Bohr-Rogosinski inequalities for bounded analytic functions

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Abstract—In this paper we first consider another version of the Rogosinski inequality for analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in the unit disk $|z| < 1$, in which we replace the coefficients a_n $(n = 0, 1, \dots, N)$ of the power series by the derivatives $f^{(n)}(z)/n!$ $(n = 0, 1, \dots, N)$. Secondly, obtain improved versions of the classical Bohr's inequality and Bohr's inequality for the harmonic mappings of the form $f = h + \overline{g}$, where the analytic part h is bounded by 1 and that $|g'(z)| \leq k|h'(z)|$ in $|z| < 1$ and for some $k \in [0, 1]$.

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1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk, and $\mathcal{A}$ denote the space of analytic functions in $\mathbb{D}$ with the topology of uniform convergence on compact sets. Define $\mathcal{B} = \{f \in \mathcal{A} : |f(z)| < 1 \text{ in } \mathbb{D}\}$. Then the Bohr radius is the largest number $r > 0$ such that if $f \in \mathcal{B}$ has the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\sum_{n=0}^{\infty} |a_n|\, |z|^n \leq 1 \ \text{ for } \ |z| \leq r$$

which is called the classical Bohr inequality for the family $\mathcal{B}$. Rogosinski radius is the largest number $r > 0$ such that, under the previous assumptions,

$$|S_N(z)| < 1 \ \text{ for } \ |z| < r,$$

where $S_N(z) = \sum_{n=0}^{N} a_n z^n$ $(N \geq 0)$ denote the partial sums of f. This inequality is called the classical Rogosinski inequality for the family $\mathcal{B}$.

If **B** and **R** denote the Bohr radius and the Rogosinski radius, respectively, then because $|S_N(z)| \leq \sum_{n=0}^{N} |a_n|\, |z|^n \leq \sum_{n=0}^{\infty} |a_n|\, |z|^n$, it is clear that $\mathbf{B} \leq \mathbf{R}$. In fact the following two classical results are well-known.

Theorem A. *Suppose that $f \in \mathcal{B}$. Then we have* $\mathbf{B} = 1/3$, and (see Rogosinski [25] and also [21, 27]) $\mathbf{R} = 1/2$.

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There is a long history about the consequences of Bohr's inequality, in particular. Indeed, Bohr [14] discovered that $\mathbf{B} \geq 1/6$ and the fact that $\mathbf{B} = 1/3$ was obtained independently by M. Riesz, I. Schur and N. Weiner. Extensions and modifications of Bohr's result can be found from [11, 14, 23, 24] and the recent articles [1–3, 9, 15–19, 22]). We refer to [5, 6, 8, 13] for the extension of the Bohr inequality to several complex variables. More recently, Kayumov and Ponnusamy [18] introduced and investigated Bohr-Rogosinski's radii for the family $\mathcal{B}$, and they discussed Bohr-Rogosinski's radius for the class of subordinations. In [7], Aizenberg, et al. generalized the Rogosinski radius for holomorphic mappings of the open unit polydisk into an arbitrary convex domain. In [17], Kayumov et al. investigated Bohr's radius for complex-valued harmonic mappings that are locally univalent in $\mathbb{D}$. Several improved versions of Bohr's inequality were given by Kayumov and Ponnusamy in [18] and these were subsequently followed by Evdoridis et al. [15] to obtain improved versions of Bohr's inequality for the class of harmonic mappings. In [16], Kayumov and Ponnusamy discussed Bohr's radius for the class of analytic functions g, when g is subordinate to a member of the class of odd univalent functions. For more information about Bohr's inequality and further related works, we refer the reader to the recent survey article [4] and the references therein.

In this paper we shall introduce and investigate another version of the Rogosinski inequality for analytic functions defined on the unit disk $\mathbb{D}$ by substituting the derivatives of the analytic function instead of the coefficients of its power series. We shall also introduce and study several new versions of the classical Bohr's inequality.

2. AN IMPROVED VERSION OF THE CLASSICAL ROGOSINSKI INEQUALITY

What could happened to the partial sums of the analytic function in the unit disk if we replaced the coefficients $a_0, a_1, \ldots, a_{N-1}$ by the functions $f(z), f'(z), \ldots, f^{(N-1)}(z)$?
In this section we give an answer in the following form.

