T_0 - Closure Operators and Pre-Orders

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Received June 06, 2017

Abstract—It is well known that the lattice of closed subsets of any topological space is isomorphic to that of a T_0-topological space. This result is extended to lattices of closed subsets with respect to arbitrary closure operator on a set. Also, we establish a one-to-one correspondence between closure operators which are both algebraic and topological on a given set X and pre-orders on X and prove that this correspondence induces a one-to-one correspondence between topological algebraic T_0-closure operators on X and partial orders on X.

2010 Mathematical Subject Classification: 06A15,06F30, 54H12.

Keywords and phrases: *closure operator, Moore class, algebraic lattice, T_0-closure operator, pre-order.*

1. INTRODUCTION AND PRELIMINARIES

A partially ordered set (poset) is a pair $(X, \leq)$, where X is a non empty set and $\leq$ is a partial order (a reflexive, transitive and antisymmetric binary relation) on X. For any subset A of X and $x \in X$, x is called a lower bound (upper bound) of A if $x \leq a$ ($a \leq x$ respectively) for all $a \in A$. A poset $(X, \leq)$ is called a lattice if every nonempty finite subset of X has greatest lower bound (or glb or infimum) and least upper bound (or lub or supremum) in X. If $(X, \leq)$ is a lattice and, for any $a, b \in X$, if we define $a \wedge b =$ infimum $\{a, b\}$ and $a \vee b =$ supremum $\{a, b\}$, then $\wedge$ and $\vee$ are binary operations on X which are commutative, associative and idempotent and satisfy the absorption laws $a \wedge (a \vee b) = a = a \vee (a \wedge b)$. Conversely, any algebraic system $(X, \wedge, \vee)$ satisfying the above properties becomes a lattice in which the partial order is defined by $a \leq b \Longleftrightarrow a = a \wedge b \Longleftrightarrow a \vee b = b$. A lattice $(X, \wedge, \vee)$ is called distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in X$ (equivalently $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in X$). A lattice $(X, \wedge, \vee)$ is called a bounded lattice if it has the smallest element 0 and largest element 1; that is, there are elements 0 and 1 in X, such that $0 \leq x \leq 1$ for all $x \in X$.

A partially ordered set in which every subset has infimum and supremum is called a complete lattice. If $(L, \leq)$ is a complete lattice and $X \subseteq L$, we write inf X or $\wedge X$ or $\bigwedge\limits_{x \in X} x$ for the infimum of X and sup X or $\vee X$ or $\bigvee\limits_{x \in X} x$ for the supremum of X. If $X = \{x_1, x_2, \cdots, x_n\}$ is a finite subset, then we write $\bigwedge\limits_{i=1}^{n} x_i$ or $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ for inf X and $\bigvee\limits_{i=1}^{n} x_i$ or $x_1 \vee x_2 \vee \cdots \vee x_n$ for sup X. Any complete lattice has the smallest element and the greatest element which are denoted by 0 and 1 respectively. Logically, the infimum and supremum of the empty set are 1 and 0 respectively. An element $a \neq 0$ in a complete lattice L is called compact if, for any $A \subseteq L$, $a \leq \sup A \Longrightarrow a \leq \sup F$ for some finite $F \subseteq A$. A complete lattice in which every element is the supremum of a set of compact elements is called an algebraic lattice.

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It is well known that any class of subsets of a set X which is closed under arbitrary intersections and finite unions gives a topology on X with respect to which the members of the class are precisely closed sets. In other words, a Moore class on X which is closed under finite unions is precisely the class of closed subsets of X with respect to a topology on X. Such Moore classes can be named as topological Moore classes. Also, a closure operator c on a set X satisfying the additional properties $c(\phi) = \phi$ and $c(A \cup B) = c(A) \cup c(B)$ induces a topology on X with respect which $c(A)$ is the closure of A, for any subset A of X. For this reason, such closure operators can be called as topological closure operators. Further the class of closed subsets of a topological space forms a complete lattice. For elementary properties of posets, lattices and topological spaces we refer to [1–5].

2. T_0-CLOSURE OPERATORS

It is proved in [6] that the lattice of closed subsets of any topological space is isomorphic to that of a T_0-topological space. In this section, we extend this to any given closure operator on a given set. First, let us recall that an extensive, idempotent and inclusion preserving mapping of the power set $\mathscr{P}(X)$ into itself is called a closure operator on a set X.

