On Two-Dimensional Power Associative Algebras Over Algebraically Closed Fields and R

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Abstract—In this paper we describe all power-associative algebra structures on a two-dimensional vector space over algebraically closed fields and R. The list of all two-dimensional left(right) unital power-associative algebras, along with their unit elements, is specified. Also we compare the result of the paper with that results obtained earlier.

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1. INTRODUCTION

The class of power-associative algebras is big enough to contain such classes of well known algebras as associative, Lie, commutative Jordan and alternative (in the last two cases it is assumed that the underlying field is of characteristic 0). In this paper we use results from [1, 6, 7, 11, 18, 19] to describe all power-associative algebra structures on a two-dimensional vector space over algebraically closed fields and $\mathbb{R}$ by providing the lists of canonical representatives of their structure constant's matrices. The description of two-dimensional power-associative algebras over the field of complex numbers and $\mathbb{R}$ has been given in [24]. However, the approach considered in the paper is totally different from that of [24] and the same time the results of the paper confirm those of [24] in the case of field of complex and real numbers, moreover, we solve the problem over the fields of finite characteristic case as well.

As to the descriptions of some other classes of two-dimensional algebras we refer the reader to [2]–[5], [8]–[17], [20]–[23].

We consider two-dimensional power-associative algebras over algebraically closed fields of characteristic not $2, 3$, characteristic 2, characteristic 3 and over $\mathbb{R}$ separately according to classification results of [1, 11]. To the best knowledge of the authors the descriptions of left(right) unital two-dimensional algebras over algebraically closed fields and $\mathbb{R}$ have not been given yet.

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2. PRELIMINARIES

Let $\mathbb{F}$ be a field, $A \otimes B$ stand for the Kronecker product of the matrices A and B over $\mathbb{F}$.

Let $(\mathbb{A}, \cdot)$ be a m-dimensional algebra over $\mathbb{F}$ and $e = (e_1, e_2, ..., e_m)$ be its basis. Then the bilinear binary operation $\cdot$ is represented by a matrix $A = (A_{ij}^k) \in M(m \times m^2; \mathbb{F})$ as follows

$$\mathbf{u} \cdot \mathbf{v} = eA(u \otimes v),$$

where $\mathbf{u} = eu, \mathbf{v} = ev$ and $u = (u_1, u_2, ..., u_m)^T, v = (v_1, v_2, ..., v_m)^T$ are the column coordinate vectors of $\mathbf{u}$ and $\mathbf{v}$, respectively. The matrix $A \in M(m \times m^2; \mathbb{F})$ defined above is called the matrix of structure constants (MSC) of $\mathbb{A}$ with respect to the basis e. Further we assume that the basis e is fixed and we do not make a difference between the algebra $\mathbb{A}$ and its MSC A (see [10]).

If $e' = (e'_1, e'_2, ..., e'_m)$ is another basis of $\mathbb{A}$, $e'g = e$ with $g \in G = GL(m; \mathbb{F})$, and A' is MSC of $\mathbb{A}$ with respect to e' then we have

$$A' = gA(g^{-1})^{\otimes 2}. \tag{1}$$

Thus, the isomorphism of algebras $\mathbb{A}$ and $\mathbb{B}$ over $\mathbb{F}$ can be given in terms of MSC as follows.

Definition 1. *Two m-dimensional algebras $\mathbb{A}$, $\mathbb{B}$ over $\mathbb{F}$, given by their matrices of structure constants A, B, are said to be isomorphic if there exists $g \in GL(m; \mathbb{F})$ such that $B = gA(g^{-1})^{\otimes 2}$.*

In the paper we deal with a class of algebras called "power associative algebras", the definition is given as follows.

Definition 2. *An algebra $\mathbb{A}$ is said to be power-associative if its every subalgebra generated by a single element is associative.*

Here are the basic results which are used in the paper. First of all we need results concerning the minimal(independent) system of identities for power-associativity of algebras. The problem has been studied in [6, 7, 19]. In [6] the author has shown that the algebra $\mathbb{A}$ over a field $\mathbb{F}$ ($Char(\mathbb{F}) = 0$) is power-associative if and only if

$$[\mathbf{u}, \mathbf{u}, \mathbf{u}] = 0, \;\; [\mathbf{u}^2, \mathbf{u}, \mathbf{u}] = 0 \text{ for all } \mathbf{u} \in \mathbb{A},$$

where $[\mathbf{u}, \mathbf{v}, \mathbf{w}] = (\mathbf{u}\mathbf{v})\mathbf{w} - \mathbf{u}(\mathbf{v}\mathbf{w})$ is the associator of $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{A}$.

If $\mathbb{F}$ is an infinite field of finite characteristic p then the variety of power associative algebras over the $\mathbb{F}$ can not be defined by a finite system of identities. This case has been considered in [19] and the final result is given as follows.

