[E-mail:]taieb5555@yahoo.com

Abstract

Abstract: *In this paper, we study singular fractional systems of nonlinear integro-differential equations. We investigate the existence and uniqueness of solutions by means of Schauder fixed point theorem and using the contraction mapping principle. Moreover, we define and study the Ulam-Hyers stability and the generalized Ulam-Hyers stability of solutions. Some applications are presented to illustrate the main results.*

Keywords: Caputo derivative, fixed point, singular fractional integro-differential equation, existence, uniqueness, Ulam-Hyers stability.

Stability of Singular Fractional Systems of Nonlinear Integro-Differential Equations

A. TAIEB

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TAIEB
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1 Introduction and Preliminaries

Recently, the field of fractional calculus has attracted interest of researchers in several areas including mathematics, physics, chemistry, engineering and even finance and social sciences. For the basic theory of fractional differential equations and its applications, see [2, 5, 7]. Moreover, some authors have established the existence and uniqueness of solutions for some fractional systems, see [1, 3, 8]. Other authors paid much attention to the boundary value problems of singular fractional differential equations, see [6, 10]. On the other hand, since it is quite useful in many applications, numerical analysis, biology and economics, the Ulam-Hyers stability problems have been attracted by many researchers, for more details, the readers can refer to the recent contributions [4, 9, 10].

The present paper is devoted to build several results on the existence, uniqueness, Ulam-Hyers stability and the generalized Ulam-Hyers stability of solutions for the following singular fractional systems of nonlinear integro-differential equations:

$$
\begin{cases}
D^{\alpha_1}u_1(t)=f_1\left(t,u_2(t),D^{\beta_1}u_2(t),\int\limits_0^t g_1(\tau,u_2(\tau))\,d\tau\right), \\
0<t\leq 1, \\
\vdots \\
D^{\alpha_{n-1}}u_{n-1}(t)=f_{n-1}\left(t,u_n(t),D^{\beta_{n-1}}u_n(t),\int\limits_0^t g_{n-1}(\tau,u_n(\tau))\,d\tau\right), \\
0<t\leq 1, \\
D^{\alpha_n}u_n(t)=f_n\left(t,u_1(t),D^{\beta_n}u_1(t),\int\limits_0^t g_n(\tau,u_1(\tau))\,d\tau\right), \\
0<t\leq 1, \\
n-1<\alpha_k<n,\ 0<\beta_k<\alpha_k, \\
u_k^{(j)}(0)=0,\ j=0,1,...,n-2,\ k=1,2,...,n, \\
D^{\mu_k}u_k(1)+J^{\delta_k}u_k(1)=0,\ n-2<\mu_k<n-1,\ \delta_k>0,
\end{cases} \tag{1}
$$

where $n\in\mathbb{N}^*\setminus\{1\}$. The operators D^{α_k} and D^{μ_k} are the derivatives in the sense of Caputo, defined by

$$D^{\gamma}u(t)=\frac{1}{\Gamma(m-\gamma)}\int_0^t (t-s)^{m-\gamma-1}u^{(m)}(s)\,ds=J^{m-\gamma}u^{(m)}(t), \tag{2}$$

with $m-1<\gamma<m$, $m\in\mathbb{N}^*$. The functions $f_k:(0,1]\times\mathbb{R}^3\to\mathbb{R}$ are continuous, singular at $t=0$, and $\lim\limits_{t\to 0^+}f_k(t)=\infty$. And J^{δ_k} are the Riemann-Liouville fractional integrals. Where the Riemann-Liouville fractional integral J of order $\alpha\geq 0$ for a continuous function f on $[0,\infty)$ is defined by:

$$J^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds,\ \alpha>0,\ t\geq 0, \tag{3}$$

with $\Gamma(\alpha):=\int_0^{\infty}e^{-x}x^{\alpha-1}dx$.

First, we list some well known properties of the fractional calculus theory which can be found in [5].

(i) : For $\alpha,\beta>0$; $n-1<\alpha<n$, we have $D^{\alpha}t^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}t^{\beta-\alpha-1},\beta>n$, and $D^{\alpha}t^{j}=0,\ j=0,1,...,n-1$.

(ii) : $D^pJ^qf(t)=J^{q-p}f(t)$, where $q>p>0$ and $f\in L^1([a,b])$.

(iii) : Let $n\in\mathbb{N}^*$, $n-1<\alpha<n$, and $D^{\alpha}u(t)=0$. Then, $u(t)=\sum\limits_{j=0}^{n-1}c_j\,t^j$, and $J^{\alpha}D^{\alpha}u(t)=u(t)+\sum\limits_{j=0}^{n-1}c_j\,t^j$, $(c_j)_{j=0,1,...,n-1}\in\mathbb{R}$.

The following Lemma is fundamental to prove our existence results

Lemma 1 *(Shauder fixed point theorem) Let (E,d) be a complete metric space, let X be a closed convex subset of E, and let $A:E\to E$ be a mapping such that the set $Y:=\{Ax:x\in X\}$ is relatively compact in E. Then A has at last one fixed point.*

From the following auxiliary result, we will impart the solution of system (1).

Lemma 2 *Let given* $n \in \mathbb{N}^* \setminus \{1\}$; $n-1 < \alpha_k < n$, $k = 1, 2, ..., n$, *and a family* $(U_k)_{k=1,...n} \in C([0,1], \mathbb{R})$. *Then, the unique solution of*

$$\begin{cases} D^{\alpha_1} u_1(t) = U_1(t), \\ \vdots \\ D^{\alpha_n} u_n(t) = U_n(t), \\ u_k^{(j)}(0) = 0, j = 0, 1, ..., n-2, \\ D^{\mu_k} u_k(1) + J^{\delta_k} u_k(1) = 0, n-2 < \mu_k < n-1, \ \delta_k > 0, \end{cases} \tag{4}$$

is given by $(u_1, u_2, ..., u_n)(t)$, *where*

$$\begin{aligned} u_k(t) = \int_0^t \tfrac{(t-s)^{\alpha_k - 1}}{\Gamma(\alpha_k)} U_k(s)\, ds - \tfrac{\Gamma(n-\mu_k)\Gamma(n+\delta_k) t^{n-1}}{(n-1)!(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \\ \times \int_0^1 \left(\tfrac{(1-s)^{\alpha_k-\mu_k-1}}{\Gamma(\alpha_k-\mu_k)} + \tfrac{(1-s)^{\alpha_k+\delta_k-1}}{\Gamma(\alpha_k+\delta_k)} \right) U_k(s)\, ds. \end{aligned} \tag{5}$$

Proof. The property (iii) allow us to write the problem (4) to an equivalent integral equations:

$$u_k(t) = \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} U_k(s)\, ds - \sum_{j=0}^{n-1} c_j^k t^j, \tag{6}$$

with,

$$\begin{pmatrix} c_0^1 & c_1^1 & \cdots & c_{n-1}^1 \\ c_0^2 & c_1^2 & \cdots & c_{n-1}^2 \\ \vdots & \vdots & \vdots & \vdots \\ c_0^n & c_1^n & \cdots & c_{n-1}^n \end{pmatrix} \in M_n(\mathbb{R}).$$

