On some Theorems of the Dunkl-Lipschitz class for the Dunkl transform

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Abstract: Using a generalized spherical mean operator, we obtain a generalization of two theorems 84 and 85 of Titchmarsh for the Dunkl transform for functions satisfying the Dunkl-Lipschitz condition in the space $\mathrm{L}^p(\mathbb{R}^d, w_k(x)dx)$, where $1 < p \leq 2$.

Keywords: Dunkl operator, Dunkl transform, generalized spherical mean operator.

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1 Intoduction and preliminaries

Dunkl [5] defined a family of first-order differential-difference operators related to some reflection groups. The theory of Dunkl operators provides generalization of various multivariable function, among others we cite the exponential function, the Fourier transform and the translation. For more details about this theory see [4, 6, 8, 9, 10, 11] and the references therein.

E.C. Titchmarsh ([12] theorems 84, 85) proved following two theorems:

Theorem 1.1 *If $f(x)$ belongs to Lipschitz class $Lip(\alpha, p)$ in the L^p norm on the real line $\mathbb{R}$, then its Fourier transform $\widehat{f}$ belongs to $\mathrm{L}^\beta(\mathbb{R})$ for*

$$\frac{p}{p+\alpha p-1} < \beta \leq q = \frac{p}{p-1},$$

where $0 < \alpha \leq 1$, $1 < p \leq 2$ and

$$Lip(\alpha, p) = \{f \in \mathrm{L}^p(\mathbb{R}),\ \|f(x+h) - f(x)\|_{\mathrm{L}^p(\mathbb{R})} = O(h^\alpha)\ as\ h \to 0\}$$

Theorem 1.2 *Let $\alpha \in (0,1)$ and assume that $f \in \mathrm{L}^2(\mathbb{R})$. Then the following are equivalents*

1. $\|f(x+h) - f(x)\|_{\mathrm{L}^2(\mathbb{R})} = O(h^\alpha)$ *as* $h \longrightarrow 0$,
2. $\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 d\lambda = O(r^{-2\alpha})$ *as* $r \longrightarrow +\infty$,

In this paper we try to explore the validity of those theorems in case of the Dunkl transform in the space $\mathrm{L}^p(\mathbb{R}^d, w_k(x)dx)$, where $1 < p \leq 2$. For this generalization, we use a generalized spherical mean operator.

In the first we collect some notations and results on Dunkl analysis (see [5], [10]).

We consider $\mathbb{R}^d$ with the Euclidean scalar product $\langle ., . \rangle$ and $|x| = \sqrt{\langle x, x \rangle}$. For $\alpha \in \mathbb{R}^d \backslash \{0\}$, let σ_α be the reflection in the hyperplane $\mathrm{H}_\alpha \subset \mathbb{R}^d$ orthogonal to α.
i.e.,

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x \rangle}{|x|^2}\alpha.$$

A finite set $\mathrm{R} \subset \mathbb{R}^d \backslash \{0\}$ is called a root system, if $\mathrm{R} \cap \mathbb{R}.\alpha = \{\alpha, -\alpha\}$ and $\sigma_\alpha \mathrm{R} = \mathrm{R}$ for all $\alpha \in \mathrm{R}$. For a given root system R the reflection σ_α, $\alpha \in \mathrm{R}$, generate a finite group $\mathrm{W} \subset O(d)$, the reflection group associated with R. We fix $\beta \in \mathbb{R}^d \backslash \cup_{\alpha \in \mathrm{R}} \mathrm{H}_\alpha$ and define a positive root system $\mathrm{R}_+ = \{\alpha \in \mathrm{R} / \ \ \langle \alpha, \beta \rangle > 0\}$.