If $Char(\mathbb{F}) = 2$ ($Char(\mathbb{F}) = 3$, $Char(\mathbb{F}) = 5$, $Char(\mathbb{F}) = p > 5$) then an algebra $\mathbb{A}$ over $\mathbb{F}$ is power-associative if and only if

$$[\mathbf{u}, \mathbf{u}^3, \mathbf{u}] = 0, \;\; [\mathbf{u}^{n-2}, \mathbf{u}, \mathbf{u}] = 0 \text{ for all } \mathbf{u} \in \mathbb{A}, \; n = 3, 2^k, \; k = 2, 3, ...$$

(respectively,

$$[\mathbf{u}^{n-2}, \mathbf{u}, \mathbf{u}] = 0 \text{ for all } \mathbf{u} \in \mathbb{A}, \; n = 4, 5, 3^k, \; k = 1, 2, 3, ...,$$

$$[\mathbf{u}^{n-2}, \mathbf{u}, \mathbf{u}] = 0 \text{ for all } \mathbf{u} \in \mathbb{A}, \; n = 3, 4, 6, 5^k, \; k = 1, 2, 3, ...,$$

$$[\mathbf{u}^{n-2}, \mathbf{u}, \mathbf{u}] = 0 \text{ for all } \mathbf{u} \in \mathbb{A}, \; n = 3, 4, p^k, \; k = 1, 2, 3, ...),$$

where $\mathbf{u}^n = \mathbf{u}^{n-1}\mathbf{u}$ for all $n = 1, 2, 3, ...$ and $\mathbf{u}^1 = \mathbf{u}$ by definition.

Further we consider only the case $m = 2$ and for the simplicity we use

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$$

for MSC, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$ stand for elements of $\mathbb{F}$ (structure constants of $\mathbb{A}$).

The next we need results from [1, 11] on classification of all algebra structures on two-dimensional vector space over a field $\mathbb{F}$ of $Char(\mathbb{F}) \neq 2, 3$, $Char(\mathbb{F}) = 2$, $Char(\mathbb{F}) = 3$ and $\mathbb{F} = \mathbb{R}$ cases, respectively.

Theorem 2.1. *Over an algebraically closed field $\mathbb{F}$ ($Char(\mathbb{F}) \neq 2$ and 3), any non-trivial 2-dimensional algebra is isomorphic to only one of the following algebras listed by their matrices of structure constants:*

- $A_1(\mathbf{c}) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2+1 & \alpha_4 \\ \beta_1 & -\alpha_1 & -\alpha_1+1 & -\alpha_2 \end{pmatrix}$, *where* $\mathbf{c} = (\alpha_1, \alpha_2, \alpha_4, \beta_1) \in \mathbb{F}^4$,
- $A_2(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ -\beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = (\alpha_1, \beta_1, \beta_2) \in \mathbb{F}^3$,
- $A_3(\mathbf{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & \beta_2 & 1 & -1 \end{pmatrix}$, *where* $\mathbf{c} = (\beta_1, \beta_2) \in \mathbb{F}^2$,
- $A_4(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = (\alpha_1, \beta_2) \in \mathbb{F}^2$,
- $A_5(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & 2\alpha_1-1 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = \alpha_1 \in \mathbb{F}$,
- $A_6(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ -\beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = (\alpha_1, \beta_1) \in \mathbb{F}^2$,
- $A_7(\mathbf{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 0 & -1 \end{pmatrix}$, *where* $\mathbf{c} = \beta_1 \in \mathbb{F}$,
- $A_8(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = \alpha_1 \in \mathbb{F}$,
- $A_9 = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 1 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix}$, $A_{10} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $A_{11} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$,
  $A_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Theorem 2.2. *Over an algebraically closed field $\mathbb{F}$ ($Char(\mathbb{F}) = 2$), any non-trivial 2-dimensional algebra is isomorphic to only one of the following algebras listed by their matrices of structure constants:*

- $A_{1,2}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2+1 & \alpha_4 \\ \beta_1 & -\alpha_1 & -\alpha_1+1 & -\alpha_2 \end{pmatrix}$, *where* $\mathbf{c} = (\alpha_1, \alpha_2, \alpha_4, \beta_1) \in \mathbb{F}^4$,
- $A_{2,2}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = (\alpha_1, \beta_1, \beta_2) \in \mathbb{F}^3$,

- $A_{3,2}(\mathbf{c})=\begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & \beta_2 & 1 & -1 \end{pmatrix}$, *where* $\mathbf{c}=(\beta_1,\beta_2)\in\mathbb{F}^2$,

- $A_{4,2}(\mathbf{c})=\begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c}=(\alpha_1,\beta_2)\in\mathbb{F}^2$,

- $A_{5,2}(\mathbf{c})=\begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & 1 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c}=\alpha_1\in\mathbb{F}$,

- $A_{6,2}(\mathbf{c})=\begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c}=(\alpha_1,\beta_1)\in\mathbb{F}^2$,

- $A_{7,2}(\mathbf{c})=\begin{pmatrix} \alpha_1 & 1 & 1 & 0 \\ 0 & 1-\alpha_1 & -\alpha_1 & -1 \end{pmatrix}$, *where* $\mathbf{c}=\alpha_1\in\mathbb{F}$,

- $A_{8,2}(\mathbf{c})=\begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c}=\alpha_1\in\mathbb{F}$,

- $A_{9,2}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, $A_{10,2}=\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $A_{11,2}=\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 \end{pmatrix}$,
  $A_{12,2}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Theorem 2.3. *Over an algebraically closed field* $\mathbb{F}$ $(Char(\mathbb{F})=3)$, *any non-trivial* 2*-dimensional algebra is isomorphic to only one of the following algebras listed by their matrices of structure constant matrices:*

- $A_{1,3}(\mathbf{c})=\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2+1 & \alpha_4 \\ \beta_1 & -\alpha_1 & -\alpha_1+1 & -\alpha_2 \end{pmatrix}$, *where* $\mathbf{c}=(\alpha_1,\alpha_2,\alpha_4,\beta_1)\in\mathbb{F}^4$,

- $A_{2,3}(\mathbf{c})=\begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}\simeq\begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ -\beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c}=(\alpha_1,\beta_1,\beta_2)\in\mathbb{F}^3$,

