Topological Analysis of the Liouville Foliation for the Kovalevskaya Integrable Case on the Lie Algebra So(4)

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Abstract—In this paper we study the topology of the Liouville foliation for the integrable case of Euler's equations on the Lie algebra so(4) discovered by I. V. Komarov, which is a generalization of the Kovalevskaya integrable case in rigid body dynamics. We generalize some results by A. V. Bolsinov, P. H. Richter and A. T. Fomenko about the topology of the classical Kovalevskaya case. We also show how the Fomenko–Zieschang invariant can be calculated for every admissible curve in the image of the momentum map.

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1. INTRODUCTION AND MAIN RESULTS

I. V. Komarov in his paper [4] showed that the Kovalevskaya integrable case in rigid body dynamics can be included in a one-parameter family of integrable Hamiltonian systems on the pencil of Lie algebras so(3, 1) − e(3) − so(4). The Kovalevskaya top was realized as a system on Lie algebra e(3). We will briefly describe this construction.

Consider the six-dimensional space $\mathbb{R}^6(\mathbf{J}, \mathbf{x})$ and the following one-parameter family of Poisson brackets depending on the real parameter κ:

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \quad \{J_i, x_j\} = \varepsilon_{ijk} x_k, \quad \{x_i, x_j\} = \kappa \varepsilon_{ijk} J_k,$$

where $\varepsilon_{ijk} = \operatorname{sign}(\{123\} \to \{ijk\})$. When $\kappa > 0, \kappa = 0$ and $\kappa < 0$ this bracket coincides with the Lie–Poisson bracket for the Lie algebras so(4), e(3) and so(3, 1) respectively. These brackets have two Casimir functions:

$$f_1 = x_1^2 + x_2^2 + x_3^2 + \kappa(J_1^2 + J_2^2 + J_3^2), \qquad f_2 = x_1 J_1 + x_2 J_2 + x_3 J_3.$$

In the case of $\kappa \geq 0,\ a > 0$, the common level surfaces of the Casimir

$$M^4_{a,b} = \{(\mathbf{J}, \mathbf{x}) | \quad f_1(\mathbf{J}, \mathbf{x}) = a, \quad f_2(\mathbf{J}, \mathbf{x}) = b\}$$

are orbits of the coadjoint representation and symplectic leaves. The Hamiltonian H of the system and the integral K are equal to

$$H = J_1^2 + J_2^2 + 2J_3^2 + 2c_1 x_1, \quad K = (J_1^2 - J_2^2 - 2c_1 x_1 + \kappa c_1^2)^2 + (2J_1 J_2 - 2c_1 x_2)^2,$$

where c_1 is an arbitrary constant. We may assume that $c_1 = 1$ and $\kappa = -1, 0$ or 1. We do not consider the case of $\kappa < 0$ in this paper.

We discuss the topology of the Liouville foliation defined by the momentum mapping $\mathfrak{F} = (H, K) : M^4_{a,b} \to \mathbb{R}^2$ on every non-singular orbit $M^4_{a,b}$. Bifurcation diagrams $\Sigma_{a,b}$ of $\mathfrak{F}$ are often denoted by Σ

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for short. They were constructed by M. Kharlamov [8] when $\kappa = 0$ and by I. Kozlov [3] when $\kappa > 0$. They consist of the smooth fragments of curves called arcs and the points of their tangency, intersection or cusps called singular points. Some families of arcs and singular points have analogs in the case of $\kappa = 0$. These "old" families are denoted by $y_1, ..., y_{13}$ and $\alpha_1, ..., \delta_2$ in [3] and [2] respectively. Families of singular points that have no such analogs are denoted by $z_1, ..., z_{11}$ in [3]. We establish five families ξ_i, $i = 1..5$ of such "new" arcs. Table 1 contains necessary information about new arcs of Σ and families of Liouville tori denoted by (1), ..., (5) in [2].

class	bifurcation diagram arcs	atom	family of tori
ξ_1	$(z_4, z_3), (z_4, z_5), (z_4, z_{11}), (z_4, z_8), (z_7, z_5), (z_7, z_8)$	A	(1)
ξ_2	$(z_3, z_2), (z_5, z_2)$	2 A	(3)
ξ_3	$(z_2, z_1), (z_{10}, z_1)$	2 A	(2)
ξ_4	$(z_6, z_5), (z_6, z_8), (z_7, z_5), (z_7, z_8)$	A	(4)
ξ_5	$(z_8, z_9), (z_8, z_{10}), (z_{11}, z_{10}), (z_{11}, z_9)$	A	(1)

Table 1. Classes of new arcs of the bifurcation diagrams

Our research is based on the theory of topological classification of integrable Hamiltonian systems developed by A. T. Fomenko and his school and discussed in [1]. Marked molecule is an invariant that classifies Liouville foliations on three-dimensional manifolds. One should know some special coordinate basic cycles on the boundary tori of the neighborhoods of the critical fibers to calculate this invariant. Such bases are called admissible coordinate systems. The definition of admissible coordinate systems for all types of the 3-atoms was given in [1].

