equations Approach

12.04

15 unknowns: $\ddot{x}_1, \ddot{y}_1, \ddot{x}_2, \ddot{y}_2, \ddot{x}_3, \ddot{y}_3, \ddot{\theta}_1, \ddot{\theta}_2, \ddot{\theta}_3, T_{0x}, T_{0y}, T_{1x}, T_{1y}, T_{2x}, T_{2y}$

FBD bar1



LMB (bar1) $\sum \vec{F}_{bar1} = m_1 \vec{a}_{G1/O}$

$$-\vec{T}_0 + \vec{T}_1 - m_1 g \hat{\jmath} = m_1 \vec{a}_{G1/O}$$

$\downarrow$

(1) $T_{0x} - T_{1x} + m_1 \ddot{x}_1 = 0$

(2) $T_{0y} - T_{1y} + m_1 g + m_1 \ddot{y}_1 = 0$

AMB /G1 $\sum \vec{M}_{/G1} = \dot{\vec{H}}_{/G1}$

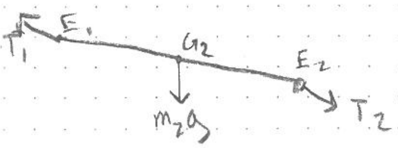
$$\vec{r}_{O/G1} \times -\vec{T}_0 + \vec{r}_{E1/G1} \times \vec{T}_1 = \cancel{\vec{r}_{G1/O} \times m_1 \ddot{\vec{r}}_{G1/O}} + I_{G1} \ddot{\theta}_1 \hat{k}$$

$$\vec{r}_{G1/O} \times \vec{T}_0 + \vec{r}_{E1/G1} \times \vec{T}_1 = I_{G1} \ddot{\theta}_1 \hat{k}$$

$$r_{G1/O(x)} \cdot T_{0y} - r_{G1/O(y)} \cdot T_{0x} + r_{E1/G1(x)} \cdot T_{1y} - r_{E1/G1(y)} \cdot T_{1x} = I_{G1} \ddot{\theta}_1$$

(3) $I_{G1} \ddot{\theta}_1 - r_{G1/O(x)} T_{0y} + r_{G1/O(y)} T_{0x} - r_{E1/G1(x)} T_{1y} + r_{E1/G1(y)} T_{1x} = 0$

FBD bar2



LMB (bar2) $\sum \vec{F}_{bar2} = m_2 \vec{a}_{G2/O}$

$$-\vec{T}_1 + T_2 - m_2 g \hat{\jmath} = m_2 \vec{a}_{G2/O}$$

$\downarrow$

(4) $T_{1x} - T_{2x} + m_2 \ddot{x}_2 = 0$

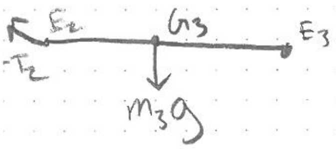
(5) $T_{1y} - T_{2y} + m_2 g + m_2 \ddot{y}_2 = 0$

AMB /G2 $\sum \vec{M}_{/G2} = \dot{\vec{H}}_{/G2}$

$$\vec{r}_{E1/G2} \times -\vec{T}_1 + \vec{r}_{E2/G2} \times \vec{T}_2 = I_{G2} \ddot{\theta}_2 \hat{k}$$

(6) $I_{G2} \ddot{\theta}_2 - r_{G2/E1(x)} T_{1y} + r_{G2/E1(y)} T_{1x} - r_{E2/G2(x)} T_{2y} + r_{E2/G2(y)} T_{2x} = 0$

FBD bar3



LMB (bar3) $\sum \vec{F}_{bar3} = m_3 \vec{a}_{G3/O}$

$$-\vec{T}_2 - m_3 g \hat{\jmath} = m_3 \vec{a}_{G3/O}$$

$\downarrow$

(7) $T_{2x} + m_3 \ddot{x}_3 = 0$

(8) $T_{2y} + m_3 g + m_3 \ddot{y} = 0$

AMB /G3 $\sum \vec{M}_{/G3} = \dot{\vec{H}}_{/G3}$

$$\vec{r}_{E2/G3} \times -\vec{T}_2 = I_{G3} \ddot{\theta}_3 \hat{k}$$

(9) $I_{G3} \ddot{\theta}_3 - r_{G3/E2(x)} T_{2y} + r_{G3/E2(y)} T_{2x} = 0$

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Triple Pendulum (cont.)

DAE's (cont.)

12.04 (cont.)

