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OPTIMAL CONTROL OF A LARGE DAM WITH COMPOUND POISSON INPUT AND COSTS DEPENDING ON WATER LEVELS

Abstract. This paper studies a discrete model of a large dam where the difference between lower and upper levels, L, is assumed to be large. Passage across the levels leads to damage, and the damage costs of crossing the lower or upper level are proportional to the large parameter L. Input stream of water is described by compound Poisson process, and the water cost depends upon current level of water in the dam. The aim of the paper is to choose the parameters of output stream (specifically defined in the paper) minimizing the long-run expenses that include the damage costs and water costs. The present paper addresses the important question *How does the structure of water costs affect the optimal solution?* We prove the existence and uniqueness of a solution. A special attention is attracted to the case of linear structure of the costs.

As well, the paper contributes to the theory of state-dependent queueing systems. The inter-relations between important characteristics of a state-dependent queueing system are established, and their asymptotic analysis that involves analytic techniques of Tauberian theory and heavy traffic approximations is provided.

Contents

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1. Introduction

1.1. Description of the system and formulation of the problem. The paper studies a discrete model of a large dam. A large dam is specified by the parameters L^{lower} and L^{upper}, which are, respectively, the lower and upper levels of the dam. If the current water level is between these bounds, the dam is assumed to be in a normal state. The difference $L = L^{\text{upper}} - L^{\text{lower}}$ is large, and this is the reason for calling the dam *large*. This feature enables us to use asymptotic analysis as $L \to \infty$ and solve different problems of optimal control. (A direct way without an asymptotic analysis is very hard.)

Let L_t denote the water level at time t. If $L^{\text{lower}} < L_t \leq L^{\text{upper}}$, then the state of the dam is called *normal*. Passage across upper level (flooding) or reaching lower level (emptiness) lead to damage. The costs per unit time of this damage are

$$J_1 = j_1 L \tag{1.1}$$

for the lower level and, respectively,

$$J_2 = j_2 L \tag{1.2}$$

for the upper level, where j_1 and j_2 are given real constants. (In real dams that are large, the damage costs are proportional to the capacity of dam.)

The water inflow is described by a compound Poisson process. Namely, the probability generating function of input amount of water (which is assumed to be an integer-valued random variable) in an interval t is given by

$$f_t(z) = \exp\left\{-\lambda t \left(1 - \sum_{i=1}^{\infty} r_i z^i\right)\right\}, \tag{1.3}$$

where r_i is the probability that at a specified moment of Poisson arrival the amount of water will increase by i units. In practice, this means that the arrival of water is registered at random instants t_1, t_2, ...; the times between consecutive instants are mutually independent and exponentially distributed with parameter λ, and quantities of water (number of water units) of input flow are specified as a quantity i with probability r_i ($r_1 + r_2 + \ldots = 1$). Clearly that this assumption is more applicable to real world problems than the assumption of [4] where the inter-arrival times of water units are exponentially distributed with parameter λ. For example, the assumption made in the present paper enables us to approach a continuous dam model, assuming that the water levels L_t take the discrete values $\{j\Delta\}$, where j is a positive integer and step Δ is a positive small real constant. In the paper, the water levels L_t are assumed to be integer-valued. The aforementioned set of values $\{j\Delta\}$ for water levels can be obtained by scaling.

The outflow of water is state-dependent as follows. If the level of water is between L^{lower} and L^{upper}, then an interval between departures of water units has the probability distribution function $B_1(x, C)$ (depending on parameter C, the meaning of which will become clear later). If the level of water exceeds L^{upper}, then an interval between departures of water units has the probability distribution function $B_2(x)$. The probability distribution function $B_2(x)$ is assumed to obey the condition $\int_0^\infty x\mathrm{d}B_2(x) < 1/(\lambda\mathsf{E}\varsigma)$, where $\mathsf{E}\varsigma$ is the mean batch size. If the level of water is L^{lower} exactly, then output of water is frozen, and it resumes again as soon as the level of water exceeds the level L^{lower}. (The exact mathematical formulation of the problem taking into account some specific details is given below.)

Let c_{L_t} denote the cost of water at level L_t. The sequence c_i is assumed to be positive and non-increasing. The problem of the present paper is to choose the parameter C (and, specifically, $\int_0^\infty x\mathrm{d}B_1(x, C)$ and $\int_0^\infty x^2\mathrm{d}B_1(x, C)$) of the dam in the normal state minimizing the objective function

$$J = p_1 J_1 + p_2 J_2 + \sum_{i=L^{\text{lower}}+1}^{L^{\text{upper}}} c_i q_i, \tag{1.4}$$

where

$$p_1 = \lim_{t\to\infty} \mathsf{Pr}\{L_t = L^{\text{lower}}\}, \tag{1.5}$$

$$p_2 = \lim_{t\to\infty} \mathsf{Pr}\{L_t > L^{\text{upper}}\}, \tag{1.6}$$

$$q_i = \lim_{t\to\infty} \mathsf{Pr}\{L_t = L^{\text{lower}} + i\},\ i = 1, 2, \ldots, L, \tag{1.7}$$

and J_1 and J_2 are defined in (1.1) and (1.2).

Usually L^{lower} is identified with an empty queue (i.e. $L^{\text{lower}} := 0$ and $L^{\text{upper}} := L$). Just this assumption is made in the paper, and the dam model is the following queueing system with service depending on queue-length. If immediately before moment of a service start the number of customers in the system exceeds the level L, then the customer is served by the probability distribution function $B_2(x)$. Otherwise, the service time distribution is $B_1(x)$. The value p_1 is the stationary probability of an empty system, the value p_2 is the stationary probability that a customer is served by probability distribution $B_2(x)$, and q_i, $i = 1, 2, \ldots, L$, are the stationary probabilities of the queue-length process, so $p_1+p_2+\sum_{i=1}^{L} q_i = 1$. (For the described queueing system, the right-hand side limits in relations (1.5)–(1.7) do exist.)

So, in the present paper we assume that $L^{\text{lower}} = 0$ and $L^{\text{upper}} = L$. We also assume that initial water level is 0.

In our study, the parameter L increases unboundedly, and we deal with the series of queueing systems. The above parameters, such as p_1, p_2, J_1, J_2 as well as other parameters are functions of L. The argument L will be often omitted in these functions.

1.2. **Motivation, discussion of the study and review of related literature.** Similarly to [4], it is assumed that the input parameter λ, the probabilities r_1, r_2,... and probability distribution function $B_2(x)$ are given, while the appropriate probability distribution function $B_1(x, C)$ should be found from the specified parametric family of functions $B_1(x, C)$, where

$$C = \lim_{L\to\infty} L\delta(L), \tag{1.8}$$

where $\delta(L)$ is a specified nonnegative vanishing parameter of the system as $L \to \infty$.

The reason of solving this specific problem, where the probability distribution function $B_2(x)$ is given while the probability distribution functioned $B_1(x, C)$ is controlled, is that in practical problems of the water resources planning, it is important to know how much water should be used per unit time in order to minimize the risks of the disasters such as emptiness or flooding the dam.

The outflow rate, should be chosen such that to minimize the objective function of (1.4) with respect to the parameter C, which results in choice of the corresponding probability distribution function $B_1(x, C)$ of that family.

A particular problem have been studied in [4]. A circle of problems associated with the results of [4] are discussed in a review paper [5].

The simplest model with Poisson input stream and the objective function having the form $J = p_1J_1 + p_2J_2$ (i.e. the water costs are not taken into account), has been studied in [4]. Denote $\rho_2 = \lambda\int_0^\infty x\mathrm{d}B_2(x)$ and $\rho_1 = \rho_1(C) = \lambda\int_0^\infty x\mathrm{d}B_1(x, C)$. (The optimal value of C is unique in a minimization problem precisely formulated in [4].) It was shown in [4] that the unique solution to the control problem has one of the three forms given there in the formulation of Theorem 5.1. The aforementioned three forms in the formulation of Theorem 5.1 in [4] fall into the category of a large area of heavy traffic analysis in queueing theory. We mention the books of Chen and Yao [7] and Whitt [33], where a reader can find many other references. It has also been shown in [4] that the solution to the control problem is insensitive to the type of probability distributions $B_1(x, C)$ and $B_2(x)$. Specifically, it is expressed via the first moment of $B_2(x)$ and the first two moments of $B_1(x, C)$.

Compared to the earlier studies in [4], the solution of the problems in the present paper requires a much deepen and delicate analysis. The results of [4] are extended in two directions: (1) the arrival process is compound Poisson rather than Poisson, and (2) structure of water costs in dependence of the level of water in the dam is included.

The first extension leads to new techniques of stochastic analysis. The main challenge in [4] was reducing the certain characteristics of the system during a busy period to the convolution type recurrence relation such as $Q_n = \sum_{i=0}^{n} Q_{n-i+1} f_i$ ($Q_0 \neq 0$), where $f_0 > 0$, $f_i \geq 0$ for all $i \geq 1$, $\sum_{i=0}^{\infty} f_i = 1$ and then using the known results on the asymptotic behaviour of Q_n as $n \to \infty$. In the case when arrivals are compound Poisson, the same characteristics of the system cannot be reduced to the aforementioned convolution type of recurrence relation. Instead, we obtain a more general scheme including as a part the aforementioned recurrence relation. In this case, asymptotic analysis of the required characteristics becomes very challenging. It is based on special stochastic domination methods.

The second extension leads to new analytic techniques of asymptotic analysis. Asymptotic methods of [4] are no longer working, and more delicate techniques should be used instead. That is, instead of Takács' asymptotic theorems [31], p. 22-23, special Tauberian theorems with remainder by Postnikov [21], Sect. 25 should be used. For different applications of the aforementioned Takács' asymptotic theorems and Tauberian theorems of Postnikov see [5].

Another challenging problem for the dam model in the present paper is the solution to the control problem, that is, the proof of a uniqueness of the optimal solution. In the case of the model in [4] the existence and uniqueness of a solution follows automatically from the explicit representations of the functionals obtained there. (The existence of a solution follows from the fact that in the case $\rho_1 = 1$ we get a bounded value of the functional, while in the cases $\rho_1 < 1$ and $\rho_1 > 1$ the functional is unbounded. Then the uniqueness of a solution reduces to elementary minimization problem for smooth convex functions.)

In the case of the model in the present paper, the solution of the problem with extended criteria (1.4) is related to the same set of solutions as in [4]. That is, it must be either $\rho_1 = 1$ or one of the two limits of $\rho_1 = 1+\delta(L)$, $\rho_1 = 1-\delta(L)$ for positive small vanishing $\delta(L)$ as the series parameter L increases unboundedly, and $L\delta(L) \to C$. In the present study, it is convenient to define the parameter C as

$$C = \lim_{L\to\infty} L[\rho_1(L) - 1], \tag{1.9}$$

and use $C(L) = L[\rho_1(L)-1]$ as a series parameter. Hence, it is quite natural to consider $\rho_1(L)$ as a control sequence, while $C(L)$ is a sequence derivative from $\rho_1(L)$. Furthermore, in this case $C(L)$ is expressed uniquely via $\rho_1(L)$ and vice versa.

The definition of the parameter C given here differs from that definition given in [4]. Unlike [4], where C was defined as a nonnegative control parameter (see (1.8)), in the present definition (1.9) the value of C can be either positive or negative.

Unlike [4], we use the notation $\rho_{1,l}(L) = \lambda^l \int_0^\infty x^l \mathrm{d}B_1(x, C(L))$, $l = 2, 3$. The existence of $\rho_{1,3}(L)$ (i.e. the moments of the third order of $B_1(x, C(L))$) will be specially assumed in the formulations of the statements corresponding to the case studies.

It is assumed in the present paper that c_i is a nonincreasing sequence. If the cost sequence c_i were an arbitrary bounded sequence, then a richer set of possible cases could be studied. However, in the case of arbitrary cost sequence, the solution does not need be unique.

A nonincreasing sequence c_i depends on L in series. This means that as L changes (increasing to infinity) we have different not increasing sequences (see example in Section 8). The initial value c_1 and final value c_L are taken fixed and strictly positive, and the limit of c_L as $L \to \infty$ is assumed to be positive as well.

Realistic models arising in practice assume that the probability distribution function $B_1(x, C)$ should also depend on i, i.e have the representation $B_{1,i}(x, C)$. The model of the present paper, where $B_1(x, C)$ is the same for all i, under appropriate additional information can approximate those more general models. Namely, one can suppose that the stationary service time distribution $B_1(x, C)$ has the representation $B_1(x, C) = \sum_{i=1}^{L} q_i B_{1,i}(x, C)$ (q_i, $i = 1, 2, \ldots, L$ are the state probabilities), and the solution to the control problem for $B_1(x, C)$ enables us to find then the approximate solution to the control problem for $B_{1,i}(x, C)$, $i = 1, 2, \ldots, L$. The additional information about $B_{1,i}(x, C)$, $i = 1, 2, \ldots, L$, might be that all these distributions belong to the same parametric family of distributions with known relationships between the values of a parameter. For instance, the simplest model can be of the form $B_1(x, C) = aB_1^*(x, C) + bB_1^{**}(x, C)$, where the distributions $B_1^*(x, C)$ and $B_1^{**}(x, C)$ are of the same type (say, two-phase Erlang distributions), $a := \sum_{i=1}^{L^0} q_i$ ($L^0 < L$), and, respectively, $b := \sum_{i=L^0+1}^{L} q_i$, and the relationship between the means,

$$\gamma = \frac{\int_0^\infty x \mathrm{d}B_1^*(x, C)}{\int_0^\infty x \mathrm{d}B_1^{**}(x, C)},$$

is known.

In the present paper we address the following questions.

• Uniqueness of an optimal solution and its structure.

• Interrelation between the parameters j_1, j_2, ρ_2, c_i $(i = 1, 2, \dots, L)$ when the optimal solution is $\rho_1 = 1$.

The uniqueness of an optimal solution is given by Theorem 3.7. In the case of the model considered in [4] the condition, when the optimal solution is $\rho_1 = 1$, the interrelation between the parameters j_1, j_2 and ρ_2 is $j_1 = j_2\rho_2/(1-\rho_2)$. In the case of the model considered in this paper, when the optimal solution is $\rho_1 = 1$, the interrelation between the aforementioned and some additional parameters gives us the inequality $j_1 \leq j_2\rho_2/(1-\rho_2)$ (see Section 7, Corollary 7.5).

A more exact result is obtained in the particular case of linearly decreasing costs as the level of water increases (for brevity, this case is called *linear costs*). In this case, a numerical solution of the problem is given.

The solution to the control problem enables us to find optimal initial condition of the system. The steady state distribution, under which the optimal value of the control parameter $\rho_1(L)$ is achieved, is associated with the optimal initial level of water in the dam.

1.3. **Organization of the paper.** The rest of the paper is organized as follows. In Section 2 the main ideas and methods of asymptotic analysis are given. In Section 2.1, we recall the basic methods related to state dependent queueing system with ordinary Poisson input that have been used in [4]. Then in Section 2.2, extensions of these methods for the model considered in this paper are given. Specifically, the methodology of constructing linear representations between mean characteristics given during a busy period is explained. Main results of the paper are formulated in Section 3.

The sections following after Section 3 are of two types. The first type of the results is presented in Sections 4 and 5. These large sections present the preliminary results of the paper and study the characteristics of the system, their asymptotic behaviour and specifically the asymptotic behaviour of different stationary probabilities. The second type of the results is presented in Sections 6, 7, 8 and 9 and related to the solution to the control problem and further study of its properties.

In Section 4, the asymptotic behavior of the stationary probabilities p_1 and p_2 for specific sets of states are studied. In Section 4.1, known Tauberian theorems that are used in the asymptotic analysis in the paper are recalled. Section 4.3 establishes explicit formulae for the probabilities p_1 and p_2. In Section 4.4 some preliminary results are established for the further study of asymptotic behaviour of stationary probabilities p_1 and p_2 as $L \to \infty$ in Sections 4.5 and 4.6. Section 5 is devoted to asymptotic analysis of the stationary probabilities q_{L-i}, $i = 1, 2, \dots$. In Section 5.1, the explicit representation for the stationary probabilities q_i is derived. On the basis of this explicit representation and Tauberian theorems, in following Sections 5.2, 5.3 and 5.4 asymptotic theorems for these stationary probabilities are established in the cases $\rho_1 = 1$, $\rho_1 = 1 + \delta(L)$ and $\rho_1 = 1 - \delta(L)$ correspondingly, where positive $\delta(L)$ is assumed vanishing such that $L[\rho_1(L) - 1] \to C$ as $L \to \infty$. In Section 6 the objective function given in (1.4) is studied. In following Sections 6.1, 6.2 and 6.3, the asymptotic theorems for this objective function are established for the cases $\rho_1 = 1$, $\rho_1 = 1+\delta(L)$ and $\rho_1 = 1-\delta(L)$, correspondingly. In Section 7, the theorem on existence and uniqueness of a solution is proved. In Section 8, the case of linear costs is studied. Numerical results relevant to Section 8 are provided in Section 9. Section 10 contains long proofs of the lemmas, theorems and propositions formulated in the paper.

2. Methodology of analysis

In this section we describe the methodology used in the present paper. This is very important for the following two reasons. The first reason is that the standard approach of a diffusion approximation (transient) of a dam process with the following computation of the stationary distribution of the diffusion is hard, because in that case we should deal with the interchange of the order of limits (see discussion on the page 514 of Whitt [34] as well as in Whitt [35]). The second reason is that the earlier methods of [4] do not work for this extended model and, hence, need in substantial revision.

We start from the model where arrivals are Poisson, and then we explain how the methods should be developed for the model where an arrival process is compound Poisson. In this and later sections we write $B_1(x)$ rather than $B_1(x, C)$. As well, the parameter L will be omitted from the related notation for the characteristics of the system.

2.1. State dependent queueing system with Poisson input and its characteristics.

In this section, we consider the simplest model in which arrival flow is Poisson with parameter λ. The service time depends upon queue-length as follows. If immediately before a service of a customer, the number of customers in the system is not greater than L, then the probability distribution function is $B_1(x)$. Otherwise, if the number of customers in the system exceeds L, then the probability distribution function is $B_2(x)$. Note, that in the case when $L = 0$, then the only first customer arrived in a busy period has the probability distribution function is $B_1(x)$; all other has the probability distribution function $B_2(x)$.

Let T_L denote the length of a busy period of this system, and let $T_L^{(1)}$, $T_L^{(2)}$ denote the cumulative times spent for service of customers arrived during that busy period with probability distribution functions $B_1(x)$ and $B_2(x)$ correspondingly. For $k=1,2$, the expectations of service times will be denoted by $1/\mu_k=\int_0^\infty x\mathrm{d}B_k(x)$, and the loads by $\rho_k=\lambda/\mu_k$. Let ν_L, $\nu_L^{(1)}$ and $\nu_L^{(2)}$ denote correspondingly the number of served customers during a busy period, and the numbers of those customers served by the probability distribution functions $B_1(x)$ and $B_2(x)$. The random variable $T_L^{(1)}$ coincides in distribution with a busy period of the $M/G/1/L$ queueing system (L is the number of waiting places excluding the place for server). The elementary explanation of this fact is based on a property of level crossings and the property of the lack of memory of exponential distribution (e.g. [4]), so the analytic representation for $\mathsf{E}T_L^{(1)}$ is the same as this for the expected busy period of the $M/G/1/L$ queueing system. The recurrence relation for the Laplace-Stieltjes transform and consequently that for the expected busy period of the $M/G/1/L$ queueing system has been derived by Tomko [32] (see also Cooper and Tilt [8]). So, for $\mathsf{E}T_L^{(1)}$ the following recurrence relation is satisfied:

$$\mathsf{E}T_L^{(1)}=\sum_{i=0}^{L}\mathsf{E}T_{L-i+1}^{(1)}\int_0^\infty \mathrm{e}^{-\lambda x}\frac{(\lambda x)^i}{i!}\mathrm{d}B_1(x), \tag{2.1}$$

where $\mathsf{E}T_0^{(1)}=1/\mu_1$.

Remark 2.1. The random variable $T_i^{(1)}$ is defined similarly to that of $T_L^{(1)}$. In that case the parameter i is the threshold value of the model, and the set $\{\mathsf{E}T_i^{(1)}\}$ may be thought as the set of mean busy periods of $M/G/1/i$ queueing systems with the same parameter of Poisson input, the same probability distribution of service time, but different number of waiting places.