Theorem 1. *Suppose that $f \in \mathcal{B}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then*

$$\left| \sum_{k=0}^{n} \frac{f^{(k)}(z)}{k!} z^k \right| \leq \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k}^2 \quad \textit{for all } |z| \leq r \leq \frac{1}{2}.$$

Proof. To prove this theorem we will use a modification of Landau's method (see [20] and [21, p. 26]).

We consider the function $g : \mathbb{D} \to \mathbb{D}$ defined by $g(\zeta) = f(\alpha(\zeta+1))$, where $|\alpha| \leq 1/2$, and use the substitution $\xi = D(\zeta) = \alpha(\zeta+1)$. In view of the Cauchy integral formula, integration along a circle γ around the origin lying in its neighborhood, we have

$$\frac{f^{(k)}(\alpha)}{k!} = \frac{1}{2\pi i} \int_{D(\gamma)} \frac{f(\xi)}{(\xi-\alpha)^{k+1}}\, d\xi = \frac{1}{2\pi i \alpha^k} \int_{\gamma} \frac{g(\zeta)}{\zeta^{k+1}}\, d\zeta$$

and thus, we can write

$$\sum_{k=0}^{n} \frac{f^{(k)}(\alpha)}{k!} \alpha^k = \frac{1}{2\pi i} \int_{\gamma} g(\zeta) \left(\sum_{k=0}^{n} \frac{1}{\zeta^{k+1}} \right) d\zeta = \frac{1}{2\pi i} \int_{\gamma} \frac{g(\zeta)}{\zeta^{n+1}} \left(\sum_{k=0}^{n} \zeta^k \right) d\zeta. \tag{1}$$

Set

$$1 + \zeta + \zeta^2 + \zeta^3 + \cdots = \frac{1}{1-\zeta} = K^2(\zeta) = (K_n(\zeta))^2 + O(z^{n+1}),$$

where we write

$$K(\zeta) = (1-\zeta)^{-1/2} = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (-\zeta)^k \text{ and } K_n(\zeta) = \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k} (-\zeta)^k.$$

In view of the above observations, (1) reduces to

$$\sum_{k=0}^{n} \frac{f^{(k)}(\alpha)}{k!}\alpha^k = \frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)}{\zeta^{n+1}}(K_n(\zeta))^2\, d\zeta \tag{2}$$

and therefore, with $\zeta = |\zeta|e^{i\phi}$, and $|g(\zeta)| \le 1$ for all $|\alpha| \le 1/2$ and $|\zeta| \le 1$, we have

$$\left|\sum_{k=0}^{n} \frac{f^{(k)}(\alpha)}{k!}\alpha^k\right| \;\le\; \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|\zeta|^{n+1}}|K_n(\zeta)|^2|\zeta|\, d\phi \;=\; \frac{1}{|\zeta|^n}\sum_{k=0}^{n}\binom{-\frac{1}{2}}{k}^2 |\zeta|^{2k}.$$

Allowing $|\zeta| \to 1$, we get

$$\left|\sum_{k=0}^{n} \frac{f^{(k)}(\alpha)}{k!}\alpha^k\right| \le \sum_{k=0}^{n}\binom{-\frac{1}{2}}{k}^2 \text{ for all } |\alpha| \le \frac{1}{2}$$

which completes the proof of the theorem. □

3. IMPROVED VERSIONS OF THE CLASSICAL BOHR'S INEQUALITY

For $f \in \mathcal{B}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the following inequalities due to Schwarz-Pick will be used frequently: for $|z| = r < 1$,

$$|f(z)| \le \frac{r+a}{1+ar} \text{ and } |f'(z)| \le \frac{1-|f(z)|^2}{1-|z|^2}, \tag{3}$$

where $|a_0| = a \in [0,1)$. Also, it is well-known that the Taylor coefficients of $f \in \mathcal{B}$ satisfy the inequalities:

$$|a_k| \le 1 - a^2 \text{ for any } k \ge 1. \tag{4}$$

More generally, we have ([26]) the sharp estimate

$$\frac{|f^{(k)}(z)|}{k!} \le \frac{1-|f(z)|^2}{(1-|z|)^k(1+|z|)} \text{ for any } k \ge 1 \text{ and } z \in \mathbb{D}, \tag{5}$$

which in particular gives second inequality in (3), and (4) by setting $z = 0$ in (5). In the following we also assume that $m \in \mathbb{N}$, and the idea of replacing a_k by $\frac{f^{(k)}(z)}{k!}$ is used in [22]. But our concern here is slightly different from theirs.

