Solution of State Space Singular Continuous-Time Fractional Linear System Using Sumudu Transform

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Abstract—This paper focuses on the development of a new process for solving state space singular continuous-time fractional linear systems based on Caputo fractional derivative-integral. The main idea of this new approach consists on using Sumudu transform since its interesting properties. State-of-the-art methods are used to compare the obtained results.

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1. INTRODUCTION

Recent years have been marked by an enormous growth of the use of the fractional calculus and fractional systems in several applications and domains [8, 13–16]. The use of the mathematical tools, theories and methods is required to solve such a system. One of the key and important problem in systems and control theory [1, 5, 17] is to find the solution of a singular continuous-time fractional linear system with regular pencil which is already studied in [4, 6, 11].

Instead of the existing methods [4, 6, 11], in this paper, we are going to use the Sumudu transform [2, 18] to solve such a singular continuous-time fractional linear system since the Sumudu transform is considered as an alternative to the Laplace transform, and has many particular and interesting properties as for instance the preserving properties of the scale and unit [2].

This paper is organized as follows. In Section 2, the basic definitions and properties used are recalled. Then, in Section 3, the application of Sumudu transform to solve singular continuous-time fractional linear system is proposed. In Section 4, academic and real examples are presented. The obtained results are compared to the existing methods. Finally, the last Section summarizes and discusses the obtained results.

2. MATHEMATICAL BACKGROUND

In this section, we recall some interesting definitions and properties which will be used.

Definition 1. [6] *The function defined by:*

$$\mathbf{D}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, \quad x^{(n)}(\tau) = \frac{d^n x(\tau)}{d\tau^n}, \tag{1}$$

is called the Caputo fractional derivative-integral of the function $x(t)$, where $n-1<\alpha\le n$, $n\in\mathbb{N}^*$, and Γ refers to the standard Gamma function.

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Definition 2. [2, 18] *Let us consider the set of functions:*

$$A = \Big\{ x(t) | \exists M,\ \tau_1,\ \tau_2\ > 0,\ |x(t)| < Me^{-\frac{|t|}{\tau_j}},\ \text{if } t \in (-1)^j \times [0,\infty) \Big\}.$$

The Sumudu transform $X(v)$ of the function $x(t)$ is defined over the set of functions A by:

$$X(v) = \mathcal{S}[x(t)](v) = v^{-1} \int_0^{\infty} x(t) e^{-\frac{t}{v}} dv, \quad v \in (-\tau_1, \tau_2). \tag{2}$$

Theorem 1. [9] *The Sumudu transform of the fractional derivative-integral* (1) *for $n-1 < \alpha \leq n$, $n \in \mathbb{N}^*$, has the form:*

$$\mathcal{S}\left[\mathbf{D}^{\alpha} x(t)\right](v) = v^{-\alpha} \left(X(v) - \sum_{k=1}^{n} v^{k-1} \left[x^{(k-1)}(t) \right]_{t=t_0} \right), \tag{3}$$

where $X(v)$ denotes the Sumudu transform of the function $x(t)$.

Some interesting properties of the Sumudu transform are described in the following proposition.

Proposition 1. [2, 9, 18] *Let $X_1(v)$ and $X_2(t)$ be the Sumudu Transforms of the functions $x_1(t)$ and $x_2(t)$ respectively. Then for any $a \in \mathbb{R}_+^*$, and $n-1 < \alpha \leq n$, $n \in \mathbb{N}^*$, we have:*

1. $\mathcal{S}\left[\dfrac{t^a}{\Gamma(a+1)}\right](v) = v^a.$

2. $\mathcal{S}\Big[\Big(x_1 \star x_2\Big)(t)\Big](v) = v\, \mathcal{S}\Big[x_1(t)\Big](v)\, \mathcal{S}\Big[x_2(t)\Big](v)$, *where $\star$ is the convolution product.*

3. $\mathcal{S}\Big[\mathbf{D}^{\alpha} \delta(t)\Big](v) = v^{-\alpha-1}$, *where δ is the Dirac delta function.*

Inspired by [10] and [12], we obtain easily the following result.