Definition 1. *Let c be a closure operator on a set X. For any $x \in X$, we write $c(x)$ for $c(\{x\})$; c is called a T_0-closure operator on X if, for any elements x and y in X, $c(x) = c(y) \Longrightarrow x = y$.*

Example 1. *Recall that a topological space X is called a T_0-space if, for any $x \neq y \in X$, there exists an open set containing x and not containing y or vice-versa. It can be easily proved that a topological space X is a T_0-space if and only if, for any $x, y \in X$, $\overline{x} = \overline{y} \Longrightarrow x = y$ and therefore the topological closure operator c defined by $c(A) = \overline{A}$, the closure of A is a T_0-closure operator on X.*

In the following, we prove that, for any closure operator c on a set X, there exists a T_0-closure operator $\overline{c}$ defined on a suitable set Y such that the Moore classes $\mathscr{M}_c$ and $\mathscr{M}_{\overline{c}}$ are isomorphic, where

$$\mathscr{M}_c = \{A \subseteq X \mid c(A) = A\}.$$

First we have the following.

Definition 2. *Let c be a closure operator on a set X. Define a relation θ_c on X by $\theta_c = \{(x, y) \in X \times X \mid c(x) = c(y)\}$. Then θ_c is an equivalence relation on X. Let us consider the quotient set $X_c = \{\theta_c(x) \mid x \in X\}$, where $\theta_c(x)$ is the equivalence class containing x; that is, $\theta_c(x) = \{y \in X \mid c(x) = c(y)\}$. Let $p_c : X \longrightarrow X_c$ be the natural map defined by $p_c(x) = \theta_c(x)$ for all $x \in X$. clearly p_c is a surjection.*

Definition 3. *Let X_c be the set constructed above, corresponding to a given closure operator c on a set X. Define*

$$\overline{c} : \mathscr{P}(X_c) \longrightarrow \mathscr{P}(X_c) \text{ by } \overline{c}(A) = p_c(c(p_c^{-1}(A))) = \{\theta_c(x) \mid x \in c(p_c^{-1}(A))\}$$

for any $A \subseteq X_c$, where $p_c : X \longrightarrow X_c$ is the natural map.

Theorem 1. *For any closure operator c on a set X, $\overline{c}$ is a T_0-closure operator on X_c.*

Proof. First we observe the following:

$$p_c^{-1}(p_c(c(Y))) = c(Y) \quad \text{for all} \quad Y \subseteq X. \tag{1}$$

Clearly $Z \subseteq p_c^{-1}(p_c(Z))$ for all $Z \subseteq X$. Now, let $Y \subseteq X$. Then

$$x \in p_c^{-1}(p_c(c(Y))) \Longrightarrow p_c(x) = p_c(a) \text{ for some } a \in c(Y)$$

$$\Longrightarrow \theta_c(x) = \theta_c(a),\ \ a \in c(Y) \Longrightarrow x \in c(x) = c(a) \subseteq c(c(Y)) = c(Y).$$

Thus $p_c^{-1}(p_c(c(Y))) = c(Y)$ and hence (1) is proved.

Now, let $A \subseteq X_c$. Then, since p_c is a surjection, $A = p_c(p_c^{-1}(A)) \subseteq p_c(c(p_c^{-1}(A))) = \overline{c}(A)$. Therefore, $\overline{c}$ is extensive. Also,

$$\overline{c}(\overline{c}(A)) = p_c c p_c^{-1}(p_c(c(p_c^{-1}(A)))) = p_c c(c(p_c^{-1}(A))) \text{ (by (1))} = p_c(c(p_c^{-1}(A))) = \overline{c}(A).$$

Therefore $\overline{c}$ is idempotent. Finally, let $A \subseteq B \subseteq X_c$. Then

$$\overline{c}(A) = p_c(c(p_c^{-1}(A))) \subseteq p_c(c(p_c^{-1}(B))) = \overline{c}(B).$$

Therefore $\overline{c}$ is inclusion preserving. Thus $\overline{c}$ is a closure operator on X_c.