Recurrence relation (2.1) is a particular form of the recurrence relation

$$Q_n=\sum_{i=0}^{n}Q_{n-i+1}f_i, \tag{2.2}$$

where $Q_0\neq 0$, $f_0>0$, $f_i\geq 0$, $i=1,2,\ldots$ and $\sum_{i=0}^\infty f_i=1$ (see Takács [31]).

Using the obvious system of equations given by (2.1) and (2.2) in [4] and Wald's equations (see [9], p.384) given by (2.3) and (2.4) in [1] one can express the quantities $\mathsf{E}T_L$, $\mathsf{E}\nu_L$, $\mathsf{E}T_L^{(2)}$, $\mathsf{E}\nu_L^{(1)}$ and $\mathsf{E}\nu_L^{(2)}$ all via $\mathsf{E}T_L^{(1)}$ as the linear functions. Since equations (2.1) – (2.4) of [4] should be mentioned many times in this paper, we find convenient to list them here for following direct references:

$$\mathsf{E}T_L = \mathsf{E}T_L^{(1)} + \mathsf{E}T_L^{(2)}, \tag{2.3}$$

$$\mathsf{E}\nu_L = \mathsf{E}\nu_L^{(1)} + \mathsf{E}\nu_L^{(2)}, \tag{2.4}$$

$$\mathsf{E}T_L^{(1)} = \frac{1}{\mu_1}\mathsf{E}\nu_L^{(1)}, \tag{2.5}$$

$$\mathsf{E}T_L^{(2)} = \frac{1}{\mu_2}\mathsf{E}\nu_L^{(2)}. \tag{2.6}$$

Then, to obtain the desired linear representations note that the number of arrivals during a busy cycle coincides with the total number of customers served during a busy period. That is, for their expectations we have

$$\lambda\left(\mathsf{E}T_L + \frac{1}{\lambda}\right) = \lambda\mathsf{E}T_L + 1 = \mathsf{E}\nu_L, \tag{2.7}$$

which together with the aforementioned relations (2.3) – (2.6) yields the linear representations required.

For example,

$$\mathsf{E}\nu_L^{(2)} = \frac{1}{1-\rho_2} - \mu_1 \cdot \frac{1-\rho_1}{1-\rho_2}\mathsf{E}T_L^{(1)}, \tag{2.8}$$

and

$$\mathsf{E}T_L^{(2)} = \frac{\rho_2}{\lambda(1-\rho_2)} - \frac{\rho_2}{\rho_1} \cdot \frac{1-\rho_1}{1-\rho_2}\mathsf{E}T_L^{(1)}. \tag{2.9}$$

As a result, the stationary probabilities p_1 and p_2 both are expressed via $\mathsf{E}\nu_L^{(1)}$ as follows:

$$p_1 = \frac{1-\rho_2}{1+(\rho_1-\rho_2)\mathsf{E}\nu_L^{(1)}},$$

$$p_2 = \frac{\rho_2+\rho_2(\rho_1-1)\mathsf{E}\nu_L^{(1)}}{1+(\rho_1-\rho_2)\mathsf{E}\nu_L^{(1)}}.$$

It is interesting to note that the coefficients in the linear representations all are insensitive to the probability distribution functions $B_1(x)$ and $B_2(x)$ and are only expressed via parameters such as μ_1, μ_2 and λ.

The asymptotic behaviour of $\mathsf{E}T_L^{(1)}$ as $L \to \infty$ that given by (2.1) is established on the basis of the known asymptotic behaviour of the sequence Q_n as $n \to \infty$ that given by (2.2) (see [31], p.22, [21] as well as recent paper [5]). To make the paper self-contained, the necessary results about the asymptotic behaviour of Q_n as $n \to \infty$ are given in Section 4.1.

2.2. **State dependent queueing system with compound Poisson input and its characteristics.**

2.2.1. *Historic background.* For $M^X/G/1/L$ queues, certain characteristics associated with busy periods have been studied by Rosenlund [22]. Developing the results of Tomko [32], Rosenlund [22] has derived the recurrence relations for the joint Laplace-Stieltjes and z-transform of two-dimensional distributions of a generalized busy period and the number of customers served during that period. In turn, both of these approaches [32] and [22] are based on a well-known Takács' method (see [29] or [30]).

For further analysis, [22] used matrix-analytic techniques and techniques of the theory of analytic functions. This type of analysis is very hard and seems cannot be easily adapted for the purposes of the present paper, where a more general model than that from [22] is studied. Busy periods and loss characteristics during busy periods for $M^X/G/1/n$ systems have also been studied by Pacheco and Ribeiro [19], [20] and Ferreira, Pacheco and Ribeiro [10].

Along with previously mentioned paper [4], the method of asymptotic analysis closely related to subject matter of this paper have been considered by Abramov [2] and [3] and further reviewed in [5].

The first studies of single server queueing systems with Poisson input and service depending on queue-length were due to Suzuki [25], [26], and the paper by Suzuki and Ebe [27] was probably the first one to consider decision rules problems. Since then there have been numerous publications related to state-dependent queueing systems, e.g. Knessl *et al* [13],[14], Mandelbaum and Pats [15], Miller [17] and Miller and McInnes [18]. Some of these publications include control problems as well. Basic results for a single-server queueing system with Poisson input and service depending on queue-length can also be found in Abramov [1].

2.2.2. *Structure of busy periods, and reasoning for the recurrence relations of convolution type.* In this section we explain how the method of Section 2.1 can be extended, and how the characteristics of the system can be expressed via the similar convolution type recurrence relations.

To explain the origin of the convolution type recurrence relations in queueing systems and further representation for the required mean characteristics, we recall what the structure of busy period in systems with Poisson input is, and how this structure is extended from relatively simple systems to more complicated ones.

Let us first recall the structure of a busy period in the $M/G/1$ queueing system (e.g. Takács [29], [30]).

The busy period starts upon arrival of a customer in the idle system. If during the service time of the customer no arrival occurs, then the length of the busy period coincides with the length of the service. If at least one arrival occurs, then the structure of busy period is as follows. Suppose that during a service time there are n arrivals. Then, the time interval from the moment of service beginning when there are n customers in the

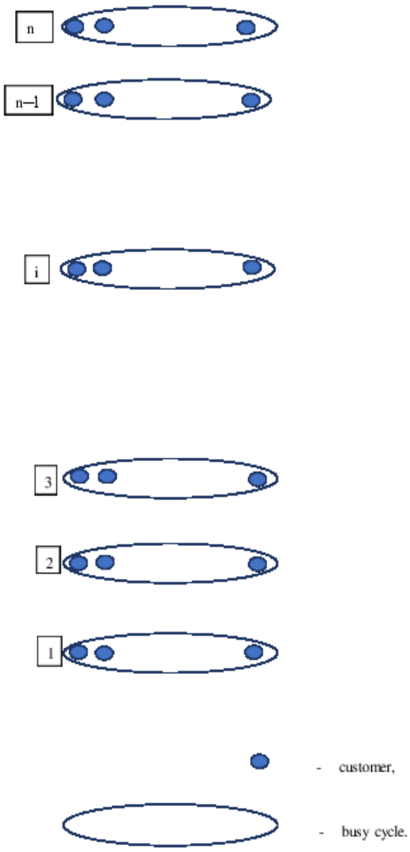

FIGURE 1. General scheme of a busy period in the $M/G/1$ queue.

system until the moment when there remain only $n-1$ customers in the system at the first time after the interval start, coincides in distribution with a busy period. In Figure 1, the typical structure of the $M/G/1$ busy period is shown.

Consider now the busy period in the $M/G/1$ queueing system with threshold level L. In this system, the service time distribution depends on the number of customers in the queue as follows. If immediately before the service start the number of customers in the system is greater than L, then the service time distribution of that customer is $B_2(x)$. Otherwise, it is $B_1(x)$. The typical structure of a busy period is indicated

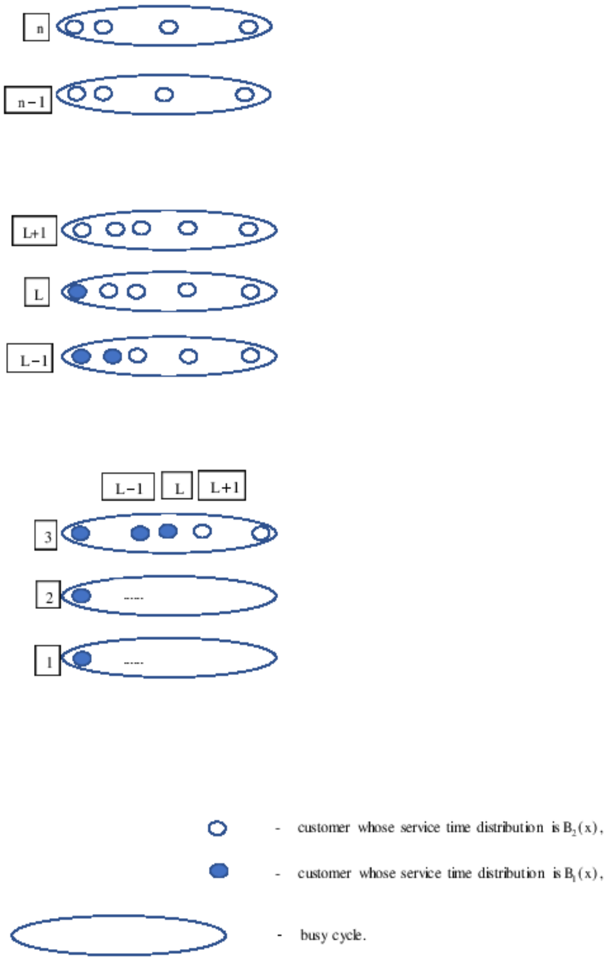

FIGURE 2. General scheme of a busy period in the state-dependent $M/G/1$ queue, where L is the threshold level.

in Figure 2, where customers that are over the threshold level (served by probability distribution $B_2(x)$) are indicated with white color, and all other customers (served by probability distribution $B_1(x)$) are indicated with dark color.

Unlike the case of the standard $M/G/1$ queue (without threshold), in the case of the state-dependent queueing systems with service depending on queue-length, a time interval from the moment of service beginning when the number of customers in the system is i until the time moment when the number of customers becomes $i-1$ at the first time since its

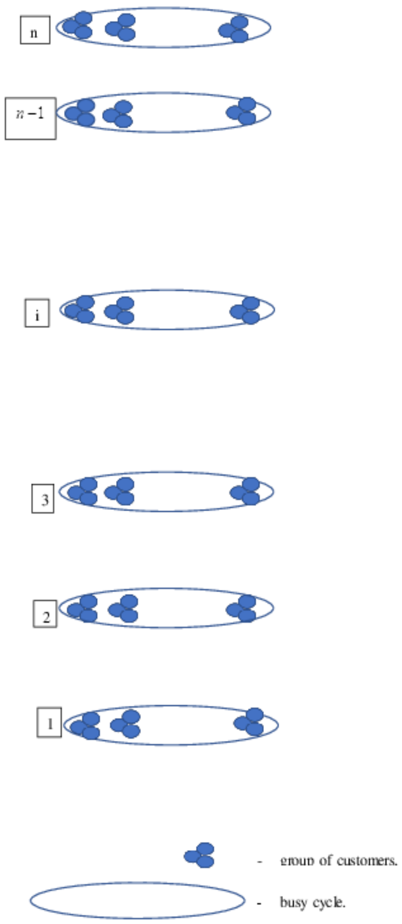

FIGURE 3. General scheme of a busy period in the $M^X/G/1$ queue.

start, depends on i. Specifically, if $i \leq L$, then the length of the aforementioned cycle is distributed as the state-dependent queueing system with the biased threshold equal to $L-i+1$. But if $i > L$, then the length of a busy cycle coincides with the length of a busy period of the standard $M/G/1$ queue, the service time of customers which is $B_2(x)$. Then, the recurrence relations for the mean busy periods are understandable.

Let us now consider the standard $M^X/G/1$ queueing system (without threshold). The structure of a busy period for this queueing system is given in Figure 3.

As we can see from the comparison of Figures 1 and 3, the only difference between them is that in the case of the system with batch arrivals, the elements of busy cycles that are indicated in Figure 3, are groups of customers rather than isolated customers as in Figure 1. That is, the number of cycles indicated in Figure 3 is associated with the number of batches arrived during the service time of the first batch of arrived customers.

However, in the case of the present queueing system that is with batch arrivals and threshold, the structure of a busy period is much more complicated than that in the previous cases. We consider the state-dependent queueing system with batch arrivals in the following formulation. The arrival of a batch is Poisson. If immediately before a service start the queue-length is not greater than L, then the probability distribution function of the customer is $B_1(x)$. Otherwise, it is $B_2(x)$. The description of the system implies that the first customer in a busy period is served with probability distribution function $B_1(x)$. This situation makes the system artificial. In the more natural situation, when the service depends on queue-length *at the moment of a service start* rather than *immediately before the service start* the structure of the process is more complicated, and its analysis is technically harder. However, the asymptotic behavior, as L increases to infinity, is the same. To avoid the technical complications, the problem is considered in the aforementioned simplified formulation.

Then, the linear representations are similar to those derived for the state dependent queueing system with ordinary Poisson input. Indeed, equations (2.3) – (2.6) all hold in the case of the present queueing system as well. The first two, (2.3) and (2.4) are obvious, and (2.5) and (2.6) follow from the same Wald's identities as in the case of ordinary Poisson arrivals. However, instead of (2.7) given in Section 2.1, the relation between $\mathsf{E}T_L$ and $\mathsf{E}\nu_L$ in the case of batch arrivals is slightly different. Specifically,

$$\lambda\mathsf{E}\varsigma\left(\mathsf{E}T_L+\frac{1}{\lambda}\right)=\lambda\mathsf{E}\varsigma\mathsf{E}T_L+\mathsf{E}\varsigma=\mathsf{E}\nu_L. \tag{2.10}$$

Note, that the left-hand side of (2.10) can be rewritten in the different way:

$$\begin{aligned}&\lambda\mathsf{E}\varsigma\left(\mathsf{E}T_L-\frac{1}{\mu_1}+\frac{1}{\mu_1}+\frac{1}{\lambda}\right)\\ &=\lambda\mathsf{E}\varsigma\left(\mathsf{E}T_L-\frac{1}{\mu_1}\right)+\rho_1+\mathsf{E}\varsigma_1,\end{aligned} \tag{2.11}$$

where ς_1 is the first batch that starts a busy period. From (2.11) we have

$$\lambda \mathsf{E}\varsigma\left(\mathsf{E}T_L - \frac{1}{\mu_1}\right) + (\rho_1 - 1 + \mathsf{E}\varsigma_1) = \mathsf{E}\nu_L - 1. \tag{2.12}$$

The meaning of the quantity $\rho_1 - 1 + \mathsf{E}\varsigma_1$ on the left-hand side of (2.12) is the expected number of customers in the system after the service completion of the first customer in the busy period. That is, the expected number of independent busy cycles after the service completion of the first customer in the busy period is

$$\mathsf{E}\zeta_1 = \rho_1 - 1 + \mathsf{E}\varsigma_1. \tag{2.13}$$

The main difficulty, however, is that the recurrence relation for $\mathsf{E}T_L^{(1)}$ (or that for the corresponding quantity $\mathsf{E}\nu_L^{(1)}$) cannot be presented as a convolution type recurrence relation in simple terms as (2.2), since, as it was indicated, the structure of a busy period in the case of batch arrivals and threshold becomes very complicated, and is not quite similar to that given in Figure 2. This is because the size of the first batch is random, and this is essentially affected to the complexity. However, under the assumption that the first batch that starts a busy period contains a single customer only, the structure of the busy period will become similar to that given in Figure 2.

In following Figure 4, the typical structure of such busy period is indicated.

The number of cycles that are indicated in Figure 4 coincides with the number of customers arrived during the service time of the first customer. For instance, suppose that during the service time of the first customer, three batches of customers arrived. If the corresponding numbers of customers in those batches are 2, 1 and 3, then the total number of cycles is 6. Each of these 6 customers is considered as a tagged customer in the corresponding cycle, and the structure of the busy period becomes similar to that in Figure 2.

2.2.3. *Analysis of the busy period.* We first start from the structure indicated in Figure 4. For this model, let $\widetilde{T}_j$, $j = 1, 2, \ldots, L$, denote the time interval starting from a moment when there are $L-j+1$ customers in the system until the moment when there remain $L-j$ customers for the first time since beginning of that interval. Similarly to the notation used in Section 2.1, we introduce the random variables $\widetilde{T}_j^{(1)}$, $\widetilde{T}_j^{(2)}$, $\widetilde{\nu}_j$, $\widetilde{\nu}_j^{(1)}$, $\widetilde{\nu}_j^{(2)}$, $j = 1, 2, \ldots, L$, which have the same meaning as before. Specifically, when j takes the value L, $\widetilde{T}_L$ is the length a busy period starting from a single customer (1-busy period); $\widetilde{\nu}_L$ is the number of customers that served during a 1-busy period, and so on.

Comparison of the busy periods structures in Figures 2 and 4 enables us to conclude that Takács' method [29], [30] that was applied previously

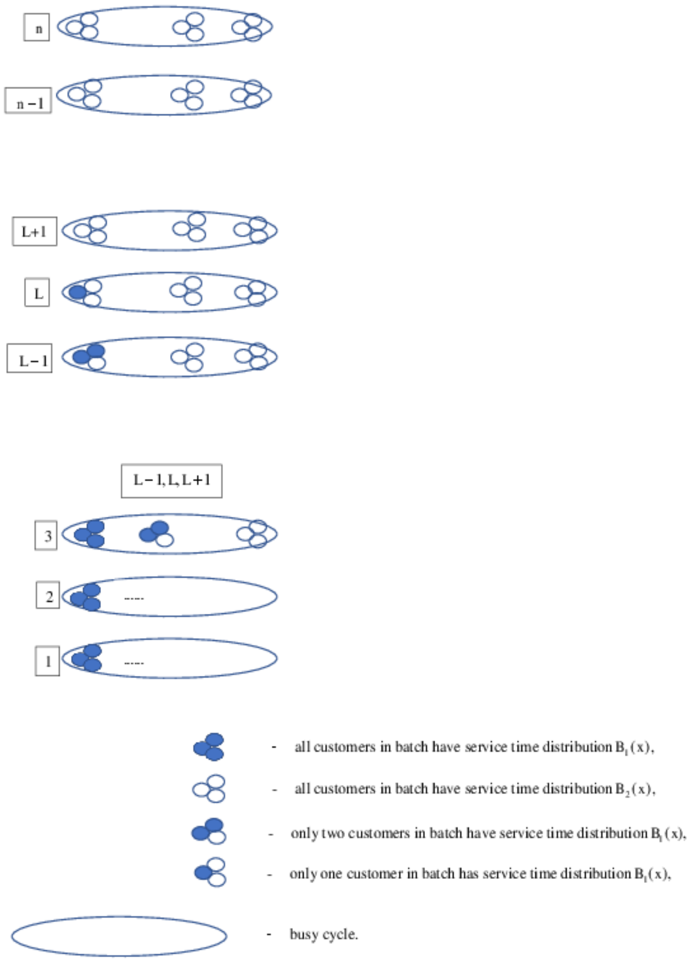

FIGURE 4. Particular scheme of a busy period in the state-dependent $M^X/G/1$ queue when the first batch contains only one customer. (L is the threshold level.)

to the $M/G/1$ state-dependent queueing system is applicable to the state-dependent $M^X/G/1$ queueing system, in which the first batch contains a single customer. Based on the aforementioned Takács' method, the recurrence relation similar to that of (2.1) is

$$\mathsf{E}\widetilde{T}_L^{(1)} = \sum_{i=0}^{L} \mathsf{E}\widetilde{T}_{L-i+1}^{(1)} \int_0^\infty \frac{1}{i!}\frac{\mathrm{d}^i f_x(z)}{\mathrm{d}z^i}\Big|_{z=0} \mathrm{d}B_1(x), \tag{2.14}$$

where $\mathsf{E}\widetilde{T}_0^{(1)} = 1/\mu_1$, and the generating function $f_x(z)$ is given by (1.3).

So, the only difference between (2.1) and (2.14) is in their integrands on the right-hand side of (2.14), and in the particular case $r_1 = 1$, $r_i = 0$, $i \geq 2$ we clearly arrive at the same expression as (2.1).

