Asymptotics for Hermite – Padé Approximants Associated with the Mittag-Leffler Functions

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Abstract—In this article, under certain restrictions, the convergence rate of type II Hermite – Padé approximants (including nondiagonal ones) for a system $\{\,_1F_1(1,\gamma;\lambda_j z)\}_{j=1}^{k}$, consisting of degenerate hypergeometric functions is found, when $\{\lambda_j\}_{j=1}^{k}$ are different complex numbers, and $\gamma \in \mathbb{C}\setminus\{0,-1,-2,...\}$. Without the indicated restrictions, similar statements were obtained for approximants of the indicated type, provided that the numbers $\{\lambda_j\}_{j=1}^{k}$ are the roots of the equation $\lambda^k = 1$. The theorems proved in this paper complement and generalize the results obtained earlier by other authors.

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1. INTRODUCTION

Let $\mathbb{Z}_+^k$ be the set of k-dimensional multi-indices (k ordered nonnegative integers). The sum $|m| = m_1 + \ldots + m_k$ is an order of the multi-index $\overrightarrow{m} = (m_1, \ldots, m_k)$. Also let us fix $n \in \mathbb{Z}_+^1$, multi-index $\overrightarrow{m} = (m_1, \ldots, m_k) \in \mathbb{Z}_+^k$ and denote $n_j = n + |m| - m_j$ for $j = 1, 2, ..., k$.

Consider the system of entire functions

$$F_\gamma^j(z) = \,_1F_1(1,\gamma;\lambda_j z) = \sum_{p=0}^{\infty} \frac{\lambda_j^p}{(\gamma)_p} z^p, \quad j = 1, 2, ..., k, \tag{1}$$

where $\gamma \in \mathbb{C}\setminus\mathbb{Z}_-$, $\mathbb{Z}_- = \{0,-1,-2,...\}$, $(\gamma)_0 = 1$, $(\gamma)_p = \gamma(\gamma+1)\cdots(\gamma+p-1)$ is the Pochhammer symbol, $\lambda = \{\lambda_j\}_{j=1}^{k}$ are different nonzero complex numbers (for $k = 1$, we assume that $\lambda_1 = 1$). Series of the form (1) are called hypergeometric series, and their sums are called degenerate hypergeometric functions. Recall (see [1, 2]) that the Mittag-Leffler function is defined by the power series

$$E_{\rho,\beta}(z) = \sum_{p=0}^{\infty} \frac{z^p}{\Gamma(p\,\rho^{-1} + \beta)} \quad (\rho > 0,\ \beta \in \mathbb{C})$$

and is a generalization of the exponential function. Taking into account the well-known equality $(\gamma)_p = \Gamma(p+\gamma)/\Gamma(\gamma)$, where, just as in the previous formula, $\Gamma(z)$ is the gamma function, we can see that the functions (1) are Mittag-Leffler functions. Therefore, the coordinates of a vector function $F_\gamma^\lambda = \left\{F_\gamma^1(z), ..., F_\gamma^k(z)\right\}$ are Mittag-Leffler functions. If $\gamma = 1$, then the vector function F_1^λ is an ordered set of exponentials $\{e^{\lambda_j z}\}_{j=1}^{k}$.

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Rational fractions

$$\pi^j_{n,\overrightarrow{m}}(z)=\pi^j_{n,\overrightarrow{m}}(z;F^\lambda_\gamma)=\frac{P^j_{n,\overrightarrow{m}}(z)}{Q_{n,\overrightarrow{m}}(z)},\ \ j=1,2,\dots,k,$$

are called *type* $(n,\overrightarrow{m})$ *Hermite – Padé approximants for the system* F^λ_γ, where algebraic polynomials $Q_{n,\overrightarrow{m}}(z)=Q_{n,\overrightarrow{m}}(z;F^\lambda_\gamma)$, $P^j_{n,\overrightarrow{m}}(z)=P^j_{n,\overrightarrow{m}}(z;F^\lambda_\gamma)$, $\deg Q_{n,\overrightarrow{m}}\leqslant|m|$, $\deg P^j_{n,\overrightarrow{m}}\leqslant n_j$ satisfy the conditions

$$R^j_{n,\overrightarrow{m}}(z)=R^j_{n,\overrightarrow{m}}(z;F^\lambda_\gamma)=Q_{n,\overrightarrow{m}}(z)\,F^j_\gamma(z)-P^j_{n,\overrightarrow{m}}(z)=A_jz^{n+|m|+1}+\dots.$$

