Multiple Interpolation by the Functions of Finite Order in the Half-plane

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Abstract—The aim of this paper is to study the multiple interpolation problem in the spaces of analytical functions of finite order $\rho > 1$ in the half-plane. The necessary and sufficient conditions for solvability of interpolation problem are obtained. These conditions are obtained in terms of the Nevanlinna product of interpolation nodes. The solution of the interpolation problem is constructed in the form of the Jones interpolation series, which is a generalization of the Lagrange interpolation series.

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1. INTRODUCTION

In 1948, A. F. Leont'ev [1] first considered the interpolation problem in the space of entire functions of finite order $\rho > 0$, which received the name *of free interpolation problem*. As known, interpolation problem is called problem of free interpolation in the case when on the values of interpolation function F in interpolation nodes only necessary restrictions are imposed, related to the fact that the function F must to belong to the considered space. In this paper we consider the problem of free multiple interpolation in the spaces of analytical functions of finite order $\rho > 1$ in the half-plane.

Let $\mathbb{C}$ be the complex plane, $\mathbb{C}_+ = \{z : \Im z > 0\}$, let $\mathbb{R}$ be the real axis and $\mathbb{N}$ be the set of positive integers. We denote by $C(a, r)$ the open disk of radius r with centre at a. Let Ω_+ be the intersection of a set Ω with the half-plane $\mathbb{C}_+$: $\Omega_+ = \Omega \cap \mathbb{C}_+$. Denote by $[\rho, \infty]^+$ the space of analytical functions of finite order $\rho > 1$ in $\mathbb{C}_+$ [2, Chapter I, §1], i.e. $f \in [\rho, \infty]^+$ if

$$\sup_{z\in\mathbb{C}_+} \frac{\ln^+ \ln^+ |f(re^{i\theta})|}{\ln r} < \infty\,, \quad \limsup_{r\to\infty} \sup_{0<\theta<\pi} \frac{\ln^+ \ln^+ |f(re^{i\theta})|}{\ln r} \leqslant \rho\,,$$

where $\ln^+ a = \begin{cases} \ln^+ a = \ln a\,, & a > 1, \\ \ln^+ a = 0\,, & a \leqslant 1\,. \end{cases}$

Cauchy inequality

$$|f(z)^{(k-1)}| \leqslant \frac{(k-1)!}{(\Im z)^{k-1}} \max_{|\zeta - z| \leqslant \Im z} |f(\zeta)|, \quad k \in \mathbb{N},$$

for derivatives of an analytic function f on $\mathbb{C}_+$ leads to the reasonableness of the following definition.

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Definition 1. *A divisor* $D = \{a_n, q_n\}_{n=1}^{\infty}$ *(i.e., a set of distinct complex numbers* $\{a_n = r_n e^{i\theta_n}\}_{n=1}^{\infty} \subset \mathbb{C}_+$ *with limit points all on the real axis* $\mathbb{R}$ *and infinity with their integer multiplicities* $\{q_n\}_{n=1}^{\infty} \subset \mathbb{N}$*) is called an interpolation divisor in the space* $[\rho, \infty]^+$ *if for any sequence of complex numbers* $\{b_{n,k}\}$, $k = 1, 2, \ldots, q_n$, $n \in \mathbb{N}$, *satisfying the conditions*:

$$\sup_{n\in\mathbb{N}} \frac{1}{\ln(|a_n|+2)} \ln^+ \ln^+ \sup_{1\leqslant k\leqslant q_n} \frac{|b_{n,k}|(\Im a_n)^{k-1}}{(k-1)!} < \infty\,, \tag{1}$$

$$\limsup_{|a|_n\to\infty} \frac{1}{\ln|a_n|} \ln^+ \ln^+ \sup_{1\leqslant k\leqslant q_n} \frac{|b_{n,k}|(\Im a_n)^{k-1}}{(k-1)!} \leqslant \rho\,, \tag{2}$$

there exists a function $F \in [\rho, \infty]^+$ *solving the interpolation problem*

$$F^{(k-1)}(a_n) = b_{n,k}, \quad k = 1, 2, \ldots, q_n,\ n \in \mathbb{N}\,. \tag{3}$$

The conditions (1) and (2) are necessary restrictions on the sequence $\{b_{n,k}\}$. These restrictions are related to the fact that function $F(z)$, solving the interpolation problem (3), must belong to the space $[\rho, \infty]^+$.

