Well-posedness and uniform approximations of the solution of a boundary value problem for a singular integro-differential equation of the first kind

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Abstract—On a real segment, we consider a boundary value problem for a singular integro-differential equation of the first kind with the Cauchy kernel in the characteristic part. The well-posedness of this problem, established by the authors on a pair of specially selected spaces, allows to use approximate methods for its solving. We propose a general projection method, establish the conditions for its convergence in the chosen spaces and estimates the error of approximate solutions. As a result, uniform error estimates are obtained. A computational scheme of the wavelet collocation method is constructed, its theoretical substantiation is carried out, the results of a numerical experiment are presented on a model example.

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1. INTRODUCTION

Numerous applied problems lead to the need to solve various classes of singular integro-differential equations (s.i.d.e.). In elasticity theory, filtration theory, control theory and stable processes (see, for example, [2], [7]), singular integro-differential equations with the Cauchy kernel of the first kind on a real segment, with some boundary conditions, arise. By present time, there are a large number of works devoted to solving this problem. Since the exact solution to the problem can only be found in particular cases, various approximate methods are widely used. To substantiate them, studies are carried out in the Hoelder spaces as long as at the spaces of quadratically summable functions. The properties of the singular integral do not allow to use the spaces of continuous functions. However, the most relevant in practice are uniform error estimates.

In this article, the authors apply a specific approach to the study this problem, which they used earlier to solve integral equations of the first kind with a logarithmic singularity in the kernel [8] and with the Cauchy kernel on the interval [5]. Following [9], here we propose a special pair of spaces, based on the restriction of spaces of continuous functions; for such spaces, the well-posedness of the problem is established. This allows us, on the basis of the general theory of approximate methods of analysis [1], [4], to substantiate theoretically various approximative methods .

The paper proposes a general projection method with establishing conditions for its convergence and uniform error estimates. Since wavelet approximations have recently been of interest, in this paper, for the problem under consideration, a wavelet collocation method is proposed with subsequent justification in the selected spaces, its numerical implementation is carried out using a model example.

2. WELL-POSEDNESS OF THE PROBLEM

We consider a singular integro-differential equation of the first kind of the form

$$Ax \equiv \frac{1}{\pi}\int\limits_{-1}^{1}\frac{x'(\tau)}{\tau - t}d\tau + \frac{1}{\pi}\int\limits_{-1}^{1} h(\tau,t)x(\tau)d\tau = y(t), \quad -1 < t < 1, \tag{1}$$

with boundary conditions

$$x(-1) = x(1) = 0, \tag{2}$$

where $x(t)$ is desired function, $y(t)$ and $h(\tau, t)$ are given continuous functions.

The singular integral

$$I\phi \equiv I(\phi; t) = \frac{1}{\pi} \int_{-1}^{1} \frac{\phi(\tau)}{\tau - t} d\tau$$

is understood in the sense of Cauchy's principal value.

As the space of the required elements X, we choose the space of functions $x(t)$ satisfying the following conditions: $x(t)$ is continuously differentiable on $[-1, 1]$, satisfies condition (2), and $\rho I x'$ is continuous. We provide X with the norm

$$\|x\|_X = \|\rho x'\|_C + \|\rho I x'\|_C \tag{3}$$

where

$$\|x\|_C = \max_{-1 \le t \le 1} |x(t)|, \quad \rho(t) = \sqrt{1 - t^2}.$$

As the space of right-hand sides Y, we choose the space of continuous functions $y(t)$ for which $I(\rho y)$ is also a continuous function. We introduce the norm in Y by the relation

$$\|y\|_Y = \|\rho y\|_C + \|I(\rho y)\|_C. \tag{4}$$

The spaces X and Y are Banach. The completeness of the space Y was established in [5]. The completeness of the space X follows from the results obtained below.

