Closed Pesudo-Riemannian gradient Ricci solitons

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In this paper, we study gradient Ricci soliton structures on compact indecomposable Lorentzian 3-manifolds that admit a parallel light-like vector field with closed orbits. There are examples of non-trivial, i.e., non-Einstein and non-gradient steady Ricci solitons on these compact Lorentzian manifolds that show the difference between closed Riemannian and pseudo-Riemannian Ricci solions. We give a necessary condition for the existence of gradient Ricci solitons on these manifolds and show the cases that gradient Ricci solitons are necessarily trivial. In particular, we indicate on a difference between the behavior of gradient Ricci solitons with non-gradient Ricci solitons on these compact manifolds. Furthermore, we show closed gradient Ricci solitons with the structure of warped products when the base is conformal to an n-dimensional, $n \geq 3$, pseudo-Euclidean torus, which are invariant under the action of an $(n-1)$-dimensional discrete translation group, are necessarily trivial. Also, we see that there is not any non-trivial closed gradient Ricci soliton with the structure of warped product Lorentzian metric.

I. INTRODUCTION

Let (M, g) be a pseudo-Riemannian manifold and X be a smooth vector field on M. We say that the triple (M, g, X) is a pseudo-Riemannian Ricci soliton if the following equation is satisfied

$$L_X(g) + Ric(g) = \lambda g, \tag{I.1}$$

where L_X is the Lie-derivative with respect to X, Ric is the Ricci tensor and λ is a real number. A Ricci soliton is called shrinking, steady or expanding according to whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. If for a smooth function f on a manifold (M, g), $X = 1/2\nabla f$, where ∇f is the gradient of f, then equation (I.1) leads to

$$Hess_f(g) + Ric(g) = \lambda g, \tag{I.2}$$

where $Hess_f$ denotes the Hessian of the function f. In this case, the soliton is called the gradient Ricci soliton and f is called the potential function. Let (M, g, f) be a gradient Ricci soliton.

Ricci solitons are natural generalizations of Einstein manifolds. If X is a Killing vector field in the Ricci soliton equation (I.1) or f is a constant function in the gradient Ricci soliton equation (I.2), then we obtain the Einstein equation $Ric(g) = \lambda g$ and the soliton is an Einstein manifold or in the steady case, it is a Ricci flat manifold. The concept of Ricci solitons was first introduced in [9] by Hamilton as a self-similar solution of Hamilton's Ricci flow, $\partial_t g(t) = -2Ric(g(t))$, on Riemannian metrics. Ricci flow is an evolutionary intrinsic geometric flow introduced by Hamilton on Riemannian metrics in 1982 in order to study the topology of three-dimensional manifolds [10].

A soliton for the Ricci flow is a metric that changes only by rescaling and pullback by a one-parameter family of diffeomorphisms as it evolves under the Ricci flow. If we have a Riemannian

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or pseudo-Riemannian metric g, a complete vector field X and a real number λ (all independent of time) that satisfy the Ricci soliton equation (I.1), then $g(t) = \sigma(t)\phi_t^*(g)$, is a solution of the Ricci flow, where $\sigma(t) := 1 - 2\lambda t$ and ϕ_t is a family of diffeomorphisms generated by the t-dependent vector field $\sigma(t)^{-1}X$. See [8] for more details.

Geometry of Riemannian Ricci solitons has been studied widely because of the role of the Ricci flow in solving the Poincaré conjecture and Thurston's geometrization conjecture that were finally proved by Perelman [14]. Ricci solitons often arise as limits of dilations of singularities in the Ricci flow [4]. Recently, the geometric structure of Ricci solitons in pseudo-Riemannian setting have been investigated by various authors. For some recent results and further references on pseudo-Riemannian Ricci solitons, we may refer to [3] and references therein. Also, solutions of Euclidean signature Einstein gravity coupled to a free massless scalar field with nonzero cosmological constant are associated to shrinking or expanding Ricci solitons [1].

By the work of Perelman, we know that closed Riemannian Ricci solitons are necessarily gradient and, moreover, closed expanding or steady Ricci solitons are necessarily Einstein [6], which is derived from maximum principles for the Laplace operator that is an elliptic operator on Riemannian manifolds. In [11], we show that there are examples of non-trivial and non-gradient steady Ricci solitons of compact indecomposable Lorentzian 3-manifolds admitting a parallel light-like vector field with closed orbits. These examples that are steady Ricci solitons with zero scalar curvature show the difference between closed Riemannian and Lorentzian Ricci solitons.

The aim of this paper is the investigation of gradient Ricci soliton structures on closed and orientable pseudo-Riemannian manifolds. For this purpose, after a preliminary in Section 2, we study gradient Ricci soliton equation on compact indecomposable Lorentzian 3-manifolds that admitting a parallel light-like vector field with closed orbits in Section 3. We give a necessary condition for the existence of non-trivial solutions and show the cases that solutions are necessarily trivial. Also, we emphasize on a difference between the behavior of gradient Ricci solitons and non-gradient Ricci solitons on these manifolds. Finally, in Section 4, we consider warped product pseudo-Riemannian Ricci solitons. We show there is not any non-trivial closed gradient Ricci soliton with the structure of warped product when the base is conformal to an n-dimensional, $n \geq 3$, pseudo-Euclidean torus that is invariant under the action of an $(n-1)$-dimensional discrete translation group. Also, we prove there is not any non-trivial closed gradient Ricci solitons with respect to the Lorentzian warped product $-dt^2 + f^2(t)g_F$ where g_F is a Riemannian metric.