In 1975, B. Ya. Levin and N. Uen [3] considered the problem of simple interpolation (i.e., $q_n = 1$, $n \in \mathbb{N}$) in the space $[\rho, \infty]^+$, $\rho > 1$, in the case when for any $\varepsilon > 0$ the inequality $\Im a_n \geqslant \exp(-|a_n|^{\rho+\varepsilon})$ holds for all $n > N(\varepsilon)$. They obtained necessary conditions and sufficient conditions for the solvability of the corresponding interpolation problems in terms of the Nevanlinna product of interpolation nodes. But between two types of these conditions there was a gap that did not allow the interpolation nodes to "accumulate" at points of the real axis. This problem without these restrictions was solved in works [4, 5]. The problem of multiple interpolation under restrictions $\Im a_n \geqslant \exp(-|a_n|^{\rho+\varepsilon})$ was considered by N. Uen [6]. N. Uen also obtained necessary conditions and sufficient conditions for the solvability of the interpolation problem which are similar to the conditions in the article [3]. The aim of this paper is to study the interpolation problem in the space $[\rho, \infty]^+$ for $\rho > 1$. We find necessary and sufficient conditions for the interpolation problem to be solvable. This conditions are formulated in the terms of canonical product determined by the interpolation nodes. According to its content, the problem is a problem of free interpolation.

Denote by $B_q(u, v)$ the Nevanlinna primary factor

$$B_q(u,v) = \begin{cases} \dfrac{\bar{v}(u-v)}{v(u-\bar{v})}\,, & q = 0, \\ B_0(u,v)\exp\left(\displaystyle\sum_{j=1}^{q} \frac{u^j}{j}\left(\frac{1}{v^j} - \frac{1}{\bar{v}^j}\right)\right), & q \in \mathbb{N}\,. \end{cases}$$

Let D be the divisor such that for any $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \frac{q_n \sin\theta_n}{1 + r_n^{\rho+\varepsilon}} < \infty, \quad \rho > 1\,, \tag{4}$$

then the function

$$E(z) = E_D(z) =: \prod_{|a_n|<1} \left(\frac{z-a_n}{z-\bar{a}_n}\right)^{q_n} \prod_{|a_n|\geqslant 1} B_q^{q_n}(z, a_n), \quad q = [\rho],$$

belongs to the space $[\rho, \infty]^+$ [7, Theorem 4] (see also [2, Chapter I, §3, Theorem 3.2]). By $[\cdot]$, we denote here the integer part of a number. The function $E(z)$ is called *the canonical function of the divisor* D.

Our main result is the theorem stated below.

Theorem 1. *The following two statements are equivalent.*

1) *The divisor D is an interpolation divisor in the space $[\rho,\infty]^+$.*

2) *The condition* (4) *holds and the canonical function $E(z)$ of the divisor D satisfies the conditions*:

$$\sup_{n\in\mathbb{N}} \frac{1}{\ln(r_n+2)} \ln^+ \ln^+ \frac{|\gamma_{n,1}|}{(\Im a_n)^{q_n}} < \infty \,, \tag{5}$$

$$\limsup_{r_n\to\infty} \frac{1}{\ln r_n} \ln^+ \ln^+ \frac{|\gamma_{n,1}|}{(\Im a_n)^{q_n}} \leqslant \rho \,, \tag{6}$$

where

$$\gamma_{n,k} = \frac{1}{(k-1)!} \left(\frac{d}{dz}\right)^{k-1} \frac{(z-a_n)^{q_n}}{E(z)}\bigg|_{z=a_n}, \quad k=1,2,\ldots,q_n,\ n\in\mathbb{N}\,. \tag{7}$$

In addition, following Titchmarsh [8], we shall use the following terms and notation. If some argument involves a number independent of the main variables, then it is called a constant. To denote absolute positive constants, not necessarily the same ones, we use the letters M and p. One can come across an assertion of the type "$|f(z)| < Mr^p$"; therefore, "$|f(z)| < 3Mr^{2p}$", which need not cause any misunderstanding.