Further we need some properties of the characteristic operator $G : X \to Y$ defined by the equality

$$Gx \equiv G(x; t) = \frac{1}{\pi} \int_{-1}^{1} \frac{x'(\tau)}{\tau - t} d\tau.$$

Lemma 1. *The operator $G : X \to Y$ is linear, bounded and*

$$\|G\|_{X \to Y} = 1.$$

Proof. We represent $x(t)$ in the form

$$x(t) = \sqrt{1 - t^2}\psi(t) = \sqrt{1 - t^2} \sum_{k=1}^{\infty} c_{k-1}^U(\psi) U_{k-1}(t), \tag{5}$$

where

$$U_k(t) = \frac{\sin(k+1)\arccos t}{\sqrt{1 - t^2}}, \quad k = 1, 2, ...$$

are the Chebyshev polynomials of the second kind, and

$$c_k^U(\psi) = \frac{2}{\pi} \int_{-1}^{1} \sqrt{1 - t^2}\psi(t) U_k(t) dt$$

are the Fourier-Chebyshev coefficients of the function $\psi(t)$.

Let us denote $\psi_k = c_{k-1}^U(\psi)$. Given that

$$x'(t) = \sum_{k=1}^{\infty} \psi_k (\sin k \arccos t)' = -\sum_{k=1}^{\infty} k\psi_k \frac{\cos k \arccos t}{\sqrt{1 - \tau^2}} \,,$$

we find

$$G(x;t) = \frac{1}{\pi}\int_{-1}^{+1}\frac{1}{\tau - t}\Big(-\sum_{k=1}^{\infty}k\psi_k\frac{\cos k\arccos\tau}{\sqrt{1-\tau^2}}\Big)d\tau = \sum_{k=1}^{\infty}k\psi_k\frac{1}{\pi}\int_{-1}^{+1}\frac{\cos k\arccos\tau}{\sqrt{1-\tau^2}(\tau - t)}d\tau.$$

Given the well-known relation [2], [7]

$$\frac{1}{\pi}\int_{-1}^{+1}\frac{T_k(\tau)}{\sqrt{1-\tau^2}(\tau - t)}d\tau = U_{k-1}(t), \tag{6}$$

where $T_k(t) = \cos k\arccos t$, $\quad k = 0, 1, 2, ...$ are the first kind Chebyshev polynomials of the $k-$th degree, we get

$$Gx = \sum_{k=1}^{\infty}k\psi_k U_{k-1}(t). \tag{7}$$

From the definition of norms (3), (4), representation (7), and the known relation

$$\frac{1}{\pi}\int_{-1}^{+1}\frac{\sqrt{1-\tau^2}U_{k-1}(\tau)}{(\tau - t)}d\tau = -T_k(t), \tag{8}$$

we find

$$\|Gx\|_Y = \|\rho Gx\|_C + \|I(\rho Gx)\|_C = \Big\|\rho(t)\sum_{k=1}^{\infty}k\psi_k U_{k-1}(t)\Big\|_C + \Big\|\sum_{k=1}^{\infty}k\psi_k I(\rho U_{k-1};t)\Big\|_C$$

$$= \Big\|\rho(t)\sum_{k=1}^{\infty}k\psi_k U_{k-1}(t)\Big\|_C + \Big\|\sum_{k=1}^{\infty}k\psi_k \cos k\arccos t\Big\|_C = \|\rho I x'\|_C + \|\rho x'\|_C.$$

Therefore,

$$\|Gx\|_Y = \|\rho x'\|_C + \|\rho I x'\|_C = \|x\|_X,$$

whence the statement to be proved follows.

Lemma 2. *The operator $G : X \to Y$ is continuously invertible and*

$$\|G^{-1}\| = 1, \quad G^{-1} : Y \to X.$$

Proof. Consider the characteristic equation

$$Gx = y, \quad x \in X, y \in Y, \tag{9}$$

under conditions (2). The solution to equation (9) will be sought in the form of (5), and the right-hand side $y(t)$ will be represented as Fourier series in the Chebyshev polynomials of the second kind. Then we have

$$\sum_{k=1}^{\infty}k\psi_k U_{k-1}(t) = \sum_{k=1}^{\infty}c_{k-1}^U(y)U_{k-1}(t).$$