II. PRELIMINARIES

Let (M, g, X) be an n-dimensional Ricci soliton. Then by tracing equation (I.1), we get

$$Div(X) + \tau = n\lambda, \tag{II.1}$$

where τ is the scalar curvature and $Div(X)$ is the divergence of the vector field X. If the manifold M is closed and orientable then by the Divergence Theorem, we have

$$\int_M \tau \, d\mu_g = n\lambda \, vol(M). \tag{II.2}$$

So $\lambda = n^{-1}r$ where $r = vol(M)^{-1} \int_M \tau \, d\mu_g$ is the mean scalar curvature.

Proposition II.1. *Let (M, g, X) be a closed pseudo-Riemannian Ricci soliton with constant scalar curvature. Then $\tau = n\lambda$ and $Div(X) = 0$. In particular, in the steady case, the scalar curvature is zero.*

Proof. If (M, g, X) is a closed pseudo-Riemannian Ricci soliton with constant scalar curvature then by (II.2) we have $\tau\, vol(M) = n\lambda\, vol(M)$. Therefore, $\tau = n\lambda$. Thus, equation (II.1) implies that $Div(X) = 0$. □

Remark II.2. Let (M, g, f) be a gradient Ricci soliton. Then by tracing equation (I.2), we have

$$\Delta_g f = n\lambda - \tau, \tag{II.3}$$

where Δ_g is the Laplace-Beltrami operator with respect to the pseudo-Rieman-nian metric g that is an ultrahyperbolic operator in pseudo-Riemannian cases or normally hyperbolic operator when the metric is given in the Lorentzian signature. Since solutions of the Laplace equation Δ_g on a closed Riemannian manifold are necessarily trivial, i.e., constant functions, then we have no non-trivial closed Riemannian Ricci solitons with constant scalar curvature. Whereas, in pseudo-Riemannian cases, the Laplace equation may have non-trivial solutions. Therefore, we may find non-trivial examples of closed pseudo-Riemannian gradient Ricci solitons with constant scalar curvature.

In the following, we give an example of a 3-dimensional Lorentzian torus with zero scalar curvature that Ricci soliton equation on it necessarily has the trivial solution. Although, the wave equation $\Delta_g f = 0$ has solutions other than constants.

Examle II.3. Consider the metric $g = 2\Lambda\, dx\, dz + L^2(z)\, dy^2$ in local coordinates (x, y, z) on a 3-dimensional torus $\mathbb{T}^3$, where Λ is a non-zero real number and $L(z)$ is a periodic function with period T that is strictly positive or negative, because the metric is non-degenerate at any point on $\mathbb{T}^3$. This metric defines a Lorentzian metric on $\mathbb{T}^3$ that admits a parallel light-like vector field with non-closed leaves [2, Proposition 3.1]. If L is a constant function then $(\mathbb{T}^3, g)$ is a flat manifold. Since the scalar curvature is zero then by Proposition (II.1), Ricci solitons can only appear in the steady case. We suppose that $f(x, y, z)$ be a smooth function on $(\mathbb{T}^3, g)$ that is periodic with respect to its variables. Then the gradient Ricci soliton equation in the steady case shows that f is a potential function if and only if $f = f(z)$ is a smooth periodic function that satisfies the following equation $f'' = L^{-1}L''$. Therefore, for a smooth periodic function $L(z)$ with period T,

$$\begin{aligned} f'(z) &= \int \frac{L''}{L}\, dz + C \\ &= h(z) + \Big(\frac{1}{T}\int_0^T \frac{L''}{L}\, dz\Big) z + C, \end{aligned} \tag{II.4}$$

where,

$$h(z) = \int_0^z L^{-1}L''\, dt - T^{-1}\Big(\frac{1}{T}\int_0^T \frac{L''}{L}\, dz\Big) z, \tag{II.5}$$

is a periodic function. See [5] for integrals of periodic functions. Thus, f is periodic if and only if $\int_0^T L^{-1}L''\, dz = 0$. But, by partial integration,

$$\int_0^T \frac{L''}{L}\, dz = \frac{L'}{L}(T) - \frac{L'}{L}(0) + \int_0^T \Big(\frac{L'}{L}\Big)^2 dz = \int_0^T \Big(\frac{L'}{L}\Big)^2 dz, \tag{II.6}$$

because $L^{-1}L'$ is periodic with period T. Therefore, f' is a periodic function if and only if L is a constant function. Hence, f is a constant function and the soliton is a flat manifold.

But, since $\Delta_g f = 2\Lambda^{-1}\partial_{xz} f + \Lambda^{-1}L^{-1}L'\partial_x f + L^{-2}\partial_{yy} f$ then all periodic functions $f = f(z)$ are solutions of the Laplace-Beltrami equation on T^3 with respect to the metric g. On the other hand, for a smooth funtion f on (T^3, g) we have $\|\nabla f\|^2 = 2\Lambda^{-1} f_x f_z + L^{-2} f_y^2$. Therefore, if $f = f(z)$ then $\|\nabla f\|^2 = 0$.

In the following, we see that for a closed pseudo-Riemannian gradient Ricci soliton with constant scalar curvature in the steady case, the associated gradient vector field is a null vector field. Whereas, the associated potential vector field with a non-gradient closed pseudo-Riemannian steady Ricci soliton with constant scalar curvature is not necessarily a null vector field. See [11].

Lemma II.4. *Let (M, g, f) be a gradient Ricci soliton with constant scalar curvature. Then*

$$\|\nabla f\|^2 - 2\lambda f = \text{const.} \tag{II.7}$$

Furthermore, in the steady case $\|\nabla f\|^2$ is constant. Whereas, in a non-steady case, If $\|\nabla f\|^2$ is constant then the Ricci soliton is an Einstein manifold.

