Determinability of Semirings of Continuous Nonnegative Functions with Max-plus by the Lattices of Their Subalgebras

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(Submitted by A. A. Editor-name)

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Received October 17, 2018

Abstract—Denote by $\mathbb{R}_+^\vee$ the semifield with zero of nonnegative real numbers with operations of max-addition and multiplication. Let X be a topological space and $C^\vee(X)$ be the semiring of continuous nonnegative functions on X with pointwise operation max-addition and multiplication of functions. By a subalgebra we mean a nonempty subset A of $C^\vee(X)$ such that $f \vee g, fg, rf \in A$ for any $f, g \in A, r \in \mathbb{R}_+^\vee$. We consider the lattice $\mathbb{A}(C^\vee(X))$ of subalgebras of the semiring $C^\vee(X)$ and its sublattice $\mathbb{A}_1(C^\vee(X))$ of subalgebras with unity. The main result of the paper is the proof of the definability of the semiring $C^\vee(X)$ both by the lattice $\mathbb{A}(C^\vee(X))$ and by its sublattice $\mathbb{A}_1(C^\vee(X))$.

2010 Mathematical Subject Classification: 06B05, 16S60, 54H99

Keywords and phrases: *Semiring of continuous functions, subalgebra, lattice of subalgebras, isomorphism, Hewitt space, max-addition.*

1. INTRODUCTION

The present paper belongs to the direction of mathematics that studies interconnections between topological spaces and associated algebraic systems of continuous functions.

Departure notions. By a semiring we mean an algebraic system $\langle S, +, \cdot, 0, 1\rangle$ in which $\langle S, +, 0\rangle$ is a commutative monoid with neutral element 0, $\langle S, \cdot, 1\rangle$ is a monoid with neutral element 1, multiplication is distributive with respect to addition on both sides, and $0 \cdot a = a \cdot 0 = 0$ for all $a \in S$. A commutative semiring that is not a ring and every nonzero element of which is invertible is called a semifield with zero. It is easy to show that if S is a semifield with zero, then $ab, a + b \neq 0$ for any $a, b \in S\backslash\{0\}$. Therefore the set $S\backslash\{0\}$ with the same operations of addition and multiplication forms an algebraic system, which is called a semifield.

Let S be the field $\mathbb{R}$ of real numbers, or the semifield with zero $\mathbb{R}_+$ of non-negative real numbers, or the semifield $\mathbb{P}$ of positive real numbers (with the interval topology). Let $C(X, S)$ denote the set of continuous S-valued functions with point-wise operations of addition and multiplication of functions defined on an arbitrary topological space X. Then $C(X) = C(X, \mathbb{R})$ is a ring, $C^+(X) = C(X, \mathbb{R}_+)$ is a semiring, and $U(X) = C(X, \mathbb{P})$ is a semifield.

The ring $C(X)$ is an algebra over $\mathbb{R}$. A subalgebra of $C(X)$ is any nonempty subset that is closed under addition and multiplication of functions and under multiplication by constants from $\mathbb{R}$. By analogy we call a nonempty subset $A \subseteq C(X, S)$ a subalgebra if $f + g, fg, rf \in A$ for any $f, g \in A$ and any $r \in S$. Thus, we are using the term «subalgebra» in a wider sense than a ring that is simultaneously a vector space.

Let $\mathbb{A}(C(X, S))$ denote the lattice of subalgebras of $C(X, S)$ with respect to inclusion $\subseteq$, and let $\mathbb{A}_1(C(X, S))$ denote its sublattice of subalgebras with unity (strict inclusion is denoted by $\subset$). It is easy to show that the infimum $A \wedge B$ of subalgebras A and B is their intersection $A \cap B$, and the supremum $A \vee B$ consists of functions equal to finite sums of products of the form $f_1 \cdot \ldots \cdot f_n$,

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where $f_1, \dots, f_n \in A \cup B$, $n \in \mathbb{N}$. Since A and B are subalgebras, we can assume without loss of generality that $n \le 2$. If $0 \notin S$, then sometimes $A \cap B = \emptyset$. Therefore we regard the empty set $\emptyset$ as an element of the lattice $\mathbb{A}(C(X,S))$ (its zero).

For any $a, b \in \mathbb{R}$, we put $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$. If for $\mathbb{R}_+$ and $\mathbb{P}$ usual addition replaced by max-addition, we get the semifield with zero $\mathbb{R}_+^\vee$ and the semifield $\mathbb{P}^\vee$. For the semiring $C^\vee(X) = C(X, \mathbb{R}_+^\vee)$ and the semifield $U^\vee(X) = C(X, \mathbb{P}^\vee)$, the notions of a subalgebra and a lattice of subalgebras (with unity) are defined as above.

A Hausdorff space X is called a Tikhonov space if for any nonempty closed set $F \subset X$ and any point $x \in X \backslash F$ there exists a function $f \in C(X)$ (or, which is equivalent, $f \in C^\vee(X)$) such that $f(F) = \{a\}$ and $f(x) = b$ for some $a \ne b$. Tikhonov spaces are, up to homeomorphism, subspaces of Tikhonov powers of $\mathbb{R}$. A topological space X is called a Hewitt space (or real-complete, or functionally closed) if it is homeomorphic to a closed subspace of some Tikhonov power of the space $\mathbb{R}$. For example, any compactum is a Hewitt space.

Main results. Suppose that with every topological space X, an algebraic structure $A(X)$ is somehow associated. This correspondence is assumed to be invariant, that is, to homeomorphic spaces there correspond isomorphic structures. The situation when the converse is true is described by the following definition. We say that a topological space $X \in K$ is defined (uniquely up to homeomorphism) in a class K of topological spaces by an algebraic structure $A(X)$ if for an arbitrary topological space $Y \in K$ an isomorphism $A(Y) \cong A(X)$ implies a homeomorphism $Y \approx X$. The notion of definability of an algebraic structure $A(X)$ in a class K of topological spaces by a derived algebraic structure $A'(X)$ is introduced in a similar fashion.

In 1939 Gel'fand and Kolmogorov proved (see [1, Theorem 2]) one of the first definability theorems for topological spaces: an arbitrary compactum X is defined by the ring $C(X)$. In 1948 Hewitt established (see [2, Theorem 57]) the definability of an arbitrary Hewitt space X by the ring $C(X)$, and in 1997 Vechtomov proved (see [3, Theorem 1]) the definability of X by the lattice $\mathbb{A}(C(X))$ of subalgebras of the ring $C(X)$.

A systematic study of semirings and semifields of continuous functions began in 1998 (see [4]). In their study, great attention is devoted to methods and results that can be transferred from theory of rings of continuous functions, icluding those related to definability. It is natural to assume spaces under these considerations to be Hewitt spaces, since the following proposition is true (see [5, Theorems 3.9 and 8.7]).

Proposition 1. *For an arbitrary space X, there exist a Thikhonov space τX and a Hewitt space $\nu\tau X$ for which the rings $C(X)$, $C(\tau X)$, and $C(\nu\tau X)$ are canonically isomorphic, and therefore so are the corresponding semirings and semifields* (*as with the usual addition and with max-addition*).

In work [6] we proved the definability of an arbitrary Hewitt space X by each of the lattices $\mathbb{A}(U^\vee(X))$ and $\mathbb{A}_1(U^\vee(X))$, and also showed that for any topological space X these lattices define the semifield $U^\vee(X)$.

The following theorems are the main results of the present paper.