$Q_{n,\overrightarrow{m}}$, $P^j_{n,\overrightarrow{m}}$ are called [3] *type II Hermite – Padé polynomials for the system* F^λ_γ. For the first time these polynomials appeared in Hermite's work [4] for the system of exponents F^λ_1 in the form of integrals, which are called *Hermite's integrals*. The decisive role of these integrals in the proof of the transcendence of the numbers e, π is well known (see [5]).

For $k=1$ (in this case $\overrightarrow{m}=m_1=m$, and $\pi_{n,m}(z;F^1_\gamma):=\pi^1_{n,\overrightarrow{m}}(z)$ are called *Padé approximants of function* F^1_γ) explicit expressions for the remainder function $R_{n,m}(z):=R^1_{n,\overrightarrow{m}}(z;F^1_\gamma)$ and the denominator $Q_m(z;F^1_\gamma)$ were found by H. van Rossum [6]: namely, for $n\geqslant m-1$

$$Q_m(z;F^1_\gamma)=\ {}_1F_1(-m,-n-m-\gamma+1;-z)\,,$$

$$R_{n,m}(z;F^1_\gamma)=\frac{(-1)^m m!\ (\gamma)_n\ z^{n+m+1}}{(\gamma)_{n+m}\ (\gamma)_{n+m+1}}\ {}_1F_1(m+1,n+m+\gamma+1;z)\,. \qquad (2)$$

Recall that

$${}_1F_1(\alpha,\beta;z)=\sum_{p=0}^{\infty}\frac{(\alpha)_p}{(\beta)_p}\ \frac{z^p}{p!}\,.$$

For the system F^γ_λ, analogues of Hermite's representations were obtained by A.I. Aptekarev [7]: namely, for $n\geqslant m_j-1$[1] and $j=1,2,...,k$,

$$Q_{n,\overrightarrow{m}}(z;F^\lambda_\gamma)=\frac{z^{n+|m|+\gamma}}{\Gamma(n+|m|+\gamma)}\int_0^{+\infty}T(x)e^{-zx}dx,$$

$$R^j_{n,\overrightarrow{m}}(z;F^\lambda_\gamma)=\frac{e^{\lambda_jz}z^{n+|m|+1}}{\lambda_j^{\gamma-1}(\gamma)_{n+|m|}}\int_0^{\lambda_j}T(x)e^{-zx}dx\,, \qquad (3)$$

where $T(x)=x^{n+\gamma-1}\prod_{\nu=1}^k(x-\lambda_\nu)^{m_\nu}$. In the integral, which defines the remainder function $R^j_{n,\overrightarrow{m}}$, we integrate along an arbitrary curve connecting the points 0 and λ_j. Henceforth, for complex numbers w and τ we assume that $w^\tau=e^{\tau\ln w}$, with a single-valued branch of the logarithm defined by the equality $\ln w=\ln|w|+i\arg_0 w$, $\arg_0 w\in(-\pi,\pi]$. In [7], the following asymptotic equality was proved: if $n+|m|\to+\infty$, then

$$Q_{n,\overrightarrow{m}}(z;F^\lambda_\gamma)=\exp\left\{-\frac{\sum_{i=1}^k\lambda_im_i}{n+|m|+\gamma-1}\,z\right\}(1+o(1))\,. \qquad (4)$$

In (4), as in other similar equalities, we assume that the estimate $o(1)$ is uniform with respect to z on compact sets in $\mathbb{C}$.

In cases when $k=1$ by De Bruin [9] and $k>1$ by A.I. Aptekarev [7] it was shown that the fractions $\pi^j_{n,\overrightarrow{m}}(z;F^\lambda_\gamma)$ converge to $F^j_\gamma(z)$ uniformly on compact sets in $\mathbb{C}$ as $n\geqslant m_j-1$

[1] For the necessary conditions of $n\geqslant m_j-1$ see [8]. Further, for $\gamma\neq1$, we assume their fulfillment.

and $j = 1, 2, \dots, k$, or as $n + |m| \to \infty$, respectively. The problem of describing the rate of this convergence is of current interest [8, 10–18].