Kinematic Constraints: equate accelerations of hinges for each bar

$O:\ \ddot{\vec{r}}_{O/O} = \ddot{\vec{r}}_{O/G_1} + \ddot{\vec{r}}_{G_1/O}$

$E_1:\ \ddot{\vec{r}}_{E_1/G_1} + \ddot{\vec{r}}_{G_1/O} = \ddot{\vec{r}}_{E_1/G_2} + \ddot{\vec{r}}_{G_2/O}$

$E_2:\ \ddot{\vec{r}}_{E_2/G_2} + \ddot{\vec{r}}_{G_2/O} = \ddot{\vec{r}}_{E_2/G_3} + \ddot{\vec{r}}_{G_3/O}$

$O:$

$$0 = \ddot{\theta}_1 \hat{K} \times \vec{r}_{O/G_1} - \dot{\theta}_1^2 \vec{r}_{O/G_1} + \ddot{x}_1 \hat{\imath} + \ddot{y}_1 \hat{\jmath}$$

$$0 = \ddot{\theta}_1 r_{O/G_1(x)} \hat{\jmath} - \ddot{\theta}_1 r_{O/G_1(y)} \hat{\imath} - \dot{\theta}_1^2 r_{O/G_1(x)} \hat{\imath} - \dot{\theta}_1^2 r_{O/G_1(y)} \hat{\jmath} + \ddot{x}_1 \hat{\imath} + \ddot{y}_1 \hat{\jmath}$$

$$0 = -\ddot{\theta}_1 r_{G_1/O(x)} \hat{\jmath} + \ddot{\theta}_1 r_{G_1/O(y)} \hat{\imath} + \dot{\theta}_1^2 r_{G_1/O(x)} \hat{\imath} + \dot{\theta}_1^2 r_{G_1/O(y)} \hat{\jmath} + \ddot{x}_1 \hat{\imath} + \ddot{y}_1 \hat{\jmath}$$

$\downarrow$

(10) $\ddot{x}_1 + \ddot{\theta}_1 r_{G_1/O(y)} = -\dot{\theta}_1^2 r_{G_1/O(x)}$

(11) $\ddot{y}_1 - \ddot{\theta}_1 r_{G_1/O(x)} = -\dot{\theta}_1^2 r_{G_1/O(y)}$

$E_1:$

$$\ddot{\theta}_1 \hat{K} \times \vec{r}_{E_1/G_1} - \dot{\theta}_1^2 \vec{r}_{E_1/G_1} + \ddot{x}_1 \hat{\imath} + \ddot{y}_1 \hat{\jmath} = \ddot{\theta}_2 \hat{K} \times \vec{r}_{E_1/G_2} - \dot{\theta}_2^2 \vec{r}_{E_1/G_2} + \ddot{x}_2 \hat{\imath} + \ddot{y}_2 \hat{\jmath}$$

$$\ddot{\theta}_1 r_{E_1/G_1(x)} \hat{\jmath} - \ddot{\theta}_1 r_{E_1/G_1(y)} \hat{\imath} - \dot{\theta}_1^2 r_{E_1/G_1(x)} \hat{\imath} - \dot{\theta}_1^2 r_{E_1/G_1(y)} \hat{\jmath} + \ddot{x}_1 \hat{\imath} + \ddot{y}_1 \hat{\jmath}$$

$$= \ddot{\theta}_2 r_{E_1/G_2(x)} \hat{\jmath} - \ddot{\theta}_2 r_{E_1/G_2(y)} \hat{\imath} - \dot{\theta}_2^2 r_{E_1/G_2(x)} \hat{\imath} - \dot{\theta}_2^2 r_{E_1/G_2(y)} \hat{\jmath} + \ddot{x}_2 \hat{\imath} + \ddot{y}_2 \hat{\jmath}$$

$\downarrow$

(12) $\ddot{x}_1 - \ddot{x}_2 - \ddot{\theta}_1 r_{E_1/G_1(y)} - \ddot{\theta}_2 r_{G_2/E_1(y)} = \dot{\theta}_1^2 r_{E_1/G_1(x)} + \dot{\theta}_2^2 r_{G_2/E_1(x)}$

(13) $\ddot{y}_1 - \ddot{y}_2 + \ddot{\theta}_1 r_{E_1/G_1(x)} + \ddot{\theta}_2 r_{G_2/E_1(x)} = \dot{\theta}_1^2 r_{E_1/G_1(y)} + \dot{\theta}_2^2 r_{G_2/E_1(y)}$