Theorem 1. *Any Hewitt space X is defined by each of the lattices $\mathbb{A}(C^\vee(X))$ and $\mathbb{A}_1(C^\vee(X))$.*

Theorem 2. *For any topological space X, the semiring $C^\vee(X)$ is defined by each of the lattices $\mathbb{A}(C^\vee(X))$ and $\mathbb{A}_1(C^\vee(X))$.*

Theorem 2 follows from theorem 1. Indeed, by Proposition 2, any isomorphism of the lattices $\mathbb{A}(C^\vee(X))$ and $\mathbb{A}(C^\vee(Y))$, implies an isomorphism of the lattices $\mathbb{A}_1(C^\vee(X))$ and $\mathbb{A}_1(C^\vee(Y))$, and therefore of the lattices $\mathbb{A}_1(C^\vee(\nu\tau X))$ and $\mathbb{A}_1(C^\vee(\nu\tau Y))$ canonically isomorphic to them (see Proposition 1). Hence, $\nu\tau X \approx \nu\tau Y$ by Theorem 1. Therefore, $C^\vee(X) \cong C^\vee(\nu\tau X) \cong C^\vee(\nu\tau Y) \cong C^\vee(Y)$.

We proceed to the proof of Theorem 1. In view of Proposition 1, all the spaces considered in what follows are assumed to be Tikhonov spaces.

2. THE SUBALGEBRA $\mathbb{R}_+^{\vee}$

The smallest subalgebra of the semiring $C^{\vee}(X)$ containing a function f is said to be one-generated and denoted by $\langle f\rangle$. Let $[f]$ denote the smallest subalgebra with unity of the semiring $C^{\vee}(X)$ that contains a function f. Clearly, $[f] = \langle f\rangle \vee \mathbb{R}_+^{\vee}$ and

$$\langle f\rangle = \{a_1 f \vee \ldots \vee a_n f^n \colon a_1, \ldots, a_n \in \mathbb{R}_+^{\vee}, n \in \mathbb{N}\},$$
$$[f] = \{a_0 \vee a_1 f \vee \ldots \vee a_n f^n \colon a_0, a_1, \ldots, a_n \in \mathbb{R}_+^{\vee}, n \in \mathbb{N}_0\}.$$

Functions of subalgebras $\langle f\rangle$ and $[f]$ we write polynomials with increasing powers of f. The coefficients of polynomials be denoted by the letters a, b, c, d, and r (often with indices).

For an arbitrary function $f \in C^{\vee}(X)$, denote by $\mathrm{Z}(f)$ and $\operatorname{coz} f$ the sets

$$\mathrm{Z}(f) = \{x \in X \colon f(x) = 0\}, \quad \operatorname{pos} f = \{x \in X \colon f(x) > 0\}.$$

Denote by e_Z the function $f \in C^{\vee}(X)$ such that $f = 0$ on Z and $f = 1$ on $X\backslash Z$ for some subset $Z \subset X$. In particular, $e_{\emptyset} = 1$.

We say that in a lattice there is a lattice characterization of some property if this property can be described in terms of this lattice. For example, since the subalgebra $\mathbb{R}_+^{\vee}$ is the zero of the lattice $\mathbb{A}_1(C^{\vee}(X))$, it follows that there is a lattice characterization of the subalgebra $\mathbb{R}_+^{\vee}$ in $\mathbb{A}_1(C^{\vee}(X))$.

A nonzero element A of a lattice with zero is called an atom if $B = 0$ for any element $B < A$.

A lattice characterization of subalgebras $\langle e_Z\rangle$ in the lattice $\mathbb{A}(C^{\vee}(X))$ is given by

Lemma 1. *Subalgebras $\langle e_Z\rangle$ are exactly atoms of the lattice $\mathbb{A}(C^{\vee}(X))$.*

Proof. Let A be an atom of the lattice $\mathbb{A}(C^{\vee}(X))$, that is, A be a minimal subalgebra. We choose nonzero function $f \in A$. Then $\langle f\rangle, \langle f^2\rangle \subseteq A$. Since A is a minimal subalgebra, it follows that $\langle f\rangle = \langle f^2\rangle = A$. Therefore, $f \in \langle f^2\rangle$ and f has the form

$$f = a_1 f^2 \vee \ldots \vee a_n (f^2)^n.$$

Therefore, $1 = a_1 f \vee \ldots \vee a_n f^{2n-1}$ on $\operatorname{pos} f \neq \emptyset$, that is $f = a$ on $\operatorname{pos} f$ for some $a > 0$. Thus, $A = \langle f\rangle = \langle e_Z\rangle$, where $Z = \operatorname{pos} f$.

Conversely, let $A = \langle e_Z\rangle$. Since $e_Z = e_Z^i$ for all $i \in \mathbb{N}$, it follows that any nonzero function $f \in \langle e_Z\rangle$ has the form ae_Z, $a > 0$. Thus, $\langle f\rangle = \langle e_Z\rangle$, that is, $\langle e_Z\rangle$ is a minimal subalgebra. $\square$

Lemma 2. *For any subalgebra $\langle e_Z\rangle$, the following conditions are equivalent*:

1) $\langle e_Z\rangle = \mathbb{R}_+^{\vee}$;

2) *if* $\langle e_{Z'}\rangle \neq \langle e_Z\rangle$ *and* $\langle e_{Z''}\rangle \subseteq \langle e_Z\rangle \vee \langle e_{Z'}\rangle$, *then* $\langle e_{Z''}\rangle = \langle e_Z\rangle$ *or* $\langle e_{Z''}\rangle = \langle e_{Z'}\rangle$.

Proof. 1) $\Rightarrow$ 2). Let $\langle e_Z\rangle = \mathbb{R}_+^{\vee}$. Consider subalgebras $\langle e_{Z'}\rangle \neq \langle e_Z\rangle$ and $\langle e_{Z''}\rangle$ such that $\langle e_{Z''}\rangle \subseteq \langle e_Z\rangle \vee \langle e_{Z'}\rangle$. Since $e_Z = 1$ and $e_{Z'}^i = e_{Z'}$ for all $i \in \mathbb{N}$, it follows that a function $e_{Z''}$ has the form $a \vee be_{Z'}$. If $a > 0$, then $e_{Z''} \geq a > 0$, that is, $e_{Z''} = 1$. If $a = 0$, then $b = 1$, that is, $e_{Z''} = e_{Z'}$.

2) $\Rightarrow$ 1). Assume condition 2) holds, but $\langle e_Z\rangle \neq \mathbb{R}_+^{\vee}$; in particular, $\emptyset \subset Z \subset X$. Put $Z' = X\backslash Z$ and $Z'' = \emptyset$. Then $\langle e_{Z''}\rangle \neq \langle e_Z\rangle, \langle e_{Z'}\rangle$ and $\langle e_{Z''}\rangle \subseteq \langle e_Z\rangle \vee \langle e_{Z'}\rangle$, since $e_{Z''} = e_Z \vee e_{Z'}$; a contradiction with condition 2). Thus, $\langle e_Z\rangle = \mathbb{R}_+^{\vee}$. $\square$

From Lemma 2, we obtain

Proposition 2. *In the lattice $\mathbb{A}(C^{\vee}(X))$ there is a lattice characterization of the subalgebra $\mathbb{R}_+^{\vee}$, and therefore of the sublattice $\mathbb{A}_1(C^{\vee}(X))$.*

Remark 1. By Proposition 2, for the proof of Theorem 1, we can restrict ourselves to the case of the lattice $\mathbb{A}_1(C^{\vee}(X))$. Therefore until the end of this work, we will work in the lattice $\mathbb{A}_1(C^{\vee}(X))$, and the subalgebras with unity we will be call subalgebras.

