RINGS WHOSE ELEMENTS ARE LINEAR COMBINATIONS OF THREE COMMUTING IDEMPOTENTS

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ABSTRACT. We classify those rings in which all elements are linear combinations over $\mathbb{Z}$ of at most three commuting idempotents. Our results improve on recent publications by the author in Albanian J. Math. (2018), Gulf J. Math. (2018), Bull. Iran. Math. Soc. (2019) and Boll. Un. Mat. Ital. (2019) as well as on publications due to Hirano-Tominaga in Bull. Austral. Math. Soc. (1988), Ying et al. in Can. Math. Bull. (2016) and Tang et al. in Lin. & Multilin. Algebra (2018).

1. INTRODUCTION AND BACKGROUND

Everywhere in the text of the present paper, all rings R are assumed to be associative, containing the identity element 1 which differs from the zero element 0 of R. The standard terminology and notations are mainly in agreement with [8]. For instance, $U(R)$ denotes the set of all units in R, $Id(R)$ the set of all idempotents in R, $Nil(R)$ the set of all nilpotents in R and $J(R)$ the Jacobson radical of R. As usual, $\mathbb{Z}$ stands for the ring of all integers, and $\mathbb{Z}_k \cong \mathbb{Z}/k\mathbb{Z}$ is its quotient modulo the principal ideal $(k) = k\mathbb{Z}$, where $k \in \mathbb{N}$ is the set of all naturals.

About the specific notions, they will be explained in detail below.

Definition 1.1. We shall say that the ring R is from the class $\mathcal{R}_1$ if, for any $r \in R$, there exist commuting each to other $e_1, e_2, e_3 \in Id(R)$ such that $r = e_1 + e_2 + e_3$ or $r = e_1 + e_2 - e_3$.

Obvious examples of such rings are the rings $\mathbb{Z}_k$, where $k = 2, 3, 4, 5, 6$. Contrasting with that, the ring $\mathbb{Z}_7$ need not be so.

Definition 1.2. We shall say that the ring R is from the class $\mathcal{R}_2$ if, for any $r \in R$, there exist commuting each to other $e_1, e_2, e_3 \in Id(R)$ such that $r = e_1 + e_2 + e_3$ or $r = e_1 - e_2 - e_3$.

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Immediate examples of such rings are the rings $\mathbb{Z}_k$, where $k = 2, 3, 4, 5, 6$. Reciprocally, the ring $\mathbb{Z}_7$ does not have that property.

The most important principally known achievements concerning the subject are as follows: Classically, a ring is said to be *boolean* if each its element is an idempotent – such a ring is known to be a subdirect product copies of the two element field $\mathbb{F}_2$. A successful attempt to generalize that concept is made in [7] to those rings whose elements are the sum of two commuting idempotents – in fact, these rings are known to be commutative being a subdirect product of family of copies of the two and three element fields $\mathbb{F}_2$ and $\mathbb{F}_3$, respectively. In particular, if every element of a ring is an idempotent or minus an idempotent, then this ring is either boolean, or $\mathbb{F}_3$, or a direct product of two such rings.

Further extensions of these notions, in terms of linear combinations over $\mathbb{Z}$ of at most four commuting idempotents, are subsequently given below in details.

- $\forall r \in R$, $r = e_1 + e_2$ or $r = e_1 - e_2$ for some two commuting $e_1, e_2 \in Id(R)$ (see [10]).
- $\forall r \in R$, $r = e_1 + e_2$ or $r = -e_1 - e_2$ for some two commuting $e_1, e_2 \in Id(R)$ (see [5]).
- $\forall r \in R$, $r = e_1 + e_2 + e_3$ for some three commuting $e_1, e_2, e_3 \in Id(R)$ (see [4] and [9]).
- $\forall r \in R$, $r = e_1 + e_2 + e_3$ or $r = -e_1$ for some three commuting $e_1, e_2, e_3 \in Id(R)$ (see [2]).
- $\forall r \in R$, $r = e_1 + e_2 + e_3$ or $r = e_1 - e_2$ for some three commuting $e_1, e_2, e_3 \in Id(R)$ (see [4]).
- $\forall r \in R$, $r = e_1 + e_2 + e_3$ or $r = -e_1 - e_2$ for some three commuting $e_1, e_2, e_3 \in Id(R)$ (see [1]).
- $\forall r \in R$, $r = e_1 + e_2 + e_3$ or $r = -e_1 - e_2 - e_3$ for some three commuting $e_1, e_2, e_3 \in Id(R)$ (see [3]).
- $\forall r \in R$, $r = e_1 + e_2 + e_3 + e_4$ for some four commuting $e_1, e_2, e_3, e_4 \in Id(R)$ (see [6]).