Proposition 2. *Let A, $E \in \mathbb{R}^{n\times n}$ be real matrices with* $\det E = 0$ *and* $\det(E - v^{\alpha} A) \neq 0$. *Then for any $v \in \mathbb{C}$, and $n-1 < \alpha \leq n$, $n \in \mathbb{N}^*$, the Laurent series on the neighborhood of ∞ is given by:*

$$(E - v^{\alpha} A)^{-1} = \sum_{i=-\mu}^{\infty} \phi_i\, v^{i\,\alpha}, \tag{4}$$

where $\mu = \mathrm{rg}(E) - \deg(v^{-\alpha}\ \det(E - v^{\alpha} A)) + 1$ *represent the index of nilpotency of $(E - v^{\alpha} A)$ and ϕ_i are the fundamental matrices.*

Proposition 3. [12] *For any real matrix $A \in \mathbb{R}^{n\times n}$, the fundamental matrices verify:*

$$\phi_i = (\phi_0\, A)^i\, \phi_0\,, \quad \forall i \in \mathbb{N}.$$

3. MAIN RESULTS

We denote by $\mathbb{R}^{m\times n}$ the set of real matrices with m rows and n columns, and by $\mathbb{R}^m$ the set of real columns vectors.

Let us consider the fractional continuous-time systems of the form:

$$\mathbf{D}^{\alpha} E x(t) = A x(t) + B u(t), \tag{5}$$

$$y(t) = C x(t) + D u(t), \tag{6}$$

where $\mathbf{D}^{\alpha}$ is the Caputo fractional derivative-integral, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the state, the input and the output vectors of the model respectively and $E,\ A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$.

The boundary conditions of the system (5) are given by:

$$x(t_0) = x_0,$$

nevertheless, the solution $x(t)$ is impulse free which is equivalent to the following compatibility conditions:

- $Ex(t_0)$ and $v^{k-i\alpha}Ex^{(k)}(t_0)$ exist, for all $i = \overline{1,\,\mu}$, $0 \le k \le n-1$, $n \in \mathbb{N}^*$ and $v \in (-\tau_1, \tau_2)$.
- $u(t)$ is provided and $u^{(k)}(t_0) = 0$, for all $0 \le k \le n-1$.

Assuming that $\det E = 0$ and the pencil of the pair $(E,\,A)$ of the system (5) is regular, i.e.;

$$\det(E - v^{\alpha}A) \neq 0, \tag{7}$$

for any $v \in \mathbb{C}$. Hence, the system (5) is called *fractional continuous-time singular system*.

Using the Caputo fractional derivative-integral (1), the system (5) becomes:

$$E\mathbf{D}^{\alpha}x(t) = Ax(t) + Bu(t). \tag{8}$$

The system (8) is also called *singular implicit fractional dynamical system*. Thus, its solution is given by the following theorem which is one of the main results of this paper.

Theorem 2. *The solution of the singular implicit fractional dynamical system* (8) *is given by:*

$$\begin{aligned} x(t) \;=\; & \sum_{i=0}^{\infty} \phi_i \frac{1}{\Gamma((i+1)\alpha)}\, B \int_0^t (t-\tau)^{(i+1)\alpha-1} u(\tau)d\tau \\ & + \sum_{i=1}^{\mu} \phi_{-i} \left(\mathbf{D}^{(i-1)\alpha} Bu(t) + \sum_{k=0}^{n-1} E\, \mathbf{D}^{i\alpha-k-1}\Big(\delta(t)\Big) x^{(k)}(t_0) \right) \\ & + \sum_{i=0}^{\infty}\sum_{k=0}^{n-1} \phi_i\, E\, \frac{t^{i\alpha+k}}{\Gamma(i\alpha+k+1)}\, x^{(k)}(t_0), \end{aligned} \tag{9}$$

where α, ϕ_i, Γ, δ, and μ represent the fractional order, the fundamental matrices, the standard Gamma function, the Dirac delta function, and the index of nilpotency of $(E - v^{\alpha}A)$ respectively.