Theorem 2. *Suppose that $f \in \mathcal{B}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then*

$$A_f(z) := |f(z^m)| + |z^m|\,|f'(z^m)| + \sum_{k=2}^{\infty}|a_k|r^k \le 1 \text{ for all } r \le R_{m,1}, \tag{6}$$

where $R_{m,1}$ is the maximal positive root of the equation $\varphi_m(x) = 0$ with

$$\varphi_m(x) = (1-x)(x^{2m} + 2x^m - 1) + 2x^2(1+x^m)^2 \tag{7}$$

and the constant $R_{m,1}$ cannot be improved.

Proof. Let $f \in \mathcal{B}$ and $|a_0| = a \in [0,1)$. It is a simple exercise to see that for $0 \le x \le x_0$ (≤ 1) and $0 \le \alpha \le 1/2$, we have

$$b(x) := x + \alpha(1-x^2) \le b(x_0).$$

m	$R_{m,1}$
1	0.280776
2	0.39149
3	0.441112
4	0.467644
5	0.482442

Table 1. $R_{m,1}$ is the maximal positive root of the equation $(1-x)(x^{2m}+2x^m-1)+2x^2(1+x^m)^2=0$

This simple fact will be used in the later theorems also. Using this inequality and (4), we easily obtain from (3) and (6) that

$$\begin{aligned} A_f(z) &\leq |f(z^m)| + \frac{r^m}{1-r^{2m}}(1-|f(z^m)|^2) + (1-a^2)\frac{r^2}{1-r} \\ &\leq \frac{r^m+a}{1+ar^m} + \frac{r^m}{1-r^{2m}}\left[1-\left(\frac{r^m+a}{1+ar^m}\right)^2\right] + (1-a^2)\frac{r^2}{1-r} \\ &= 1-\frac{(1-a)(1-r^m)}{1+ar^m} + (1-a^2)\frac{r^m}{(1+ar^m)^2} + (1-a^2)\frac{r^2}{1-r} \\ &= 1+\frac{(1-a)\varphi_m(a,r)}{(1+ar^m)^2(1-r)}, \end{aligned}$$

where

$$\begin{aligned} \varphi_m(a,r) &= -(1-r^m)(1+ar^m)(1-r) + (1+a)r^m(1-r) + r^2(1+a)(1+ar^m)^2 \\ &= (1-r)(ar^{2m}+2r^m-1) + r^2(1+a)(1+ar^m)^2. \end{aligned}$$

The second inequality above is justified because of the fact that $\frac{r^m}{1-r^{2m}} \leq \frac{1}{2}$ for $r \leq \sqrt[m]{\sqrt{2}-1}$. Also, $R_{m,1} \leq \sqrt[m]{\sqrt{2}-1}$, where $R_{m,1}$ is as in the statement. Now, since $\varphi_m(a,r)$ is an increasing function of a in $[0,1)$, it follows that

$$\varphi_m(a,r) \leq \varphi_m(1,r) = (1-r)(r^{2m}+2r^m-1) + 2r^2(1+r^m)^2 = \varphi_m(r),$$

where $\varphi_m(r)$ is given by (7). Clearly, $A_f(z) \leq 1$ if $\varphi_m(r) \leq 0$, which holds for $r \leq R_{m,1}$.

To show the sharpness of the radius $R_{m,1}$, we let $a \in [0,1)$ and consider the function

$$f(z) = \frac{a+z}{1+az} = a + (1-a^2)\sum_{k=1}^{\infty}(-a)^{k-1}z^k, \quad z \in \mathbb{D} \tag{8}$$

so that

$$\frac{f^{(k)}(z)}{k!} = (1-a^2)\frac{(-a)^{k-1}}{(1+az)^{k+1}} \quad \text{for } k \geq 1 \text{ and } z \in \mathbb{D}.$$

