FIXED POINT THEOREM FOR F-CONTRACTION MAPPINGS IN PARTIAL METRIC SPACES

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Abstract. In this paper, we prove some fixed point theorems for F-contraction mappings in partial metric spaces. In particular, the main results generalizes a fixed point theorem due to Wardowski [8] in which F-contraction was introduced as a generalization of Banach Contraction Principle. An illustrative example is provided to validate the results.
Keywords. Partial metric spaces; F-contraction mappings; Fixed point theorem.
2010 Mathematics Subject Classification. 47H10, 54H25.

1. INTRODUCTION

The notion of F-contraction mappings was introduced by Wardowski [8] as a generalization of Banach Contraction [5] and proved to be very useful in the existing metric fixed point theory. This gave rise to numerous fixed point results for F-contraction mappings as extension and its generalizations. Wardowski [8] introduced an F-contraction mapping and defined it as follows:

Definition 1.1. [8] Let (M,d) be a metric space, a mapping $T : M \longrightarrow M$ is said to be an F-contraction on M, if there exists $\tau > 0$ such that for all $x, y \in M$,

$$d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)). \tag{1.1}$$

and $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ a mapping satisfying the following conditions:
$F1$: F is strictly increasing, that is for all $x, y \in \mathbb{R}_+$ such that $x \leq y \Rightarrow F(x) \leq F(y)$.
$F2$: For each sequence $\{\alpha_n\}_{n\geq 1}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$, if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$.
$F3$: There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We denote the set of all functions satisfying the conditions $F1 - F3$ by Δ_F

Remark 1.2. From $F1$ and the contractive condition 1.1, we observe that every F-contraction is necessarily continuous.

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The following example is one amongst the examples of F-contractions:

Example 1.3. [7] Let $F:\mathbb{R}_+ \longrightarrow \mathbb{R}$ be defined by $F(\alpha)=\ln(\alpha)$. It is clear that F satisfies $F1-F3$ for any $k\in(0,1)$. Each mapping $T:M\longrightarrow M$ satisfying $d(Tx,Ty)>0\Rightarrow \tau+F(d(Tx,Ty))\leq F(d(x,y))$ is an F-contraction such that $d(Tx,Ty)\leq e^{-\tau}d(x,y)$ for all $x,y\in M, Tx\neq Ty$. Obviously, for all $x,y\in M$ such that $Tx=Ty$, the inequality $d(Tx,Ty)\leq e^{-\tau}d(x,y)$ holds and T is a Banach contraction. One can find more examples in [8] .

A partial metric space was introduced by Matthews [6], which is a generalization of metric space and admits non-zero self distance:

Definition 1.4. [6] Let X be a non-empty set. A partial metric space is a pair (X,p), where p is a function $p:X\times X\to\mathbb{R}^+$, called the partial metric, such that for all $x,y,z\in X$ the following axioms hold:

(P1) $x=y\Leftrightarrow p(x,y)=p(x,x)=p(y,y)$;
(P2) $p(x,x)\leq p(x,y)$;
(P3) $p(x,y)=p(y,x)$; and
(P4) $p(x,y)\leq p(x,z)+p(z,y)-p(z,z)$.

Clearly, by (P1)-(P3), if $p(x,y)=0$, then $x=y$. But the converse is in general not true.

One among the classical examples of partial metric spaces is a pair $([0,\infty),p)$ where $p(x,y)=\max\{x,y\}$ for all $x,y\in[0,\infty)$. One can find more examples in [3, 6].

Each partial metric p on X generates a T_0 topology τ_p on X whose basis is the collection of all open p-balls $\{B_p(x,\varepsilon):x\in X,\varepsilon>0\}$, where
$B_p(x,\varepsilon)=\{y\in X:p(x,y)<p(x,x)+\varepsilon\}$ for all $x\in X$, and ε is a positive real number.

2. Preliminaries

The following are some basic concepts and preliminaries useful for the establishment of our results:

Bukatin et. al. [3] provided the following definition which is useful in our discussion:

Definition 2.1. [6] Let (X,p) be a partial metric space. Then,

(i) a sequence $\{x_n\}$ in (X,p) is said to be convergent to $x\in X$ if and only if $p(x,x)=\lim_{n\to\infty}p(x,x_n)$.
(ii) a sequence $\{x_n\}$ in (X,p) is a Cauchy sequence if and only if $\lim_{n,m\to\infty}p(x_n,x_m)$ exists and is finite.
(iii) a partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to the topology τ_p to a point $x\in X$ such that $p(x,x)=\lim_{n,m\to\infty}p(x_n,x_m)$.