A function $k : \mathrm{R} \longrightarrow \mathbb{C}$ be a multiplicity function on R. For brevity, we introduce the index

$$\gamma = \gamma(k) = \sum_{\alpha \in \mathrm{R}_+} k(\alpha).$$

Moreover, let w_k denote the weight function

$$w_k(x) = \prod_{\alpha \in \mathrm{R}_+} |\langle \alpha, x \rangle|^{2k(\alpha)},$$

where w_k is W-invariant and homogeneous of degree 2γ

We let η be the normalized surface measure on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ and set

$$d\eta_k(y) = w_k(y) d\eta(y).$$

Then η_k is a W-invariant measure on $\mathbb{S}^{d-1}$, we let $d_k = \eta_k(\mathbb{S}^{d-1})$.

Introduced by C.F. Dunkl in [5] the Dunkl operators T_j, $1 \leq j \leq d$, on $\mathbb{R}^d$ associated with the reflection group W and the multiplicity function k are the first-order differential-difference operators given by

$$\mathrm{T}_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in \mathrm{R}_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}, \quad f \in \mathrm{C}^1(\mathbb{R}^d),$$

where $\alpha_j = \langle \alpha, e_j \rangle$; $(e_1, ..., e_d)$ being the canonical basis of $\mathbb{R}^d$ and $\mathrm{C}^1(\mathbb{R}^d)$ is the space of functions of class C^1 on $\mathbb{R}^d$.

The Dunkl kernel E_k on $\mathbb{R}^d \times \mathbb{R}^d$ has been introduced by C.F. Dunkl in [7]. For $y \in \mathbb{R}^d$ the function $x \longmapsto \mathrm{E}_k(x, y)$ can be viewed as the analytic solution on $\mathbb{R}^d$ of the following initial problem

$$\begin{cases} \mathrm{T}_j u(x, y) = y_j u(x, y) & \text{for} \quad 1 \leq j \leq d \\ u(0, y) = 1 & \text{for all} \quad y \in \mathbb{R}^d \end{cases}$$

This kernel has unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

M. Rösler has proved in [11] the following integral representation for the Dunkl kernel

$$\mathrm{E}_k(x, z) = \int_{\mathbb{R}^d} e^{\langle y, z \rangle} d\mu_x(y), \; x \in \mathbb{R}^d, \; z \in \mathbb{C}^d,$$

where μ_x is a probability measure on $\mathbb{R}^d$ with support in the closed ball $\mathrm{B}(0, |x|)$ of center 0 and raduis $|x|$.

Proposition 1.3 *[10]: Let $z, w \in \mathbb{C}^d$ and $\lambda \in \mathbb{C}$. Then*

1. $\mathrm{E}_k(z, 0) = 1$,
2. $\mathrm{E}_k(z, w) = \mathrm{E}_k(w, z)$,
3. $\mathrm{E}_k(\lambda z, w) = \mathrm{E}_k(z, \lambda w)$,
4. *For all $\nu = (\nu_1, ..., \nu_d) \in \mathbb{N}$, $x \in \mathbb{R}^d$, $z \in \mathbb{C}^d$, we have*

$$|\mathrm{D}_z^\nu \mathrm{E}_k(x, z)| \leq |x|^{|\nu|} exp(|x||Re z|),$$

where

$$\mathrm{D}_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} .....\partial z_d^{\nu_d}}; \quad |\nu| = \nu_1 + ... + \nu_d.$$

In particulier

$$|\mathrm{D}_z^\nu \mathrm{E}_k(ix, z)| \leq |x|^{|\nu|},$$

for all $x, z \in \mathbb{R}^d$.

We denote $\mathrm{L}_k^p = \mathrm{L}^p(\mathbb{R}^d, w_k(x)dx)$ with $1 < p \leq 2$ the Banach space of measurable functions on $\mathbb{R}^d$ such that

$$\|f\|_{p,k} = \left(\int_{\mathbb{R}^d} |f(x)|^p w_k(x)dx\right)^{1/p} < +\infty$$