Proof. See [3, Lemma 11.14]. □

Remark II.5. In a non-steady case, by replacing the potential function f with $f - c(2\lambda)^{-1}$ for a constant c in equation (II.7), we have $\|\nabla f\|^2 - 2\lambda f = 0$. A gradient Ricci soliton with such a potential function is simply called a gradient Ricci soliton with the normalized potential function.

Lemma II.6. *Let (M, g, f) be a closed gradient Ricci soliton with constant scalar curvature. Then*

$$\int_M \|\nabla f\|^2 \, d\mu_g = 0. \tag{II.8}$$

Proof. For an arbitrary function f on a pseudo-Riemannian manifold (M, g), we have

$$\Delta_g f^2 = 2f\Delta_g f + \|\nabla f\|^2. \tag{II.9}$$

See [14, p.94]. Now, we suppose that f be a potential function that defines a gradient Ricci soliton on a closed manifold with constant scalar curvature. Then, by proposition (II.1), $\Delta_g f = 0$. Thus, by integration of (II.9) over (M, g), we have $\int_M \|\nabla f\|^2 \, d\mu_g = \int_M \Delta_g f^2 \, d\mu_g$. But, $\int_M \Delta_g f^2 \, d\mu_g$ is zero because of the divergence theorem. □

Corollary II.7. *If (M, g, f) is a closed steady gradient Ricci soliton with constant scalar curvature, then the gradient vector field is a null vector field. Furthermore, in this case, the Ricci operator is 2 or 3 step nilpotent operator.*

Proof. According to Lemma (II.4), $\|\nabla f\|^2$ is constant. Thus, (II.8) is satisfied if and only if $\|\nabla f\|^2 = 0$. For proving the second part, we refer to [3, Theorem 11.22]. The proof is independent of the assumption that the manifold is locally homogeneous. □

Corollary II.8. *If (M, g, f) is a non-trivial and non-steady closed gradient Ricci soliton with constant scalar curvature, then the associated gradient vector field is not null, timelike or spacelike. Furthermore, if f is the normalized potential function, then $\int_M f d\mu_g = 0$.*

Proof. Let (M, g, f) be a non-trivial and non-steady gradient Ricci soliton. Therefore, by lemma (II.4), $\|\nabla f\|^2$ is not constant. Hence, ∇f is not a null vector field. Also, by lemma (II.6), $\|\nabla f\|^2$ cannot be strictly positive or negative. Therefore, ∇f is not a timelike or spacelike vector field. Now, if f is the normalized potential function, Remark (II.5), then $f = (2\lambda)^{-1}\|\nabla f\|^2$. Hence, $\int_M f \, d\mu_g = (2\lambda)^{-1} \int_M \|\nabla f\|^2 \, \mu_g$. Thus, by lemma (II.6), $\int_M f \, d\mu_g = 0$. □

Lemma II.9. *If (M, g, f) is an n-dimensional closed gradient Ricci soliton with constant scalar curvature, then $\|Hess_f\|^2 = 0$.*

Proof. By [3, Lemma 11.14], for an n-dimensional pseudo-Riemannian gradient Ricci soliton with constant scalar curvature we have $\|Hess_f\|^2 = \lambda(n\lambda - \tau)$. Therefore, if $\lambda = 0$ then $\|Hess_f\|^2 = 0$. Also, if (M, g, f) is an n-dimensional closed pseudo-Riemannian gradient Ricci soliton with constant scalar curvature in a non-steady case then by proposition (II.1), $\tau = n\lambda$. Hence, in this case, $\|Hess_f\|^2 = 0$. □

III. GRADIENT RICCI SOLITNS ON COMPACT INDECOMPOSABLE LORENTZIAN 3-MANIFOLDS THAT ADMITTING A PARALLEL LIGHT-LIKE VECTOR FIELD WITH CLOSED ORBITS

For any $n \in \mathbb{N}$, let Γ_n be the group of diffeomorphisms of $\mathbb{R}^3$ generated by the maps $\tau_x(x,y,z) = (x+1,y,z)$, $\tau_y(x,y,z) = (x,y+1,z)$ and $\tau_{z,n}(x,y,z) = (x+ny,y,z+1)$ that preserve the moving frame $(\partial_x, \partial_y + nz\partial_x, \partial_z)$. See [2] for more details.

Proposition III.1. *If (M,g) is an orientable, compact indecomposable Lorentzian 3-manifold endowed with a parallel light-like vector field with closed orbits, then (M,g) is isometric to $(\mathbb{R}^3/\Gamma_n, g)$ where g is the metric induced by a metric $\tilde{g}$ on $\mathbb{R}^3$ whose matrix in the Γ_n-invariant moving frame $(\partial_x, \partial_y + nz\partial_x + \theta\partial_z, \partial_z)$, θ is a real number, is*

$$\begin{pmatrix} 0 & 0 & \Lambda \\ 0 & L^2(y,z) & \nu(y,z) \\ \Lambda & \nu(y,z) & \mu(y,z) \end{pmatrix}, \tag{III.1}$$

where L, μ and ν are $(1,1)$-biperiodic functions and Λ is a non-zero real number. Also, L is a non-vanishing function because of the non-degeneracy of the metric g at any point on the manifold. $\mathbb{R}^3/\Gamma_n$ is called parabolic torus.

