MAE 6700 : Final Project

Motion of a Disk

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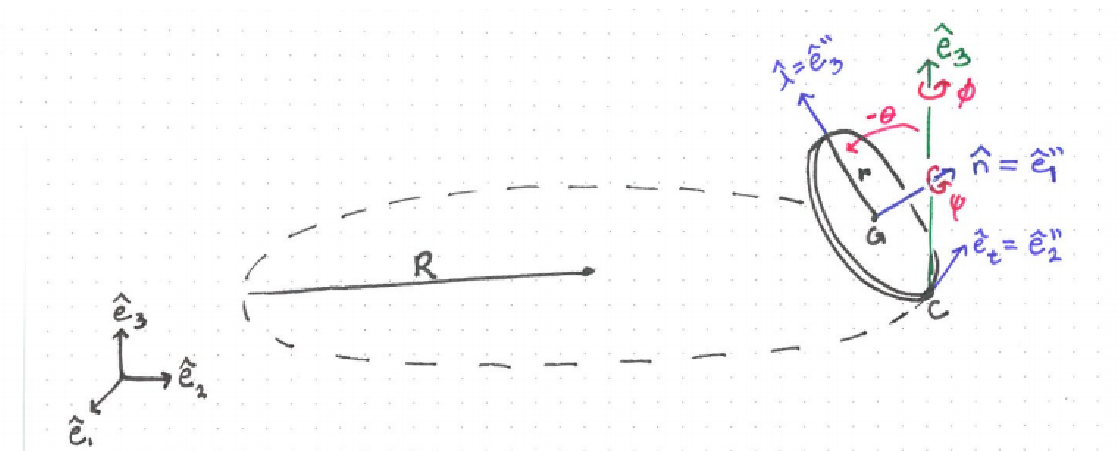
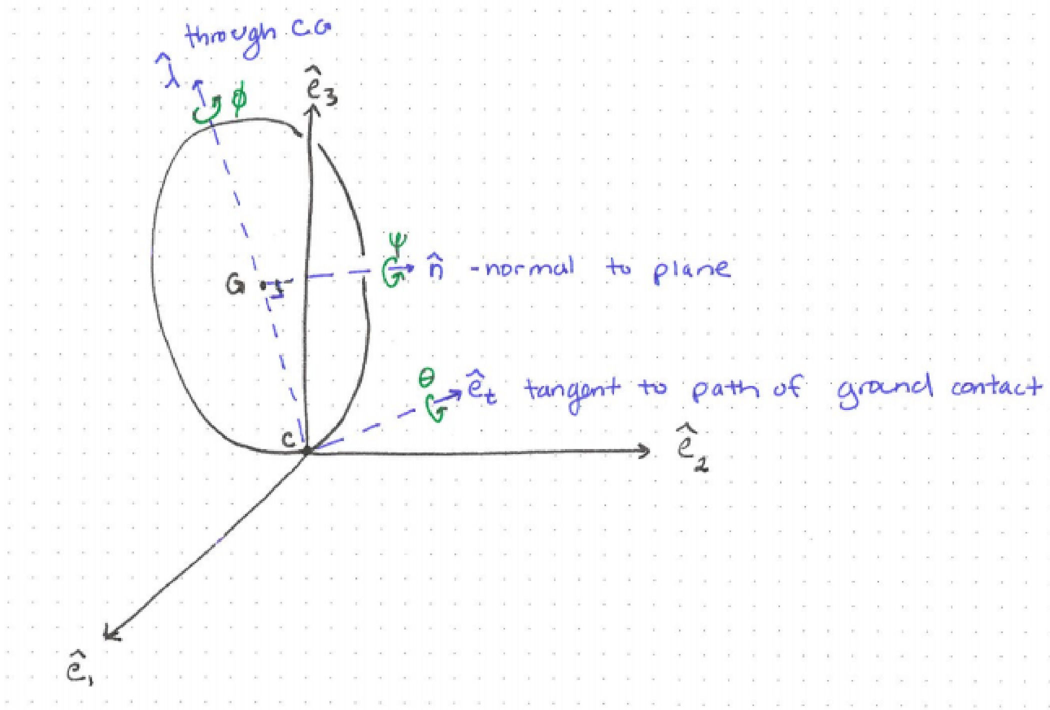
January 12, 2015

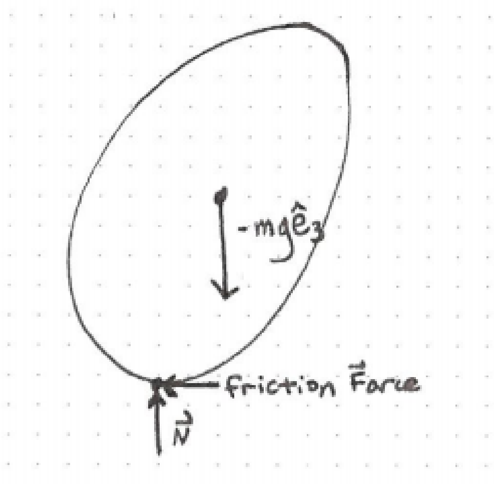


Our Disk:

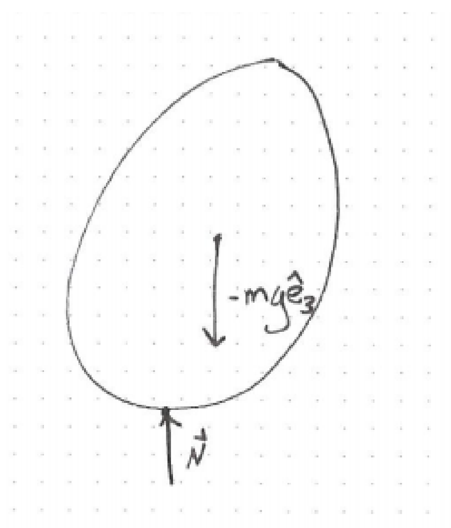


Free-Body Diagrams:

With friction:



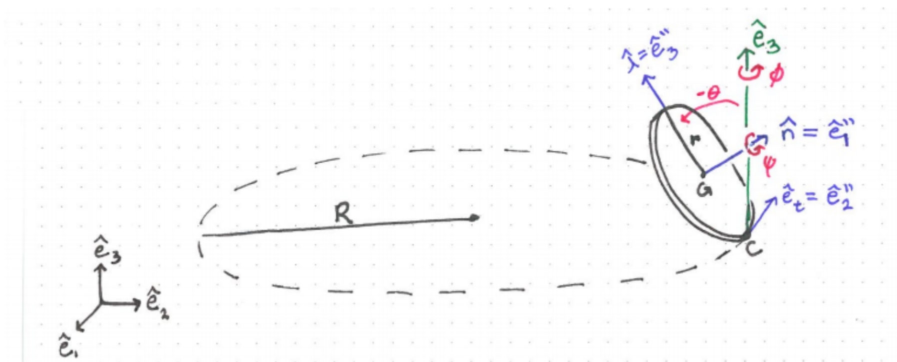
Without:



Problem 1 - Simple Rolling Precession

Find the equations of motion for simple precession of a rolling disk.

Answer



To find the equations of motion for simple precession of a disk, angular momentum balance about the contact point can be used. See full algebraic calculation in the appendix, A1. The final equation of motion obtained is:

$$mrgsin\theta = \dot{\phi}\dot{\psi}cos\theta(mr^2 + I_1) + \dot{\phi}^2 cos\theta sin\theta(I_1 + I_3 - mr^2) \tag{1}$$

This equation has three possible inputs: θ, $\dot{\phi}$, and $\dot{\psi}$. Given a desired precession radius, R, we can find a relationship between $\dot{\phi}$, and $\dot{\psi}$:

$$\dot{\phi} = -\dot{\psi}\frac{r}{R} \tag{2}$$

the explanation for which can be found in the appendix, A2. Plugging this value into our equation (1), we can solve for either $\dot{\phi}$ or $\dot{\psi}$:

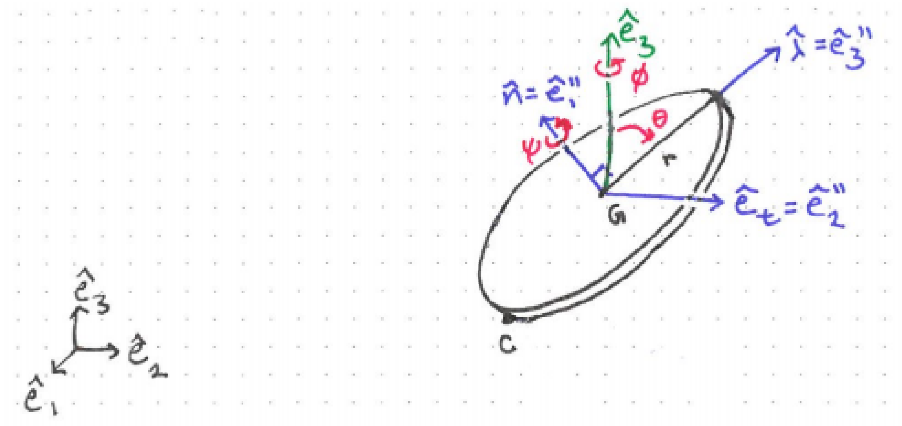
$$\dot{\psi}^2 = \frac{R^2 mgtan\theta}{rsin\theta(I_1 + I_3) - mr^3 sin\theta - mr^2 R - RI_1} \tag{3}$$

This allows us to find simple precession motion by simply defining a precession radius, R, and a tip angle, θ. Simulation of such motion can be found on page 9 of this report, under "Simulation 3".

Problem 2 - Simple Sliding Precession

Find the equations of motion for steady precession of a sliding disk.

Answer



For steady precession of a disk on a frictionless surface, the tip angle cannot change. $\dot{\theta} = 0$. This means that the height of the center of mass, G, never changes. $\ddot{z}_G = 0$. Since there are no horizontal forces on this disk, the acceleration in the x and y directions is also zero. $\ddot{x}_G = 0$, $\ddot{y}_G = 0$. Therefore, for steady precession, $\vec{a}_G = \vec{0}$. Looking at angular momentum balance about the center of gravity, we find that:

$$mrgtan\theta = \dot{\phi}\dot{\psi}I_1 + \dot{\phi}^2 sin\theta(I_1 + I_3) \tag{4}$$

We can solve for $\dot{\psi}$:

$$\dot{\psi} = \frac{mrgtan\theta - \dot{\phi}^2 sin\theta(I_1 + I_3)}{\dot{\phi}I_1}; \tag{5}$$

By defining a steer rate and tip angle, we can get the needed rolling rate for precession. See page 13, "Simulation 6" for an animation of such motion.

For the full derivation, see the appendix, A3.

Problem 3

Find the common subset of solutions to problems 1 and 2.

Answer

Equation of motion for rolling case:

$$mrgsin\theta = \dot{\phi}\dot{\psi}cos\theta(mr^2 + I_1) + \dot{\phi}^2 cos\theta sin\theta(I_1 + I_3 - mr^2) \tag{6}$$

Equation of motion for sliding case:

$$mrgsin\theta = \dot{\phi}\dot{\psi}cos\theta I_1 + \dot{\phi}^2 cos\theta sin\theta(I_1 + I_3) \tag{7}$$

When $\dot{\psi}$ approaches $\dot{\phi}$, and θ approaches $\frac{\pi}{2}$, the two equations both approach:

$$mrgsin\theta = \dot{\phi}^2\theta(2I_1 + I_3) \tag{8}$$

$\dot{\psi}$ approaching $\dot{\phi}$ corresponds to the precession radius approaching the radius of the disk. (See equation 2.)