$E_2:$

$$\ddot{\theta}_2 \hat{K} \times \vec{r}_{E_2/G_2} - \dot{\theta}_2^2 \vec{r}_{E_2/G_2} + \ddot{x}_2 \hat{\imath} + \ddot{y}_2 \hat{\jmath} = \ddot{\theta}_3 \hat{K} \times \vec{r}_{E_2/G_3} - \dot{\theta}_3^2 \vec{r}_{E_2/G_3} + \ddot{x}_3 \hat{\imath} + \ddot{y}_3 \hat{\jmath}$$

$$\ddot{\theta}_2 r_{E_2/G_2(x)} \hat{\jmath} - \ddot{\theta}_2 r_{E_2/G_2(y)} \hat{\imath} - \dot{\theta}_2^2 r_{E_2/G_2(x)} \hat{\imath} - \dot{\theta}_2^2 r_{E_2/G_2(y)} \hat{\jmath} + \ddot{x}_2 \hat{\imath} + \ddot{y}_2 \hat{\jmath}$$

$$= \ddot{\theta}_3 r_{E_2/G_3(x)} \hat{\jmath} - \ddot{\theta}_3 r_{E_2/G_3(y)} \hat{\imath} - \dot{\theta}_3^2 r_{E_2/G_3(x)} \hat{\imath} - \dot{\theta}_3^2 r_{E_2/G_3(y)} \hat{\jmath} + \ddot{x}_3 \hat{\imath} + \ddot{y}_3 \hat{\jmath}$$

$\downarrow$

(14) $\ddot{x}_2 - \ddot{x}_3 - \ddot{\theta}_2 r_{E_2/G_2(y)} - \ddot{\theta}_3 r_{G_3/E_2(y)} = \dot{\theta}_2^2 r_{E_2/G_2(x)} + \dot{\theta}_3^2 r_{G_3/E_2(x)}$

(15) $\ddot{y}_2 - \ddot{y}_3 + \ddot{\theta}_2 r_{E_2/G_2(x)} + \ddot{\theta}_3 r_{G_3/E_2(x)} = \dot{\theta}_2^2 r_{E_2/G_2(y)} + \dot{\theta}_3^2 r_{G_3/E_2(y)}$

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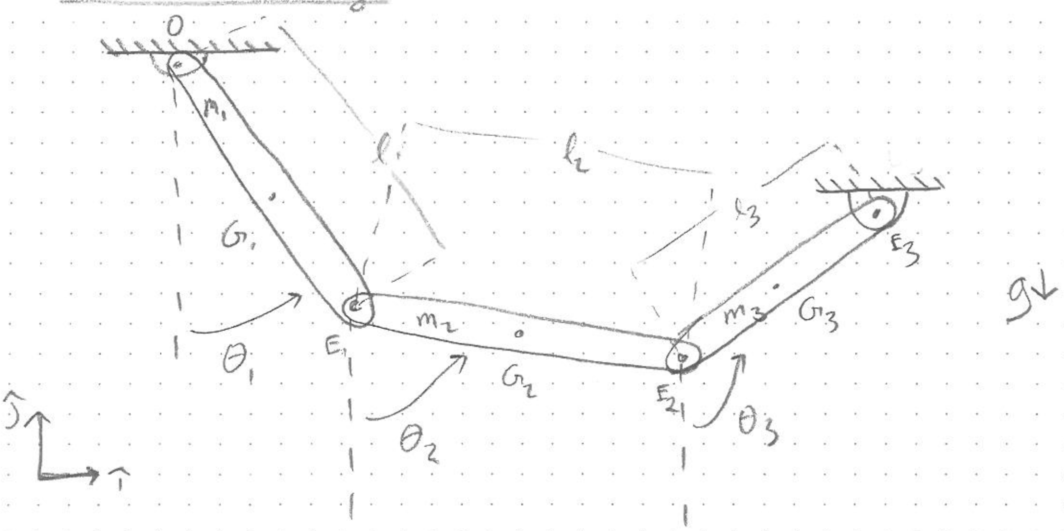
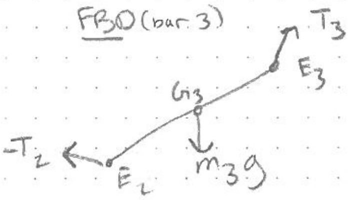
4.01

Four-Bar Linkage



We can use the same equations from DAE's for the triple pendulum, with some changes, and two new equations to accommodate for our two new unknowns: T_{3x} and T_{3y}, the tensions on the fourth hinge.