In all of the aforementioned variations, the ring is of necessity commutative.

Our major tactic is to develop the technique exploited in [1]-[6] as well as to utilize some new tricks inspired by the specification of the ring structure. Specifically, we shall carefully study the rings from Definitions 1.1 and 1.2 characterizing them up to an isomorphism.

2. Main Results

For the reader's convenience and readability of the exposition, we distribute our results into the following two subsections.

2.1. Rings from the class $\mathcal{R}_1$.

We start here with the following technicality.

Proposition 2.1. *Any ring R from the class $\mathcal{R}_1$ decomposes as $R_1 \times R_2 \times R_3$, where R_1, R_2, R_3 are rings also lying in the class $\mathcal{R}_1$ such that $2 \in Nil(R_1)$, $3 \in Nil(R_2)$ and $5 \in Nil(R_3)$.*

Proof. Writing $-3 = e_1 + e_2 + e_3$, we infer that $-4 = e_1 + e_2 - (1 - e_3) = e_1 + e_2 - e_3'$. Since $e_2 - e_3' = e_2(1 - e_3') - e_3'(1 - e_2)$ is a difference of two orthogonal idempotents, we may assume without loss of generality that $e_2e_3' = 0$. Consequently, multiplying both sides of $-4 = e_1 + e_2 - e_3'$ by e_2, we get that $5e_2 = -e_1e_2$ and hence $6e_1e_2 = 0$ implying that $30e_2 = 0$. After that, squaring $-4 = e_1 + e_2 - e_3'$ and taking into account that $e_1 + e_2 + e_3' = -4 + 2e_3'$, we derive that $20 = 2e_3' + 2e_1e_2 - 2e_1e_3'$. The last equality being multiplied by e_1 leads to $20e_1 = 2e_1e_2$ and so $60e_1 = 0$. However, $20 = 2e_3' + 2e_1e_2 - 2e_1e_3'$ ensures that $600 = 60e_3'$ and thus a simple manipulation in $-4 = e_1 + e_2 - e_3'$ leads to $360 = 8.9.5 = 0$ whence $30^3 = 0$. Thus $30 = 2.3.5 \in Nil(R)$.

Let us now $-3 = e_1 + e_2 - e_3$. Observing that $e_2 - e_3 = e_2(1 - e_3) - e_3(1 - e_2)$ is a difference of two orthogonal idempotents, we can assume with no harm in generality that $e_2e_3 = 0$. Thus $-3e_2 = e_1e_2 + e_2$ means that $-4e_2 = e_1e_2$, and $-3e_1e_2 = e_1e_2 + e_1e_2$ gives that $5e_1e_2 = 0$ which assures that $20e_2 = 0$. Similarly, repeating the same procedure for e_1, one detects that $20e_1 = 0$. We further square $-3 = e_1 + e_2 - e_3$ and deduce that $12 = 2e_3 + 2e_1e_2$ because $e_1 + e_2 + e_3 = -3 + 2e_3$. Multiplying both sides of the equality about 12 by e_3, we obtain that $10e_3 = 0$. Therefore, $-3.20 = (e_1 + e_2 - e_3).20 = 0$, i.e., $60 = 4.3.5 = 0$.

Finally, in both cases, 2.3.5 must be a nilpotent and thus the Chinese Remainder Theorem applies to conclude that $R \cong R_1 \times R_2 \times R_3$, where R_1, R_2, R_3 are rings again from the class $\mathcal{R}_1$ with $2 \in Nil(R_1)$, $3 \in Nil(R_2)$ and $5 \in Nil(R_3)$, as formulated. □

The following assertion is pivotal, strengthening [1, Proposition 2.2].

Lemma 2.2. *Suppose that R is a ring of characteristic 5. Then the following three conditions are equivalent:*

(i) $x^3 = x$ *or* $x^4 = 1$, $\forall x \in R$.

(ii) $x^3 = -x$ *or* $x^4 = 1$, $\forall x \in R$.

(iii) $x^3 = x$ *or* $x^3 = -x$, $\forall x \in R$.

(iv) R *is isomorphic to the field* $\mathbb{Z}_5$.