Here is an animation of both cases, simultaneously: http://screencast.com/t/VD5RF0bj

The pink disk is the frictionless case, the green is with friction.
Initial conditions: r = 40, R = 44, $\theta = .95\frac{\pi}{2}$

Another common solution is when $\theta = 0$ and the only non-zero angular velocity is $\dot{\phi}$. In other words, a perfectly spun coin on a table. http://screencast.com/t/ENqfUTYST3T

In both of these cases, $\vec{a}_G$ equals or is close to $\vec{0}$, which is another condition for which these two cases exhibit the same motion. This can be shown mathematically by looking at the original angular momentum balance equations and writing out ${}^F\dot{\vec{H}}$ for each rather than using the transport theorem.

Problem 4a - General Motion of a Rolling Disk

Test as many cases as possible. Check against your solution to problem 1. Check for conservation of energy.

Answer

These simulations were made using Euler Angles around the basis vectors $\hat{e}_3$, $\hat{e}'_2$, and $\hat{e}''_1$, respectively.

To obtain equations of motion for the general motion of a rolling disk, we can use angular momentum balance about the ground contact point, c. Doing so gives us the following equation:

$$\vec{r}_{G/c} \times -mg\hat{e}_3 = \vec{r}_{G/c} \times m\vec{a}_G + \underline{\underline{I}}\dot{\vec{\omega}} + \vec{\omega} \times \underline{\underline{I}}\vec{\omega} \tag{9}$$

We know all of these terms except $\vec{a}_G$. To get this term, we can use what we know about rolling, and look at the velocity of the point c, relative to two different frames: the ground, and the disk. The velocity of point c relative to the ground is zero, while relative to the disk is nonzero:

$$\vec{0} = \vec{v}_G + \vec{v}_{c/G} \tag{10}$$

Solving for $\vec{v}_G$ and taking the time derivative of the equation gives us an expression for $\vec{a}_G$, which we can plug into our angular momentum balance equation. This gives us three equations, and three unknowns (the three Euler angles,) so we can plug these equations into Matlab to solve.

Simulation 1 - Spinning quarter - thickness = 1, omega0=[2 .2 .2]

View animation here: http://screencast.com/t/uFl7rUudl

Below is the difference in total energy and initial total energy over time. This shows that energy is conserved, as expected:

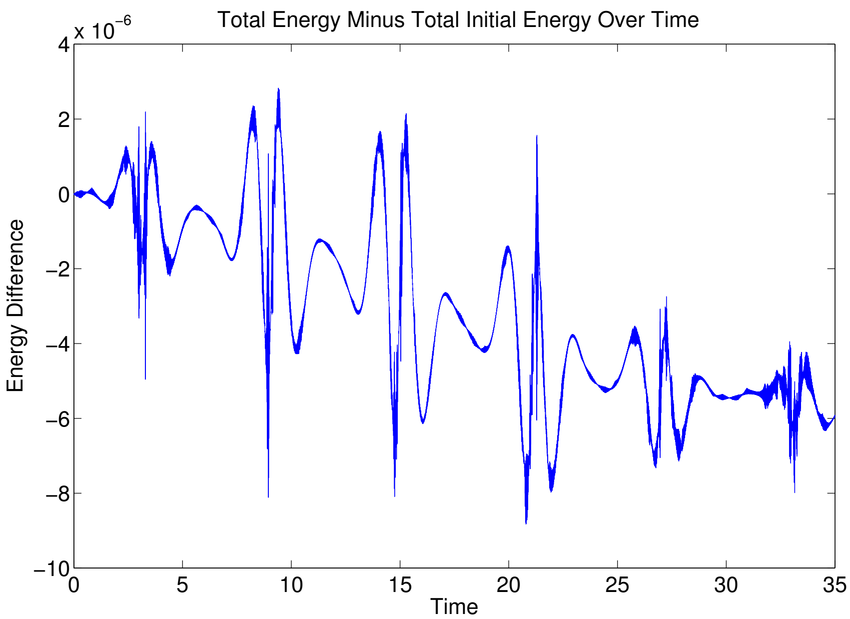


Figure 1: Difference over time between initial energy and current energy

Below is a plot of the x, y, and z positions of the center of mass over time:

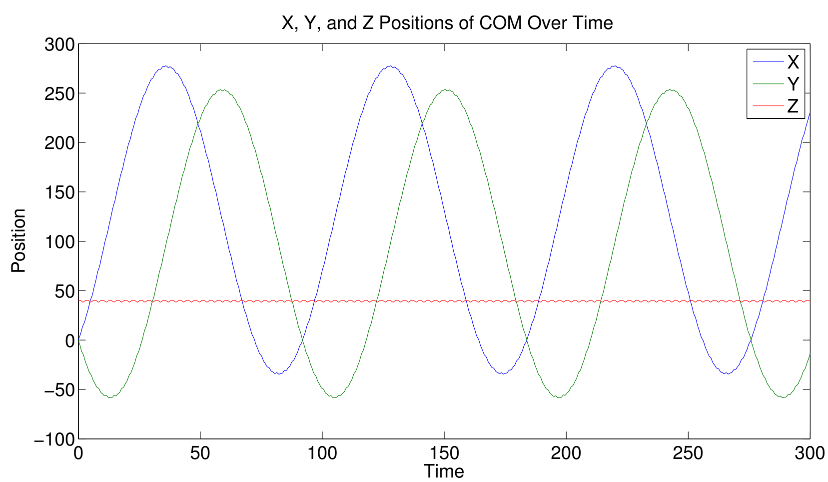


Figure 2: Motion of the Center of Mass

Simulation 2 - Rolling plates with small perturbation - thickness = 1, omega0=[-.2 -.2 (.55 or 1.5)]