FBD (bar 3)



LMB (bar 3)

$$-\vec{T}_2 + \vec{T}_3 - m_3 g\hat{j} = m_3 \vec{a}_{G3/O}$$

(7) $-T_{2x} + T_{3x} = m_3 \ddot{x}_3$

(8) $-T_{2y} + T_{3y} - m_3 g = m_3 \ddot{y}_3$

AMB/G3

$$\vec{r}_{E2/G3} \times -\vec{T}_2 + \vec{r}_{E3/G3} \times T_3 = I_{G3}\ddot{\theta}_3\hat{k}$$

$$-r_{E2/G3(x)} T_{2y} + r_{E2/G3(y)} T_{2x} + r_{E3/G3(x)} T_{3y} - r_{E3/G3(y)} T_{3x} = I_{G3}\ddot{\theta}_3$$

(9) $r_{G3/E2(x)} T_{2y} - r_{G3/E2(y)} T_{2x} + r_{E3/G3(x)} T_{3y} - r_{E3/G3(y)} T_{3x} = I_{G}\ddot{\theta}_3$

New Constraint:

$$\ddot{\vec{r}}_{E3/E3} = \ddot{\vec{r}}_{E3/G3} + \ddot{\vec{r}}_{G3/O}$$

$$0 = \ddot{\theta}_3\hat{k} \times \vec{r}_{E3/G3} - \dot{\theta}_3^2 \vec{r}_{E3/G3} + \ddot{x}_3\hat{i} + \ddot{y}_3\hat{j}$$

$$\ddot{x}_3\hat{i} + \ddot{y}_3\hat{j} + \ddot{\theta}_3 r_{E3/G3(x)}\hat{j} - \ddot{\theta}_3 r_{E3/G3(y)}\hat{i} = \dot{\theta}_3^2 r_{E3/G3(x)}\hat{i} + \dot{\theta}_3^2 r_{E3/G3(y)}\hat{j}$$

(16) $\ddot{x}_3 - \ddot{\theta}_3 r_{E3/G3(y)} = \dot{\theta}_3^2 r_{E3/G3(x)}$

(17) $\ddot{y}_3 + \ddot{\theta}_3 r_{E3/G3(x)} = \dot{\theta}_3^2 r_{E3/G3(y)}$

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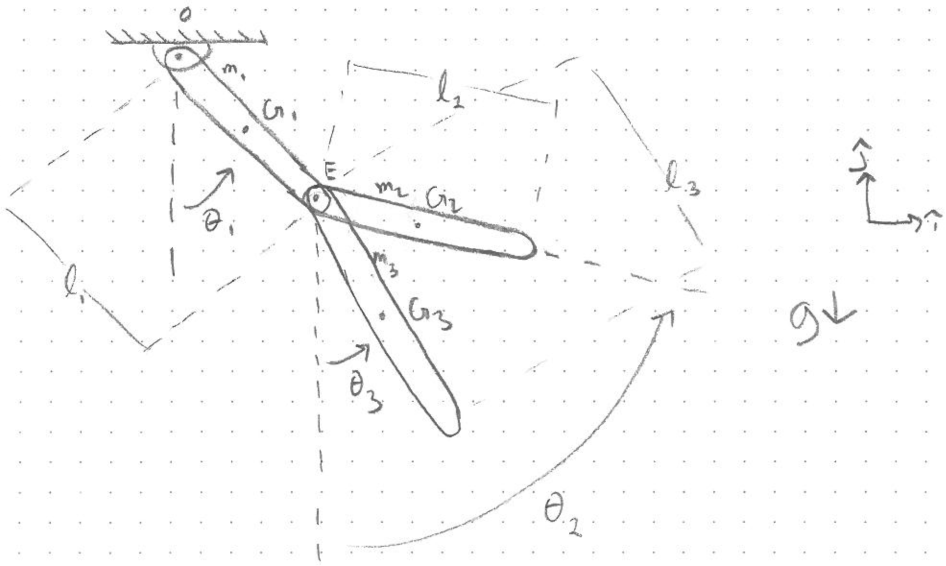
Branched Double Pendulum

6.01



$$\vec{r}_{G1/O} = \frac{l_1}{2}\sin\theta_1\hat{i} - \frac{l_1}{2}\cos\theta_1\hat{j}$$

$$\vec{r}_{G2/O} = 2\vec{r}_{G1/O} + \frac{l_2}{2}\sin\theta_2\hat{i} - \frac{l_2}{2}\cos\theta_2\hat{j}$$

$$\vec{r}_{G3/O} = 2\vec{r}_{G1/O} + \frac{l_3}{2}\sin\theta_3\hat{i} - \frac{l_3}{2}\cos\theta_3\hat{j}$$

$$\vec{v}_{G1/O} = \frac{l_1}{2}\dot{\theta}_1\cos\theta_1\hat{i} + \frac{l_1}{2}\dot{\theta}_1\sin\theta_1\hat{j}$$