Proof. By applying the Sumudu transform (formulas (2) and (3)) to the system (8), we obtain:

$$\mathcal{S}\left[E\mathbf{D}^{\alpha}x(t)\right](v) = \mathcal{S}\left[Ax(t) + Bu(t)\right](v).$$

Using $X(v)$ and $U(v)$ as the Sumudu transforms of $x(t)$ and $u(t)$ respectively, and the fact that both the pencil $(E,\,A)$ is regular (formula (7)), and the system (8) is singular and impulse free, we get:

$$X(v) = (E - v^{\alpha}A)^{-1}\left(v^{\alpha}BU(v) + E\sum_{k=1}^{n} v^{k-1}x^{(k-1)}(t_0)\right).$$

Substituting the term $(E - v^{\alpha}A)^{-1}$ by the one given by the expression (4), yields:

$$\begin{aligned} X(v) \;=\; & \left(\sum_{i=-\mu}^{\infty} \phi_i\, v^{i\,\alpha}\right)\Big(v^{\alpha}BU(v)\Big) + \left(\sum_{i=-\mu}^{\infty} \phi_i\, v^{i\,\alpha}\right)\left(E\sum_{k=0}^{n-1} v^k x^{(k)}(t_0)\right), \\ =\; & v\sum_{i=0}^{\infty} \phi_i\, v^{(i+1)\,\alpha-1}BU(v) + \sum_{i=1}^{\mu} \phi_{-i}\, v^{(1-i)\,\alpha}BU(v) \\ & + \sum_{i=0}^{\infty}\sum_{k=0}^{n-1} E\,\phi_i\, v^{i\,\alpha+k}\, x^{(k)}(t_0) + \sum_{i=1}^{\mu}\sum_{k=0}^{n-1} \phi_{-i}\, E\, v^{k-i\,\alpha}\, x^{(k)}(t_0). \end{aligned}$$

Finally, the result is derived by using the inverse Sumudu transform and the convolution theorem (Proposition 1):

$$\begin{aligned} x(t) \;=\; & \sum_{i=0}^{\infty} \phi_i \, \frac{1}{\Gamma((i+1)\alpha)} \, B \int_0^t (t-\tau)^{(i+1)\alpha-1} u(\tau) d\tau \\ & + \sum_{i=1}^{\mu} \phi_{-i} \left(\mathbf{D}^{(i-1)\alpha} B u(t) + \sum_{k=0}^{n-1} E \, \mathbf{D}^{i\alpha-k-1} \Big(\delta(t) \Big) x^{(k)}(t_0) \right) \\ & + \sum_{i=0}^{\infty} \sum_{k=0}^{n-1} \phi_i \, E \, \frac{t^{i\alpha+k}}{\Gamma(i\alpha+k+1)} \, x^{(k)}(t_0). \end{aligned}$$

□

For $\alpha = 1$, we get:

Corollary 1. *For $\alpha = 1$, the solution of the singular implicit fractional dynamical system* (8) *is given by:*

$$\begin{aligned} x(t) \;=\; & \sum_{i=0}^{\infty} \phi_i \, B \, \frac{1}{\Gamma(i+1)} \int_0^t (t-\tau)^i u(\tau) d\tau \\ & + \sum_{i=1}^{\mu} \phi_{-i} \left(B u^{(i-1)}(t) + \sum_{k=0}^{n-1} E \, \delta^{(i-k-1)}(t) \, x^{(k)}(t_0) \right) \\ & + \sum_{i=0}^{\infty} \sum_{k=0}^{n-1} \phi_i \, E \, \frac{t^{i+k}}{\Gamma(i+k+1)} \, x^{(k)}(t_0). \end{aligned} \tag{10}$$

4. EXPERIMENTAL RESULTS

Academic and real examples are presented in this section. The obtained results are compared to the existing ones.