- $A_{3,3}(\mathbf{c})=\begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & \beta_2 & 1 & -1 \end{pmatrix}$, *where* $\mathbf{c}=(\beta_1,\beta_2)\in\mathbb{F}^2$,

- $A_{4,3}(\mathbf{c})=\begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c}=(\alpha_1,\beta_2)\in\mathbb{F}^2$,

- $A_{5,3}(\mathbf{c})=\begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & -1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c}=\alpha_1\in\mathbb{F}$,

- $A_{6,3}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ -\beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = (\alpha_1, \beta_1) \in \mathbb{F}^2$,
- $A_{7,3}(\mathbf{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 0 & -1 \end{pmatrix}$, *where* $\mathbf{c} = \beta_1 \in \mathbb{F}$,
- $A_{8,3}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = \alpha_1 \in \mathbb{F}$,
- $A_{9,3} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$, $A_{10,3} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$,
- $A_{11,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{pmatrix}$, $A_{12,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Theorem 2.4. *Any non-trivial* 2*-dimensional real algebra is isomorphic to only one of the following listed, by their matrices of structure constants, algebras:*

$A_{1,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2+1 & \alpha_4 \\ \beta_1 & -\alpha_1 & -\alpha_1+1 & -\alpha_2 \end{pmatrix}$, *where* $\mathbf{c} = (\alpha_1, \alpha_2, \alpha_4, \beta_1) \in \mathbb{R}^4$,

$A_{2,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\beta_1 \geq 0,\ \mathbf{c} = (\alpha_1, \beta_1, \beta_2) \in \mathbb{R}^3$,

$A_{3,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & -1 \\ \beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\beta_1 \geq 0,\ \mathbf{c} = (\alpha_1, \beta_1, \beta_2) \in \mathbb{R}^3$,

$A_{4,r}(\mathbf{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & \beta_2 & 1 & -1 \end{pmatrix}$, *where* $\mathbf{c} = (\beta_1, \beta_2) \in \mathbb{R}^2$,

$A_{5,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = (\alpha_1, \beta_2) \in \mathbb{R}^2$,

$A_{6,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & 2\alpha_1-1 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = \alpha_1 \in \mathbb{R}$,

$A_{7,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, *where* $\beta_1 \geq 0,\ \mathbf{c} = (\alpha_1, \beta_1) \in \mathbb{R}^2$,

$A_{8,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & -1 \\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, *where* $\beta_1 \geq 0,\ \mathbf{c} = (\alpha_1, \beta_1) \in \mathbb{R}^2$,

$A_{9,r}(\mathbf{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 0 & -1 \end{pmatrix}$, *where* $\mathbf{c} = \beta_1 \in \mathbb{R}$,

$$A_{10,r}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix},\ \textit{where } \mathbf{c}=\alpha_1\in\mathbb{R},$$

$$A_{11,r}=\begin{pmatrix}\frac{1}{3} & 0 & 0 & 0\\ 1 & \frac{2}{3} & -\frac{1}{3} & 0\end{pmatrix},\quad A_{12,r}=\begin{pmatrix}0 & 1 & 1 & 0\\ 1 & 0 & 0 & -1\end{pmatrix},$$

$$A_{13,r}=\begin{pmatrix}0 & 1 & 1 & 0\\ -1 & 0 & 0 & -1\end{pmatrix},\quad A_{14,r}=\begin{pmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix},\quad A_{15,r}=\begin{pmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}.$$

3. THE MAIN RESULTS

One of the identities needed for power associativity of any algebra $\mathbb{A}$ is

$$[\mathbf{u},\mathbf{u},\mathbf{u}]=\mathbf{u}^2\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{u}^2=0 \tag{2}$$

which means that the cube of any element $\mathbf{u}\in\mathbb{A}$ is well defined. The description of a such class of algebras is an interesting problem. Let us describe all such two-dimensional algebras first.

3.1. Two-dimensional algebras with well defined cubes

Using $\mathbf{u}\cdot\mathbf{v}=eA(u\otimes v)$, where $\mathbf{u}=eu,\ \mathbf{v}=ev$, we can rewrite (2) as follows

$$A((A\otimes I)-(I\otimes A))\otimes u^{\otimes 3}=0.$$

Let $\mathbb{A}$ be a 2-dimensional algebra given by its MSC as $A=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\\ \beta_1 & \beta_2 & \beta_3 & \beta_4\end{pmatrix}$. The condition above in terms of x,y, where $u=(x,y)$, and the structure constants is given as a system of equations as follows:

$$x^2y\alpha_1\alpha_2+xy^2\alpha_2^2-x^2y\alpha_1\alpha_3-xy^2\alpha_3^2+y^3\alpha_2\alpha_4-y^3\alpha_3\alpha_4-x^3\alpha_2\beta_1+x^3\alpha_3\beta_1-x^2y\alpha_2\beta_2$$
$$+x^2y\alpha_3\beta_2-x^2y\alpha_2\beta_3+x^2y\alpha_3\beta_3-xy^2\alpha_2\beta_4+xy^2\alpha_3\beta_4=0,$$
$$x^2y\alpha_1\beta_2+xy^2\alpha_2\beta_2+xy^2\alpha_3\beta_2+y^3\alpha_4\beta_2-x^3\beta_1\beta_2-x^2y\beta_2^2-x^2y\alpha_1\beta_3-xy^2\alpha_2\beta_3$$
$$-xy^2\alpha_3\beta_3-y^3\alpha_4\beta_3+x^3\beta_1\beta_3+x^2y\beta_3^2-xy^2\beta_2\beta_4+xy^2\beta_3\beta_4=0.$$