3. SUBALGEBRAS $[f]$

For an arbitrary function $f \in C^{\vee}(X)$, denote by $\operatorname{Max} f$ and $\operatorname{Min} f$ the sets

$$\operatorname{Max} f = \{x \in X \colon f(x) = \sup f\}, \quad \operatorname{Min} f = \{x \in X \colon f(x) = \inf f\}.$$

Lemma 3. *For any subalgebras $[f], [g] \subseteq [f]$ and any points $x, y \in X$, we have:*

1) *if $f(x) < f(y)$, then $g(x) \leq g(y)$;*
2) *if $g(x) < g(y)$, then $f(x) < f(y)$; in particular, $|\operatorname{Im} g| \leq |\operatorname{Im} f|$;*
3) *if $[f] = [g]$ and $[f], [g] \neq \mathbb{R}_{+}^{\vee}$, then $\mathrm{Z}(f) = \mathrm{Z}(g)$;*
4) *if $[f] = [g]$, then $|\operatorname{Im} f| = |\operatorname{Im} g|$;*
5) *if $[f], [g] \neq \mathbb{R}_{+}^{\vee}$, then $\operatorname{Max} f = \operatorname{Max} g$ and $\operatorname{Min} f \subseteq \operatorname{Min} g$;*
6) *if $f(x) = g(x) = 1$ and $g = a_0 \vee a_1 f \vee \ldots \vee a_n f^n$, then $a_0 \vee a_1 \vee \ldots \vee a_n = 1$;*
7) *if $f(x) = 1$ and $f = a_0 \vee a_1 f \vee \ldots \vee a_n f^n$, then $f = a_0$ for $f < 1$ or $a_1 = 1$;*
8) $\big(a_0 \vee a_1 f \vee \ldots \vee a_n f^n\big)^m = a_0^m \vee a_1^m f^m \vee \ldots \vee a_n^m f^{nm}$.

Proof. Since $[g] \subseteq [f]$, it follows that a function g is presented as

$$g = a_0 \vee a_1 f \vee \ldots \vee a_n f^n. \tag{1}$$

Moreover, for any $i \geq 1$ and any $a > 0$, the inequalities $f(x) < f(y)$ and $af^i(x) < af^i(y)$ are equivalent. Therefore, statements 1) and 2) are true.

3) Let $[f] = [g]$ and $[f], [g] \neq \mathbb{R}_{+}^{\vee}$; in particular, $a_1 \vee \ldots \vee a_n > 0$. For any $i \geq 1$ and any $a > 0$, $\mathrm{Z}(af^i) = \mathrm{Z}(f)$. Therefore, by (1), we obtain that $\mathrm{Z}(g) \subseteq \mathrm{Z}(f)$. Similarly, $\mathrm{Z}(f) \subseteq \mathrm{Z}(g)$. Thus, $\mathrm{Z}(f) = \mathrm{Z}(g)$.

4) If $[f] = [g]$, then $[g] \subseteq [f]$ and $[f] \subseteq [g]$. Now, by statement 2), we obtain that $|\operatorname{Im} f| = |\operatorname{Im} g|$.

Summing statements 1) and 2), we get statement 5). Statements 6) and 8) are trivial.

7) Let $f(x) = 1$ and $f = a_0 \vee a_1 f \vee \ldots \vee a_n f^n$. Then $a_0, a_1, \ldots, a_n \leq 1$. If $0 < f(y) < 1$ and $a_1 < 1$, then $f(y) > a_1 f(y) \vee \ldots \vee a_n f^n(y)$, that is, $f(y) = a_0$. ☐

We now resolve the question of equality of subalgebras $[f]$.

Lemma 4. *For any subalgebras $[f]$ and $[g]$ are different from $\mathbb{R}_{+}^{\vee}$, we have:*

1) *if $|\operatorname{Im} f| = 2$, then $[f] = [g]$ if and only if*

$$|\operatorname{Im} g| = 2, \quad \operatorname{Max} f = \operatorname{Max} g, \quad \mathrm{Z}(f) = \mathrm{Z}(g). \tag{2}$$

2) *if $|\operatorname{Im} f| > 2$, then $[f] = [g]$ if and only if $f = rg$ for some $r > 0$.*

Proof. 1) Let $\operatorname{Im} f = \{a, b\}$, $a > b \geq 0$. If $[f] = [g]$, then by statements 3)–5) of Lemma 3 condition (2) holds.

Conversely, assume condition (2) holds. Then $\operatorname{Im} g = \{c, d\}$ for some $c > d \geq 0$, where

$$f^{-1}(a) = g^{-1}(c), \quad f^{-1}(b) = g^{-1}(d), \quad b = 0 \Longleftrightarrow d = 0.$$

We can assume without loss of generality that $a = c = 1$. If $b = d = 0$, then $f = g$, that is, $[f] = [g]$. If $b, d > 0$, then consider $m, n \in \mathbb{N}$ such that $b^m \leq d$ and $d^n \leq b$. Therefore, $f = b \vee g^n \in [g]$ and $g = d \vee f^m \in [f]$. Thus, $[f] \subseteq [g]$ and $[g] \subseteq [f]$, that is, $[f] = [g]$.

2) Let $|\operatorname{Im} f| > 2$. If $[f] = [g]$, then $|\operatorname{Im} f| = |\operatorname{Im} g|$ by statement 4) of Lemma 3. Since $f \in [g]$ and $f \in [g]$, $f = u(g)$ and $g = v(f)$ for some polynomials

$$u(g) = a_0 \vee a_1 g \vee \ldots \vee a_n g^n, \quad v(f) = b_0 \vee b_1 f \vee \ldots \vee b_m f^m. \tag{3}$$

Therefore,

$$f = (u \circ v)(f), \quad (u \circ v)(f) = c_0 \vee c_1 f \vee \ldots \vee c_k f^k, \quad c_1 = a_1 b_1. \tag{4}$$

We can assume without loss of generality that

$$1 = g(x) = f(x) > f(y) > f(z) \text{ for some } x, y, z \in X. \tag{5}$$

Then by statement 6) of Lemma 3 from (3), (4), and (5), we obtain that

$$a_0 \vee a_1 \vee \ldots \vee a_n = b_0 \vee b_1 \vee \ldots \vee b_m = c_0 \vee c_1 \vee \ldots \vee c_k = 1.$$

If $c_1 < 1$, then by statement 7) of Lemma 3 from (4), we obtain that $f(y) = f(z) = c_0$; a contradiction with $f(y) > f(z)$. Therefore, $c_1 = 1$. Since $a_1, b_1 \leq 1$, it follows that $a_1 = b_1 = 1$. Thus, by (3), we obtain that $f = u(g) \geq g$ and $g = v(f) \geq f$, that is, $f = g$.

Converse is obvious. □

Denote by $\mathbb{A}_1([f])$ the lattice of subalgebras $A \subseteq [f]$.

Lemma 5. *For any subalgebra* $[f]$, *we have*:

1) $|\operatorname{Im} f| = 1$ *if and only if* $|\mathbb{A}_1([f])| = 1$;
2) $|\operatorname{Im} f| = 2$ *and* $\mathrm{Z}(f) = \emptyset$ *if and only if* $|\mathbb{A}_1([f])| = 2$;
3) $|\operatorname{Im} f| = 2$ *and* $\mathrm{Z}(f) \neq \emptyset$ *if and only if* $|\mathbb{A}_1([f])| = 3$;
4) $|\operatorname{Im} f| \geq 3$ *if and only if* $|\mathbb{A}_1([f])| = \infty$.

In particular, there is a lattice characterization of the inequality $|\operatorname{Im} f| \geq 3$.

Proof. It suffices to prove the implications $\Rightarrow$.

1) Obviously, if $|\operatorname{Im} f| = 1$, then $\mathbb{A}_1([f]) = \{\mathbb{R}_+^{\vee}\}$.

2), 3). Let $|\operatorname{Im} f| = 2$. If $[g] \neq \mathbb{R}_+^{\vee}$ and $[g] \subseteq [f]$, then $|\operatorname{Im} g| \neq 1$, $\operatorname{Max} f = \operatorname{Max} g$, and $|\operatorname{Im} g| \leq |\operatorname{Im} f|$ by statements 2) and 5) of Lemma 3; in particular, $|\operatorname{Im} g| = 2$.