Proof. See [2, Section 4]. □

Proposition III.2. *If $(\mathbb{R}^3/\Gamma_n, g, f)$ is a gradient Ricci soliton, where $(\mathbb{R}^3/\Gamma_n, g)$ is defined in proposition (III.1), then $f = f(z - \theta y)$ is a periodic function with period 1 and θ is an integer number that satisfies the following partial differential equation*

$$Lf_{zz} = \partial_y H + \theta\partial_z H + L_{zz}, \tag{III.2}$$

where,

$$H(y,z) := \frac{n\Lambda}{L} + \frac{\theta}{2}\frac{\partial_z \mu}{L} + \frac{1}{2}\frac{\partial_y \mu}{L} - \frac{\partial_z \nu}{L}. \tag{III.3}$$

Proof. Let f be a smooth real-valued function on $\mathbb{R}^3/\Gamma_n$ that defines a gradient Ricci soliton with respect to the metric g defined by proposition (III.1). Since the scalar curvature is zero then by proposition (II.1) Ricci solitons can only appear in the steady case. Thus, by considering the gradient Ricci soliton equation in the steady case we get $\partial_{xx} f = 0$ that implies $f = f_1(y,z)x + f_2(y,z)$. But, f is Γ_n-invariant. Hence, $f = f(y,z)$ is a $(1,1)$-biperiodic function. On the other hand, by corollary (II.7), ∇f is a null vector field. Therefore, $\|\nabla f\|^2 = L^{-2}(\theta\partial_z f + \partial_y f) = 0$ that implies $f = f(z - \theta y)$. Since $f(y,z)$ is a $(1,1)$-biperiodic function then $f = f(z - \theta y)$ is periodic with period 1 and θ is an integer number. Finally, we get equations (III.2) and (III.3) by gradient Ricci soliton equation. □

By integrating of equation (III.2) with respect to z we have

$$\theta H + \partial_z L = \int L\partial_{zz} f \, dz - \int \partial_y H \, dz + G(z). \tag{III.4}$$

Since the left-hand side of equation (III.4) is $(1,1)$-biperiodic then $G(z)$ is periodic and

$$\int_0^1 L\partial_{zz} f \, dz - \int_0^1 \partial_y H \, dz = 0, \tag{III.5}$$

see [5] for integrals of periodic functions, that implies

$$\int_0^1\int_0^1 L\partial_{zz}f\,dz\,dy-\int_0^1\int_0^1\partial_y H\,dz\,dy=0. \tag{III.6}$$

But,

$$\int_0^1\int_0^1\partial_y H\,dz\,dy=\int_0^1\int_0^1\partial_y H\,dy\,dz=\int_0^1\big[H(1,z)-H(0,z)\big]\,dz=0, \tag{III.7}$$

because, $H(y,z)$ is a $(1,1)$-biperiodic function. Thus,

$$\int_0^1\int_0^1 L\partial_{zz}f\,dz\,dy=0. \tag{III.8}$$

Equation (III.8) is a necessary but not sufficient condition for the existence of a gradient Ricci soliton on the closed Lorentzian manifold $(\mathrm{R}^3/\Gamma_n,g)$. If L is constant, $L=L(y)$ is a periodic function, or $L=L(z)$ is a periodic function then equation (III.8) is satisfied. But, in the following, we see that in none of these cases for arbitrary biperiodic functions $\nu(y,z)$ and $\mu(y,z)$, there are not any non-trivial gradient Ricci solitons.

By equation (III.2), we have an inhomogeneous transport equation

$$\partial_y H+\theta\partial_z H=Lf''(z-\theta y)-\partial_{zz}L. \tag{III.9}$$

If we consider equation (III.9) with the initial condition $H(0,z)=h(z)$, then by the method of characteristics we get

$$\begin{aligned}H(y,z)&=h(z-\theta y)+f_{zz}(z-\theta y)\int_0^y L(s,z+(s-y)\theta)\,ds\\&\quad+\int_0^y L_{zz}(s,z+(s-y)\theta)\,ds.\end{aligned} \tag{III.10}$$

See [7]. Since H is a $(1,1)$-biperiodic function then h is a periodic function with period 1 and

$$f_{zz}(z-\theta y)\int_0^1 L(z+(s-y)\theta,s)\,ds+\int_0^1 L_{zz}(z+(s-y)\theta,s)\,ds=0. \tag{III.11}$$

Let $L\equiv 1$. Then by integral equation (III.10), $H(y,z)=h(z-\theta y)+f''(z-\theta y)y$. Hence, H cannot be a biperiodic function unless $f''(z-\theta y)=0$ that implies f be a constant function.

Now, we suppose that $L=L(y)$ be a periodic function. Then by integral equation (III.10),

$$\begin{aligned}H(y,z)&=h(z-\theta y)+f''(z-\theta y)\int_0^y L(s)\,ds\\&=h(z-\theta y)+f''(z-\theta y)\Big(K(y)+\Big(\int_0^1 L(s)\,ds\Big)y+C\Big),\end{aligned} \tag{III.12}$$

where $K(y)$ is a periodic function. Hence, $H(y,z)$ is a biperiodic function if and only if $\int_0^1 L(s)ds=0$. Since $L(y)$ is a non-vanishing function then $\int_0^1 L(y)\,dy$ is a non-zero real number. Thus, H is not a biperiodic function. In the following we suppose that $L = L(z)$ be a periodic function. Then by integral equation (III.10),

$$\begin{aligned} H(y,z) &= h(z-\theta y) + f''(z-\theta y)\int_0^y L(z+(s-y)\theta)\,ds \\ &+ \int_0^y L''(z+(s-y)\theta)\,ds \\ &= h(z-\theta y) + f''(z-\theta y)\int_0^y L(z+(s-y)\theta)\,ds \\ &+ \theta^{-1}L'(z) - \theta^{-1}L'(z-\theta y). \end{aligned} \tag{III.13}$$