Using the method of uncertain coefficients, we find

$$\psi_k = c_{k-1}^U(\psi) = \frac{c_{k-1}^U(y)}{k},$$

and therefore, the solution $x^*(t)$ of equation (9) exists for any right-hand side $y \in Y$ and can be represented in the form

$$x^*(t) = G^{-1}y = \sqrt{1-t^2}\sum_{k=1}^{\infty}\frac{c_{k-1}^U(y)}{k}\,U_{k-1}(t). \tag{10}$$

Using (3), (4), (10), and (6), we obtain

$$\|G^{-1}y\|_X = \|\rho(G^{-1}y)'\|_C + \|\rho I(G^{-1}y)\|_C = \left\|\sqrt{1-t^2}\sum_{k=1}^{\infty}\frac{c_{k-1}^U(y)}{k}(\sin k\arccos t)'\right\|_C$$

$$+\left\|\sqrt{1-t^2}\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\tau-t}\sum_{k=1}^{\infty}\frac{c_{k-1}^U(y)}{k}(\sin k\arccos\tau)'d\tau\right\|_C$$

$$=\left\|\sum_{k=1}^{\infty}c_{k-1}^U(y)T_k(t)\right\|_C + \left\|\sqrt{1-t^2}\sum_{k=1}^{\infty}c_{k-1}^U(y)U_{k-1}(t)\right\|_C = \|I(\rho y)\|_C + \|\rho y\|_C = \|y\|_Y,$$

whence the assertion of the lemma follows.

We write down problem (1)–(2) in the operator form

$$Ax \equiv Gx + Rx = y, \quad x \in X,\ y \in Y, \tag{11}$$

where

$$Rx \equiv \frac{1}{\pi}\int_{-1}^{1} h(\tau,t)x(\tau)d\tau.$$

By Lemma 2, equation (11) is reduced to an equation of the second kind

$$x + G^{-1}Rx = G^{-1}y, \quad x, G^{-1}y \in X.$$

Then, taking into account the known result of functional analysis [4], we have

Lemma 3. *If the linear operator $R : X \to Y$ satisfies the inequality*

$$q \equiv \|R\| < 1, \quad R : X \to Y,$$

then the operator $A \equiv G + R : X \to Y$ is continuously invertible and

$$\|A^{-1}\| \le (1-q)^{-1}, \quad A^{-1} : Y \to X.$$

Lemma 4. *Let the kernel $h(t,\tau)$ be a continuous function in both the variables. Then, for the operator $R : X \to Y$ the estimate*

$$\|R\|_{X\to Y} \le 2\|h\|_{C\otimes C}$$

holds.

Proof. Given the definitions of norms (3), (4) and formula (6), we have

$$\|Rx\|_Y = \left\|\sqrt{1-t^2}\frac{1}{\pi}\int_{-1}^{1}h(\tau,\tau)x(\tau)d\tau\right\|_C + \left\|\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}}{\tau-t}\frac{1}{\pi}\int_{-1}^{1}h(t,\tau_1)x(\tau_1)d\tau_1 d\tau\right\|_C$$

$$\le \frac{2}{\pi}\|x\|_C\|h\|_{C\otimes C} + \left\|\frac{1}{\pi}\int_{-1}^{1}h(\tau,\tau_1)x(\tau_1)\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}}{\tau-t}d\tau_1 d\tau\right\|_C = \frac{2}{\pi}\|x\|_C\|h\|_{C\otimes C}$$

$$+\left\|\frac{1}{\pi}\int_{-1}^{1} h(\tau,\tau_1)x(\tau_1)td\tau_1\right\|_C \le \frac{2}{\pi}\|x\|_C\|h\|_{C\otimes C}+\frac{2}{\pi}\|x\|_C\|h\|_{C\otimes C}\le \frac{4}{\pi}\|h\|_{C\otimes C}\|x\|_C.$$

Now we will show that

$$\|x\|_C \le \frac{\pi}{2}\|x\|_X. \tag{12}$$

Due to conditions (2),

$$|x(t)| = \left|\int_{-1}^{t} x'(\tau)d\tau\right| = \left|\int_{-1}^{t}\frac{\rho(\tau)x'(\tau)}{\rho(\tau)}d\tau\right| \le \|\rho x'\|_C\left|\int_{-1}^{t}\frac{d\tau}{\sqrt{1-\tau^2}}\right|$$

$$= \|\rho x'\|_C\left(\arcsin t+\frac{\pi}{2}\right) \le \left(\arcsin t+\frac{\pi}{2}\right)\|x\|_X, \quad t\in[-1,1].$$

Similarly,

$$|x(t)| = \left|\int_{t}^{1} x'(\tau)d\tau\right| \le \left(\frac{\pi}{2}-\arcsin t\right)\|x\|_X, \quad t\in[-1,1].$$

This implies (12).