$$\vec{v}_{G2/O} = 2\vec{v}_{G1/O} + \frac{l_2}{2}\dot{\theta}_2\cos\theta_2\hat{i} + \frac{l_2}{2}\dot{\theta}_2\sin\theta_2\hat{j}$$

$$\vec{v}_{G3/O} = 2\vec{v}_{G1/O} + \frac{l_3}{2}\dot{\theta}_3\cos\theta_3\hat{i} + \frac{l_3}{2}\dot{\theta}_3\sin\theta_3\hat{j}$$

$$KE_{TOT} = \sum_{1}^{3} \frac{1}{2}m_i v_{Gi/O}^2 + \frac{1}{2}I_{Gi}\dot{\theta}_i^2$$

6.02

$$\begin{aligned} PE_{TOT} = {} & m_1 g \cdot \frac{1}{2} l_1(1-\cos\theta_1) \\ & + m_2 g \cdot \left[l_1(1-\cos\theta_1) + \frac{1}{2} l_2(1-\cos\theta_2)\right] \\ & + m_3 g \cdot \left[l_1(1-\cos\theta_1) + \frac{1}{2} l_3(1-\cos\theta_3)\right] \end{aligned}$$

$$\mathcal{L} = KE_{TOT} - PE_{TOT}$$

Lagrange Equations:

$$\frac{\partial\mathcal{L}}{\partial\theta_1} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\theta_2} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\theta_3} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_3} = 0$$

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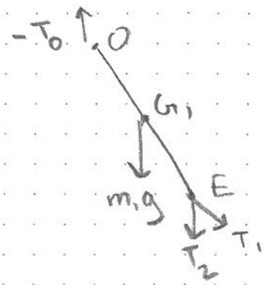
Branched Double Pendulum (cont.)

DAEs

As with the triple pendulum, we have 15 unknowns.

FBD bar1



LMB (bar1) $\quad \sum \vec{F}_{bar1} = m_1 \vec{a}_{G_1/O}$

$$-\vec{T}_0 + \vec{T}_1 + \vec{T}_2 - m_1 g \hat{\jmath} = m_1 \vec{a}_{G_1/O}$$

$\downarrow$

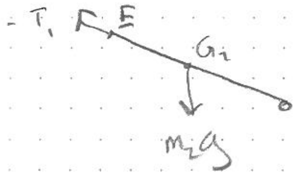
① $\quad T_{0x} - T_{1x} - T_{2x} + m_1 \ddot{x}_1 = 0$

② $\quad T_{0y} - T_{1y} - T_{2y} + m_1 \ddot{y}_1 = -m_1 g$

AMB$/G_1$ $\quad \sum \vec{M}_{/G_1} = \dot{\vec{H}}_{/G_1}$

③ $\quad \vec{r}_{O/G_1} \times -\vec{T}_0 + \vec{r}_{E/G_1} \times \vec{T}_1 + \vec{r}_{E/G_1} \times \vec{T}_2 = I_{G_1} \ddot{\theta}_1 \hat{k}$

FBD bar2



LMB bar2

$$-\vec{T}_1 - m_2 g \hat{\jmath} = m_2 \vec{a}_{G_2/O}$$

$\downarrow$

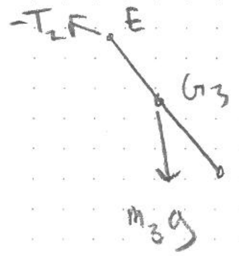
④ $\quad T_{1x} + m_2 \ddot{x}_2 = 0$

⑤ $\quad T_{1y} + m_2 \ddot{y}_2 = -m_2 g$

AMB$/G_2$

⑥ $\quad \vec{r}_{E/G_2} \times -\vec{T}_1 = I_{G_2} \ddot{\theta}_2 \hat{k}$

FBD bar3



LMB bar3

$$-\vec{T}_2 - m_3 g \hat{\jmath} = m_3 \vec{a}_{G_3/O}$$

$\downarrow$

⑦ $\quad T_{2x} + m_3 \ddot{x}_3 = 0$

⑧ $\quad T_{2y} + m_3 \ddot{y}_3 = -m_3 g$

AMB$/G_3$

⑨ $\quad \vec{r}_{E/G_3} \times -\vec{T}_2 = I_{G_3} \ddot{\theta}_3 \hat{k}$

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Branched Double Pendulum (cont.)

DAEs (cont.)