The explicit results associated with recurrence relation (2.14) is given later in the paper. Apparently, the similar system of equations as (2.3) – (2.6) is satisfied for the characteristics of the state dependent queueing system $M^X/G/1$. Namely,

$$\mathsf{E}\widetilde{T}_L = \mathsf{E}\widetilde{T}_L^{(1)} + \mathsf{E}\widetilde{T}_L^{(2)}, \tag{2.15}$$

$$\mathsf{E}\widetilde{\nu}_L = \mathsf{E}\widetilde{\nu}_L^{(1)} + \mathsf{E}\widetilde{\nu}_L^{(2)}, \tag{2.16}$$

$$\mathsf{E}\widetilde{T}_L^{(1)} = \frac{1}{\mu_1}\mathsf{E}\widetilde{\nu}_L^{(1)}, \tag{2.17}$$

$$\mathsf{E}\widetilde{T}_L^{(2)} = \frac{1}{\mu_2}\mathsf{E}\widetilde{\nu}_L^{(2)}. \tag{2.18}$$

Therefore, the same linear representations via $\mathsf{E}\widetilde{T}_L^{(1)}$ hold for characteristics of these systems, where by ρ_1 and ρ_2 one now should mean the expected numbers of *arrived customers* per service time (not the expected number of arrivals) having the probability distribution function $B_1(x)$ and, respectively, $B_2(x)$. In other words, for all $L = 1, 2, \ldots$, we have $\mathsf{E}\widetilde{T}_L = a + b\mathsf{E}\widetilde{T}_L^{(1)}$ with

$$a = \frac{\rho_2}{\lambda(1-\rho_2)}, \quad b = \frac{\rho_1 - \rho_2}{\rho_1(1-\rho_2)}. \tag{2.19}$$

Let us now consider the length of a busy period T_L and associated random variables $T_L^{(1)}$, $T_L^{(2)}$, ν_L, $\nu_L^{(1)}$ and $\nu_L^{(2)}$. In the following consideration, all these characteristics are associated with queueing models, in which a batch size that starts a busy period (initial batch size) can be different. The original batch size that starts a busy period, ς_1, has the distribution $\mathsf{Pr}\{\varsigma_1 = i\} = r_i$. Then, the random variable $\zeta_1 = \kappa_1 - 1 + \varsigma_1$ is the total number of customers in the system after the service completion of the first customer in a busy period (or the number of independent busy cycles after the service completion of the first customer in a busy period), where the random variable κ_1 denotes the total number of customers arrived during the service time of the first customer in a busy period. Recall (see relation (2.13)) that $\mathsf{E}\zeta_1 = \rho_1 - 1 + \mathsf{E}\varsigma_1$.

Based on this, a busy period will be denoted $T_L(\zeta_1)$ and the basic characteristics of the queueing system associated with the busy period will be denoted $T_L^{(1)}(\zeta_1)$, $T_L^{(2)}(\zeta_1)$, $\nu_L(\zeta_1)$, $\nu_L^{(1)}(\zeta_1)$ and $\nu_L^{(2)}(\zeta_1)$.

Another value of the initial characteristic of these random functions that is considered below is $\zeta_1 \wedge L$, where $a \wedge b$ denotes $\min\{a, b\}$. That is, instead of the argument ζ_1 in the random functions we will consider the argument $\zeta_1 \wedge L$, by restricting the space of possible events in which

$\varsigma_1+\kappa_1 \le L+1$. Then, the queueing models with different initial characteristics ζ_1 and $\zeta_1 \wedge L$ are assumed to be given on the same probability space, and the corresponding notation for the characteristics of queueing system, in which that argument is $\zeta_1 \wedge L$, is $T_L(\zeta_1 \wedge L)$, $T_L^{(1)}(\zeta_1 \wedge L)$, $T_L^{(2)}(\zeta_1 \wedge L)$, $\nu_L(\zeta_1 \wedge L)$, $\nu_L^{(1)}(\zeta_1 \wedge L)$ and $\nu_L^{(2)}(\zeta_1 \wedge L)$.

The busy period $T_L(\zeta_1)$ can be represented

$$T_L(\zeta_1)\overset{d}{=}\chi_1+\sum_{i=1}^{\zeta_1\wedge L}\widetilde{T}_{L-i+1}+\sum_{i=1}^{\zeta_1-L}\widetilde{T}_{0,i}, \tag{2.20}$$

where χ_1 is the service time of the first customer;

1-busy periods $\widetilde{T}_{L-i+1}$, $i=1,2,\ldots,L$ are mutually independent, and ς_1 and κ_1 are independent of these 1-busy periods; hence, ζ_1 is also independent of the aforementioned 1-busy periods;

$\widetilde{T}_{0,i}$, $i=1,2,\ldots$, is a sequence of independent and identically distributed 1-busy periods of the $M^X/G/1$ queueing system, the service times of which all are independent and identically distributed random variables having the probability distribution function $B_2(x)$, and the distributions of interarrival times and batch sizes are the same as in the original state dependent queueing system;

$\overset{d}{=}$ denotes the equality in distribution;

in the case where $\zeta_1-L\le 0$, the empty sum in (2.20) is assumed to be zero.

In turn, the representation for $T_L^{(1)}(\zeta_1)$ is as follows:

$$T_L^{(1)}(\zeta_1)\overset{d}{=}\chi_1+\sum_{i=1}^{\zeta_1\wedge L}\widetilde{T}^{(1)}_{L-i+1}. \tag{2.21}$$

Notice, that along with (2.21) we also have

$$T_L^{(1)}(\zeta_1\wedge L)\overset{d}{=}\chi_1+\sum_{i=1}^{(\zeta_1\wedge L)\wedge L}\widetilde{T}^{(1)}_{L-i+1}=\chi_1+\sum_{i=1}^{\zeta_1\wedge L}\widetilde{T}^{(1)}_{L-i+1}. \tag{2.22}$$

That is, $T_L^{(1)}(\zeta_1)$ and $T_L^{(1)}(\zeta_1\wedge L)$ coincide in distribution.

Whereas $\mathsf{E}\widetilde{T}_L^{(1)}(\zeta_1)$ is determined by recurrence relation (2.14), which is a particular case of (2.2), the convolution type recurrence relation, as it is mentioned in Section 2.2.2, is no longer valid for $\mathsf{E}T_L^{(1)}(\zeta_1)$.

2.2.4. *Techniques of asymptotic analysis for $\mathsf{E}T_L^{(1)}$ when L large and associated characteristics.* For the following asymptotic analysis of $\mathsf{E}T_L^{(1)}$ and other mean characteristics such as $\mathsf{E}\nu_L^{(1)}$, $\mathsf{E}\nu_L^{(2)}$ we will use the following techniques. Let $\mathcal{F}_L$ denotes the σ-algebra of the random variable $\zeta_1\wedge L$.

Then we have an increasing family of σ-algebras $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}$, where $\mathcal{F}$ is the σ-algebra of the random variable ζ_1.

Apparently, $\mathsf{Pr}\{\zeta_1 \wedge n > N\} \leq \mathsf{Pr}\{\zeta_1 \wedge (n+1) > N\}$ for all $n = 1, 2, \ldots$ and any fixed N, and hence,

$$\lim_{L\to\infty} \mathsf{Pr}\{\zeta_1 \wedge L \leq N\} = \mathsf{Pr}\{\zeta_1 \leq N\}$$

and consequently, for all $n = 1, 2, \ldots$ we have $\mathsf{E}\{\zeta_1 \wedge n\} \leq \mathsf{E}\{\zeta_1 \wedge (n+1)\}$, and consequently

$$\lim_{L\to\infty} \mathsf{E}\{\zeta_1 \wedge L\} = \mathsf{E}\zeta_1.$$

From (2.21) and (2.22) we have

$$\mathsf{E}T_L^{(1)}(\zeta_1 \wedge L) = \mathsf{E}T_L^{(1)}(\zeta_1) \tag{2.23}$$

for any $L \geq 1$.

Based on (2.23) let us now study the mean characteristics $\mathsf{E}T_L(\zeta_1 \wedge L)$, $\mathsf{E}T_L^{(1)}(\zeta_1 \wedge L)$, $\mathsf{E}T_L^{(2)}(\zeta_1 \wedge L)$, $\mathsf{E}\nu_L^{(1)}(\zeta_1 \wedge L)$ and $\mathsf{E}\nu_L^{(2)}(\zeta_1 \wedge L)$ showing first the justice of the linear representations that are similar to those (2.8) and (2.9). Writing $\mathsf{E}\widetilde{T}_i = a + b\mathsf{E}\widetilde{T}_i^{(1)}$, $i = 1, 2, \ldots, L$, where a and b are specified constants, by the total expectation formula we obtain:

$$\begin{aligned}
\mathsf{E}T_L(\zeta_1 \wedge L) &= \mathsf{E}\mathsf{E}\{T_L(\zeta_1 \wedge L)|\mathcal{F}_L\} \\
&= \frac{1}{\mu_1} + \sum_{i=1}^{L} \mathsf{Pr}\{\zeta_1 \wedge L = i\} \sum_{j=1}^{i} \mathsf{E}\widetilde{T}_{L-j+1} \\
&= \frac{1}{\mu_1} + \sum_{i=1}^{L} \mathsf{Pr}\{\zeta_1 \wedge L = i\} \sum_{j=1}^{i} (a + b\mathsf{E}\widetilde{T}_{L-i+1}^{(1)}) \\
&= \frac{1}{\mu_1} + a\sum_{i=1}^{L} i\mathsf{Pr}\{\zeta_1 \wedge L = i\} \\
&\quad\quad + b\sum_{i=1}^{L} \mathsf{Pr}\{\zeta_1 \wedge L = i\} \sum_{j=1}^{i} \mathsf{E}\widetilde{T}_{L-i+1}^{(1)} \\
&= \frac{1}{\mu_1} + a\mathsf{E}(\zeta_1 \wedge L) + b\mathsf{E}\mathsf{E}\{T_L^{(1)}(\zeta_1 \wedge L)|\mathcal{F}_L\} \\
&= \frac{1}{\mu_1} + a\mathsf{E}(\zeta_1 \wedge L) + b\mathsf{E}T_L^{(1)}(\zeta_1 \wedge L).
\end{aligned}$$

Hence, we obtained the representation

$$\mathsf{E}T_L(\zeta_1 \wedge L) = \frac{1}{\mu_1} + a\mathsf{E}(\zeta_1 \wedge L) + b\mathsf{E}T_L^{(1)}(\zeta_1 \wedge L). \tag{2.24}$$

Keeping in mind (2.19) we obtain

$$\mathsf{E}T_L(\zeta_1 \wedge L) = \frac{1}{\mu_1} + \frac{\rho_2}{\lambda(1-\rho_2)}\mathsf{E}(\zeta_1 \wedge L) + \frac{\rho_1 - \rho_2}{\rho_1(1-\rho_2)}\mathsf{E}T_L^{(1)}(\zeta_1 \wedge L).$$

3. Main result

In this section, we formulate the main result of this paper. The main result of the paper is based on heavy-traffic conditions. They are motivated by the fact (mentioned later in Remark 4.12 and based on the statement of Theorem 4.10) that the only case $\rho_1 = 1$ gives finite limit of the functional $J(L)$, as L increases to infinity. In all other cases where ρ_1 is fixed, the functional is not bounded in limit, and only in the cases where $L[\rho_1(L) - 1]$ converges to finite limit, that is positive, negative or zero, may give the optimal solution to the control problem.

So, we start from the heavy traffic conditions and explicit representations for the objective function under these conditions.

3.1. **Heavy traffic conditions.**

Condition 3.1. *Assume that* $L[\rho_1(L) - 1] \to C > 0$, *as* $L \to \infty$. *Assume also that* $\rho_{1,3}(L)$ *is a bounded sequence,* $\mathsf{E}\varsigma^3 < \infty$ *and the limit* $\lim_{L\to\infty} \rho_{1,2}(L) = \widetilde{\rho}_{1,2}$ *exists.*

Condition 3.2. *Assume that* $L[\rho_1(L) - 1] \to C < 0$, *as* $L \to \infty$. *Assume also that* $\rho_{1,3}(L)$ *is a bounded sequence,* $\mathsf{E}\varsigma^3 < \infty$ *and the limit* $\lim_{L\to\infty} \rho_{1,2}(L) = \widetilde{\rho}_{1,2}$ *exists.*

Remark 3.3. The case when $C = 0$ is also considered and is related to both of these conditions.

Let $\widehat{B}_1(s) = \int_0^\infty \mathrm{e}^{-sx}\mathrm{d}B_1(x)$, $s \geq 0$.

Condition 3.4. *Under Condition 3.2 let* $\delta(L) = 1 - \rho_1(L)$, *and assume that there exists* $\delta_0 > 0$ *such that for all* $\delta(L) < \delta_0$ *as* $L \to \infty$, *each of the functional equations* $z = \widehat{B}_1(\lambda - \lambda z)$ *(depending on the parameter* δ*) has a unique solution in the interval* $(1, \infty)$.

Remark 3.5. Conditions 3.1 and 3.2 contain some technical assumptions such as $\rho_{1,3}(L) < \infty$, $\mathsf{E}\varsigma^3 < \infty$ and the existence of the limit $\lim_{L\to\infty} \rho_{1,2}(L) = \widetilde{\rho}_{1,2}$. The aforementioned assumptions are originated from the analytic approach that is based on Taylor's expansion and application of Tauberian theorems. We do think that these assumptions can be avoided by using the direct approach that considers a diffusion approximation of the transient dam process and computes then the stationary distribution of the diffusion. On this way, these technical assumptions can be avoided, however one would have deal with interchange of limits (large L vs. large t). This problem is very hard in general (see discussion of a similar problem in Whitt [34]). For solution of the limits interchange problem in generalized Jackson networks in heavy traffic see Gamarnik and Zeevi [11] and Braverman, Dai and Miyazawa [6]. The further references can be found in [6].

Condition 3.4 is originated from an application of the analytic method of [36]. It requires to consider the class of probability distributions, the Laplace-Stieljes transform of which is analytic in some negative area of Res and use Taylor's expansion for small values of δ. This class of distributions is smaller than that under Conditions 3.1 or 3.2 and implies the existence of all moments of the distributions.

3.2. **Series of objective functions.** Let $\widehat{C}_L(z)=\sum_{j=0}^{L-1}c_{L-j}z^j$ denote a backward generating cost function, and let

$$C(L)=L[\rho_1(L)-1]$$

be the function of L.

We introduce the following series of objective functions corresponding to the cases

(i) $$C(L)>0,$$

(ii) $$C(L)<0,$$

and

(iii) $$C(L)=0,$$

which are subject to minimization.

Below we define three series of objective functions $J^{\text{upper}}[L,C(L)]$ and $J^{\text{lower}}[L,C(L)]$ and $J^0(L,C(L))$ depending on large parameter L and the objective function in the aforementioned marginal case. The first series, $J^{\text{upper}}[L,C(L)]$, is associated with condition (i), the second one, $J^{\text{lower}}[L,C(L)]$, with condition (ii) and the last case is associated with condition (iii).

The problem is to find a function $\rho_1(L)$ under which, as $L\to\infty$, the functionals $J^{\text{upper}}[L,C(L)]$ or $J^{\text{lower}}[L,C(L)]$ (in dependence which of the conditions is satisfied) converges in limit, as $L\to\infty$, to minimum. If $\rho_1(L)$ converges to 1, then we arrive at the limiting case associated with (iii), where $\widetilde{\rho}_{1,2}=\lim_{L\to\infty}\rho_{1,2}(L)$ and $c^*=\lim_{L\to\infty}c^0(L)$ (see also Remark 3.6 below).

3.2.1. *Series of objective functions corresponding to the case (i).*

$$\begin{aligned}
&J^{\text{upper}}[L,C(L)]\\
&=C(L)\left[j_1\frac{1}{\exp\left(\frac{2C(L)\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1}\right.\\
&\quad\left.+j_2\frac{\rho_2\exp\left(\frac{2C(L)\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)}{(1-\rho_2)\left(\exp\left(\frac{2C(L)\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1\right)}\right]\\
&\quad+c^{\text{upper}}[L,C(L)],
\end{aligned}\tag{3.1}$$

where

$$
\begin{aligned}
&c^{\mathrm{upper}}[L, C(L)] \\
&= \frac{2C(L)\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma} \cdot \frac{\exp\left(\frac{2C(L)\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)}{\exp\left(\frac{2C(L)\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right) - 1} \\
&\quad \times \frac{1}{L}\,\widehat{C}_L\left(1 - \frac{2C(L)\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma]L}\right).
\end{aligned}
\tag{3.2}
$$

The series of objective functions given by (3.1) and (3.2) is used in Proposition 6.2.

3.2.2. *Series of objective functions corresponding to the case (ii).*

$$
\begin{aligned}
&J^{\mathrm{lower}}[L, C(L)] \\
&= -C(L)\left[j_1 \exp\left(-\frac{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma}{2C(L)\mathsf{E}\varsigma}\right)\right. \\
&\quad \left. + j_2 \frac{\rho_2}{1-\rho_2}\left(\exp\left(-\frac{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma}{2C(L)\mathsf{E}\varsigma}\right) - 1\right)\right] \\
&\quad + c^{\mathrm{lower}}[L, C(L)],
\end{aligned}
\tag{3.3}
$$

where

$$
\begin{aligned}
&c^{\mathrm{lower}}[L, C(L)] \\
&= -\frac{2C(L)\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma} \cdot \frac{1}{\exp\left(-\frac{2C(L)\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right) - 1} \\
&\quad \times \frac{1}{L}\,\widehat{C}_L\left(1 - \frac{2C(L)\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma]L}\right).
\end{aligned}
\tag{3.4}
$$

The series of objective functions given by (3.3) and (3.4) is used in Proposition 6.3.

3.2.3. *Series of objective functions corresponding to the case (iii).*

$$
\begin{aligned}
J^0(L) = & j_1 \frac{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma}{2} \\
& + j_2 \frac{\rho_2}{1-\rho_2}\frac{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma}{2} \\
& + c^0(L),
\end{aligned}
\tag{3.5}
$$

where

$$
c^0(L) = \frac{1}{L}\sum_{i=1}^{L} c_i
$$

Remark 3.6. The series of the second moments $\rho_{1,2}(L)$ that are used in (3.1), (3.2), (3.3), (3.4) and (3.5) is assumed to converge to the limit $\widetilde{\rho}_{1,2}$.

3.3. **Formulation of the main result.**

Theorem 3.7. *Under the assumption that the costs c_i are nonincreasing, and under Conditions 3.1, 3.2 and 3.4, a solution to the control problem do exist and unique in the sense explained below. The solution to the control problem in* (1.4) – (1.7) *is defined as follows.*

Let $\overline{J}$ be the minimum value of the possible limits

$$\overline{J}^{\mathrm{upper}}(C)=\lim_{L\to\infty}J^{\mathrm{upper}}[L,C(L)]$$

for the series of objective function $J^{\mathrm{upper}}[L,C(L)]$ defined in (3.1) *and* (3.2) *and, respectively, let $\underline{J}$ be the minimum value of the possible limits*

$$\underline{J}^{\mathrm{lower}}(C)=\lim_{L\to\infty}J^{\mathrm{lower}}[L,C(L)]$$

or

$$\underline{J}^{0}(0)=\lim_{L\to\infty}J^{0}(L)$$

for the series of objective functions $J^{\mathrm{lower}}[L,C(L)]$ defined in (3.3) *and* (3.4) *or $J^0(L)$ defined in* (3.5)*. Then there is a function $\rho_1(L)$ satisfying the following properties.*

If $j_1>j_2\rho_2/(1-\rho_2)$ is satisfied, then only for a positive limit

$$\lim_{L\to\infty}L[\rho_1(L)-1]=\overline{C}>0,$$

the optimal value of the objective function, $\overline{J}$, is reached.

Otherwise, for the optimal value of the objective function, the limit

$$\lim_{L\to\infty}L[\rho_1(L)-1]$$

can be positive, negative or zero.

4. Asymptotic theorems for the stationary probabilities p_1 and p_2

In this section, the explicit expressions are derived for the stationary probabilities, and their asymptotic behavior is studied. These results will be used in our further findings of the optimal solution.