Proof. "(i) $\Rightarrow$ (iii)". For an arbitrary $y \in R$ satisfying $y^4 = 1$ but $y^3 \neq y$, considering the element $y^2 - 1$, it must be that $(y^2-1)^4 = 1$ or $(y^2-1)^3 = y^2 - 1$. In the first case we receive $y^2 = -1$ and thus $y^3 = -y$, as required, while in the second one we arrive at $y^2 = 1$ and so $y^3 = y$ which is against our initial assumption.

"(ii) $\Rightarrow$ (iii)". The same trick as that in the previous implication will work, assuming now that $y^3 \neq -y$.

"(iii) $\iff$ (iv)". Let P be the subring of R generated by 1, and thus note that $P \cong \mathbb{Z}_5$. We claim that $P = R$, so we assume in a way of contradiction that there exists $b \in R \setminus P$. With no loss of generality, we shall also assume that $b^3 = b$ since $b^3 = -b$ obviously implies that $(2b)^3 = 2b$ as $5 = 0$ and $b \notin P \iff 2b \notin P$.

Let us now $(1+b)^3 = -(1+b)$. Hence $b = b^3$ along with $5 = 0$ enable us that $b^2 = 1$. This allows us to conclude that $(1+2b)^3 \neq \pm(1+2b)$, however. In fact, if $(1+2b)^3 = 1+2b$, then one deduces that $2b = 3 \in P$ which is manifestly untrue. If now $(1+2b)^3 = -1-2b$, then one infers that $2b = 2 \in P$ which is obviously false. That is why, only $(1+b)^3 = 1+b$ holds. This, in turn, guarantees that $b^2 = -b$. Moreover, $b^3 = b$ is equivalent to $(-b)^3 = -b$ as well as $b^3 = -b$ to $(-b)^3 = -(-b)$ and thus, by what we have proved so far applied to $-b \notin P$, it follows that $-b = b^2 = (-b)^2 = -(-b) = b$. Consequently, $2b = 0 = 6b = b \in P$ because $5 = 0$, which is the wanted contradiction. We thus conclude that $P = R$, as expected.

Conversely, it is trivial that the elements of $\mathbb{Z}_5 = \{0,1,2,3,4 \mid 5 = 0\}$ are solutions of one of the equations $x^3 = x$ or $x^3 = -x$.

"(iv) $\Rightarrow$ (i),(ii)". It is self-evident that all elements of $\mathbb{Z}_5 = \{0,1,2,3,4 \mid 5 = 0\}$ satisfy one of the equations $x^3 = x$ or $x^4 = 1$ as well as one of $x^3 = -x$ or $x^4 = 1$. $\square$

We now come to the following.

Theorem 2.3. *A ring* R *lies in the class* $\mathcal{R}_1$ *if, and only if, it is commutative and* $R \cong R_1 \times R_2 \times R_3$, *where* R_1, R_2, R_3 *are rings for which*

(1) $R_1 = \{0\}$ *or* $R_1/J(R_1)$ *is a boolean factor-ring with nil* $J(R_1) = 2Id(R_1)$ *such that* $4 = 0$;

(2) $R_2 = \{0\}$ *or* R_2 *is a subdirect product of copies of the fields* $\mathbb{Z}_2$ *and* $\mathbb{Z}_3$;

(3) $R_3 = \{0\}$ *or* $R_3 \cong \mathbb{Z}_5$.

Proof. "**Necessity.**" Appealing to Proposition 2.1, there is a decomposition $R \cong R_1 \times R_2 \times R_3$, where the direct factors R_1, R_2 and R_3 still belong to the class $\mathcal{R}_1$. What we need to do is to describe explicitly these three rings separately.