Then, we can state that

$$\begin{cases} u_k^{(j)}(0) = -j! c_j^k, \ j = 0, 1, ..., n-2, \\ D^{\mu_k} u_k(1) = \int_0^1 \tfrac{(1-s)^{\alpha_k-\mu_k-1}}{\Gamma(\alpha_k-\mu_k)} U_k(s)\, ds - \tfrac{\Gamma(n)}{\Gamma(n-\mu_k)} c_{n-1}^k, \\ J^{\delta_k} u_k(1) = \int_0^1 \tfrac{(1-s)^{\alpha_k+\delta_k-1}}{\Gamma(\alpha_k+\delta_k)} U_k(s)\, ds - \tfrac{\Gamma(n)}{\Gamma(n+\delta_k)} c_{n-1}^k. \end{cases}$$

Applying the conditions given in (4), we get

$$c_j^k = 0, \ j = 0, 1, ..., n-2,$$

$$c_{n-1}^k = \tfrac{\Gamma(n-\mu_k)\Gamma(n+\delta_k)}{(n-1)!(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \int_0^1 \left(\tfrac{(1-s)^{\alpha_k-\mu_k-1}}{\Gamma(\alpha_k-\mu_k)} + \tfrac{(1-s)^{\alpha_k+\delta_k-1}}{\Gamma(\alpha_k+\delta_k)} \right) U_k(s)\, ds. \tag{7}$$

Substituting (7) in (6), we get (5). This completes the proof. ■

Now, we introduce the Banach space

$$B := \left\{ (u_1, u_2, ..., u_n) : u_k \in C\left([0,1], \mathbb{R}\right),\ D^{\beta_{k-1}} u_k \in C\left([0,1], \mathbb{R}\right),\ k = 1, 2, ..., n \right\},$$

where $n \in \mathbb{N}^* \setminus \{1\}$, $\beta_0 = \beta_n$, endowed with the norm:

$$\|(u_1, u_2, ..., u_n)\|_B = \max_{1 \le k \le n} \left(\|u_k\|_\infty, \left\| D^{\beta_{k-1}} u_k \right\|_\infty \right).$$

2 Existence and Uniqueness

This section is devoted to establish sufficient conditions for the existence and uniqueness of solutions to system (1). Then, we will give some applications of our Theorems.

Define the nonlinear operator $T : B \to B$ by

$$T\left(u_1, ..., u_n\right)(t) := \left(T_1\left(u_2\right)(t), ..., T_{n-1}\left(u_n\right)(t), T_n\left(u_1\right)(t)\right),$$

with

$$\begin{gathered} T_k\left(u_{k+1}\right)(t) := \\ \int_0^t \frac{(t-s)^{\alpha_k - 1}}{\Gamma(\alpha_k)} f_k \left(s, u_{k+1}(s), D^{\beta_k} u_{k+1}(s), \int\limits_0^s g_k\left(\tau, u_{k+1}(\tau)\right) d\tau \right) ds \\ - \frac{\Gamma(n-\mu_k)\Gamma(n+\delta_k) t^{n-1}}{(n-1)!(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \\ \times \int_0^1 \left(\frac{(1-s)^{\alpha_k - \mu_k - 1}}{\Gamma(\alpha_k - \mu_k)} + \frac{(1-s)^{\alpha_k + \delta_k - 1}}{\Gamma(\alpha_k + \delta_k)} \right) f_k \left(\begin{array}{c} s, u_{k+1}(s), D^{\beta_k} u_{k+1}(s), \\ \int\limits_0^s g_k\left(\tau, u_{k+1}(\tau)\right) d\tau \end{array} \right) ds. \end{gathered} \tag{8}$$

for all $t \in [0,1]$, $k = 1, 2, ..., n$, $n \in \mathbb{N}^* \setminus \{1\}$, where $u_{n+1} = u_1$.

Lemma 3 *Let $n-1 < \alpha_k < n$, $k = 1, 2, ..., n$, $n \in \mathbb{N}^* \setminus \{1\}$, $\Upsilon_k : (0,1] \to \mathbb{R}$ are continuous, and $\lim\limits_{t \to 0^+} \Upsilon_k(t) = \infty$. Suppose that there exist constants $0 < \eta_k < 1$, such that $t^{\eta_k} \Upsilon_k(t)$ are continuous for all $t \in [0,1]$. Then*

$$\begin{gathered} u_k(t) = \int_0^t \frac{(t-s)^{\alpha_k - 1}}{\Gamma(\alpha_k)} \Upsilon_k(s)\, ds - \frac{\Gamma(n-\mu_k)\Gamma(n+\delta_k) t^{n-1}}{(n-1)!(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \\ \times \int_0^1 \left(\frac{(1-s)^{\alpha_k - \mu_k - 1}}{\Gamma(\alpha_k - \mu_k)} + \frac{(1-s)^{\alpha_k + \delta_k - 1}}{\Gamma(\alpha_k + \delta_k)} \right) \Upsilon_k(s)\, ds, \end{gathered}$$

are continuous on $[0,1]$.

Proof. By the continuity of $t^{\eta_k} \Upsilon_k$ and

$$\begin{gathered} u_k(t) = \int_0^t \frac{(t-s)^{\alpha_k - 1} s^{-\eta_k}}{\Gamma(\alpha_k)} s^{\eta_k} \Upsilon_k(s)\, ds - \frac{\Gamma(n-\mu_k)\Gamma(n+\delta_k) t^{n-1}}{(n-1)!(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \\ \times \int_0^1 \left(\frac{(1-s)^{\alpha_k - \mu_k - 1}}{\Gamma(\alpha_k - \mu_k)} + \frac{(1-s)^{\alpha_k + \delta_k - 1}}{\Gamma(\alpha_k + \delta_k)} \right) s^{-\eta_k} s^{\eta_k} \Upsilon_k(s)\, ds, \end{gathered} \tag{9}$$

it is clear that $u_k(0)=0,\ k=1,2,...,n.$

Now, we splite the proof into three cases. Case 1: For $t_0=0$ and $\forall t\in(0,1]$, for all $k=1,2,...,n$, $t^{\eta_k}\Upsilon_k(t)$ are continuous. So, there exist positive constants C_k, such that: $|t^{\eta_k}\Upsilon_k(t)|\leq C_k$, $\forall t\in[0,1]$. Then,