4.1. **Preliminaries.**

In this section we recall the main properties of recurrence relation (2.2). The detailed theory of these recurrence relations can be found in Takács [31]. For the generating function $Q(z)=\sum_{j=0}^{\infty}Q_jz^j$, $|z|\leq 1$, we have

$$Q(z)=\frac{Q_0F(z)}{F(z)-z},\tag{4.1}$$

where $F(z)=\sum_{j=0}^{\infty}f_jz^j$.

Asymptotic behavior of Q_n as $n\to\infty$ has been studied by Takács [31] and Postnikov [21]. Recall the theorems that are needed in this paper.

Denote $\gamma_m=\lim_{z\uparrow 1}\mathrm{d}^mF(z)/\mathrm{d}z^m$.

Lemma 4.1. *(Takács [31], p.22-23). If $\gamma_1 < 1$ then*

$$\lim_{n\to\infty} Q_n = \frac{Q_0}{1-\gamma_1}. \tag{4.2}$$

If $\gamma_1 = 1$ and $\gamma_2 < \infty$, then

$$\lim_{n\to\infty} \frac{Q_n}{n} = \frac{2Q_0}{\gamma_2}.$$

If $\gamma_1 > 1$, then

$$\lim_{n\to\infty} \left(Q_n - \frac{Q_0}{\delta^n[1-F'(\delta)]} \right) = \frac{Q_0}{1-\gamma_1}, \tag{4.3}$$

where δ is the least in absolute value root of the functional equation $z = F(z)$ and $F'(z)$ is the derivative of $F(z)$.

Lemma 4.2. *(Postnikov [21], Sect.25). Let $\gamma_1 = 1$, $\gamma_2 < \infty$ and $f_0 + f_1 < 1$. Then, as $n \to \infty$,*

$$Q_{n+1} - Q_n = \frac{2Q_0}{\gamma_2} + o(1). \tag{4.4}$$

4.2. **Extension of Takács' lemma.** For the following considerations we also need in extended version of Takács' Lemma 4.1 in the case when $\gamma_1 > 1$.

Let $Q_n(L)$ be a series of number satisfying for each L the system of recurrence relations

$$Q_n(L) = \sum_{i=0}^{n} Q_{n-i+1}(L) f_n(L), \tag{4.5}$$

where $Q_0(L)$ is an arbitrary positive number, and $\sum_{i=0}^{\infty} f_i(L) = 1$, $f_i(L) \geq 0$, and let

$$f_n^* = \lim_{L\to\infty} f_n(L) \tag{4.6}$$

exist. So, (4.5) is an extended version of recurrence relations given by (2.2), where the series parameter L is added, with the limiting sequence given by (4.6).

Assume that for all L

$$\gamma_1(L) = \sum_{i=1}^{\infty} i f_i(L) > 1.$$

So, as L increases to infinity, from (4.6) we have

$$\lim_{L\to\infty} \gamma_1(L) = \gamma_1^*. \tag{4.7}$$

Denote by $F_L(z)$ the series $F_L(z) = \sum_{i=0}^{\infty} f_i(L) z^i$, and by $\delta(L)$ the least in absolute value root of the functional equation $z = F_L(z)$.

Lemma 4.3. *Assume that* $\gamma_1^* > 1$,

$$\lim_{L\to\infty} Q_0(L) = Q_0^*. \tag{4.8}$$

Then,

$$\lim_{L\to\infty}\lim_{n\to\infty} Q_n(L)[\delta(L)]^n = \lim_{n\to\infty} Q_n^*(\delta^*)^n, \tag{4.9}$$

where $Q_n^* = \lim_{L\to\infty} Q_n(L)$ *and* δ^* *is the least in absolute value root of the functional equation* $z = F^*(z)$, $F^*(z) = \lim_{L\to\infty} F_L(z)$.

Proof. It follows from (4.5) that $Q_n(L)$ are defined for all L, and there exists the limit Q_n^* of $Q_n(L)$ as $L \to \infty$. As well, the sequence $\delta(L)$ converges to its limit δ^* which is, due to the assumption $\gamma_1^* > 1$, strictly less than 1. The limit

$$\lim_{n\to\infty} Q_n(L)[\delta(L)]^n$$

is defined and, according to (4.3) of Lemma 4.1, expressed via the quantity $Q_0(L)$ divided by $1 - F_L'[\delta(L)]$, where $F_L'(z)$ is the derivative of $F_L(z)$. Hence, taking into account assumption (4.8) and the convergence $F^*(z) = \lim_{L\to\infty} F_L(z)$, we obtain

$$\lim_{n\to\infty}\lim_{L\to\infty} Q_n(L)[\delta(L)]^n = \lim_{n\to\infty} Q_n^*(\delta^*)^n. \tag{4.10}$$

The convergence in L, depending on γ_1^* and Q_0^* only, is uniform. Hence, the Moore-Osgood theorem on interchanging order of limits is applicable, and (4.9) follows from (4.10). □

4.3. **Exact formulae for p_1 and p_2.** In this section we derive the exact formulae for p_1 and p_2. These formulae follow from the following two steps (see the proof of Lemma 4.4).

1. We establish the linear representations for $\mathsf{E}\nu_L^{(2)}(\zeta_1)$ in terms of $\mathsf{E}\nu_L^{(1)}(\zeta_1)$.
2. Then, the explicit formulae for p_1 and p_2 follow from renewal reward theorem.

Lemma 4.4. *We have:*

$$p_1 = \frac{(1-\rho_2)\mathsf{E}\zeta_1}{\mathsf{E}\zeta_1 + (\rho_1 - \rho_2)\left[\mathsf{E}\nu_L^{(1)}(\zeta_1) - 1\right]}, \tag{4.11}$$

and

$$p_2 = \frac{\rho_2\mathsf{E}\zeta_1 + \rho_2(\rho_1 - 1)\left[\mathsf{E}\nu_L^{(1)}(\zeta_1) - 1\right]}{\mathsf{E}\zeta_1 + (\rho_1 - \rho_2)\left[\mathsf{E}\nu_L^{(1)}(\zeta_1) - 1\right]}, \tag{4.12}$$

where ρ_1 *and* ρ_2 *mean the load parameters of the system, that is, the expected numbers of arrived customers per service time having the probability distribution function* $B_1(x)$ *and, respectively,* $B_2(x)$.

Proof. First, derive the linear representation of $\mathsf{E}\nu_L^{(2)}(\zeta_1)$ via $\mathsf{E}\nu_L^{(1)}(\zeta_1)$. From relation (2.10) and equations (2.3) – (2.6), which also hold true in the case of the present queueing system with batch arrivals, we obtain:

$$\mathsf{E}\nu_L^{(2)}(\zeta_1) = \frac{\mathsf{E}\zeta_1}{1-\rho_2} - \frac{1-\rho_1}{1-\rho_2}\left[\mathsf{E}\nu_L^{(1)}(\zeta_1) - 1\right]. \tag{4.13}$$

Using renewal arguments (e.g. [23]) and relation (2.10), we have:

$$p_1 = \frac{\frac{1}{\lambda}}{\mathsf{E}T_L^{(1)}(\zeta_1) + \mathsf{E}T_L^{(2)}(\zeta_1) + \frac{1}{\lambda}} = \frac{\mathsf{E}\zeta_1}{\mathsf{E}\nu_L^{(1)}(\zeta_1) + \mathsf{E}\nu_L^{(2)}(\zeta_1)} \tag{4.14}$$

and

$$p_2 = \frac{\mathsf{E}T_L^{(2)}(\zeta_1)}{\mathsf{E}T_L^{(1)}(\zeta_1) + \mathsf{E}T_L^{(2)}(\zeta_1) + \frac{1}{\lambda}} = \frac{\rho_2\mathsf{E}\nu_L^{(2)}(\zeta_1)}{\mathsf{E}\nu_L^{(1)}(\zeta_1) + \mathsf{E}\nu_L^{(2)}(\zeta_1)}. \tag{4.15}$$

Now, substituting (4.13) for the right sides of (4.14) and (4.15) we obtain relations (4.11) and (4.12) of this lemma. □

4.4. **Preliminary asymptotic expansions for large L.** The explicit results for p_1 and p_2 that are established in Lemma 4.4 are expressed via the unknown quantity $\mathsf{E}\nu_L^{(1)}(\zeta_1)$. So, the aim is to find the asymptotic behaviour of $\mathsf{E}\nu_L^{(1)}(\zeta_1)$ as L increases to infinity and thus to find the asymptotic behaviour of p_1 and p_2.

In this section we obtain some preliminary asymptotic representations that follow from the explicit results. Those asymptotic representations will be used in the sequel. The results of this section are as follows.

1. We first establish the linear representation for $\mathsf{E}\widetilde{\nu}_L^{(2)}$ in terms of $\mathsf{E}\widetilde{\nu}_L^{(1)}$ (see Lemma 4.5).

2. By similar way, we derive the representation for $\mathsf{E}\nu_L^{(2)}(\zeta_1 \wedge L)$ in terms of $\mathsf{E}\nu_L^{(1)}(\zeta_1 \wedge L)$ (see Lemma 4.6).

3. Based on that representation, we prove that $\mathsf{E}\nu_L^{(2)}(\zeta_1) - \mathsf{E}\nu_L^{(2)}(\zeta_1 \wedge L) = o(1)$ as $L \to \infty$ (see Lemma 4.7).

Lemma 4.5. *For $\mathsf{E}\widetilde{\nu}_L^{(2)}$, $L = 1, 2, \ldots$, we have the following representation*

$$\mathsf{E}\widetilde{\nu}_L^{(2)} = \frac{\mathsf{E}\varsigma}{1-\rho_2} - \frac{1-\rho_1}{1-\rho_2}\mathsf{E}\widetilde{\nu}_L^{(1)}, \tag{4.16}$$

where $\rho_1 = \lambda\mathsf{E}\varsigma/\mu_1$ and $\rho_2 = \lambda\mathsf{E}\varsigma/\mu_2 < 1$, and $\mathsf{E}\widetilde{\nu}_L^{(1)}$ is given by

$$\mathsf{E}\widetilde{\nu}_L^{(1)} = \sum_{i=0}^{L} \mathsf{E}\widetilde{\nu}_{L-i+1}^{(1)} \int_0^\infty \frac{1}{i!}\frac{\mathrm{d}^i f_x(z)}{\mathrm{d}z^i}\Big|_{z=0} \mathrm{d}B_1(x), \tag{4.17}$$

$\mathsf{E}\widetilde{\nu}_0^{(1)} = 1.$

Proof. Taking into account that the number of arrivals during 1-busy cycle (1-busy period plus idle period) coincides with the number of customers served during the same 1-busy period, according to Wald's identity we have:

$$\lambda \mathsf{E}\varsigma\left(\mathsf{E}\widetilde{T}_L + \frac{1}{\lambda}\right) = \lambda \mathsf{E}\varsigma \mathsf{E}\widetilde{T}_L + \mathsf{E}\varsigma = \mathsf{E}\widetilde{\nu}_L = \mathsf{E}\widetilde{\nu}_L^{(1)} + \mathsf{E}\widetilde{\nu}_L^{(2)}.$$

This equality together with (2.15) – (2.18) yields the desired statement of the lemma, where (4.17) in turn follows from (2.14) and Wald's identity (2.17). □

The next step is to derive representations for $\mathsf{E}\nu_L^{(1)}(\zeta_1\wedge L)$ and $\mathsf{E}\nu_L^{(2)}(\zeta_1\wedge L)$. We have the following lemma.

Lemma 4.6. *For $\mathsf{E}\nu_L^{(2)}(\zeta_1 \wedge L)$ we have*

$$\mathsf{E}\nu_L^{(2)}(\zeta_1 \wedge L) = \frac{\mathsf{E}(\zeta_1 \wedge L)}{1-\rho_2} - \frac{1-\rho_1}{1-\rho_2}\left[\mathsf{E}\nu_L^{(1)}(\zeta_1 \wedge L) - 1\right], \tag{4.18}$$

where similarly to (2.22)

$$\mathsf{E}\nu_L^{(1)}(\zeta_1 \wedge L) = 1 + \mathsf{E}\sum_{i=1}^{\zeta_1\wedge L}\widetilde{\nu}_{L-i+1}^{(1)}, \tag{4.19}$$

and $\mathsf{E}\widetilde{\nu}_{L-i+1}^{(1)}$, $i = 1, 2, \ldots, L$, are given by (4.17).

Proof. Following the same arguments as in the proof of (2.24), one can write

$$\mathsf{E}\nu_L^{(2)}(\zeta_1 \wedge L) = a\mathsf{E}(\zeta_1 \wedge L) + b\left[\mathsf{E}\nu_L^{(1)}(\zeta_1 \wedge L) - 1\right] \tag{4.20}$$

for the specified constants a and b, for which the linear representation $\mathsf{E}\widetilde{\nu}_L^{(2)} = a + b\mathsf{E}\widetilde{\nu}_L^{(1)}$ is satisfied. Hence, according to relation (4.16) of Lemma 4.5, $a = 1/(1-\rho_2)$ and $b = -(1-\rho_1)/(1-\rho_2)$. The proof is completed. □

The following lemma yields an estimate for the difference $\mathsf{E}\nu_L^{(2)}(\varsigma_1) - \mathsf{E}\nu_L^{(2)}(\varsigma_1 \wedge L)$.

Lemma 4.7. *As $L \to \infty$,*

$$\mathsf{E}\nu_L^{(2)}(\zeta_1) - \mathsf{E}\nu_L^{(2)}(\zeta_1 \wedge L) = o(1). \tag{4.21}$$

Proof. It follows from (2.23) and Wald's identity that

$$\mathsf{E}\nu_L^{(1)} = \mathsf{E}\nu_L^{(1)}(\zeta_1 \wedge L).$$

Hence, (4.20) can be rewritten in the form

$$\mathsf{E}\nu_L^{(2)}(\zeta_1 \wedge L) = \frac{1}{1-\rho_2}\mathsf{E}(\zeta_1 \wedge L) - \frac{1-\rho_1}{1-\rho_2}\left[\mathsf{E}\nu_L^{(1)}(\zeta_1) - 1\right],$$

and (4.21) follows. □

4.5. **Asymptotic theorems for p_1 and p_2 under 'usual assumptions'.** By 'usual assumption' we mean the standard cases as $\rho_1 < 1$ or $\rho_1 > 1$ for the asymptotic behaviour as $L \to \infty$. In the following sections the heavy traffic assumptions are assumed. By the heavy-traffic assumptions, we mean such a case where for some positive constant c

$$-c < \lim_{L\to\infty} L[\rho_1(L) - 1] < c, \tag{4.22}$$

and $\rho_2 < 1$.

The main result of Section 4.3 is Lemma 4.4, where the stationary probabilities p_1 and p_2 are expressed explicitly via $\mathsf{E}\nu_L^{(1)}(\zeta_1)$. The aim of this section is to obtain the analogue of asymptotic Theorem 3.1 of [4]. To this end, we do as follows.

1. We first study the asymptotic behavior of $\mathsf{E}\widetilde{\nu}_L^{(1)}$ as $L \to \infty$. For this purpose derive the representation for the generating function $\sum_{j=0}^{\infty} \mathsf{E}\widetilde{\nu}_j^{(1)} z^j$. Then, we obtain the asymptotic behaviour of $\mathsf{E}\widetilde{\nu}_L^{(1)}$ as $L \to \infty$ by using Takács' theorem (Lemma 4.1 or Lemma 4.3 containing an extension of Takács' theorem) and Postnikov's theorem (Lemma 4.2).

2. Then, we derive asymptotic representation for $\mathsf{E}\nu_L^{(1)}(\zeta_1 \wedge L)$ as $L \to \infty$, and on the basis of this representation and renewal reward theorem (e.g. [23]) we find asymptotic behaviour of stationary probabilities p_1 and p_2 as $L \to \infty$.

To derive the generative function $\sum_{j=0}^{\infty} \mathsf{E}\widetilde{\nu}_j^{(1)} z^j$, we use representation (4.17). This yields:

$$\begin{aligned}\sum_{j=0}^{\infty} \mathsf{E}\widetilde{\nu}_j^{(1)} z^j &= \sum_{j=0}^{\infty} z^j \sum_{i=0}^{j} \mathsf{E}\widetilde{\nu}_{L-i+1}^{(1)} \int_0^{\infty} \frac{1}{i!} \frac{\mathrm{d}^i f_x(u)}{\mathrm{d}u^i}\Big|_{u=0} \mathrm{d}B_1(x)\\ &= \frac{U(z)}{U(z) - z},\end{aligned} \tag{4.23}$$

where

$$\begin{aligned}U(z) &= \int_0^{\infty} \exp\left\{-\lambda x \left(1 - \sum_{i=1}^{\infty} r_i z^i\right)\right\} \mathrm{d}B_1(x)\\ &= \widehat{B}_1(\lambda - \lambda \widehat{R}(z)).\end{aligned} \tag{4.24}$$

(By $\widehat{B}_1(s)$, $s \geq 0$, we denote the Laplace-Stieltjes transform of $B_1(x)$, and $\widehat{R}(z) = \sum_{i=1}^{\infty} r_i z^i$, $|z| \leq 1$.) Hence, from (4.24) and (4.23) we obtain:

$$\sum_{j=0}^{\infty} \mathsf{E}\widetilde{\nu}_j^{(1)} z^j = \frac{\widehat{B}_1(\lambda - \lambda \widehat{R}(z))}{\widehat{B}_1(\lambda - \lambda \widehat{R}(z)) - z}. \tag{4.25}$$

Notice, that the right-hand side of (4.23) and, hence, that of (4.25) has the same form as (4.1). Therefore we can use Lemmas 4.1 and 4.2, and according to these lemmas, the asymptotic behaviour of $\mathsf{E}\widetilde{\nu}_L^{(1)}$, as $L \to \infty$, is given by the following statements.

Lemma 4.8. *If* $\rho_1 < 1$, *then*

$$\lim_{L\to\infty} \mathsf{E}\widetilde{\nu}_L^{(1)} = \frac{1}{1-\rho_1}. \tag{4.26}$$

If $\rho_1 = 1$, *and additionally* $\rho_{1,2} = \int_0^\infty (\lambda x)^2 dB_1(x) < \infty$ *and* $\mathsf{E}\varsigma^2 < \infty$, *then*

$$\mathsf{E}\widetilde{\nu}_L^{(1)} - \mathsf{E}\widetilde{\nu}_{L-1}^{(1)} = \frac{2\mathsf{E}\varsigma}{\rho_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma} + o(1). \tag{4.27}$$

If $\rho_1 > 1$, *then*

$$\lim_{L\to\infty}\left[\mathsf{E}\widetilde{\nu}_L^{(1)} - \frac{1}{\varphi^L[1+\lambda\widehat{B}_1'(\lambda-\lambda\widehat{R}(\varphi))\widehat{R}'(\varphi)]}\right] = \frac{1}{1-\rho_1}, \tag{4.28}$$

where $\varphi < 1$ *is the least positive root of the functional equation* $z = \widehat{B}_1(\lambda - \lambda\widehat{R}(z))$.

The proof of this lemma is given in Section 10.

Remark 4.9. In the case when ρ_1 is the function of L, relation (4.28) of Lemma 4.8 is rewritten as follows.

$$\lim_{L\to\infty} \varphi^L \mathsf{E}\widetilde{\nu}_L^{(1)} = \frac{1}{\varphi^L[1+\lambda\widehat{B}_1'(\lambda-\lambda\widehat{R}(\varphi))\widehat{R}'(\varphi)]}. \tag{4.29}$$

where φ is the root of functional equation associated with the limiting functions depending of L, as $L \to \infty$. Limit relation (4.29) is based on application of Lemma 4.3 instead of Lemma 4.1.

With the aid of Lemma 4.8 one can obtain the statements on asymptotic behavior of $\mathsf{E}\nu_L^{(1)}(\zeta_1)$, $\mathsf{E}\nu_L^{(1)}(\zeta_1 \wedge L)$ and, consequently, p_1 and p_2. The theorem below characterizes asymptotic behavior of the probabilities p_1 and p_2 as $L \to \infty$.