To prove that $\overline{c}$ is a T_0-closure operator on X_c, first let us prove that $c(\theta_c(x)) = c(x)$ for any $x \in X$. Since $x \in \theta_c(x)$, we clearly have $c(x) \subseteq c(\theta_c(x))$. Also, $z \in \theta_c(x) \Longrightarrow (x, z) \in \theta_c \Longrightarrow c(z) = c(x) \Longrightarrow z \in c(x)$ and hence $\theta_c(x) \subseteq c(x)$, so that $c(\theta_c(x)) \subseteq c(x)$. Therefore, we get that

$$c(\theta_c(x)) = c(x) \quad \text{for all} \quad x \in X. \tag{2}$$

Also, for any x and $y \in X$,

$$y \in p_c^{-1}(\theta_c(x)) \Longleftrightarrow p_c(y) = \theta_c(x) \Longleftrightarrow \theta_c(y) = \theta_c(x) \Longleftrightarrow y \in \theta_c(x).$$

Thus,

$$p_c^{-1}(\theta_c(x)) = \theta_c(x) \text{ for all } x \in X. \tag{3}$$

Note here that, on the left of (3), $\theta_c(x)$ is treated as an element of X_c and on the right $\theta_c(x)$ is treated as a subset of X.

Now, for any $\theta_c(x)$ and $\theta_c(y) \in X_c$, where x and $y \in X$,

$$\overline{c}(\theta_c(x)) = \overline{c}(\theta_c(y)) \Longrightarrow p_c(c(p_c^{-1}(\theta_c(x)))) = p_c(c(p_c^{-1}(\theta_c(y))))$$

$$\Longrightarrow p_c(c(\theta_c(x))) = p_c(c(\theta_c(y))) \text{ (by (3))} \Longrightarrow p_c(c(x)) = p_c(c(y)) \text{ (by (2))}$$

$$\Longrightarrow p_c^{-1}(p_c(c(x))) = p_c^{-1}(p_c(c(y))) \Longrightarrow c(x) = c(y) \text{ (by (1))} \Longrightarrow (x, y) \in \theta_c \Longrightarrow \theta_c(x) = \theta_c(y).$$

Thus $\overline{c}$ is a T_0-closure operator on X_c. □

From Venkateswarlu et al. [7] that, for any closure operator c on a set X, the Moore class corresponding to c is given by $\mathscr{M}_c = \{A \subseteq X \mid c(A) = A\}$ and that $\mathscr{M}_c$ is a complete lattice under the inclusion ordering.

Theorem 2. *Let c be a closure operator on X and $\overline{c}$ be the corresponding T_0-closure operator on X_c. Then $\mathscr{M}_c \cong \mathscr{M}_{\overline{c}}$ as lattices under the inclusion orders.*

Proof. We have $\mathscr{M}_c = \{Y \subseteq X \mid c(Y) = Y\}$ and $\mathscr{M}_{\overline{c}} = \{A \subseteq X_c \mid \overline{c}(A) = A\}$. Now, define $f : \mathscr{M}_c \longrightarrow \mathscr{M}_{\overline{c}}$ by $f(Y) = p_c(Y)$, for any $Y \in \mathscr{M}_c$, where $p_c : X \longrightarrow X_c$ is the natural map. First, note that

$$Y \in \mathscr{M}_c \Longrightarrow c(Y) = Y \Longrightarrow \overline{c}(p_c(Y))$$

$$= p_c c p_c^{-1} p_c(Y) = p_c\, c(Y) \text{ (by (1) in the above Theorem 1)} = p_c(Y) \Longrightarrow p_c(Y) \in \mathscr{M}_{\overline{c}}$$

and hence f is well defined and clearly f is order preserving. Now define

$$g : \mathscr{M}_{\overline{c}} \longrightarrow \mathscr{M}_c \text{ by } g(A) = p_c^{-1}(A)$$

for all $A \in \mathscr{M}_{\overline{c}}$. Note that

$$A \in \mathscr{M}_{\overline{c}} \Longrightarrow A = \overline{c}(A) = p_c(c(p_c^{-1}(A)))$$

$$\Longrightarrow p_c^{-1}(A) = c(p_c^{-1}(A)) \text{ (by (1) in Theorem 1)} \Longrightarrow p_c^{-1}(A) \in \mathscr{M}_c.$$