Describing R_1: Here $2 \in J(R_1)$. We assert that $4 = 0$. To establish this, the proof of Proposition 2.1 assures that $8 = 0$. Next, write that $4 = e_1 + e_2 + e_3$ or that $4 = e_1 + e_2 - e_3$. In the first case, the multiplication by $e_1e_2e_3$ gives that $e_1e_2e_3 = 0$. Squaring the equality of 4, one infers that $4 = 2e_1e_2 + 2e_2e_3 + 2e_3e_1$. Multiplying this by e_1e_2, we derive that $2e_1e_2 = 0$. Similarly, $2e_2e_3 = 2e_3e_1 = 0$. So, $4 = 0$, as promised. In the second case, the multiplication by $e_1e_2e_3$ guarantees that $3e_1e_2e_3 = 0$. Again squaring the equation for 4, one deduces that $4 = 2e_3 + 2e_1e_2 - 2e_2e_3 - 2e_3e_1$. Multiplying this by e_1e_2, we detect that $2e_1e_2 = -2e_1e_2e_3$, so that $2e_1e_2e_3 = -2e_1e_2e_3$ which means that $4e_1e_2e_3 = 0$. The last, combined with $3e_1e_2e_3 = 0$, allows us to conclude that $e_1e_2e_3 = 0 = 2e_1e_2$. Furthermore, $(4 + e_3)^2 = (e_1 + e_2)^2$ is a guarantor that $e_3 = e_1 + e_2$ and thus $4 = e_1 + e_2 - e_3 = 0$, as promised, and the claim really sustained.

We observe also that $R_1/J(R_1)$ is of characteristic 2 ring from the class $\mathcal{R}_1$. Thus it has to be a boolean ring. What needs to show is the equality $J(R_1) = 2Id(R_1)$. In fact, given $z \in J(R_1)$, we write $z = e_1 + e_2 + e_3$ or $z = e_1 - e_2 - e_3$ for some three commuting idempotents e_1, e_2, e_3 in R_1. In the first case $z - 2e_3 = e_1 + e_2 - e_3$ still lies in $J(R_1)$. Since $e_2 - e_3 = e_2(1 - e_3) - e_3(1 - e_2)$ and $e_2(1 - e_3).e_3 = 0$, we may assume that $e_2.e_3 = 0$. As in the proof of [4, Theorem 2.2], one obtains that $z - 2e_3 = 2e_2$. Hence $z = 2(e_2 + e_3) \in 2Id(R_1)$, as expected, as the sum $e_2 + e_3$, because e_2, e_3 are mutually orthogonal idempotents. In the second case, we may process as in the proof of [4, Theorem 2.2]. So, we receive the pursued description of R_1 after all.

Describing R_2: Here $3 \in J(R_2)$. We claim that $J(R_2) = \{0\}$ and that $3 = 0$. In fact, as in the preceding case for R_1, we have that $J(R_2) = 2Id(R_2)$ or $J(R_2) = -2Id(R_2)$. If for any $j \in J(R_2)$ we write $j = 2i$ for some $i \in Id(R_2)$, then $-j + 3i = i \in J(R_2) \cap Id(R_2) = \{0\}$ whence $i = 0 = j$. Symmetrically, if $j = -2i$, then $j + 3i = i \in J(R_2) \cap Id(R_2) = \{0\}$ and hence $i = 0 = j$, as required. Furthermore, since $3 = 0$, it easily follows that $x^3 = x$ for all $x \in R_2$ and thus [7] is applicable to get the wanted description of R_2.

Describing R_3: Here $5 \in J(R_3)$. We assert that $J(R_3) = \{0\}$ and that $5 = 0$. Indeed, as in the previous case for R_1, we have that $J(R_3) = \pm 2Id(R_3)$. If for any $j \in J(R_3)$ we write $j = 2i$ for some $i \in Id(R_3)$, then $-2j + 5i = i \in$

$J(R_3) \cap Id(R_3) = \{0\}$ and hence $i = 0 = j$. By a reason of symmetry, if $j = -2i$, then $2j + 5i = i \in J(R_3) \cap Id(R_3) = \{0\}$ whence $i = 0 = j$, as required. Furthermore, as $5 = 0$, it readily follows that $x^5 = x$ for all $x \in R_3$. We now intend to show that there is a more precise situation, that is, any element x from R_3 satisfies $x^4 = 1$ or $x^3 = x$. To that goal, writing $x = e_1 + e_2 + e_3$ for some three commuting idempotents e_1, e_2, e_3 from R_3, we are processing in the same manner as in [1, Theorem 2.3, Case 3] to get that $x^4 = 1$; actually this equality holds under the substitution $x \to x - 1$. Writing next that $x = e_1 + e_2 - e_3$, again for some three commuting idempotents e_1, e_2, e_3 of R_3, we may assume with no harm of generality that $e_1e_3 = e_2e_3 = 0$. Indeed, $x = e_1 + e_2(1 - e_3) - e_3(1 - e_2)$ as the last two members are orthogonal idempotents, so further $x = e_1[1 - e_3(1 - e_2)] + e_2(1 - e_3) - e_3(1 - e_2)(1 - e_1)$ where all members retain idempotents as well as the first and the second ones are orthogonal with the third one, as claimed. Squaring the initial equality, we deduce that $x^2 - x = 2(e_1e_2 + e_3) \in 2Id(R_3)$ as it is easily checked that the sum $e_1e_2 + e_3$ is an idempotent, whence $(x^2 - x)^2 = 2(x^2 - x)$. This enables us that $x^4 - 2x^3 - x^2 + 2x = 0$ which, by the substitution $x \to x - 1$, leads to $x^4 - x^3 + x^2 - x = 0$. Now replacing x by $2x$ and, respectively, by $3x$ in $x^4 = 1$, we once again detect that $x^4 = 1$ since $5 = 0$, so that doing that in $x^4 - x^3 + x^2 - x = 0$ we have that $x^4 + 2x^3 - x^2 - 2x = 0$ and $x^4 - 2x^3 - x^2 + 2x = 0$, respectively. These two equalities, in combination, imply that $x^3 = x$, as desired. Therefore, Lemma 2.2 (i) is now in use and we, consequently, obtain the wanted description of R_3 as being isomorphic to $\mathbb{Z}_5$.