The Dunkl transform is defined for $f \in \mathrm{L}_k^p$ by the formula

$$\mathcal{F}_k(f)(\xi) = c_k^{-1} \int_{\mathbb{R}^d} f(x)\mathrm{E}_k(-i\xi, x)w_k(x)dx,$$

where the constant c_k is given by

$$c_k = \int_{\mathbb{R}^d} e^{-\frac{|z|^2}{2}} w_k(z)dz.$$

The inverse formula

$$f(x) = \int_{\mathbb{R}^d} \mathcal{F}_k(f)(\xi)\mathrm{E}_k(ix, \xi)w_k(\xi)d\xi, \quad x \in \mathbb{R}^d,$$

The Dunkl transform shares several properties with its counterpart in the classical case, we mention here in particular that Parseval Theorem holds in L_k^2.
By Plancherel's theorem and the Marcinkiewicz's interpolation theorem ([12]), we get for $f \in \mathrm{L}_k^p$ with $1 < p \leq 2$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\mathcal{F}_k(f)\|_{q,k} \leq C\|f\|_{p,k}, \tag{1}$$

where C is a positive constant.

The Dunkl Laplacian Δ_k is defined by the formula

$$\Delta_k = \sum_{i=1}^{d} \mathrm{T}_i^2.$$

K. Trimèche has introduced [13] the Dunkl translation operators τ_x, $x \in \mathbb{R}^d$. For $f \in \mathrm{L}_k^p$ and we have

$$\mathcal{F}_k(\tau_x(f))(\xi) = \mathrm{E}_k(ix, \xi)\mathcal{F}_k(f)(\xi),$$

In L_k^p, Consider the generalized spherical mean operator defined by

$$\mathrm{M}_h f(x) = \frac{1}{d_k} \int_{\mathbb{S}^{d-1}} \tau_x(hy)d\eta_k(y), \ x \in \mathbb{R}^d, h > 0.$$

For $\nu \geq -\frac{1}{2}$, we introduce the normalized Bessel functuion j_ν defined by

$$j_\nu(z) = \Gamma(\nu+1)\sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n}}{n!\Gamma(n+\nu+1)},\ z\in\mathbb{C},$$

where Γ is the gamma-function.

Lemma 1.4 *[1] The following inequalities are fulfilled*

1. $|j_\nu(x)| \leq 1$,
2. $|1-j_\nu(x)| \leq |x|$.

Lemma 1.5 *The following inequality is true*

$$|1-j_\nu(x)| \geq c,$$

with $|x|\geq 1$, *where* $c>0$ *is a certain constant.*

Proof. (Analog of lemma 2.9 in [2]) ■

From [3], we have

$$1-j_\alpha(r) \asymp \min(1,r^2) \tag{2}$$

where the symbol $\asymp$ means the left-hand side is bounded above and below by a positive constant times the right hand side.

For $f\in \mathrm{L}^p_k$, we have

$$\mathcal{F}_k(\mathrm{M}_h f)(\xi) = j_{\gamma+\frac{d}{2}-1}(h|\xi|)\mathcal{F}_k(f)(\xi).$$

The first and higher order finite differences of $f(x)$ are defined as follows

$$\mathrm{Z}_h f(x) = (\mathrm{M}_h - \mathrm{I})f(x),$$

where I is the identity operator in L^p_k.

$$\mathrm{Z}^m_h f(x) = \mathrm{Z}_h(\mathrm{Z}^{m-1}_h f(x)) = (\mathrm{M}_h-\mathrm{I})^m f(x) = \sum_{i=0}^{m}(-1)^{m-i}\tbinom{m}{i}\mathrm{M}^i_h f(x),$$

where $\mathrm{M}^0_h f(x) = f(x)$, $\mathrm{M}^i_h f(x) = \mathrm{M}_h(\mathrm{M}^{i-1}_h f(x))$ for $i=1,2,....,m$ and $m=1,2,......$

Let $\mathrm{W}^m_{p,k}$ be the Sobolev space constructed the Dunkl Laplacian Δ_k

$$\mathrm{W}^m_{p,k} = \{f \in \mathrm{L}^p_k,\ \Delta^j_k f \in \mathrm{L}^p_k,\ j = 1, 2, ..., m\}$$