In [11], the rate of convergence of Padé approximants $\pi_{n,m}(z; F^1_\gamma)$ was established: for $n \geqslant m - 1$ and $n + m \to \infty$,

$$F^1_\gamma(z) - \pi_{n,m}(z; F^1_\gamma) = (-1)^m \frac{m!\,(\gamma)_n\, e^{2mz/(n+m)}}{(\gamma)_{n+m}\,(\gamma)_{n+m+1}}\, z^{n+m+1}\,(1 + o(1)). \tag{5}$$

From the equalities (4), (5) and the identity (2) it follows that

$${}_1F_1(m+1, n+m+\gamma+1; z) = \exp\left\{\frac{mz}{n+m}\right\}(1 + o(1)) \tag{6}$$

as $n + m \to \infty$. In case $k > 1$, the available results on the rate of convergence of Hermite – Padé approximants pertain mainly to the diagonal case and are obtained under the condition that the numbers $\{\lambda_j\}_{j=1}^k$ are real, and $\gamma = 1$. Essentially, the only method in such studies is the saddle-point method. For complex numbers $\{\lambda_j\}_{j=1}^k$ and in the nondiagonal case, the use of the saddle-point method is extremely difficult. In such a situation, in [8], a new method that is based on the Taylor theorem and heuristic considerations underlying the Laplace and saddle-point methods was applied.

In this article, we prove a multidimensional analogue of Theorem 4 from [8], in which the case $k = 2$ was considered. When proving we use the methods of this paper and the analogue of van Rossum's identity established by us. Besides, under certain conditions on $\overrightarrow{m}$ and λ the main restriction $\lim_{n\to\infty} m(n)/\sqrt{n} = 0$ of Theorem 4 can be removed.

Also we note the paper by H. Stahl [3], in which for $k = 2$, $\lambda_1 = -1$, $\lambda_2 = 1$ the rate of convergence of "rescaled" diagonal Hermite – Padé approximants was established using the method of the matrix Riemann – Hilbert problem. With the rescaling of variable $z = n\zeta$, the zeros and poles of such rational approximants fill some curves in the complex plane $\mathbb{C}_\zeta$. Today, the questions related to the description of these curves and the asymptotics of the rescaled approximants attract considerable interest of specialists (see, for example, [12–14]).

2. MAIN RESULT: $|m| = o(\sqrt{n})$, $\lambda = \{\lambda_j\}_{j=1}^k \subset \mathbb{C}$

Theorem 1. *Let $n \in \mathbb{Z}^1_+$, $\overrightarrow{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k_+$, $\{\lambda_j\}_{j=1}^k$ be different nonzero complex numbers and $n \geqslant m_j - 1$, $j = 1, 2, \dots, k$. If $\lim_{n\to\infty} m(n)/\sqrt{n} = 0$, then uniformly with respect to all $\overrightarrow{m}$, for which $0 \leqslant |m| \leqslant m(n)$,*

$$F^j_\gamma(z) - \pi^j_{n,\overrightarrow{m}}(z; F^\lambda_\gamma) =$$
$$= (-1)^{|m|}\,\lambda_j^{n+m_j+1}\,\Omega_j(k)\,\frac{m_j!\,(\gamma)_n\, z^{n+|m|+1}}{(\gamma)_{n+|m|}\,(\gamma)_{n+m_j+1}}\,(1 + o(1)),$$

as $n \to +\infty$, where $\Omega_j(1) = 1$, $\Omega_j(k) = \prod_{\substack{\nu=1\\ \nu\neq j}}^k (\lambda_\nu - \lambda_j)^{m_\nu}$, if $k > 1$.

Before proceeding to the proof of Theorem 1, we note, that under the assumptions made in it, from (4) it follows that $Q_{n,\overrightarrow{m}}(z) = (1 + o(1))$ for $n + |m| \to \infty$. Therefore, it is sufficient to find the asymptotics of the functions $R^j_{n,\overrightarrow{m}}$. First, we prove an analogue of the van Rossum identity (2) for $k > 1$.