Concerning the commutativity of the whole ring R, since R_2 and R_3 are obviously commutative, what remains to show is that this property holds for R_1. This, however, follows by the utilization of [4, Theorem 2.2].

"**Sufficiency.**" A direct consultation with [7] informs us that every element of R_2 is a sum of two idempotents. Likewise, as in [1], [4] or [5], each element in R_1 is a sum of three idempotents. Since R_3 has only five elements, we are, therefore, in a position to exploit the same manipulation as that in the corresponding results from [1], [4] and [5] getting that the direct product $R_1 \times R - 2 \times R_3$ belongs to the class $\mathcal{R}_1$, as expected. □

The next comments shed some more light on the examined object.

Remark 2.4. The direct product $\mathbb{Z}_5 \times \mathbb{Z}_5$ does not lie in the class $\mathcal{R}_1$, whereas the direct products $\mathbb{Z}_4 \times \mathbb{Z}_5$ and $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_5$ still lie. Moreover, concerning the element-wise presentation in these rings, the crucial elements $(1, 4)$ and $(2, 4)$, having problems with their representation in [1] and [4], here work positively. In fact, one has that $(1,4) = (1,0) + (0,0) - (0,1)$, $(2,4) = (1,0) + (1,0) - (0,1)$ and $(1,2,4) = (1,1,0) + (0,1,0) - (0,0,1)$.

2.2. **Rings from the class $\mathcal{R}_2$.** We begin here with the following technicality.

Proposition 2.5. *Any ring R from the class $\mathcal{R}_2$ decomposes as $R_1 \times R_2 \times R_3$, where R_1, R_2, R_3 are rings also lying in the class $\mathcal{R}_2$ such that $2 \in Nil(R_1)$, $3 \in Nil(R_2)$ and $5 \in Nil(R_3)$.*

Proof. Let firstly $-3 = e_1 + e_2 + e_3$. So $-4 = e_1 + e_2 - (1 - e_3) = e_1 + e_2 - e_3'$. Since $e_2 - e_3' = e_2(1 - e_3') - e_3'(1 - e_2)$ and this is a difference of two orthogonal idempotents, we may assume with no harm of generality that $e_2e_3' = 0$. Therefore, multiplying by e_2 both sides of $-4 = e_1 + e_2 - e_3'$, we obtain that $5e_2 = -e_1e_2$ whence $6e_1e_2 = 0$ and thus $30e_2 = 0$. Furthermore, squaring $-4 = e_1+e_2-e_3'$ and bearing in mind that $e_1+e_2+e_3' = -4+2e_3'$, we infer that $20 = 2e_3'+2e_1e_2-2e_1e_3'$ which being multiplied by e_1 leads to $20e_1 = 2e_1e_2$, so that $60e_1 = 0$. However, $20 = 2e_3'+2e_1e_2-2e_1e_3'$ means then that $600 = 60e_3'$ and so a plain manipulation in $-4 = e_1 + e_2 - e_3'$ gives that $360 = 8.9.5 = 0$ and hence $30^3 = 0$. Thus $30 = 2.3.5 \in Nil(R)$.