Therefore g is well defined and clearly g is an order preserving map. Also, for any $Y \in \mathscr{M}_c$, $(g \circ f)(Y) = p_c^{-1}\ p_c(Y) = Y$ (by (1) in Theorem 1) and, for any $A \in \mathscr{M}_{\overline{c}}$, $(f \circ g)(A) = p_c\ p_c^{-1}(A) = A$ (since p_c is a surjection). Therefore $f \circ g$ and $g \circ f$ are identities on $\mathscr{M}_{\overline{c}}$ and $\mathscr{M}_c$ respectively and hence f and g are order isomorphisms. Thus $\mathscr{M}_c \cong \mathscr{M}_{\overline{c}}$. □

Theorem 3. *Let c be a closure operator on a set X and $\overline{c}$ be the corresponding T_0-closure operator on X_c. Then $\overline{c}$ is a topological closure operator if and only if so is c.*

Proof. Suppose that $\overline{c}$ is topological. Then $\overline{c}(\phi) = \phi$ and $\overline{c}(A \cup B) = \overline{c}(A) \cup \overline{c}(B)$ for all subsets A and B of X_c. We have $\phi = \overline{c}(\phi) = p_c(c(p_c^{-1}(\phi))) = p_c(c(\phi))$ and therefore $c(\phi) = \phi$. Next, let $Y, Z \subseteq X$. Clearly we have $c(Y) \cup c(Z) \subseteq c(Y \cup Z)$. On the other hand, since $Y \cup Z \subseteq p_c^{-1}(p_c(Y \cup Z))$, we have

$$\begin{aligned} p_c(c(Y \cup Z)) &\subseteq p_c\, c\, p_c^{-1}\ (p_c(Y \cup Z)) = \overline{c}(p_c(Y \cup Z)) = \overline{c}(p_c(Y) \cup p_c(Z)) \\ &= \overline{c}(p_c(Y)) \cup \overline{c}(p_c(Z)) \ \text{ (since } \overline{c} \text{ is topological)} \end{aligned}$$

and therefore

$$c(Y \cup Z) \subseteq p_c^{-1}[\overline{c}(p_c(Y)) \cup \overline{c}(p_c(Z))] = p_c^{-1}(\overline{c}\,(p_c\,(Y))) \cup p_c^{-1}(\overline{c}\,(p_c\,(Z)))$$

$$= c(p_c^{-1}\, p_c\,(Y)) \cup c(p_c^{-1}\, p_c\,(Z)) \ \text{ (by (1) of Theorem 1)} \subseteq c(Y) \cup c(Z),$$

since $p_c^{-1}\ p_c\ (Y) \subseteq c(Y)$ and $p_c^{-1}\ p_c\ (Z) \subseteq c(Z)$. Therefore $c(Y \cup Z) = c(Y) \cup c(Z)$. Thus c is a topological closure operator on X.

Conversely, suppose that c is topological. Then $\overline{c}(\phi) = p_c(c(p_c^{-1}(\phi))) = p_c(c(\phi)) = p_c(\phi) = \phi$. Let A and B be any subsets of X_c. Since each of p_c, c and p_c^{-1} are union preserving, we have

$$\begin{aligned} \overline{c}(A \cup B) &= p_c(c(p_c^{-1}(A \cup B))) = p_c[c(p_c^{-1}(A) \cup p_c^{-1}(B))] = p_c[c(p_c^{-1}(A)) \cup c(p_c^{-1}(B))] \\ &= p_c[c(p_c^{-1}(A))] \cup p_c[c(p_c^{-1}(B))] = \overline{c}(A) \cup \overline{c}(B). \end{aligned}$$

Thus $\overline{c}$ is a topological closure operator on X_c . □

Let us recall that a closure operator c on X is called algebraic if $c(Y) = \cup\{c(F) \mid F \subseteq Y \text{ and } F \text{ is finite }\}$ for all $Y \subseteq X$.

Theorem 4. *A closure operator c on a set X is algebraic if and only if the corresponding closure operator $\overline{c}$ on X_c is algebraic.*

Proof. Suppose that c is algebraic. Let $A \subseteq X_c$. Then

$$\overline{c}(A) = p_c(c(p_c^{-1}(A))) = p_c\left(\cup\{c(F) \mid F \subseteq p_c^{-1}(A) \text{ and } F \text{ is finite }\}\right) = \cup\{p_c(c(F)) \mid F \subseteq p_c^{-1}(A), F \text{ is finite}\}.$$

Now, let $a \in \overline{c}(A)$. Then $a \in p_c(c(F))$, for some $F = \{x_1, x_2, \cdots, x_n\} \subseteq p_c^{-1}(A)$. Put $F' = \{p_c(x_1), p_c(x_2), \cdots, p_c(x_n)\}$. Then F' is a finite subset of A and $a \in p_c\,(c(\,p_c^{-1}(F'))) = \overline{c}(F')$. Therefore $\overline{c}(A) \subseteq \cup\{\,\overline{c}(F') \mid F' \subseteq A \text{ and } F' \text{ is finite }\}$. The other inclusion is trivial. Thus $\overline{c}$ is an algebraic closure operator.