Consequently, we have $\|Rx\|_Y \le 2\|h\|_{C\otimes C}\|x\|_X$ and, therefore, $\|R\|_{X\to Y}\le 2\|h\|_{C\otimes C}$.

Lemma 5. *Let the kernel of R have the form $h(t,\tau)=\frac{1}{\sqrt{1-\tau^2}(\tau-t)}$. Then*

$$\|R\|_{X\to Y} \le \frac{\pi+1}{2\pi}.$$

Proof. Using the definitions of norms (3) and (4), the formula for the inverse of a singular integral with the Cauchy kernel on the interval [3] and relation (6), we have

$$\|Rx\|_Y = \left\|\sqrt{1-t^2}\frac{1}{\pi}\int_{-1}^{1}\frac{x(\tau)}{(\tau-t)\sqrt{1-\tau^2}}d\tau\right\|_C + \left\|\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}}{\tau-t}\frac{1}{\pi}\int_{-1}^{1}\frac{x(\tau_1)d\tau_1}{(\tau_1-\tau)\sqrt{1-\tau_1^2}}d\tau\right\|_C$$

$$= \left\|\sqrt{1-t^2}\frac{1}{\pi}\int_{-1}^{1}\frac{(x(\tau)-x(t))d\tau}{(\tau-t)\sqrt{1-\tau^2}}\right\|_C + \frac{1}{\pi^2}\|x\|_C.$$

Applying the Lagrange formula and taking into account the estimate (12), we obtain

$$\|Rx\|_Y \le \left\|\sqrt{1-t^2}\frac{1}{\pi}\int_{-1}^{1}\frac{x'(\xi)(\tau-t)d\tau}{(\tau-t)\sqrt{1-\tau^2}}\right\|_C + \frac{1}{2\pi}\|x\|_X \le \frac{1}{2}\|\rho x'\|_C+\frac{1}{2\pi}\|x\|_X \le \frac{\pi+1}{2\pi}\|x\|_X,$$

$\xi\in(-1,1)$. Whence the assertion of the lemma comes.

If the operator $R: X\to Y$ is completely continuous, then the operator $G^{-1}R$ is also completely continuous. Then from the Riesz-Schauder theory [4] it follows

Theorem 1. *Let $R: X\to Y$ be a completely continuous operator and the homogeneous problem corresponding to (1), (2) have only a trivial solution. Then the operator $A=G+R: X\to Y$ is continuously invertible.*

3. GENERAL PROJECTION METHOD

Let $X_n \subset X$ be a subspace of elements of the form

$$x_n(t) = \sqrt{1-t^2}\sum_{k=1}^{n}\alpha_k U_{k-1}(t) = \sum_{k=1}^{n}\alpha_k \sin(k\arccos t), \tag{13}$$

where $\alpha_k \in \mathbf{R}$,$Y_n = \mathbf{H}_{n-1} \subset Y$; here $\mathbf{H}_n$ is the space of algebraic polynomials of degree at most n.

An approximate solution to problem (1)–(2) will be sought in the form of element (13), which we will define as a solution to the equation

$$A_n x_n \equiv P_n G x_n + P_n R x_n = P_n y, \quad x_n \in X_n, P_n y \in Y_n, \tag{14}$$

where $P_n : Y \to Y_n$ is a linear projection operator.

As shown above, $Gx_n = \sum\limits_{k=1}^{n} k\alpha_k U_{k-1}(t)$, $P_n^2 = P_n$, so $P_n G x_n = G x_n$ for any $x_n \in X_n$.