Let secondly $-3 = e_1-e_2-e_3$. Since $e_1-e_2 = e_1(1-e_2)-e_2(1-e_1)$ is a difference of two orthogonal idempotents, we with no loss of generality may assume that $e_1.e_2 = 0$. Therefore, by multiplying with e_1 both sides of the given equality, we deduce that $4e_1 = e_1e_3$ whence $3e_1e_3 = 0$ and so $12e_1 = 0$. Further, by squaring $-3 = e_1 - e_2 - e_3$ and taking into account that $e_1 + e_2 + e_3 = 2e_1 + 3$, we derive that $6 = 2e_1-2e_1e_3+2e_2e_3$ and multiplying this by $e_3$and next by e_2, we get that $4e_2e_3 = 0$ and hence $12e_3 = 0$. Consequently, the equality $6 = 2e_1 - 2e_1e_3 + 2e_2e_3$ allows us to conclude that $36 = 6^2 = 0$, so that $6 \in Nil(R)$.

Finally, in both cases, 2.3.5 must be a nilpotent and so the Chinese Remainder Theorem yields that $R \cong R_1 \times R_2 \times R_3$, where R_1, R_2, R_3 are rings again from the class $\mathcal{R}_2$ with $2 \in Nil(R_1)$, $3 \in Nil(R_2)$ and $5 \in Nil(R_3)$, as stated. □

We now arrive at the following.

Theorem 2.6. *A ring R lies in the class $\mathcal{R}_2$ if, and only if, it is commutative and $R \cong R_1 \times R_2 \times R_3$, where R_1, R_2, R_3 are rings for which*

(1) $R_1 = \{0\}$ or $R_1/J(R_1)$ is a boolean quotient-ring with nil $J(R_1) = 2Id(R_1)$ such that $4 = 0$;

(2) $R_2 = \{0\}$ or R_2 is a subdirect product of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$;

(3) $R_3 = \{0\}$ or $R_3 \cong \mathbb{Z}_5$.

Proof. "**Necessity.**" According to Proposition 2.5, there is a decomposition $R \cong R_1 \times R_2 \times R_3$, where the direct factors R_1, R_2 and R_3 still belong to the class $\mathcal{R}_2$. What we need to do is to describe in an explicit form these three rings separately.

Describing R_1: Here $2 \in J(R_1)$. We assert that $4 = 0$. To establish this, the proof of Proposition 2.5 insures that $8 = 0$. Next, we may write that $4 = e_1+e_2+e_3$, which follows identically to case (1) of Theorem 2.3, and that $4 = e_1 - e_2 - e_3$. The latter after multiplying with $e_1e_2e_3$ forces that $5e_1e_2e_3 = 0$, and by squaring implies that $4 = 2e_1 - 2e_1e_2 - 2e_1e_3 + 2e_2e_3$. The multiplication of the last equation with e_1e_2 yields that $4e_1e_2 = 0$ whence $4e_1e_2e_3 = 0$ whence $e_1e - 2e_3 = 0$. Again multiplying the equality of 4 by e_2e_3 now tells us that $2e_2e_3 = 0$. Consequently, $(e_1 - 4)^2 = (e_2 + e_3)^2$, that is, $e_1 = e_2 + e_3$ and hence $4 = e_1 - e_2 - e_3 = 0$, as promised. This substantiates our assertion, indeed.

We see also that $R_1/J(R_1)$ is of characteristic 2 ring from the class $\mathcal{R}_2$. Thus it has to be a boolean ring. What suffices to prove is the equality $J(R_1) = 2Id(R_1)$. In fact, given $z \in J(R_1)$, one writes that $z = e_1 + e_2 + e_3$, which can be handled as in Theorem 2.3 above, or that $z = e_1 - e_2 - e_3$ which can be written as $J(R_1) \ni z + 2e_2 = e_1 + e_2 - e_3$. Again as above tricked, $z + 2e_2 = 2e_2$, so that $z = 0 \in 2Id(R_1)$, as expected. So, we finally receive the pursued description of R_1.

Describing R_2: Here $3 \in J(R_2)$. We claim that $J(R_2) = \{0\}$. In fact, as in the preceding case, we have that $J(R_2) = 2Id(R_2)$ or $J(R_2) = -2Id(R_2)$. If for any $j \in J(R_2)$ we write $j = 2i$ for some $i \in Id(R_2)$, then $-j + 3i = i \in J(R_2) \cap Id(R_2) = \{0\}$ whence $i = 0 = j$. Symmetrically, if $j = -2i$, then $j+3i = i \in J(R_2) \cap Id(R_2) = \{0\}$ and hence $i = 0 = j$, as required. Furthermore, since $3 = 0$, it easily follows that $x^3 = x$ for all $x \in R_2$ and thus [7] is applicable to get the wanted description of R_2.

