An Inequality for Projections and Convex Functions

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Abstract—We propose the conditions for a continuous function to be projection-convex, i.e. $f(\lambda p + (1-\lambda)q) \le \lambda f(p) + (1-\lambda)f(q)$ for any projections p and q and any real $\lambda \in (0,1)$. Also we obtain the characterization of projections commutativity and the characterization of trace in terms of equalities for non-flat functions.

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1. INTRODUCTION

The present paper is inspired by [1] and [2]. We establish new criteria for the commutation of projections in terms of operator equalities involving functional calculus. We obtain a trace characterization for the class of all positive normal functionals on a von Neumann algebra. Other trace characterizations may be found in [3]–[8] and [11].

2. PRELIMINARIES

Let H be a Hilbert space over the field $\mathbb{C}$ and I be the identity operator on H, let $B(H)$ be the $*$-algebra of all linear bounded operators on H. The *commutant* of a set $X \subset B(H)$ is the set

$$X' = \{y \in B(H) :\ xy = yx \ \text{ for all } \ x \in X\}.$$

A $*$-subalgebra $\mathcal{M}$ of the algebra $B(H)$ is called a *von Neumann algebra* acting in the Hilbert space H if $\mathcal{M} = \mathcal{M}''$. If $X \subset B(H)$, then X' is a von Neumann algebra and X'' is the least von Neumann algebra containing X. For a von Neumann algebra $\mathcal{M}$ of operators on H, let $\mathcal{M}^{\rm pr}$, $\mathcal{M}^{+}$, $\mathcal{Z}(\mathcal{M})$ be the lattice of projections, the cone of positive operators and the center of the algebra $\mathcal{M}$, respectively. For $p \in \mathcal{M}^{\rm pr}$ put $p^{\perp} = I - p$ and let $\mathcal{M}_p = \{px|_{pH} :\ x \in \mathcal{M}\}$ be a reduced von Neumann algebra. Let $\mathcal{M}^*$ denote the set of all $\|\cdot\|$-continuous linear functionals on $\mathcal{M}$. A linear functional φ on $\mathcal{M}$ is *positive*, if $\varphi(x) \ge 0$ for any $x \in \mathcal{M}^+$. A positive functional φ on $\mathcal{M}$ is *tracial* if $\varphi(xx^*) = \varphi(x^*x)$ for any $x \in \mathcal{M}$.

The set $K = \{f \in C[0,1] :\ f(x) \le f(1)x + f(0)(1-x) \text{ for all } x \in [0,1]\}$ is a subcone in $C[0,1]$. The set of all convex functions $f \in C[0,1]$ and $K_1 = \{f \in C[0,1] :\ f(x) < f(1)x + f(0)(1-x) \text{ for all } x \in (0,1)\}$ are subcones in K. The non-convex function

$$f(x) = \begin{cases} 1/6 + x/3, & 0 \le x \le 1/3, \\ 5/18, & 1/3 < x < 2/3, \\ 31x/18 - 19/18, & 2/3 \le x \le 1 \end{cases}$$

lies in K_1.

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Lemma 1 ([10], Chap. 5, Theorem 1.41, item (ii)). *If the von Neumann algebra $\mathcal{N}$ is generated by two projections $p, q \in B(H)^{\mathrm{pr}}$ then there exists a unique projection $z \in \mathcal{Z}(\mathcal{N})$ such that the algebra $\mathcal{N}_z$ is of type I_2 and $\mathcal{N}_{z^\perp}$ is Abelian with $\dim_{\mathbb{C}} \mathcal{N}_{z^\perp} \leq 4$.*

Lemma 2 ([9], Theorem 2.3.3). *Let a von Neumann algebra $\mathcal{N}$ be of type I_n (n is a cardinal). Then the algebra $\mathcal{N}$ is $*$-isomorphic to the tensor product $\mathcal{Z}(\mathcal{N})\overline{\otimes}B(K)$, where K is a Hilbert space with $\dim K = n$.*