Thus, H is a biperiodic function if and only if

$$f''(z-\theta y)\int_0^1 L(z+(s-y)\theta)\,ds = 0. \tag{III.14}$$

Since L is a non-vanishing function then $f''(z-\theta y) = 0$ that implies f be a constant function. Therefore, we get the following theorem

Theorem III.3. *If a periodic function $f = f(z-\theta y)$ defines a gradient Ricci soliton on the parabolic torus $(\mathrm{R}^3/\Gamma_n, g)$, defined in proposition (III.1), then*

$$\int_0^1\int_0^1 f''(z-\theta y)L(y,z)\,dy\,dz = \int_0^1\int_0^1 f(z-\theta y)\partial_{zz}L(y,z)\,dy\,dz = 0. \tag{III.15}$$

Furthermore, for arbitrary biperiodic functions $\nu(y,z)$ and $\mu(y,z)$, there is not any non-trivial gradient Ricci soliton when L is constant, $L = L(y)$ is a periodic function or $L = L(z)$ is a periodic function. In particular, there is not any non-trivial gradient Ricci soliton when $L = L(z-\theta y)$, $\nu = \nu(z-\theta y)$, and $\mu = \mu(z-\theta y)$ are arbitrary periodic functions.

Proof. We can get equation (III.15) from (III.10) by partial integration. Also, we saw that there is not any non-trivial gradient Ricci soliton when L is constant, $L = L(y)$ is a periodic function, or $L = L(z)$ is a periodic function by considering the initial value problem for the inhomogeneous transport equation

$$\partial_y H + \theta\partial_z H = Lf''(z-\theta y) - \partial_{zz}L, \;\; H(0,z) = h(z).$$

Now, we suppose that $L = L(z-\theta y)$, $\nu = \nu(z-\theta y)$, and $\mu = \mu(z-\theta y)$ are periodic functions. Then equation (III.2) leads to $L(z-\theta y)f''(z-\theta y) = L''(z-\theta y)$. Therefore, $f'(z-\theta y) = \int L^{-1}L''(z-\theta y)dz$. But, $f'(z-\theta y)$ is periodic if and only if $\int_0^1 (L^{-1}L'')(z-\theta y)\,dz = 0$. By partial integration

$$\int_0^1 \frac{L''}{L}(z-\theta y)\,dz = \int_0^1 (\frac{L'}{L})^2(z-\theta y)\,dz, \tag{III.16}$$

that is a non-negative function. Thus, $\int_0^1 (L^{-1}L'')(z-\theta y)\,dz = 0$ if and only if L is a constant function that implies f be constant. □

Remark III.4. If $\theta = 0$ then by proposition (III.2), we have a gradient Ricci soliton on $(\mathrm{R}^3/\Gamma_n, g)$ if there exist a periodic function $f = f(z)$ that satisfies the following partial differential equation

$$Lf''(z) = \partial_y H + \partial_{zz}L, \tag{III.17}$$

where,

$$H(y,z)=\frac{n\Lambda}{L}+\frac{\partial_y\mu}{2L}-\frac{\partial_z\nu}{L}. \tag{III.18}$$

Also, by integrating of equation (III.17) with respect to y we have $H=\int\partial_{zz}L\,dy-f''(z)\int L\,dy+G(z)$. Since H is a biperiodic function then $G(z)$ is a periodic function and $\int_0^1\partial_{zz}L\,dy-f''(z)\int_0^1 L\,dy=0$. Therefore,

$$f''(z)=\frac{\int_0^1\partial_{zz}L\,dy}{\int_0^1 L\,dy}. \tag{III.19}$$

Thus, if L is a constant function or $L=L(y)$ is a periodic function then $f''(z)=0$ that implies f be a constant function. Also, if $L=L(z)$ is a periodic function then $f''(z)=L^{-1}L''(z)$. But, as we saw in example (II.3), there are not any periodic functions other than constanst that satisfy in this equation.

In the following, we see that for an arbitrary biperiodic function $L(y,z)$ if ν and μ are constant functions then there is not any non-trivial gradient Ricci soliton.

Theorem III.5. *If we consider the metric g defined in proposition (III.1) on the parabolic torus $\mathbb{R}^3/\Gamma_n$ when $L=L(y,z)$ is an arbitrary $(1,1)$-biperiodic function, ν and μ are constant functions then there are not any non-trivial gradient Ricci solitons.*

Proof. Let $\nu\equiv\mu\equiv 1$. Then by proposition (III.2), we have a gradient Ricci soliton equation if and only if for a periodic function $f(z-\theta y)$, the following partial differential equation is satisfied

$$\partial_{zz}f=\frac{-n\Lambda}{L^3}(\partial_yL+\theta\partial_zL)+\frac{\partial_{zz}L}{L}. \tag{III.20}$$

By integrating of equation (III.20) with respect to z we have

$$\partial_zf=\int\frac{-n\Lambda}{L^3}(\partial_yL+\theta\partial_zL)\,dz+\int\frac{\partial_{zz}L}{L}\,dz+F(y). \tag{III.21}$$

Since ∂_zf is $(1,1)$-biperiodic then

$$\int_0^1\frac{-n\Lambda}{L^3}(\partial_yL+\theta\partial_zL)\,dz+\int_0^1\frac{\partial_{zz}L}{L}\,dz=0, \tag{III.22}$$

that implies

$$\int_0^1\int_0^1\frac{-n\Lambda}{L^3}(\partial_yL+\theta\partial_zL)\,dz\,dy+\int_0^1\int_0^1\frac{\partial_{zz}L}{L}\,dz\,dy=0. \tag{III.23}$$