Equating the coefficients at x^3,x^2y,xy^2,y^3 to zero we get the system of equations with respect to $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4$ as follows

$$\begin{aligned}
-\alpha_2\beta_1+\alpha_3\beta_1 &=0,\\
\alpha_1\alpha_2-\alpha_1\alpha_3-\alpha_2\beta_2+\alpha_3\beta_2-\alpha_2\beta_3+\alpha_3\beta_3 &=0,\\
\alpha_2^2-\alpha_3^2-\alpha_2\beta_4+\alpha_3\beta_4 &=0,\\
\alpha_2\alpha_4-\alpha_3\alpha_4 &=0,\\
-\beta_1\beta_2+\beta_1\beta_3 &=0,\\
\alpha_1\beta_2-\beta_2^2-\alpha_1\beta_3+\beta_3^2 &=0,\\
\alpha_2\beta_2+\alpha_3\beta_2-\alpha_2\beta_3-\alpha_3\beta_3-\beta_2\beta_4+\beta_3\beta_4 &=0,\\
\alpha_4\beta_2-\alpha_4\beta_3 &=0,
\end{aligned} \tag{3}$$

i.e., in terms of the structure constant the identity (2) is equivalent to (3).

Theorem 3.1. *Over an algebraically closed field $\mathbb{F}$, ($Char(\mathbb{F})\neq 2,\ 3$) any nontrivial 2-dimensional algebra with well defined cubes, is isomorphic to only one of the following such algebras listed by their matrices of structure constants*

- $A_1\left(\frac{1}{3},-\frac{1}{3},0,0\right)=\begin{pmatrix}\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0\\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3}\end{pmatrix}$.
- $A_2\left(\alpha_1,\beta_1,1-\alpha_1\right)=\begin{pmatrix}\alpha_1 & 0 & 0 & 1\\ \beta_1 & 1-\alpha_1 & 1-\alpha_1 & 0\end{pmatrix}\simeq\begin{pmatrix}\alpha_1 & 0 & 0 & 1\\ -\beta_1 & 1-\alpha_1 & 1-\alpha_1 & 0\end{pmatrix}$,
- $A_3\left(\beta_1,1\right)=\begin{pmatrix}0 & 1 & 1 & 0\\ \beta_1 & 1 & 1 & -1\end{pmatrix}$,
- $A_4\left(\alpha_1,1-\alpha_1\right)=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & 1-\alpha_1 & 1-\alpha_1 & 0\end{pmatrix}$, *where* $\alpha_1\neq\frac{2}{3}$,
- $A_4\left(\alpha_1,2\alpha_1-1\right)=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & 2\alpha_1-1 & 1-\alpha_1 & 0\end{pmatrix}$,
- $A_5\left(\frac{2}{3}\right)=\begin{pmatrix}\frac{2}{3} & 0 & 0 & 0\\ 1 & \frac{1}{3} & \frac{1}{3} & 0\end{pmatrix}$,
- $A_8\left(\frac{1}{3}\right)=\begin{pmatrix}\frac{1}{3} & 0 & 0 & 0\\ 0 & \frac{2}{3} & -\frac{1}{3} & 0\end{pmatrix}$,
- $A_{10}=\begin{pmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$,
- $A_{11}=\begin{pmatrix}0 & 1 & 1 & 0\\ 1 & 0 & 0 & -1\end{pmatrix}$,
- $A_{12}=\begin{pmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}$.

Proof. For $A_1(\alpha_1,\alpha_2,\alpha_4,\beta_1)$ due to (3) we have

$$\beta_1=0,\quad 1-3\alpha_1=0,\quad -1-3\alpha_2=0,\quad \alpha_4=0,$$

that is in the algebra $A_1\left(\frac{1}{3},-\frac{1}{3},0,0\right)$ the cube of any element is well defined.

For $A_2(\alpha_1,\beta_1,\beta_2)$ case one has

$$\begin{aligned}\beta_1-\alpha_1\beta_1-\beta_1\beta_2 &=0,\\ 1-3\alpha_1+2\alpha_1^2+\alpha_1\beta_2-\beta_2^2 &=0,\\ -1+\alpha_1+\beta_2 &=0,\end{aligned}$$

so in the algebra $A_2\left(\alpha_1,\beta_1,-\alpha_1+1\right)$ the cube of any element is well defined.

For $A_3(\beta_1, \beta_2)$ case one has

$$\beta_1 - \beta_1\beta_2 = 0, \quad 1 - \beta_2^2 = 0, \quad -3 + 3\beta_2 = 0,$$

so in the algebra $A_3\left(\beta_1, 1\right)$ the cube of any element is well defined.

For $A_4(\alpha_1, \beta_2)$ case, using (3) we have

$$1 - 3\alpha_1 + 2\alpha_1^2 + \alpha_1\beta_2 - \beta_2^2 = 0,$$

which implies that $\beta_2 = 1 - \alpha_1$ or $\beta_2 = 2\alpha_1 - 1$ and therefore in the algebras

$$A_4\left(\alpha_1, 1-\alpha_1\right), \text{where } \alpha_1 \neq \frac{2}{3}, \ A_4(\alpha_1, 2\alpha_1 - 1)$$

the cube of any element is well defined.

For $A_5(\alpha_1)$ case, due to (3) one gets $2 - 3\alpha_1 = 0$, so in the case of $A_5(\frac{2}{3})$ the cube of any element is well defined.