Then equation (14) takes the form

$$A_n x_n \equiv G x_n + P_n R x_n = P_n y, \quad x_n \in X_n, P_n y \in Y_n. \tag{15}$$

This equation, written in the operator form, is a system of linear algebraic equations (SLAE) with respect to unknown coefficients $\alpha_1, ..., \alpha_n \in \mathbf{R}$.

Theorem 2. *Assume that the conditions of Theorem 1 are satisfied. If*

$$q_n = \|A^{-1}\| \|R - P_n R\|_{X_n \to Y} < 1,$$

then the operators $A_n : X_n \to Y_n$ are also linearly invertible,

$$\|A_n^{-1}\| \le \frac{\|A^{-1}\|}{1-q_n},$$

and, for the approximate solution error, the following estimate holds:

$$\|x^* - x_n^*\| \le \frac{\|A^{-1}\|}{1-q_n}\Big[\|y - P_n y\| + q_n\|y\|\Big]$$

Here, $x^* = A^{-1}y$ denotes the exact solution of the boundary value problem (1)–(2), $x_n^* = A_n^{-1}P_n y$ is the exact solution of the approximating equation (15).

The statement of the theorem follows from the results obtained above and [1, ch.1, thrm. 14].

Remark. In addition, under the conditions of Theorem 2, the error can be estimated by the following inequalities

$$\|x^* - x_n^*\| \le \|E - A_n^{-1}P_n R\|_{X\to X}\|Gx^* - P_n G x^*\|_Y;$$

$$\|x^* - x_n^*\| \le \|(G + P_n R)^{-1}\|_{Y\to X}\|Gx^* - P_n G x^*\|_Y.$$

Theorem 3. *Let the conditions be satisfied:*

a) s.i.d.e. (1) with boundary conditions (2) is uniquely solvable in the space X for any right-hand side $y \in Y$;

b) the kernel of $h(t,\tau)$ is such that the operator $R : X \to Y$ is completely continuous;

c) the operators $P_n^2 = P_n$, $P_n \to E$ are strong in Y, where $E : Y \to Y$ is the unit operator.

Then, starting from a certain $n \in \mathbf{N}$, the approximating equations (15) are also uniquely solvable, and approximate solutions x_n^* converge to the exact one, x^*, in the space X with the velocity

$$\|x^* - x_n^*\|_X = O\Big\{\|y - P_n y\|_Y + \|h - P_n^t h\|_{Y,C}\Big\};$$

here the notation P_n^t means that the operator P_n is applied to $h(t,\tau)$ in the variable t.

Proof. We estimate the proximity of the exact operator A and its approximating operators A_n. For any $x_n \in \mathbf{X}_n$ we find

$$\|Ax_n - A_n x_n\|_Y = \|Rx_n - P_n R x_n\|_Y = \left\| \frac{1}{\pi} \int_{-1}^{1} \rho(\tau)[h(t,\tau) - P_n^t(h(t,\tau))] x_n(\tau) d\tau \right\|_Y$$

$$\leq \|h - P_n^t h\|_{Y \otimes C} \|x_n\|_C \leq \frac{\pi}{2} \|h - P_n^t h\|_{Y \otimes C} \|x_n\|_X,$$

here we denote by $\|h\|_{Y \otimes C}$ the norm of the function $h(t,\tau)$ in the first variable in the space Y and in the second variable in the space $\mathbf{C}$.

Then, taking into consideration the conditions of the theorem, we have

$$\|A - A_n\|_{X_n \to Y} \to 0, \quad n \to \infty.$$

Therefore, by virtue of [1, ch.1,thrm.7], for all $n \in \mathbf{N}$, for which the inequality

$$\|A^{-1}\|_{Y \to X} \|A - A_n\|_{X_n \to Y} < 1,$$

holds, equation (15) is uniquely solvable. In addition, by hypothesis of the theorem, we have

$$\|y - P_n y\|_Y \to 0, \quad n \to \infty,$$

therefore, the approximate solutions $x_n^*(t)$ converge to the exact one, $x^*(t)$, in the norm of the space X with the velocity

$$\|x^* - x_n^*\|_X = O\Big\{ \|y - P_n y\|_Y + \|h - P_n^t h\|_{Y \otimes C} \Big\}.$$