Theorem 4.10. *If* $\rho_1 < 1$, *then*

$$\lim_{L\to\infty} p_1(L) = 1 - \rho_1, \tag{4.30}$$

$$\lim_{L\to\infty} p_2(L) = 0. \tag{4.31}$$

If $\rho_1 = 1$, *and additionally* $\rho_{1,2} = \int_0^\infty (\lambda x)^2 dB_1(x) < \infty$ *and* $\mathsf{E}\varsigma^2 < \infty$, *then*

$$\lim_{L\to\infty} Lp_1(L) = \frac{\rho_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma}{2\mathsf{E}\varsigma}, \tag{4.32}$$

$$\lim_{L\to\infty} Lp_2(L) = \frac{\rho_2}{1-\rho_2} \cdot \frac{\rho_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma}{2\mathsf{E}\varsigma}. \tag{4.33}$$

If $\rho_1 > 1$, *then*

$$\lim_{L\to\infty} \frac{p_1(L)}{\varphi^L} = \frac{(1-\rho_2)[1+\lambda\widehat{B}_1'(\lambda-\lambda\widehat{R}(\varphi))\widehat{R}'(\varphi)]}{(\rho_1-\rho_2)}, \tag{4.34}$$

$$\lim_{L\to\infty} p_2(L) = \frac{\rho_2(\rho_1-1)}{\rho_1-\rho_2}, \tag{4.35}$$

where φ *is defined in the formulation of Lemma 4.8.*

The proof of this theorem is given in Section 10.

Remark 4.11. In the case when ρ_1 is the function of L, relations (4.34) and (4.35) of Lemma 4.10 remain the same, since an application of Lemma 4.3 instead of Lemma 4.1 gives the same result. In this case, φ is the root of functional equation associated with the limiting functions depending of L, as $L \to \infty$.

Remark 4.12. It follows from Theorem 4.10, under the assumption $\rho_1 < 1$ we have (4.30) and (4.31). The probability $p_1(L)$ tends to the positive limit as $L \to \infty$, while the probability $p_2(L)$ vanishes as $L \to \infty$. Then, for large L, the functional $J = J(L)$ in (1.4) is estimated as $J \approx (1-\rho_1)J_1 = (1-\rho_1)j_1L$, that is, it increases proportionally to large parameter L.

Under the assumption $\rho_1 > 1$ we have (4.34) and (4.35). Then, for large L, the probability $p_1(L)$ is estimated as

$$p_1(L) \asymp \frac{(1-\rho_2)[1+\lambda\widehat{B}_1'(\lambda-\lambda\widehat{R}(\varphi))\widehat{R}'(\varphi)]}{(\rho_1-\rho_2)}\varphi^L,$$

that is, tends to zero since $\varphi < 1$. The probability $p_2(L)$ converges to the positive limit as $L \to \infty$. This means, that for large L, the functional $J = J(L)$ in (1.4) is estimated as $J \approx \rho_2(\rho_1-1)/(\rho_1-\rho_2)J_2 = \rho_2(\rho_1-1)/(\rho_1-\rho_2)j_2L$. That is, similarly to the case $\rho_1 < 1$ it tends to infinity with the rate proportional to L.

Under the assumption $\rho_1 = 1$ following the limits (4.32) and (4.33) the limit of $J(L)$ is finite. So, the only case $\rho_1 = 1$ among these three cases $\rho_1 < 1$, $\rho_1 = 1$ and $\rho_1 > 1$ can be a "candidate" to the optimal solution. In fact, the case $\rho_1 = 1$ belongs to the class of heavy traffic conditions given by (4.22), an optimal solution must belong to the set of traffic parameters $\rho_1(L)$ such as there is the limit $L\rho_1(L)$ depending on parameters j_1, j_2 and the sequence of costs $\{c_i\}$ depending of water levels in the dam.

4.6. **Asymptotic theorems for p_1 and p_2 under special heavy traffic conditions.** In this section we establish asymptotic theorems for p_1 and p_2 under heavy traffic assumptions where (j) $\rho_1 = 1 + \delta(L)$ or (jj) $\rho_1 = 1 - \delta(L)$, and $\delta(L)$ is a vanishing positive parameter as $L \to \infty$.

The theorems presented in this section are analogues of the theorems of [4] given in Section 4 of that paper. The conditions are special, because these heavy traffic conditions include the change of the parameter ρ_1 as L increases to infinity and $\delta(L)$ vanishes, but the other load parameter ρ_2 remains unchanged.

In case (j) we have the following two theorems.

Theorem 4.13. *Under Condition 3.1 we have*

$$Lp_1 = \frac{C}{\exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1}[1+o(1)], \tag{4.36}$$

$$Lp_2 = \frac{C\rho_2\exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)}{(1-\rho_2)\left[\exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1\right]}[1+o(1)]. \tag{4.37}$$

The proof of this theorem is given in Section 10.

Theorem 4.14. *Under Condition 3.1 assume that $L\delta(L)\to 0$. Then,*

$$\lim_{L\to\infty} Lp_1(L) = \frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2\mathsf{E}\varsigma}, \tag{4.38}$$

$$\lim_{L\to\infty} Lp_2(L) = \frac{\rho_2}{1-\rho_2}\frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2\mathsf{E}\varsigma}. \tag{4.39}$$

Proof. The statement of the theorem follows by expanding the main terms of asymptotic relations (4.36) and (4.37) for small C. ☐

In case (jj) we have the following two theorems.

Theorem 4.15. *Under Condition 3.2 we have:*

$$p_1 = \delta\exp\left(-\frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2C\mathsf{E}\varsigma}\right)[1+o(1)], \tag{4.40}$$

$$p_2 = \frac{\delta\rho_2\left[\exp\left(-\frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2C\mathsf{E}\varsigma}\right)-1\right]}{1-\rho_2}[1+o(1)]. \tag{4.41}$$

The proof of this theorem is given in Section 10.

Theorem 4.16. *Under Condition 3.2 assume that $L\delta\to 0$. Then we have* (4.38) *and* (4.39).

Proof. The proof of the theorem follows by expanding the main terms of the asymptotic relations (4.40) and (4.41) for small C. ☐

5. Asymptotic theorems for the stationary probabilities q_i

The aim of this section is asymptotic analysis of the stationary probabilities q_i, $i=1,2,\ldots,L$ as $L\to\infty$. The challenge is to first obtain the explicit representation for q_i in terms of $\mathsf{E}\nu_i^{(1)}(\zeta_1)$, and then to study the asymptotic behavior of q_i as $L \to \infty$ on the basis of the known asymptotic results for $\mathsf{E}\nu_L^{(1)}(\zeta_1)$ as $L \to \infty$.

To find asymptotic behaviour of stationary probabilities we need the following major steps.

1. First, we derive representation for stationary probabilities in terms of $\mathsf{E}\nu_i^{(1)}(\zeta_1)$, $i = 0, 1, \ldots$, where $\mathsf{E}\nu_i^{(1)}(\zeta_1)$ with lower index i has the similar meaning as $\mathsf{E}\nu_L^{(1)}(\zeta_1)$ with the replacement of the level L by i (see Lemma 5.1).

2. Then, we study asymptotic behaviour of the difference $\mathsf{E}\nu_{L-j}^{(1)}(\zeta_1) - \mathsf{E}\nu_{L-j-1}^{(1)}(\zeta_1)$ as $L \to \infty$. This difference is an important part of the formula that defines the asymptotic behaviour of the stationary probabilities.

3. The asymptotic behaviour of the aforementioned difference should be studied for the following three cases $\rho_1 = 1$, $\rho_1 = 1 + \delta(L)$ and $\rho_1 = 1 - \delta(L)$, where $\delta(L)$ is a positive vanishing value. The first two cases are based on a standard study based on asymptotic behaviour of the difference $\mathsf{E}\nu_{L-j}^{(1)}(\zeta_1) - \mathsf{E}\nu_{L-j-1}^{(1)}(\zeta_1)$, and the main results for this study are Theorems 5.2 and 5.3. The third case is more complicated and involves special results on asymptotic behaviour of this type of sequences [36]. As well, some special additional assumptions are required (see Theorem 5.4).

5.1. **Explicit representation for the stationary probabilities q_i.** The aim of this section is to prove the following statement.

Lemma 5.1. *For $i = 1, 2, \ldots, L$ we have*

$$q_i = \rho_1 p_1 \left(\mathsf{E}\nu_i^{(1)}(\zeta_1) - \mathsf{E}\nu_{i-1}^{(1)}(\zeta_1) \right). \tag{5.1}$$

Proof. Rewriting relation (2.12) in the form

$$\mathsf{E}T_L(\zeta_1) - \frac{1}{\mu_1} + \frac{\mathsf{E}\zeta_1}{\lambda \mathsf{E}\varsigma} = \frac{[\mathsf{E}\nu_L(\zeta_1) - 1]}{\lambda \mathsf{E}\varsigma}, \tag{5.2}$$

and now using renewal arguments (e.g. [23]), relation (5.2) and Wald's identities

$$\mathsf{E}T_i^{(1)}(\zeta_1) - \frac{1}{\mu_1} = \frac{\rho_1}{\lambda \mathsf{E}\varsigma} \left[\mathsf{E}\nu_i^{(1)}(\zeta_1) - 1 \right], \ i = 1, 2, \ldots, L,$$

we obtain:

$$q_i = \frac{\mathsf{E}T_i^{(1)}(\zeta_1) - \mathsf{E}T_{i-1}^{(1)}(\zeta_1)}{\mathsf{E}T_L(\zeta_1) + \frac{1}{\lambda}} = \rho_1 \frac{\mathsf{E}\nu_i^{(1)}(\zeta_1) - \mathsf{E}\nu_{i-1}^{(1)}(\zeta_1)}{\mathsf{E}\nu_L(\zeta_1)}, \tag{5.3}$$
$$i = 1, 2, \ldots, L.$$

Taking into account that $\mathsf{E}\nu_L(\zeta_1) = \mathsf{E}\nu_L^{(1)}(\zeta_1) + \mathsf{E}\nu_L^{(2)}(\zeta_1)$ and then applying the linear representation for $\mathsf{E}\nu_L^{(2)}(\zeta_1)$ given by (4.13), from (5.3) we obtain:

$$q_i = \frac{\rho_1(1-\rho_2)\mathsf{E}\zeta_1}{\mathsf{E}\zeta_1 + (\rho_1-\rho_2)\left[\mathsf{E}\nu_L^{(1)}(\zeta_1) - 1\right]}\left(\mathsf{E}\nu_i^{(1)}(\zeta_1) - \mathsf{E}\nu_{i-1}^{(1)}(\zeta_1)\right),$$
$$i = 1, 2, \ldots, L.$$

Hence, representation (5.1) follows from (4.11) (see Lemma 4.4), and Lemma 5.1 is proved. □

5.2. **Asymptotic analysis of the stationary probabilities q_i: The case $\rho_1 = 1$.** Let us study asymptotic behavior of the stationary probabilities q_i. We start from the following modified version of (4.27) (Lemma 4.8):

$$\mathsf{E}\widetilde{\nu}_{L-j}^{(1)} - \mathsf{E}\widetilde{\nu}_{L-j-1}^{(1)} = \frac{2\mathsf{E}\varsigma}{\rho_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma} + o(1), \tag{5.4}$$

which is assumed to be satisfied under the conditions $\rho_{1,2} < \infty$ and $\mathrm{E}\varsigma^2 < \infty$. Under the same conditions, similarly to (10.3) we obtain:

$$\begin{aligned}\mathsf{E}\nu_{L-j}^{(1)}(\zeta_1 \wedge L) - &\mathsf{E}\nu_{L-j-1}^{(1)}(\zeta_1 \wedge L)\\ &= \frac{2\mathsf{E}\varsigma}{\rho_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma}\\ &\quad\times \sum_{i=1}^{L} i\mathsf{Pr}\{\zeta_1 \wedge L = i\} + o(1)\\ &= \frac{2\mathsf{E}\varsigma\mathsf{E}\zeta_1}{\rho_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma} + o(1).\end{aligned} \tag{5.5}$$

Hence, since for any $j < L$,

$$\mathsf{E}\nu_{L-j}^{(1)}(\zeta_1 \wedge L) = \mathsf{E}\nu_{L-j}^{(1)}(\zeta_1 \wedge (L-j)) = \mathsf{E}\nu_{L-j}^{(1)}(\zeta_1),$$

then from (5.5) we have the estimate

$$\mathsf{E}\nu_{L-j}^{(1)}(\zeta_1) - \mathsf{E}\nu_{L-j-1}^{(1)}(\zeta_1) = \frac{2\mathsf{E}\varsigma\mathsf{E}\zeta_1}{\rho_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma} + o(1). \tag{5.6}$$

Asymptotic relations (5.6), (4.32) together with explicit relation (5.1) of Lemma 5.1 leads to the following theorem.

Theorem 5.2. *In the case $\rho_1 = 1$ under the additional conditions $\rho_{1,2} < \infty$ and $\mathrm{E}\varsigma^2 < \infty$ for large values j such that $j/L \to 1$ as $L \to \infty$, we have*

$$\lim_{L\to\infty} Lq_j = 1. \tag{5.7}$$

Note, that the asymptotic relation given by (5.7) is not expressed via $\mathsf{E}\varsigma$ and, therefore, it is invariant and hence the same as that for the queueing system with ordinary Poisson arrivals.

5.3. **Asymptotic analysis of the stationary probabilities q_i: The case $\rho_1 = 1 + \delta(L)$.** In the case $\rho_1 = 1 + \delta(L)$, $\delta > 0$ the asymptotic behaviour of q_i is specified by the following theorem.

Theorem 5.3. *Assume that Condition 3.1 is satisfied and* $\mathrm{E}\varsigma^3 < \infty$. *Then, for all $j \geq 0$, we have*

$$q_{L-j} = \frac{\exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)}{\exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1} \times \left(1 - \frac{2\delta\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma}\right)^j \times \frac{2\delta\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma} + o(\delta). \tag{5.8}$$

The proof of this theorem is given in Section 10.

5.4. **Asymptotic analysis of the stationary probabilities q_i: The case $\rho_1 = 1 - \delta(L)$.** In the case $\rho_1 = 1 - \delta(L)$, $\delta > 0$, the study is more delicate and based on special analysis. The additional assumption of this case is that the class of probability distribution functions $\{B_1(x)\}$ and $\mathsf{Pr}\{\varsigma = i\}$ are given such that there exists a unique root $\tau \in (1, \infty)$ of the equation

$$z = \widehat{B}_1(\lambda - \lambda\widehat{R}(z)), \tag{5.9}$$

and there exists the first derivative $\widehat{B}_1'(\lambda - \lambda\widehat{R}(\tau))$.

Under the assumption that $\rho_1 < 1$ the root of (5.9) is not necessarily exists and if exists it is not necessarily unique. Such type of condition has been considered by Willmot [36] to obtain the asymptotic behavior for high queue-level probabilities in stationary $M/G/1$ queues.

The analysis provided here consists of the following three parts.

1. We consider $M^X/G/1$ queueing system with $\rho_1 < 1$, and under the assumption that the first batch in a busy period consists of only one customer, we derive the asymptotic formula for the stationary probability q_i for large i. The idea is first to extend the asymptotic result of Willmot [36] obtained for the $M/G/1$ queueing system. Similarly to Willmot [36], we show that the stationary probability q_i is presented as $c\tau^{-i}[1 + o(1)]$ with exact explicit expression for the constant c.

2. Based on this result, we then express τ via the numerical characteristics $\mathsf{E}\widetilde{\nu}_{L-j}^{(1)}$, $L \to \infty$, which in turn were studied in Section 4.

3. Then we provide asymptotic study of τ as ρ_1 approaches 1. The study is based on Taylor's expansion for an explicit analytic expression.

Denote the stationary probabilities in the $M/G/1$ queueing system by $q_i[M/G/1]$, $i = 0, 1, \ldots$. It was shown in [36] that

$$q_i[M/G/1] = \frac{(1-\rho_1)(1-\tau)}{\tau^i[1+\lambda\widehat{B}_1'(\lambda-\lambda\tau)]}[1+o(1)] \text{ as } i\to\infty, \tag{5.10}$$

where $\widehat{B}_1(s)$ denotes the Laplace-Stieltjes transform of the service time distribution in the $M/G/1$ queueing system, and τ denotes a root of the equation $z = \widehat{B}_1(\lambda - \lambda z)$ greater than 1, which is assumed to be unique. On the other hand, according to the Pollaczek-Khintchine formula (e.g. Takács [30], p.242), $q_i[M/G/1]$ can be represented explicitly

$$q_i[M/G/1] = (1-\rho_1)\left(\mathsf{E}\nu_i^{(1)} - \mathsf{E}\nu_{i-1}^{(1)}\right), i = 1, 2, \ldots, \tag{5.11}$$

where the random variable $\nu_i^{(1)}$ in this formula is associated with the number of served customers during a busy period of the state dependent $M/G/1$ queueing system, where the value of the system parameter, where the service is changed, is i (see Section 2.1). Representation (5.11) can be easily checked, since in this case

$$\sum_{j=0}^{\infty}\mathsf{E}\nu_j^{(1)}z^j = \frac{\widehat{B}_1(\lambda-\lambda z)}{\widehat{B}_1(\lambda-\lambda z)-z}, \tag{5.12}$$

and multiplication of the right-hand side of (5.12) by $(1-\rho_1)(1-z)$ leads to the well-known Pollaczek-Khintchine formula. Then, from (5.10) and (5.11) there is the asymptotic proportion for large L and any $j \geq 0$:

$$\frac{\mathsf{E}\nu_{L-j}^{(1)} - \mathsf{E}\nu_{L-j-1}^{(1)}}{\mathsf{E}\nu_{L}^{(1)} - \mathsf{E}\nu_{L-1}^{(1)}} = \tau^j[1+o(1)]. \tag{5.13}$$

In the case of batch arrivals the results are similar. One can prove that the same proportion as (5.13) holds in this case as well, where τ in the case of batch arrivals denotes a unique real root of the equation of (5.9), which is greater than 1. (Recall that our convention is an existence of a unique real solution of (5.9) greater than 1.) Indeed, the arguments of [36] are elementary extended for the queueing system with batch arrivals. The simplest way to extend these results straightforwardly is to consider the stationary queueing system with batch Poisson arrivals, in which the first batch in each busy period is equal to 1. Denote this system by $M^{1,X}/G/1$. For this specific system, similarly to (5.10) we obtain:

$$q_i[M^{1,X}/GI/1] = \frac{(1-\rho_1)(1-\tau)}{\tau^i[1+\lambda\widehat{B}_1'(\lambda-\lambda\widehat{R}(\tau))\widehat{R}'(\tau)]}[1+o(1)], \tag{5.14}$$
$$\text{as } i\to\infty,$$

where $q_i[M^{1,X}/GI/1]$, $i=0,1,\ldots$, denotes the stationary probabilities in this system. Then, taking into account (4.25), similarly to (5.11) one can write

$$q_i[M^{1,X}/GI/1]=(1-\rho_1)\left(\mathsf{E}\widetilde{\nu}_i^{(1)}-\mathsf{E}\widetilde{\nu}_{i-1}^{(1)}\right),\ i=1,2,\ldots. \tag{5.15}$$

From (5.14) and (5.15) we obtain

$$\frac{\mathsf{E}\widetilde{\nu}_{L-j}^{(1)}-\mathsf{E}\widetilde{\nu}_{L-j-1}^{(1)}}{\mathsf{E}\widetilde{\nu}_{L}^{(1)}-\mathsf{E}\widetilde{\nu}_{L-1}^{(1)}}=\tau^j[1+o(1)]. \tag{5.16}$$

From (5.16) and the results of Sections 4.4 and 4.5 (see (4.26), the statement on asymptotic behaviour of $\mathsf{E}\nu_L^{(1)}(\zeta_1\wedge L)$ as $L\to\infty$ given in Section 10 (relation (10.2)) and the equality $\mathsf{E}\nu_L^{(1)}(\zeta_1\wedge L)=\mathsf{E}\nu_L^{(1)}(\zeta_1)$), we also have the estimate

$$\frac{\mathsf{E}\nu_{L-j}^{(1)}(\zeta_1)-\mathsf{E}\nu_{L-j-1}^{(1)}(\zeta_1)}{\mathsf{E}\nu_{L}^{(1)}(\zeta_1)-\mathsf{E}\nu_{L-1}^{(1)}(\zeta_1)}=\tau^j[1+o(1)], \tag{5.17}$$

which coincides with (5.13).

Now we formulate and prove a theorem on asymptotic behavior of the stationary probabilities q_i in the case $\rho_1=1-\delta$, $\delta>0$. The special assumption in this theorem is that the class of probability distributions $\{B_1(x)\}$ is defined according to the above convention. More precisely, in the case $\rho_1=1-\delta$, $\delta>0$, and vanishing δ as $L\to\infty$ this means that Condition 3.4 should be satisfied.