It is easy to see that for algebras $A_6(\alpha_1, \beta_1)$, $A_7(\beta_1)$ the system (3) is inconsistent.

For $A_8(\alpha_1)$ the system (3) is equivalent to $-1 + 3\alpha_1 = 0$, therefore $A_8\left(\frac{1}{3}\right)$ is an algebra with well-defined cubes.

It is easy to find out that for the case of A_9 the system (3) is inconsistent and for the case of algebras A_{10}, A_{11}, A_{12} their structure constants satisfy the system of equations (3). □

Due to the similarities of proofs we state the corresponding results in $Char(\mathbb{F}) = 2, 3$ and $\mathbb{F} = \mathbb{R}$ cases without proofs as follows.

Theorem 3.2. *Over an algebraically closed field* $\mathbb{F}$, $(Char(\mathbb{F}) = 2)$ *any nontrivial two-dimensional algebra with well-defined cubes is isomorphic to only one of the following such algebras listed by their matrices of structure constants*

- $A_{1,2}\left(1, -1, 0, 0\right) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$,
- $A_{2,2}\left(\alpha_1, \beta_1, 1-\alpha_1\right) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix}$,
- $A_{3,2}\left(\beta_1, 1\right) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 1 & -1 \end{pmatrix}$,
- $A_{4,2}\left(\alpha_1, 1-\alpha_1\right) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\alpha_1 \neq 0$,
- $A_{4,2}\left(\alpha_1, 1\right) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1 & 1-\alpha_1 & 0 \end{pmatrix}$,
- $A_{5,2}(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$,

- $A_{8,2}(1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$,

- $A_{10,2} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$,

- $A_{11,2} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 \end{pmatrix}$,

- $A_{12,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Theorem 3.3. *Over an algebraically closed field $\mathbb{F}$, $(Char(\mathbb{F}) = 3)$ any nontrivial 2-dimensional algebra with well-defined cubes is isomorphic to only one of the following such algebras listed by their matrices of structure constants*

- $A_{2,3}(\alpha_1, \beta_1, 1-\alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ -\beta_1 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix}$,

- $A_{3,3}(\beta_1, 1) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 1 & -1 \end{pmatrix}$,

- $A_{3,3}(0, -1) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix}$,

- $A_{4,3}(\alpha_1, 1-\alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix}$,

- $A_{4,3}(\alpha_1, 2\alpha_1-1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 2\alpha_1-1 & 1-\alpha_1 & 0 \end{pmatrix}$,

- $A_{9,3} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$,

- $A_{10,3} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$,

- $A_{11,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{pmatrix}$,

- $A_{12,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Theorem 3.4. *Over the real field $\mathbb{R}$ there exist, up to isomorphism, only the following non trivial two-dimensional algebras with well-defined cubes.*

- $A_{1,r}\left(\frac{1}{3}, -\frac{1}{3}, 0, 0\right) = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$,
- $A_{2,r}(\alpha_1, \beta_1, 1-\alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\beta_1 \geq 0$,
- $A_{3,r}(\alpha_1, \beta_1, 1-\alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & -1 \\ \beta_1 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\beta_1 \geq 0$,
- $A_{4,r}(\beta_1, 1) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 1 & -1 \end{pmatrix}$,
- $A_{5,r}(\alpha_1, 1-\alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\alpha_1 \neq \frac{2}{3}$,
- $A_{5,r}(\alpha_1, 2\alpha_1 - 1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 2\alpha_1 - 1 & 1-\alpha_1 & 0 \end{pmatrix}$,
- $A_{6,r}\left(\frac{2}{3}\right) = \begin{pmatrix} \frac{2}{3} & 0 & 0 & 0 \\ 1 & -\frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$,
- $A_{10,r}\left(\frac{1}{3}\right) = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix}$,
- $A_{12,r} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$,
- $A_{13,r} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}$,
- $A_{14,r} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$,
- $A_{15,r} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

3.2. Two-dimensional power-associative algebras

Now to describe two-dimensional power-associative algebras let us recall the necessary and sufficient conditions stated earlier for power associativity of two-dimensional algebras.

Proposition 3.5. *A two-dimensional algebra $\mathbb{A}$ is power-associative if and only if*

$$[\mathbf{u},\mathbf{u},\mathbf{u}]=0,\ \ [\mathbf{u}^2,\mathbf{u},\mathbf{u}]=0 \textit{ for all } \mathbf{u}\in\mathbb{A}.$$

Proof. It is evident that one needs only the proof of "if" part. So let $[\mathbf{u},\mathbf{u},\mathbf{u}]=0,\ \ [\mathbf{u}^2,\mathbf{u},\mathbf{u}]=0$ for all $\mathbf{u}\in\mathbb{A}$. Due to the dimension reason for any element $\mathbf{u}\in\mathbb{A}$ there exist $a,b\in\mathbb{F}$ such that $\mathbf{u}^3=a\mathbf{u}^2+b\mathbf{u}$. Therefore for any $n\geq 3$ the equality $[\mathbf{u}^{n-2},\mathbf{u},\mathbf{u}]=0$ is true, that is in $Char(\mathbb{F})\neq 2$ case the proposition is true.

