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RICCI SOLITONS IN KENMOTSU MANIFOLD UNDER GENERALIZED D-CONFORMAL DEFORMATION

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ABSTRACT. In this paper we study Ricci solitons in generalized D-conformally deformed Kenmotsu manifold and we analyzed the nature of Ricci solitons when associated vector field is orthagonal to Reeb vector field.

1. INTRODUCTION

A Ricci soliton is a Riemannian metric g on a manifold M together with a vector field V such that

$$(\mathscr{L}_{\mathrm{V}} g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{1.1}$$

where $\mathscr{L}_{\mathrm{V}}$ denotes the Lie derivative along V and S and λ are respectively Ricci tensor and a constant. A Ricci soliton is said to be shrinking or steady or expanding according as λ is negative, zero or positive. A Ricci soliton is said to be a gradient Ricci soliton if the vector field V is gradient of some smooth function f on M.

R. Sharma [12] initiated the study of Ricci solitons in contact Riemannian geometry. Ghosh and R. Sharma [6] [7], R. Sharma [12] established results by considering K-contact, Kenmotsu, Sasakian and (κ, μ)- contact metrics as Ricci solitons. Bejan and Crasmarenu [2] extended the study of Ricci solitons to paracontact manifolds. De and others [16][8] studied Ricci solitons in f-Kenmotsu manifolds. In [14] authors analyze, the behaviour of Generalized Sasakian space form and generalized (κ, μ) space form under generalized D-conformal deformation. Several authors Nagaraja and Premalatha [9], De and Ghosh [5] and Shaik et al [11] studied the behaviour of normal almost contact metric, (κ, μ) contact metric and trans-Sasakian manifolds under D-homothetic deformations. We make use of the invariance of certain contact structures under generalized D-conformal and D-homothetic deformations to study Ricci solitons.

This paper structures as follows: after a brief review of Kenmotsu manifolds in section 2, we study generalized D-conformally and D homothetically deformed Kenmotsu metrics as Ricci solitons in section 3.

2. PRELIMINARIES

A $(2n+1)$-dimensional smooth manifold M is said to be an almost contact metric manifold if it admits an almost contact metric structure (ϕ, ξ, η, g) consisting of a tensor field ϕ of type $(1,1)$, a vector field ξ, a 1-form η and a Riemannian metric g compatible with (ϕ, ξ, η) satisfying

$$\phi^2 X = -X + \eta(X)\xi,\ \phi\xi = 0,\ g(X,\xi) = \eta(X),\ \eta(\xi) = 1, \eta \circ \phi = 0, \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.2}$$

2010 *Mathematics Subject Classification.* 53D10, 53D15.

Key words and phrases. Ricci solitons, Kenmotsu, generalized D-Conformal, D-homothetic, Conformal, shrinking, expanding, steady.

An almost contact metric manifold is said to be a Kenmotsu manifold [3] if

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{2.3}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.4}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y), \tag{2.5}$$

where ∇ denotes the Riemannian connection of g.
In a Kenmotsu manifold the following relations hold [4].

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.6}$$

$$S(X,\xi) = -2n\eta(X), \tag{2.7}$$

$$S(\phi X, \phi Y) = S(X,Y) + 2n\eta(X)\eta(Y), \tag{2.8}$$

for any vector fields X, Y, Z on M, where R and S denote, respectively, the curvature tensor of type $(1,3)$ and the Ricci tensor of type $(0,2)$ on M.

3. Ricci solitons in Kenmotsu manifolds under generalized D-Conformal Deformations

Let (M, ϕ, ξ, η, g) be an almost contact metric manifold, where g is a Ricci soliton. The generalized D-conformal deformation [1] on M is given by

$$\phi^* = \phi,\ \xi^* = \frac{1}{a}\xi,\ \eta^* = a\eta,\ g^* = bg + (a^2 - b)\eta \otimes \eta. \tag{3.1}$$