Bukatin et. al. [3] proved the following lemma which is useful in our discussion:

Lemma 2.2. [3] *Let (X,p) be a partial metric space. Then the mapping $p^s:X\times X\to[0,\infty)$ given by*

$$p^s(x,y)=2p(x,y)-p(x,x)-p(y,y)$$

for all $x,y\in X$ defines a metric on X.

Bukatin et. al. [3] also proved the following lemma:

Lemma 2.3. [3] *Let (X,p) be a partial metric space. Then:*

(i) *a sequence $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).*

(ii) *a partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete.*

Some researchers have attempted to generalize, improve and extend the notion of F-contraction mappings to partial metric spaces. For instance Nazam et. al. [7] introduced an improved F-contraction of rational type in partial metric spaces and used it to prove a common fixed point theorem for a pair of self mappings.
The following Theorem was given by Nazam et. al. [7]:

Theorem 2.4. [7] *The contractions $H,T : X \to X$ on a complete partial metric space X such that either H or T is continuous and (H,T) is a pair of improved F-contraction of rational type have a common fixed point z of (H,T) in X such that $P(z,z) = 0$.*

Wardowski [8] proved the following results using F-contraction mapping:

Theorem 2.5. [8] *Let (M,d) be a complete metric space and let $T : M \longrightarrow M$ be an F-contraction on M. Then T has a unique fixed point $x_0 \in M$ and for every $x \in M$ the sequence $\{T^n x\}_{n\in N}$ converges to x_0.*

In this article we generalize Theorem 2.5 to partial metric spaces to obtain some fixed point theorems for The following is the extension of the Definition 1.1 to partial metric spaces:

3. Main results

Definition 3.1. Let (X,p) be a partial metric space. The mapping $T : X \longrightarrow X$ is said to be an F-contraction on X if there exists $\tau > 0$ and $F \in \Delta_F$ such that for all $x,y \in X$,

$$p(Tx,Ty) > 0 \Rightarrow \tau + F(p(Tx,Ty)) \leq F(p(x,y)). \tag{3.1}$$

The following theorem is the extension of Theorem 2.5 to partial metric spaces:

Theorem 3.2. *Let (X,p) be a complete partial metric space and let $T : X \longrightarrow X$ be an F-contraction map on X, then T has a unique fixed point $v \in X$ such that $p(v,v) = 0$.*

Proof. : Let $x_0 \in X$ be any arbitrary point and fixed. We define a sequence $\{x_n\}_{n\in\mathbb{N}} \in X$ such that $x_{n+1} = Tx_n$, for all $n = 0,1,2,\ldots$. Also we denote $a_n = p(x_{n+1},x_n)$, for all $n = 0,1,2,\ldots$
If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then $Tx_{n_0} = x_{n_0}$. This will end the proof. Now suppose $a_n > 0$ for all $n \in \mathbb{N}$ with $x_{n+1} \neq x_n$.
Then by the contractive condition (3.1) of Definition 3.1 we have,

$$F(a_n) \leq F(a_{n-1}) - \tau \leq F(a_{n-2}) - 2\tau \leq F(a_{n-3}) - 3\tau \leq \ldots \leq F(a_0) - n\tau. \tag{3.2}$$

From (3.2) we obtain $\lim\limits_{n\to\infty} F(a_n) = -\infty$.
Since $F \in \Delta_F$, then by $(F2)$ of Definition 1.1 we get,

$$\lim_{n\to\infty} a_n = 0. \tag{3.3}$$

By $(F3)$ of Definition 1.1, there exists $k \in (0,1)$ such that,

$$\lim_{n\to\infty} {a_n}^k F(a_n) = 0. \tag{3.4}$$

Following (3.2) for all $n \in \mathbb{N}$ we have,

$${a_n}^k (F(a_n) - F(a_0)) \leq -{a_n}^k n\tau \leq 0. \tag{3.5}$$

Taking into account (3.3) and (3.4) and letting $n \to \infty$ in (3.5) we obtain,

$$\lim_{n\to\infty} n{a_n}^k = 0. \tag{3.6}$$

Since (3.6) holds, then there exists $n_1 \in \mathbb{N}$ such that $n{a_n}^k \leq 1$, for all $n \geq n_1$.
this implies that,