Describing R_3: Here $5 \in J(R_3)$. We assert that $J(R_3) = \{0\}$ and that $5 = 0$. Indeed, as in the previous case for R_1, we have that $J(R_3) = \pm 2Id(R_3)$. If for any $j \in J(R_3)$ we write $j = 2i$ for some $i \in Id(R_3)$, then $-2j + 5i = i \in J(R_3) \cap Id(R_3) = \{0\}$ and hence $i = 0 = j$. By a reason of symmetry, if $j = -2i$, then $2j + 5i = i \in J(R_3) \cap Id(R_3) = \{0\}$ whence $i = 0 = j$, as required. Furthermore, as $5 = 0$, it readily follows that $x^5 = x$ for all $x \in R_3$. However, we intend to prove the more exact equations $x^4 = 1$ or $x^3 = -x$. In order to do that, write $x = e_1 + e_2 + e_3$ or $x = e_1 - e_2 - e_3$. In the first case we adapt the corresponding situation from Theorem 2.3 yielding that $x^4 = 1$ under the substitution $x \to x - 1$. In the second case, we can assume without loss of generality that $e_1e_2 = e_1e_3 = 0$. Squaring the second equation, one gets that $x^2 + x = 2(e_1 + e_2e_3) \in 2Id(R_3)$ as it is elementarily to verify that the sum $e_1 + e_2e_3$ is an idempotent, and hence $(x^2 + x)^2 = 2(x^2 + x)$. This means that $x^4 + 2x^3 - x^2 - 2x = 0$ which, under the substituting $x \to x - 1$, makes sense that $x^4 - 2x^3 - x^2 + 2x = 0$. Furthermore, as already above demonstrated, the two substitutions $x \to 2x$ and $x \to 3x$ in $x^4 = 1$ give nothing new, while in the other valid equality $x^4 - 2x^3 - x^2 + 2x = 0$ they ensure that $x^4 - x^3 + x^2 - x = 0$ and $x^4 + x^3 + x^2 + x = 0$. These two equations imply in turn together that $x^3 = -x$, as wanted. Therefore, Lemma 2.2 (ii) is now in use and we, consequently, obtain the desired description of R_3 as being isomorphic to $\mathbb{Z}_5$.

The commutativity of the former ring R follows in the same way as in Theorem 2.3 above.

"**Sufficiency.**" It follows by adapting the same idea as in the "sufficiency" of Theorem 2.3. □

Remark 2.7. The direct product $\mathbb{Z}_5 \times \mathbb{Z}_5$ does not lie in the class $\mathcal{R}_2$, whereas the direct products $\mathbb{Z}_4 \times \mathbb{Z}_5$ and $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_5$ still lie. Besides, concerning the element-wise presentation in these rings, the crucial elements $(1, 4)$ and $(2, 4)$, having problems with their representation in [1] and [4], here work positively. In fact, one has that $(1, 4) = (1, 0) - (0, 0) - (0, 1)$, $(2, 4) = (0, 0) - (1, 0) - (1, 1)$ and $(1, 2, 4) = (1, 0, 0) - (0, 1, 0) - (0, 1, 1)$.

Curiously, it seems that the classes $\mathcal{R}_1$ and $\mathcal{R}_2$ coincide, which is somewhat a similarity with a relationship illustrated in [1].

As for the sentence "*It is pretty easy to establish that a (finite) direct product of rings from the class $\mathcal{C}$ is also a ring from the class $\mathcal{C}$*", stated in the proof of "Sufficiency" of Theorem 2.4 in [5], it certainly makes sense under the validation of points (1), (2) and (3) there as [5, Example 2.6] unambiguously showed.

We end our work with the following two queries:

Problem 2.8. Describe the isomorphism structure of those rings R from the class $\mathcal{R}_3$ in which for each element r there are three commuting idempotents such that $r = e_1 + e_2 - e_3$ or $r = e_1 - e_2 - e_3$.

The next statement is of general interest in the present topic.

Problem 2.9. If every element of a ring is expressed as a (finite) linear combination over $\mathbb{Z}$ of commuting idempotents, is that ring necessarily commutative?

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