In the case of $Char(\mathbb{F})=2$ we have

$$[\mathbf{u},\mathbf{u}^3,\mathbf{u}]=[\mathbf{u},a\mathbf{u}^2+b\mathbf{u},\mathbf{u}]=a[\mathbf{u},\mathbf{u}^2,\mathbf{u}]+b[\mathbf{u},\mathbf{u},\mathbf{u}]=a[\mathbf{u},\mathbf{u}^2,\mathbf{u}]$$

and

$$[\mathbf{u},\mathbf{u}^2,\mathbf{u}]=\mathbf{u}^3\mathbf{u}-\mathbf{u}\mathbf{u}^3=(a\mathbf{u}^2+b\mathbf{u})\mathbf{u}-\mathbf{u}(a\mathbf{u}^2+b\mathbf{u})=0.$$

This completes the proof. □

Due to Proposition 3.5 to describe all two-dimensional power-associative algebras it is enough to find out which of the algebra among those in Theorems 3.1—3.4 satisfies the identity

$$(\mathbf{u}^2\cdot\mathbf{u})\cdot\mathbf{u}=\mathbf{u}^2\cdot\mathbf{u}^2. \tag{4}$$

In terms of MSC the identity (4) is equivalent to

$$A((A\otimes A)-(A(A\otimes I))\otimes I)u^{\otimes 4}=0,$$

where $u=(x,y)$. The identity above produces the following system of equations with respect to the structure constants

$$\begin{aligned}
\alpha_1\alpha_2\beta_1-\alpha_1\alpha_3\beta_1+\alpha_4\beta_1^2-\alpha_3\beta_1\beta_3 &=0,\\
-\alpha_1^2\alpha_2+\alpha_1^2\alpha_3+\alpha_2^2\beta_1-2\alpha_1\alpha_4\beta_1+\alpha_1\alpha_2\beta_2-\alpha_1\alpha_3\beta_2+2\alpha_4\beta_1\beta_2&\\
+\alpha_1\alpha_2\beta_3+\alpha_4\beta_1\beta_3-\alpha_3\beta_2\beta_3-\alpha_3\beta_3^2-\alpha_3\beta_1\beta_4 &=0,\\
-2\alpha_1\alpha_2^2+\alpha_1\alpha_3^2+\alpha_1^2\alpha_4-\alpha_2\alpha_4\beta_1-\alpha_3\alpha_4\beta_1+\alpha_2^2\beta_2-2\alpha_1\alpha_4\beta_2+\alpha_4\beta_2^2+\alpha_2^2\beta_3&\\
+\alpha_2\alpha_3\beta_3+\alpha_3^2\beta_3-\alpha_1\alpha_4\beta_3+\alpha_4\beta_2\beta_3+\alpha_1\alpha_2\beta_4+\alpha_4\beta_1\beta_4-\alpha_3\beta_2\beta_4-2\alpha_3\beta_3\beta_4 &=0,\\
-\alpha_2^3-\alpha_2^2\alpha_3+2\alpha_1\alpha_3\alpha_4-\alpha_4^2\beta_1-\alpha_2\alpha_4\beta_2-\alpha_3\alpha_4\beta_2+\alpha_3\alpha_4\beta_3&\\
+\alpha_2^2\beta_4+\alpha_2\alpha_3\beta_4+\alpha_3^2\beta_4-\alpha_1\alpha_4\beta_4+\alpha_4\beta_2\beta_4-\alpha_3\beta_4^2 &=0,\\
-\alpha_2^2\alpha_4+\alpha_1\alpha_4^2-\alpha_4^2\beta_2+\alpha_3\alpha_4\beta_4 &=0,\\
-\alpha_3\beta_1^2+\alpha_1\beta_1\beta_2-\beta_1\beta_3^2+\beta_1^2\beta_4 &=0,\\
\alpha_1\alpha_3\beta_1-\alpha_4\beta_1^2-\alpha_1^2\beta_2+\alpha_2\beta_1\beta_2-\alpha_3\beta_1\beta_2+\alpha_1\beta_2^2-\alpha_3\beta_1\beta_3+\alpha_1\beta_2\beta_3&\\
+\alpha_1\beta_3^2-\beta_2\beta_3^2-\beta_3^3-\alpha_1\beta_1\beta_4+2\beta_1\beta_2\beta_4 &=0,\\
\alpha_2\alpha_3\beta_1+\alpha_3^2\beta_1+\alpha_1\alpha_4\beta_1-2\alpha_1\alpha_2\beta_2-\alpha_1\alpha_3\beta_2-\alpha_4\beta_1\beta_2+\alpha_2\beta_2^2-\alpha_4\beta_1\beta_3+\alpha_2\beta_2\beta_3&\\
+\alpha_2\beta_3^2+\alpha_3\beta_3^2-\alpha_2\beta_1\beta_4-2\alpha_3\beta_1\beta_4+\beta_2^2\beta_4+\alpha_1\beta_3\beta_4-2\beta_3^2\beta_4+\beta_1\beta_4^2 &=0,\\
\alpha_2\alpha_4\beta_1+2\alpha_3\alpha_4\beta_1-\alpha_2^2\beta_2-\alpha_2\alpha_3\beta_2-\alpha_1\alpha_4\beta_2+\alpha_4\beta_3^2-2\alpha_4\beta_1\beta_4&\\
-\alpha_3\beta_2\beta_4+\alpha_2\beta_3\beta_4+\alpha_3\beta_3\beta_4+\beta_2\beta_4^2-\beta_3\beta_4^2 &=0,\\
\alpha_4^2\beta_1-\alpha_2\alpha_4\beta_2-\alpha_4\beta_2\beta_4+\alpha_4\beta_3\beta_4 &=0.
\end{aligned} \tag{5}$$

Theorem 3.6. *Over an algebraically closed field $\mathbb{F}$, $(Char(\mathbb{F}) \neq 2,\ 3)$ any nontrivial 2-dimensional power-associative algebra is isomorphic to only one of the following power-associative algebras listed by their matrices of structure constants*

- $A_1\left(\frac{1}{3}, -\frac{1}{3}, 0, 0\right) = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$,
- $A_2\left(\frac{1}{2}, 0, \frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$,
- $A_4\left(\frac{1}{2}, \frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$,
- $A_4(1, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,
- $A_4\left(\alpha_1, 2\alpha_1 - 1\right) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 2\alpha_1 - 1 & -\alpha_1 + 1 & 0 \end{pmatrix}$,
- $A_8\left(\frac{1}{3}\right) = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix}$,
- $A_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Proof. As it was mentioned above to prove the theorem we should verify that the structure constants each of the algebras from the classification results of Theorems 3.1—3.4 to satisfy the system of equations (5).