$$a_n \leq \frac{1}{n^{\frac{1}{k}}}, \text{ for all } n \geq n_1. \tag{3.7}$$

Next we show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.
Consider $n,m \in \mathbb{N}$ such that $m > n \geq n_1$, then by (3.7) and axiom $(P3)$ of Definition (1.4) we have,

$$\begin{aligned}
p(x_n,x_m) \leq& p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) + \dots + p(x_{m-1},x_m) - \sum_{j=n+1}^{m-1} p(x_j,x_j) \\
\leq& p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) + \dots + p(x_{m-1},x_m) \\
=& a_n + a_{n+1} + \dots + a_{m-1} \\
=& \sum_{i=n}^{m-1} a_i \leq \sum_{i=n}^{\infty} a_i \\
\leq& \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.
\end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ implies that $\lim_{n\to\infty} p(x_n,x_m) = 0$. By Lemma 2.2 we obtain that for any $n,m \in \mathbb{N}$, $p^s(x_n,x_m) \leq 2p(x_n,x_m) \to 0$ as $n \to \infty$. This implies that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to p^s and hence converges by Lemma 2.3. Thus there exists $v \in X$ such that,

$$p(v,v) = \lim_{n\to\infty} p(x_n,v) = \lim_{n,m\to\infty} p(x_n,x_m). \tag{3.8}$$

Since $\lim_{n,m\to\infty} p(x_n,x_m) = 0$, then from (2.8) we deduce that

$$p(v,v) = 0 = \lim_{n\to\infty} p(x_n,v). \tag{3.9}$$

By (3.9) it follows that $x_{n+1} \to v$ as $n \to \infty$ with respect to τ_p . By the continuity of T it implies that,

$$v = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tv.$$

From the contractive condition (3.1) we can write, $\tau + F(p(v,Tv)) \leq F(p(v,v))$.
This implies that $p(v,Tv) = 0$ and by axioms $(P1)$ and $(P2)$ of Definition 1.4 we conclude that $v = Tv$.
Hence v is a fixed point of T.
We now prove that a fixed point is unique.

Suppose, there exists $u \in X$ such that $u \neq v$ and $u = Tu$. From the contractive condition (3.1) we can have,

$$\begin{aligned} p(Tu,Tv) = p(u,v) > 0 \Rightarrow F(p(u,v)) <& \tau + F(p(u,v)) \\ =& \tau + F(p(Tu,Tv)), \\ \leq& F(p(u,v)). \end{aligned}$$

This implies that $F(p(u,v)) < F(p(u,v))$ which is a contradiction. Hence $u = v$.
Thus the fixed point of T is unique. □

Corollary 3.3. *Let (X,p) be a complete partial metric space and a contraction $T : X \to X$ satisfies all conditions in Theorem 3.2. If we take F as defined in Example 1.3, then T is a Banach Contraction as generalized by Matthews* [6].

We now present a fixed point theorem for a pair of maps satisfying F-contraction condition as an extension of Theorem 3.2:

Theorem 3.4. *Let (X,p) be a complete partial metric space and $H,T : X \to X$ be a pair of F-contraction mappings, such that for all $x,y \in X$ we have,*

$$p(Hx,Ty) > 0 \Rightarrow \tau + F(p(Hx,Ty)) \leq F(\mathbb{M}(x,y)), \tag{3.10}$$

where,
$\mathbb{M} = max\left\{ p(x,y), p(x,Hx), p(y,Ty), \frac{p(x,Ty)+p(y,Hx)}{2} \right\}$. *Then there exists a unique fixed point which is common for both H and T.*

Proof. : We first show the existence of a fixed point for both H and T.
Let $x_0 \in X$ be any arbitrary point and fixed. We define a sequence $\{x_n\} \in X$, for all $n \in \mathbb{N}$ such that $x_{n+1} = Hx_n$ and $x_{n+2} = Tx_{n+1}$ for $n = 0,1,2,\ldots$.
Now suppose that $p(x_{n+1},x_{n+2}) > 0$, for all $n \in \mathbb{N} \cup \{0\}$ with $x_{n+1} \neq x_{n+2}$.
Then by the contractive condition (3.10) we have,