2. PRELIMINARIES

Denote by

$$A_n(z) = \prod_{0<|a_n-a_k|\leqslant r_n/2} \left[\frac{a_k(z-\bar a_k)}{\bar a_k(z-a_k)}\right]^{q_k},\ E_n(z) = E(z)\left[\frac{a_n(z-\bar a_n)}{\bar a_n(z-a_n)}\right]^{q_n}.$$

$$B_n(z) = E(z)\left[\frac{a_n(z-\bar a_n)}{\bar a_n(z-a_n)}\right]^{q_n} A_n(z)\,.$$

We need the following statements.

Lemma 1. *If the divisor D satisfies* (4) *then*

$$\sup_{z\in\mathbb{C}_+} \frac{1}{\ln(|z|+2)} \sup_{n\in\mathbb{N}} \ln^+\ln^+|E_n(z)| < \infty, \quad \limsup_{|z|\to\infty} \frac{1}{\ln|z|} \sup_{n\in\mathbb{N}} \ln^+\ln^+|E_n(z)| \leqslant \rho\,, \tag{8}$$

$$\sup_{z\in\mathbb{C}_+} \frac{1}{\ln(|z|+2)} \sup_{n\in\mathbb{N}} \ln^+\ln^+|B_n(z)| < \infty, \quad \limsup_{|z|\to\infty} \frac{1}{\ln|z|} \sup_{n\in\mathbb{N}} \ln^+\ln^+|B_n(z)| \leqslant \rho\,. \tag{9}$$

The lemma is proved by standard methods for estimating canonical products (see e.g. [2], [9]), and we omit the proof.

Lemma 2. *If the divisor D satisfies* (4), (5) *and* (6), *then*

$$\sup_{n\in\mathbb{N}} \frac{\ln q_n}{\ln(r_n+2)} < \infty\,, \quad \limsup_{r_n\to\infty} \frac{\ln q_n}{\ln r_n} \leqslant \rho\,. \tag{10}$$

Proof. If we use the fact that

$$(2\Im a_n)^{q_n} = |E_n(a_n)\gamma_{n,1}|, \quad n=1,2,\ldots\,,$$

we get the assertion of the lemma from (5), (6) and (8). □

Lemma 3. *If the divisor D satisfies* (4), (5) *and* (6), *then*

$$\sup_{n\in\mathbb{N}}\sum_{k=1}^{\infty}\frac{q_k\Im a_k\Im a_n}{|a_n-\bar{a}_k|^2(1+r_k^2)^{\frac{\rho+1}{2}}}<\infty\,.$$

Proof. We get from (5), (6) and (9) that

$$\sup_{n\in\mathbb{N}}\frac{1}{\ln(|a_n|+2)}\ln^+\ln^+|A_n(a_n)|<\infty,\quad \limsup_{|a_n|\to\infty}\frac{1}{\ln|a_n|}\ln^+\ln^+|A_n(a_n)|\leqslant\rho\,. \tag{11}$$

From the last inequalities, the identity

$$\left|\frac{a-b}{\bar{a}-b}\right|^2=1-\frac{4\Im a\Im b}{|\bar{a}-b|^2}\,,$$

and the elementary inequality $x\leqslant-\ln(1-x)$ $(0\leqslant x<1)$ we get the next relations

$$\begin{gathered}\sup_{n\in\mathbb{N}}\frac{1}{\ln|r_n|+2}\ln\sum_{0<|a_n-a_k|\leqslant r_n/2}\frac{q_k\Im a_k\Im a_n}{|a_n-\bar{a}_k|^2}<\infty,\\ \limsup_{r_n\to\infty}\frac{1}{\ln|r_n|}\ln\sum_{0<|a_n-a_k|\leqslant r_n/2}\frac{q_k\Im a_k\Im a_n}{|a_n-\bar{a}_k|^2}\leqslant\rho\,.\end{gathered} \tag{12}$$