Example 1. *Let us consider the singular system* (8) *for $\alpha = 1$ and,*

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

with the initial conditions:

$$x_0 = \begin{pmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \end{pmatrix}.$$

We have, $\forall v \in \mathbb{C}$:

$$\det(E - vA) = -v^2 \neq 0,$$

thus, the system is regular, moreover, the index of nilpotency is $\mu = 2$. Therefore, the fundamental matrices are:

$$\phi_{-2} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \phi_{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \phi_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Using (10)*, leads to the following result:*

$$x(t) = \begin{pmatrix} -u(t) - u'(t) - x_{2,0}\,\delta(t) \\ -u(t) \\ x_{3,0} \end{pmatrix},$$

which is the same result obtained in [3].

Example 2. *Let us consider the equation* (8) *with* $0 < \alpha \leq 1$*, and the matrices*

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \tag{11}$$

with the boundary conditions:

$$x_0 = \begin{pmatrix} x_{0,1} \\ x_{0,2} \end{pmatrix}.$$

For almost all $v \in \mathbb{C}$*:*

$$\det(E - v^{\alpha}A) = 2v^{\alpha}(1 + v^{\alpha}) \neq 0,$$

then, the pencil of the system (11) *is regular and* $\mu = 1$*. Hence, the fundamental matrices are:*

$$\phi_{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \phi_{2m} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \phi_{2m+1} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad m \in \mathbb{N}.$$

Finally, the solution is given by:

$$x(t) = \begin{pmatrix} \displaystyle\sum_{i=0}^{\infty}(-1)^i \left[\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}x_{1,0} + \frac{1}{\Gamma((i+1)\alpha)}\int_0^t (t-\tau)^{(i+1)\alpha-1}\,u(\tau)\,d\tau\right] \\ \\ u(t) \end{pmatrix}.$$

Using Laplace transform, the same result is obtained [4].

Example 3. *Let us consider the systems* (5) *and* (6) *with* $0 < \alpha \leq 1$ *and the matrices*

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{12}$$

$$C = \begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix}, \quad D = 0, \tag{13}$$

and the initial conditions:

$$x_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

For almost all $v \in \mathbb{C}$*:*

$$\det(E - v^\alpha A) = -v^{2\alpha} \neq 0,$$

then pencil of the system (12) *is regular and* $\mu = 2$*. Hence, the fundamental matrices are:* $\phi_{-2} = 0_{\mathbb{R}^{4\times4}}$*, and,*

$$\phi_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ \phi_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \phi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows then,

$$x(t) = \begin{pmatrix} 1 \\ 0 \\ \dfrac{t^\alpha}{\Gamma(\alpha)} \\ -u(t) \end{pmatrix}.$$

Finally, using the systems (6) *and* (13) *and the state* $x(t)$*, the output result is:*

$$y(t) = 1 - u(t),$$

which is the same one obtained in [11].

Example 4. *Figure* 1 *shows the fractional descriptor electrical circuit with given resistances* R_1*,* R_2 *; inductances* L_1*,* L_2*, and source current* u*.*

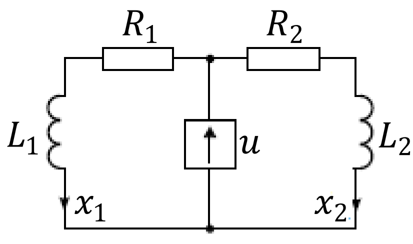

Figure 1. Fractional electrical circuit [6].