From formula (1), we obtain

$$\int_{\mathbb{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^q |\mathcal{F}_k(f)(\xi)|^q w_k(\xi) d\xi \le C^q \|\mathrm{M}_h f(x) - f(x)\|^q_{p,k} \tag{3}$$

We know that

$$\int_{\mathbb{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^q |\mathcal{F}_k(f)(\xi)|^q w_k(\xi) d\xi = \int_0^\infty s^{2\gamma+d-1} |1 - j_{\gamma+\frac{d}{2}-1}(hs)|^q F^q(s) ds, \tag{4}$$

where

$$F(s) = \left(\int_{\mathrm{S}^{d-1}} |\mathcal{F}_k(f)(sy)|^q w_k(y) dy \right)^{1/q}.$$

For $f \in \mathrm{W}^m_{2,k}$ we have from [9] the following formula

$$\|\mathrm{Z}^m_h \Delta^r_k f(x)\|^2_{2,k} = \int_{\mathbb{R}^d} |\xi|^{4r} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{2m} |\mathcal{F}_k(f)(\xi)|^2 w_k(\xi) d\xi. \tag{5}$$

2 Main result

Theorem 2.1 *Let $f(x) \in \mathrm{L}^p_k$, $1 < p \le 2$, and let*

$$\|\mathrm{M}_h f(x) - f(x)\|_{p,k} = O(h^\delta),\ 0 < \delta \le 1 \ as\ h \to 0.$$

Then $F \in \mathrm{L}^\beta((0,\infty), s^{2\gamma+d-1} ds)$ provided

$$\frac{dp + 2\gamma p}{(p-1)(2\gamma + d) + \delta p} < \beta \le q = \frac{p}{p-1}.$$

Proof. Assume that

$$\|\mathrm{M}_h f(x) - f(x)\|_{p,k} = O(h^\delta),\ 0 < \delta \le 1 \ as\ h \to 0.$$

By formulas (2) and (3), we have

$$\begin{aligned} \int_0^\infty s^{2\gamma+d-1} |1 - j_{\gamma+\frac{d}{2}-1}(hs)|^q F^q(s) ds &\le C^q \|\mathrm{M}_h f(x) - f(x)\|^q_{p,k} \\ &\le C_0 h^{q\delta}, \end{aligned}$$

where C_0 is a positive constant.

From formula (2), we obtain

$$\int_0^{1/h} |sh|^{2q} s^{2\gamma+d-1} F^q(s) ds \leq C_0 h^{q\delta}$$

hence

$$\int_0^{1/h} s^{2q+2\gamma+d-1} F^q(s) ds \leq C_0 h^{q(\delta-2)}$$

We put the fonction ψ defined by

$$\psi(t) = \int_1^t (s^2 F(s))^\beta s^{2\gamma+d-1} ds$$

we write

$$\psi(t) = \int_1^t (s^2 F(s))^\beta s^{(2\gamma+d-1)\frac{\beta}{q}} s^{(2\gamma+d-1)(1-\frac{\beta}{q})} ds$$

Then, if $\beta < q$ by Hölder inequality we obtain

$$\begin{aligned}\psi(t) &\leq \left(\int_1^t s^{2q} F^q(s) s^{2\gamma+d-1} ds\right)^{\frac{\beta}{q}} \left(\int_1^t s^{2\gamma+d-1} ds\right)^{1-\frac{\beta}{q}} \\ &\leq C_0 (t^{2q-\delta q})^{\frac{\beta}{q}} . (1+t^{2\gamma+d})^{1-\frac{\beta}{q}} \\ &= O(t^{2\beta-\delta\beta} . t^{2\gamma+d-\frac{(2\gamma+d)\beta}{q}})\end{aligned}$$

Therefore

$$\psi(t) = O\left(t^{2\beta-\delta\beta+2\gamma+d-\frac{(2\gamma+d)\beta}{q}}\right)$$