The condition (4) implies that for any $\varepsilon>0$ the series

$$\sum_{n=1}^{\infty}\frac{q_n\Im a_n}{r_n^{\rho+1+\varepsilon}} \tag{13}$$

converges. From (12) and (13) we obtain the statement of the lemma. □

Lemma 4. *If the divisor D satisfies* (4), (5) *and* (6), *then*

$$\begin{gathered}\sup_{n\in\mathbb{N}}\frac{1}{\ln(|a_n|+2)}\ln^+\ln^+\max_{1\leq k\leq q_n}\frac{|\gamma_{n,k}|}{\Im a_n^{q_n-k-1}}\leqslant\infty\,,\\ \limsup_{|a|_n\to\infty}\frac{1}{\ln|a_n|}\ln^+\ln^+\max_{1\leq k\leq q_n}\frac{|\gamma_{n,k}|}{(\Im a_n)^{q_n-k-1}}\leqslant\rho\,.\end{gathered} \tag{14}$$

Proof. Let's fix $\varepsilon>0$. From (10) and (11), it follows that there exists $p>0$ such that

$$\left|\frac{a_n-a_k}{a_n-\bar{a}_k}\right|\geqslant\exp\left[-\frac{r_n^p}{q_n}\right],k\neq n,\ \text{and}\ \left|\frac{a_n-a_k}{a_n-\bar{a}_k}\right|\geqslant\exp\left[-\frac{r_n^{\rho+\varepsilon}}{q_n}\right]\,, \tag{15}$$

for all $r_n>r_0=r_0(\varepsilon)$.

Next, let $l_n=\min_{k\neq n}|a_n-a_k|$. It follows from (15) that

$$l_n\geqslant\Im a_n\exp\left[-\frac{r_n^p}{q_n}\right]\ (n\in\mathbb{N}),\ \text{and}\ l_n\geqslant\Im a_n\exp\left[-\frac{r_n^{\rho+\varepsilon}}{q_n}\right]\,, \tag{16}$$

if $r_n>r_0(\varepsilon)$.

We define an analytic function $\psi(t)$ on the disk $C(0,1)$ by the equality $t^n\psi(t)=E(a_n+l_nt)$. It follows from (5), (6) and (16), that for some $p_1>0$

$$|\psi(0)|=\frac{|E^{(q_n)}(a_n)|}{q_n!}l_n^{q_n}\geqslant\exp\left[-r_n^{p_1}\right](n\in\mathbb{N}),\ \text{and}\ |\psi(0)|\geqslant\exp\left[-r_n^{\rho+\varepsilon}\right]\,,$$

for all $r_n > r_0 = r_0(\varepsilon)$. Moreover,

$$|\psi(t)| \leqslant \max_{|z-a_n|\leqslant l_n} |E(z)| \leqslant \exp\left[r_n^{p_2}\right] (n \in \mathbb{N}), \text{ and } |\psi(t)| \leqslant \exp\left[r_n^{\rho+\varepsilon}\right] ,$$

for some $p_2 > 0$ and for all $r_n > r_0 = r_0(\varepsilon)$.

Let $g(t) = \psi(t)/\psi(0)$. Since $g(t)$ does not have roots in the disk $C(0, 1/2)$ and $g(0) = 1$, we can apply the Carathéodory inequality (see [9, Chapter I, Theorem 9]), which gives for some $p_3 > 0$, if $|t| \leqslant 1/4$ then $|g(t)| \geqslant \exp\left[-r_n^{p_3}\right]$ $(n \in \mathbb{N})$ and $|g(t)| \geqslant \exp\left[-r_n^{\rho+\varepsilon}\right]$, for all $r_n > r_0 = r_0(\varepsilon)$. From this for $|\tau| \leqslant l_n/4$, $|\tau| \geqslant l_n/8$, for some $p_4 > 0$

$$|E(a_n+\tau)| \geqslant \left(\frac{\tau}{l_n}\right)^{q_n} \exp\left[-r_n^{p_4}\right] (n \in \mathbb{N}), \text{ and } |E(a_n+\tau)| \geqslant \left(\frac{\tau}{l_n}\right)^{q_n} \exp\left[-r_n^{\rho+\varepsilon}\right] , \tag{17}$$

for all $r_n > r_0 = r_0(\varepsilon)$. Further, by definition,

$$\gamma_{n,k} = \frac{1}{2\pi i} \int\limits_{|z-a_n|=l_n/4} \frac{(\zeta-a_n)^{q_n-k}}{E(\zeta)}\, d\zeta, \quad n \in \mathbb{N},\ k = 1, \dots, q_n\,.$$

Therefor, (14) follows from (16), (17) and (10).