3. A CONVEXITY INEQUALITY FOR PROJECTIONS

Lemma 3. *For each pair $p, q \in B(H)^{\mathrm{pr}}$ such that $pq = qp$ and any function $f \in K$ the inequality $f(\lambda p + (1-\lambda)q) \leq \lambda f(p) + (1-\lambda)f(q)$ holds for any $\lambda \in [0, 1]$.*

Proof. For projections p, q we consider the von Neumann algebra $\{p,\ q\}''$. It is Abelian, therefore $\{p, q\}'' \cong L_\infty(\Omega, \Sigma, \mu)$ for some measure space (Ω, Σ, μ) and there exist $A, B \in \Sigma$ such that $p \sim I_A$, $q \sim I_B$. Clearly,

$$f(\lambda I_A + (1-\lambda)I_B) = f(1)I_{A\cap B} + f(0)I_{(A\cup B)^c} + f(\lambda)I_{A\setminus B} + f(1-\lambda)I_{B\setminus A},$$

also

$$\lambda f(I_A) + (1-\lambda)f(I_B) = f(1)I_{A\cap B} + f(0)I_{(A\cup B)^c} + (\lambda f(1) + (1-\lambda)f(0))I_{A\setminus B} + ((1-\lambda)f(1) + \lambda f(0))I_{B\setminus A}.$$

$\square$

Remark 1. For each commutative pair $p, q \in B(H)^{\mathrm{pr}}$ and any $f \in K_1$ the equality $f(\lambda p + (1-\lambda)q) = \lambda f(p) + (1-\lambda)f(q)$ holds if and only if either $\lambda = 1$ or $\lambda = 0$, or $I_{A\setminus B} = I_{B\setminus A} = 0$ (the latter means that $p = q$).

Lemma 4. *For $\lambda \in [0, 1]$ and each pair $p, q \in \mathbb{M}_2(C(\Omega))^{\mathrm{pr}}$ and any function $f \in K$ the inequality $f(\lambda p + (1-\lambda)q) \leq \lambda f(p) + (1-\lambda)f(q)$ holds for all $\lambda \in [0, 1]$.*

Proof. It suffices to consider $p = \mathrm{diag}(1, 0)$ and

$$q = \begin{pmatrix} t & \delta\sqrt{t(1-t)} \\ \overline{\delta}\sqrt{t(1-t)} & 1-t \end{pmatrix}, \tag{1}$$

where $t \in [0, 1]$ and $\delta \in \mathbb{C}$ with $|\delta| = 1$ (see [1]). There exists $r \in \mathbb{M}_2(\mathbb{C})^{\mathrm{pr}}$ such that $\lambda p + (1-\lambda)q = \mu_1 r + \mu_2 r^\perp$, $\mu_1,\ \mu_2 \in [0, 1]$, $\mu_1 + \mu_2 = 1$. Therefore, $f(\lambda p + (1-\lambda)q) = f(\mu_1)r + f(\mu_2)r^\perp$. On the other hand,

$$\lambda f(p) + (1-\lambda)f(q) = \lambda f(1)p + \lambda f(0)(I-p) + (1-\lambda)f(1)q + (1-\lambda)f(0)(I-q)$$

$$= f(0)I + (f(1) - f(0))(\lambda p + (I-\lambda)q) = (f(1)\mu_1 + f(0)(1-\mu_1))r + (f(1)\mu_2 + f(0)(1-\mu_2))r^\perp.$$

$\square$

Remark 2. Note that $\mu_{1,2} = \frac{1}{2}(1 \mp \sqrt{1 - 4\lambda(1-\lambda)(1-t)})$. For each pair $p, q \in \mathbb{M}_2(C(\Omega))^{\mathrm{pr}}$ and any $f \in K_1$ the equality $f(\lambda p + (1-\lambda)q) = \lambda f(p) + (1-\lambda)f(q)$ holds if and only if either $\lambda = 1$ or $\lambda = 0$ or $p = q$.