Here is an animation of two disks whose initial conditions only differ in the initial roll angular velocity: http://screencast.com/t/2rCUnVoWhs

Below is the difference in total energy and initial total energy over time. This shows that energy is conserved, as expected:

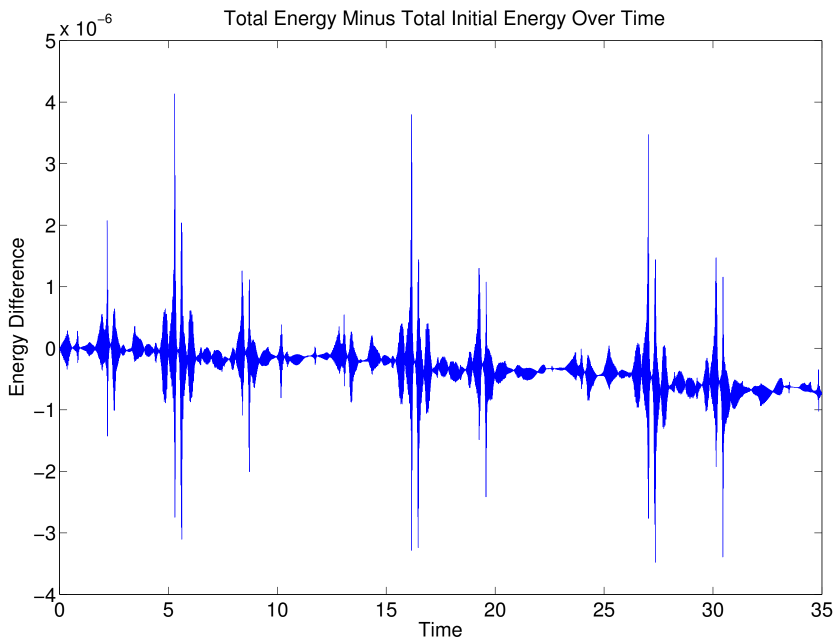


Figure 3: Difference over time between initial energy and current energy

Below are plots of the x, y, and z positions of the center of mass for each disk over time:

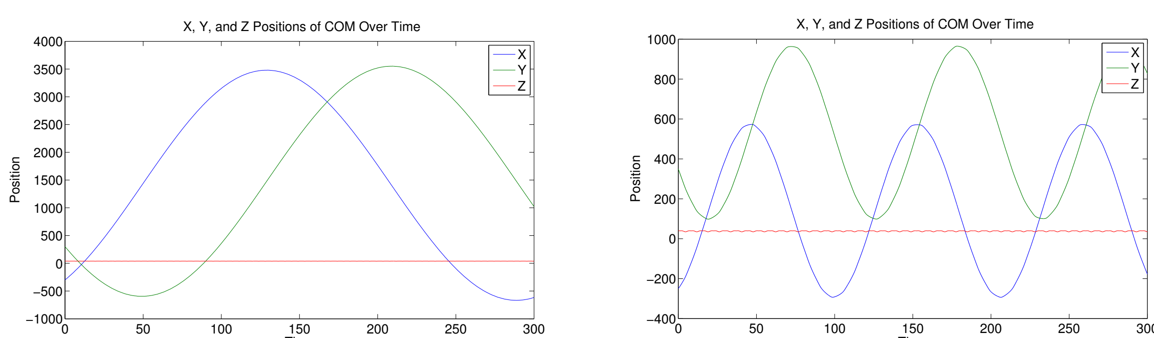


Figure 4: Motion of the Center of Mass

Simulation 3 - Simple Precession - thickness = 1, precession radius = 300, tip angle = -pi/4

View animation here: http://screencast.com/t/oi3LzxIo

Below is the difference in total energy and initial total energy over time for both simulations. This shows that energy is conserved, as expected:

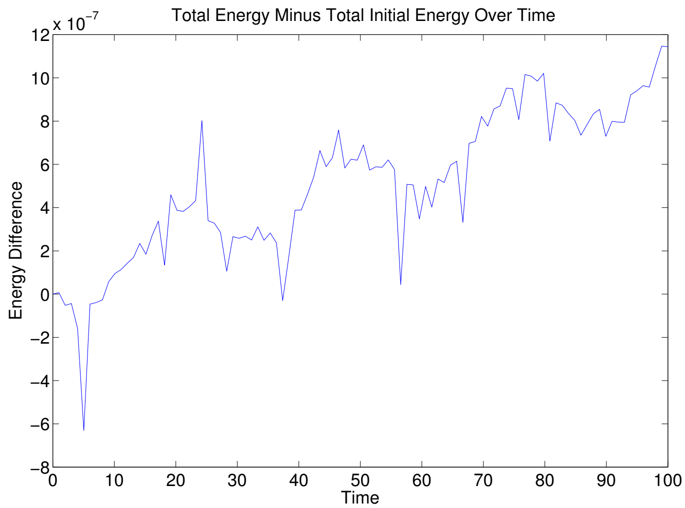


Figure 5: Difference over time between initial energy and current energy

Below is a plot of the x, y, and z positions of the center of mass over time:

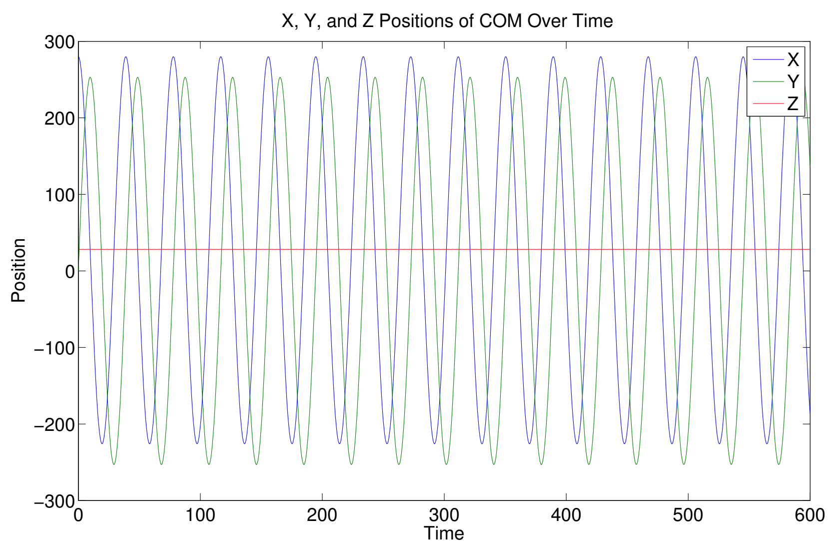


Figure 6: Motion of the Center of Mass

Problem 4b - General Motion of a Sliding Disk

Test as many cases as possible. Check against your solution to problem 2. Check for conservation of energy.

Answer

These simulations were made using Euler Angles around the basis vectors $\hat{e}_3$, $\hat{e}_2'$, and $\hat{e}_1''$, respectively.

To obtain equations of motion for the general motion of a sliding disk, we can use the same approach as with the rolling disk, by taking angular momentum balance about the ground contact point, c. In order to get our value for the acceleration of G this time, we'll need to use linear momentum balance:

$$N\hat{e}_3 - mg\hat{e}_3 = m\vec{a}_G \tag{11}$$

Since both the normal force and the force due to gravity are in the vertical direction, we know that $\vec{a}_G$ only has a vertical component:

$$\vec{a}_G = \ddot{z}_G\hat{e}_3 \tag{12}$$

We know that $z_G = rcos\theta$, so differentiating twice, we get that:

$$\vec{a}_G = -r(\ddot{\theta}sin\theta + \dot{\theta}^2cos\theta)\hat{e}_3 \tag{13}$$

We now have our three ODEs to plug into Matlab.

Simulation 4 - Spinning & translating frictionlessly - thickness = 1, omega0=[0 0 2], [2 0 0]

View animation here: http://screencast.com/t/2fDYGN5XegaB

Below is the difference in total energy and initial total energy over time for both simulations. This shows that energy is conserved, as expected:

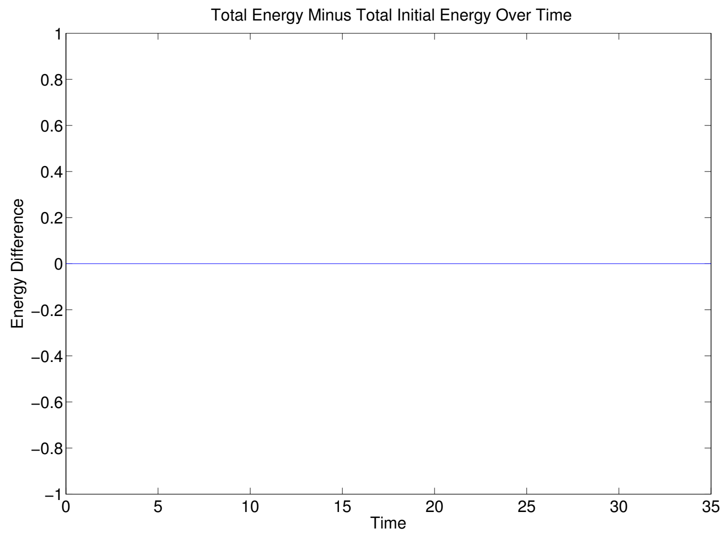


Figure 7: Difference over time between initial energy and current energy

Below is a plot of the angles of one of the disks over time. The plot for the other disk is identical, with the non-zero slope corresponding to psi instead of phi:

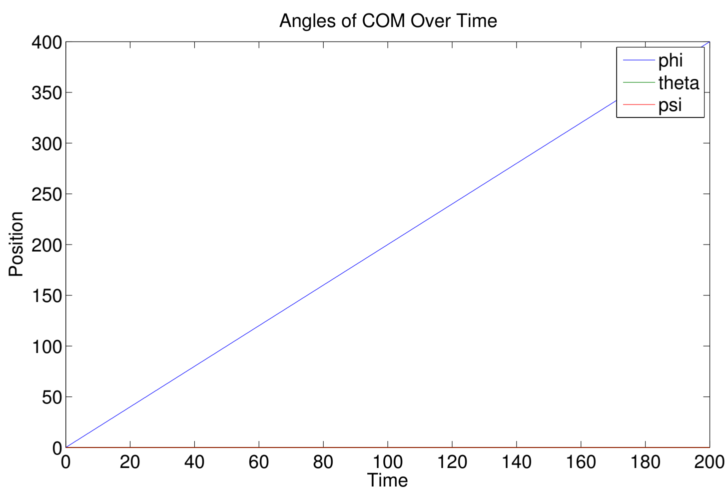


Figure 8: Motion of the Center of Mass

Simulation 5 - General frictionless rotation & translation - thickness = 1 , omega0=[1 1 1]

View animation here: http://screencast.com/t/Z5kUi8FLF

Below is the difference in total energy and initial total energy over time for both simulations. This shows that energy is conserved, as expected:

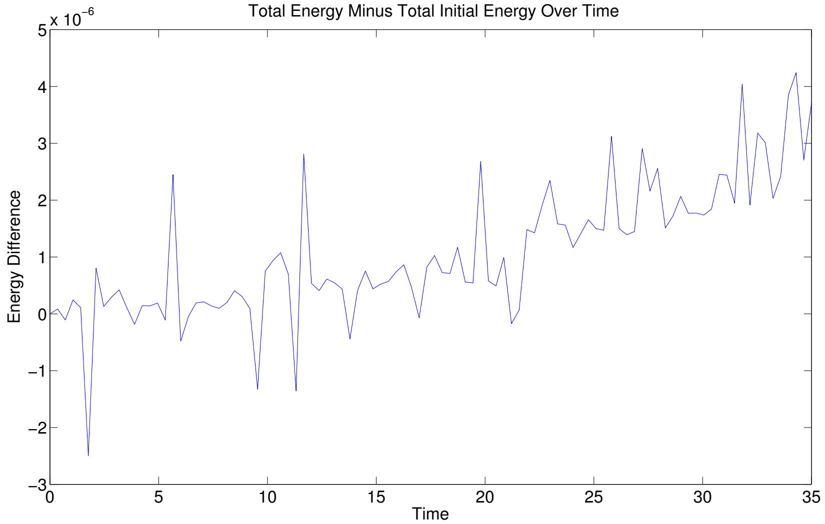


Figure 9: Difference over time between initial energy and current energy

Below is a plot of the height of the center of mass over time:

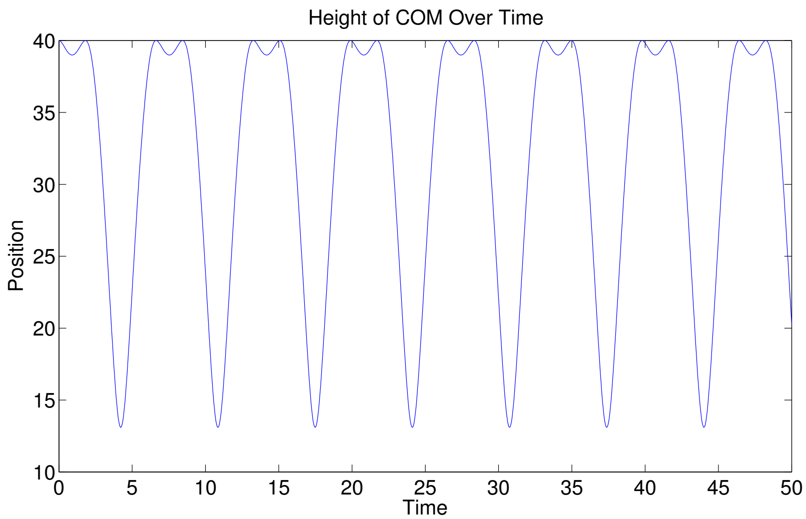


Figure 10: Motion of the Center of Mass

Simulation 6 - Steady Precession of a Sliding Disk - thickness = 1 , $\theta = \frac{\pi}{5}$, $\dot{\phi} = \frac{\pi}{3}$

View animation here: http://screencast.com/t/luqilnOlyLQ

Below is the difference in total energy and initial total energy over time. This shows that energy is conserved, as expected:

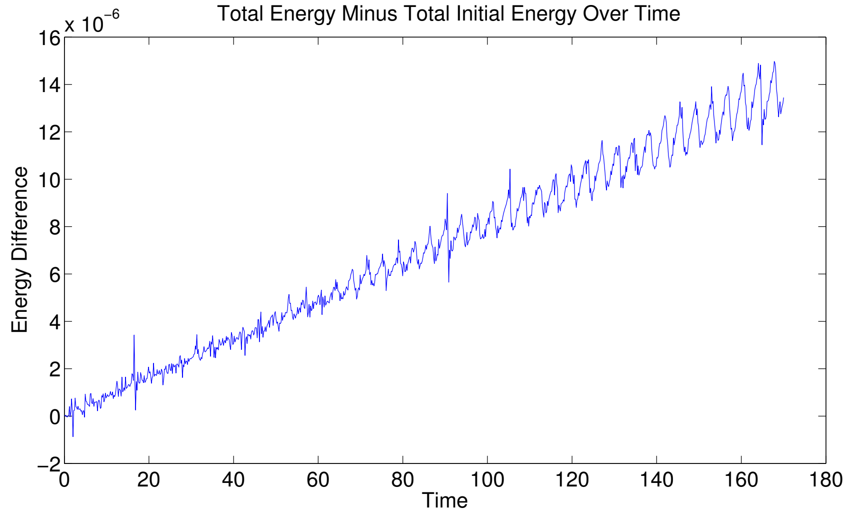


Figure 11: Difference over time between initial energy and current energy

Below is a plot of the height of the center of mass over time:

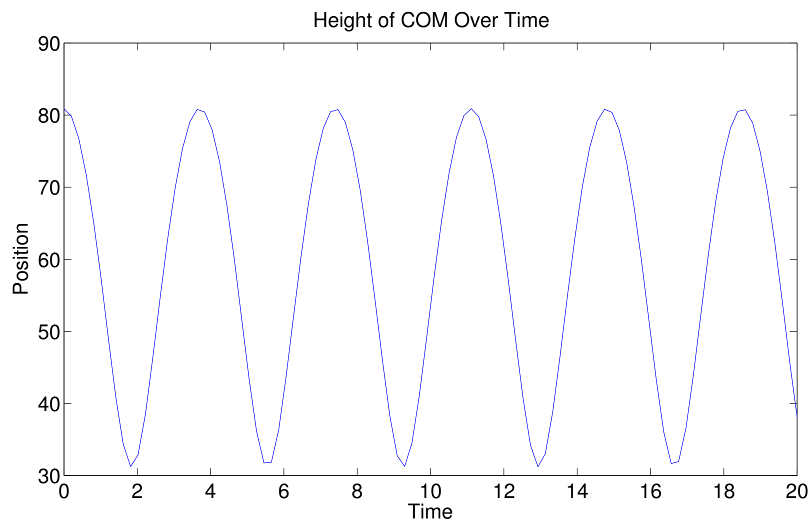


Figure 12: Motion of the Center of Mass

Rolling Simulation Code:

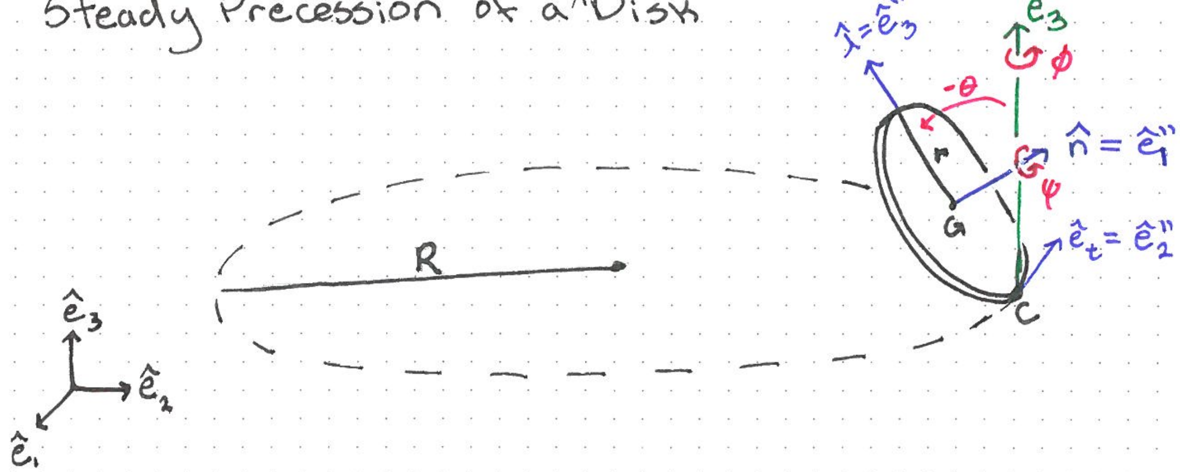
Rolling Derivation Code:

Rolling in Circles Code:

Sliding Simulation Code:

Sliding Derivation Code:

Steady Precession of a Rolling Disk



θ is constant

$\vec\omega = \dot\phi\,\hat e_3 + \dot\psi\,\hat e_1''$, $\dot\phi$ & $\dot\psi$ are constant

$\mathcal{F} \equiv$ fixed frame : $\hat e_1, \hat e_2, \hat e_3$

$\mathcal{T} \equiv$ tipped frame: $\hat e_1'', \hat e_2'', \hat e_3''$, $\vec\omega_{\mathcal{T}/\mathcal{F}} = \dot\phi\,\hat e_3$

$\boxed{A1}$

Angular Momentum Balance: $\sum \vec M_{/c} = {}^{\mathcal F}\dot{\vec H}_{/c}$

$$\sum \vec M_{/c} = \vec r_{G/c} \times -mg\,\hat e_3 = mrg\sin\theta\,\hat e_2$$

$${}^{\mathcal F}\dot{\vec H}_{/c} = \underbrace{\cancel{{}^{\mathcal T}\dot{\vec H}_{/c}}}_{\vec 0} + \vec\omega_{\mathcal T/\mathcal F} \times \vec H_{/c} \quad \text{(Transport Theorem)}$$

$\vec 0$ because $\vec\omega_{B/\mathcal T} = \dot\psi\,\hat e_1''$, which is constant in $\mathcal T$ and $\vec r_{G/c}$ is also constant in the ~~tipped~~ frame

$$= \dot\phi\,\hat e_3 \times \left(\vec r_{G/c} \times m\vec v_{G/c} + \underline{\underline{I}}^{G}\vec\omega\right)$$

$$= \dot\phi\,\hat e_3 \times \left[r\hat e_3'' \times m\left(\vec\omega \times r\hat e_3''\right) + \underline{\underline{I}}\,\vec\omega\right]$$

using BAC-CAB...

$$= \dot\phi\,\hat e_3 \times \left[m\left(\vec\omega[r^2\hat e_3''\cdot\hat e_3''] - r^2\hat e_3''[\hat e_3''\cdot\vec\omega]\right) + \underline{\underline{I}}\,\vec\omega\right]$$

$$= \dot\phi\,\hat e_3 \times \left[mr^2\left(\dot\phi\,\hat e_3 + \dot\psi\,\hat e_1'' - \dot\phi\cos\theta\,\hat e_3''\right) + \underline{\underline{I}}\,\vec\omega\right]$$

${}^F\dot{\vec{H}}_{/c} =$ con't...

$$= \dot{\phi}\,\hat{e}_3 \times \left[mr^2 \left(\dot{\phi}\,\hat{e}_3 + \dot{\varphi}\,\hat{e}_1'' - \dot{\phi}\cos\theta\,\hat{e}_3'' \right) + \underline{\underline{I}}\,\vec{\omega} \right]$$

$$= mr^2 \left[\dot{\phi}\dot{\varphi}\cos\theta\,\hat{e}_2'' - \dot{\phi}^2\cos\theta\sin\theta\,\hat{e}_2'' \right] + \underbrace{\dot{\phi}\,\hat{e}_3 \times (\underline{\underline{I}}\,\vec{\omega})}$$

$$\dot{\phi}\,\hat{e}_3 \times \left[I_1 \hat{e}_1''\hat{e}_1'' + I_2 \hat{e}_2''\hat{e}_2'' + I_3 \hat{e}_3''\hat{e}_3'' \right] \cdot \left(\dot{\phi}\,\hat{e}_3 + \dot{\varphi}\,\hat{e}_1'' \right)$$

$$= \dot{\phi}\,\hat{e}_3 \times \left(I_1 \dot{\phi}\sin\theta\,\hat{e}_1'' + \vec{0} + I_3 \dot{\phi}\cos\theta\,\hat{e}_3'' + I_1 \dot{\varphi}\,\hat{e}_1'' \right)$$

$$= I_1 \dot{\phi}^2 \sin\theta\cos\theta\,\hat{e}_2'' + I_3 \dot{\phi}^2 \cos\theta\sin\theta\,\hat{e}_2'' + I_1 \dot{\phi}\dot{\varphi}\cos\theta\,\hat{e}_2''$$

$$= \dot{\phi}\dot{\varphi}\cos\theta\,(mr^2 + I_1)\,\hat{e}_2'' + \dot{\phi}^2\cos\theta\sin\theta\,(-mr^2 + I_1 + I_3)\,\hat{e}_2''$$