□

We will also use the Govorov theorems [2, Chapter I, §1, Theorem 3.2].

Theorem 2. (Govorov) *Any function $f \in [\rho, \infty]^+$, $\rho > 1$, is represented as*

$$\begin{aligned} f(z) = e^{i(\alpha_0+\alpha_1 z+\dots+\alpha_q z^q)} \prod_{|z_n|\leqslant 1} B_0(z, z_n) \prod_{|z_n|>1} B_q(z, z_n) \\ \times \exp\left\{\frac{1}{\pi i} \int\limits_{-\infty}^{+\infty} \frac{(tz+1)^{q+1}}{(t^2+1)^{q+1}(t-z)} (\ln|f(t)|\, dt + d\sigma(t))\right\} , \end{aligned} \tag{18}$$

where $\alpha_0, \alpha_1, \dots, \alpha_q$ are real constants, $z_n = \tau_n e^{\varphi_n}$ are roots of $f(z)$. All infinite products and integrals in (18) *converge absolutely. For any $\varepsilon > 0$, the conditions hold*:

$$\sum_{\tau_n\leqslant 1} \tau_n \sin\varphi_n < \infty, \quad \sum_{\tau_n>1} \frac{\sin\varphi_n}{\tau_n^{\rho+\varepsilon}} < \infty, \quad \int\limits_{-\infty}^{+\infty} \frac{|\ln|f(t)||\, dt + |d\sigma(t)|}{1+|t|^{\rho+1+\varepsilon}} < \infty\,.$$

We need the following lemma.

Lemma 5. *Let $\{w_n\}$ and $\{r_n\}$ be two sequences of positive numbers satisfying the conditions*:

$$\limsup_{n\to\infty} r_n = \infty\,, \quad \sup_n \frac{\ln^+\ln^+ w_n}{\ln r_n} < \infty\,, \quad \limsup_{r_n\to+\infty} \frac{\ln^+\ln^+ w_n}{\ln r_n} = \rho < \infty\,,$$

then the sequence $\{\varepsilon_n\}$, $\lim\limits_{n\to\infty} \varepsilon_n = 0$, of positive numbers can be chosen so that

$$\ln^+ w_n \leqslant r_n^{\rho+\varepsilon_n}\,,$$
$$r_k^{\rho+\varepsilon_k} \leqslant r_n^{\rho+\varepsilon_n} \quad if \quad r_k < r_n\,,$$

and for some sequence of values r_{n_k} $(k = 1, 2, \dots)$ tending to infinity

$$\ln^+ w_{n_k} = r_{n_k}^{\rho+\varepsilon_{n_k}}\,.$$

This lemma is a consequence of the theorem on the existence of a proximate order of an entire function [9, Chapter I, Theorem 16].

3. THE PROOF OF IMPLICATION 1) $\Rightarrow$ 2) OF THEOREM 1.

Let the divisor D be interpolation divisor in the space $[\rho,\infty]^+$. There exists a function $F\in[\rho,\infty]^+$ such that $F^{(k-1)}(a_1)=1$, $k=1,\dots q_1$, and $F^{(k-1)}(a_n)=0$ if $n\geqslant 2$, $k=1,\dots q_n$. Therefore the divisor $D\backslash\{a_1,q_1\}$ belongs to the set of zeros of F and satisfies condition (4) [2, Chapter I, §3]. It means that and the divisor D also satisfies (4).

We prove implication 1) $\Rightarrow$ (5) and (6) by contradiction. We now prove (5). Assume the contrary, that there exists a subsequence $\{c_n\}\subset\{a_n\}$ such that

$$\lim_{n\to\infty}(\ln(|c_n|+2))^{-1}\ln^+\ln^+(\Im c_n)^{-p_n}|\gamma_{n,1}|=\infty\,, \tag{19}$$

where p_n are the multiplicities of the numbers c_n in D.

Using the Carleson interpolation theorem in the space $\mathbf{H}^\infty$ [11] and passing if necessary to a subsequence, we can also assume that the set $\{c_n\}$ is sparse enough that

$$\inf_n\{\Im c_n|B'(c_n)|\}\geqslant\delta>0\,,\quad n=1,2,\dots\,, \tag{20}$$

where $B(z)=\prod\limits_{n=1}^{\infty}\dfrac{\bar c_n(z-c_n)}{c_n(z-\bar c_n)}$ is Blaschke product corresponding to $\{c_n\}$.