Corollary. *Under the conditions of Theorem 3, the approximate solutions x_n^* converge to the exact one, x^*, uniformly with speed*

$$\|x^* - x_n^*\|_Y = O\Big\{ \|y - P_n y\|_C + \|h - P_n^t h\|_{Y \otimes C} \Big\};$$

$$\|\rho(x^* - x_n^*)'\|_C = O\Big\{ \|y - P_n y\|_Y + \|h - P_n^t h\|_{Y \otimes C} \Big\}.$$

Taking into account the estimates of the approximation of a function, obtained in [5], by segments of the Fourier–Chebyshev series and Lagrange interpolation polynomials in the space Y for various classes of functions, we can state that the methods of orthogonal polynomials, collocations, and subdomains are correctly applied to the problem (1)–(2) under consideration with appropriate theoretical justification and obtaining constructive error estimates that take into account the structural properties of the source data. For this, in the general projection method, the Fourier–Chebyshev operator, the Lagrange operator and subdomains, respectively, should be considered as the operator P_n.

4. WAVELET APPROXIMATIONS

Recently, approximate methods based on wavelet approximation have been of interest.

An approximate solution of equation (1) will be sought in the form

$$x_m(t) = a_0 \varphi_{0,0}(t) + a_1 \varphi_{0,1}(t) + \sum_{j=0}^{m-1} \sum_{k=0}^{2^j - 1} b_{j,k} \psi_{j,k}(t)$$

where

$$\varphi_{m,k}(t) = \sqrt{1-t^2} \sum_{j=0}^{2^m} U_j(t) U_j\left(t_k^{2^m+1}\right) \frac{2\,|\sin[\pi(k+1)/(2^m+2)]|}{\sqrt{\pi(2^m+2)}}, \quad m = 0, 1, ..., \ 0 \leq k \leq 2^m,$$

$$\psi_{m,k}(t)=\sqrt{1-t^2}\sum_{j=2^m+1}^{2^{m+1}}U_j(t)U_j\left(t_k^{2^m}\right)\frac{2\left|\sin[\pi(k+1)/(2^m+1)]\right|}{\sqrt{\pi(2^m+1)}},\quad m=1,2,...,\ 0\le k\le 2^m-1,$$

are the scaling function and the Chebyshev wavelet function of the second kind, respectively [10], t_k^n $(0\le k\le n-1$, are zeros of the Chebyshev polynomial of the second kind $U_n(t)$. It is obvious that

$$x_m(t)=a_0\frac{\sin(\arccos t)+\sin(2\arccos t)}{\sqrt{\pi}}+a_1\frac{\sin(\arccos t)-\sin(2\arccos t)}{\sqrt{\pi}}$$
$$+\sum_{j=0}^{m-1}\sum_{k=0}^{2^j-1}\sum_{i=2^m+1}^{2^{m+1}}b_{j,k}\sin[(i+1)\arccos t]\,U_i\left(t_k^{2^m}\right)\frac{2\left|\sin[\pi(k+1)/(2^m+1)]\right|}{\sqrt{\pi(2^m+1)}},\quad(16)$$

$$x_m'(t)=-a_0\frac{T_1(t)+2T_2(t)}{\sqrt{\pi}\sqrt{1-t^2}}-a_1\frac{T_1(t)-2T_2(t)}{\sqrt{\pi}\sqrt{1-t^2}}$$
$$-\sum_{j=0}^{m-1}\sum_{k=0}^{2^j-1}\sum_{i=2^m+1}^{2^{m+1}}b_{j,k}\frac{(i+1)T_{i+1}(t)}{\sqrt{1-t^2}}U_i\left(t_k^{2^m}\right)\frac{2\left|\sin[\pi(k+1)/(2^m+1)]\right|}{\sqrt{\pi(2^m+1)}}.\quad(17)$$

We substitute (16) and (17) in the s.i.d.e. (1). We will search unknown coefficients a_0, a_1, $b_{j,k}$ $(0\le j\le m-1,\ 0\le k\le 2^j-1)$, from the condition that the residuals at the collocation nodes to be equal to zero:

$$t_k^{2^m+1}=\cos\frac{\pi\,(k+1)}{2^m+2},\quad m=1,2,...,\quad 0\le k\le 2^m.\quad(18)$$