$$
\begin{aligned}
&|u_k(t)-u_k(0)| \\
&=\left|\begin{array}{c} \int_0^t \frac{(t-s)^{\alpha_k-1}s^{-\eta_k}}{\Gamma(\alpha_k)}s^{\eta_k}\Upsilon_k(s)\,ds-\frac{\Gamma(n-\mu_k)\Gamma(n+\delta_k)t^{n-1}}{(n-1)!(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \\ \times\int_0^1\left(\frac{(1-s)^{\alpha_k-\mu_k-1}}{\Gamma(\alpha_k-\mu_k)}+\frac{(1-s)^{\alpha_k+\delta_k-1}}{\Gamma(\alpha_k+\delta_k)}\right)s^{-\eta_k}s^{\eta_k}\Upsilon_k(s)\,ds \end{array}\right| \\
&\leq \frac{C_k t^{\alpha_k-\eta_k}}{\Gamma(\alpha_k)}\int_0^1(1-v)^{\alpha_k-1}v^{-\eta_k}dv+\frac{C_k\Gamma(n-\mu_k)\Gamma(n+\delta_k)t^{n-1}}{(n-1)!(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \\
&\quad\times\int_0^1\left(\frac{(1-s)^{\alpha_k-\mu_k-1}s^{-\eta_k}}{\Gamma(\alpha_k-\mu_k)}+\frac{(1-s)^{\alpha_k+\delta_k-1}s^{-\eta_k}}{\Gamma(\alpha_k+\delta_k)}\right)ds \\
&\leq \frac{C_k Be(\alpha_k,1-\eta_k)t^{\alpha_k-\eta_k}}{\Gamma(\alpha_k)}+\frac{C_k\Gamma(n-\mu_k)\Gamma(n+\delta_k)t^{n-1}}{(n-1)!(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \\
&\quad\times\left(\frac{Be(\alpha_k-\mu_k,1-\eta_k)}{\Gamma(\alpha_k-\mu_k)}+\frac{Be(\alpha_k+\delta_k,1-\eta_k)}{\Gamma(\alpha_k+\delta_k)}\right),
\end{aligned} \tag{10}
$$

where Be denotes Beta function. Therefore,

$$
\begin{aligned}
&|u_k(t)-u_k(0)| \\
\leq\ & \frac{C_k\Gamma(1-\eta_k)t^{\alpha_k-\eta_k}}{\Gamma(\alpha_k+1-\eta_k)}+\frac{C_k\Gamma(n-\mu_k)\Gamma(n+\delta_k)t^{n-1}}{(n-1)!(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \\
&\times\left(\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k-\mu_k+1-\eta_k)}+\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k+\delta_k+1-\eta_k)}\right), \\
\to\ & 0,\quad \text{as } t\to 0,\ k=1,2,...,n.
\end{aligned} \tag{11}
$$

Case 2: For $t_0\in(0,1)$ and $\forall t\in(t_0,1]$,

$$
\begin{aligned}
&|u_k(t)-u_k(t_0)| \\
\leq\ & \frac{C_k\Gamma(1-\eta_k)\left(t^{\alpha_k-\eta_k}-t_0^{\alpha_k-\eta_k}\right)}{\Gamma(\alpha_k+1-\eta_k)}+\frac{\Gamma(n-\mu_k)\Gamma(n+\delta_k)\left(t^{n-1}-t_0^{n-1}\right)}{(n-1)!(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \\
&\times\left(\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k-\mu_k+1-\eta_k)}+\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k+\delta_k+1-\eta_k)}\right), \\
\to\ & 0,\quad \text{as } t\to t_0,\ k=1,2,...,n.
\end{aligned} \tag{12}
$$

Case 3: For t_0 $(0,1]$ and $\forall t\in[0,t_0]$, the proof is similar to that of case 2, we leave it. The proof is thus completed. ■

Lemma 4 *Let $n-1<\alpha_k<n$, $k=1,2,...,n$, $n\in\mathbb{N}^*\setminus\{1\}$, $g_k:[0,1]\times\mathbb{R}\to\mathbb{R}$ and $f_k:(0,1]\times\mathbb{R}^3\to\mathbb{R}$ are continuous, and $\lim\limits_{t\to 0^+} f_k(t,.,.,.)=\infty$. Assume that there exist constants $0<\eta_k<1$, such that $t^{\eta_k}f_k(t,.,.,.)$ are continuous on $[0,1]\times\mathbb{R}^3$. Then,*

$$
\begin{aligned}
D^{\beta_{k-1}}T_k(u_{k+1})(t) &= \int_0^t \frac{(t-s)^{\alpha_k-\beta_{k-1}-1}}{\Gamma(\alpha_k-\beta_{k-1})} f_k\left(\begin{array}{c} s,u_{k+1}(s),D^{\beta_k}u_{k+1}(s), \\ \int\limits_0^s g_k(\tau,u_{k+1}(\tau))\,d\tau \end{array}\right) ds \\
&\quad -\frac{\Gamma(n-\mu_k)\Gamma(n+\delta_k)t^{n-1-\beta_{k-1}}}{\Gamma(n-\beta_{k-1})(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \\
&\quad \times\int_0^1\left(\frac{(1-s)^{\alpha_k-\mu_k-1}}{\Gamma(\alpha_k-\mu_k)}+\frac{(1-s)^{\alpha_k+\delta_k-1}}{\Gamma(\alpha_k+\delta_k)}\right) f_k\left(\begin{array}{c} s,u_{k+1}(s),D^{\beta_k}u_{k+1}(s), \\ \int\limits_0^s g_k(\tau,u_{k+1}(\tau))\,d\tau \end{array}\right) ds,
\end{aligned} \tag{13}
$$

are continuous on $[0,1]$, where $u_{n+1}=u_1$, $\beta_0=\beta_n$, and $0<\beta_{k-1}<\alpha_{k-1}$.

Proof. Let $(u_1,u_2,...,u_n)\in B$, so $u_{k+1}(t)\in C([0,1])$, $D^{\beta_k}u_{k+1}(t)\in C([0,1])$, where $u_{n+1}=u_1, k=1,2,...,n$. Thus, there exist positives constants a_{k+1}, b_k : $|u_{k+1}(t)|\le a_{k+1}$, $\left|D^{\beta_k}u_{k+1}(t)\right|\le b_k, \forall t\in[0,1]$. On the other hand, g_k are continuous on $[0,1]\times\mathbb{R}$, which implies that there exist $c_k>0$: $|g_k(t,u_{k+1})|\le c_k$. Since $t^{\eta_k}f_k(t,.,.,.)$ are continuous on $[0,1]\times\mathbb{R}^3$, let

$$
M_k=\left\| t^{\eta_k}f_k\left(t,u_{k+1},D^{\beta_k}u_{k+1},\int\limits_0^t g_k(\tau,u_{k+1}(\tau))\,d\tau\right)\right\|_\infty, \tag{14}
$$

for $-a_{k+1}\le u_{k+1}\le a_{k+1}$, and $-b_k\le D^{\beta_k}u_{k+1}(t)\le b_k$, and $-c_k\le g_k(t,u_{k+1})\le c_k$. Then, tanks to (14), we have