Suppose further that $F\in[\rho,\infty]^+$ is such that

$$F^{(k-1)}(a_n)=0,\quad a_n\neq c_n,\ k=1,\dots,q_n;$$
$$F^{(k-1)}(c_n)=0,\quad k=1,\dots,p_n-1;\,F^{(p_n-1)}(c_n)=(p_n-1)!(\Im c_n)^{1-p_n}\,.$$

By Theorem 2, the function $f(z)=\dfrac{F(z)B(z)}{E(z)}$ belongs to $[\rho,\infty]^+$. We have

$$f(c_n)=((p_n-1)!)^{-1}F^{(p_n-1)}\gamma_{n,1}B'(c_n)=(\Im c_n)^{-p_n}\gamma_{n,1}\Im c_nB'(c_n)\,.$$

The last equality with (20) and $f\in[\rho,\infty]^+$ contradicts (19).

The inequality (5) is proved. The inequality (6) is proved similarly.

4. THE PROOF OF IMPLICATION 2) $\Rightarrow$ 1) OF THEOREM 1.

We remark that if $|a_n|\leqslant 1$, then the multiplicities q_n are bounded by virtue of conditions (10). Therefore, there exists a bounded function $g\in\mathbf{H}^\infty$ solving the interpolation problem (3) [12]. Let $b'_{n,k}=b_{n,k}-g^{(k-1)}(a_n)$ for $r_n>1$, and $b'_{n,k}=b_{n,k}$ for $r_n\leqslant 1$, $n=1,2,\dots$, $k=1,\dots q_n$. It is clear that the numbers $b'_{n,k}$ satisfy conditions (1) and (2).

Since the series (13) converges (after a renumbering of the points a_n if necessary), it can be assumed that

$$\frac{\Im a_{n+1}}{1+r^2_{n+1}}\leqslant\frac{\Im a_n}{1+r^2_n},\quad n=1,2,\dots\,. \tag{21}$$

Next, for $r_n>1$ let

$$\alpha_{n,m}=\frac{(-1)^{m-1}}{(m-1)!}\sum_{j=o}^{q_n-m}\frac{1}{j!}\,\gamma_{n,q_n+1-m-j}\,b'_{n,j+1},\quad m=1,\dots,q_n\,,$$

$$\alpha_n(z)=\sum_{k=n}^{\infty}\frac{1+\bar a_k(z+i\Im a_n)}{i(\bar a_k-z-i\Im a_n)}\cdot\frac{\Im a_k}{(1+r_k^2)^{\frac{[\rho]+3}{2}}}\,.$$

The series defining the functions $\alpha_n(z)$ converges uniformly in each domain $D^n_{r,\delta} = \{z : |z| \le r, \Im z \ge -\Im a_n + \delta,\ \delta > 0\}$, because for $z \in D^n_{r,\delta}$, $r \ge 2$

$$\left|\frac{1+\bar{a_k}(z+i\Im a_n)}{i(\bar{a_k}-z-i\Im a_n)}\right| \frac{\Im a_k}{(1+r_k^2)^{\frac{[\rho]+3}{2}}} \le \frac{(1+r)(1+r_k)}{\delta}\frac{\Im a_k}{(1+r_k^2)^{\frac{[\rho]+3}{2}}}\,,$$

and from (13) it follows that the series

$$\sum_{k=1}^{\infty} \frac{q_k \Im a_k}{(1+r_k^2)^{\frac{\rho+1}{2}+\varepsilon}}$$

converges for any $\varepsilon > 0$.