Theorem 1. *For each pair $p, q \in B(H)^{\mathrm{pr}}$ and any function $f \in K$ the inequality $f(\lambda p + (1-\lambda)q) \leq \lambda f(p) + (1-\lambda)f(q)$ holds for any $\lambda \in [0, 1]$.*

Proof. Consider the von Neumann algebra $\mathcal{N} = \{p, q\}''$. By Lemma 1 there exists a projection $z \in \mathcal{Z}(\mathcal{N})$ such that an algebra $\mathcal{N}_{z^\perp}$ is Abelian and $\mathcal{N} = \mathcal{N}_z \oplus \mathcal{N}_{z^\perp}$. Now by Lemma 3 we have $f(\lambda p z^\perp + (1-\lambda)q z^\perp) \leq \lambda f(p z^\perp) + (1-\lambda)f(q z^\perp)$. Since $\mathcal{N}_z \cong \mathbb{M}_2(C(\Omega))$ (see Lemmas 1, 2) by Lemma 4 we have $f(\lambda pz + (1-\lambda)qz) \leq \lambda f(pz) + (1-\lambda)f(qz)$. To finish the proof it suffices to note that $f(p) = f(pz \oplus pz^\perp) = f(pz) \oplus f(pz^\perp)$. $\square$

Remark 3. For each pair $p, q \in \mathbb{M}_2(C(\Omega))^{\mathrm{pr}}$ and any function $f \in K_1$ the equality $f(\lambda p + (1-\lambda)q) = \lambda f(p) + (1-\lambda)f(q)$ holds if and only if either $\lambda = 1$ or $\lambda = 0$ or $p = q$.

Corollary 1. *For each pair $p, q \in B(H)^{\mathrm{pr}}$ and a convex function $f \in C[0,1]$ the inequality $f(\lambda p + (1-\lambda)q) \le \lambda f(p) + (1-\lambda)f(q)$ holds for any $\lambda \in [0,1]$.*

Corollary 2. *For each pair $p, q \in B(H)^{\mathrm{pr}}$, any strictly convex function $f \in C[0,1]$ and all $\lambda \in (0,1)$ the equality $f(\lambda p + (1-\lambda)q) = \lambda f(p) + (1-\lambda)f(q)$ holds if and only if $p = q$.*

Proof. Consider the von Neumann algebra $\mathcal{N} = \{p, q\}''$. By Lemma 1 there exists a projection $z \in \mathcal{Z}(\mathcal{N})$ such that the algebra $\mathcal{N}_{z^\perp}$ is Abelian, $\mathcal{N} = \mathcal{N}_z \oplus \mathcal{N}_{z^\perp}$. Note that $f(p) = f(pz \oplus pz^\perp) = f(pz) \oplus f(pz^\perp)$, hence

$$f((\lambda p + (1-\lambda)q)z) \oplus f((\lambda p + (1-\lambda)q)z^\perp) = f((\lambda p + (1-\lambda)q)z \oplus (\lambda p + (1-\lambda)q)z^\perp)$$

$$= f(\lambda p + (1-\lambda)q) = \lambda f(p) + (1-\lambda)f(q) = \lambda f(pz \oplus pz^\perp) + (1-\lambda)f(qz \oplus qz^\perp)$$

$$= \lambda(f(pz) \oplus f(pz^\perp)) + (1-\lambda)(f(qz) \oplus f(qz^\perp))$$

$$= (\lambda f(pz)) \oplus (\lambda f(pz^\perp)) + ((1-\lambda)f(qz)) \oplus ((1-\lambda)f(qz^\perp))$$

$$= (\lambda f(pz) + (1-\lambda)f(qz)) \oplus (\lambda f(pz^\perp) + (1-\lambda)f(qz^\perp)).$$

Thus $f((\lambda p + (1-\lambda)q)z) = (\lambda f(pz) + (1-\lambda)f(qz))$ and $f((\lambda p + (1-\lambda)q)z^\perp) = (\lambda f(pz^\perp) + (1-\lambda)f(qz^\perp))$. The algebra $\mathcal{N}_{z^\perp}$ is Abelian, hence $pz^\perp = qz^\perp$ by Remark 1. Since $\mathcal{N}_z \cong \mathbb{M}_2(C(\Omega))$ we have $pz = qz$ by Remark 2. Finally, we note that $p = pz + pz^\perp = qz + qz^\perp = q$. $\square$

We may directly prove the inequality for some functions.