In $A_1\left(\frac{1}{3}, -\frac{1}{3}, 0, 0\right)$ case its structure constants satisfy the system of equations (5), therefore $A_1\left(\frac{1}{3}, -\frac{1}{3}, 0, 0\right)$ is power-associative algebra.

In $A_2\left(\alpha_1, \beta_1, -\alpha_1 + 1\right)$ case the system (5) implies that

$$\beta_1^2 = 0,\ 3\beta_1 - 5\alpha_1\beta_1 = 0,\ 2 - 7\alpha_1 + 6\alpha_1^2 = 0,\ -\beta_1 = 0,\ -1 + 2\alpha_1 = 0,\ -\beta_1 + 3\alpha_1\beta_1 - 2\alpha_1^2\beta_1 = 0,$$

$$-2 + 9\alpha_1 - 13\alpha_1^2 + 6\alpha_1^3 - \beta_1^2 = 0,\ -2\beta_1 + 3\alpha_1\beta_1 = 0,\ 1 - 3\alpha_1 + 2\alpha_1^2 = 0,$$

that is, $\alpha_1 = \frac{1}{2},\ \beta_1 = 0$ and so $A_2\left(\frac{1}{2}, 0\right)$ is a power-associative algebra.

The structure constants of $A_3\left(\beta_1, 1\right)$ do not satisfy (5).

In $A_4\left(\alpha_1, -\alpha_1 + 1\right)$ case, where $\alpha_1 \neq \frac{2}{3}$, the system of equations (5) is equivalent to $-2 + 9\alpha_1 - 13\alpha_1^2 + 6\alpha_1^3 = 0$. Therefore $\alpha_1 = \frac{1}{2}$ or $\alpha_1 = 1$ and one gets the following power-associative algebras $A_4\left(\frac{1}{2}, \frac{1}{2}\right),\ A_4(1, 0)$.

The structure constants of $A_4\left(\alpha_1, 2\alpha_1 - 1\right)$ satisfy the equations (5), therefore $A_4\left(\alpha_1, 2\alpha_1 - 1\right)$ are power-associative algebras.

The system of components of $A_5\left(\frac{2}{3}\right)$ does not satisfy (5).

In $A_8\left(\frac{1}{3}\right)$ case the system (5) is satisfied, so $A_8\left(\frac{1}{3}\right)$ is a power-associative algebra.

The structure constants of A_{10} and A_{11} do not satisfy the system of equations (5). Evidently, A_{12} satisfies the system of equations (5), therefore A_{12} also is a power-associative algebra. □

The proofs of the theorems below are similar to that above.

Theorem 3.7. *Over an algebraically closed field $\mathbb{F}$, $(Char(\mathbb{F}) = 2)$ any nontrivial 2-dimensional power-associative algebra is isomorphic to only one of the following power-associative algebras listed by their matrices of structure constants*

- $A_{1,2}\left(1,-1,0,0\right) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$,
- $A_{4,2}(1,0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,
- $A_{4,2}\left(\alpha_1,1\right) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1 & -\alpha_1+1 & 0 \end{pmatrix}$,
- $A_{8,2}\left(1\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$,
- $A_{10,2} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$,
- $A_{12,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Theorem 3.8. *Over an algebraically closed field $\mathbb{F}$, $(Char(\mathbb{F}) = 3)$ any nontrivial 2-dimensional power-associative algebra is isomorphic to only one of the following power-associative algebras listed by their matrices of structure constants*

- $A_{2,3}\left(\frac{1}{2},0,\frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$,
- $A_{3,3}\left(0,-1\right) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix}$,
- $A_{4,3}\left(\frac{1}{2},\frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$,
- $A_{4,3}(1,0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,
- $A_{4,3}\left(\alpha_1,2\alpha_1-1\right) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 2\alpha_1-1 & -\alpha_1+1 & 0 \end{pmatrix}$,

- $A_{10,3} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$

- $A_{12,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$

Theorem 3.9. *Over the real field $\mathbb{R}$ there exist, up to isomorphism, only the following non trivial two-dimensional power-associative algebras*

- $A_{1,r}\left(\frac{1}{3}, -\frac{1}{3}, 0, 0\right) = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$

- $A_{2,r}\left(\frac{1}{2}, 0, \frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$

- $A_{3,r}\left(\frac{1}{2}, 0, \frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$

- $A_{5,r}\left(\frac{1}{2}, \frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$

- $A_{5,r}(1, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$

- $A_{5,r}\left(\alpha_1, 2\alpha_1 - 1\right) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 2\alpha_1 - 1 & 1 - \alpha_1 & 0 \end{pmatrix},$

- $A_{10,r}\left(\frac{1}{3}\right) = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix},$

- $A_{15,r} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$

The derivation algebras and automorphism groups of all the algebras listed above have been described in [5].