Conversely, suppose that $\overline{c}$ is algebraic. Let $A \subseteq X$ and $x \in c(A)$. Then

$$p_c(x) \in p_c(c(A)) \subseteq p_c(c(p_c^{-1}(p_c(A)))), \text{ since } A \subseteq p_c^{-1}(p_c(A))$$

$$= \overline{c}(p_c(A)) = \cup\,\{\,\overline{c}(K) \mid K \subseteq p_c(A), K \text{ is finite }\}$$

and hence $p_c(x) \in \overline{c}(K) = p_c\, c\, p_c^{-1}(K)$ for some finite subset of K of $p_c(A)$.

Let $K = \{p_c(a_1), p_c(a_2), \cdots, p_c(a_n)\}$, where $a_1, a_2, \cdots, a_n \in A$. Put $F = \{a_1, a_2, \cdots, a_n\}$. Since $p_c(x) \in p_c(c(p_c^{-1}(K)))$, we get that $p_c(x) = p_c(y)$ for some $y \in p_c^{-1}(K)$. Therefore $\theta_c(x) = \theta_c(y)$ and $y \in c\, p_c^{-1}(K) = c(p_c^{-1}(p_c(F))) = c(F)$. From this, we get that $c(x) = c(y),\ y \in c(F)$. Now, $x \in c(x) = c(y) \subseteq c(c(F)) = c(F)$ and F is a finite subset of A. Thus c is an algebraic closure operator on X. □

The following corollaries are immediate consequences of Theorems 2, 3 and 4.

Corollary 1. *For any given topological space X, there exists a T_0-space Y such that the lattice of closed subsets of X is isomorphic to that of Y.*

Corollary 2. *For any given algebraic closure operator c on a set X, there exists an algebraic T_0-closure operator $\overline{c}$ on a suitable set Y such that the Moore classes of c and $\overline{c}$ are isomorphic to each other.*

3. PRE-ORDERS AND CLOSURE OPERATORS

In this section we establish a one-to-one correspondence between pre-orders on a set X and closure operators, which are both algebraic and topological, on the set X and prove that this induces a one-to-one correspondence between partial orders on X and topological algebraic T_0-closure operators on X. Let us begin with the following

Definition 4. *Let X be a non-empty set. A binary relation θ on X is said to be a pre-order on X if θ is reflexive and transitive.*

An antisymmetric pre-order on X is called a partial order on X. As usual, pre-orders or partial orders are denoted by $\leq$, $\geq$, $\subseteq$ etc. We write $a \leq b$ for $(a, b) \in \leq$.

Theorem 5. *Let $\leq$ be a pre-order on a set X. For any $A \subseteq X$, define $c(A) = \{x \in X \mid x \leq a$ for some $a \in A\}$. Then c is a closure operator on X which is both algebraic and topological. Also, c is a T_0-closure operator on X if and only if $\leq$ is a partial order on X.*

Proof. Clearly $c : \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$ is a mapping and $c(\phi) = \phi$. Also, for any $A, B \in \mathscr{P}(X)$,

$$A \subseteq c(A),\ A \subseteq B \Longrightarrow c(A) \subseteq c(B),\ c(c(A)) = c(A) \quad \text{and} \quad c(A \cup B) = c(A) \cup c(B).$$

Therefore c is a topological closure operator on X. Further $c(A) = \bigcup\limits_{a \in A} c(\{a\})$ for any $A \subseteq X$, and hence c is algebraic also. Thus c is a closure operator on X which is both algebraic and topological. Next, for any x and $y \in X$, we have

$$c(x) = c(y) \Longleftrightarrow c(x) \subseteq c(y) \text{ and } c(y) \subseteq c(x) \Longleftrightarrow x \in c(y) \text{ and } y \in c(x) \Longleftrightarrow x \leq y \text{ and } y \leq x.$$

From this, it follows that c is a T_0-closure operator on X if and only if $\leq$ is antisymmetric also; that is, $\leq$ is a partial order on X. □

The following is a converse of the above theorem, in the sense that every algebraic and topological closure operator on X is induced by a pre-order on X.