Example 1. Let $f(x) = x^3$ for $x \in \mathbb{R}$. Then

$$f(\lambda p + (1-\lambda)q) = \lambda^3 p + \lambda^2(1-\lambda)pqp + \lambda(1-\lambda)^2 qpq + \lambda(1-\lambda)(pq + qp) + (1-\lambda)^3 q.$$

Since $(p-q)^2 \ge 0$, we have $pq + qp \le p + q$. Also $pqp \le p$, $qpq \le q$, hence

$$f(\lambda p + (1-\lambda)q) \le \lambda^3 p + \lambda^2(1-\lambda)p + \lambda(1-\lambda)^2 q + \lambda(1-\lambda)(p+q) + (1-\lambda)^3 q$$

$$= \lambda p + (1-\lambda)q = \lambda f(p) + (1-\lambda)f(q).$$

The equality $f(\lambda p + (1-\lambda)q) = \lambda f(p) + (1-\lambda)f(q)$ holds only if $pq + qp = p + q$, $pqp = p$ and $qpq = q$. Thus $p = q$.

Theorem 1 leads us to another class of functions.

Example 2. The function $f(x) = e^x$ lies in K_1. Therefore, $e^{\lambda p + (1-\lambda)q} \le \lambda e^p + (1-\lambda)e^q$ for any $\lambda \in [0,1]$ and the equality holds only if $p = q$.

4. THE COMMUTATIVITY OF PROJECTIONS

Lemma 5. *Let $f \in C[0,1]$, $\lambda \in (0,1)$ and $p, q \in \mathbb{M}_2(\mathbb{C})^{\mathrm{pr}}$, then the equality $f(\lambda p + (1-\lambda)q) = \lambda f(p) + (1-\lambda)f(q)$ yields $p + q = I$ if and only if $\frac{f(\lambda)-f(0)}{\lambda} = \frac{f(1-\lambda)-f(1)}{1-\lambda} = f(1) - f(0)$ and either $f(\mu) \ne \mu f(1) + (1-\mu)f(0)$ or $f(1-\mu) \ne (1-\mu)f(1) + \mu f(0)$ for any $\mu \in (0,1) \setminus \{\lambda,\ 1-\lambda\}$.*

Proof. Consider $p = \mathrm{diag}(1,0)$, $t \in [0,1]$ and $\delta \in \mathbb{C}$ with $|\delta| = 1$, and let q be as in (1). Let $\lambda p + (1-\lambda)q = \mu_1 r + \mu_2 r^\perp$, then $f(\lambda p + (1-\lambda)q) = \lambda f(p) + (1-\lambda)f(q)$ implies that $f(\mu_1)r + f(\mu_2)r^\perp = \lambda(f(1)p + f(0)p^\perp) + (1-\lambda)(f(1)q + f(0)q^\perp)$. We have $f(\mu_k) = f(0) + (f(1) - f(0))\mu_k$ for $k = 1, 2$. Therefore,

$$\frac{f(\mu_1) - f(0)}{\mu_1} = \frac{f(\mu_2) - f(0)}{\mu_2} = f(1) - f(0).$$

If these equalities hold and for any $\mu \in (0,1) \setminus \{\lambda,\ 1-\lambda\}$ either $f(\mu) \neq \mu f(1) + (1-\mu)f(0)$ or $f(1-\mu) \neq (1-\mu)f(1) + \mu f(0)$ then $\{\mu_1,\ \mu_2\} = \{\lambda, 1-\lambda\}$ since $\mu_1 + \mu_2 = 1$, see Remark 2.