For this function, we observe that for $z=r$ and $a \in [0,1)$,

$$\begin{aligned} |f(z^m)| + |z^m|\,|f'(z^m)| + \sum_{k=2}^{\infty}|a_k|r^k &= \frac{a+2r^m+ar^{2m}}{(1+ar^m)^2} + (1-a^2)\frac{ar^2}{1-ar} \\ &= 1+\frac{(1-a)P_m(a,r)}{(1+ar^m)^2(1-ar)}, \end{aligned} \tag{9}$$

where

$$P_m(a,r) = (1-ar)(ar^{2m} + 2r^m - 1) + ar^2(1+a)(1+ar^m)^2$$

and the last expression (9) is larger than 1 if and only if $P_m(a,r) > 0$. By a simple calculation, we find that

$$\frac{\partial P_m(a,r)}{\partial a} = r^{2m} + 2a(r^2 - r^{2m+1}) + r(1+r-2r^m) + 3a^2r^{2m+2} + 4ar^{m+2} + 2ar^2 + 4a^3r^{2m+2} + 6a^2r^{m+2}$$

which is clearly non-negative for each $r \in [0,1)$ and thus, for each $r \in [0,1)$, $P_m(a,r)$ is an increasing function of a. This fact gives

$$0 < r^{2m} + r(1+r-2r^m) = P_m(0,r) \le P_m(a,r) \le P_m(1,r) = \varphi_m(r),$$

where $\varphi_m(r)$ is given by (7). Therefore, the right hand side of (9) is smaller than or equal to 1 for all $a \in [0,1)$, only in the case $r \le R_{m,1}$. Finally, it also suggests that the right hand side of (9) is larger than 1 if $r > R_{m,1}$. This completes the proof. □

Remark 1. *In Tables 1 and 2, we listed the values of $R_{m,1}$ and $R_{m,2}$ for certain values of m. If we allow $m \to \infty$ in Theorem 2 (resp. Theorem 3 below), we note that $R_{m,1} \to 1/2$ (resp. $R_{m,2} \to 1/2$ below) and thus if $f \in \mathcal{B}$, then we have the inequality*

$$|f(0)| + \sum_{k=2}^{\infty} \left| \frac{f^{(k)}(0)}{k!} \right| r^k \le 1 \text{ for all } r \le 1/2,$$

and the number $1/2$ is sharp.

Theorem 3. *If $f \in \mathcal{B}$, then*

$$B_f(z) := |f(z^m)| + \sum_{k=2}^{\infty} \left| \frac{f^{(k)}(z^m)}{k!} \right| r^k \le 1 \text{ for all } r \le R_{m,2}, \tag{10}$$

where $R_{m,2}$ is the minimum positive root of the equation $\psi_m(x) = 0$ with

$$\psi_m(x) = 2x^2 - (1-x^{2m})(1-x^m-x) \tag{11}$$

and the constant $R_{m,2}$ cannot be improved.

m	$R_{m,2}$
1	0.355416
2	0.430586
3	0.464327
4	0.481418
5	0.490359

Table 2. $R_{m,2}$ is the maximal positive root of the equation $2x^2 - (1-x^{2m})(1-x^m-x) = 0$

Proof. As before we let $f \in \mathcal{B}$ and $a = |a_0|$. By assumption, (3) and (5) (with z^m in place of z), we have

$$\begin{aligned} B_f(z) &\leq |f(z^m)| + \frac{1-|f(z^m)|^2}{1+r^m} \sum_{k=2}^{\infty} \left(\frac{r}{1-r^m}\right)^k \\ &= |f(z^m)| + \frac{r^2}{(1-r^{2m})(1-r^m-r)}(1-|f(z^m)|^2) \\ &\leq \frac{r^m+a}{1+ar^m} + \frac{r^2}{(1-r^{2m})(1-r^m-r)}\left[1-\left(\frac{r^m+a}{1+ar^m}\right)^2\right] \\ &= 1 - \frac{(1-a)(1-r^m)}{1+ar^m} + \frac{r^2(1-a^2)}{(1+ar^m)^2(1-r^m-r)} \\ &= 1 + \frac{(1-a)\psi_m(a,r)}{(1+ar^m)^2(1-r^m-r)}, \end{aligned}$$

where

$$\psi_m(a,r) = -(1-r^m)(1+ar^m)(1-r^m-r) + r^2(1+a).$$

The second inequality above is a consequence of our earlier observation used in Theorem 2 but this time with $\alpha = r^2/[(1-r^{2m})(1-r^m-r)]$. It is a simple exercise to see that $\psi_m(a,r)$, for each $m \geq 1$, is an increasing function of a in $[0,1)$, and thus, it follows that

$$\psi_m(a,r) \leq \psi_m(1,r) = \psi_m(r),$$

where $\psi_m(r)$ is given by (11). Clearly, $\psi_m(a,r) \leq 0$ if $\psi_m(r) \leq 0$, which holds for $r \leq R_{m,2}$, where $R_{m,2}$ is the minimum positive root of the equation $\psi_m(r) = 0$. Thus, $B_f(z) \leq 1$ for $r \leq R_{m,2}$ and the inequality (10) follows.