Kinematic Constraints:

$$O: \ddot{\vec{r}}_{O/O} = \ddot{\vec{r}}_{O/G1} + \ddot{\vec{r}}_{G1/O}$$

$$E: \ddot{\vec{r}}_{E/G1} + \ddot{\vec{r}}_{G1/O} = \ddot{\vec{r}}_{E/G2} + \ddot{\vec{r}}_{G2/O}$$

$$\ddot{\vec{r}}_{E/G1} + \ddot{\vec{r}}_{G1/O} = \ddot{\vec{r}}_{E/G3} + \ddot{\vec{r}}_{G3/O}$$

The above equations lead to the following equations:

(10) $\ddot{x}_1 + \ddot{\theta}_1 r_{G1/O(y)} = -\dot{\theta}_1^2 r_{G1/O(x)}$

(11) $\ddot{y}_1 - \ddot{\theta}_1 r_{G1/O(x)} = -\dot{\theta}_1^2 r_{G1/O(y)}$

(12) $\ddot{x}_1 - \ddot{x}_2 - \ddot{\theta}_1 r_{E1/G1(y)} - \ddot{\theta}_2 r_{G2/E1(y)} = \dot{\theta}_1^2 r_{E1/G1(x)} + \dot{\theta}_2^2 r_{G2/E1(x)}$

(13) $\ddot{y}_1 - \ddot{y}_2 + \ddot{\theta}_1 r_{E1/G1(x)} + \ddot{\theta}_2 r_{G2/E1(x)} = \dot{\theta}_1^2 r_{E1/G1(y)} + \dot{\theta}_2^2 r_{G2/E1(y)}$

(14) $\ddot{x}_1 - \ddot{x}_3 - \ddot{\theta}_1 r_{E1/G1(y)} - \ddot{\theta}_3 r_{G3/E1(y)} = \dot{\theta}_1^2 r_{E1/G1(x)} + \dot{\theta}_3^2 r_{G3/E1(x)}$

(15) $\ddot{y}_1 - \ddot{y}_3 + \ddot{\theta}_1 r_{E1/G1(x)} + \ddot{\theta}_3 r_{G3/E1(x)} = \dot{\theta}_1^2 r_{E1/G1(y)} + \dot{\theta}_3^2 r_{G3/E1(y)}$

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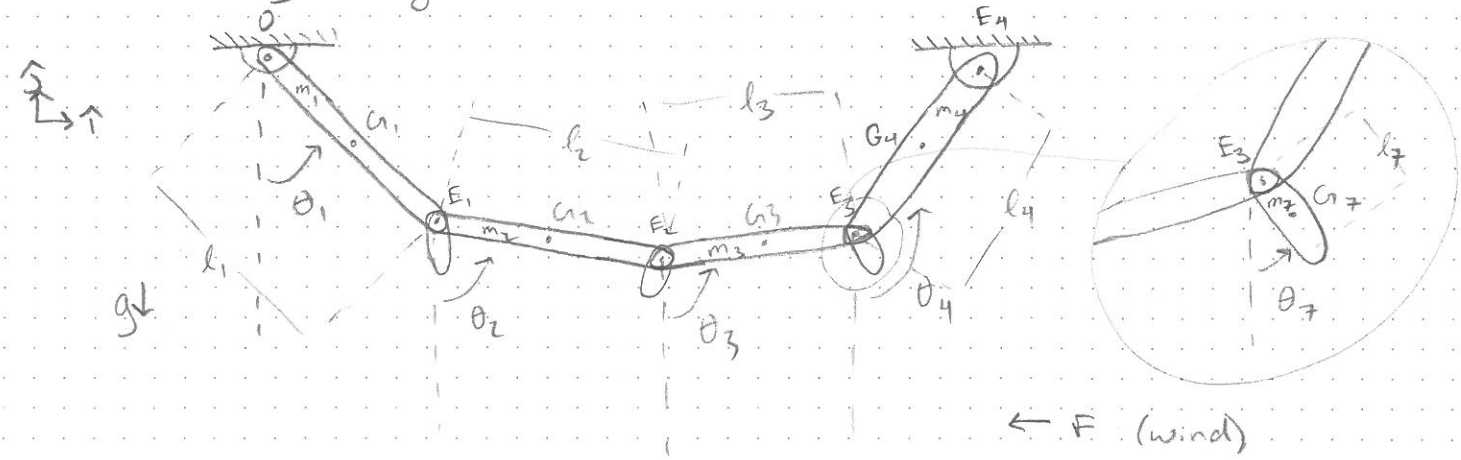
String of Lights



To find the motion, I will use DAEs: 7.01

We have 37 unknowns:

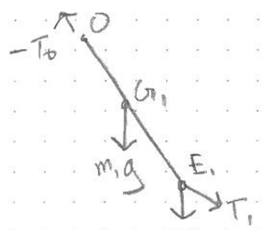
$\ddot{x}_1, \ddot{y}_1, \ddot{x}_2, \ddot{y}_2, \ddot{x}_3, \ddot{y}_3, \ddot{x}_4, \ddot{y}_4, \ddot{x}_5, \ddot{y}_5, \ddot{x}_6, \ddot{y}_6, \ddot{x}_7, \ddot{y}_7$

$\ddot{\theta}_1, \ddot{\theta}_2, \ddot{\theta}_3, \ddot{\theta}_4, \ddot{\theta}_5, \ddot{\theta}_6, \ddot{\theta}_7,$

$T_{0x}, T_{0y}, T_{1x}, T_{1y}, T_{2x}, T_{2y}, T_{3x}, T_{3y}, T_{4x}, T_{4y}, T_{5x}, T_{5y}, T_{6x}, T_{6y}, T_{7x}, T_{7y}$

Using LMB and AMB of each individual bar, I will get 21 equations. The remaining 16 equations will come from constraints at each hinge.

FBD bar1



LMB

$$-\vec{T}_0 + \vec{T}_1 + \vec{T}_5 - m_1 g \hat{\jmath} = m_1 \vec{a}_{G1/O}$$

↓

① $m_1 \ddot{x}_1 + T_{0x} - T_{1x} - T_{5x} = 0$

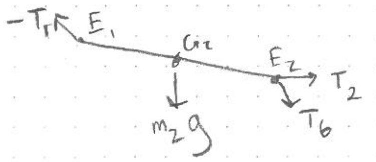
② $m_1 \ddot{y}_1 + T_{0y} - T_{1y} - T_{5y} = -m_1 g$

AMB/G_1

$$\vec{r}_{O/G1} \times -\vec{T}_0 + \vec{r}_{E1/G1} \times \vec{T}_5 + \vec{r}_{E1/G1} \times \vec{T}_1 = I_{G1} \ddot{\theta}_1 \hat{k}$$

③ $I_{G1}\ddot{\theta}_1 - r_{G1/O(x)} T_{0y} + r_{G1/O(y)} T_{0x} - r_{E1/G1(x)} T_{1y} + r_{E1/G1} T_{1x} - r_{E1/G1(x)} T_{5y} + r_{E1/G1(y)} T_{5x} = 0$

FBD bar2



LMB

$$-\vec{T}_1 + \vec{T}_2 + \vec{T}_6 - m_2 g \hat{\jmath} = m_2 \vec{a}_{G2/O}$$

↓

④ $m_2 \ddot{x}_2 = T_{1x} - T_{2x} - T_{6x} = 0$

⑤ $m_2 \ddot{y}_2 = T_{1y} - T_{2y} - T_{6y} = -m_2 g$

AMB/G_2

$$\vec{r}_{E1/G2} \times -\vec{T}_1 \quad + \vec{r}_{E2/G2} \times (\vec{T}_2 + \vec{T}_6) = I_{G2} \ddot{\theta}_2 \hat{k}$$

⑥ $I_{G2}\ddot{\theta}_2 - r_{G2/E1(x)} T_{1y} + r_{G2/E1(y)} T_{1x} +$

$- r_{E2/G2(x)} (T_{2y} + T_{6y}) + r_{E2/G2(y)} (T_{2x} + T_{6x}) = 0$

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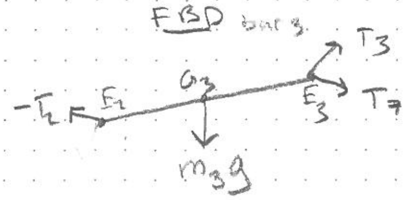
String of Lights (cont.)

17.01 (cont.)

FBD bar 3



LMB

$$-\vec{T}_2 + \vec{T}_3 + \vec{T}_7 - m_3 g\hat{\jmath} = m_3 \vec{a}_{G3/O}$$

$\downarrow$

(7) $m_3 \ddot{x}_3 + T_{2x} - T_{3x} - T_{7x} = 0$

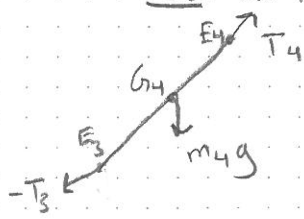
(8) $m_3 \ddot{y}_3 + T_{2y} - T_{3y} - T_{7y} = -m_3 g$

AMB/G3

$$\vec{r}_{E2/G3} \times -\vec{T}_2 + \vec{r}_{E3/G3} \times (\vec{T}_3 + \vec{T}_7) = I_{G3}\ddot{\theta}_3 \hat{k}$$