$$
\begin{aligned}
&\left|D^{\beta_{k-1}}T_k(u_{k+1})(t)\right| \\
&\le \frac{M_k}{\Gamma(\alpha_k-\beta_{k-1})}\int_0^t (t-s)^{\alpha_k-\beta_{k-1}-1}s^{-\eta_k}ds+\frac{\Gamma(n-\mu_k)\Gamma(n+\delta_k)t^{n-1-\beta_{k-1}}}{\Gamma(n-\beta_{k-1})(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \\
&\quad \times\int_0^1\left(\frac{(1-s)^{\alpha_k-\mu_k-1}s^{-\eta_k}}{\Gamma(\alpha_k-\mu_k)}+\frac{(1-s)^{\alpha_k+\delta_k-1}s^{-\eta_k}}{\Gamma(\alpha_k+\delta_k)}\right)ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left|D^{\beta_{k-1}}T_k(u_{k+1})(t)\right| \\
&\le \frac{M_k\Gamma(1-\eta_k)t^{\alpha_k-\beta_{k-1}-\eta_k}}{\Gamma(\alpha_k-\beta_{k-1}+1-\eta_k)}+\frac{\Gamma(n-\mu_k)\Gamma(n+\delta_k)t^{n-1-\beta_{k-1}}}{\Gamma(n-\beta_{k-1})(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \\
&\quad \times\left(\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k-\mu_k+1-\eta_k)}+\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k+\delta_k+1-\eta_k)}\right).
\end{aligned} \tag{15}
$$

From the inequality (15), we see that $t^{\alpha_k-\beta_{k-1}-\eta_k}$ and $t^{n-1-\beta_{k-1}}$ are continuous on $[0,1]$. By the same method as in Lemma 3, we can show that $D^{\beta_{k-1}}T_k$, $k=1,2,...,n$, are continuous on $[0,1]$, where $u_{n+1}=u_1,\beta_0=\beta_n$. ■

Lemma 5 *Let $n-1<\alpha_k<n$, $k=1,2,...,n$, $n\in\mathbb{N}^*\setminus\{1\}$, $g_k:[0,1]\times\mathbb{R}\to\mathbb{R}$ and $f_k:(0,1]\times\mathbb{R}^3\to\mathbb{R}$ are continuous, and $\lim\limits_{t\to 0^+} f_k(t,.,.,.)=\infty$. Assume that there exist constants $0<\eta_k<1$, such that $t^{\eta_k}f_k(t,.,.,.)$ are continuous on $[0,1]\times\mathbb{R}^3$. Then, the operator $T:B\to B$ is completely continuous.*

Proof. For $(u_1,u_2...,u_n)\in B$,

$$T(u_1,u_2...,u_n)(t)=(T_1(u_2),T_2(u_3),...,T_n(u_1))(t),$$

such that $T_k(u_{k+1})(t)$ is given by (8). By Lemma 3 and Lemma 4, we have $T:B\to B$. Now, we divide the proof into three steps.

Step 1 : We show that $T:B\to B$ is a continuous operator.

Let $\left(u_1^0,u_2^0,...,u_n^0\right)\in B:\left\|\left(u_1^0,u_2^0,...,u_n^0\right)\right\|_B=l_0$, and let $(u_1,u_2,...,u_n)\in B$; $\left\|(u_1,u_2,...,u_n)-\left(u_1^0,u_2^0,...,u_n^0\right)\right\|_B<1$, that is $\|(u_1,u_2,...,u_n)\|_B<1+l_0=l$. Since g_k are continuous on $[0,1]\times\mathbb{R}$, there exist $m_k:\|g_k(t,u_{k+1})\|_\infty=m_k$. From the continuity of $t^{\eta_k}f_k(t,.,.,.)$, we get $t^{\eta_k}f_k(t,.,.,.)$ are uniformly continuous on $[0,1]\times[-l,l]^2\times[-m_k,m_k]$. Hence, for all $t\in[0,1]$, and all $\epsilon>0$, there exists $\gamma>0$ $(\gamma<1)$:

$$\left|\begin{array}{c} t^{\eta_k}f_k\left(t,u_{k+1}(t),D^{\beta_k}u_{k+1}(t),\int\limits_0^t g_k(\tau,u_{k+1}(\tau))\,d\tau\right)\\ -t^{\delta_k}f_k\left(t,u_{k+1}^0(t),D^{\beta_k}u_{k+1}^0(t),\int\limits_0^t g_k\left(\tau,u_{k+1}^0(\tau)\right)d\tau\right)\end{array}\right|<\epsilon, \tag{16}$$

where $(u_1,u_2,...,u_n)\in B$ and

$$\left\|(u_1,u_2,...,u_n)-\left(u_1^0,u_2^0,...,u_n^0\right)\right\|_B<\gamma.$$

Thanks to (16), we get

$$\left\|T_k(u_{k+1})-T_k\left(u_{k+1}^0\right)\right\|_\infty$$

$$\leq\epsilon\left(\begin{array}{c}\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k+1-\eta_k)}\sup\limits_{t\in[0,1]}t^{\alpha_k-\eta_k}+\frac{\Gamma(n-\mu_k)\Gamma(n+\delta_k)}{(n-1)!(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))}\\ \times\left(\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k-\mu_k+1-\eta_k)}+\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k+\delta_k+1-\eta_k)}\right)\end{array}\right). \tag{17}$$

We pose:

$$\begin{aligned}F_k:=&\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k+1-\eta_k)}+\frac{\Gamma(n-\mu_k)\Gamma(n+\delta_k)}{(n-1)!(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))}\\ &\times\left(\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k-\mu_k+1-\eta_k)}+\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k+\delta_k+1-\eta_k)}\right).\end{aligned} \tag{18}$$

Then for all $k=1,2,...,n$,

$$\left\|T_k(u_{k+1})-T_k\left(u_{k+1}^0\right)\right\|_\infty\leq\epsilon_k. \tag{19}$$

Similarly using (16), we get

$$\left\|D^{\beta_{k-1}}T_k(u_{k+1})-D^{\beta_{k-1}}T_k\left(u_{k+1}^0\right)\right\|_\infty\leq\epsilon_k^*, \tag{20}$$

where

$$F_k^* := \frac{\Gamma(1-\eta_k)}{\Gamma\left(\alpha_k-\beta_{k-1}+1-\eta_k\right)} + \frac{\Gamma(n-\mu_k)\Gamma(n+\delta_k)}{\Gamma\left(n-\beta_{k-1}\right)(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \times \left(\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k-\mu_k+1-\eta_k)} + \frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k+\delta_k+1-\eta_k)}\right). \tag{21}$$

Using (19) and (20) , we get

$$\left\|T\left(u_1,u_2,...,u_n\right) - T\left(u_1^0,u_2^0,...,u_n^0\right)\right\|_B \leq \epsilon \max_{1\leq k\leq n}\left({}_k,\ {}_k^*\right). \tag{22}$$

Thus, $\left\|T\left(u_1,u_2,...,u_n\right) - T\left(u_1^0,u_2^0,...,u_n^0\right)\right\|_B \to 0$ as $\left\|\left(u_1,u_2,...,u_n\right) - \left(u_1^0,u_2^0,...,u_n^0\right)\right\|_B \to 0$. Hence, $T : B \to B$ is continuous.