Theorem 6. *Let c be an algebraic and topological closure operator on a set X. For any x and $y \in X$, define $x \leq_c y$ if and only if $c(x) \subseteq c(y)$. Then $\leq_c$ is a pre-order on X such that, for any $A \subseteq X$,*

$$c(A) = \{x \in X \mid x \leq_c a \text{ for some } a \in A\}.$$

Also, $\leq_c$ is a partial order if and only if c is a T_0-closure operator on X.

Proof. Clearly $x \leq_c x$ for all $x \in X$. Also,

$$x \leq_c y \text{ and } y \leq_c z \Longrightarrow c(x) \subseteq c(y) \subseteq c(z) \Longrightarrow x \leq_c z.$$

Therefore $\leq_c$ is a pre-order on X. Since c is an algebraic and topological closure operator on X, it follows that for any $A \subseteq X$,

$$c(A) = \cup\{c(F) \mid F \subseteq A \text{ and } F \text{ is finite}\} = \cup\left\{\bigcup_{i=1}^{n} c(a_i) \mid a_1, a_2, \cdots, a_n \in A\right\} = \bigcup_{a \in A} c(a).$$

Since $x \in c(a) \Longleftrightarrow c(x) \subseteq c(a) \Longleftrightarrow x \leq_c a$, we have

$$c(A) = \{x \in X \mid x \leq_c a \text{ for some } a \in A\}.$$

Also, since $c(x) = c(y) \Longleftrightarrow x \leq_c y$ and $y \leq_c x$, it follows that $\leq_c$ is a partial order on X if and only if c is a T_0-closure operator on X. □

The following is an immediate consequence of Theorems 5 and 6.

Corollary 3. *Let X be any non-empty set. Then $c \mapsto \leq_c$ is a one-to-one correspondence between algebraic and topological closure operators on X and pre-orders on X such that c is a T_0-closure operator if and only if $\leq_c$ is a partial order on X.*

The following is an easy verification using the definitions of algebraic closure operators and topological closure operators.

Theorem 7. *A closure operator c on X is both algebraic and topological if and only if, for any $A \subseteq X$, $c(A) = \bigcup_{a \in A} c(a)$.*

Next, we prove that any function defined from a set X into any set Y induces an algebraic and topological closure operator X. First, we have the following.

Theorem 8. *Let $f : X \to Y$ be a function. For any $A \subseteq X$, define*

$$c_f(A) = f^{-1}(f(A)) = \{x \in X \mid f(x) = f(a) \text{ for some } a \in A\}.$$

Then c_f is an algebraic and topological closure operator on X and $\{c_f(a) \mid a \in X\}$ is a partition of X.

Proof. Clearly $c_f(\phi) = \phi$ and $A \subseteq f^{-1}(f(A)) = c_f(A)$ for any $A \subseteq X$. Also,

$$x \in c_f(c_f(A)) \Longrightarrow f(x) \in f(c_f(A)) \Longrightarrow f(x) = f(y) \text{ for some } y \in c_f(A)$$

$$\Longrightarrow f(x) = f(y) \text{ and } f(y) \in f(A) \Longrightarrow f(x) = f(y) = f(a) \text{ for some } a \in A \Longrightarrow x \in f^{-1}(f(A)) = c_f(A).$$

Therefore $c_f(c_f(A)) = c_f(A)$. Further,

$$A \subseteq B \subseteq X \Longrightarrow f(A) \subseteq f(B) \Longrightarrow f^{-1}(f(A)) \subseteq f^{-1}(f(B)) \Longrightarrow c_f(A) \subseteq c_f(B).$$

Thus c_f is a closure operator on X.