On the other hand, if $\mu \in (0,1) \setminus \{\lambda,\ 1-\lambda\}$ is such that $f(\mu) = \mu f(1) + (1-\mu)f(0)$ and $f(1-\mu) = (1-\mu)f(1) + \mu f(0)$ then the equality $f(\lambda p + (1-\lambda)q) = \lambda f(p) + (1-\lambda)f(q)$ also holds for p and q such that $\lambda p + (1-\lambda)q = \mu r + (1-\mu)r^\perp$. The equality

$$\frac{1}{2}(1 - \sqrt{1 - 4\lambda(1-\lambda)(1-t)}) = \mu$$

holds for $t = 1 - \dfrac{\mu(1-\mu)}{\lambda(1-\lambda)}$, hence the equality $f(\lambda p + (1-\lambda)q) = \lambda f(p) + (1-\lambda)f(q)$ does not imply that $q = I - p$. □

Lemma 6. *For each pair $p, q \in B(H)^{\mathrm{pr}}$ such that $pq = qp$, $\lambda \in (0,1)$ and any function $f \in C[0,1]$ such that*

$$\frac{f(\lambda) - f(0)}{\lambda} = \frac{f(1-\lambda) - f(0)}{1-\lambda} = f(1) - f(0)$$

the equality $f(\lambda p + (1-\lambda)q) = \lambda f(p) + (1-\lambda)f(q)$ holds.

Proof. For p, q we consider the von Neumann algebra $\{p,q\}''$. It is Abelian, so $\{p,q\}'' \cong L_\infty(\Omega, \Sigma, \mu)$ for some measure space (Ω, Σ, μ) and there exist $A, B \in \Sigma$ such that $p \sim I_A$, $q \sim I_B$. We have

$$f(\lambda p + (1-\lambda)q) = f(\lambda I_A + (1-\lambda)I_B) = f(1)I_{A\cap B} + f(0)I_{A^c\cap B^c} + f(\lambda)I_{A\setminus B} + f(1-\lambda)I_{B\setminus A} =$$

$$= f(1)I_{A\cap B} + f(0)I_{A^c\cap B^c} + (\lambda f(1) + (1-\lambda)f(0))I_{A\setminus B} + ((1-\lambda)f(1) + \lambda f(0))I_{B\setminus A}$$

$$= f(0) + (f(1) - f(0))(\lambda I_A + (1-\lambda)I_B) = (\lambda + 1 - \lambda)f(0) + (f(1) - f(0)(\lambda I_A + (1-\lambda)I_B)$$

$$= \lambda(f(1)I_A + f(0)I_{A^c}) + (1-\lambda)(f(1)I_B + f(0)I_{B^c}) = \lambda f(I_A) + (1-\lambda)f(I_B) = \lambda f(p) + (1-\lambda)f(q).$$

□

Theorem 2. *Let $p, q \in B(H)^{\mathrm{pr}}$, $\lambda \in [0,1]$ and $f \in C[0,1]$ be such that*

$$\frac{f(\lambda) - f(0)}{\lambda} = \frac{f(1-\lambda) - f(0)}{1-\lambda} = f(1) - f(0)$$

and either $f(\mu) \neq \mu f(1) + (1-\mu)f(0)$ or $f(1-\mu) \neq \mu f(1) + (1-\mu)f(0)$ for $\mu \in (0,1) \setminus \{\lambda, 1-\lambda\}$. Then $f(\lambda p + (1-\lambda)q) = \lambda f(p) + (1-\lambda)f(q)$ if and only if $pq = qp$.

Proof. Consider the von Neumann algebra $\mathcal{N} = \{p,q\}''$. By Lemma 1 there exists a projection $z \in \mathcal{Z}(\mathcal{N})$ such that $\mathcal{N}_{z^\perp}$ is Abelian. By Lemma 6 we have

$$f(\lambda pz^\perp + (1-\lambda)qz^\perp) = \lambda f(pz^\perp) + (1-\lambda)f(qz^\perp).$$

Since $\mathcal{N}_z \cong \mathbb{M}_2(C(\Omega))$ by Lemma 5 the equality $f(\lambda pz + (1-\lambda)qz) = \lambda f(pz) + (1-\lambda)f(qz)$ yields $pzqz = qzpz$. Finally, note that $\mathcal{N} = \mathcal{N}_z \oplus \mathcal{N}_{z^\perp}$ and $f(p) = f(pz \oplus pz^\perp) = f(pz) \oplus f(pz^\perp)$. □

For some functions we may prove the implication $f(\lambda p + (1-\lambda)q) = \lambda f(p) + (1-\lambda)f(q) \implies pq = qp$ without Theorem 2.