Using Kirchhoff's laws, we can write the equation:

$$E\mathbf{D}^\alpha x(t) = Ax(t) + Bu(t), \tag{14}$$

where $0 < \alpha \leq 1$ *and,*

$$E = \begin{pmatrix} L_1 & -L_2 \\ 0 & 0 \end{pmatrix}, \ A = \begin{pmatrix} -R_1 & R_2 \\ -1 & -1 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and the initial conditions:

$$x_0 = \begin{pmatrix} x_{0,1} \\ x_{0,2} \end{pmatrix}.$$

The system (14) *is singular, more then that, its pencil is regular:*

$$\det(E - v^\alpha A) = (R_1 + R_2)v^{2\alpha} + (L_1 + L_2)v^\alpha \neq 0, \quad \text{for } v \in \mathbb{C}.$$

Then, $\mu = 1$. *The corresponding fundamental matrices are:*

$$\phi_{-1} = \begin{pmatrix} 0 & \dfrac{L_2}{L_1 + L_2} \\ \\ 0 & \dfrac{L_1}{L_1 + L_2} \end{pmatrix},$$

$$\forall i \in \mathbb{N}, \ \phi_i = \begin{pmatrix} \dfrac{(-1)^i (R_1 + R_2)^i}{(L_1 + L_2)^{i+1}} & \dfrac{(-1)^i R_2 (R_1 + R_2)^i}{(L_1 + L_2)^{i+1}} + \dfrac{(-1)^{i+1} L_2 (R_1 + R_2)^{i+1}}{(L_1 + L_2)^{i+2}} \\ \\ \dfrac{(-1)^{i+1} (R_1 + R_2)^i}{(L_1 + L_2)^{i+1}} & \dfrac{(-1)^i R_1 (R_1 + R_2)^i}{(L_1 + L_2)^{i+1}} + \dfrac{(-1)^{i+1} L_1 (R_1 + R_2)^{i+1}}{(L_1 + L_2)^{i+2}} \end{pmatrix}.$$

By using (9)*, we find:*

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

where,

$$\begin{aligned} x_1(t) \ &= \ \sum_{i=1}^{\infty} \frac{1}{\Gamma((i+1)\alpha)} \left(\frac{(-1)^i R_2 (R_1 + R_2)^i}{(L_1 + L_2)^{i+1}} + \frac{(-1)^{i+1} L_2 (R_1 + R_2)^{i+1}}{(L_1 + L_2)^{i+2}} \right) \times \\ & \int_0^t (t - \tau)^{(i+1)\alpha - 1} u(\tau) d\tau + \frac{L_2}{L1 + L2} u(t) \\ & + \ \sum_{i=1}^{\infty} \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)} \left(\frac{(-1)^i L_1 (R_1 + R_2)^i}{(L_1 + L_2)^{i+1}} x_{0,1} + \frac{(-1)^{i+1} L_2 (R_1 + R_2)^i}{(L_1 + L_2)^{i+1}} x_{0,2} \right), \end{aligned}$$

and,

$$x_2(t) = \sum_{i=1}^{\infty} \frac{1}{\Gamma((i+1)\alpha)} \left(\frac{(-1)^i R_1 (R_1+R_2)^i}{(L_1+L_2)^{i+1}} + \frac{(-1)^{i+1} L_1 (R_1+R_2)^{i+1}}{(L_1+L_2)^{i+2}} \right) \times$$

$$\int_0^t (t-\tau)^{(i+1)\alpha-1} u(\tau) d\tau + \frac{L_1}{L1+L2} u(t)$$

$$+ \sum_{i=1}^{\infty} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} \left(\frac{(-1)^{i+1} L_1 (R_1+R_2)^i}{(L_1+L_2)^{i+1}} x_{0,1} + \frac{(-1)^{i+2} L_2 (R_1+R_2)^i}{(L_1+L_2)^{i+1}} x_{0,2} \right).$$

5. DISCUSSION AND CONCLUSION

Prior to this work, Several methods have been used to solve singular continuous-time fractional linear system with regular pencil.

In this paper, singular continuous-time fractional linear systems with regular pencil have been solved using Sumudu transform due to its interesting properties. However, the case where E is invertible ($\det E \neq 0$) has been already studied using Sumudu transform in [7].

The obtained results are promising and encourage us to extend and to use this approach for other type of system and circuit, and also for other applications as for example the crone suspension which is one of our future research topics.

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