Theorem 2. *For any $k \geqslant 1$ and $j = 1, 2, \dots, k$*

$$R^j_{n,\overrightarrow{m}}(z; F^\lambda_\gamma) = (-1)^{|m|}\,\lambda_j^{n+m_j+1}\,\Omega_j(k)\,\frac{\Gamma(n+\gamma)\, z^{n+|m|+1}}{(\gamma)_{n+|m|}}\times$$
$$\times \sum_{l=0}^{|m|-m_j} a_l\,\frac{(m_j + l)!}{\Gamma(n + m_j + l + \gamma + 1)!}\,{}_1F_1(m_j + l + 1, n + m_j + l + \gamma + 1; \lambda_j z), \tag{7}$$

where $a_0 = 1$ *and for* $l \geqslant 1$

$$a_l = \sum_{\substack{t_1+\ldots+t_k-t_j=l \\ t_\nu \geqslant 0}} \left\{ \prod_{\substack{\nu=1 \\ \nu\neq j}}^{k} C_{m_\nu}^{t_\nu} \left(\frac{\lambda_j}{\lambda_\nu - \lambda_j} \right)^{t_\nu} \right\}. \tag{8}$$

Proof. For $k=1$ equalities (2) and (7) coincide. Therefore, further we assume that $k>1$. In the integral (3), which defines the remainder function, we change the variable $x = \lambda_j t$ and obtain

$$R^j_{n,\overrightarrow{m}}(z) = \lambda_j^{n+m_j+1} \frac{z^{n+|m|+1}}{(\gamma)_{n+|m|}} \int_0^1 t^{n+\gamma-1}(t-1)^{m_j} \prod_{\substack{\nu=1 \\ \nu\neq j}}^{k} (\lambda_j t - \lambda_\nu)^{m_\nu}\, e^{\lambda_j(1-t)z} dt. \tag{9}$$

The integral in (9) we denote by $I_j(z)$.

In this integral we substitute $u = 1-t$ and then factor out $\Omega_j(k)$. Then

$$I_j(z) = (-1)^{|m|}\, \Omega_j(k) \int_0^1 (1-u)^{n+\gamma-1}\, u^{m_j} \prod_{\substack{\nu=1 \\ \nu\neq j}}^{k} \Big(1 + \frac{\lambda_j\, u}{\lambda_\nu - \lambda_j}\Big)^{m_\nu} e^{\lambda_j u z} du. \tag{10}$$

Denote the integral in (10) by $J_j(z)$. Applying the binomial theorem and using a well-known identity (see, for example, [7])

$$\prod_{\substack{\nu=1 \\ \nu\neq j}}^{k} \left\{ \sum_{t_\nu=0}^{m_\nu} C_{m_\nu}^{t_\nu} \left(\frac{\lambda_j u}{\lambda_\nu - \lambda_j} \right)^{t_\nu} \right\} = \sum_{l=0}^{|m|-m_j} a_l\, u^l, \tag{11}$$

the integral $J_j(z)$ can be represented as

$$J_j(z) = \int_0^1 (1-u)^{n+\gamma-1}\, u^{m_j} \left\{ \sum_{l=0}^{|m|-m_j} a_l\, u^l \right\} \sum_{p=0}^{\infty} \frac{(\lambda_j\, z)^p}{p!}\, u^p\, du =$$

$$= \sum_{l=0}^{|m|-m_j} a_l \left\{ \sum_{p=0}^{\infty} B(m_j+p+l+1; n+\gamma) \frac{(\lambda_j\, z)^p}{p!} \right\} =$$

$$= \Gamma(n+\gamma) \sum_{l=0}^{|m|-m_j} a_l \frac{(m_j+l)!}{\Gamma(n+m_j+l+\gamma+1)}\, {}_1F_1(m_j+l+1, n+m_j+l+\gamma+1; \lambda_j z)\,.$$

Here and further, $B(u;v)$ is the Euler beta function. The last equality, together with (9) and (10), implies (7). Theorem 2 is proved. □