Assume that $\mathrm{Z}(f) = \emptyset$. Then $\mathrm{Z}(g) = \emptyset$, since $[g] \neq \mathbb{R}_+^{\vee}$ and $[g] \subseteq [f]$. Therefore, $[f] = [g]$ by Lemma 4. Thus, the lattice $\mathbb{A}_1([f])$ is the two-element chain $\mathbb{R}_+^{\vee} \subset [f]$.

Assume that $\operatorname{Im} f = \{a, 0\}$ for some $a > 0$. By lemma 4, if $\mathrm{Z}(g) = \emptyset$, then $[g] = [a/2 \vee f]$, and if $\mathrm{Z}(g) \neq \emptyset$, then $[f] = [g]$. Moreover, $[a/2 \vee f] \subset [f]$, since $a/2 \vee f \in [f]$ and $[a/2 \vee f] \neq [f]$ by Lemma 4. Thus, the lattice $\mathbb{A}_1([f])$ is the three-element chain $\mathbb{R}_+^{\vee} \subset [a/2 \vee f] \subset [f]$.

4) Let $|\operatorname{Im} f| \geq 3$. Then $[f^{2n}] \subset [f^n]$ for all $n \in \mathbb{N}$, since $f^{2n} = (f^n)^2 \in [f^n]$ and $[f^{2n}] \neq [f^n]$ by Lemma 4. Thus, the lattice $\mathbb{A}_1([f])$ is infinite. □

The lattice $\mathbb{A}_1(C^{\vee}(X))$ is complete, since every nonempty family $\{A_i\}_{i \in I}$ of its elements has an infimum — the intersection of subalgebras of the family, and a supremum — the subalgebra consisting of all possible finite sums of functions of the form $f_1 \cdot \ldots \cdot f_n$, where $f_1, \ldots, f_n \in \bigcup_{i \in I} A_i$, $n \in \mathbb{N}$. An element A of a complete lattice is said to be compact if for any nonempty family $\{A_i\}_{i \in I}$ of its elements, $A \leq \bigvee_{i \in I} A_i$ implies $A \leq \bigvee_{i \in J} A_i$ for some finite subset $J \subseteq I$. An element A of a lattice is said to be $\vee$-indecomposable if the fact that $A = B \vee C$ for some elements B and C of the lattice implies $A = B$ or $A = C$.

We give a lattice characterization of the subalgebras $[f]$.

Proposition 3. *The subalgebras* $[f]$ *are exactly* $\vee$*-indecomposable compact elements of the lattice* $\mathbb{A}_1(C^{\vee}(X))$.

Proof. Let $[f] \subseteq \bigvee_{i \in I} A_i$ for some family of subalgebras $\{A_i\}_{i \in I}$. Then there exist subalgebras $A_{i_1}, \ldots, A_{i_m}$, $m \in \mathbb{N}$ and functions $f_1, \ldots, f_n \in A_{i_1} \cup \ldots \cup A_{i_m}$, $n \in \mathbb{N}$ such that $f \in \mathbb{R}_+^{\vee}[f_1, \ldots, f_n]$. Thus, $[f] \subseteq A_{i_1} \vee \ldots \vee A_{i_m}$, that is, any subalgebra $[f]$ is a compact element.

Prove that any subalgebra $[f]$ is a $\vee$-indecomposable. Suppose the opposite, that is, $[f] = A \vee B$ for some subalgebras $A, B \subset [f]$. Then subalgebras $\mathbb{R}_+^{\vee}, A, B$, and $[f]$ are distinct. Thus, $|\operatorname{Im} f| \geq 3$ by Lemma 5, since $|\mathbb{A}_1([f])| \geq 4$.

Since $f \in A \vee B$, it follows that a function f has the form

$$f = a_1 u_1(f) v_1(f) \vee \ldots \vee a_n u_n(f) v_n(f), \tag{6}$$

where

$$u_1, \dots, u_n \in A \cap \mathbb{R}_+^{\vee}[f], \quad v_1, \dots, v_n \in B \cap \mathbb{R}_+^{\vee}[f].$$

We can assume without loss of generality that

$$1 = u_i(f(x)) = v_i(f(x)) = f(x) > f(y) > f(z) \text{ for some } x, y, z \in X \text{ and any } i = 1, \dots, n. \tag{7}$$

Since $1 > f(y) > f(z)$, by statement 7) of Lemma 3 from (6), we obtain that a polynomial $a_j u_j(f) v_j(f)$ contains a monomial f for some $j \in \{1, \dots, n\}$. Assume that

$$u_j(f) = b_0 \vee b_1 f \vee \dots \vee b_m f^m, \quad v_j(f) = c_0 \vee c_1 f \vee \dots \vee c_k f^k. \tag{8}$$

Then $a_j(b_0c_1 \vee b_1c_0) = 1$. Moreover, $a_j, b_0, b_1, c_0, c_1 \leq 1$, since from (6), (7), and (8), we obtain that

$$a_0 \vee a_1 \vee \dots \vee a_n = b_0 \vee b_1 \vee \dots \vee b_m = c_0 \vee c_1 \vee \dots \vee c_k = 1.$$

Therefore, $a_j = b_0 = c_1 = 1$ or $a_j = b_1 = c_0 = 1$.

If $a_j = b_0 = c_1 = 1$, then from (6) and (8), we obtain that

$$f \geq a_j u_j(f) v_j(f) \geq a_j b_0 v_j(f) = v_j(f), \quad v_j(f) \geq c_1 f = f.$$

Whence $f = v_j(f) \in B$, that is, $[f] \subseteq B$; a contradiction with $B \subset [f]$. Similarly, if $a_j = b_1 = c_0 = 1$, then $[f] \subseteq A$; a contradiction with $A \subset [f]$. Thus, a subalgebra $[f]$ is a $\vee$-indecomposable element.

Conversely, suppose that a subalgebra A is a $\vee$-indecomposable compact element of the lattice $\mathbb{A}_1(C^{\vee}(X))$. Since $A = \bigvee_{f \in A}[f]$, it follows that $A = [f_1] \vee \dots \vee [f_n]$ for some $f_1, \dots, f_n \in A$, $n \in \mathbb{N}$. We can assume without loss of generality that n takes the least possible value. If $n \geq 2$, then $A = [f_1] \vee ([f_2] \vee \dots \vee [f_n])$, which fact by the $\vee$-indecomposability of A means $A = [f_1]$ or $A = [f_2] \vee \dots \vee [f_n]$; a contradiction with the choice of n. Thus, $n = 1$. □

4. THE SUBALGEBRAS $[e_Z]$ AND $[\overline{e}_Z]$

Denote by $\overline{e}_Z$ a function $1/2 \vee e_Z$. By Lemma 4 for any subalgebra $[f]$, we have

$$\begin{aligned} &[f] = [e_Z] \iff (|\operatorname{Im} f| = 2,\ \mathrm{Z}(f) = Z), \\ &[f] = [\overline{e}_Z] \iff (|\operatorname{Im} f| = 2,\ \operatorname{Min} f = Z,\ \mathrm{Z}(f) = \emptyset). \end{aligned} \tag{9}$$

An element A of a lattice with zero is called a predatom if in a lattice there are exactly two elements less than A: zero and an atom of a lattice.

From Lemma 5, we obtain the following.

Lemma 6. *The subalgebras $[e_Z]$ and $[\overline{e}_Z]$ are exactly the predatoms and the atoms of the lattice $\mathbb{A}_1(C^{\vee}(X))$, respectively.*

Denote by $f|_Z$ the restriction of a function f on $Z \subseteq X$. If sets $Z_1, \dots, Z_n$ form a partition of a set X, then we will write $X = Z_1 \sqcup \dots \sqcup Z_n$.