To show the sharpness of the radius $R_{m,2}$, we let $a \in (0,1)$ and consider the function $g(z) = f(-z)$, where f is given by (8). Then for g, we easily have

$$\frac{g^{(k)}(z)}{k!} = -(1-|a|^2)\frac{a^{k-1}}{(1-az)^{k+1}}, \quad z \in \mathbb{D}.$$

Now, we choose a as close to 1 as we please and set $z = r < \sqrt[m]{a}$. By a simple calculation, the corresponding $B_g(z)$ takes the form

$$\begin{aligned} B_g(z) &= \frac{a-r^m}{1-ar^m} + \frac{ar^2(1-a^2)}{(1-ar^m)^2(1-ar^m-ar)} \\ &= 1 - \frac{(1-a)(1+r^m)}{1-ar^m} + \frac{ar^2(1-a^2)}{(1-ar^m)^2(1-ar^m-ar)} \\ &= 1 + \frac{(1-a)P_m(a,r)}{(1-ar^m)^2(1-ar^m-ar)}, \end{aligned} \tag{12}$$

where

$$P_m(a,r) = a(1+a)r^2 - (1+r^m)(1-ar^m)(1-ar^m-ar).$$

Clearly, $B_g(z) < 1$ if and only if $P_m(a,r) < 0$. By elementary calculations, we find that

$$\begin{aligned} \frac{\partial P_m(a,r)}{\partial a} &= 2a[r^2 - r^m(1+r^m)(r^m+r)] + r^2 + (1+r^m)(2r^m+r) \\ &= 2a(r^2 - r^{2m} - r^{m+1} - r^{3m} - r^{2m+1}) + (2r^{2m} + r^{m+1} + 2r^m + r^2 + r) \end{aligned}$$

which is easily seen to be greater than or equal to 0 for any $r \in [0,1)$ and $m \geq 1$. Consequently,

$$P_m(a,r) \leq P_m(1,r) = \psi_m(r) = 2r^2 - (1-r^{2m})(1-r^m-r).$$

Therefore, the expression on the right of (12) is smaller than or equal to 1 for all $a \in (0,1)$, only in the case when $r \le R_{m,2}$. Finally, it also suggests that $a \to 1$ in the right hand side of (12) shows that the expression (12) is larger than 1 if and only if $r > R_{m,2}$. This completes the proof. $\square$

Theorem 4. *Suppose that $f \in \mathcal{B}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then*

$$C_f(z) := |f(z^m)| + |z|\,|f'(z^m)| + \sum_{k=2}^{\infty} |a_k| r^k \le 1 \quad \text{for all} \quad r \le R_{m,3}, \tag{13}$$

where $R_{m,3}$ is the maximal positive root of the equation $\Phi_m(r) = 0$ with

$$\Phi_m(r) = 3x - 1 + x^m \left[2x^2(x^m+2) + x^m(1-x)\right] \tag{14}$$

and the constant $R_{m,3}$ cannot be improved.

m	$R_{m,3}$
1	0.280776
2	0.316912
3	0.327911
4	0.33152
5	0.332726

Table 3. $R_{m,3}$ is the maximal positive root of the equation $3x - 1 + x^m[2x^2(x^m+2) + x^m(1-x)] = 0$