Theorem 5.4. *Assume that Conditions 3.2 and 3.4 are satisfied and* $\mathrm{E}\varsigma^3<\infty$. *Then,*

$$\begin{aligned}
q_{L-j}&=\frac{1}{\exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1}\\
&\quad\times\frac{2\delta\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\\
&\quad\times\left(1+\frac{2\delta\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)^j[1+o(1)],
\end{aligned} \tag{5.18}$$

for any $j\geq 0$.

The proof of this theorem is given in Section 10.

6. Derivations for the objective function

6.1. **The case $\rho_1=1$.** In this section we prove the following result.

Proposition 6.1. *In the case $\rho_1 = 1$, under the additional conditions $\rho_{1,2} < \infty$ and* $\mathrm{E}\varsigma^2 < \infty$ *we have:*

$$
\begin{aligned}
\lim_{L\to\infty} J(L) =& j_1 \frac{\rho_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma}{2\mathsf{E}\varsigma} \\
&+ j_2 \frac{\rho_2}{1-\rho_2} \cdot \frac{\rho_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma}{2\mathsf{E}\varsigma} + c^*,
\end{aligned}
\tag{6.1}
$$

where

$$c^* = \lim_{L\to\infty} \frac{1}{L} \sum_{i=1}^{L} c_i.$$

Proof. The first two terms in the right-hand side of (6.1) follow from asymptotic relations (4.32) and (4.33) (Theorem 4.10). The last term c^* of the right-hand side of (6.1) follows from (5.7) (Theorem 5.2), since

$$\lim_{L\to\infty} \sum_{i=1}^{L} q_i c_i = \lim_{L\to\infty} \frac{1}{L} \sum_{i=1}^{L} c_i = c^*.$$

□

6.2. **The case $\rho_1 = 1 + \delta(L)$.** In the case $\rho_1 = 1 + \delta(L)$, $\delta > 0$ we have the following statement.

Proposition 6.2. *Under the assumptions of Theorem 5.3 for the series of objective functions $J(L)$ we have representation* (3.1). *Then, a solution to the control problem is found in the set of possible limits $\overline{J}^{\mathrm{upper}}(C)$.*

The proof of Proposition 6.2 is given in Section 10.

6.3. **The case $\rho_1 = 1 - \delta(L)$.** In the case $\rho_1 = 1 - \delta(L)$, $\delta > 0$ we have the following statement.

Proposition 6.3. *Under the assumptions of Theorem 5.4 for the series of objective functions $J(L)$ we have representation* (3.3). *Then, a solution to the control problem is found in the set of possible limits $\underline{J}^{\mathrm{lower}}(C)$.*

The proof of Proposition 6.3 is given in Section 10.

7. A solution to the control problem and its properties

In this section we discuss the solution to the control problem and study its properties.

7.1. **Alternative representations for the last terms in the objective functions and their properties.** The series of objective functions $J^{\rm upper}[L, C(L)]$ and $J^{\rm lower}[L, C(L)]$ are given by (3.1) and, respectively, by (3.3), and the last terms in these functionals are given by (3.2) and, respectively, by (3.4). For our further analysis we need in other representations for these last terms.

Recall that when $\rho_1(L) = 1+\delta(L)$, the parameter C defined in (1.9) is positive, while in the opposite case $\rho_1(L) = 1-\delta(L)$ it is negative. For the following study of the properties of the possible limits of $c^{\rm upper}[L, C(L)]$ and $c^{\rm lower}[L, C(L)]$ as $L \to \infty$, we need to introduce the function

$$\psi(C) = \lim_{L\to\infty} \frac{\sum_{j=0}^{L-1} c_{L-j}\left(1 - \frac{2C\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j}{\sum_{j=0}^{L-1}\left(1 - \frac{2C\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j} \tag{7.1}$$

and establish its connection with the aforementioned limits. However, for the purposes it is profitable to split this function into two different functions in order to distinguish two case studies. So, instead of (7.1), we consider two functions both defined for a positive argument. Specifically, denoting $D = |C|$ consider the following two functions

$$\psi(D) = \lim_{L\to\infty} \frac{\sum_{j=0}^{L-1} c_{L-j}\left(1 - \frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j}{\sum_{j=0}^{L-1}\left(1 - \frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j}, \tag{7.2}$$

and

$$\eta(D) = \lim_{L\to\infty} \frac{\sum_{j=0}^{L-1} c_{L-j}\left(1 + \frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j}{\sum_{j=0}^{L-1}\left(1 + \frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j}. \tag{7.3}$$

Since $\{c_i\}$ is a nonincreasing and bounded sequence, then the limits of (7.2) and (7.3) do exist.

Denote

$$\overline{c}^{\rm upper}(C) = \lim_{L\to\infty} c^{\rm upper}[L, C(L)],$$

and

$$\underline{c}^{\rm lower}(C) = \lim_{L\to\infty} c^{\rm lower}[L, C(L)],$$

The relations between $\overline{c}^{\rm upper}(C)$ and $\psi(D)$ and, respectively, between $\underline{c}^{\rm lower}(C)$ and $\eta(D)$ are given in the lemma below.

Lemma 7.1. *We have:*

$$\overline{c}^{\rm upper}(C) = \psi(D), \tag{7.4}$$

and

$$\underline{c}^{\rm lower}(C) = \eta(D). \tag{7.5}$$

The proof of this lemma is given in Section 10.

The next lemma establishes the main properties of functions $\psi(D)$ and $\eta(D)$.

Lemma 7.2. *The function $\psi(D)$ is a nonincreasing function, and its maximum is $\psi(0)=c^*$. The function $\eta(D)$ is a nondecreasing function, and its minimum is $\eta(0)=c^*$.*

(Recall that $c^*=\lim_{L\to\infty}(1/L)\sum_{i=1}^{L}c_i$ is defined in Proposition 6.1.)

The proof of this lemma is given in Section 10.

In the following we need in stronger results that is given by Lemma 7.2. Namely, we prove the following lemmas.

Lemma 7.3. *If the sequence $\{c_i\}$ contains at least two distinct values, then the function $\psi(D)$ is a strictly decreasing function, and the function $\eta(D)$ is a strictly increasing function.*

The proof of this lemma is given in Section 10.

Lemma 7.4. *Under assumption of Lemma 7.3 the function $\psi(D)$ is convex and the function $\eta(D)$ is concave.*

The proof of this lemma is given in Section 10.

7.2. **Proof of the main result and discussion of optimal solution.**

7.2.1. *Preliminaries.* Before starting the proof we discuss the structure of an optimal solution if it exists.

As it has already been discussed in Remark 4.12, that a possible optimal solution falls into the category of the heavy traffic conditions, which are specified by the relation between the parameters j_1 and j_2 and the structure of costs c_i. Following Remark 4.12, the existence of a solution is intuitively understandable, since according to relations (4.32) and (4.33) of Theorem 4.10, under the conditions $\rho_1=1$, $\widetilde{\rho}_{1,2}<\infty$ and $\mathsf{E}\varsigma^2<\infty$ both limits $\lim_{L\to\infty}Lp_1(L)$ and $\lim_{L\to\infty}Lp_2(L)$ are finite, and the functional 1.4 must be finite. More rigorous arguments are given in Section 7.2.2.

Here we classify different cases of an optimal solution.

The three possible cases of the heavy traffic conditions are as follows.

Case 1: $\lim_{L\to\infty}L[\rho_1(L)-1]=C>0$. Analysis of this case is based on heavy-traffic assumption (j) of Condition 3.1 and the statements of Theorem 4.13. This case is associated with Condition (i) and series of objective functions defined by (3.1) and (3.2).

Case 2: $\lim_{L\to\infty}L[\rho_1(L)-1]=C<0$. Analysis of this case is based on heavy-traffic assumption (jj) of Conditions 3.2 and 3.4 and the statements of Theorem 4.15. This case is associated with Condition (ii) and series of objective functions defined by (3.3) and (3.4).

Case 3: $\lim_{L\to\infty} L[\rho_1(L)-1]=0$. Analysis of this case is based the statements of Theorem 4.14. Note that asymptotic results under this heavy-traffic condition coincides with limit relations (4.32) and (4.33) of Theorem 4.10. This case is associated with Condition (iii) and series of objective functions defined by (3.5).

In the proof given below we consider Case 1. The other cases can be studied similarly.

7.2.2. *Existence of a solution.* Note first, that under the assumptions made, there is a solution to the control problem considered in this paper. Indeed, a solution contains two terms one of them corresponds to the expression for $p_1J_1+p_2J_2$ in (1.4) and another one corresponds to the term $\sum_{i=L^{\text{lower}}+1}^{L^{\text{upper}}} c_iq_i$ in (1.4).

The first term of a solution is related to the models where the water costs are not taken into account. For Case 1, this term can be extracted from the function $J^{\text{upper}}[L,C(L)]$ in (3.1) by setting $c^{\text{upper}}[L,C(L)]=0$ and passing to the limit as $L\to\infty$. Denoting this function by $J^*(C)$ we have the following explicit expression

$$\begin{aligned}J^*(C)=&C\left[j_1\frac{1}{\exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1}\right.\\&\left.+\,j_2\frac{\rho_2}{1-\rho_2}\cdot\frac{\exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)}{\exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1}\right].\end{aligned}\tag{7.6}$$

Taking derivative in C and equating it to zero, we obtain the equation

$$\begin{aligned}&j_1\left[1-\frac{\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)}{\exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1}\right]\\&+j_2\frac{\rho_2}{1-\rho_2}\exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)\\&\times\left[1-\frac{\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)}{\exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1}\right]=0.\end{aligned}\tag{7.7}$$

Equation (7.7) has a solution. Indeed, setting $C=0$ for the left-hand side of (7.7) transforms it to the inequality

$$j_1+j_2\frac{\rho_2}{1-\rho_2}>0.$$

On the other hand, setting

$$C=\frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2\mathsf{E}\varsigma}$$

we obtain the inequality

$$j_1\left[1-\frac{\mathrm{e}}{\mathrm{e}-1}\right]+j_2\frac{\rho_2}{1-\rho_2}\left[1-\frac{\mathrm{e}}{\mathrm{e}-1}\right]<0.$$

Thus, (7.7) has a solution.

7.2.3. *Uniqueness of a solution.* To prove that the solution that is discussed in Section 7.2.2 is unique, we are to prove that the second derivative of the function $J^*(C)$ defined in (7.6) is positive. Indeed, the derivative of the function on left-hand side of (7.7) is

$$\left(j_1+j_2\frac{\rho_2}{1-\rho_2}\right)\frac{\frac{2C(\mathsf{E}\varsigma)^2}{(\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma)^2}\exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)}{\left[\exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1\right]^2}>0,$$

and taking into account that the left-hand side of (7.7) is presented as

$$\left[\exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1\right]\frac{\mathrm{d}J^*}{\mathrm{d}C},$$

we arrive at the conclusion that the second derivative of the function $J^*(C)$ is positive. Hence, the function $J^*(C)$ is convex.

The second term, which is the limit $\overline{c}^{\text{upper}}(C)$ is defined in Section 7.1. According to Lemma 7.1, $\overline{c}^{\text{upper}}(C)=\psi(C)$. According to Lemma 7.4 the function $\psi(C)$ is a convex function in C and its maximum is $\psi(0)=c^*$. The function $\eta(D)=\eta(-C)$ is a concave function in D (convex in C) with $\eta(0)=c^*$. Hence, the solution to control problem is unique.

7.2.4. *Structure of the optimal solution and corollary.* Now we discuss the structure of the optimal solution to the control problem. It is associated with three possible cases considered in Section 7.2.1.

Case 1: $\lim_{L\to\infty}L[\rho_1(L)-1]>0$. This case is associated with Condition (i). In this case, the minimum of $\overline{J}^{\text{upper}}(C)$ occurs for $C=\overline{C}>0$. Then, $\overline{c}^{\text{upper}}(\overline{C})<c^*$, and the value of the limiting term for $p_1J_1+p_2J_2+\sum_{i=L^{\text{lower}+1}}^{L^{\text{upper}}}c_iq_i$ of the series of objective functions in (1.4) is given by $\overline{J}^{\text{upper}}(\overline{C})$.

Case 2: $\lim_{L\to\infty}L[\rho_1(L)-1]=C<0$. This case is associated with Condition (ii) and under this condition the inequality $j_1<j_2\rho_2/(1-\rho_2)$ is satisfied. In this case, the minimum of $\underline{J}^{\text{lower}}(C)$ occurs for $C=\underline{C}>0$. Then, $\underline{c}^{\text{lower}}(0)>c^*$, and the value of the limiting term for $p_1J_1+p_2J_2+\sum_{i=L^{\text{lower}+1}}^{L^{\text{upper}}}c_iq_i$ of the series of objective function in (1.4) is given by $\underline{J}^{\text{under}}(\underline{C})$.

Case 3: $\lim_{L\to\infty} L[\rho_1(L)-1]=0$. Since $\overline{c}^{\text{upper}}(0)=c^*>\overline{c}^{\text{upper}}(\overline{C})$ for any positive C, then this case must belong to the case $j_1\leq j_2\rho_2/(1-\rho_2)$, where the only equality holds in the only case of same constant water cost for any level of the dam. In this case, the minimum of $\underline{J}^{\text{lower}}(C)$ occurs for $C=\underline{C}=0$. Then, $\underline{c}^{\text{lower}}(0)=c^*$ and the limiting term for $p_1J_1+p_2J_2+\sum_{i=L^{\text{lower}}+1}^{L^{\text{upper}}}c_iq_i$ of the series of objective functions in (1.4) is

$$j_1\frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2\mathsf{E}\varsigma}+j_2\frac{\rho_2}{1-\rho_2}\cdot\frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2\mathsf{E}\varsigma}+c^*.$$

(The result is mentioned in (6.1).)

Note, that as in the first case so in the third one, the optimal value of the limiting term for $p_1J_1+p_2J_2+\sum_{i=L^{\text{lower}}+1}^{L^{\text{upper}}}c_iq_i$ of the series of objective functions must be less then it is given by (6.1), since otherwise the strategy with $\rho_1=1$ would be also selected, which is impossible since the optimal solution is unique.

So, Theorem 3.7 is proved and the cases are discussed.

From Theorem 3.7 we have the following evident property of the optimal control.

Corollary 7.5. *The solution to the control problem can be* $\rho_1=1$ *only in the case* $j_1\leq j_2\rho_2/(1-\rho_2)$. *Specifically, the only equality is achieved for* $c_i\equiv c$, $i=1,2,\dots,L$, *where* c *is an arbitrary positive constant.*

Although Corollary 7.5 provides a result in the form of simple inequality, this result is not really useful, since it is an evident extension of the result of [4]. A more constructive result is obtained for the special case considered in the next section.

8. Example of linear costs

In this section we study an example related to the case of linear costs.

Assume that c_1 and $c_L<c_1$ are given. Then the assumption that the costs are linear means, that

$$c_i=c_1-\frac{i-1}{L-1}(c_1-c_L),\quad i=1,2,\dots,L. \tag{8.1}$$

It is assumed that as L is changed, the costs are recalculated as follows. The first and last values of the cost c_1 and c_L remains the same. Other costs in the intermediate points are recalculated according to (8.1).

Therefore, to avoid confusing with the appearance of the index L for the fixed (unchangeable) values of cost c_1 and c_L, we use the other notation: $c_1=\overline{c}$ and $c_L=\underline{c}$. Then (8.1) has the form

$$c_i=\overline{c}-\frac{i-1}{L-1}(\overline{c}-\underline{c}),\quad i=1,2,\dots,L. \tag{8.2}$$

In the following we shall also use the inverse form of (8.2). Namely,

$$c_{L-i}=\underline{c}+\frac{i}{L-1}(\overline{c}-\underline{c}),\quad i=0,1,\ldots,L-1. \tag{8.3}$$

Apparently,

$$c^{*}=\frac{\overline{c}+\underline{c}}{2}. \tag{8.4}$$

For $\overline{c}^{\mathrm{upper}}(C)$ we have

$$\begin{aligned}
\overline{c}^{\mathrm{upper}}(C)&=\psi(D)\\
&=\lim_{L\to\infty}\frac{\sum_{j=0}^{L-1}\left(\underline{c}+\frac{j}{L-1}(\overline{c}-\underline{c})\right)\left(1-\frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j}{\sum_{j=0}^{L-1}\left(1-\frac{2C\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j}\\
&=\underline{c}+(\overline{c}-\underline{c})\lim_{L\to\infty}\frac{1}{L-1}\cdot\frac{\sum_{j=0}^{L-1}j\left(1-\frac{2C\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j}{\sum_{j=0}^{L-1}\left(1-\frac{2C\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j}\\
&=\underline{c}+(\overline{c}-\underline{c})\frac{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2C\mathsf{E}\varsigma}\\
&\quad\times\frac{-\frac{2C\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}+\exp\left(\frac{2C\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1}{\exp\left(\frac{2C\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1}.
\end{aligned} \tag{8.5}$$

For example, as $C=D=0$ in (8.5), then $\overline{c}^{\mathrm{upper}}(0)$ is $\underline{c}+1/2(\overline{c}-\underline{c})=c^{*}$. This is in agreement with the statement of Proposition 6.1.

In turn, for $\underline{c}^{\mathrm{lower}}(C)$ (C is negative), we have

$$\begin{aligned}
\underline{c}^{\mathrm{lower}}(C)&=\eta(D)\\
&=\lim_{L\to\infty}\frac{\sum_{j=0}^{L-1}\left(\underline{c}+\frac{j}{L-1}(\overline{c}-\underline{c})\right)\left(1-\frac{2C\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j}{\sum_{j=0}^{L-1}\left(1-\frac{2C\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j}\\
&=\underline{c}+(\overline{c}-\underline{c})\lim_{L\to\infty}\frac{1}{L-1}\cdot\frac{\sum_{j=0}^{L-1}j\left(1-\frac{2C\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j}{\sum_{j=0}^{L-1}\left(1-\frac{2C\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j}\\
&=\underline{c}-(\overline{c}-\underline{c})\frac{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2C\mathsf{E}\varsigma}\\
&\quad\times\frac{-\frac{2C\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}-1+\exp\left(\frac{2C\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)}{1-\exp\left(\frac{2C\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)}.
\end{aligned} \tag{8.6}$$

Again, as $C=0$ in (8.6), then $\underline{c}^{\mathrm{lower}}(0)$ is $\underline{c}+1/2(\overline{c}-\underline{c})=c^{*}$. So, we arrive at the agreement with the statement of Proposition 6.1.

TABLE 1. The values of parameter j_2 and corresponding arguments of optimal value C

Parameter j_2	Argument of optimal value C	Difference $C(j_2+1)-C(j_2)$
1.06	0.200	
1.08	0.180	0.020
1.10	0.162	0.018
1.12	0.144	0.018
1.14	0.128	0.016
1.16	0.112	0.016
1.18	0.096	0.016
1.20	0.081	0.015
1.22	0.067	0.014
1.24	0.054	0.013
1.26	0.042	0.012
1.28	0.030	0.012
1.30	0.019	0.011
1.32	0.009	0.010
1.34	0	0.009

9. NUMERICAL STUDY

The explicit solution in the case of linear costs is very routine and cumbersome. We provide below a numerical study. For simplicity, the input flow in the numerical example is assumed to be ordinary Poisson, that is we set $\mathrm{E}\varsigma = 1$ and $\mathrm{E}\varsigma^2 = 1$ in our calculations.

Following Corollary 7.5, take first $j_1 = j_2\rho_2/(1-\rho_2)$. Clearly, that for this relation between parameters j_1 and j_2 the minimum of $\underline{J}^{\mathrm{lower}}(C)$ must be achieved for $C = 0$, while the minimum of $\overline{J}^{\mathrm{upper}}(C)$ must be achieved for a positive C. Now, keeping j_1 fixed assume that j_2 increases. Then, the problem is to find the value of parameter j_2 such that the value C corresponding to the minimization of $\overline{J}^{\mathrm{upper}}(C)$ reaches zero.