$$\begin{aligned} \tau + F(p(x_{n+1},x_{n+2})) =& \tau + F(p(Hx_n,Tx_{n+1})) \\ \leq& F(\mathbb{M}(x_n,x_{n+1})). \end{aligned}$$

where,

$$\begin{aligned} \mathbb{M}(x_n,x_{n+1}) =& \max\left\{ p(x_n,x_{n+1}), p(x_n,Hx_n), p(x_{n+1},Tx_{n+1}), \frac{p(x_n,Tx_{n+1}) + p(x_{n+1},Hx_n)}{2} \right\}, \\ =& \max\left\{ p(x_n,x_{n+1}), p(x_n,x_{n+1}), p(x_{n+1},x_{n+2}), \frac{p(x_n,x_{n+2}) + p(x_{n+1},x_{n+1})}{2} \right\}, \\ =& \max\left\{ p(x_n,x_{n+1}), p(x_{n+1},x_{n+2}), \frac{p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2})}{2} \right\}, \\ =& \max\left\{ p(x_n,x_{n+1}), p(x_{n+1},x_{n+2}) \right\}. \end{aligned}$$

Suppose $\max\{p(x_n,x_{n+1}),p(x_{n+1},x_{n+2})\} = p(x_{n+1},x_{n+2})$ then,
$\tau + F(p(x_{n+1},x_{n+2})) \leq F(p(x_{n+1},x_{n+2}))$, $\Rightarrow F(p(x_{n+1},x_{n+2})) \leq F(p(x_{n+1},x_{n+2})) - \tau$ which is a contradiction. Thus $\max\{p(x_n,x_{n+1}),p(x_{n+1},x_{n+2})\} = p(x_n,x_{n+1})$.
Hence we can write,

$$F(p(x_{n+1},x_{n+2})) \leq F(p(x_n,x_{n+1})) - \tau, \text{ for all } n \in \mathbb{N}\cup\{0\}. \tag{3.11}$$

Hence from (3.11) we have,

$$F(p(x_n,x_{n+1})) \leq F(p(x_{n-1},x_{n-2})) - 2\tau. \tag{3.12}$$

Repeating these steps n-times we get,

$$F(p(x_n,x_{n+1})) \leq F(p(x_0,x_1)) - n\tau. \tag{3.13}$$

Then from 3.13 we obtain,

$$\lim_{n\to\infty} F(p(x_n,x_{n+1})) = -\infty. \tag{3.14}$$

By $(F1)$ of Definition 1.1 we obtain,

$$\lim_{n\to\infty}(p(x_n,x_{n+1})) = 0. \tag{3.15}$$

By $(F3)$, there exists $k \in (0,1)$ such that,

$$\lim_{n\to\infty}(p(x_n,x_{n+1}))^k F(p(x_n,x_{n+1})) = 0. \tag{3.16}$$

Following 3.13, for all $n \in \mathbb{N}$ we have,

$$(p(x_n,x_{n+1}))^k (F(p(x_n,x_{n+1})) - F(p(x_0,x_1))) \leq -(p(x_n,x_{n+1}))^k n\tau \leq 0. \tag{3.17}$$

Considering (3.14),(3.15) and letting $n \to \infty$ in (3.16), we get,

$$\lim_{n\to\infty} n(p(x_n,x_{n+1}))^k = 0. \tag{3.18}$$

Since (3.18) holds, there exists $n_1 \in \mathbb{N}$, such that,

$$n(p(x_n,x_{n+1}))^k \leq 1, \text{ for all } n \geq n_1. \tag{3.19}$$

Next we show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Consider $n,m \in \mathbb{N}$ such that $m > n \geq n_1$, then by (3.19) and axiom $(P3)$ of Definition 1.4 we have,

$$\begin{aligned}
p(x_n,x_m) \leq& p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) + \ldots + p(x_{m-1},x_m) - \sum_{j=n+1}^{m-1} p(x_j,x_j) \\
\leq& p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) + \ldots + p(x_{m-1},x_m) \\
=& a_n + a_{n+1} + \ldots + a_{m-1} \\
=& \sum_{i=n}^{m-1} a_i \leq \sum_{i=n}^{\infty} a_i \\
\leq& \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.
\end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}$ implies that $\lim_{n\to\infty}p(x_n,x_m)=0$. By Lemma 2.2 we get that, for any $n,m\in\mathbb{N}, p^s(x_n,x_m)\leq 2p(x_n,x_m)\to 0$ as $n\to\infty$. This implies that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to p^s. Since (X,p) is complete then so is (X,p^s), then there exists $u\in X$ such that,