Example 3. Let $f(x) = x(x - \frac{1}{3})(x - \frac{2}{3})(x - 1)$. The equality $f(\frac{1}{3}p + \frac{2}{3}q) = \frac{1}{3}f(p) + \frac{2}{3}f(q)$ implies that $pq = qp$. Note that $f(p) = f(q) = 0$. Then the equality $f(\frac{1}{3}p + \frac{2}{3}q) = 0$ holds, hence

$$(p + 2q)(p + 2q - I)(p + 2q - 2I)(p + 2q - 3I) = 0.$$

Thus $(pq - qp)^2 = 0$ and $pq = qp$.

Example 4. Let $f(x) = x(x - \frac{1}{2})(x - 1)$. The equality $f(\frac{1}{2}p + \frac{1}{2}q) = \frac{1}{2}f(p) + \frac{1}{2}f(q)$ implies that $pq = qp$. Since $f(p) = f(q) = 0$ we have $f(\frac{1}{2}p + \frac{1}{2}q) = 0$. But by straightforward calculations

$$f\left(\frac{1}{2}p + \frac{1}{2}q\right) = \frac{1}{8}(pqp - qp - pq + qpq),$$

then $pq = pqpq$ and $qp = qpqp$. Hence $(pq - qp)^2 = 0$ and $qp = pq$.

Example 5. Let $f(x) = x(x - \frac{1}{3})(x - \frac{1}{2})(x - 1)$. The equality $f(\frac{1}{2}p + \frac{1}{2}q) = \frac{1}{2}f(p) + \frac{1}{2}f(q)$ yields $pq = qp$. Since $f(p) = f(q) = 0$ we have $f(\frac{1}{2}p + \frac{1}{2}q) = 0$. By straightforward calculations

$$f\left(\frac{1}{2}p + \frac{1}{2}q\right) = \frac{1}{16}\left((qp - pq)^2 + \frac{1}{3}(pqp + qpq - pq - qp)\right),$$

therefore, $3q(qp - pq)^2q + (qpqpq - qpq) = 0$. Note that $(pq - qp)^2 \leq 0$ and $q(pq - qp)^2q \leq 0$. Also, $qpqpq - qpq \leq 0$, thus $qpqpq = qpq$. Analogously, $pqpqp = pqp$. Since $(ipq - iqp)^3 = i(qpqp - qpqpqp + pqpqpq - pqpq) = 0$ we have $pq = qp$.

Theorem 2 leads us to more complicated examples.

Example 6. The function $f(x) = \sin(3\pi x)$ meets the conditions of Theorem 2. The equality $f(\frac{1}{3}p + \frac{2}{3}q) = \frac{1}{3}f(p) + \frac{2}{3}f(q)$ implies that $pq = qp$.

Example 7. The function

$$f(x) = \begin{cases} x\sin(\frac{\pi}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(or $f(x) = \sin(2\pi x)$) meets the conditions of Theorem 2. The equality $f(\frac{1}{2}p + \frac{1}{2}q) = \frac{1}{2}f(p) + \frac{1}{2}f(q)$ yields $pq = qp$.

5. CHARACTERIZATION OF TRACIAL FUNCTIONALS

Lemma 7. *Let $p, q \in \mathbb{M}_2(C(\Omega))^{\mathrm{pr}}$, $f \in C[0,1]$ be such a nonlinear function that $f(1-x) + f(x) = f(1) + f(0)$ for all $x \in [0,1]$, and $f(x) \neq f(0) + (f(1) - f(0))x$ for any $x \in (0,1) \setminus \{\frac{1}{2}\}$. Also, let φ be a positive functional on $\mathbb{M}_2(C(\Omega))$. Then the following conditions are equivalent:*