$\left\{ \underline{AMB}_{/c} \right\} \cdot \hat{e}_2''$

1 $\Rightarrow$ $$\boxed{mrg\sin\theta = \dot{\phi}\dot{\varphi}\cos\theta\,(mr^2 + I_1) + \dot{\phi}^2\cos\theta\sin\theta\,(-mr^2 + I_1 + I_3)}$$

What is the relationship between $\dot{\phi}$ and $\dot{\varphi}$? $\boxed{A2}$

By inspection, we can see that as the disk goes around the circle, ϕ increases at the same rate.

Let s_c be the arc length of the circular path drawn by c.

Then $s_c = \phi R$.

The disk is also rolling along this path with no slip, so

$$s_c = -r\varphi.$$

Therefore

$$\phi R = -r\varphi$$

$$\Rightarrow \dot{\phi} R = -\dot{\varphi} r$$

$$\Rightarrow \dot{\phi} = -\dot{\varphi}\frac{r}{R}$$

Plugging this $\dot{\phi}$ into (1)...

$$mrg\sin\theta = -\dot{\varphi}^2 \frac{r}{R}\cos\theta(mr^2 + I_1) + \dot{\varphi}^2 \frac{r^2}{R}\cos\theta\sin\theta(mr^2 + I_1 + I_3)$$

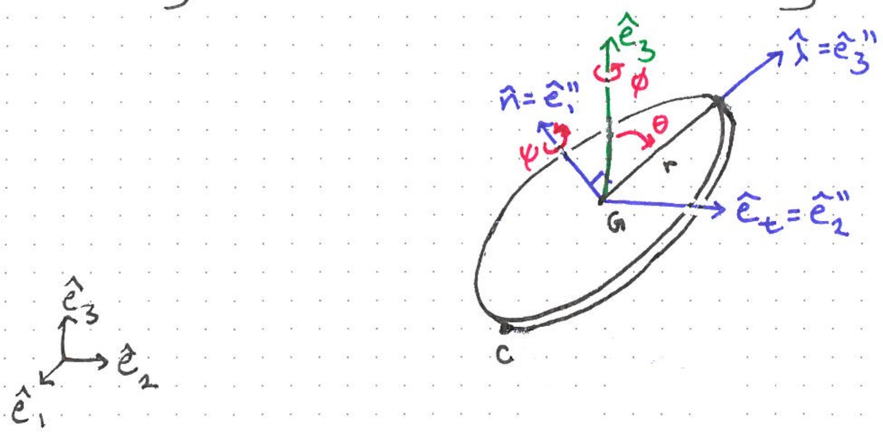
$$\Rightarrow \dot{\varphi}^2 = \frac{mgr\sin\theta}{\frac{r^2}{R^2}\cos\theta\sin\theta(mr^2 + I_1 + I_3) - \frac{r}{R}\cos\theta(mr^2 + I_1)}$$

$$= \frac{R^2 m \cancel{r} g \tan\theta}{-r^2\sin\theta\, m r^{\cancel{2}} + r^{\cancel{2}}\sin\theta\, I_1 + r^{\cancel{2}}\sin\theta\, I_3 - \cancel{r}Rmr^2 - \cancel{r}R\, I_1}$$

$$\boxed{\dot{\varphi}^2 = \frac{R^2 mg\tan\theta}{r\sin\theta I_1 + r\sin\theta I_3 - mr^3\sin\theta - mr^2R - RI_1}}$$

We can use this relationship to obtain initial conditions that will simulate a disk rolling in a circle of constant radius.

Steady Precession of a Sliding Disk



$\vec{\omega} = \dot{\phi}\hat{e}_3 + \dot{\psi}\hat{e}_1''$, $\dot{\phi}, \dot{\psi}, \theta$ are constant

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A3

Angular Momentum Balance: $\sum \vec{M}_{/G} = {}^F\dot{\vec{H}}_{/G}$

$$\sum \vec{M}_{/G} = \vec{r}_{C/G} \times \vec{N}$$

$$ {}^F\dot{\vec{H}}_{/G} = \underbrace{{}^T\dot{\vec{H}}_{/G}}_{\vec{0}\ \text{(see rolling case)}} + \vec{\omega}_{T/F} \times \vec{H}_{/G} \quad \text{(Transport Theorem)}$$

We know all of the above terms except $\vec{N}$. We can use linear momentum balance to find it.

$$\vec{N} + -mg\hat{e}_3 = m\vec{a}_G$$

$\vec{a}_G = \vec{0}$, because our tip angle, θ is constant, and there are no horizontal forces. Therefore:

$$\vec{N} = mg\hat{e}_3$$

Plugging this into AMB...

$$\vec{r}_{C/G} \times mg\hat{e}_3 = \dot{\phi}\hat{e}_3 \times \underline{\underline{I}}\vec{\omega}$$

which simplifies to:

(2) $$\boxed{mrg\sin\theta = \dot{\phi}\dot{\psi}\cos\theta I_1 + \dot{\phi}^2\cos\theta\sin\theta(I_1 + I_3)}$$