But,

$$\int_0^1\int_0^1\frac{-n\Lambda}{L^3}(\partial_yL+\theta\partial_zL)\,dz\,dy=\frac{n\Lambda}{2}\Big(\int_0^1\Big[\frac{1}{L^2}(1,z)-\frac{1}{L^2}(0,z)\Big]\,dz+\int_0^1\Big[\frac{1}{L^2}(y,1)-\frac{1}{L^2}(y,0)\Big]\,dy\Big). \tag{III.24}$$

Since L^{-2} is a $(1,1)$-biperiodic function then equation (III.23) is satisfied if and only if

$$\int_0^1\int_0^1\frac{\partial_{zz}L}{L}\,dz\,dy=0. \tag{III.25}$$

But, by partial integrattion

$$\int_0^1 \frac{\partial_{zz}L}{L}\,dz = \int_0^1 \left(\frac{\partial_z L}{L}\right)^2 dz, \tag{III.26}$$

that is a non-negative function. Thus, equation (III.23) is satisfied if and only if $\partial_z L = 0$ that implies $L = L(y)$ be a periodic function of y. Now, by equation (III.20), $f''(z-\theta y) = -n\Lambda L^{-3}L'$ that shows f'' and consequently f are not periodic functions of $\xi = z - \theta y$. Hence, L is constant that implies f be a constant function. □

Question III.6. Is there any compact Lorentzian 3-manifold that admitting a parallel light-like vector field with the structure of gradient Ricci soliton, or gradient Ricci solitons on these compact Lorentzian 3-manifolds are necessarily Einstein manifolds?

A. A difference between the behavior of gradient and non-gradient Ricci solitons

In [11], we construct examples of non-gradient Ricci solitons on R^3/Γ_n with respect to the metric g, defined in proposition (III.1), when θ is zero and $L = L(z)$ is a periodic function while $\nu \equiv \mu \equiv 0$. In the following, we see that if θ is a non-zero real number then there is not any non-gradient Ricci soliton when $L = L(z)$ is a periodic function and $\nu \equiv \mu \equiv 0$. While, in gradient cases, as we see before, when $L = L(z)$ is a periodic function, $\nu(y,z)$ and $\mu(y,z)$ are arbitrary biperiodic functions, we have the same result about the existence of solutions in both cases when θ is zero, or θ is a non-zero real number.

Theorem III.7. *There is not any non-trivial and non-gradient Ricci soliton on parabolic torus R^3/Γ_n with respect to the metric g defined in proposition (III.1) when $L = L(z)$ is a periodic function, $\nu \equiv \mu \equiv 0$ and $\theta \neq 0$.*

Proof. Let $X = (X_1, X_2, X_3)$ be a vector field on R^3/Γ_n that defines a Ricci soliton structure with respect to the metric g. Then by Ricci soliton equation (I.1) we have the following system of partial differential equations

$$\begin{aligned}
&2\Lambda\partial_x X_3 = 0, && \text{(III.27a)}\\
&\Lambda(\partial_x X_1 + \partial_z X_3) = 0, && \text{(III.27b)}\\
&L^2(z)\partial_x X_2 + \Lambda(nz\partial_x X_3 + \partial_y X_3 + \theta\partial_z X_3) = 0, && \text{(III.27c)}\\
&nzL(z)\partial_x X_2 + \theta\partial_z(L(z)X_2) + \partial_y(L(z)X_2) + L'(z)X_3 = 0, && \text{(III.27d)}\\
&L^2(z)\partial_z X_2 + \Lambda(nz\partial_x X_1 + \partial_y X_1 + \theta\partial_z X_1 - nX_3) = 0, && \text{(III.27e)}\\
&\frac{n\theta\Lambda L'(z) + L^2(z)L''(z)}{L^3(z)} + 2\Lambda(\partial_z X_1 + nX_2) = 0. && \text{(III.27f)}
\end{aligned}$$

By equation (III.27a), $X_3 = X_3(y,z)$ is a biperiodic function. If we consider the partial derivative of equation (III.27b) with respect x then $\partial_{xx}X_1 = 0$ that implies $X_1 = F_2(y,z)x + F_3(y,z)$. But, X_1 is Γ_n-invariant. Therefore, $X_1 = X_1(y,z)$ is a biperiodic function. Hence, equation (III.27b) implies that $X_3 = X_3(y)$ is a periodic function. Now, by differentiating of equation (III.27c) with respect to x, $\partial_{xx}X_2 = -L^{-2}(z)\Lambda X_3'(y)$. But, since X_2 is Γ_n-invariant then $X_2 = X_2(y,z)$ is a biperiodic function. Therefore, equation (III.27c) implies that X_3 is a constant function. Thus, the vector field $X = (X_1, X_2, X_3)$ defines a Ricci soliton if $X_3 = c$ is a constant function, $X_2 = X_2(y,z)$ and

$X_3 = X_3(y,z)$ are biperiodic functions that satisfy the following partial differential equations

$$\theta\partial_z(L(z)X_2) + \partial_y(L(z)X_2) + cL'(z) = 0, \tag{III.28}$$

$$L^2(z)\partial_z X_2 + \Lambda(\partial_y X_1 + \theta\partial_z X_1 - nc) = 0, \tag{III.29}$$

$$\frac{n\theta\Lambda L'(z) + L^2(z)L''(z)}{L^3(z)} + 2\Lambda(\partial_z X_1 + nX_2) = 0. \tag{III.30}$$