In our example we take $j_1 = 1$, $\rho_2 = 1/2$, $\underline{c} = 1$, $\overline{c} = 2$, $\widetilde{\rho}_{1,2} = 1$. In the table below we outline some values j_2 and the corresponding value C for the optimal solution of $\overline{J}^{\mathrm{upper}}(C)$. It is seen from the table that the optimal value is achieved in the case $j_2 \approx 1.34$. Therefore, in the present example $j_1 = 1$ and $j_2 \approx 1.34$ lead to the optimal solution $\rho_1 = 1$.

It is seen from Table 1 that as j_2 increases, the value of cost C is monotonically decreases. The inverse proportion between j_2 and C implies that the optimal value of $\overline{J}^{\mathrm{upper}}(C)$ will be lower when the value of j_2 is larger. That is, when the damage of flooding is kept increased, the value C and consequently the value of $\overline{J}^{\mathrm{upper}}(C)$ must be decreased. The third column in the table shows how the the parameter C decreases

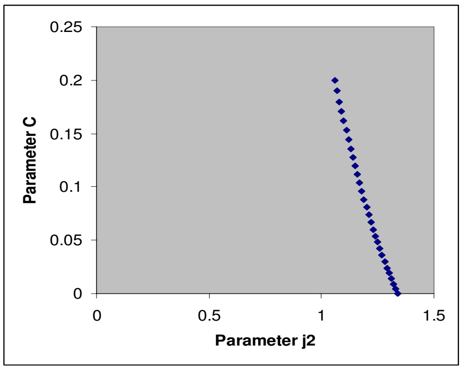

FIGURE 5. Graph of the dependence of C on j_2

when the parameter j_2 increases. It shows almost linear dependence of these parameters. The graph of the dependence of C on j_2 is shown in Figure 5.

10. Proofs

Proof of Lemma 4.8. Asymptotic relations (4.26) and (4.28) follow by application of those (4.2) and, respectively, (4.3) of Lemma 4.1.

In order to prove asymptotic relation (4.27) we should apply a Tauberian theorem by Postnikov (Lemma 4.2). Then asymptotic relation (4.27) is to follow from (4.4) if we prove that the Tauberian condition $f_0+f_1<1$ of Lemma 4.2 is satisfied. In the case of the present model, we must prove that for some $\lambda_0>0$ the equality

$$\int_0^\infty \mathrm{e}^{-\lambda_0 x}(1+\lambda_0 r_1 x)\mathrm{d}B_1(x)=1 \tag{10.1}$$

is not the case. Without loss of generality r_1 in (10.1) can be set to be equal to 1, since

$$\int_0^\infty \mathrm{e}^{-\lambda_0 x}(1+\lambda_0 r_1 x)\mathrm{d}B_1(x)\leq\int_0^\infty \mathrm{e}^{-\lambda_0 x}(1+\lambda_0 x)\mathrm{d}B_1(x).$$

Thus, we have to prove the inequality

$$\int_0^\infty \mathrm{e}^{-\lambda x}(1+\lambda x)\mathrm{d}B_1(x) < 1.$$

Indeed, $\int_0^\infty \mathrm{e}^{-\lambda x}(1+\lambda x)\mathrm{d}B_1(x)$ is an analytic function in λ, and hence, according to the theorem on the maximum module of an analytic function, equality (10.1) where $r_1 = 1$ must hold for *all* $\lambda_0 \geq 0$. This means that (10.1) is valid if and only if

$$\int_0^\infty \mathrm{e}^{-\lambda_0 x}\frac{(\lambda_0 x)^i}{i!}\mathrm{d}B_1(x) = 0$$

for all $i \geq 2$ and $\lambda_0 \geq 0$. Since $\int_0^\infty \mathrm{e}^{-\lambda x}(-x)^i \mathrm{d}B_1(x)$ is the ith derivative of the Laplace-Stieltjes transform $\widehat{B}_1(\lambda)$, then in this case the Laplace-Stieltjes transform $\widehat{B}_1(\lambda)$ must be a linear function in λ, i.e. $\widehat{B}_1(\lambda) = d_0 + d_1\lambda$, where d_0 and d_1 are some constants. However, since $|\widehat{B}_1(\lambda)| \leq 1$, we must have $d_0 = 1$ and $d_1 = 0$. This is a trivial case where $B_1(x)$ is concentrated in point 0, and therefore it is not a probability distribution function having a positive mean. Thus (10.1) is not the case, and the aforementioned Tauberian conditions are satisfied.

Now, the final part of the proof of (4.27) reduces to an elementary algebraic calculations:

$$\gamma_2 := \frac{\mathrm{d}^2}{\mathrm{d}z^2}\widehat{B}_1(\lambda - \lambda\widehat{R}(z))|_{z=1} = \frac{\mathsf{E}\varsigma^2}{\mathsf{E}\varsigma} - 1 + \rho_{1,2}(\mathsf{E}\varsigma)^2.$$

The lemma is proved.

Proof of Theorem 4.10. Let us first find asymptotic representation for $\mathsf{E}\nu_L^{(1)}(\zeta_1 \wedge L)$ as $L \to \infty$. According to Lemma 4.8 and explicit representation (4.19), we obtain as follows.

If $\rho_1 < 1$, then

$$\begin{aligned}\lim_{L\to\infty}\mathsf{E}\nu_L^{(1)}(\zeta_1 \wedge L) &= \frac{1}{1-\rho_1}\lim_{L\to\infty}\sum_{i=1}^{L} i\mathsf{Pr}\{\zeta_1 \wedge L = i\}\\ &= 1 + \frac{\mathsf{E}\zeta_1}{1-\rho_1}.\end{aligned} \tag{10.2}$$

If $\rho_1 = 1$, $\rho_{1,2} < \infty$ and $\mathsf{E}\varsigma^2 < \infty$, then

$$\begin{aligned}\lim_{L\to\infty}\frac{\mathsf{E}\nu_L^{(1)}(\zeta_1 \wedge L)}{L} &= \frac{2\mathsf{E}\varsigma}{\rho_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma}\\ &\quad\times\lim_{L\to\infty}\sum_{i=1}^{L} i\mathsf{Pr}\{\zeta_1 \wedge L = i\}\\ &= \frac{2\mathsf{E}\varsigma\mathsf{E}\zeta_1}{\rho_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma}.\end{aligned} \tag{10.3}$$

If $\rho_1 > 1$, then

$$\begin{aligned}\lim_{L\to\infty}\mathsf{E}\nu_L^{(1)}(\zeta_1\wedge L)\varphi^L &= \frac{1}{1+\lambda\widehat{B}_1'(\lambda-\lambda\widehat{R}(\varphi))\widehat{R}'(\varphi)}\\ &\quad\times\lim_{L\to\infty}\sum_{i=1}^{L} i\mathsf{Pr}\{\zeta_1\wedge L=i\}\\ &= \frac{\mathsf{E}\zeta_1}{1+\lambda\widehat{B}_1'(\lambda-\lambda\widehat{R}(\varphi))\widehat{R}'(\varphi)}.\end{aligned}\tag{10.4}$$

Therefore, taking into account these limiting relations (10.2), (10.3) and (10.4) by virtue of the equality $\mathsf{E}\nu_L^{(1)}(\zeta_1\wedge L) = \mathsf{E}\nu_L^{(1)}(\zeta_1)$ and explicit representations (4.11) and (4.12) (Lemma 4.4) for p_1 and p_2, we finally arrive at the statements of the theorem. The theorem is proved.

Proof of Theorem 4.13. Note first, that under Condition 3.1 there is the following expansion for φ:

$$\varphi = 1-\frac{2\delta\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}+O(\delta^2),\tag{10.5}$$

where φ itself is the limiting root of the series of functional equations, based on application of Lemma 4.3. This expansion is similar to that given originally in the book of Subhankulov [24], p.362, and its proof is provided as follows. Write the equation $\varphi = \widehat{B}_1(\lambda-\lambda\widehat{R}(\varphi))$ and expand the right-hand side by Taylor's formula taking $\varphi = 1-z$, where z is small enough when δ is small. We obtain:

$$1-z = 1-(1+\delta)z+\frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+(1+\delta)\left(\mathsf{E}\varsigma^2-\mathsf{E}\varsigma\right)}{2\mathsf{E}\varsigma}z^2+O(z^3).\tag{10.6}$$

Disregarding the small term $O(z^3)$ in (10.6) we arrive at the quadratic equation

$$\delta z-\frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+(1+\delta)\left(\mathsf{E}\varsigma^2-\mathsf{E}\varsigma\right)}{2\mathsf{E}\varsigma}z^2 = 0.\tag{10.7}$$

The positive solution of (10.7),

$$z = \frac{2\delta\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+(1+\delta)\left(\mathsf{E}\varsigma^2-\mathsf{E}\varsigma\right)},$$

leads to the expansion given by (10.5).

Let us now expand the right-hand side of (10.4) when δ is small. For the term $1+\lambda\widehat{B}_1'(\lambda-\lambda\widehat{R}(\varphi))\widehat{R}'(\varphi)$ we have the expansion

$$1+\lambda\widehat{B}_1'(\lambda-\lambda\widehat{R}(\varphi))\widehat{R}'(\varphi) = \delta+O(\delta^2),\tag{10.8}$$

Then, according to the l'Hospitale rule

$$\lim_{u\uparrow 1}\frac{1-\widehat{R}(u)}{1-u}=\mathsf{E}\varsigma.$$

Hence

$$(10.9)\qquad \frac{1-\widehat{R}(\varphi)}{1-\varphi}=\mathsf{E}\varsigma[1+o(1)].$$

Substituting (10.5), (10.8) and (10.9) into (10.4) we obtain the expansion

$$(10.10)\qquad \mathsf{E}\nu_L^{(1)}(\zeta_1\wedge L)=\frac{\exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1}{\delta}\mathsf{E}\varsigma[1+o(1)].$$

Hence, relations (4.36) and (4.37) of the theorem follow by virtue of the equality $\mathsf{E}\nu_L^{(1)}(\zeta_1\wedge L)=\mathsf{E}\nu_L^{(1)}(\zeta_1)$ and explicit representations (4.11) and (4.12) (Lemma 4.4) for p_1 and p_2.

Proof of Theorem 4.15. The explicit representation for the generating function for $\mathsf{E}\widetilde{\nu}_j^{(1)}$ is given by (4.25). Since the sequence $\{\mathsf{E}\widetilde{\nu}_j^{(1)}\}$ is increasing, then the asymptotic behavior of $\mathsf{E}\nu_L^{(1)}(\zeta_1)$ as $L\to\infty$ under the assumptions $L[\rho_1(L)-1]\to C$ as $L\to\infty$ can be found according to a Tauberian theorem of Hardy and Littlewood (see e.g. [21], [31], p.203 and [28]). Namely, according to that theorem, the behaviour of $\mathsf{E}\widetilde{\nu}_L^{(1)}$ as $L\to\infty$ and $L[\rho_1(L)-1]\to C$ can be found from the asymptotic expansion of

$$(10.11)\qquad (1-z)\frac{\widehat{B}_1(\lambda-\lambda\widehat{R}(z))}{\widehat{B}_1(\lambda-\lambda\widehat{R}(z))-z}$$

as $z\uparrow 1$. Similarly to the evaluation given in the proof of Theorem 4.3 [4], we have:

$$\begin{aligned}
&(1-z)\frac{\widehat{B}_1(\lambda-\lambda\widehat{R}(z))}{\widehat{B}_1(\lambda-\lambda\widehat{R}(z))-z}\\
&=\frac{1-z}{1-z-\rho_1(1-z)+\frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2\mathsf{E}\varsigma}(1-z)^2+O((1-z)^3)}\\
(10.12)\qquad &=\frac{1}{\delta+\frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2\mathsf{E}\varsigma}(1-z)+O((1-z)^2)}\\
&=\frac{1}{\delta\left[1+\frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2\delta\mathsf{E}\varsigma}(1-z)\right]+O((1-z)^2)}\\
&=\frac{1}{\delta\exp\left[\frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2\delta\mathsf{E}\varsigma}(1-z)\right]}[1+o(1)],
\end{aligned}$$

where $\delta=\delta(L)$ denoted the difference $1-\rho_1(L)$.

Therefore, assuming that $z=\frac{L-1}{L}\to 1$ as $L\to\infty$, from (10.12) we arrive at the following estimate:

$$\mathsf{E}\widetilde{\nu}_L^{(1)}=\frac{1}{\delta}\exp\left(-\frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2C\mathsf{E}\varsigma}\right)[1+o(1)]. \tag{10.13}$$

Comparing (4.28) with (10.4) and taking into account (10.9), which holds true in the case of this theorem as well, we obtain:

$$\mathsf{E}\nu_L^{(1)}(\zeta_1\wedge L)=\frac{\mathsf{E}\varsigma}{\delta}\exp\left(-\frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2C\mathsf{E}\varsigma}\right)[1+o(1)]. \tag{10.14}$$

Hence, relations (4.40) and (4.41) of the theorem follow by virtue of the equality $\mathsf{E}\nu_L^{(1)}(\zeta_1\wedge L)=\mathsf{E}\nu_L^{(1)}(\zeta_1)$ and explicit representations (4.11) and (4.12) (Lemma 4.4) for p_1 and p_2.

Proof of Theorem 5.3. Based on Lemma 4.3, we have the following expansion of (4.28) for large L:

$$\mathsf{E}\widetilde{\nu}_{L-j}^{(1)}=\frac{\varphi^j}{\varphi^L[1+\lambda\widehat{B}_1'(\lambda-\lambda\widehat{R}(\varphi))\widehat{R}'(\varphi)]}[1+o(1)]. \tag{10.15}$$

In turn, from (10.15) for large L we obtain:

$$\mathsf{E}\widetilde{\nu}_{L-j}^{(1)}-\mathsf{E}\widetilde{\nu}_{L-j-1}^{(1)}=\frac{(1-\varphi)\varphi^j}{\varphi^L[1+\lambda\widehat{B}_1'(\lambda-\lambda\widehat{R}(\varphi))\widehat{R}'(\varphi)]}[1+o(1)]. \tag{10.16}$$

From (10.16), similarly to (10.4), we further have:

$$\begin{aligned}&\mathsf{E}\nu_{L-j}^{(1)}(\zeta_1\wedge L)-\mathsf{E}\nu_{L-j-1}^{(1)}(\zeta_1\wedge L)\\&=\frac{(1-\widehat{R}(\varphi))(1-\varphi)\varphi^j}{[1+\lambda\widehat{B}_1'(\lambda-\lambda\widehat{R}(\varphi))\widehat{R}'(\varphi)](1-\varphi)}[1+o(1)],\end{aligned}$$

and, according to the equality $\mathsf{E}\nu_L^{(1)}(\zeta_1\wedge L)=\mathsf{E}\nu_L^{(1)}(\zeta_1)$,

$$\begin{aligned}&\mathsf{E}\nu_{L-j}^{(1)}(\zeta_1)-\mathsf{E}\nu_{L-j-1}^{(1)}(\zeta_1)\\&=\frac{(1-\widehat{R}(\varphi))(1-\varphi)\varphi^j}{[1+\lambda\widehat{B}_1'(\lambda-\lambda\widehat{R}(\varphi))\widehat{R}'(\varphi)](1-\varphi)}[1+o(1)].\end{aligned} \tag{10.17}$$

Next, under the conditions of the theorem, asymptotic expansions (10.5) (10.8) and (10.9) hold. Taking into consideration these expansions we arrive at the following asymptotic relations for $j = 0, 1, \ldots$:

$$\begin{aligned}\mathsf{E}\nu_{L-j}^{(1)}(\zeta_1) - \mathsf{E}\nu_{L-j-1}^{(1)}(\zeta_1) = \exp\left(\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma}\right)& \\ \times\left(1 - \frac{2\delta\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma}\right)^j& \\ \times\frac{2\delta\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma}[1 + o(1)].&\end{aligned}$$

Now, taking into account asymptotic relation (4.36) of Theorem 4.13 and the explicit formula given by (5.1) (Lemma 5.1) we arrive at the statement of the theorem.

Proof of Theorem 5.4. Under the assumptions of this theorem let us first derive the following asymptotic expansion:

$$\tau = 1 + \frac{2\delta\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma} + O(\delta^2). \tag{10.18}$$

Asymptotic expansion (10.18) is similar to that of (10.5), and its proof is also similar. Namely, taking into account that the equation $z = \widehat{B}_1(\lambda - \lambda\widehat{R}(z))$ has a unique solution in the set $(1, \infty)$, and this solution approaches 1 as δ vanishes. Therefore, by the Taylor expansion of this equation around the point $z = 1$, we have:

$$1 + z = 1 + (1-\delta)z + \frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3 + (1-\delta)(\mathsf{E}\varsigma^2 - \mathsf{E}\varsigma)}{2\mathsf{E}\varsigma}z^2 + O(z^3). \tag{10.19}$$

Disregarding the term $O(z^3)$, from (10.19) we arrive at the quadratic equation

$$\delta z - \frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3 + (1-\delta)(\mathsf{E}\varsigma^2 - \mathsf{E}\varsigma)}{2\mathsf{E}\varsigma}z^2 = 0,$$

and obtain a positive solution

$$z = \frac{2\delta\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3 + (1+\delta)\left(\mathsf{E}\varsigma^2 - \mathsf{E}\varsigma\right)}.$$

This proves (10.18).

Next, from (5.17), (10.18) and explicit formula (5.1) we obtain

$$q_{L-j} = q_L\left(1 + \frac{2\delta\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma}\right)^j [1 + o(1)]. \tag{10.20}$$

Taking into consideration

$$\begin{aligned}
&\sum_{j=0}^{L-1}\left(1+\frac{2\delta\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)^j\\
&=\frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2\delta\mathsf{E}\varsigma}\left[\left(1+\frac{2\delta\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)^L-1\right]\\
&=\frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2\delta\mathsf{E}\varsigma}\left[\exp\left(-\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1\right]\\
&\quad\times[1+o(1)],
\end{aligned}$$

from the normalization condition $p_1+p_2+\sum_{i=1}^{L}q_i=1$ and the fact that both p_1 and p_2 have the order $O(\delta)$, we obtain:

$$\begin{aligned}
q_L=&\frac{2\delta\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\\
&\times\frac{1}{\exp\left(-\frac{2C\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1}[1+o(1)].
\end{aligned}\tag{10.21}$$

The desired statement of the theorem follows from (10.21).

Proof of Proposition 6.2. The representation for the term

$$\begin{aligned}
C\Bigg[&j_1\frac{1}{\exp\left(\frac{2C\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1}\\
&+j_2\frac{\rho_2\exp\left(\frac{2C\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)}{(1-\rho_2)\left(\exp\left(\frac{2C\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1\right)}\Bigg]
\end{aligned}$$

of the right-hand side of (3.1) follows from (4.36) and (4.37) (Theorem 4.13). This term is similar to that of (5.2) in [4]. The new term which takes into account the water costs is $\overline{c}^{\text{upper}}(C)=\lim_{L\to\infty}c^{\text{upper}}[L,C(L)]$.

Taking into account representation (5.8), for this term we obtain:

$$\begin{aligned}
\overline{c}^{\text{upper}}(C) &= \lim_{L\to\infty}\sum_{j=0}^{L-1} q_{L-j}c_{L-j}\\
&= \lim_{L\to\infty}\sum_{j=0}^{L-1} c_{L-j}\cdot\frac{\exp\left(\frac{2C\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)}{\exp\left(\frac{2C\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1}\\
&\quad\times\left(1-\frac{2\delta L\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j\\
&\quad\times\frac{2\delta L\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma)L},
\end{aligned}$$

and, since $\lim_{L\to\infty}\delta L = C$, representation (3.2) follows.

Proof of Proposition 6.3. The representation for the term

$$\begin{aligned}
&-C\left[j_1\exp\left(-\frac{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2C\mathsf{E}\varsigma}\right)\right.\\
&\quad\left.+\,j_2\frac{\rho_2}{1-\rho_2}\left(\exp\left(-\frac{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2C\mathsf{E}\varsigma}\right)-1\right)\right]
\end{aligned}$$

of the right-hand side of (3.3) follows from (4.40) and (4.41) (Theorem 4.15). This term is similar to that (5.3) in [4]. The new term, which takes into account the water costs, is $\underline{c}^{\text{lower}}(C) = \lim_{L\to\infty} c^{\text{lower}}[L, C(L)]$. Taking into account representation (5.18), for this term we obtain:

$$\begin{aligned}
\underline{c}^{\text{lower}}(C) &= \lim_{L\to\infty}\sum_{j=0}^{L-1} q_{L-j}c_{L-j}\\
&= \lim_{L\to\infty}\sum_{j=0}^{L-1} c_{L-j}\cdot\frac{1}{\exp\left(-\frac{2C\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)-1}\\
&\quad\times\left(1+\frac{2\delta L\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j\\
&\quad\times\frac{2\delta L\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L},
\end{aligned}$$

and, because of

$$\lim_{L\to\infty} L[\rho_1(L)-1] = C \le 0,$$

representation (3.4) follows.