Let us estimate $\Re\alpha_n(z)$. We have

$$\Re\alpha_n(z) = \sum_{k=n}^{\infty} \frac{(\Im a_k + \Im z + \Im a_n + r_k^2(\Im z + \Im a_n) + |z+i\Im a_n|^2 \Im a_k)}{|\bar{a}_k - z - i\Im a_n|^2} \cdot \frac{\Im a_k}{(1+r_k^2)^{\frac{[\rho]+3}{2}}}\,. \tag{22}$$

Since $\Im a_n > 0$, $\Im \bar{a}_k < 0$, then $|\bar{a}_k - a_n - i\Im a_n| > |\bar{a}_k - a_n|$. From this, by Lemma 3, by the inequality (21) and the equality (22) we obtain, in particular,

$$\begin{aligned}\Re\alpha_n(a_n) &\le \sum_{k=n}^{\infty} \frac{\Im a_k(\Im a_k(1+|a_n+i\Im a_n|^2) + 2\Im a_n(1+r_k^2))}{|\bar{a}_k - a_n|^2(1+r_k^2)^{\frac{[\rho]+3}{2}}} \le \\ &\leqslant \sum_{k=n}^{\infty}\left(\frac{\Im a_k}{1+r_k^2} + \frac{2\Im a_n}{1+4r_n^2}\right)\frac{\Im a_k(1+r_k^2)(1+4r_n^2)}{|\bar{a}_k - a_n|^2(1+r_k^2)^{\frac{[\rho]+3}{2}}} \le \\ &\leqslant 5\frac{1+4r_n^2}{1+r_n^2}\sum_{k=n}^{\infty}\frac{\Im a_n}{|\bar{a}_k - a_n|^2}\frac{\Im a_k}{(1+r_k^2)^{\frac{[\rho]+1}{2}}} \leqslant M < \infty\,,\end{aligned} \tag{23}$$

for some $M > 0$, and

$$\Re\alpha_n(z) \geqslant \sum_{k=n}^{\infty} \frac{(\Im a_k)^2}{(1+r_k^2)^{\frac{[\rho]+3}{2}}}\frac{1}{|\bar{a}_k - z - i\Im a_n|^2}\,. \tag{24}$$

Next for $r_n > 1$ let

$$P_n(z) = \sum_{m=1}^{q_n} \alpha_{n,m}\left[\frac{\varphi_n(z)}{z-a_n}\right]^{(m-1)}\,, \tag{25}$$

where

$$\varphi_n(z) = \left(\frac{1+z\bar{a}_n}{1+r_n^2}\right)^{S_n+[\rho]+3}\left(\frac{2\Im a_n}{z-\bar{a}_n}\right)^2 \exp[\alpha_n(a_n) - \alpha_n(z)]\,,$$

and S_n is a sequence of natural numbers, which we choose below. Notice, that

$$\varphi_n(a_n) = 1\,. \tag{26}$$

In addition, using the elementary inequality $1 + x \leqslant \sqrt{2(1+x^2)}$, we obtain for $|z| \geqslant 1$:

$$\left|\frac{1+z\bar{a}_n}{1+r_n^2}\right| \leqslant \frac{|z|(1+r_n)}{1+r_n^2} \leqslant \frac{\sqrt{2}|z|}{\sqrt{1+r_n^2}}\,.$$

From this

$$|\varphi_n(z)| \le 4\left(\frac{\sqrt{2}|z|}{\sqrt{1+r_n^2}}\right)^{S_n+[\rho]+3}\frac{(\Im a_n)^2}{|z-\bar{a}_n|^2}\exp\{\Re[\alpha_n(a_n) - \alpha_n(z)]\}\,. \tag{27}$$

Let us show that the formal series

$$F(z)=E(z)\sum_{n=1}^{\infty}P_n(z)+g(z) \tag{28}$$

solves the interpolation problem (3). We show that

$$F_1^{(k-1)}(a_n)=b'_{n,k}, \quad r_n>1, k=1,2,\ldots,q_n\,, \tag{29}$$

where $F_1(z)=F(z)-g(z)$. We have

$$\frac{(z-a_n)^{q_n}F_1(z)}{E(z)}=(z-a_n)^{q_n}P_n(z)+(z-a_n)^{q_n}\sum_{k\neq n}P_k(z)\,. \tag{30}$$

From (26), it follows

$$\frac{\varphi_n(z)}{z-a_n}=\frac{1}{z-a_n}+\widetilde{\varphi_n}(z), \quad n=1,2,\ldots\,,$$

where $\widetilde{\varphi_n}(z)$ is a analytic function on $\mathbb{C}_+$, $\widetilde{\varphi_n}(a_n)=\varphi'_n(a_n)$.