(9) $I_{G3}\ddot{\theta}_3 - r_{G3/E2(x)} T_{2y} + r_{G3/E2(y)} T_{2x}$
$\quad - r_{E3/G3(x)}(T_{3y} + T_{7y}) + r_{E3/G3(y)}(T_{3x} + T_{7x}) = 0$

FBD bar 4



LMB

$$-\vec{T}_3 + \vec{T}_4 - m_4 g\hat{\jmath} = m_4 \vec{a}_{G4/O}$$

$\downarrow$

(10) $m_4 \ddot{x}_4 + T_{3x} - T_{4x} = 0$

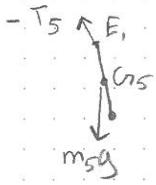
(11) $m_4 \ddot{y}_4 + T_{3y} - T_{4y} = -m_4 g$

AMB/G4

$$\vec{r}_{E3/G4} \times -\vec{T}_3 + \vec{r}_{E4/G4} \times \vec{T}_4 = I_{G4}\ddot{\theta}_4 \hat{k}$$

(12) $I_{G4}\ddot{\theta}_4 - r_{G4/E3(x)} T_{3y} + r_{G4/E3(y)} T_{3x}$
$\quad - r_{E4/G4(x)} T_{4y} + r_{E4/G4(y)} T_{4x} = 0$

FBD bar 5



LMB

$$-\vec{T}_5 - m_5 g\hat{\jmath} = m_5 \vec{a}_{G5/O}$$

$\downarrow$

(13) $m_5 \ddot{x}_5 + T_{5x} = 0$

(14) $m_5 \ddot{y}_5 + T_{5y} = -m_5 g$

AMB/G5

$$\vec{r}_{E1/G5} \times -\vec{T}_5 = I_{G5}\ddot{\theta}_5 \hat{k}$$

(15) $I_{G5}\ddot{\theta}_5 - r_{G5/E1(x)} T_{5y} + r_{G5/E1(y)} T_{5x} = 0$

The FBD, LMB, and AMB for bars 6 and 7 look like those for bar 5, and produce the following equations:

(16) $m_6 \ddot{x}_6 + T_{6x} = 0$

(17) $m_6 \ddot{y}_6 + T_{6y} = -m_6 g$

(18) $I_{G6}\ddot{\theta}_6 - r_{G6/E2(x)} T_{6y} + r_{G6/E2(y)} T_{6x} = 0$

(19) $m_7 \ddot{x}_7 + T_{7x} = 0$

(20) $m_7 \ddot{y}_7 + T_{7y} = -m_7 g$

(21) $I_{G7}\ddot{\theta}_7 - r_{G7/E3(x)} T_{7y} + r_{G7/E3(y)} T_{7x} = 0$

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String of Lights (cont.)

7.01 (cont.)

Kinematic Constraints:

$$O: \ddot{\vec{r}}_{O/O} = \ddot{\vec{r}}_{O/G1} + \ddot{\vec{r}}_{G1/O}$$

$$E_1: \ddot{\vec{r}}_{E1/G1} + \ddot{\vec{r}}_{G1/O} = \ddot{\vec{r}}_{E1/G2} + \ddot{\vec{r}}_{G2/O}$$

$$\ddot{\vec{r}}_{E1/G1} + \ddot{\vec{r}}_{G1/O} = \ddot{\vec{r}}_{E1/G5} + \ddot{\vec{r}}_{G5/O}$$

$$E_2: \ddot{\vec{r}}_{E2/G2} + \ddot{\vec{r}}_{G2/O} = \ddot{\vec{r}}_{E2/G3} + \ddot{\vec{r}}_{G3/O}$$

$$\ddot{\vec{r}}_{E2/G2} + \ddot{\vec{r}}_{G2/O} = \ddot{\vec{r}}_{E2/G6} + \ddot{\vec{r}}_{G6/O}$$

$$E_3: \ddot{\vec{r}}_{E3/G3} + \ddot{\vec{r}}_{G3/O} = \ddot{\vec{r}}_{E3/G4} + \ddot{\vec{r}}_{G4/O}$$

$$\ddot{\vec{r}}_{E3/G3} + \ddot{\vec{r}}_{G3/O} = \ddot{\vec{r}}_{E3/G7} + \ddot{\vec{r}}_{G7/O}$$

$$E_4: \ddot{\vec{r}}_{E4/E4} = \ddot{\vec{r}}_{E4/G4} + \ddot{\vec{r}}_{G4/O}$$