Proof. As in the proofs of Theorems 2 and 3, it follows from (3), (4) and (5) that

$$\begin{aligned} C_f(z) &\le |f(z^m)| + \frac{r}{1-r^{2m}}(1 - |f(z^m)|^2) + (1-a^2)\frac{r^2}{1-r} \\ &\le \frac{r^m + a}{1 + ar^m} + \frac{r}{1-r^{2m}}\left[1 - \left(\frac{r^m+a}{1+ar^m}\right)^2\right] + (1-a^2)\frac{r^2}{1-r} \\ &= 1 - \frac{(1-a)(1-r^m)}{1+ar^m} + (1-a^2)\frac{r}{(1+ar^m)^2} + (1-a^2)\frac{r^2}{1-r} \\ &= 1 + \frac{(1-a)\Phi_m(a,r)}{(1+ar^m)^2(1-r)}, \end{aligned}$$

where

$$\begin{aligned} \Phi_m(a,r) &= -(1-r^m)(1+ar^m)(1-r) + r(1+a)(1-r) + r^2(1+a)(1+ar^m)^2 \\ &= r(1+a) + ar^{m+2}(1+a)(2+ar^m) - (1-r^m)(1+ar^m)(1-r) \\ &\le \Phi_m(1,r) = \Phi_m(r), \end{aligned}$$

where $\Phi_m(a,r)$ is seen to be an increasing function of a in $[0,1)$, and $\Phi_m(r)$ is given by (14). Note that the second inequality above holds since $\max \frac{2r}{1-r^{2m}} < 1$ and so for any $r < R_m$, where R_m is the maximal positive root of the equation $2r - (1-r^{2m}) = 0$, and $R_{m,3} < R_m$ for $m \in \mathbb{N}$, where $R_{m,3}$ is the maximal positive root of the equation $\Phi_m(r) = 0$. Since $\Phi_m(r) \le 0$ for $r \le R_{m,3}$, we obtain $C_f(z) \le 1$ for $r \le R_{m,3}$ and the desired inequality (13) follows.

It remains to show the sharpness of the radius $R_{m,3}$. To do this we let $a \in [0,1)$ and consider the function f is given by (8). For this function, we observe that for $z = r$,

$$C_f(z) = \frac{(r^m + a)(1 + ar^m) + r(1 - a^2)}{(1 + ar^m)^2} + (1 - a^2)\frac{ar^2}{1 - ar} = 1 + \frac{(1 - a)Q_m(a, r)}{(1 + ar^m)^2(1 - r)}, \tag{15}$$

where

$$Q_m(a, r) = r(1 + a) + a^2 r^{m+2}(1 + a)(2 + ar^m) - (1 - r^m)(1 + ar^m)(1 - ar).$$

We see that $C_f(z) > 1$ for $a \in [0,1)$ if and only if $Q_m(a,r) > 0$. By a simple calculation, we find that $Q_m(a,r)$ is an increasing function of a in $[0,1)$ and therefore, we have

$$Q_m(a, r) \le Q_m(1, r) = 2r + 2r^{m+2}(2 + r^m) - (1 - r^m)(1 + r^m)(1 - r) = \Phi_m(r),$$

$\Phi_m(r)$ is given by (14). Therefore, the expression (15) is smaller than or equal to 1 for all $a \in [0,1)$, only when $r \le R_{m,3}$. Finally, it also suggests that $a \to 1$ in (15) shows that the expression (15) is larger than 1 if $r > R_{m,3}$. This completes the proof. □

Remark 2. *In Table 3, we listed the values of $R_{m,3}$ for certain values of m. If we allow $m \to \infty$ in Theorem 4, we see that $R_{m,3} \to \frac{1}{3}$ and hence we have the classical Bohr inequality for $f \in \mathcal{B}$:*

$$|f(0)| + \sum_{k=1}^{\infty} |a_k|\,|z|^k \le 1 \ \ \textit{for all } r \le 1/3.$$

and 1/3 is sharp.

4. TWO IMPROVED VERSIONS OF BOHR'S INEQUALITY FOR HARMONIC MAPPINGS

Theorem 5. *Suppose that $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ is a harmonic mapping of $\mathbb{D}$ such that $|g'(z)| \le k|h'(z)|$ for some $k \in [0,1]$ and $h \in \mathcal{B}$. Then we have*

$$D_f(z) := |h(z^m)| + \sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n \le 1 \ \ \textit{for all } r \le R^k_{m,1}, \tag{16}$$

where $R^k_{m,1}$ is the maximal positive root of the equation $\lambda_m(r) = 0$ with

$$\lambda_m(r) = 2x(1 + k)(1 + x^m) - (1 - x)(1 - x^m) \tag{17}$$

and the constant $R^k_{m,1}$ cannot be improved.