where a and b are two positive functions on M.
It is well known that $(M, \phi^*, \xi^*, \eta^*, g^*)$ is also an almost contact metric manifold [1]. We note that the transformation (3.1) reduces to D-homothetic [15] or conformal according as $a = b =$constant or $a^2 = b$ [13].
Let (M, ϕ, ξ, η, g) be a Kenmotsu manifold and $(M, \phi^*, \xi^*, \eta^*, g^*)$ be almost contact metric manifold obtained by generalized D-conformal deformation (3.1). It is well known that $(M, \phi^*, \xi^*, \eta^*, g^*)$ is also a Kenmotsu manifold [13] and a, b are in the constant direction of ξ. Throughout this paper the quantity with $*$ denote the quantities in $(M, \phi^*, \xi^*, \eta^*, g^*)$ and quqntities without $*$ are from (M, ϕ, ξ, η, g).
The relation between the connections ∇ and ∇^* is given by [1]

$$\nabla^*_X Y = \nabla_X Y + \frac{(a^2 - b)}{a^2} g(\phi X, \phi Y)\xi, \tag{3.2}$$

for any vector fields X, Y on M.
Using (3.2) we now calculate the Riemann curvature tensor R^* of $(M, \phi^*, \xi^*, \eta^*, g^*)$ as follows;

$$\begin{aligned} R^*(X,Y)Z =& R(X,Y)Z + \frac{a^2 - b}{a^2}\left[g(\phi Y, \phi Z)X - g(\phi X, \phi Z)Y\right] \\ &+ g(\phi Y, \phi Z)\xi\,[\frac{2bX(a)}{a^3} - \frac{X(b)}{a^2}] - g(\phi X, \phi Z)\xi[\frac{2bY(a)}{a^3} - \frac{Y(b)}{a^2}], \end{aligned} \tag{3.3}$$

for any X, Y, Z on M.
On contracting (3.3), we obtain the Ricci tensor S^* of generalized D-conformally deformed Kenmotsu manifold as

$$S^*(Y,Z) = S(Y,Z) + \frac{2n(a^2 - b)}{a^2}\left[g(Y,Z) - \eta(Y)\eta(Z)\right]. \tag{3.4}$$

Taking the Lie derivative of $g^* = bg + (a^2 - b)\eta \otimes \eta$ along V and using (3.1)and (3.4), we obtain

$$\begin{aligned}
&(\mathscr{L}_{\mathrm{V}} g^*)(X,Y) + 2S^*(X,Y) + 2\lambda g^*(X,Y)\\
&= V(b)g(X,Y) + b(\mathscr{L}_{\mathrm{V}} g)(X,Y) + [2aV(a) - V(b)]\eta(X)\eta(Y)\\
&+ (a^2 - b)\{(\mathscr{L}_{\mathrm{V}}\eta)(X)\eta(Y) + \eta(X)\mathscr{L}_{\mathrm{V}}\eta)(Y)\} + 2S(X,Y)\\
&+ \frac{4n(a^2-b)}{a^2}\{g(X,Y) - \eta(X)\eta(Y)\} + 2\lambda\{bg(X,Y) + (a^2-b)\eta(X)\eta(Y)\}.
\end{aligned} \tag{3.5}$$

We Lie-differentiate $\eta(\xi) = 1$ along V, to get

$$(\mathscr{L}_{\mathrm{V}}\eta)(\xi) + \eta(\mathscr{L}_V \xi) = 0. \tag{3.6}$$

Also Lie-differentiation of $g(\xi,\xi) = 1$ along V gives

$$(\mathscr{L}_{\mathrm{V}} g)(\xi,\xi) + 2\eta(\mathscr{L}_{\mathrm{V}}\xi) = 0. \tag{3.7}$$

Further Setting $X = Y = \xi$ in (1.1) and using (2.7), we obtain

$$(\mathscr{L}_{\mathrm{V}} g)(\xi,\xi) = 4n - 2\lambda. \tag{3.8}$$

Using (3.8), equation (3.7) yields

$$\eta(\mathscr{L}_V \xi) = \lambda - 2n. \tag{3.9}$$

Now, (3.6) yields

$$(\mathscr{L}_{\mathrm{V}}\eta)(\xi) = 2n - \lambda. \tag{3.10}$$

By putting $Y = \xi$ in (1.1), it follows that

$$(\mathscr{L}_{\mathrm{V}}\eta)(X) = g(X, \mathscr{L}_{\mathrm{V}}\xi) - 2S(X,\xi) - 2\lambda\eta(X) \tag{3.11}$$