$$\lim_{n\to\infty}p^s(x_n,u)=0. \tag{3.20}$$

Moreover by Lemma (2.3),

$$p(u,u)=\lim_{n\to\infty}p(x_n,u)=\lim_{n,m\to\infty}p(x_n,x_m). \tag{3.21}$$

Then, from (3.21) we deduce that,

$$p(u,u)=0=\lim_{n\to\infty}p(x_n,u). \tag{3.22}$$

It follows that, $x_{n+1}\to u$ and $x_{n+2}\to u$ as $n\to\infty$ with respect to $\tau(p)$. Hence by the continuity of T it implies that,
$u=\lim_{n\to\infty}x_n=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}x_{n+2}=\lim_{n\to\infty}Tx_{n+1}=T\lim_{n\to\infty}x_{n+1}=Tu.$
Hence from (3.10) we have,

$$\begin{aligned}\tau+F(p(u,Hu))=&\tau+F(p(Hu,Tu))\\ &\leq F(\mathbb{M}(u,u))=F(p(u,u)).\end{aligned}$$

This yields that $p(u,Hu)=0$ and by $(P1)$, $(P2)$ of Definition 1.4 we obtain that $u=Hu$. Thus $Hu=Tu=u$. Hence (H,T) has a common fixed point $u\in X$.
Next we will show that u is unique common fixed point of H and T.
Suppose that by contradiction there exists $z\in X$ such that $u\neq z$ and $z=Tz$. From the contractive condition (3.10) we have,

$$\tau+F(p(Hu,Tz))\leq F(\mathbb{M}(u,z)), \tag{3.23}$$

where,

$$\begin{aligned}\mathbb{M}(u,z)=&\max\left\{p(u,z),p(u,Hu),p(z,Tz),\frac{p(u,Tz)+p(z,Hu)}{2}\right\}\\ =&\max\left\{p(u,z),p(u,u),p(z,z),\frac{p(u,z)+p(z,u)}{2}\right\}\\ =&\max\left\{p(u,z),\frac{p(u,z)+p(z,u)}{2}\right\}\\ =&p(u,z).\end{aligned}$$

From (3.23) we have,

$$\tau+F(p(Hu,Tz))\leq F(p(u,z)), \tag{3.24}$$

From (3.24) we obtain that, $p(u,z)<p(u,z)$ which is a contradiction. Hence $u=z$ and u is a unique common fixed point of (H,T).

□

Now we provide an illustrative example for Theorem 3.2.

Example 3.5. Let $X=[0,1]$ and define $p(x,y)=\max\{x,y\}$ for all $x,y\in X$. Then (X,p) is a complete partial metric space. Define the mapping $T:X\to X$ such that for all $x\in X$,

$$T(x)=\begin{cases}\frac{x}{3} & \text{if} \quad x\in[0,1)\\ 0 & \text{if} \quad x=1\end{cases}.$$

Clearly T is a self mapping. Define the function $F:\mathbb{R}^+\to\mathbb{R}$ by $F(r)=\ln(r)$ for all $r\in\mathbb{R}^+$.
Let $x,y\in X$ such that $p(Tx,Ty)>0$, this implies that

$\tau+F(p(Tx,Ty))=\tau+\ln(\max(\{\frac{x}{3},\frac{y}{3}\}))$. Now suppose that $y\geq x$ without loss of generality and taking $\tau\leq\ln(3)$ we obtain that,

$$\begin{aligned}\tau+\ln(\max(\{\frac{x}{3},\frac{y}{3}\}))&\leq\ln(3)+\ln(\frac{y}{3})\\&=\ln(p(x,y))\\&=F(p(x,y)).\end{aligned}$$

Similarly, if $x\geq y$ we obtain that,

$$\tau+F(p(Tx,Ty))\leq F(p(x,y)).$$

Thus, the contractive condition (3.1) is satisfied for all $x,y\in X$. Hence all hypotheses of the Theorem 3.2 are satisfied and note that T has a unique fixed point $x=0$.

Remark 3.6. We also notice that T in Example 3.5 is not an F-contraction in (X,d) and consequently Theorem 2.5 cannot be applied.

Acknowledgements

Authors have no conflicts of interest.

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