By linear inhomogeneous transport equation (III.28) and the method of characteristic, we get $X_2(y,z) = L^{-1}(z)F(z-\theta y) - c\theta^{-1}$. Now, by substituting X_2 in equation (III.29), we have the following linear inhomogeneous transport equation

$$\Lambda(\partial_y X_1 + \theta\partial_z X_1 - nc) + L(z)F'(z-\theta y) - L'(z)F(z-\theta y) = 0. \tag{III.31}$$

Now, by the characteristic method for the inhomogeneous transport equation (III.31)

$$\begin{aligned} X_1(y,z) = {} & ncy + F_1(z-\theta y) + \Lambda^{-1}L(z)F(z-\theta y) \\ & - \Lambda^{-1}F'(z-\theta y)\int_0^y L(z+(s-y)\theta)\,ds. \end{aligned} \tag{III.32}$$

Since X_1 is a $(1,1)$-biperiodic function then

$$\Lambda^{-1}F'(z-\theta y)\int_0^1 L(z+(s-y)\theta)\,ds - nc = 0. \tag{III.33}$$

Hence,

$$F'(z-\theta y) = \frac{nc\Lambda}{\int_0^1 L(z+(s-y)\theta)\,ds}, \tag{III.34}$$

because $\int_0^1 L(z+(s-y)\theta)\,ds$ is a non-vanishing function. But, if $c \neq 0$ then $F'(z-\theta y)$ is non-vanishing periodic function that implies F cannot be a periodic function. Hence, $c = 0$ and F is a constant function. Since F is constant then $X_2 = X_2(z)$. Now, by equation (III.28), $X_2 = aL^{-1}(z)$ where a is constant. On the other hand, equation (III.30) shows that $\partial_z X_1$ is a function of z. Therefore, $X_1 = F(y) + G(z)$. But, by equation (III.29), $F'(y) = -\Lambda^{-1}L^2(z)X_2'(z) - \theta G'(z)$ that implies F be constant and $X_1 = X_1(z)$. Then, by equation (III.29), $X_1'(z) = a(\Lambda\theta)^{-1}L'(z)$. Now, by substituting X_1' and X_2 in equation (III.30) we have the following ordinary differential equation

$$L''(z) + \frac{n\theta\Lambda L'(z)}{L^2(z)} + \frac{2a}{\theta}L(z)L'(z) + 2na\Lambda = 0. \tag{III.35}$$

By integrating of equation (III.35), we have

$$L'(z) - \frac{n\theta\Lambda}{L} + \frac{a}{\theta}L^2 = -2na\Lambda z + C. \tag{III.36}$$

Since the left-hand side of equation (III.36) is periodic then $a = 0$ and X_2 is the zero function. Therefore, X_1 is constant. Thus, $L_X g = 0$ and solitons are necessarily trivial. □

IV. CLOSED PSEUDO-RIEMANNIAN GRADIENT RICCI SOLITONS WITH THE STRUCTURE OF WARPED PRODUCTS

In this section, we first analyze the gradient Ricci soliton structure on a closed pseudo-Riemannin warped product manifold that the base is conformal to an n-dimensional torus which is invarient under the action of $(n-1)$-dimensional discrete translation group and then we investigate this structure on a closed Lorentzian warped product manifold.

A. Closed pseudo-Riemannian gradient warped product Ricci solitons when the base is conformal to an n-dimensional pseudo-Eucleadian torus, which are invariant under the action of $(n-1)$-dimensional discrete translation group

Let $M = B \times_f F$ be a warped product pseudo-Riemannian manifold with the warped product metric $\tilde{g} = g_B + f^2 g_F$, where f is a positive function. If $(M, \tilde{g}, h)$ is a gradient Ricci soliton with h, as potential function, and f is non-constant then h depends only on the base and the fiber is necessarily an Einstein manifold [12, Theorem 1.1 & Corollary 1.1]. In what follows, we consider closed warped product gradient Ricci solitons with the base conformal to an n-dimensional pseudo-Euclidean torus, and the fiber chosen to be a closed Einstein manifold. We investigate solutions of the gradient Ricci soliton (I.2) in this case which are invariant under the action of an $(n-1)$-dimensional discrete translation group.

Let $(\mathbb{R}^n, g)$ be a pseudo-Euclidean space, $n \geq 3$, with coordinates $x = (x_1, \cdots, x_n)$ and $g_{ij} = \delta_{ij}\varepsilon_i$ where δ_{ij} is the delta Kronecker, $\varepsilon_i = \pm 1$, with at least one $\varepsilon_i = 1$ and $M = (\mathbb{R}^n, \bar{g}) \times_f F^m$ be the warped product space where $\bar{g} = \varphi^{-2} g$ and F^m, $m \geq 1$, is a pseudo-Riemannian Einstein manifold with the metric g_F. In [12], solutions of the gradient Ricci soliton equation with respect to the warped product metric g_M that are invariant under the action of an $(n-1)$-dimensional translation group, are studied. If $\varphi(\xi)$, $f(\xi)$ and $h(\xi)$ are smooth real-valued functions where $\xi = \sum_{i=1}^{n} \alpha_i x_i$, $\alpha_i \in \mathbb{R}$ and $h(\xi)$ defines a gradient Ricci solitons on $M = (\mathbb{R}^n, \bar{g}) \times_f F^m$ with respect to the metric $\tilde{g} = \bar{g} + f^2 g_F$ then the functions $f(\xi)$, $\varphi(\xi)$ and $h(\xi)$ satisfy the following ordinary differential equation

$$(n-2)f\varphi'' + f\varphi h'' - m\varphi f'' - 2m\varphi' f' + 2f\varphi' h' = 0. \tag{IV.1}$$

For more details see [12, Theorem 1.3].