As in [13], we know that $\mathscr{L}_{\mathrm{V}}\xi = \eta(\mathscr{L}_V\xi)\xi$ and using (2.7), (3.9) in (3.11), we get

$$(\mathscr{L}_{\mathrm{V}}\eta)(X) = (2n - \lambda)\eta(X). \tag{3.12}$$

Finally with the use of (3.12), (3.5) reduces to

$$\begin{aligned}
&(\mathscr{L}_{\mathrm{V}} g^*)(X,Y) + 2S^*(X,Y) + 2\lambda g^*(X,Y)\\
&= 2(1-b)S(X,Y) + \{V(b) + \frac{4n(a^2-b)}{a^2}\}g(X,Y)\\
&+ \{2aV(a) - V(b) + \frac{4n(a^2-b)(a^2-1)}{a^2}\}\eta(X)\eta(Y).
\end{aligned} \tag{3.13}$$

i.e g^* is a Ricci soliton iff

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y), \tag{3.14}$$

where $\alpha = \frac{1}{2(b-1)}\{V(b) + \frac{4n(a^2-b)}{a^2}\}$, $\beta = \frac{1}{2(b-1)}\{2aV(a) - V(b) + \frac{4n(a^2-b)(a^2-1)}{a^2}.\}$

By substituting (3.14) in (3.4) and using (3.1) we obtain

$$S^*(X,Y) = \frac{1}{(b-1)}[\{\frac{V(b)}{2b} + \frac{2n(a^2-b)}{a^2}\}g^*(X,Y) + \{\frac{V(a)}{a} - \frac{V(b)}{2b}\}\eta^*(X)\eta^*(Y)]. \tag{3.15}$$

Therefore, we have the following theorem

Theorem 3.1. *Under generalized D-conformal deformation of a Kenmotsu manifold (M,ϕ,ξ,η,g), η-Einstein Ricci soliton remains η-Einstein Ricci soliton.*

When $a = b =$ constant, equation (3.15) reduces to

$$S^*(X,Y) = \frac{2n}{a} g^*(X,Y). \tag{3.16}$$

Corollary 3.1. *Under D-homothetic deformation of a Kenmotsu manifold (M,ϕ,ξ,η,g), η-Einstein Ricci soliton deforms to an Einstein metric.*

When $a^2 = b$, equation (3.15) reduces to

$$S^*(X,Y) = \frac{V(a)}{a(a^2-1)} g^*(X,Y). \tag{3.17}$$

Corollary 3.2. *Under a conformally deformed Kenmotsu manifold (M,ϕ,ξ,η,g), η-Einstein Ricci soliton deforms to an Einstein metric.*

A vector field V on a Riemannian manifold is said to be concurrent [10] if

$$(\nabla_X V) = \rho X, \tag{3.18}$$

for all X, where ρ is a constant.
We have

$$(\mathscr{L}_V g)(X,Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V). \tag{3.19}$$

By virtue of (3.18), (3.19) becomes

$$(\mathscr{L}_V g)(X,Y) = 2\rho g(X,Y). \tag{3.20}$$

Setting $Y = \xi$ in (3.20), we get

$$(\mathscr{L}_V \eta)(X) = g(X, \mathscr{L}_V \xi) + 2\rho\eta(X). \tag{3.21}$$

It is well known that [13] $\mathscr{L}_V \xi = \eta(\mathscr{L}_V \xi)\xi$, therefore (3.21) yields

$$(\mathscr{L}_V \eta)(X) = (\lambda - 2n + 2\rho). \tag{3.22}$$

In view of (3.20) and (3.22), (3.5) becomes

$$\begin{aligned} &(\mathscr{L}_V g^*)(X,Y) + 2S^*(X,Y) + 2\lambda g^*(X,Y) \\ &= 2S(X,Y) + \{V(b) + 2\rho b + \frac{4n(a^2-b)}{a^2} + 2\lambda b\} g(X,Y) \\ &\{2aV(a) - V(b) + 4(a^2-b)(\lambda+\rho-n) - \frac{4n(a^2-b)}{a^2}\}\eta(X)\eta(Y). \end{aligned} \tag{3.23}$$