For any $A \subseteq X$, we have

$$x \in c_f(A) \Longleftrightarrow x \in f^{-1}(f(A)) \Longleftrightarrow f(x) = f(a) \text{ for some } a \in A$$
$$\Longleftrightarrow x \in f^{-1}(f(a)) = c_f(a) \text{ for some } a \in A$$

and hence $c_f(A) = \bigcup_{a \in A} c_f(a)$. Thus c_f is both algebraic and topological. In particular, $X = c_f(X) = \bigcup_{x \in X} c_f(x)$. Note that, for any $x \in X$, $c_f(x) = \{a \in X \mid f(a) = f(x)\}$. For any x and $y \in X$, we have

$$f(x) \neq f(y) \Longleftrightarrow c_f(x) \cap c_f(y) = \phi \Longleftrightarrow c_f(x) \neq c_f(y)$$

and therefore, any two distinct $c_f(x)'$s are disjoint. Thus $\{c_f(x) \mid x \in X\}$ forms a partition of X. □

The following is a converse of the above theorem, in the sense that any algebraic topological closure operator c on a set X is induced by a mapping of X into a suitable set, provided $\{c(a) \mid a \in X\}$ is a partition of X.

Theorem 9. *Let c be an algebraic topological closure operator on a set X such that $\{c(a) \mid a \in X\}$ forms a partition of X. Then there exist a set Y and a function $f : X \longrightarrow Y$ such that $c(A) = c_f(A)$ for all $A \subseteq X$.*

Proof. Since c is given to be an algebraic and topological closure operator on X, we have $c(A) = \bigcup\limits_{a \in A} c(a)$, for all $A \subseteq X$. Recall, from Definition 2, that we have $X_c = X/_{\theta_c} = \{\theta_c(x) \mid x \in X\}$, where θ_c is the equivalence relation $\{(x, y) \in X \times X \mid c(x) = c(y)\}$. Now, let $f : X \longrightarrow X_c$ be the natural map given by $f(x) = \theta_c(x)$ for all $x \in X$. First we observe that, for any $A \subseteq X$,

$$x \in c(A) \Longleftrightarrow x \in c(a) \text{ for some } a \in A \Longleftrightarrow c(x) = c(a) \text{ for some } a \in A$$

(since $c(x) \cap c(a) \neq \phi$ and $\{c(a) \mid a \in X\}$ is a partition of X). Therefore, we have

$$\begin{aligned} c_f(A) = f^{-1}(f(A)) &= \{x \in X \mid f(x) = f(a) \text{ for some } a \in A\} \\ &= \{x \in X \mid \theta_c(x) = \theta_c(a) \text{ for some } a \in A\} = \{x \in X \mid (x, a) \in \theta_c \text{ for some } a \in A\} \\ &= \{x \in X \mid c(x) = c(a) \text{ for some } a \in A\} = c(A). \end{aligned}$$

□

Let us recall that a closure operator c on X is called a T_0-closure operator if, for any x and $y \in X$, $c(x) = c(y) \Longrightarrow x = y$. In the following, we exhibit certain equivalent conditions for c_f be a T_0-closure operator, where f is a given function defined on X.

Theorem 10. *The following are equivalent to each other for any function $f : X \longrightarrow Y$.*

(1) c_f is a T_0-closure operator on X;

(2) f is an injection;

(3) $c_f(x) = \{x\}$ for all $x \in X$;

(4) c_f is trivial; that is, $c_f(A) = A$ for all $A \subseteq X$.

Proof. (1) $\Longrightarrow$ (2): for any x and $y \in X$, we have

$$f(x) = f(y) \Longrightarrow x \in f^{-1}(f(y)) \text{ and } y \in f^{-1}(f(x))$$

$$\Longrightarrow x \in c_f(y) \text{ and } y \in c_f(x) \Longrightarrow c_f(x) \subseteq c_f(y) \text{ and } c_f(y) \subseteq c_f(x) \Longrightarrow c_f(x) = c_f(y) \Longrightarrow x = y.$$

Therefore f is an injection.

(2) $\Longrightarrow$ (3): for any x and $y \in X$, we have

$$y \in c_f(x) \Longrightarrow y \in f^{-1}(f(x)) \Longrightarrow f(y) = f(x) \Longrightarrow y = x$$

and therefore $c_f(x) = \{x\}$.

(3) $\Longrightarrow$ (4): since c_f is algebraic and topological, we have

$$c_f(A) = \bigcup_{a \in A} c_f(a) = \bigcup_{a \in A} \{a\} = A \text{ for any } A \subseteq X.$$

(4) $\Longrightarrow$ (1) is trivial. □

Acknowledgments. We are grateful to the reviewers for careful reading of the manuscript and helpful remarks.

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