Lemma 7. *For any subalgebras $[e_Z] \neq [e_{Z'}]$, we have $X = Z \sqcup Z'$ if and only if*

$$|\operatorname{Im} f| \leq 2 \text{ for any subalgebra } [f] \subseteq [e_Z] \vee [e_{Z'}]. \tag{10}$$

Proof. The necessity is obvious. Conversely, assume condition (10) holds. Consider the subalgebra $[e_Z \vee 2e_{Z'} \vee 3e_Z e_{Z'}] \subseteq [e_Z] \vee [e_{Z'}]$. It is easy to see that if $X \neq Z \sqcup Z'$, then $|\{0, 1, 2, 3\} \cap \operatorname{Im}(e_Z \vee 2e_{Z'} \vee 3e_Z e_{Z'})| \geq 3$; a contradiction with (10). Thus, $X = Z \sqcup Z'$. □

Remark 2. By Lemma 5 the inequality $|\operatorname{Im} f| \leq 2$ of condition (10) is a purely lattice one. From this fact, Proposition 3, and Lemma 6, we get that Lemma 10 contains a lattice characterization of subalgebras $[e_Z]$ and $[e_{Z'}]$ such that $X = Z \sqcup Z'$. Generally, we will repeatedly encounter a situation where properties for which lattice characterizations were obtained earlier are used for a lattice characterization of some property.

We give a lattice characterization of subalgebras $[f], |\operatorname{Im} f| < \infty$.

Proposition 4. *For any subalgebra* $[f] \neq \mathbb{R}_+^\vee$, *we have*:

1) $|\operatorname{Im} f| < \infty$ *if and only if* $[f] \subseteq [e_{Z_1}] \vee \ldots \vee [e_{Z_n}]$ *for some subalgebras* $[e_{Z_1}], \ldots, [e_{Z_n}]$;

2) $|\operatorname{Im} f| < \infty$ *and* $\mathrm{Z}(f) = \emptyset$ *if and only if* $[f] \subseteq [\overline{e}_{Z_1}] \vee \ldots \vee [\overline{e}_{Z_n}]$ *for some subalgebras* $[\overline{e}_{Z_1}], \ldots, [\overline{e}_{Z_n}]$.

Proof. 1) Let $\operatorname{Im} f = \{r_1, \ldots, r_n\}$, $n \geq 2$. Put $Z_1 = X \backslash f^{-1}(r_1), \ldots, Z_n = X \backslash f^{-1}(r_n)$. Then $[f] \subseteq [e_{Z_1}], \ldots, [e_{Z_n}]$, since $f = r_1 e_{Z_1} \vee \ldots \vee r_n e_{Z_n}$. The converse is obvious.

2) Let $\operatorname{Im} f = \{r_1, \ldots, r_n\}$, where $r_n > \ldots > r_1 > 0$ and $n \geq 2$. For each $i \in \{1, \ldots, n\}$, we choose $m_i \in \mathbb{N}$ such that $r_i/2^{m_i} < r_1$. Put $Z_1 = X \backslash f^{-1}(r_1), \ldots, Z_n = X \backslash f^{-1}(r_n)$. Then $[f] \subseteq [\overline{e}_{Z_1}], \ldots, [\overline{e}_{Z_n}]$, since $f = r_1 \overline{e}_{Z_1}^{m_1} \vee \ldots \vee r_n \overline{e}_{Z_n}^{m_n}$. The converse is obvious. □

5. SUBALGEBRAS OF SPECIAL TYPES

A subalgebra A is called: b-subalgebra, if $\sup f < \infty$ for any function $f \in A$; sp-subalgebra, if $\inf f > 0$ for any nonzero function $f \in A$; u-subalgebra, if $f > 0$ for any nonzero function $f \in A$. The sets of b-subalgebras, sp-subalgebras, and u-subalgebras are denote by *bSet*, *spSet*, and *uSet*, respectively.

The next statements are obvious:

1) let $pSet \in \{bSet, spSet, uSet\}$; then $A \in pSet$ if and only if $[f] \in pSet$ for all $[f] \subseteq A$;

2) $spSet \subseteq uSet$ and $\mathbb{R}_+^\vee \in bSet \cap spSet \cap uSet$;

3) if $|\operatorname{Im} f| < \infty$, then $[f] \in bSet$;

4) if $[f] \neq \mathbb{R}_+^\vee$, $|\operatorname{Im} f| < \infty$, and $\mathrm{Z}(f) = \emptyset$, then $[f] \in spSet$.

Remark 3. Prove that properties of a subalgebra A of being b-subalgebra, sp-subalgebra, and u-subalgebra are lattice. By the above and Lemma 4, it suffices to consider the case when $A = [f]$, $|\operatorname{Im} f| = \infty$.

We give a lattice characterization of sp-subalgebras $[f]$.

Case 1: $X = Z \sqcup Z'$ for some subalgebras $[e_Z]$ and $[e_{Z'}]$ such that $X = Z \sqcup Z'$ (see Lemma 7). We give lattice chracterizations of the sets E_Z and $E_{Z'}$, where

$$E_Z = \{[f] \colon \mathrm{Z}(f) = Z,\ \inf f|_{Z'} > 0\}, \quad E_{Z'} = \{[f] \colon \mathrm{Z}(f) = Z',\ \inf f|_Z > 0\}.$$

Lemma 8. $[f] \in E_Z$ *if and only if*

$$[e_Z] \subseteq [f] \vee [g] \text{ and } [\overline{e}_{Z'}] \subseteq [g] \text{ for some subalgebra } [g]. \tag{11}$$

Proof. Let $[f] \in E_Z$. We can assume without loss of generality that $\inf f|_{Z'} > 2$. Consider a function $g \in C^\vee(X)$ such that $g = 1/f$ on Z' and $g = 1$ on Z. Then condition (11) holds, since $[e_Z] = [fg] \subseteq [f] \vee [g]$ and $[\overline{e}_{Z'}] = [1/2 \vee g] \subseteq [g]$.

Conversely, assume condition (11) holds. Since $[\overline{e}_{Z'}] \subseteq [g]$, by statement 5) of Lemma 3, we have $\operatorname{Max} g = \operatorname{Max} \overline{e}_{Z'} = Z$; in particular, $\inf g > 0$ on Z. A function $e_Z \in [f] \vee [g]$ has the form

$$e_Z = a_{00} \vee a_{10} f \vee a_{01} g \vee \ldots \vee a_{mn} f^m g^n,$$

where $a_{0i} = 0$ for all $i \geq 0$, since $e_Z = 0$ on Z and $g > 0$ on Z. Therefore,

$$e_Z = f\big(a_{10} \vee a_{11} g \vee a_{20} f \vee \ldots \vee a_{mn} f^{m-1} g^n\big).$$

Since $\mathrm{Z}(e_Z) = Z$ and $\inf g > 0$ on Z, we get $\inf f > 0$ on Z' and $\mathrm{Z}(f) = Z$, Thus, $[f] \in E_Z$. □

We give a lattice characterization sp-subalgebras $[f] \neq \mathbb{R}_+^\vee$.

Lemma 9. *For any* $[f] \neq \mathbb{R}_+^\vee$, $[f] \in spSet$ *if and only if*

$$[f] \notin E_Z \cup E_{Z'} \text{ and } [f] \subseteq [g] \vee [h] \text{ for some } [g] \in E_Z, [h] \in E_{Z'}. \tag{12}$$

Proof. Let $[f] \in spA$. Then $[f] \notin E_Z \cup E_{Z'}$ and $\inf f > 0$, since $[f] \neq \mathbb{R}_+^\vee$. Put $g = fe_Z$ and $h = fe_{Z'}$. Then $[g] \in E_Z$, $[h] \in E_{Z'}$, and $[f] \subseteq [g] \vee [h]$, since $f = g \vee h$.