i.e g^* is a Ricci soliton iff

$$S(X,Y) = Ag(X,Y) + B\eta(X)\eta(Y) \tag{3.24}$$

where $A = -\{\frac{V(b)}{2} + \rho b + \frac{2n(a^2-b)}{a^2} + \lambda b\}$, $B = -\{aV(a) - \frac{V(b)}{2} + 2(a^2-b)(\lambda+\rho-n) - \frac{2n(a^2-b)}{a^2}\}$.
Putting $X = Y = V = \xi$ in (3.24) we obtain

$$\lambda = \frac{2n(a^2-b+1)}{(2a^2-b)} - \rho \tag{3.25}$$

Theorem 3.2. *Let (M,ϕ,ξ,η,g) be a Kenmotsu manifold and $(M,\phi^*,\xi^*,\eta^*,g^*)$ be obtained by generalized D-conformal deformation with the vector field V a concurrent vector field. If (M,g) is η-Einstein then (M,g^*) is a Ricci soliton. And this Ricci soliton is shrinking or expanding or steady accordingly ρ is $>$ or $<$ or $=$ to $\frac{2n(a^2-b+1)}{(2a^2-b)}$ respectively.*

Contracting (3.14), we have

$$r = \frac{n}{b-1}\ \{V(b) + \frac{4n(a^2-b)}{a^2} + 2(a^2-b)\} + \frac{aV(a)}{(b-1)} \tag{3.26}$$

Let us now use the formula [12]

$$\mathscr{L}_V r = -\triangle r + 2R_{ij}R^{ij} + 2\lambda r. \tag{3.27}$$

As r is constant, we get

$$R_{ij}R^{ij} = -\lambda r. \tag{3.28}$$

Using (3.26), (3.28) becomes

$$R_{ij}R^{ij} = -\frac{n\lambda}{b-1}\ [V(b) + \frac{4n(a^2-b)}{a^2} + 2(a^2-b)] + \frac{\lambda aV(a)}{(b-1)}. \tag{3.29}$$

Using the formula (3.27), we write

$$\mathscr{L}_V r^* = -\triangle r^* + 2R^*_{ij}(R^*)^{ij} + 2\lambda r^*. \tag{3.30}$$

Since r^* is a constant, we get

$$R^*_{ij}(R^*)^{ij} = -\lambda r^*. \tag{3.31}$$

On contracting (3.4), we obtain

$$r^* = r + \frac{4n^2(a^2-b)}{a^2}. \tag{3.32}$$

By making use of (3.32) and (3.26), (3.31) becomes

$$R^*_{ij}(R^*)^{ij} = -\lambda\ [\frac{n}{b-1}\ \{V(b) + \frac{4n(a^2-b)}{a^2} + 2(a^2-b)\} + \frac{aV(a)}{(b-1)} + \frac{4n^2(a^2-b)}{a^2}]. \tag{3.33}$$

Comparing this with (3.4), we get

$$R^*_{ij}(R^*)^{ij} = R_{ij}R^{ij} + \frac{16n^4(a^2-b)^2}{a^4}[(g_{i,j} - \eta_i\eta_j)(g^{i,j} - \eta^i\eta^j)]. \tag{3.34}$$

After simplification equation (3.34) gives

$$R^*_{ij}(R^*)^{ij} = R_{ij}R^{ij} + \frac{32n^5(a^2-b)^2}{a^4} \tag{3.35}$$

In view of (3.33) and (3.35) , using (3.29), we obtain

$$\lambda = -\frac{8n^3(a^2-b)}{a^2}. \tag{3.36}$$

Thus we can state the following.