Step 2 : Assume that $\Delta := \left\{\left(u_1,u_2,...,u_n\right) \in B : \left\|\left(u_1,u_2,...,u_n\right)\right\|_B \leq \kappa\right\}$, where $\kappa > 0$. We shall show that $T(\Delta)$ is bounded. Then, $t^{\eta_k} f_k(t,.,.,.)$ are continuous on $[0,1]\times[-l,l]^2\times[-m_k,m_k]$, there exist $L_k > 0$, $k = 1,2,...,n$, such that

$$\left|t^{\eta_k} f_k\left(t, u_{k+1}(t), D^{\beta_k}u_{k+1}(t), \int_0^t g_k\left(\tau, u_{k+1}(\tau)\right)d\tau\right)\right| \leq L_k, \tag{23}$$

$\forall t \in [0,1]$, $\forall\left(u_1,u_2,...,u_n\right) \in \Delta$. By (23) , we have

$$\left\|T_k\left(u_{k+1}\right)\right\|_\infty \leq L_k F_k,\ \ \left\|D^{\beta_{k-1}}T_k\left(u_{k+1}\right)\right\|_\infty \leq L_k F_k^*. \tag{24}$$

From the inequalities (24) , we obtain

$$\left\|T\left(u_1,u_2,...,u_n\right)\right\|_B \leq \max_{1\leq k\leq n} L_k\left(F_k, F_k^*\right). \tag{25}$$

Thus, $T(\Delta)$ is bounded.

Step 3 : We show that $T(\Delta)$ is equicontinuous. Let $\left(u_1,u_2...,u_n\right) \in \Omega$, and $t_1,t_2 \in [0,1] : t_1 < t_2$, then,

$$\left\|T_k\left(u_{k+1}\right)(t_2) - T_k\left(u_{k+1}\right)(t_1)\right\|_\infty$$

$$\leq \sup_{t\in[0,1]} \left| \begin{array}{c} \int_0^{t_2} \frac{(t_2-s)^{\alpha_k-1}s^{-\eta_k}}{\Gamma(\alpha_k)} s^{\eta_k} f_k\left(s, u_{k+1}(s), D^{\beta_k}u_{k+1}(s), \int\limits_0^s g_k(\tau, u_{k+1}(\tau))\, d\tau\right) ds \\ -\int_0^{t_1} \frac{(t_1-s)^{\alpha_k-1}s^{-\eta_k}}{\Gamma(\alpha_k)} s^{\eta_k} f_k\left(s, u_{k+1}(s), D^{\beta_k}u_{k+1}(s), \int\limits_0^s g_k(\tau, u_{k+1}(\tau))\, d\tau\right) ds \end{array} \right|$$

$$+\frac{\Gamma(n-\mu_k)\Gamma(n+\delta_k)\left(t_2^{n-1}-t_1^{n-1}\right)}{(n-1)!\left(\Gamma(n-\mu_k)+\Gamma(n+\delta_k)\right)}$$

$$\times \int_0^1 \left(\begin{array}{c} \frac{(1-s)^{\alpha_k-\mu_k-1}}{\Gamma(\alpha_k-\mu_k)} \\ +\frac{(1-s)^{\alpha_k+\delta_k-1}}{\Gamma(\alpha_k+\delta_k)} \end{array} \right) s^{-\eta_k} \left| s^{\eta_k} f_k\left(s, u_{k+1}(s), D^{\beta_k}u_{k+1}(s), \int\limits_0^s g_k(\tau, u_{k+1}(\tau))\, d\tau\right)\right| ds$$

$$\leq L_k \left(\begin{array}{c} \frac{\Gamma(1-\eta_k)\left(t_2^{\alpha_k-\eta_k}-t_1^{\alpha_k-\eta_k}\right)}{\Gamma(\alpha_k+1-\eta_k)} + \frac{\Gamma(n-\mu_k)\Gamma(n+\delta_k)\left(t_2^{n-1}-t_1^{n-1}\right)}{(n-1)!(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \\ \times\left(\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k-\mu_k+1-\eta_k)} + \frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k+\delta_k+1-\eta_k)}\right) \end{array} \right). \tag{26}$$

On the other hand, we obtain

$$\left\|D^{\beta_{k-1}}T_k(u_{k+1})(t_2) - D^{\beta_{k-1}}T_k(u_{k+1})(t_2)\right\|_\infty$$

$$\leq L_k \left(\begin{array}{c} \frac{\Gamma(1-\eta_k)\left(t_2^{\alpha_k-\beta_{k-1}-\eta_k}-t_1^{\alpha_k-\beta_{k-1}-\eta_k}\right)}{\Gamma\left(\alpha_k-\beta_{k-1}-\eta_k\right)} \\ +\frac{\Gamma(n-\mu_k)\Gamma(n+\delta_k)\left(t_2^{n-1-\beta_{k-1}}-t_1^{n-1-\beta_{k-1}}\right)}{\Gamma\left(n-\beta_{k-1}\right)(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \\ \times\left(\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k-\mu_k+1-\eta_k)} + \frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k+\delta_k+1-\eta_k)}\right) \end{array} \right). \tag{27}$$

It follows (26) and (27), that

$$\|T(u_1,u_2,...,u_n)(t_2) - T(u_1,u_2,...,u_n)(t_1)\|_B \leq$$

$$\max_{1\leq k\leq n} L_k \left(\begin{array}{c} \frac{\Gamma(1-\eta_k)\left(t_2^{\alpha_k-\eta_k}-t_1^{\alpha_k-\eta_k}\right)}{\Gamma(\alpha_k+1-\eta_k)} + \frac{\Gamma(n-\mu_k)\Gamma(n+\delta_k)\left(t_2^{n-1}-t_1^{n-1}\right)}{(n-1)!(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \\ \times\left(\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k-\mu_k+1-\eta_k)} + \frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k+\delta_k+1-\eta_k)}\right), \\ \frac{\Gamma(1-\eta_k)\left(t_2^{\alpha_k-\beta_{k-1}-\eta_k}-t_1^{\alpha_k-\beta_{k-1}-\eta_k}\right)}{\Gamma\left(\alpha_k-\beta_{k-1}-\eta_k\right)} \\ +\frac{\Gamma(n-\mu_k)\Gamma(n+\delta_k)\left(t_2^{n-1-\beta_{k-1}}-t_1^{n-1-\beta_{k-1}}\right)}{\Gamma\left(n-\beta_{k-1}\right)(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \\ \times\left(\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k-\mu_k+1-\eta_k)} + \frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k+\delta_k+1-\eta_k)}\right) \end{array} \right). \tag{28}$$