Definition 3. *An element $\mathbf{1}_L$ ($\mathbf{1}_R$) of an algebra $\mathbb{A}$ is called a left (respectively, right) unit if $\mathbf{1}_L \cdot \mathbf{u} = \mathbf{u}$ (respectively, $\mathbf{u} \cdot \mathbf{1}_R = \mathbf{u}$) for all $\mathbf{u} \in \mathbb{A}$. An algebra with the left(right) unit element is said to be left(right) unital algebra, respectively.*

Now we can describe, up to isomorphism, all two-dimensional left(right) unital power-associative algebras over an algebraically closed field $\mathbb{F}$ and $\mathbb{R}$. To do this it is sufficient to compare the algebras listed in Theorems 3.6 − 3.9 with the two-dimensional left(right) unital algebras given in [2]. As an immediate consequence of the theorems the lists of left(right) unital power-associative algebras structures on a two-dimensional vector space and their left(right) units we give in the table below.

	Algebra	$\mathbf{1}_L$	Algebra	$\mathbf{1}_R$
$Ch(\mathbb{F}) \neq 2$	$A_2(\frac{1}{2},0,\frac{1}{2})$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$A_2(\frac{1}{2},0,\frac{1}{2})$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
	$A_4(\frac{1}{2},\frac{1}{2})$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$A_4(\frac{1}{2},\frac{1}{2})$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
	$A_4(1,1)$	$\begin{pmatrix} 1 \\ t \end{pmatrix}, t \in \mathbb{F}$	$A_4(\frac{1}{2},0)$	$\begin{pmatrix} 2 \\ t \end{pmatrix}, t \in \mathbb{F}$
$Ch(\mathbb{F}) = 2$	$A_{4,2}(1,1)$	$\begin{pmatrix} 1 \\ t \end{pmatrix}, t \in \mathbb{F}$	$A_{8,2}(1)$	$\begin{pmatrix} 1 \\ t \end{pmatrix}, t \in \mathbb{F}$
	$A_{10,2}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$A_{10,2}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$\mathbb{F} = \mathbb{R}$	$A_{2,r}(\frac{1}{2},0,\frac{1}{2})$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$A_{2,r}(\frac{1}{2},0,\frac{1}{2})$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
	$A_{3,r}(\frac{1}{2},0,\frac{1}{2})$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$A_{3,r}(\frac{1}{2},0,\frac{1}{2})$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
	$A_{5,r}(\frac{1}{2},\frac{1}{2})$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$A_{5,r}(\frac{1}{2},\frac{1}{2})$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
	$A_{5,r}(1,1)$	$\begin{pmatrix} 1 \\ t \end{pmatrix}, t \in \mathbb{R}$	$A_{5,r}(\frac{1}{2},0)$	$\begin{pmatrix} 2 \\ t \end{pmatrix} t \in \mathbb{R}$

Remark 3.10. *Now we also can compare the results of the present paper with that of [24]. According to [24] the lists of two-dimensional power-associative real (complex) algebras are given as follows* $A_1,\ A_2,\ A_3,\ A_4,\ A(\sigma), B_1, B_2, B_3, B_4, B_5$ *(note that there is no need for* A_2 *as far as* $A_2 = A(0)$*)(respectively,* $A_1,\ A_3,\ A_4,\ A(\sigma), B_1, B_2, B_3, B_5$*) algebras, where* $\sigma \in \mathbb{R}$ *(respectively,* $\sigma \in \mathbb{C}$*). In the notations of the present paper they are:*

A_1 *is the trivial algebra,* $A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \simeq A_{5,r}(1,1),$

$$A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \simeq A_{5,r}(\tfrac{1}{2},0),\ A_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \simeq A_{1,r}\left(\tfrac{1}{3},-\tfrac{1}{3},0,0\right),$$

$$A(\sigma) = \begin{pmatrix} 1+\sigma & 0 & 0 & 0 \\ 0 & 1 & \sigma & 0 \end{pmatrix} \simeq A_{5,r}\left(\alpha_1,-1+2\alpha_1\right), \textit{where } \alpha_1 \neq \tfrac{1}{2}, \textit{and } \sigma = \tfrac{1-\alpha_1}{2\alpha_1-1}\ (\textit{so } \sigma \neq -\tfrac{1}{2}),$$

$$A(-\tfrac{1}{2}) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \end{pmatrix} \simeq A_{10,r}\left(\tfrac{1}{3}\right), \ B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \simeq A_{15,r},$$

$$B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \simeq A_{5,r}(1,0), \quad B_3 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \simeq A_{2,r}\left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right),$$

$$B_4 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \simeq A_{3,r}\left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right), \ B_5 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \simeq A_{5,r}(\tfrac{1}{2}, \tfrac{1}{2}).$$

(*respectively, A_1 is the trivial algebra, $A_3 \simeq A_4(\frac{1}{2}, 0)$, $A_4 \simeq A_1(\frac{1}{3}, -\frac{1}{3}, 0, 0)$, $A(\sigma) \simeq A_4(\alpha_1, 2\alpha_1 - 1)$, where $\alpha_1 \neq \frac{1}{2}$, and $\sigma = \frac{1-\alpha_1}{2\alpha_1 - 1}$, $A(-\frac{1}{2}) \simeq A_8(\frac{1}{3})$, $B_1 \simeq A_{12}, B_2 \simeq A_4(1,0)$, $B_3 \simeq A_2(\frac{1}{2}, 0, \frac{1}{2}), B_5 \simeq A_4(\frac{1}{2}, \frac{1}{2})$*).

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