Now we proceed to the proof of Theorem 1. Denote the sum in (7) by $H_j(z)$. We factor out the first term of this sum and obtain:

$$H_j(z) = \frac{m_j!}{\Gamma(n+m_j+\gamma+1)}\, {}_1F_1(m_j+1, n+m_j+\gamma+1; \lambda_j z) \Bigg\{ 1 +$$

$$+ \sum_{l=1}^{|m|-m_j} a_l \frac{(m_j+l)!}{\Gamma(n+m_j+l+\gamma+1)} \frac{\Gamma(n+m_j+\gamma+1)}{m_j!} \frac{{}_1F_1(m_j+l+1, n+m_j+l+\gamma+1; \lambda_j z)}{{}_1F_1(m_j+1, n+m_j+\gamma+1; \lambda_j z)} \Bigg\}.$$

From (6) it follows that the ratio of two hypergeometric functions on the right-hand side of the last equality converges to 1 uniformly on compact sets in $\mathbb{C}$ as $n \to \infty$. Therefore, for sufficiently large

n, the absolute value of the second term of sum in the braces in the previous equality does not exceed

$$2\sum_{l=1}^{|m|-m_j} a_l^*\,\frac{m_j+1}{n+m_j+\gamma_1+1}\,\frac{m_j+2}{n+m_j+\gamma_1+2}\cdots\frac{m_j+l}{n+m_j+\gamma_1+l}\leqslant$$
$$\leqslant 2\left\{\sum_{l=0}^{|m|-m_j} a_l^*\left(\frac{|m|}{n+|m|+\gamma_1}\right)^l-1\right\},$$

where γ_1 is the real part of γ, a_l^* is defined in the same way as a_l, with the only difference being that in (8) instead of $\lambda_j/(\lambda_\nu-\lambda_j)$ should take $|\lambda_j|/|\lambda_\nu-\lambda_j|$. When proving the last inequality, we used the fact that function $\varphi(t)=(m_j+t)/(n+m_j+1+t)$ is monotonically increasing for $t\geqslant 1$, and the well-known equality $\Gamma(z+1)=z\Gamma(z)$. Now, applying the identity (11) one more time with $\lambda_j/(\lambda_\nu-\lambda_j)$ replaced by $|\lambda_j|/|\lambda_\nu-\lambda_j|$, we obtain

$$\sum_{l=0}^{|m|-m_j} a_l^*\left(\frac{|m|}{n+|m|+\gamma_1}\right)^l=\prod_{\substack{\nu=1\\ \nu\neq j}}^{k}\left(1+\frac{|\lambda_j|}{|\lambda_\nu-\lambda_j|}\,\frac{|m|}{n+|m|+\gamma_1}\right)^{m_\nu}.$$

It remains to note that, since $\lim_{n\to\infty}|m|/\sqrt{n}=0$, the right-hand side of the last equality tends to 1 as $n\to\infty$. Theorem 1 is proved.

3. MAIN RESULT: $\lambda=\{\lambda_j\}_{j=1}^k$ ARE THE ROOTS OF THE EQUATION $z^k=1$

In the statement of Theorem 1 we have significant constraints on the growth of the multi-index order: $|m|=o(\sqrt{n})$ as $n\to\infty$. Consider one particular case when these restrictions can be removed.

Let $\{\lambda_j\}_{j=1}^k$ be the roots of the equation $z^k=1$, i. e.

$$\lambda_j=e^{i\frac{2\pi(j-1)}{k}},\quad j=1,2,\ldots,k, \tag{12}$$

where i is the imaginary unit. Note, that for every $j=1,2,\ldots,k$

$$\lambda_j\prod_{\substack{\nu=1\\ \nu\neq j}}^{k}(\lambda_\nu-\lambda_j)=\prod_{\nu=2}^{k}(\lambda_\nu-1)=(-1)^{k-1}\,k. \tag{13}$$

Equalities (13) can be easily proved if in the both sides of identity

$$\frac{z^k-\lambda_j^k}{z-\lambda_j}=\prod_{\substack{\nu=1\\ \nu\neq j}}^{k}(z-\lambda_\nu)$$

we pass to the limit as $z\to\lambda_j$.