Theorem 3.3. *Ricci soliton in a generalized D-conformally deformed Kenmotsu manifold is shrinking or expanding accordingly as $a^2 > b$ or $a^2 < b$ respectively.*

When $a = b =$ constant, equation (3.36) reduces to

$$\lambda = -\frac{8n^3(a-1)}{a}. \tag{3.37}$$

Corollary 3.3. *A Ricci soliton in a D-homothetic deformed Kenmotsu manifold is shrinking or expanding according as $a > 1$ or $a < 1$ respectively.*

By substituting $a^2 = b$, equation (3.36) gives

$$\lambda = 0 \tag{3.38}$$

Corollary 3.4. *Ricci soliton in a conformally deformed Kenmotsu manifold is steady provided a and b are constants along ξ direction.*

By substituting (3.26) in (3.32) we have

$$r^* = \frac{1}{(b-1)}[aV(a) + n\{V(b) + \frac{2(a^2-b)(a^2+2n)}{a^2}\}]. \tag{3.39}$$

Theorem 3.4. *Under generalized D-conformal deformation η-Einstein Ricci soliton remains η-Einstein and the scalar curvature for a generalized D-conformally deformed Kenmotsu manifold is given by* (3.39)*.*

When $a = b$=constant, equation (3.39) becomes

$$r^* = \frac{2n(a^2+2n)}{a}. \tag{3.40}$$

Corollary 3.5. *D-homothetically deformed Kenmotsu manifold has a constant postive scalar curvature.*

Now (3.5) can be written as

$$\begin{aligned}&(\mathscr{L}_V g^*)(X,Y) + 2S^*(X,Y) + 2\lambda g^*(X,Y)\\ &= g^*(\nabla^*_X V, Y) + g^*(X, \nabla^*_Y V) + 2S(X,Y)\\ &+ \frac{4n(a^2-b)}{a^2}\{g(X,Y) - \eta(X)\eta(Y)\} + 2\lambda\{bg(X,Y) + (a^2-b)\eta(X)\eta(Y)\}.\end{aligned} \tag{3.41}$$

Making use of (2.1), (3.3) in (3.41), we obtain

$$\begin{aligned}&= b(\mathscr{L}_V g)(X,Y) + \{g(X,V)\eta(Y) + g(Y,V)\eta(X) - 2\eta(X)\eta(Y)\eta(V)\\ &+ \eta(\nabla_X V)\eta(Y) + \eta(\nabla_Y V)\eta(X)\}(a^2-b) + 2S(X,Y)\\ &+ \frac{4n(a^2-b)}{a^2}\{g(X,Y) - \eta(X)\eta(Y)\} + 2\lambda\{bg(X,Y) + (a^2-b)\eta(X)\eta(Y)\}.\end{aligned} \tag{3.42}$$

If $V \perp \xi$. it provides $\eta(\nabla_X V) = -g(\phi V, \phi X)$. Hence by using (1.1), the above equation becomes

$$= 2(1-b)S(X,Y) + \frac{4n(a^2-b)}{a^2}\{g(X,Y) - \eta(X)\eta(Y)\} + 2\lambda(a^2-b)\eta(X)\eta(Y). \tag{3.43}$$

i.e g^* is a Ricci soliton iff

$$S(X,Y) = Cg(X,Y) + D\eta(X)\eta(Y). \tag{3.44}$$

Where $C = \frac{2n(a^2-b)}{a^2(b-1)}$ and $D = \frac{(a^2-b)(\lambda a^2-2n)}{a^2(b-1)}$.

By putting $X = Y = \xi$ in (3.44) we obtain

$$\lambda = -\frac{2n(b-1)}{(a^2-b)}. \tag{3.45}$$

Theorem 3.5. *Let (M, ϕ, ξ, η, g) be a Kenmotsu manifold and $(M, \phi^*, \xi^*, \eta^*, g^*)$ be obtained by generalized D-conformal deformation with the associated vector field V is orthagonal to Reeb vector field. If (M, g) is η-Einstein then (M, g^*) is a Ricci soliton. And If $b > 1$, this Ricci soliton is shrinking or expanding accordingly as $a^2 > b$ or $a^2 < b$ respectively.*

When $a = b$ =constant, equation (3.45) becomes

$$\lambda = -\frac{2n}{a}. \tag{3.46}$$

Corollary 3.6. *A Ricci soliton in a D-homothetic deformed Kenmotsu manifold is shrinking.*

If $a^2 = b$, then from (3.45), It follows that

Corollary 3.7. *For constants a and b along ξ, there does not exist a conformally deformed Kenmotsu metric which is a Ricci soliton.*

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