A. V. Bolsinov, P. H. Richter and A. T. Fomenko proved in [2] that for the Kovalevskaya top the admissible coordinate systems near of every arc of $\Sigma_{a,b}$ can be expressed via the uniquely defined λ-cycles. The following theorem states that this result remains true for the case of so(4):

Theorem 1. *The following coordinate systems $(\lambda_{\xi_i}, \mu_{\xi_i})$ are the admissible coordinate systems for the arcs $\xi_i, i = 1..5$:*

$$\begin{pmatrix}\lambda_{\xi_1}\\ \mu_{\xi_1}\end{pmatrix} = \begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\lambda_{\gamma_1}\\ \lambda_{\gamma_3}\end{pmatrix}, \quad \begin{pmatrix}\lambda_{\xi_2}\\ \mu_{\xi_2}\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\lambda_{\gamma_2}\\ -\lambda_{\beta_2}\end{pmatrix}, \quad \begin{pmatrix}\lambda_{\xi_3}\\ \mu_{\xi_3}\end{pmatrix} = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\lambda_{\delta_1}\\ \lambda_{\beta_1}\end{pmatrix},$$

$$\begin{pmatrix}\lambda_{\xi_4}\\ \mu_{\xi_4}\end{pmatrix} = \begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\lambda_{\gamma_4}\\ \lambda_{\gamma_3}\end{pmatrix}, \quad \begin{pmatrix}\lambda_{\xi_5}\\ \mu_{\xi_5}\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\lambda_{\gamma_3}\\ \lambda_{\beta_1}\end{pmatrix}.$$

Proof. Calculation of admissible coordinate systems for all $\xi_i, i = 1..5$, is based on well-known facts about the structure of Liouville foliation in neighborhoods of nondegenerate zero-rank critical points. The label r is equal to infinity for a point of the centre-saddle type and therefore the λ-cycles for the corresponding arcs are equal up to a sign. We also use Theorem 2 proved later about the admissible coordinate system for a critical point of centre-centre type.

Let us show the calculation for the arc ξ_1. The arcs ξ_3, ξ_4, ξ_5 can be considered analogously. Since z_5 is the image of a centre-saddle point, we have $\lambda_{\xi_1} = \pm\lambda_{\beta_2} = \pm(\lambda_{\gamma_3} - \lambda_{\gamma_1})$. The point z_1 is a critical point of centre-centre type, hence we choose the negative sign, and $\mu_{\xi_1} = \lambda_{\gamma_1}$.

We consider the point z_3 to calculate cycles for the arc ξ_2. This critical point has the centre-saddle type, hence we can take $\lambda_{\xi_2} = \sigma\lambda_{\beta_2}$ and $\mu_{\xi_2} = -\sigma\lambda_{\gamma_2}$, where $\sigma = \pm 1$. Since z_2 is the image of a degenerate critical orbit of rank 1, we need that $\mu_{\xi_2} = \lambda_{\gamma_2}$ and $\sigma = 1$ (because both cycles are determined by sgrad H). $\square$

A smooth curve without self-intersections in $\mathbb{R}^2$ is called *admissible* if it intersects the arcs of bifurcation diagram Σ transversely and does not pass through the singular points of Σ.

Remark 1. *Admissible coordinate systems for arcs $\alpha_1, ..., \delta_2$ found in [2] and the results of our theorem are sufficient to compute the marked molecule of every admissible curve for the Kovalevskaya system on* $\mathrm{so}(4)$.

2. THEOREM ABOUT A POINT OF CENTRE-CENTRE TYPE

In this section we describe the marked loop molecule for the $\mathfrak{F}$ image of a *critical point of centre-centre type*. The loop molecule has the form A $-$ A, the label $r = 0$ and the label $\varepsilon = \pm 1$. So one should calculate this sign. Small neighborhood of this point has the structure of Cartesian product of two 2-atoms of the type A. The formal definition of such point can be found in [1]. Its image will be called a *singular point of centre-centre type*. For every arc of bifurcation diagram we consider a small (vertical) interval I on the straight line $H = \mathrm{const}$ that intersects this arc transversely. The preimage of such interval is also the 3-atom A. Recall that it is a solid torus $S^1 \times D^2$ foliated by Liouville tori and one singular elliptic orbit and the 2-atom A is its base.

Definition 1. *Basis (λ, μ) of $\pi_1(T^2)$ on the boundary torus is called admissible iff:*

1. *the cycle λ is contractible;*
2. *the orientation of the cycle μ is determined by the Hamiltonian vector field* $\mathrm{sgrad}\, H$ *on the singular fibre;*
3. *basis (λ, μ) in $\pi_1(T^2)$ is positive on the boundary torus.*

We assume that basis (u, v) in T_xT^2 is positive if the quadruple of vectors $(\mathrm{grad}\, H, N, u, v)$ is positive w.r.t the volume form $\omega \wedge \omega$. Here N is the outward-pointing normal vector to the 3-atom.