In the following we show that for given periodic functions $\varphi(\xi)$ and $f(\xi)$ with period T, there is not any periodic function $h(\xi)$ with period T other than constants that satisfies in equation (IV.1). Also, in order to defining the functions $\varphi(\xi)$, $f(\xi)$ and $h(\xi)$ on an n-dimensional torus we need to consider the coefficients α_i, $1 \leq i \leq n$ in the set of integers. This fact implies that there is not any non-trivial closed gradient Ricci soliton with the structure of warped product when the base is conformal to an n-dimensional torus which is invariant under the action of an $(n-1)$-dimensional discrete translation group.

By equation (IV.1),

$$h'' + 2\frac{\varphi'}{\varphi}h + (n-2)\frac{\varphi''}{\varphi} - m\frac{f''}{f} - 2m\frac{\varphi' f'}{\varphi f} = 0, \tag{IV.2}$$

that implies

$$h' = \frac{1}{\varphi^2}\left[(2-n)\int \varphi\varphi''\, d\xi + m\int \frac{(\varphi^2 f')'}{f}\, d\xi\right]. \tag{IV.3}$$

But, by partial integration,

$$\int_0^T \varphi\varphi''\, d\xi = -\int_0^T \varphi'^2\, d\xi \qquad \text{and} \qquad \int_0^T \frac{(\varphi^2 f')'}{f}\, d\xi = \int_0^T \frac{\varphi^2 f'^2}{f^2}\, d\xi.$$

Therefore,

$$(2-n)\int_0^T \varphi\varphi''\, d\xi + m\int_0^T \frac{(\varphi^2 f')'}{f}\, d\xi = (n-2)\int_0^T \varphi'^2\, d\xi + m\int_0^T \frac{\varphi^2 f'^2}{f^2}\, d\xi,$$

that is a non-negative number. Therefore, h' is a periodic function if and only if φ and f are constant functions that imply h be a constant function. Therefore, in this case, gradient Ricci solitons are necessarily trivial.

B. Closed Lorentzian warped pruduct Ricci solitons

Let (F, g_F) be an n-dimensional Riemannian manifold and h be the potential function of a gradient Ricci soliton on the warped product space $M = \mathrm{R} \times_f F$ with respect to the Lorentzian warped metric $g_M = -dt^2 + f^2 g_F$, where f is a non-constant positive real valued function on $\mathbb{R}$. Then by [12, Theorem 1.1 & Corollary 1.1] (F, g_F) is an Einstein manifold with Einstein constant λ_F and h is a real-valued function defined on R. Therefore, by following [14, Chapter 7], we see that $h(t)$ defines a gradient Ricci soliton on (M, g_M) if and only if the following system of ordinary differential equations is satisfied

$$-ff'' - (n-1)f'^2 + \lambda f^2 + ff'h' = \lambda_F, \tag{IV.4a}$$

$$h'' = -\lambda + \frac{nf''(t)}{f(t)}. \tag{IV.4b}$$

In the following, we show that there are not any periodic functions other than constants that satisfy in equations (IV.4a) and (IV.4b).

By equation (IV.4b), $h'(t) = -\lambda t + n \int f^{-1} f''\, dt$ where $\lambda = n \int_0^1 f^{-1} f''\, dt$ that implies h' be a periodic function. Thus, λ is a positive number because $\int_0^1 f^{-1} f''\, dt = \int_0^1 f^{-2} f'^{-2}\, dt$ is a positive number. Therefore, if there exist periodic functions that satisfy in equations (IV.4a) and (IV.4b) then we get a closed shrinking Lorentzian gradient Ricci soliton.

By multiplying equation (IV.4a) in nf^{-2}, we have

$$\frac{nf''}{f} + \frac{n(n-1)f'^2}{f^2} - n\lambda - \frac{nf'h'}{f} = \frac{-n\lambda_F}{f^2}. \tag{IV.5}$$

But, from equation (IV.4b), $nf^{-1}f'' = \lambda + h''$. Now, by substituting $\lambda + h''$ instead of $nf^{-1}f''$ in equation (IV.5), we get

$$h'' - \frac{nf'}{f}h' = \frac{-n\lambda_F}{f^2} - \frac{n(n-1)f'^2}{f^2} + (n-1)\lambda. \tag{IV.6}$$

But, the left-hand side of equation (IV.6) is $\Delta_{g_M} h$. On the other hand, by equation (II.3), $\Delta_{g_M} h = (n+1)\lambda - \tau_M$, where

$$\tau_M = \frac{\tau_F}{f^2} + \frac{2nf''}{f} + \frac{n(n-1)f'^2}{f^2}, \tag{IV.7}$$

that τ_M and τ_F are the scalar curvature of M and F, respectively. Thus,

$$\frac{-n\lambda_F}{f^2} - \frac{n(n-1)f'^2}{f^2} + (n-1)\lambda = (n+1)\lambda - \frac{\tau_F}{f^2} - \frac{2nf''}{f} - \frac{n(n-1)f'^2}{f^2}. \tag{IV.8}$$

Since $\tau_F = n\lambda_F$ then we get

$$\lambda = \frac{nf''}{f}. \tag{IV.9}$$

Therefore, $h'' = 0$ that implies h be a constant function. Thus, solitons are necessarily trivial.

By examples in [11], we know that closed Lorentzian Ricci solitons are not necessarily gradient. Furthermore, They are not necessarily trivial in the steady case. But there are many questions about closed pseudo-Riemannian Ricci solitons that need to be answered, including:

Question IV.1. Are there any examples of non-trivial closed pseudo-Riemannian gradient Ricci solitons in the steady, or expanding cases?

Question IV.2. How can we find examples of non-trivial closed pseudo-Riemannian shrinking gradient Ricci solitons?

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