(i) $\forall \lambda \in (0,1)\ \forall p, q \in \mathbb{M}_2(C(\Omega))^{\mathrm{pr}}$ $(\varphi(f(\lambda p + (1-\lambda)q)) = \varphi(\lambda f(p) + (1-\lambda)f(q)))$;

(ii) $\exists \lambda \in (0,1)\ \forall p, q \in \mathbb{M}_2(C(\Omega))^{\mathrm{pr}}$ $(\varphi(f(\lambda p + (1-\lambda)q)) = \varphi(\lambda f(p) + (1-\lambda)f(q)))$;

(iii) *φ is tracial.*

Proof. The implication (i) $\Longrightarrow$ (ii) seems to be clear. It suffices to consider $p, q \in \mathbb{M}_2(\mathbb{C})^{\mathrm{pr}}$. Then $\lambda p + (1-\lambda)q = \mu_1 r + \mu_2 r^{\perp}$, where $\mu_1 + \mu_2 = 1$. We have $\varphi(f(\lambda p + (1-\lambda)q)) = \varphi(f(\mu_1)r + f(\mu_2)r^{\perp})$ and

$$\varphi(\lambda f(p) + (1-\lambda)f(q)) = \lambda f(1)\varphi(p) + \lambda f(0)\varphi(1-p) + (1-\lambda)f(1)\varphi(q) + (1-\lambda)f(0)\varphi(1-q) =$$

$$= f(0)(\varphi(r) + \varphi(r^{\perp})) + (f(1) - f(0))\varphi(\lambda p + (1-\lambda)q) =$$

$$= f(0)\varphi(r + r^{\perp}) + (f(1) - f(0))\varphi(\mu_1 r + \mu_2 r^{\perp}) =$$

$$= (f(0) + (f(1) - f(0))\mu_1)\varphi(r) + (f(0) + (f(1) - f(0))\mu_2)\varphi(r^\perp).$$

(iii) $\implies$ (i). If f is linear the equality is evident. If φ is tracial then $\varphi(r) = \varphi(r^\perp)$ and

$$(f(0) + (f(1) - f(0))\mu_1)\varphi(r) + (f(0) + (f(1) - f(0))\mu_2)\varphi(r^\perp) =$$

$$= (f(1) + f(0))\varphi(r) = (f(\mu_1) + f(\mu_2))\varphi(r) = \varphi(f(\mu_1)r + f(\mu_2)r^\perp).$$

(ii) $\implies$ (iii). If $\varphi(f(\lambda p + (1-\lambda)q)) = \varphi(\lambda f(p) + (1-\lambda)f(q))$ then the equality $f(1-x) + f(x) = f(1) + f(0)$ implies that

$$f(1-x) - (f(1) - f(0))(1-x) + f(x) - (f(1) - f(0))x = 0.$$

Therefore,

$$(f(\mu_1) - f(0) - (f(1) - f(0))\mu_1)\varphi(r) + (f(\mu_2) - f(0) - (f(1) - f(0))\mu_2)\varphi(r^\perp) = 0$$

and $\varphi(r) = \varphi(r^\perp)$ for any one dimensional $r \in \mathbb{M}_2(\mathbb{C})^{\mathrm{pr}}$, this is equivalent to φ being tracial. □

Theorem 3. *Let $\mathcal{M}$ be a von Neumann algebra, φ be a positive functional on $\mathcal{M}$, and $f \in C[0,1]$ be such that $f(x) + f(1-x) = f(0) + f(1)$ for all $x \in [0,1]$, and $f(x) \neq f(0) + (f(1) - f(0))x$ for all $x \in (0,1) \setminus \{\frac{1}{2}\}$. Then the following conditions are equivalent:*

(i) $\forall \lambda \in (0,1)\ \forall p,q \in \mathcal{M}^{\mathrm{pr}}\ \ (\varphi(f(\lambda p + (1-\lambda)q)) = \varphi(\lambda f(p) + (1-\lambda)f(q)))$;

(ii) $\exists \lambda \in (0,1)\ \forall p,q \in \mathcal{M}^{\mathrm{pr}}\ \ (\varphi(f(\lambda p + (1-\lambda)q)) = \varphi(\lambda f(p) + (1-\lambda)f(q))$;

(iii) *φ is tracial.*

Proof. The implications (i) $\implies$ (ii) and (iii) $\implies$ (i) are evident.