It is important that this way allows to determine the admissible coordinate systems near all arcs *simultaneously* as it was made in [2]. It means that *every* gluing 2×2 matrix can be calculated as the transition matrix from the basis of the first arc to the basis near the other arc. In particular determinants of all gluing matrices for the *isoenergy* surfaces $H = \mathrm{const}$ are equal to -1. If we consider a surface under the condition $K = \mathrm{const}$ then the determinants of all gluing matrices are equal to 1.

The bifurcation diagram is an angle near the singular point L under consideration. We consider inward-pointing tangent vectors to arcs γ_1, γ_2 of Σ that intersect in L. Let the derivation of H in the direction of a tangent vector be called *the derivation of H in the direction of the corresponding arc*.

Theorem 2. *Let point L be a singular point of a bifurcation diagram of centre-centre type. Let $\varepsilon_i = \pm 1, i = 1, 2$ be the signs of the derivatives of H in the direction of the intersecting arcs $\gamma_i, i = 1, 2$ respectively. Admissible coordinate systems (λ_i, μ_i) for these arcs can be chosen s.t.*

$$\begin{pmatrix} \lambda_2 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_1 \\ \varepsilon_2 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \mu_1 \end{pmatrix}.$$

Proof. 1. It is well-known that there exist local coordinates p_1, p_2, q_1, q_2 such that the symplectic form ω is given by the following formula in a neighborhood of this critical point of centre-centre type:

$$\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2,$$

and the functions H, K have the form $H = H(\alpha_1, \alpha_2)$, $K = K(\alpha_1, \alpha_2)$ in this neighborhood, where $\alpha_1 = (p_1^2 + q_1^2)$ and $\alpha_2 = (p_2^2 + q_2^2)$. We assume that $\alpha_i = 0$ on the arc γ_i, $i = 1, 2$.

Liouville tori near the critical point of centre-centre type are the Cartesian product of two circles φ_1, φ_2 with tangent vector fields u_1, u_2.

$$u_1 = (-q_1, 0, p_1, 0), \quad u_2 = (0, -q_2, 0, p_2).$$

2. We claim that (λ_i, μ_i) for the arcs γ_i have the following form:

$$(\lambda_1, \mu_1) = \left(\varphi_1, \operatorname{sgn}\left(\frac{\partial H}{\partial \alpha_1}\right)\varphi_2\right), \quad (\lambda_2, \mu_2) = \left(\varphi_2, \operatorname{sgn}\left(\frac{\partial H}{\partial \alpha_2}\right)\varphi_1\right).$$

3. Let us denote tangent vector fields to the cycles μ_1, μ_2 as v_1, v_2:

$$v_1 = \frac{\partial H}{\partial \alpha_1} u_2 \quad v_2 = \frac{\partial H}{\partial \alpha_2} u_1.$$

The orientations of cycles μ_i are determined by the vector field sgrad H:

$$\operatorname{sgrad} H = \omega^{-1} dH = \left(-2q_1 \frac{\partial H}{\partial \alpha_1}, -2q_2 \frac{\partial H}{\partial \alpha_2}, 2p_1 \frac{\partial H}{\partial \alpha_1}, 2p_2 \frac{\partial H}{\partial \alpha_2}\right).$$

To check the orientation of the λ-cycles we have to verify that two quadruples of vectors $(\operatorname{grad} H, N_i, u_i, v_i)$, $i = 1, 2$ are positive w.r.t. the volume form $\omega \wedge \omega$. Here N_i is the outward-pointing normal vector to the isoenergy 3-atom A for the arc $\gamma_i, i = 1, 2$. It can be easily checked that

$$N_1 = -\operatorname{sgn}\left(\frac{\partial H}{\partial \alpha_2}\right) N, \quad N_2 = \operatorname{sgn}\left(\frac{\partial H}{\partial \alpha_1}\right) N,$$

where vector N is the orthogonal to grad H, u_i, v_i :

$$N = \left(-p_1\alpha_2 \frac{\partial H}{\partial \alpha_2},\ p_2\alpha_1 \frac{\partial H}{\partial \alpha_1},\ -q_1\alpha_2 \frac{\partial H}{\partial \alpha_2},\ q_2\alpha_1 \frac{\partial H}{\partial \alpha_1}\right).$$

□

Corollary 1. *All elements of the gluing matrix in Theorem 2 change their signs if one change the required orientation of the quadruple of vector fields in the definition of admissible coordinate system.*

We assume that the critical point of centre-centre is the only point in the fibre. General case can be reduced to this one by considering the appropriate leaf of the bifurcation complex. A.T. Fomenko suggested the concept of bifurcation complex for systems of different dimensions in [5] and [6]. Their properties also were described there. Some applications and links between it and other topological invariants are contained in [7].

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