Given (6), (17), we obtain SLAE

$$-\frac{a_0}{\sqrt{\pi}}\left(U_0\left(t_k^{2^m+1}\right)+2U_1\left(t_k^{2^m+1}\right)\right)-\frac{a_1}{\sqrt{\pi}}\left(U_0\left(t_k^{2^m+1}\right)-2U_1\left(t_k^{2^m+1}\right)\right)$$
$$-\sum_{j=0}^{m-1}\sum_{k=0}^{2^j-1}\sum_{i=2^j+1}^{2^{j+1}}b_{j,k}(i+1)U_i\left(t_k^{2^m+1}\right)U_i\left(t_k^{2^j}\right)\frac{2\left|\sin[\pi(k+1)/(2^j+1)]\right|}{\sqrt{\pi(2^j+1)}}+\gamma_{km}=f\left(t_k^{2^m+1}\right),\quad(19)$$

$0\le k\le 2^m$, where $\gamma_{km}=R\left(x_m;t_k^{2^m+1}\right)$.

To prove the unique solvability of SLAE (19), which is a computational scheme of the wavelet collocation method, as well as the convergence of approximate solutions to the exact one, we shoud use the approach given in the previous section. But as a subspace $X_n\subset X$ we take the set of elements of the form (16), while $Y_n=H_{2^m}\subset Y$, and $P_n=L_{2^m}$ where L_{2^m} is the Lagrange operator, associating the function $\phi\in C[-1,1]$ with its Lagrange interpolation polynomial over nodes (18).

We denote by $H_\omega^r=W_\omega^r[-1,1]$ the set of functions having a continuous derivative of $r-$th order whose modulus of continuity does not exceed a given modulus of continuity $\omega(\delta)$, $r\ge 0$, $\delta\in(0,2]$, and satisfies for $r=0$ the additional condition $\ln m\,\omega\left(\frac{1}{m}\right)\to 0$, $m\to\infty$.

Theorem 4. *Let the following conditions be satisfied:*

a) equation (1) with boundary conditions (2) has a unique solution $x^*\in X$ for any right-hand side $y\in Y$;

b) the function $y(t)\in H_\omega^r$, and the kernel $h(t,\tau)$ belongs to class H_ω^r with respect to the variable t uniformly with respect to the variable τ.

Then, starting with some $m\in\mathbf{N}$, the system of the wavelet collocation method (19) has a unique solution a_0^*, a_1^*, $b_{j,k}^*$ $(0\le j\le m-1,\ 0\le k\le 2^j-1)$, and the approximate solutions x_m^* converge to the exact solution, x^*, in the space X with the velocity

$$\|x^*-x_m^*\|_X=O\left\{\frac{\ln(2^m+1)}{(2^m+1)^r}\omega\left(\frac{1}{2^m+1}\right)\right\},\quad r\ge 0.$$

5. NUMERICAL EXPERIMENT

Consider a singular integro-differential equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{x'(\tau)}{\tau-t}d\tau+\frac{1}{\pi}\int_{-1}^{1}x(\tau)d\tau=\frac{2}{3\pi}\left(3t\ln\left|\frac{1+t}{1-t}\right|-4\right),\quad |t|<1,\quad x(-1)=x(1)=0,$$

whose exact solution is $x^*(t)=1-t^2$.

Let $m=1$. SLAE of the wavelet collocation method (19) have the form

$$-\frac{a_0}{\sqrt{\pi}}\left(1+2U_1\left(t_k^3\right)\right)+\frac{a_1}{\sqrt{\pi}}\left(2U_1\left(t_k^3\right)-1\right)+3b_{0,0}\sqrt{\frac{2}{\pi}}U_2\left(t_k^3\right)+\frac{3a_0-a_1}{2\sqrt{\pi}}=f\left(t_k^3\right),$$

$$t_k^3=\cos\frac{\pi\left(k+1\right)}{4},\quad 0\le k\le 2.$$

We find its solution using the Wolfram Mathematica package. By formulas (16), the approximate solution can be written as