(ii) $\implies$ (iii). Consider $p,q \in \mathcal{M}^{\mathrm{pr}}$, then the von Neumann algebra $\mathcal{N} = \{p,q\}''$ is a subalgebra of $\mathcal{M}$. By Lemma 1 there exists a projection $z \in \mathcal{Z}(\mathcal{N})$ such that the algebra $\mathcal{N}_{z^\perp}$ is Abelian. Thus $\varphi|_{\mathcal{N}_{z^\perp}}(pz^\perp qz^\perp) = \varphi|_{\mathcal{N}_{z^\perp}}(qz^\perp pz^\perp)$. The restriction $\varphi|_{\mathcal{N}_z} \in \mathcal{N}_z^*$ and

$$\varphi|_{\mathcal{N}_z}(f(\lambda p + (1-\lambda)q)) = \lambda\varphi|_{\mathcal{N}_z}(f(p)) + (1-\lambda)|_{\mathcal{N}_z}\varphi(f(q))$$

for any $\lambda \in [0,1]$. Since $\mathcal{N}_z \cong \mathbb{M}_2(C(\Omega))$ (see Lemmas 1, 2) by Lemma 7 either f is linear or $\varphi|_{\mathcal{N}_z}$ is tracial and $\varphi|_{\mathcal{N}_z}(pzqz) = \varphi|_{\mathcal{N}_z}(qzpz)$. Finally we note that $\mathcal{N} = \mathcal{N}_z \oplus \mathcal{N}_{z^\perp}$ and

$$\varphi(pq) = \varphi|_{\mathcal{N}_z}(pzqz) + \varphi|_{\mathcal{N}_{z^\perp}}(pz^\perp qz^\perp) = \varphi|_{\mathcal{N}_z}(qzpz) + \varphi|_{\mathcal{N}_{z^\perp}}(qz^\perp pz^\perp) = \varphi(qp).$$

Every positive functional on $\mathcal{M}$ belongs to $\mathcal{M}^*$. Since $\varphi(pq) = \varphi(qp)$ for all $p,q \in \mathcal{M}^{\mathrm{pr}}$ φ is tracial. □

Example 8. Let $f(x) = x(x-1)(2x-1)$ (or $f(x) = \sin(2\pi x)$), then $f(x) + f(1-x) = 0 = f(1) + f(0)$. For any $p,q \in \mathcal{M}^{\mathrm{pr}}$ the equality $\varphi(f(\frac{1}{2}p + \frac{1}{2}q)) = \frac{1}{2}\varphi(f(p)) + \frac{1}{2}\varphi(f(q))$ holds if any only if a positive functional $\varphi \in \mathcal{M}^*$ is tracial.

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REFERENCES

1. A. M. Bikchentaev, Math. Notes **89** (4), 461–471 (2011).
2. A. M. Bikchentaev, Int. J. Theor. Physics **54** (12), 4482–4493 (2015).
3. A. M. Bikchentaev and O. E. Tikhonov, J. Inequal. Pure Appl. Math. **6** (2), article 49 (2005).
4. A. M. Bikchentaev and O. E. Tikhonov, Linear Algebra Appl. **422** (1), 274–278 (2007).
5. A. M. Bikchentaev, Math. Notes **64** (2), 159–163 (1998).
6. A. M. Bikchentaev, Lobachevskii J. Math. **32** (3), 175–179 (2011).
7. G. K. Pedersen and E. Størmer, Canad. J. Math. **34** (2), 370–373 (1982).
8. D. Petz and J. Zemánek, Linear Algebra Appl. **111**, 43–52 (1988).
9. S. Sakai, *C*-algebras and W*-algebras* (Springer-Verlag, New York, 1971).
10. M. Takesaki, *Theory of Operator algebras* (Springer, Berlin, 1979), Vol. 1.
11. O. E. Tikhonov, Positivity **9** (2), 259–264 (2005).