Conversely, assume condition (12) holds. Since $[g] \in E_Z$ and $[h] \in E_{Z'}$, we have $gh = 0$. From this fact and $[f] \subseteq [g] \vee [h]$, we obtain that a function f has the form

$$f = a_0 \vee a_1 g \vee \ldots \vee a_m g^m \vee b_1 h \vee \ldots \vee b_n h^n.$$

If $a_0 > 0$, then $\inf f \geq a_0$, that is, $[f] \in spA$.

Suppose $a_0 = 0$. Then $a_1 \vee \ldots \vee a_m \vee b_1 \vee \ldots \vee b_n > 0$, since $[f] \neq \mathbb{R}_+^\vee$. If $a_1 = \ldots = a_m = 0$ or $b_1 = \ldots = b_n = 0$, then $[f] \in E_{Z'}$ or $[f] \in E_Z$; a contradiction with (12). Therefore, $a_i, b_j > 0$ for some i, j; in particular, $\inf a_i g^i > 0$ on Z' and $\inf b_j h^j > 0$ on Z, since $[g] \in E_Z$ and $[h] \in E_{Z'}$. Thus, $[f] \in spA$, since $\inf f \geq \inf(a_i g^i \vee b_j h^j) > 0$. $\square$

Case 2: $X \neq Z \sqcup Z'$ for any subalgebras $[e_Z]$ and $[e_{Z'}]$ or, which is equivalent,

$$\text{there are not subalgebras } [e_Z] \neq \mathbb{R}_+^\vee. \tag{13}$$

Thus,

$$|\operatorname{Im}(r \vee f)| = \infty \text{ for any } [f] \neq \mathbb{R}_+^\vee, r < \sup f. \tag{14}$$

Indeed, suppose that $|\operatorname{Im}(r \vee f)| < \infty$; then put $Z = f^{-1}(a)$ for an arbitrary $a \in \operatorname{Im}(r \vee f)$ such that $a < \sup(r \vee f)$. Then Z is an open-closed set and $Z \subset X$. Thus, $e_Z \in C^\vee(X) \backslash \mathbb{R}_+^\vee$; a contradiction with (13).

Consider arbitrary subalgebras $[f]$ and $[g] \subseteq [f]$. Then $g = u(f)$ for some polynomial

$$u(f) = a_0 \vee a_1 f \vee \ldots \vee a_n f^n.$$

Suppose $g \notin \mathbb{R}_+^\vee$; in particular, $\operatorname{coz} g \neq \emptyset, g \neq a_0, n \geq 1$, and $a_1 \vee \ldots \vee a_n > 0$. Then there exists $m \leq n$ such that

$$g \neq a_0 \vee a_1 f \vee \ldots \vee a_{m-1} f^{m-1}.$$

The largest such m is called the degree of the polynomial u and denoted by $\deg u$. Denote by X_u the set

$$X_u = \bigl\{x \in X \colon a_m f^m(x) > a_0 \vee a_1 f(x) \vee \ldots \vee a_{m-1} f^{m-1}(x)\bigr\}.$$

It is easy to see that $X_u \neq \emptyset$ and $X_u \subseteq \operatorname{coz} g$.

Note that if $0 \leq i \leq m-1$, then

$$a_m f^m > a_i f^i \iff f > \sqrt[n-i]{a_i/a_m}.$$

Denote by r_u the number $\bigvee_{i=0}^{m-1} \sqrt[n-i]{a_i/a_m}$. Then

$$r_u < \sup f, \quad X_u = \{x \in X \colon f(x) > r_u\}.$$

If $g \in \mathbb{R}_+^\vee$, then put $\deg u = 0$.

Lemma 10. *For any subalgebras $[f]$ and $[g] \subseteq [f]$, we have:*

1) *if $g = u(f) = v(f) \in [f]$, then $\deg u = \deg v$;*
2) *if $g = u(f) \in [f]$ and $v(g) \in [g]$, then $\deg v \circ u = \deg v \deg u$.*

Proof. 1) If $g \in \mathbb{R}_+^\vee$, then $\deg u = \deg v = 0$. Let $g \notin \mathbb{R}_+^\vee$. Then $\deg u = n \geq 1$, $\deg v = m \geq 1$, and $[f] \neq \mathbb{R}_+^\vee$. Moreover, $r_u \vee r_v < \sup f$, and if $f > r_u \vee r_v$, then $u(f) = a_n f^n = b_m f^m = v(f)$ for some $a_n, b_m > 0$. If $n \neq m$, then $|\operatorname{Im}(r_u \vee r_v \vee f)| \leq 2$; a contradiction with (14). Thus, $n = m$.

2) If $\deg u = 0$ or $\deg v = 0$, then $\deg v \circ u = \deg v \deg u = 0$, since $v \circ u \in \mathbb{R}_+^\vee$.

Suppose $\deg u = n \geq 1$ and $\deg v = m \geq 1$, where

$$u(f) = a_0 \vee a_1 f \vee \ldots \vee a_n f^n, \quad v(g) = b_0 \vee b_1 g \vee \ldots \vee b_m g^m.$$

Then

$$(v \circ u)(f) = (b_0 \vee b_1 a_0 \vee \ldots \vee b_m a_0^m) \vee b_1 a_1 f \vee \ldots \vee b_m a_n^m f^{mn}. \tag{15}$$

Moreover, $g = a_n f^n$ on $f > r_u$ and $v(g) = b_m g^m$ on $g > r_v$, where $\sup f > r_u$ and $\sup g > r_v$. Therefore, if $f > r_u$, then $g > r_v$ and $f > \sqrt[n]{r_v/a_n}$ are equivalent inequalities; in particular, $\sup f > \sqrt[n]{r_v/a_n}$, since $\sup g > r_v$. Therefore, $\sup f > r_u \vee \sqrt[n]{r_v/a_n}$. If $f > r_u \vee \sqrt[n]{r_v/a_n}$, then $(v \circ u)(f) = b_m a_n^m f^{mn}$. From this fact and (15), we obtain that $\deg v \circ u = mn = \deg v \deg u$. □

Lemma 11. *For any subalgebra $[f] \neq \mathbb{R}_+^{\vee}$, the following conditions are equivalent*:

1) *$f = r \vee g$ for some $g \in C^{\vee}(X)$, $r > \inf g$;*

2) *there is a family of subalgebras $\{[g_i]\}_{i \in \mathbb{N}}$ such that*

$$[f] \subset \ldots \subset [g_{i+1}] \subset [g_i] \subset \ldots \subset [g_1]. \tag{16}$$

Proof. 1) $\Rightarrow$ 2). Assume condition 1) holds. Then $r < \sup g = \sup f$, since $[f] \neq \mathbb{R}_+^{\vee}$. Put

$$r_i = \frac{\inf g + ir}{1+i}, \quad g_i = g \vee r_i, \ i \in \mathbb{N}.$$

Then

$$\inf g < r_i < r_{i+1} < r, \quad \sup g_i = \sup g, \quad f = r \vee g_{i+1}, \quad g_{i+1} = r_{i+1} \vee g_i, \quad \frac{f}{g_{i+1}}, \frac{g_{i+1}}{g_i} \notin \mathbb{R}_+^{\vee}.$$

From this fact and Lemma 4, we get $[f] \subset [g_{i+1}] \subset [g_i]$, that is, condition (16) holds.