Consider the system of functions F_γ^λ, where $\lambda=\{\lambda_j\}_{j=1}^k$ and λ_j are defined by the equalities (12). In [8], in the diagonal case, when $n=m_1=\ldots=m_k$, the following asymptotic equalities were obtained using the saddle-point method: for $k>1$ and $j=1,2,...,k$

$$F_\gamma^j(z)-\pi^j_{n,\overrightarrow{m}}(z;F_\gamma^\lambda)=$$
$$=(-1)^n\,\lambda_j^{n+1}\left(\frac{1}{\sqrt[k]{k+1}}\right)^{\gamma-1}G_k(n)\,\frac{z^{n+kn+1}}{(\gamma)_{n+kn}}\,e^{\lambda_j\left(1-\sqrt[k]{1/(k+1)}\right)z}\,(1+o(1)), \tag{14}$$

where

$$G_k(n):=\sqrt{\frac{2\pi}{n\sqrt[k]{(k+1)^{k+2}}}}\left(\frac{k}{\sqrt[k]{(k+1)^{k+1}}}\right)^n.$$

Theorem 3. *Let $\gamma \in \mathbb{R} \setminus \mathbb{Z}_-$, $m_1 = \ldots = m_k = m$, and $n \in \mathbb{Z}^1_+$. Then for any $k \geqslant 1$ and $j = 1, 2, \ldots, k$*

$$F^j_\gamma(z) - \pi^j_{n,\overrightarrow{m}}(z; F^\lambda_\gamma) = (-1)^m \, \lambda_j^{n+1} \times$$
$$\times \frac{1}{k} \, B\Big(m+1; \frac{n+\gamma}{k}\Big) \, \frac{z^{n+km+1}}{(\gamma)_{n+km}} \, e^{\lambda_j \left(1 - \sqrt[k]{n/(n+km)}\ \right) z} \, e^{(m \sum_{\nu=1}^k \lambda_\nu) z/(n+km)} (1 + o(1)) \qquad (15)$$

as $n + m \to \infty$.

Proof. For $k = 1$ the asymptotic equality (15) coincides with (5). Therefore, further we assume that $k > 1$. In this case $\sum_{j=1}^k \lambda_j = 0$ and from (4) it follows that $Q_{n,\overrightarrow{m}}(z) = 1 + o(1)$ as $n + m \to \infty$. It is necessary to find the asymptotic of the remainder function

$$R^j_{n,\overrightarrow{m}}(z) = (-1)^m \, \frac{e^{\lambda_j z} \, z^{n+km+1}}{\lambda_j^{\gamma-1} \, (\gamma)_{n+km}} \int_0^{\lambda_j} x^{n+\gamma-1} \, (1 - x^k)^m \, e^{-zx} dx \, . \qquad (16)$$

Denote the integral in (16) by $I_j(z)$. Using substitution $x = \lambda_j u$ in this integral, we obtain

$$I_j(z) = \lambda_j^{n+\gamma} \int_0^1 u^{n+\gamma-1} \, (1 - u^k)^m \, e^{-\lambda_j u z} du \, . \qquad (17)$$

Consider the integrals

$$J_p = \int_0^1 (1 - u^k)^m \, u^{n+p+\gamma-1} \, du \, , \quad p = 0, 1, 2, \ldots \ .$$

It is easy to notice that

$$J_p = \frac{1}{k} \int_0^1 (1 - u^k)^m \, (u^k)^{\frac{n-k+p+\gamma}{k}} \, du^k = \frac{1}{k} \, B\Big(m+1; \frac{n+p+\gamma}{k}\Big). \qquad (18)$$

Now, we find u_0 from the equality $J_1 - u_0 J_0 = 0$. Expressing the Euler beta function in terms of the gamma function and using the Stirling formula, we obtain that

$$u_0 = \frac{J_1}{J_0} = \sqrt[k]{\frac{n}{n+km}} \, (1 + o(1))$$

as $n + m \to \infty$. In particular, from this it follows, that for sufficiently large $n + m$ we have $u_0 \in (0, 1)$.