The right-hand side of (28) is independent of $(u_1, u_2..., u_n)$ and tend to zero as $t_1 \to t_2$. Hence, $T(\Delta)$ is equicontinuous. By Arzela-Ascoli theorem, we deduce that T is completely continuous. ■

Theorem 6 *Assume that*

(H_1) : *There exist nonnegative constants* $\left(\lambda_j^k\right)_{j=1,2,3}^{k=1,...,n}$ *, such that*

$$t^{\eta_k}\left|f_k\left(t,x_1,x_2,x_3\right)-f_k\left(t,y_1,y_2,y_3\right)\right|\leq\sum_{j=1}^{3}\lambda_j^k\left|x_j-y_j\right|,$$

$\forall\ t\in[0,1]\,,\ \forall\left(x_1,x_2,x_3\right),\ \left(y_1,y_2,y_3\right)\in\mathbb{R}^3.$

(H_2) : *There exist nonnegative constants* $\omega_j,\ j=1,2,...,n,$ *such that*

$$\left|g_j\left(t,x_{j+1}\right)-g_j\left(t,y_{j+1}\right)\right|\leq\omega_j\left|x_{j+1}-y_{j+1}\right|,$$

$\forall\ t\in[0,1]\,,\ \forall x_j,y_j\in\mathbb{R},\ j=1,2,...,n.$

(H_3) : *The following inequality holds*

$$\Lambda:=\max_{1\leq k\leq n}\Sigma_k\left(F_k,F_k^*\right)<1;\ \Sigma_k:=\lambda_1^k+\lambda_2^k+\lambda_3^k\omega_k.$$

Then, system (1) *has a unique solution on* $[0,1]$.

Proof. We prove that T is a contractive operator on B. Let $(u_1,u_2...,u_n)\,,(v_1,v_2...,v_n)\in B$ and $t\in[0,1]$. Using (H_1) and (H_2), we obtain

$$\begin{gathered}\left\|T_k\left(u_{k+1}\right)(t)-T_k\left(v_{k+1}\right)(t)\right\|_\infty\\ \leq\begin{pmatrix}\lambda_1^k\left\|u_{k+1}-v_{k+1}\right\|_\infty+\lambda_2^k\left\|D^{\beta_k}\left(u_{k+1}-v_{k+1}\right)\right\|_\infty\\ +\lambda_3^k\omega_k\left\|u_{k+1}-v_{k+1}\right\|_\infty\sup\limits_{s\in[0,1]}\int\limits_0^s d\tau\end{pmatrix}\\ \times\begin{pmatrix}\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k+1-\eta_k)}\sup\limits_{t\in[0,1]}t^{\alpha_k-\eta_k}\\ +\frac{\Gamma(n-\mu_k)\Gamma(n+\delta_k)}{(n-1)!(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))}\left(\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k-\mu_k+1-\eta_k)}+\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k+\delta_k+1-\eta_k)}\right)\end{pmatrix}.\end{gathered}\tag{29}$$

Therefore,

$$\left\|T_k\left(u_{k+1}\right)(t)-T_k\left(v_{k+1}\right)(t)\right\|_\infty\leq\Sigma_kF_k\max\begin{pmatrix}\left\|u_{k+1}-v_{k+1}\right\|_\infty,\\ \left\|D^{\beta_k}\left(u_{k+1}-v_{k+1}\right)\right\|_\infty\end{pmatrix}.\tag{30}$$

Also by (H_1) and (H_2), we get

$$\left\|D^{\beta_{k-1}}\left(T_k\left(u_{k+1}\right)(t)-T_k\left(v_{k+1}\right)(t)\right)\right\|_\infty\quad\leq\quad\Sigma_kF_k^*\max\begin{pmatrix}\left\|u_{k+1}-v_{k+1}\right\|_\infty,\\ \left\|D^{\beta_k}\left(u_{k+1}-v_{k+1}\right)\right\|_\infty\end{pmatrix}.\tag{31}$$

Thanks to (30) and (31), we get

$$\left\|T\left(u_1,u_2...,u_n\right)-T\left(v_1,v_2...,v_n\right)\right\|_B\leq\Lambda\left\|\left(u_1-v_1,...,u_n-v_n\right)\right\|_B.\tag{32}$$

By (H_3), we have $\Lambda<1$, Hence, T is a contractive operator. By Banach fixed point theorem, T has a fixed point which is the unique solution of system (1). The proof is completed. ■

Example 1 *Consider the following coupled system:*

$$\begin{cases} D^{\frac{11}{4}}u_1(t) = \dfrac{\sin u_2(t)+\cos D^{\frac{8}{3}}u_2(t)+\int\limits_0^t \frac{\sin(u_2(\tau))}{\pi(\tau+1)}d\tau}{16\pi t^{\frac{1}{3}}},\ 0<t\leq 1, \\ D^{\frac{8}{3}}u_2(t) = \dfrac{\left|u_3(t)+D^{\frac{7}{3}}u_3(t)+\int\limits_0^t \frac{\cos u_3(\tau)}{4}d\tau\right|}{12\pi\sqrt{t}\left(1+\left|u_3(t)+D^{\frac{7}{3}}u_3(t)+\int\limits_0^t \frac{\cos u_3(\tau)}{4}d\tau\right|\right)},\ 0<t\leq 1, \\ D^{\frac{11}{5}}u_3(t) = \dfrac{\cos u_1(t)+\cos D^{\frac{7}{4}}u_1(t)+\int\limits_0^t \frac{\sin(u_1(\tau))}{\tau+\pi}d\tau}{16\pi t^{\frac{3}{7}}},\ 0<t\leq 1, \\ u_1(0)=u_1^{'}(0)=0,\ D^{\frac{5}{4}}u_1(1)+J^{\frac{4}{3}}u_1(1)=0, \\ u_2(0)=u_2^{'}(0)=0,\ D^{\frac{9}{5}}u_2(1)+J^{\frac{7}{2}}u_2(1)=0, \\ u_3(0)=u_3^{'}(0)=0,\ D^{\frac{6}{5}}u_3(1)+J^{\frac{5}{3}}u_3(1)=0. \end{cases} \tag{33}$$