Proof of Lemma 7.1. From (7.2) and (7.3) we correspondingly have the representations:

$$\begin{aligned}&\lim_{L\to\infty}\frac{1}{L}\sum_{j=0}^{L-1}c_{L-j}\left(1-\frac{2C\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j\\ &=\psi(D)\lim_{L\to\infty}\frac{1}{L}\sum_{j=0}^{L-1}\left(1-\frac{2C\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j,\end{aligned}\tag{10.22}$$

and since $D=-C$ (C is negative),

$$\begin{aligned}&\lim_{L\to\infty}\frac{1}{L}\sum_{j=0}^{L-1}c_{L-j}\left(1+\frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j\\ &=\lim_{L\to\infty}\frac{1}{L}\sum_{j=0}^{L-1}c_{L-j}\left(1-\frac{2C\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j\\ &=\eta(D)\lim_{L\to\infty}\frac{1}{L}\sum_{j=0}^{L-1}\left(1-\frac{2C\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j.\end{aligned}\tag{10.23}$$

The desired results follow by direct transformations of the corresponding right-hand sides of (10.22) and (10.23).

Indeed, for the right-hand side of (10.22) we obtain:

$$\begin{aligned}&\psi(D)\lim_{L\to\infty}\frac{1}{L}\sum_{j=0}^{L-1}\left(1-\frac{2C\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j\\ &=\psi(D)\lim_{L\to\infty}\left[1-\exp\left(-\frac{2C\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)\right]\\ &\quad\times\frac{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]}{2C\mathsf{E}\varsigma}.\end{aligned}\tag{10.24}$$

On the other hand, from (3.2) we have:

$$\begin{aligned}&\overline{c}^{\mathrm{upper}}(C)\left[1-\exp\left(-\frac{2C\mathsf{E}\varsigma}{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}\right)\right]\\ &\quad\times\frac{\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma}{2C\mathsf{E}\varsigma}\\ &=\frac{1}{L}\sum_{j=0}^{L-1}c_{L-j}\left(1-\frac{2C\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j.\end{aligned}\tag{10.25}$$

Hence, from (10.22), (10.24) and (10.25) we obtain (7.2). The proof of (7.5) is completely analogous and uses the representations of (3.4) and (10.23).

Proof of Lemma 7.2. Let us first prove that $\psi(0) = c^*$ is a maximum of $\psi(D)$. For this purpose we use the following well-known inequality (e.g. Hardy, Littlewood and Polya [12] or Marschall and Olkin [16]). Let $\{a_n\}$ and $\{b_n\}$ be arbitrary sequences, one of them is increasing and another decreasing. Then for any finite sum we have

$$\sum_{n=1}^{l} a_n b_n \leq \frac{1}{l} \sum_{n=1}^{l} a_n \sum_{n=1}^{l} b_n. \tag{10.26}$$

Applying inequality (10.26) to the finite sums of the left-hand side of (10.22) and passing to the limit as $L \to \infty$, we have

$$\begin{aligned}
&\lim_{L\to\infty} \frac{1}{L} \sum_{j=0}^{L-1} c_{L-j} \left(1 - \frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma]L}\right)^j \\
&\leq \lim_{L\to\infty} \frac{1}{L} \sum_{j=0}^{L-1} c_{L-j} \\
&\quad \times \lim_{L\to\infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma]L}\right)^j \\
&= \psi(0) \lim_{L\to\infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma]L}\right)^j.
\end{aligned} \tag{10.27}$$

Then, comparing (10.22) with (10.27) enables us to conclude,

$$\psi(0) = c^* \geq \psi(D),$$

i.e. $\psi(0) = c^*$ is the maximum value of $\psi(D)$.

A more delicate result we are going to prove is that $\psi(D)$ is a non-increasing convex function. We first prove that the function $\psi(D)$ is nonincreasing and then, following formulated Lemmas 7.3 and 7.4, we later prove that $\psi(D)$ is a *nontrivial* convex function, if the sequence of costs c_i is decreasing (that is, the values c_i all are not equal to a same constant).

To prove the fact that $\psi(D)$ is a nonincreasing convex function we are to derive the function $\psi(D)$ in D and show that the derivative is negative. From the explicit representation for $\psi(D)$

$$\psi(D) = \lim_{L\to\infty} \frac{\frac{1}{L}\sum_{j=0}^{L-1} c_{L-j} \left(1 - \frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma]L}\right)^j}{\frac{1}{L}\sum_{j=0}^{L-1} \left(1 - \frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma]L}\right)^j}$$

we have

$$
\begin{aligned}
\frac{\mathrm{d}\psi}{\mathrm{d}D} &= \lim_{L\to\infty}\left(\frac{1}{L}\sum_{j=0}^{L-1}\left(1-\frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j\right)^{-2}\\
&\times\left\{\lim_{L\to\infty}\frac{1}{L}\sum_{j=0}^{L-1}\left(1-\frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j\right.\\
&\times\lim_{L\to\infty}\frac{\mathrm{d}}{\mathrm{d}D}\left[\frac{1}{L}\sum_{j=0}^{L-1}c_{L-j}\left(1-\frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j\right]\\
&-\lim_{L\to\infty}\frac{1}{L}\sum_{j=0}^{L-1}c_{L-j}\left(1-\frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j\\
&\left.\times\lim_{L\to\infty}\frac{\mathrm{d}}{\mathrm{d}D}\left[\frac{1}{L}\sum_{j=0}^{L-1}\left(1-\frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j\right]\right\}.
\end{aligned}
\tag{10.28}
$$

The task is to prove that the expression in the arc brackets of (10.28) is negative or zero. That is, we are to prove that for sufficiently large L

$$
\begin{aligned}
&\sum_{j=0}^{L-1}\left(1-\frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j\\
&\times\frac{\mathrm{d}}{\mathrm{d}D}\left[\sum_{j=0}^{L-1}c_{L-j}\left(1-\frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j\right]\\
&-\sum_{j=0}^{L-1}c_{L-j}\left(1-\frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j\\
&\times\frac{\mathrm{d}}{\mathrm{d}D}\left[\sum_{j=0}^{L-1}\left(1-\frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j\right]
\end{aligned}
\tag{10.29}
$$

is negative.

Note, that (10.29) is associated with the representation

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}D}\left[\frac{\sum_{j=0}^{L-1}c_{L-j}\left(1-\frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j}{\sum_{j=0}^{L-1}\left(1-\frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j}\right]\\
&\times\left(\sum_{j=0}^{L-1}\left(1-\frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3+\mathsf{E}\varsigma^2-\mathsf{E}\varsigma]L}\right)^j\right)^2.
\end{aligned}
\tag{10.30}
$$

So, the technical task reduces to the following. Let

$$f_L(z) = \frac{\sum_{i=0}^{L} a_i z^i}{\sum_{i=0}^{L} z^i} = \frac{(1-z)\sum_{i=0}^{L} a_i z^i}{1-z^{L+1}} = \frac{a_0 + \sum_{i=1}^{L}(a_i - a_{i-1})z^i}{1-z^{L+1}} \tag{10.31}$$

be the function, $z < 1$, in which a_i is an increasing sequence. For the derivative of this function we have

$$\frac{\mathrm{d}f_L}{\mathrm{d}z} = \frac{\sum_{i=1}^{L} i(a_i - a_{i-1})z^{i-1}}{1-z^{L+1}} + \frac{(L+1)z^L\left(a_0 + \sum_{i=1}^{L}(a_i - a_{i-1})z^i\right)}{(1-z^{L+1})^2}. \tag{10.32}$$

Since a_i is an increasing sequence, then the derivative $\mathrm{d}f_L/\mathrm{d}z$ is positive. Then, for the function $f_L(1-y/L)$, in which the argument z is replaced with $1-y/L$, $0 < y < L$, now the derivative of this function in y is negative. So, as L increases to infinity, the derivative $\mathrm{d}f_L(1-y/L)/\mathrm{d}y$ tends to the negative value or zero. This enables us to arrive at the conclusion that (10.28) is negative and, as $L \to \infty$, it tends to the negative value or zero.

The first statement of Lemma 7.2 is proved. The proof of the second statement of this lemma is similar.

Proof of Lemma 7.3. In order to prove this lemma it is sufficient to prove that if the sequence $\{c_i\}$ is nontrivial, that is there are at least two distinct values of this sequence, then for any distinct positive real numbers $C_1 \neq C_2$ the values of functions are also distinct, that is, $\psi(C_1) \neq \psi(C_2)$ and $\eta(C_1) \neq \eta(C_2)$. Let us prove the first inequality: $\psi(C_1) \neq \psi(C_2)$. Rewrite (7.2) as

$$\psi(D) = \lim_{L\to\infty} \frac{\frac{1}{L}\sum_{j=0}^{L-1} c_{L-j}\left(1 - \frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma]L}\right)^j}{\frac{1}{L}\sum_{j=0}^{L-1}\left(1 - \frac{2D\mathsf{E}\varsigma}{[\rho_{1,2}(L)(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma]L}\right)^j}. \tag{10.33}$$

The limit of the denominator is equal to

$$\left[1 - \exp\left(-\frac{2D\mathsf{E}\varsigma}{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma}\right)\right] \times \frac{\widetilde{\rho}_{1,2}(\mathsf{E}\varsigma)^3 + \mathsf{E}\varsigma^2 - \mathsf{E}\varsigma}{2D\mathsf{E}\varsigma}.$$

The limit of the numerator does exist and finite, since the sequence $\{c_i\}$ is assumed to be nonincreasing and bounded. As well, according to the other representation following from Lemma 7.1 and relation (3.2), it is an analytic function in D taking a nontrivial set of values.

The analytic function $\psi(C)$ is defined for all real $C \geq 0$ and it can be extended analytically for the whole complex plane. Extension to the negative values of C enables us to arrive at the function $\eta(C) = \psi(-C)$, which is now an analytic function for all real values C and due to analytic continuation is an analytic function in whole complex plane. According to the maximum principle for the module of an analytic function, if an analytic function takes the same values in two distinct points inside the domain of its definition, then the function must be the constant. If $c_i = c^*$ for all $i = 1, 2, \ldots$, then this is just the case where $\psi(C) = c^*$ for all C. If there exist i_0 and i_1 such that $c_{i_0} \neq c_{i_1}$, then the function $\psi(C)$ cannot be a constant, because the analytic function is uniquely defined by the coefficients in the series expansion. So, the inequality $\psi(C_1) \neq \psi(C_2)$ for distinct values C_1 and C_2 follows. The proof of the second inequality $\eta(C_1) \neq \eta(C_2)$ is the part of the first one, since $\eta(C) = \psi(-C)$ is the same analytic function.

Proof of Lemma 7.4. To prove the lemma, we are to derive the function $\psi(D)$ in D twice and show that the second derivative is nonnegative. Note that in the nontrivial case when the sequence $\{c_i\}$ is a decreasing sequence containing distinct values, the first derivative of $\psi(D)$, according to Lemmas 7.2 and 7.3, is strictly negative. The proof given in Lemma 7.2 cannot guarantee this, since the limit, as $L \to \infty$, may reach 0. But the statement of Lemma 7.3 does guarantee this, since the analytic function that is not a constant must take distinct values only.

The expression for the second derivative is hardly observable. Therefore, we do not write down the exact derivations, and instead of them we provide the scheme of calculations only.

We consider the function defined by (10.31) that is used in the proof of Lemma 7.2. Its derivative is defined by (10.32). Deriving this function the second time, show that the second derivative is positive. Denote the expression in (10.32) by $I_L^{(1)}(z) + I_L^{(2)}(z)$, where $I_L^{(1)}(z)$ is the first term-fraction of the expression and $I_L^{(2)}(z)$ is the second one. For the derivative of the first term, we obtain

$$\begin{aligned} \frac{\mathrm{d}I_L^{(1)}}{\mathrm{d}z} &= \frac{\sum_{i=2}^{L} i(i-1)(a_i - a_{i-1})z^{i-2}}{1 - z^{L+1}} \\ &\quad + \frac{(L+1)z^L \sum_{i=1}^{L} i(a_i - a_{i-1})z^{i-1}}{(1 - z^{L+1})^2}, \end{aligned} \tag{10.34}$$

and for the derivative of the second term, we obtain

$$\begin{aligned}\frac{\mathrm{d}I_L^{(2)}}{\mathrm{d}z} &= \frac{(L+1)z^L\sum_{i=1}^{L} i(a_i-a_{i-1})z^{i-1}}{(1-z^{L+1})^2}\\ &+\frac{(L+1)Lz^{L-1}\left(a_0+\sum_{i=1}^{L}(a_i-a_{i-1})z^i\right)}{(1-z^{L+1})^2}\\ &+\frac{2(L+1)z^L\left(a_0+\sum_{i=1}^{L}(a_i-a_{i-1})z^i\right)}{(1-z^{L+1})^3}.\end{aligned}\tag{10.35}$$

Keeping in mind that the sequence a_i is increasing (we assume that all of $a_1, a_2, \dots$ of the sequence are not equal to a same constant), it is readily seen that all the terms-fractions on the right-hand sides of (10.34) and (10.35) are positive. That is, the derivatives $\mathrm{d}I_L^{(1)}/\mathrm{d}z$ and $\mathrm{d}I_L^{(2)}/\mathrm{d}z$ both are positive. With the following change of argument, the derivatives $\mathrm{d}I_L^{(1)}(1-y/L)/\mathrm{d}y$ and $\mathrm{d}I_L^{(2)}(1-y/L)/\mathrm{d}y$, where $0<y<L$, both are negative. This means that the required second derivative $\mathrm{d}^2f_L(1-y/L)/\mathrm{d}y^2$ is positive. This statement implies that $\psi(D)$ is a convex function. The proof of the fact that $\eta(D)$ is a concave function is similar.

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References

[1] Abramov, V.M. (1991). *Investigation of a Queueing System with Service Depending on Queue-Length.* Donish, Dushanbe. (Russian with Tajik resume.)

[2] Abramov, V.M. (2002). Asymptotic analysis of the $GI/M/1/n$ loss system as n increases to infinity. *Annals of Operations Research*, **112**, 35-41.

[3] Abramov, V.M. (2004). Asymptotic behavior of the number of lost messages. *SIAM Journal on Applied Mathematics*, **64**, 746-761.

[4] Abramov, V.M. (2007). Optimal control of a large dam. *Journal of Applied Probability*, **44**, 249-258.

[5] Abramov, V.M. (2010). Takács' asymptotic theorem and its applications: A survey. *Acta Applicandae Mathematicae*, **109**, 609-651.

[6] Braverman, A., Dai, J.G. and Miyazawa, M. (2017). Heavy traffic approximation for the stationary distribution of a generalized Jackson network: The BAR approach. *Stochastic Systems*, **7**, 143-196.

[7] Chen, H. and Yao, D.D. (2001). *Fundamentals of Queueing Networks, Performance, Asymptotics, and Optimization.* Springer, New-York.

[8] Cooper, R.B. and Tilt, B. (1976). On the relationship between the distribution of maximal queue-length in the $M/G/1$ queue and the mean busy period in the $M/G/1/n$ queue. *Journal of Applied Probability*, **13** (1), 195-199.

[9] Feller, W. (1966). *An Introduction to Probability Theory and Its Applications*, vol. 2. John Wiley, New York.

[10] Ferreira, F., Pacheco, A. and Ribeiro, H. (2017). Moments of losses during busy periods of regular and non-preemptive oscilating $M^X/G/1/n$ systems. *Annals of Operations Reseach*, **252**, 191-211.

[11] Gamarnik, D. and Zeevi, A. (2006). Validity of heavy traffic steady state approximation in generalized Jackson networks. *The Annals of Applied Probability*, **16**, 56-90.

[12] Hardy, G.H., Littlewood, J.E. and Polya, G. (1952). *Inequalities.* Second edn. Cambridge Univ. Press, London.

[13] Knessl, C., Matkowsky, B.J., Schuss, Z. and Tier, C. (1986). On the performance of state-dependent single-server queues. *SIAM Journal on Applied Mathematics*, **46** (4), 657-697.

[14] Knessl, C., Matkowsky, B.J., Schuss, Z. and Tier, C. (1994). A state-dependent $GI/G/1$ queue. *European Journal of Applied Mathematics*, **5** (2), 217-241.

[15] Mandelbaum, A. and Pats, G. (1995). State-dependent queues: Approximations and applications. In: *Stochastic Networks*, F.P.Kelly and R.J.Williams eds, IMA, vol.71, Springer, New York, pp. 239-282.

[16] Marschall, A.W. and Olkin, I. (2011). *Inequalities: Theory of Majorization and Its Applications.* Second edition. Springer, New York.

[17] Miller, B.M. (2009). Optimization of queueing system via stochastic control. *Automatica J. IFAC*, **45**, 1423-1430.

[18] Miller, B.M. and McInnes, D.J. (2011). Management of a large dam via optimal price control. In: *Proc. 18th World Congress IFAC*, Milano (Italy), August 28-September 2, 2011, pp. 12432-12438.

[19] Pacheco, A. and Ribeiro, H. (2008). Moments of duration of busy periods of $M^X/G/1/n$ systems. *Probability in the Engineering and Informational Sciences*, **22**, 1-8.

[20] Pacheco, A. and Ribeiro, H. (2008). Consecutive customer losses in regular and oscillating $M^X/G/1/n$ systems. *Queueing Systems*, **58**, 121-136.

[21] Postnikov, A.G. (1980). *Tauberian Theory and Its Application.* Proceedings of the Steklov Mathematical Institute (2) **144** (AMS Transl. from Russian.)

[22] Rosenlund, S.I. (1973). On the length and number of served customers of the busy period of a generalised $M/G/1$ queue with finite waiting room. *Advances in Applied Probability*, **5**, 379-389.

[23] Ross, S.M. (2000). *Introduction to Probability Models*, Seventh edn, Harcourt/Academic Press, Burlington.

[24] Subhankulov, M.A. (1976). *Tauberian Theorems with Remainder*, Nauka, Moscow. (Russian.)

[25] Suzuki, T. (1961). On a queueing process with service depending on queue-length. *Commentarii Mathematici Univ. St. Pauli*, **10**, 1-12.

[26] Suzuki, T. (1962). A queueing system with service depending on queue-length. *Journal of the Operations Research Society of Japan*, **4**, 147-169.

[27] Suzuki, T. and Ebe, M (1967). Decision rules for the queueing system $M/G/1$ with service depending on queue-length. *Memoirs of the Defence Academy (Japan)*, **7**, no.3, 7-13.

[28] Sznajder, R. and Filar, J.A. (1992). Some comments on a theorem of Hardy and Littlewood. *Journal of Optimization Theory and Applications*, **75**, 201-208.

[29] Takács, L. (1955). Investigation of waiting time problems by reduction to Markov processes. *Acta Mathematica Hungarica*, **6**, 101-129.

[30] Takács, L. (1962). *Introduction to the Theory of Queues*. Oxford University Press, New York/London.

[31] Takács, L. (1967). *Combinatorial Methods in the Theory of Stochastic Processes*. John Wiley, New York.

[32] Tomko, J. (1967). A limit theorem in the queueing problem with indefinitely increasing intensity of flow. *Studia Scientiarum Mathematicarum Hungarica*, **2**, 447-454. (In Russian.)

[33] Whitt, W. (2001). *Stochastic Process Limits: An Introduction to Stochastic Process Limits and Their Application to Queues*. Springer, New York.

[34] Whitt, W. (2004). Heavy-traffic limits for loss proportions in single-server queues. *Queueing Systems*, **46**, 507-736.

[35] Whitt, W. (2005). Heavy-traffic limits for the $G/H_2^*/m/n$ queue. *Mathematics of Operations Research*, **30**, 1-27.

[36] Willmot, G.E. (1988). A note on the equilibrium $M/G/1$ queue length. *Journal of Applied Probability*, **25**, 228-231 and 839.