2) $\Rightarrow$ 1). Assume condition 2) holds. Suppose

$$f \neq r \vee g \text{ and } g_i \neq r \vee g \text{ for any } g \in C^{\vee}(X), r > \inf g, i \in \mathbb{N}. \tag{17}$$

Since $\mathbb{R}_+^{\vee} \neq [f] \subset [g_{i+1}] \subset [g_i]$, there exist polynomials $u_i(g_i), v_i(g_i) \in [g_i]$ such that

$$f = u_i(g_i), \quad g_{i+1} = v_i(g_i), \quad \deg u_i \geq 2, \quad \deg v_i \geq 2.$$

Take $k \in \mathbb{N}$ such that $2^k > \deg u_1$. Since

$$u_1(g_1) = f = (u_k \circ v_{k-1} \circ \ldots \circ v_2 \circ v_1)(g_1),$$

by Lemma 10, we obtain

$$\deg u_1 = \deg(u_k \circ v_{k-1} \circ \ldots \circ v_2 \circ v_1) = \deg u_k \deg v_{k-1} \cdot \ldots \cdot \deg v_1 \geq 2^k > \deg u_1; \tag{18}$$

a contradiction. Therefore, condition (17) does not hold. Hence, $[f] \subset [r \vee g]$ for some $g \in C^{\vee}(X)$, $r > \inf g$. Therefore, a function f is presented as

$$f = a_0 \vee a_1(r \vee g) \vee \ldots \vee a_n(r \vee g)^n,$$

where $a_n > 0$ and $n \geq 1$, since $[f] \neq \mathbb{R}_+^{\vee}$. Thus, $f = r' \vee g'$, where

$$r' = a_0 \vee a_1 r \vee \ldots \vee a_n r^n, \quad g' = a_1 g \vee \ldots \vee a_n g^n,$$

and $r' > \inf g'$, since $r > \inf g$. □

We give a lattice characterization of sp-subalgebras $[f]$.

Lemma 12. *For any subalgebra $[f] \neq \mathbb{R}_+^{\vee}$, the following conditions are equivalent*:

1) *$[f] \in spSet$;*

2) *there are subalgebras $[g]$ and $[h]$ are different from $\mathbb{R}_+^{\vee}$ such that $[f] \subseteq [g] \vee [h]$, $g = g' \vee r_g$, and $h = h' \vee r_h$ for some $g', h' \in C^{\vee}(X)$, r_g, r_h, where $r_g > \inf g'$ and $r_h > \inf h'$.*

Proof. Note that by Lemma 11, condition 2) is lattice.

1) $\Rightarrow$ 2). Let $[f] \neq \mathbb{R}_+^\vee$ and $[f] \in spSet$. We can assume without loss of generality that $f > 1$. By condition (14), $|\operatorname{Im} f| = \infty$. Therefore, there exist $x, y \in X$ and $a, b, c \in \mathbb{R}_+^\vee$ such that

$$a < f(x) < b < f(y) < c < \sup f.$$

Put

$$E = \{z \in X\colon f(z) \notin (a, b)\}, \quad F = \{z \in X\colon f(z) \notin (b, c)\}.$$

Then $y \in E \backslash F$, $x \in F \backslash E$, and E, F are a closed sets. Since X is a Tikhonov space, it follows that there exist functions $u, v \in C^\vee(X)$ such that

$$u, v \leq 1, \quad u(x) = \frac{1}{2f(x)}, \quad v(y) = \frac{1}{2f(y)}, \quad u|_E = v|_F = 1.$$

Since $X = E \cup F$, we have $v \vee u = 1$. Put

$$g = 1 \vee vf, \quad h = 1 \vee uf, \quad g' = vf, \quad h' = uf, \quad r_g = r_h = 1.$$

Therefore, condition 2) holds, since

$$f = g \vee h, \quad g'(x) = f(x) > 1 > 1/2 = g'(y) \geq \inf g', \quad h'(y) = f(y) > 1 > 1/2 = h'(x) \geq \inf h'.$$

Implication 2) $\Rightarrow$ 1) is obvious. $\square$

From Remark 3) and Lemmas 9 and 12, we obtain following.

Proposition 5. *There is a lattice characterization of sp-subalgebras.*

Remark 4. We give a lattice characterization of b-subalgebras A. It is easy to see that $[f] \in bSet$ if and only if $[g] \in bSet$ for any sp-subalgebra $[g] \subseteq [f]$. From this fact, Remark 3), and Proposition 5, we obtain that it suffices to consider the case when $A = [f] \in spSet$, $|\operatorname{Im} f| = \infty$.

Lemma 13. *For any sp-subalgebra $[f], |\operatorname{Im} f| = \infty$, the following conditions are equivalent:*

1) $[f] \in bSet$;

2) *for any sp-subalgebra $[g] \in spSet \backslash \mathbb{R}_+^\vee$, there exist subalgebras $[u], [v], [h] \in spSet$ such that*

$$[u] \subseteq [f] \vee [v], \quad [u] \nsubseteq [v], \quad [g] \subseteq [u] \vee [h], \quad [g] \nsubseteq [h].$$

Proof. 1) $\Rightarrow$ 2). Let $[f]$ is an sp-, b-subalgebra and $|\operatorname{Im} f| = \infty$. If $[g]$ is an sp-subalgebra and $[g] \neq \mathbb{R}_+^\vee$, then $g(y) > g(x)$ for some $x, y \in X$. Since $|\operatorname{Im} f| = \infty$, it follows that there exist points $x', y' \in X \backslash \{x, y\}$ such that $f(y') > f(x')$. Since X is a Tikhonov space, it follows that there exists a function $u \in C^\vee(X)$ such that

$$u(y) > u(x), \quad u(y') > u(x'), \quad \inf u > 0, \quad \sup u < \infty.$$

Put $h = g/u^m$ and $v = u/f^n$, where $m, n \in \mathbb{N}$ such that $h(y) < h(x)$ and $v(y) < v(x)$. Since

$$u(y') > u(x'), \quad v(y') < v(x'), \quad g(y) > g(x), \quad h(y) < h(x),$$

it follows that by statement 1) of Lemma 3, we have $[u] \nsubseteq [v]$ and $[g] \nsubseteq [h]$. Moreover, $[u] \subseteq [f] \vee [v]$ and $[g] \subseteq [u] \vee [h]$, since $u = vf^n \in [f] \vee [v]$ and $g = hu^m \in [u] \vee [h]$. Finally, $[u]$, $[v]$, and $[h]$ are sp-subalgebras, since $\inf g, \inf u > 0$, $\sup u$, and $\sup f < \infty$. Thus, condition 2) holds.

2) $\Rightarrow$ 1). Let $[f] \in spSet$ and condition 2) holds. Suppose that $[f] \notin bSet$. Put $g = r \vee 1/f$ for some r such that $\sup 1/f > r > 0$. Then $[g] \neq \mathbb{R}_+^\vee$ and $[g] \in bSet \cap spSet$. Since $[u] \subseteq [f] \vee [v]$, it follows that a function u has the form

$$u = a_{00} \vee a_{10} f \vee a_{01} v \vee \ldots \vee a_{mn} f^m v^n$$

and $a_{ij} > 0$ for some $i \geq 1$, $j \geq 0$, since $[u] \nsubseteq [v]$. Then $\sup u \geq \sup a_{ij} f^i v^j = \infty$, since $\sup f = \infty$ and $\inf v > 0$. Therefore, $\sup u = \infty$. From this fact, $[h] \in spSet$, $[g] \subseteq [u] \vee [h]$, and $[g] \nsubseteq [h]$, we get that $\sup g = \infty$; a contradiction. Thus, $[f] \in bSet$. $\square$

From Remark 4) and Lemma 13, we obtain following.

Proposition 6. *There is a lattice characterization of b-subalgebras.*

Remark 5. We give a lattice characterization of u-subalgebras A. Since $spSet \subseteq uSet$, from Remark 3) and Proposition 5, we obtain that it suffices to consider the case when $A = [f] \in uSet\backslash spSet$.