Integration by parts yield the identity

$$\begin{aligned}\int_1^t |F(s)|^\beta s^{2\gamma+d-1} ds &= \int_1^t s^{-2\beta} \psi'(s) ds \\ &= s^{-2\beta}\psi(s) + 2\beta \int_1^t s^{-2\beta-1}\psi(s) ds \\ &= O(s^{-2\beta} s^{2\beta-\delta\beta+2\gamma+d-\frac{(2\gamma+d)\beta}{q}}) + O(\int_1^t s^{-2\beta-1} s^{2\beta-\delta\beta+2\gamma+d-\frac{(2\gamma+d)\beta}{q}} ds) \\ &= O(s^{-2\beta+2\beta-\delta\beta+2\gamma+d-\frac{(2\gamma+d)\beta}{q}})\end{aligned}$$

and it follows that $F \in \mathrm{L}^{\beta}((0,\infty), s^{2\gamma+d-1}ds)$ provided $-\delta\beta+2\gamma+d-\frac{(2\gamma+d)\beta}{q}<0$, this ends the proof. $\blacksquare$

For the second result of this paper, we need first to define the Dunkl-Lipschitz class.

Definition 2.2 *Let $\delta \in (0,m)$, $\forall m \in \mathbb{N}$. A function $f \in \mathrm{W}^m_{2,k}$ is said to be in the Dunkl-Lipschitz class, denoted by $Lip_k(\delta,2)$, if*

$$\|\mathrm{Z}^m_h \Delta^r_k f(x)\|^2_{2,k} = O(h^{\delta}) \ as \ h \to 0.$$

Theorem 2.3 *Let $f \in \mathrm{W}^m_{2,k}$. Then the following are equivalents*

1. $f \in Lip_k(\delta,2)$,
2. $\int_{|\xi|\geq s} |\xi|^{4r}|\mathcal{F}_k(f)(\xi)|^2 w_k(\xi)d\xi = O(s^{-2\delta})$ *as* $\delta \to \infty$.

Proof. 1) $\Longrightarrow$ 2): Suppose now that $f \in Lip_k(\delta,2)$. Then

$$\|\mathrm{Z}^m_h \Delta^r_k f(x)\|^2_{2,k} = O(h^{\delta}) \ as \ h \to 0.$$

From (5), we have

$$\|\mathrm{Z}^m_h \Delta^r_k f(x)\|^2_{2,k} = \int_{\mathbb{R}^d} |\xi|^{4r}|1-j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{2m}|\mathcal{F}_k(f)(\xi)|^2 w_k(\xi)d\xi.$$

If $|\xi| \in [\frac{1}{h},\frac{2}{h}]$ then $h|\xi| \geq 1$ and Lemma 1.5 implies that

$$1 \leq \frac{1}{c^{2m}}|1-j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{2m}$$

Then

$$\begin{aligned}
\int_{\frac{1}{h}\leq|\xi|\leq\frac{2}{h}} |\xi|^{4r}|\mathcal{F}_k(f)(\xi)|^2 w_k(\xi)d\xi &\leq \frac{1}{c^{2m}}\int_{\frac{1}{h}\leq|\xi|\leq\frac{2}{h}} |\xi|^{4r}|1-j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{2m}|\mathcal{F}_k(f)(\xi)|^2 w_k(\xi)d\xi \\
&\leq \frac{1}{c^{2m}}\int_{\mathbb{R}^d} |\xi|^{4r}|1-j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{2m}|\mathcal{F}_k(f)(\xi)|^2 w_k(\xi)d\xi \\
&\leq \frac{1}{c^{2m}}\|\mathrm{Z}^m_h \Delta^r_k f(x)\|^2_{2,k}
\end{aligned}$$

Hence

$$\int_{\frac{1}{h}\leq|\xi|\leq\frac{2}{h}} |\xi|^{4r}|\mathcal{F}_k(f)(\xi)|^2 w_k(\xi)d\xi = O(h^{2\delta}) \ as \ h \to 0$$

It follows that

$$\int_{s\leq|\xi|\leq 2s}|\xi|^{4r}|\mathcal{F}_k(f)(\xi)|^2w_k(\xi)d\xi \leq Ks^{-2\delta}\ as\ s\to\infty,$$

where $K>0$ is some constant.