Then, $\forall t\in[0,1]$ *and* $\forall(x_1,x_2,x_3),\ (y_1,y_2,y_3)\in\mathbb{R}^3$, *we have:*

$$t^{\frac{2}{3}}\left|f_1(t,x_1,x_2,x_3)-f_1(t,y_1,y_2,y_3)\right| \leq \frac{t^{\frac{1}{3}}}{16\pi}|x_1-y_1|+\frac{t^{\frac{1}{3}}}{16\pi}|x_2-y_2|+\frac{t^{\frac{1}{3}}}{16\pi}|x_3-y_3|,$$

$$t^{\frac{3}{4}}\left|f_2(t,x_1,x_2,x_3)-f_2(t,y_1,y_2,y_3)\right| \leq \frac{t^{\frac{1}{4}}}{12\pi}|x_1-y_1|+\frac{t^{\frac{1}{4}}}{12\pi}|x_2-y_2|+\frac{t^{\frac{1}{4}}}{12\pi}|x_3-y_3|,$$

$$t^{\frac{6}{7}}\left|f_3(t,x_1,x_2,x_3)-f_3(t,y_1,y_2,y_3)\right| \leq \frac{t^{\frac{3}{7}}}{16\pi}|x_1-y_1|+\frac{t^{\frac{3}{7}}}{16\pi}|x_2-y_2|+\frac{t^{\frac{3}{3}}}{16\pi}|x_3-y_3|,$$

where $\eta_1=\frac{2}{3}$, $\eta_2=\frac{3}{4}$ *and* $\eta_3=\frac{6}{7}$. *We can take*

$$\omega_1=\frac{1}{\pi},\ \omega_2=\frac{1}{4},\ \omega_3=\frac{1}{\pi},\left(\lambda_j^1\right)_{j=1,2,3}=\frac{1}{16\pi},\ \ \left(\lambda_j^2\right)_{j=1,2,3}=\frac{1}{12\pi},\ \ \left(\lambda_j^3\right)_{j=1,2,3}=\frac{1}{16\pi}.$$

Indeed,

$$\Sigma_1=0.0461,\ \Sigma_2=0.0597,\ \Sigma_3=0.0461,$$

$F_1=2.5371, F_2=1.9551, F_3=9.0052, F_1^*=5.8652, F_2^*=1.0001, F_3^*=5.2883.$

$\Sigma_1F_1=0.1170,\ \Sigma_2F_2=0.1167,\ \Sigma_3F_3=0.4151, \Sigma_1F_1^*=0.2704,\ \Sigma_2F_2^*=0.0597,\ \Sigma_3F_3^*=0.2438.$

Wich implies that $\Lambda<1$. *So, system* (33) *has a unique solution on* $[0,1]$.

Theorem 7 *Assume that* $n-1<\alpha_k<n$, $0<\eta_k<1$, $k=1,2,...,n$, $n\in\mathbb{N}^*\setminus\{1\}$, $g_k:[0,1]\times\mathbb{R}\to\mathbb{R}$ *and* $f_k:(0,1]\times\mathbb{R}^3\to\mathbb{R}$ *are continuous,* $\lim\limits_{t\to 0^+} f_k(t,.,.,.)=\infty$, *and* $t^{\eta_k}f_k(t,.,.,.)$ *are continuous on* $[0,1]\times\mathbb{R}^3$. *Then, system* (1) *has at least one solution on* $[0,1]$.

Proof. Let

$$R_k = \sup_{t\in[0,1]} t^{\eta_k} \left| f_k\left(t,.,.,.\right)\right|, \tag{34}$$

and $r = \max\limits_{1\leq k\leq n} R_k\left(F_k, F_k^*\right)$. We consider $\Omega := \left\{(u_1,u_2,...,u_n) \in B : \left\|(u_1,u_2,...,u_n)\right\|_B \leq r\right\}$, and we show that $T:\Omega\to\Omega$. Let $(u_1,u_2,...,u_n)\in\Omega$ and $t\in[0,1]$. By (34), we get

$$\begin{aligned}
\left\|T_k\left(u_{k+1}\right)\right\|_\infty &\leq R_k \left(\begin{array}{c} \frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k+1-\eta_k)} \sup\limits_{t\in[0,1]} t^{\alpha_k-\eta_k} + \frac{\Gamma(n-\mu_k)\Gamma(n+\delta_k)}{(n-1)!(\Gamma(n-\mu_k)+\Gamma(n+\delta_k))} \\ \times\left(\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k-\mu_k+1-\eta_k)}+\frac{\Gamma(1-\eta_k)}{\Gamma(\alpha_k+\delta_k+1-\eta_k)}\right)\end{array}\right) \\
&\leq R_k F_k.
\end{aligned} \tag{35}$$

Similarly by (34), we obtain

$$\left\|D^{\beta_{k-1}}T_k\left(u_{k+1}\right)\right\|_\infty \leq R_k{}_k^*. \tag{36}$$

Thanks to (35) and (36), we get

$$\left\|T\left(u_1,u_2,...,u_n\right)\right\|_B \leq \max_{1\leq k\leq n} R_k\left(F_k,F_k^*\right). \tag{37}$$

That is, $\left\|T\left(u_1,u_2,...,u_n\right)\right\|_B \leq r$. So, $(u_1,u_2,...,u_n)\in\Omega$, implies that $T\left(u_1,u_2,...,u_n\right)\in\Omega$. Then, Lemma 3 and Lemma 4 yield that $T_k\left(u_{k+1}\right)\in C\left([0,1]\right)$ and $D^{\beta_{k-1}}T_k\left(u_{k+1}\right)\in C\left([0,1]\right)$. Thus, $T:\Omega\to\Omega$. From Lemma 5, we state that T is completely continuous. Finally by Lemma 2, we deduce that system (1) has at least one solution on $[0,1]$. This ends the proof. ■

Example 2 *Consider the following system:*

$$\left\{\begin{array}{c}
D^{\frac{10}{3}}u_1\left(t\right)=\frac{\cos\left(u_2(t)+D^{\frac{5}{2}}u_2(t)\right)}{\sqrt{t}e^{t+1}+\int_0^t\sin(u_2(\tau))d\tau},\ 0<t\leq1, \\
D^{\frac{7}{2}}u_2\left(t\right)=t^{-\frac{1}{4}}\left(\frac{\sin u_3(t)D^{\frac{7}{3}}u_3(t)}{2\pi}+\int_0^t\cos\tau^2u_3\left(\tau\right)d\tau\right),\ 0<t\leq1, \\
D^{\frac{16}{5}}u_3\left(t\right)=t^{-\frac{4}{9}}\left(\frac{e^t\cos u_4(t)}{e+\left|\cos D^{\frac{9}{4}}u_4(t)\right|}-\int_0^t\frac{|u_4(\tau)|}{\pi e^\tau+|u_4(\tau)|}d\tau\right),\ 0<t\leq1, \\
D^{\frac{13}{4}}u_4\left(t\right)=\frac{t^{-\frac{2}{9}}\sin u_1(t)+\int_0^t\left(\tau^2+1\right)\cos u_1(\tau)d\tau}{2\pi+e\cos D^{\frac{12}{5}}u_1(t)},\ 0<t\leq1, \\
\\
u_k^{(j)}\left(0\right)=\ 0,\ j=0,1,2,\ k=1,2,3,4, \\
D^{\frac{7}{3}}u_1\left(1\right)+J^{\frac{8}{3}}u_1\left(1\right)=0,\ D^{\frac{9}{4}}u_2\left(1\right)+J^{\frac{10}{3}}u_2\left(1\right)=0, \\
D^{\frac{11}{5}}u_3\left(1\right)+J^{\frac{11}{4}}u_3\left(1\right)=0,\ D^{\frac{15}{7}}u_4 1+J^{\frac{1}{2}}u_4\left(1\right)=0.
\end{array}\right. \tag{38}$$

Taking $\eta_1=\frac{3}{4}$, $\eta_2=\frac{1}{2}$, $\eta_3=\frac{2}{3}$, $\eta_4=\frac{1}{3}$, *all the assumptios of Theorem 8 are satisfied. Hence, system* (38) *has at least one solution on* $[0,1]$.