m	$R^1_{m,1}$
1	0.154701
2	0.188829
3	0.197544
4	0.199494
5	0.199898

Table 4. $R^1_{m,1}$ is the maximal positive root of the equation $4x(1 + x^m) - (1 - x)(1 - x^m) = 0$

Proof. Recall that, as $h \in \mathcal{B}$,

$$h(z) \prec \frac{a_0 + z}{1 + \overline{a_0} z} = a_0 + (1 - |a_0|^2) \sum_{n=1}^{\infty} (-1)^{n-1} (\overline{a_0})^{n-1} z^n, \quad z \in \mathbb{D},$$

which gives [10, 12]

$$\sum_{n=1}^{\infty} |a_n| r^n \leq (1 - |a_0|^2) \sum_{n=1}^{\infty} |a_0|^{n-1} r^n = \frac{1 - a^2}{1 - ar} \text{ for all } r \leq \frac{1}{3},$$

where $a = |a_0| \in [0, 1)$. Moreover, by assumption, we obtain that $g'(z) \prec_q kh'(z)$ which quickly gives from [10] that

$$\sum_{n=1}^{\infty} n |b_n| r^{n-1} \leq \sum_{n=1}^{\infty} kn\, |a_n| r^{n-1} \text{ for all } r \leq \frac{1}{3}$$

and integrating this with respect to r gives

$$\sum_{n=1}^{\infty} |b_n| r^n \leq k \sum_{n=1}^{\infty} |a_n| r^n \text{ for all } r \leq \frac{1}{3}.$$

Here $\prec_q$ denotes the quasi-subordination. Using these and the first inequality in (3) for $h(z)$ one can obtain that for $|z| = r \leq 1/3$,

$$\begin{aligned} D_f(z) &\leq \frac{r^m + a}{1 + ar^m} + (1 + k) r \frac{1 - a^2}{1 - ar} \\ &= 1 - \frac{(1 - a)(1 - r^m)}{1 + ar^m} + (1 + k) r \frac{1 - a^2}{1 - ar} \\ &= 1 + \frac{(1 - a) \lambda_m(a, r)}{(1 + ar^m)(1 - ar)}, \end{aligned}$$

where $\lambda_m(a, r) = r(1 + k)(1 + a)(1 + ar^m) - (1 - r^m)(1 - ar)$, which is indeed an increasing function of $a \in [0, 1)$ so that

$$\lambda_m(a, r) \leq \lambda_m(1, r) = \lambda_m(r),$$

where $\lambda_m(r)$ is given by (17). We see that $D_f(z) \leq 1$ if $\lambda_m(r) \leq 0$, which holds for $r \leq R^k_{m,1}$, where $R^k_{m,1}$ is the maximal positive root of the equation $\lambda_m(r) = 0$. This proves the inequality (16).

Finally, to show the sharpness of the radius $R^k_{m,1}$, we consider the function

$$f(z) = h(z) + \lambda \overline{h(z)}, \quad h(z) = \frac{z + a}{1 + az}, \tag{18}$$

where $\lambda(0, 1]$. For this function, we get that (for $z = r$ and $a \in [0, 1)$)

$$D_f(z) = \frac{r^m + a}{1 + ar^m} + (\lambda + 1) r \frac{1 - a^2}{1 - ra}$$

and the last expression shows the sharpness of $R^k_{m,1}$ with $\lambda \to 1$. This completes the proof of the theorem. $\square$

Remark 3. *In Table 4, we listed the values of $R^k_{m,1}$ for $k = 1$ and for certain values of m. When $m \to \infty$, we have from Theorem 5 that $R^k_{m,1} \to \frac{2}{4k+6}$. Thus, under the hypotheses of Theorem 5, we have*

$$|h(0)| + \sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n \leq 1 \textit{ for all } r \leq \frac{2}{4k + 6}$$

which for $k=0$ gives the classical Bohr's inequality and for $k=1$, this inequality contains the Bohr inequality for sense-preserving harmonic mapping $f(z)=h(z)+\overline{g(z)}$ of the disk $\mathbb{D}$ with the Bohr radius $1/5$ (see [17]).