To determine the asymptotic behaviour of the integral $I_j(z)$, we expand the function $\exp\{-\lambda_j u z\}$ in the Taylor series in a neighborhood of u_0. Then

$$e^{-\lambda_j u z} = e^{-\lambda_j u_0 z} e^{-\lambda_j z (u - u_0)} = e^{-\lambda_j u_0 z} \{1 - \lambda_j z (u - u_0) + \rho_u(z)\},$$

where for $|z| < L$ and $u \in [0, 1]$

$$| \, \rho_u(z) \, | \leqslant |\lambda_j|^2 \, | \, u - u_0 \, |^2 \left\{ \frac{L^2}{2!} + \ldots + \frac{L^n}{n!} + \ldots \right\} \leqslant L_1 \, | \, u - u_0 \, |^2 \, .$$

Here and further, L, L_1 are absolute constants. Taking into account the choice of u_0, (17) and (18), we get

$$I_j(z) = \lambda_j^{n+\gamma} \, e^{-\lambda_j u_0 z} \left\{ \int_0^1 (1 - u^k)^m \, u^{n+\gamma-1} \, du + \int_0^1 (1 - u^k)^m \, u^{n+\gamma-1} \, \rho_u(z) \, du \right\} =$$
$$= \lambda_j^{n+\gamma} \, e^{-\lambda_j u_0 z} \left\{ \frac{1}{k} \, B\Big(m+1; \frac{n+\gamma}{k}\Big) \, + \, A_\rho(z) \right\},$$

where

$$|A_\rho(z)| \leqslant L_1 \int_0^1 (1-u^k)^m\, u^{n+\gamma-1}\, (u-u_0)^2\, du = L_1 \int_0^1 (1-u^k)^m\, u^{n+\gamma-1}\, (u^2-uu_0)\, du =$$
$$= L_1 \left(\frac{J_2}{J_0} - \left(\frac{J_1}{J_0}\right)^2\right) J_0.$$

When proving we used the representation $(u-u_0)^2 = (u^2-uu_0) - u_0(u-u_0)$ and the equality $J_1 - u_0 J_0 = 0$. Applying the equality (18), and then expressing the Euler beta functions in terms of the gamma functions and using the Stirling formula, we obtain:

$$\frac{J_2}{J_0} \sim \left(\frac{n-k+\gamma+2}{n+km+\gamma+2}\right)^{2/k}, \quad \left(\frac{J_1}{J_0}\right)^2 \sim \left(\frac{n-k+\gamma+1}{n+km+\gamma+1}\right)^{2/k}$$

as $n+m\to\infty$. From these asymptotic equalities and the previous inequality for $n+m\to\infty$, we have

$$I_j(z) = \lambda_j^{n+\gamma}\, e^{-\lambda_j u_0 z}\, \frac{1}{k}\, B\Big(m+1; \frac{n+\gamma}{k}\Big)\, (1+o(1)).$$

Therefore, the asymptotic equality (15) follows from (16). Theorem 3 is proved. □

In conclusion, we make two remarks.

For $n\to\infty$

$$\frac{1}{k}\, B\Big(n+1; \frac{n+\gamma}{k}\Big) \sim \left(\frac{1}{\sqrt[k]{k+1}}\right)^{\gamma-1} G_k(n).$$

Therefore, if $m=n$, then the asymptotic equalities (14) and (15) coincide. Thus, Theorem 1 of [8] is a corollary of Theorem 3. Note that these theorems are proved using completely different methods.

Moreover, we can easily show, that if $m = o(\sqrt{n})$, then for $n\to\infty$

$$\frac{1}{k}\, B\Big(m+1; \frac{n+\gamma}{k}\Big) \sim k^m \frac{m!\,(\gamma)_n}{(\gamma)_{n+m+1}}. \tag{19}$$

Taking into account the equalities (13), for $m_1 = \ldots = m_k = m$ we obtain, that

$$(-1)^{|m|}\lambda_j^{n+m_j+1} \prod_{\substack{\nu=1\\ \nu\neq j}}^{k} (\lambda_\nu - \lambda_j)^{m_\nu} = (-1)^m \lambda_j^{n+1} k^m\,.$$

Therefore, with the corresponding parameters m_j and $\gamma \in \mathbb{R}\setminus\mathbb{Z}_-$, the Theorems 1 and 3 are consistent. Also we note, that if the condition $m = o(\sqrt{n})$ as $n\to\infty$ is not satisfied, then the equivalence in (19) is broken. It means, that the condition $m = o(\sqrt{n})$ in the Theorem 1 is necessary.

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