3 Ulam-Hyers Stability

In this section, we will study the Ulam-Hyers stability and the generalized Ulam-Hyers stability for system (1).

Definition 1 *The singular fractional system* (1) *is Ulam-Hyers stable if there exists a real number* $\theta > 0$, *such that for all* $(\epsilon_1, \epsilon_2, ..., \epsilon_n) > 0$, *and for all solution* $(u_1, u_2, ..., u_n) \in B$ *of*

$$\begin{cases} \left| D^{\alpha_1} u_1(t) - f_1\left(t, u_2(t), D^{\beta_1} u_2(t), \int\limits_0^t g_1(\tau, u_2(\tau))\, d\tau\right)\right| < \epsilon_1, \\ 0 < t \leq 1, \\ \vdots \\ \left| D^{\alpha_{n-1}} u_{n-1}(t) - f_{n-1}\left(t, u_n(t), D^{\beta_{n-1}} u_n(t), \int\limits_0^t g_{n-1}(\tau, u_n(\tau))\, d\tau\right)\right| < \epsilon_2, \\ 0 < t \leq 1, \\ \left| D^{\alpha_n} u_n(t) - f_n\left(t, u_1(t), D^{\beta_n} u_1(t), \int\limits_0^t g_n(\tau, u_1(\tau))\, d\tau\right)\right| < \epsilon_n, \\ 0 < t \leq 1, \end{cases} \tag{39}$$

there exists solution $(v_1, v_2, ..., v_n) \in B$ *satisfying system* (1), *where*

$$\|(u_1 - v_1, ..., u_n - v_n)\|_B < \theta\epsilon, \ \epsilon > 0. \tag{40}$$

Definition 2 *The singular fractional system* (1) *has the generalized Ulam-Hyers stablity if there exist* $\Psi \in C(\mathbb{R}^+, \mathbb{R}^+)$, *such that for all* $\epsilon > 0$, *and for each solution* $(u_1, u_2, ..., u_n) \in B$ *of* (39), *there exists* $(v_1, v_2, ..., v_n) \in B$ *of* (1) *with*

$$\|(u_1 - v_1, ..., u_n - v_n)\|_B < \Psi(\epsilon), \ \epsilon > 0.$$

Theorem 8 *Let* $n - 1 < \alpha_k < n$, $0 < \eta_k < 1$, $k = 1, 2, ..., n$, $n \in \mathbb{N}^* \setminus \{1\}$. *Suppose that:*

(S_1) : $g_k : [0,1] \times \mathbb{R} \to \mathbb{R}$ *and* $f_k : (0,1] \times \mathbb{R}^3 \to \mathbb{R}$ *are continuous,* $\lim\limits_{t \to 0^+} f_k(t, ., ., .) = \infty$, *and* $t^{\eta_k} f_k(t, ., ., .)$ *are continuous on* $[0,1] \times \mathbb{R}^3$.

(S_2) : $\|t^{\eta_k} D^{\alpha_k} u_k\|_\infty \geq \max\limits_{1 \leq k \leq n} R_k(F_k, F_k^*)$.

(S_3) : *All the assumptions* $(H_i)_{i=1,2,3}$ *of Theorem 6 hold.*

(S_4) : $\Sigma_k < 1$.

Then, the singular fractional coupled system (1) *is generalized Ulam-Hyers stable in* B.

Proof. *Let* $(u_1, u_2, ..., u_n) \in B$ *a solution of* (39). *Using* (S_1), *we receive* (37) *and we can write:*

$$\|(u_1, u_2, ..., u_n)\|_B \leq \max_{1 \leq k \leq n} R_k(F_k, F_k^*). \tag{41}$$

Combine (S_2) *with* (41) *, yields*

$$\|(u_1,u_2,...,u_n)\|_B \leq \|t^{\eta_k} D^{\alpha_k} u_k\|_\infty . \tag{42}$$

The hypothesis (S_3) *implies that there exists a solution* $(v_1,v_2,...,v_n) \in B$ *satisfying system* (1) *. By* (42) *, we get*

$$\|(u_1-v_1,...,u_n-v_n)\|_B \leq \|t^{\eta_k} D^{\alpha_k}(u_k-v_k)\|_\infty$$

$$\leq \left(\begin{array}{c} \|t^{\eta_k}\|_\infty \left\| \begin{array}{c} D^{\alpha_k} u_k - \\ f_k\left(t, u_{k+1}, D^{\beta_k} u_{k+1}, \int\limits_0^t g_k(\tau, u_{k+1}(\tau))\, d\tau\right) \end{array} \right\|_\infty \\ + \|t^{\eta_k}\|_\infty \left\| \begin{array}{c} D^{\alpha_k} v_k - \\ f_k\left(t, v_{k+1}, D^{\beta_k} v_{k+1}, \int\limits_0^t g_k(\tau, v_{k+1}(\tau))\, d\tau\right) \end{array} \right\|_\infty \\ + \left\| t^{\eta_k} \left(\begin{array}{c} f_k\left(t, u_{k+1}, D^{\beta_k} u_{k+1}, \int\limits_0^t g_k(\tau, u_{k+1}(\tau))\, d\tau\right) \\ - f_k\left(t, v_{k+1}, D^{\beta_k} v_{k+1}, \int\limits_0^t g_k(\tau, v_{k+1}(\tau))\, d\tau\right) \end{array} \right) \right\|_\infty \end{array} \right) . \tag{43}$$

Using (S_3) *and thanks to* (1) *and* (39) *, we get*

$$\|(u_1-v_1,...,u_n-v_n)\|_B \leq \epsilon_k + \Sigma_k \max \left(\begin{array}{c} \|(u_{k+1}-v_{k+1})\|_\infty , \\ \left\|D^{\beta_{k-1}}(u_{k+1}-v_{k+1})\right\|_\infty \end{array} \right) . \tag{44}$$

Thus,

$$\|(u_1-v_1,...,u_n-v_n)\|_B \leq \frac{\epsilon}{1-\Sigma_k} := \theta\epsilon,\ \epsilon = \max_{1\leq k\leq n} \epsilon_k,\ \theta = \max_{1\leq k\leq n} \frac{1}{1-\Sigma_k}. \tag{45}$$

By (S_4) *, we see that* $\theta > 0$*. So, system* (1) *is Ulam-Hyers stable. Putting* $\Psi(\epsilon) = \theta\epsilon$*, implies that the coupled system* (1) *has the generalized Ulam-Hyers stability. This completes the proof.* ■

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