Theorem 6. *Assume the hypotheses of Theorem 5. Then we have*

$$E_f(z) := |h(z^m)| + |z^m|\,|h'(z^m)| + \sum_{n=2}^{\infty}|a_n|r^n + \sum_{n=1}^{\infty}|b_n|r^n \leq 1 \text{ for all } r \leq R_{m,2}^k,$$

where $R_{m,2}^k$ is the maximal positive root of the equation $\Lambda_m(r)=0$ with

$$\Lambda_m(r) = (1-x)(x^{2m}+2x^m-1) + 2x(x+k)(1+x^m)^2$$

and the constant $R_{m,2}^k$ cannot be improved.

m	$R_{m,2}^1$
1	0.1671
2	0.240751
3	0.267472
4	0.276691
5	0.279585

Table 5. $R_{m,2}^1$ is the maximal positive root of the equation $(1-x)(x^{2m}+2x^m-1)+2x(x+1)(1+x^m)^2=0$

Proof. As in the proofs of Theorem 5 and earlier theorems, we easily have

$$\begin{aligned} E_f(z) &\leq \frac{r^m+a}{1+ar^m} + \frac{r^m}{1-r^{2m}}\left[1-\left(\frac{r^m+a}{1+ar^m}\right)^2\right] + (1-a^2)\frac{r^2}{1-r} + k(1-a^2)\frac{r}{1-r} \\ &= 1 - \frac{(1-a)(1-r^m)}{1+ar^m} + \frac{(1-a^2)r^m}{(1+ar^m)^2} + \frac{(1-a^2)(r+k)r}{1-r} \\ &= 1 + \frac{(1-a)\Lambda_m(a,r)}{(1+ar^m)^2(1-r)}, \end{aligned}$$

where

$$\begin{aligned} \Lambda_m(a,r) &= -(1-r^m)(1-r)(1+ar^m) + r^m(1+a)(1-r) + (1+a)r(r+k)(1+ar^m)^2 \\ &= (1-r)(ar^{2m}+2r^m-1) + r(1+a)(r+k)(1+ar^m)^2 \\ &\leq \Lambda_m(1,r) = \Lambda_m(r). \end{aligned}$$

The first inequality above is justified with the same reasoning as in the proofs of earlier theorems.

Now, we see that $E_f(z) \leq 1$ whenever $\Lambda_m(r) \leq 0$, which holds for $r \leq R_{m,2}^k$, where $R_{m,2}^k$ is the maximal positive root of the equation $\Lambda_m(r)=0$.

To show the sharpness of the radius $R_{m,2}^k$, consider the function f defined by (18) with $\lambda \in (0,1]$. For this function, the corresponding expression for $E_f(z)$ with $z=r$ turned out to be

$$E_f(z) = \frac{a+2r^m+ar^{2m}}{(1+ar^m)^2} + \frac{ar(1-a^2)(r+\lambda)}{1-ar}. \tag{19}$$

The last expression is larger than 1 if and only if $P_m^k(a,r) > 0$, where

$$P_m^k(a,r) = (1-ar)(ar^{2m} + 2r^m - 1) + ar(1+a)(r+\lambda)(1+ar^m)^2. \tag{20}$$

By a simple calculation, we find that $P_m^k(a,r)$ is an increasing function of $a \in [0,1)$, and for each $r \in [0,1)$, so that

$$P_m^k(a,r) \le P_m^k(1,r) = (1-r)(r^{2m} + 2r^m - 1) + 2r(r+\lambda)(1+r^m)^2.$$

Therefore, the expression (19) is smaller than or equal to 1 for all $a \in [0,1)$, only in the case when $r \le R_{m,2}^k$ $(\lambda = k)$. Finally, it also suggests that $a \to 1$ in (20) shows that the expression (19) is larger than 1 if $r > R_{m,2}^k$. This completes the proof. □

Remark 4. *In Table 5, we listed the values of $R_{m,2}^k$ for $k = 1$ and for certain values of m. If we allow $m \to \infty$ in Theorem 6, we obtain that*

$$R_{m,2}^k \to R_2^k := \frac{1}{4}\left(\sqrt{(2k+1)^2 + 8} - (2k+1)\right)$$

where R_2^k is the positive root of the equation $2x(x+k) + x - 1 = 0$ and the conclusion of Theorem 6 takes the following form:

$$|h(0)| + \sum_{n=2}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n \le 1 \ \text{ for all } \ r \le \frac{1}{4}\left(\sqrt{(2k+1)^2 + 8} - (2k+1)\right).$$

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