$$x_1^*(t)=1.036\sqrt{1-t^2}-0.529t^2\sqrt{1-t^2}.$$

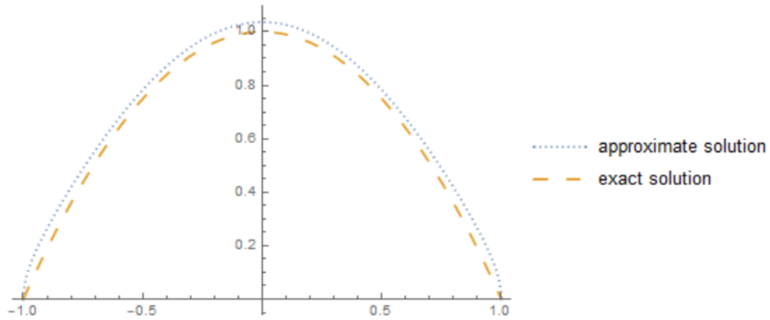

Figure 1. Graphs of approximate $x_1^*(t)$ and exact $x^*(t)$ solutions, $m=1$

Let $m=2$. Using the computational scheme of the wavelet collocation method (16)–(19) and the Wolfram Mathematica package, we find that the approximate solution:

$$x_2^*(t)=1.012\sqrt{1-t^2}-0.484t^2\sqrt{1-t^2}-0.215t^4\sqrt{1-t^2}\,.$$

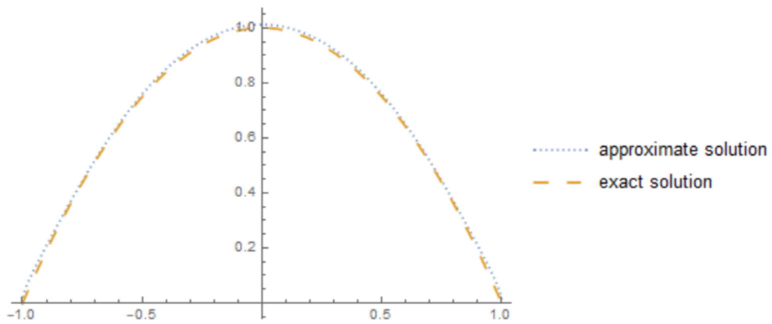

Figure 2. Graphs of approximate $x_2^*(t)$ and exact $x^*(t)$ solutions, $m=2$

REFERENCES

1. B. G. Gabdulkhaev, *Direct methods for solving singular integral equations of the first kind* (Kazan University Press, Kazan, 1980) [in Russian].
2. B. G. Gabdulkhaev, *Optimal approximations of solutions of linear problems* (Kazan University Press, Kazan, 1994) [in Russian].
3. F. D. Gakhov, *Boundary value problems* (Nauka, Moscow, 1977) [in Russian]
4. L. V. Kantorovich and G. P. Akilov, *Functional analysis* (BHV-Petersburg, St. Petersburg, 2004) [in Russian].
5. L. E. Khayrullina, *Uniform convergence of approximate solutions of a singular integral equation of the first kind with a Cauchy kernel* (Kazan, 2011) [in Russian].
6. L. E. Khayrullina and A. V. Ozhegova, "Uniform Wavelet-Approximation of Singular Integral Solutions," Lobachevskii J. Math. **39**(6), 781–786 (2018).
7. I. K. Lifanov, *The method of singular integral equations and numerical experiment* (Janus, Moscow, 1995) [in Russian].
8. A. V. Ozhegova, *Uniform approximations of solutions of weakly singular integral equations of the first kind* (Kazan, 1996) [in Russian].
9. A. V. Ozhegova and L. E. Khayrullina, "On a uniform approximation of the solution of a singular integro-differential equation of the I kind," Vestnik of Samara University. Natural Science Series. **6**(107), 54–60 (2013) [in Russian].
10. M. S. Sultanakhmedov, "Approximative properties of Chebyshev wavelets of the second kind," Vladikavkaz Math. J. **3**(17), 56–64 (2015).