We give a lattice characterization subalgebras $[f] \in (bSet \cap uSet)\backslash spSet$.

Lemma 14. *For any subalgebra $[f] \in bSet\backslash spSet$, the following conditions are equivalent*:
1) $[f] \in uSet$;
2) *there exists subalgebra $[g] \in spSet\backslash bSet$ such that the following conditions are hold*:
2.1) $[g] \subseteq [f] \vee [v]$ *for some subalgebra* $[v] \subset [g]$, $[v] \neq \mathbb{R}_+^{\vee}$;
2.2) $[u] \subseteq [f]$ *for any subalgebra* $[u] \subseteq [f] \vee [g]$, $[u] \notin spSet$.

Proof. 1) $\Rightarrow$ 2). Let $[f] \in bSet\backslash spSet$ and $[f] \in uSet$. Then $\inf f = 0$, $\sup f < \infty$, and $f > 0$. Put $g = 1/f$ and $v = 1/f^2$. Then $[g] \in spSet\backslash bSet$, $[g] \subseteq [f] \vee [v]$, and $[v] \subset [g]$, since $g = fv$, $v = g^2$, and $[v] \neq [g]$ by Lemma 4. Thus, condition 2.1) holds.

Let $[u] \notin spSet$ and $[u] \subseteq [f] \vee [g]$. Since $fg = 1$, $\inf u = 0$, and $\inf g > 0$, it follows that a function u has the form

$$u = a_1 f \vee \ldots \vee a_m f^m.$$

Thus, $[u] \subseteq [f]$, that is, condition 2.2) holds.

2) $\Rightarrow$ 1). Let condition 2) holds. Assume that $[f] \notin uSet$. Since $[f] \notin spSet$, it follows that $f(x) = 0$ for some point $x \in X$; in particular, $[f^i v^j] \notin spSet$ for any $i \geq 1$ and any $j \geq 0$. Moreover, $[f^i v^j] \subseteq [f] \vee [g]$, since $[v] \subset [g]$. Whence $[f^i v^j] \subseteq [f]$ by condition 2.2), that is, $f^i v^j \in [f]$. From this fact, $\sup f < \infty$, $\sup g = \infty$, and $[g] \subseteq [f] \vee [v]$, we obtain that a function g has the form

$$g = a_0 \vee a_1 f \vee \ldots \vee a_m f^m \vee b_1 v \vee \ldots \vee b_n v^n,\ b_n > 0,\ n \geq 1. \tag{19}$$

Therefore,

$$b_n v^n / g \leq 1. \tag{20}$$

Moreover, since $[g] \in spSet$, $[v] \subset [g]$, and $[v] \neq \mathbb{R}_+^{\vee}$, we have $\inf v > 0$.

Since $[v] \subset [g]$ and $[v] \neq \mathbb{R}_+^{\vee}$, it follows that a function v has a form

$$v = c_0 \vee c_1 g \vee \ldots \vee c_k g^k,\ c_k > 0, k \geq 1.$$

Whence $v/g = c_0/g \vee c_1 \vee \ldots \vee c_k g^{k-1}$.

If $k \geq 2$, then $\sup v/g \geq c_k g^{k-1} = \infty$, since $\sup g = \infty$ and $\inf v > 0$. Therefore, $\sup b_n v^n / g = \infty$, since $\inf v > 0$; a contradiction with (20).

Assume that $k = 1$. Then $v = c_0 \vee c_1 g$, where $c_0 > c_1 g(y)$ for some point $y \in X$, since $[v] \neq [g]$ and $c_1 > 0$.

If $n \geq 2$, then $\sup b_n v^n / g \geq \sup b_n c_1^n g^{n-1} = \infty$, since $\sup g = \infty$; a contradiction with (20). Thus, $n = 1$. Together with (19) and $v = c_0 \vee c_1 g$ this gives that

$$g = (a_0 \vee b_1 c_0) \vee a_1 f \vee \ldots \vee a_m f^m \vee b_1 c_1 g. \tag{21}$$

Whence $g(y) \geq b_1 c_0$ and $b_1 c_1 = 1$, since $\sup f < \infty$ and $\sup g = \infty$. Moreover, $b_1 c_0 > b_1 c_1 g(y) = g(y)$, since $c_0 > c_1 g(y)$ and $b_1 c_1 = 1$; a contradiction. Thus, $[f] \in uSet$. $\square$

We give a lattice characterization of subalgebras $[f] \in uSet\backslash(spSet \cup bSet)$.

Lemma 15. *For any subalgebra $[f] \notin spSet \cup bSet$, the following conditions are equivalent*:
1) $[f] \in uSet$;
2) *if* $[h] \subseteq [f] \vee [g]$, *where* $[g] \neq \mathbb{R}_+^{\vee}$, $[g] \in bSet \cap uSet$, *and* $[h] \in bSet\backslash spSet$, *then* $[h] \in uSet$.

Proof. 1) $\Rightarrow$ 2). Let $[f] \notin spSet \cup bSet$, $[f] \in uSet$, $[g] \in (bSet \cap uSet) \backslash \mathbb{R}_{+}^{\vee}$; in particular, $f, g > 0$. Therefore, if $[h] \in bSet \backslash spSet$ and $[h] \subseteq [f] \vee [g]$, then $h > 0$. Thus, $[h] \in uSet$.

2) $\Rightarrow$ 1). Assume that condition 2) holds. Put $g = 1 \wedge 1/(1 \vee f)$. Then $1 \geq g > 0$, $\inf g = 0$, and $|\operatorname{Im} g| = \infty$, since $\sup f = \infty$. Thus, $[g] \in bSet \cap uSet$ and $[g] \neq \mathbb{R}_{+}^{\vee}$.

Put $h = fg$. Then $[h] \subseteq [f] \vee [g]$. Moreover $[h] \in bSet \backslash spSet$, since $\inf f = 0$, $h = 1$ on $f \geq 1$, and $h = f$ on $f < 1$. Therefore, $[h] \in uSet$ by condition 2), that is, $h > 0$. Whence $f > 0$, since $h = fg$. Thus, $[f] \in uSet$. $\square$

From Remark 5) and Lemmas 14 and 15, we obtain following.

Proposition 7. *There is a lattice characterization of u-subalgebras.*

Denote by $\mathbb{A}_1(uC^{\vee}(X))$ the lattice of u-subalgebras. It is easy to see that the correspondence

$$\beta_X \colon \mathbb{A}_1(uC^{\vee}(X)) \to \mathbb{A}_1(U^{\vee}(X)),\ \beta_X \colon A \mapsto A \backslash \{0\}$$

is an isomorphism.

The proof of Theorem 1 (see Remark 1). Let α be an isomorphism of the latttice $\mathbb{A}_1(C^{\vee}(X))$ onto the lattice $\mathbb{A}_1(C^{\vee}(Y))$, where X and Y are Hewitt spaces. By Proposition 7, the restriction of α on the lattice $\mathbb{A}_1(uC^{\vee}(X))$ is an isomorphism of the lattice $\mathbb{A}_1(uC^{\vee}(X))$ onto the lattice $\mathbb{A}_1(uC^{\vee}(Y))$, Therefore, the map $\beta_Y \circ \alpha \circ \beta_X^{-1}$ is an isomorphism of the lattice $\mathbb{A}_1(U^{\vee}(X))$ onto the lattice $\mathbb{A}_1(U^{\vee}(Y))$. Thus (see [6, Theorem 2]), $X \approx Y$.

Acknowledgments. The paper was prepared within the framework of the state comission of the Ministry of Science and Higher Education of the Russian Federation, project no. 1.5879.2017/8.9.

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