Furthermore, we have

$$\begin{aligned}\int_{|\xi|\geq s}|\xi|^{4r}|\mathcal{F}_k(f)(\xi)|^2w_k(\xi)d\xi &= \sum_{j=0}^{\infty}\int_{2^js\leq|\xi|\leq 2^{j+1}s}|\xi|^{4r}|\mathcal{F}_k(f)(\xi)|^2w_k(\xi)d\xi\\ &\leq C\sum_{j=0}^{\infty}(2^js)^{-2\delta}\\ &\leq Cs^{-2\delta}.\end{aligned}$$

This proves that

$$\int_{|\xi|\geq s}|\xi|^{4r}|\mathcal{F}_k(f)(\xi)|^2w_k(\xi)d\xi = O(s^{-2\delta})\ as\ s\to\infty$$

2) $\Longrightarrow$ 1): Asume that

$$\int_{|\xi|\geq s}|\xi|^{4r}|\mathcal{F}_k(f)(\xi)|^2w_k(\xi)d\xi = O(s^{-2\delta})\ as\ s\to\infty$$

We have to show that

$$\int_0^{\infty}t^{4r+2\gamma+d-1}|1-j_{\gamma+\frac{d}{2}-1}(ht)|^{2m}\phi(t)dt = O(h^{2\delta})\ as\ h\to 0,$$

where

$$\phi(t)=\int_{\mathrm{S}^{d-1}}|\mathcal{F}_k(f)(ty)|^2w_k(y)dy.$$

We write

$$\int_0^{\infty}t^{4r+2\gamma+d-1}|1-j_{\gamma+\frac{d}{2}-1}(ht)|^{2m}\phi(t)dt = \mathrm{I}_1+\mathrm{I}_2$$

with

$$\mathrm{I}_1=\int_0^{1/h}t^{4r+2\gamma+d-1}|1-j_{\gamma+\frac{d}{2}-1}(ht)|^{2m}\phi(t)dt$$

and

$$\mathrm{I}_2 = \int_{1/h}^{\infty} t^{4r+2\gamma+d-1}|1 - j_{\gamma+\frac{d}{2}-1}(ht)|^{2m}\phi(t)dt$$

Firstly, from (1) of Lemma 1.4 we see that

$$\mathrm{I}_2 \leq 4^m \int_{1/h}^{\infty} t^{4r+2\gamma+d-1}\phi(t)dt = O(h^{2\delta}) \ as\ h \to 0$$

Let

$$g(t) = \int_t^{\infty} s^{4r+2\gamma+d-1}\phi(s)ds$$

We use integration by parts and formula (2) of Lemma 1.4

$$\begin{aligned}
\mathrm{I}_1 &= \int_0^{1/h} t^{4r+2\gamma+d-1}|1 - j_{\gamma+\frac{d}{2}-1}(ht)|^{2m}|\phi(t)dt \\
&\leq -h^{2m} \int_0^{1/h} t^{2m} g'(t)dt \\
&\leq -g(\frac{1}{h}) + 2mh^{2m} \int_0^{1/h} t^{2m-1} g(t)dt \\
&\leq Ch^{2m} \int_0^{1/h} t^{2m-1} t^{-2\delta} dt \\
&\leq Ch^{2m} \int_0^{1/h} t^{2m-2\delta-1} dt \\
&\leq Ch^{2\delta}.
\end{aligned}$$

which completes the proof of this theorem. ■

Corollary 2.4 *Let $f \in \mathrm{W}_{2,k}^m$, and let $f \in Lip_k(\delta, 2)$. Then*

$$\int_{|\xi| \geq s} |\mathcal{F}_k(f)(\xi)|^2 w_k(\xi) d\xi = O(s^{-4r-2\delta}) \ as\ s \to \infty.$$

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