Then

$$P_n(z)=\sum_{m=1}^{q_n}\alpha_{n,m}\left[\frac{(-1)^{m-1}(m-1)!}{(z-a_n)^m}+\widetilde{\varphi_n}^{(m-1)}(z)\right]\,,$$

From this

$$(z-a_n)^{q_n}P_n(z)=\sum_{m=1}^{q_n}\alpha_{n,m}[(-1)^{m-1}(m-1)!(z-a_n)^{q_n-m}+ \\ +(z-a_n)^{q_n}\widetilde{\varphi_n}^{(m-1)}(z)], \quad n=1,2,\ldots\,.$$

Differentiating both sides of (30) (q_n-m) times at the point $z=a_n$, we get

$$\sum_{j=0}^{q_n-m}C^j_{q_n-m}F_1^{(j)}(a_n)\left(\frac{d}{dz}\right)^{q_n-m-j}\left.\frac{(z-a_n)^{q_n}}{E(z)}\right|_{z=a_n}= \\ =(-1)^{m-1}(m-1)!(q_n-m)!\,\alpha_{n,m}, \quad m=1,\ldots,q_n\,,$$

or

$$\sum_{j=0}^{q_n-m}\frac{1}{j!}\gamma_{n,q_n+1-m-j}F_1^{(j)}(a_n)=\sum_{j=0}^{q_n-m}\frac{1}{j!}\gamma_{n,q_n+1-m-j}b'_{n,j+1}\,.$$

Consequently,

$$\begin{cases}\sum\limits_{j=0}^{q_n-m}\frac{1}{j!}\gamma_{n,q_n+1-m-j}(F_1^{(j)}(a_n)-b'_{n,j+1})=0, \\ \qquad m=1,\ldots,q_n\,, n=1,2,\ldots\,.\end{cases} \tag{31}$$

The matrix made up of the coefficients of $F_1^{(j)}(a_n) - b'_{n,j+1}$, $i = 0, \dots, q_n - 1$, in system (31) has the form

$$\Delta_n = \begin{pmatrix} \gamma_{n,q_n} & \frac{1}{1!}\gamma_{n,q_n-1} & \cdots & \frac{1}{(q_n-2)!}\gamma_{n,2} & \frac{1}{(q_n-1)!}\gamma_{n,1} \\ \gamma_{n,q_n-1} & \frac{1}{1!}\gamma_{n,q_n-2} & \cdots & \frac{1}{(q_n-2)!}\gamma_{n,1} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \gamma_{n,1} & 0 & \dots & 0 & 0 \end{pmatrix}.$$

It is clear that $\det \Delta_n \neq 0$, and hence (29) holds.

We now show that $F_1 \in [\rho, \infty]^+$ for a suitable choice of the sequence $\{S_n\}$ of positive integers. From conditions (1), (2), inequalities (14) and the definition of $\alpha_{n,m}$, we obtain for all $m = 1, \dots, q_n$, $n = 1, 2, \dots$,

$$\sup_{n\in\mathbb{N}} \frac{1}{\ln(r_n+2)} \ln^+ \ln^+ \sup_{1\leqslant m\leqslant q_n} \frac{(m-1)!|\alpha_{n,m}|}{(q_n-m+1)(\Im a_n)^m} < \infty\,,$$
$$\limsup_{r_n\to\infty} \frac{1}{\ln r_n} \ln^+ \ln^+ \sup_{1\leqslant m\leqslant q_n} \frac{(m-1)!|\alpha_{n,m}|}{(q_n-m+1)(\Im a_n)^m} \leqslant \rho\,.$$

From this, we obtain by Lemma 5

$$|\alpha_{n,m}| \leqslant \exp(r_n^{\rho+\varepsilon_n}) \frac{(q_n-m+1)(\Im a_n)^m}{(m-1)!}, \quad m = 1, \dots, q_n, n = 1, 2, \dots\,, \tag{32}$$

for some sequence $\{\varepsilon_n\}$ of positive numbers such that

$$\lim_{n\to\infty} \varepsilon_n = 0\,, \tag{33}$$

$$r_k^{\rho+\varepsilon_k} \leqslant r_n^{\rho+\varepsilon_n} \quad if \quad r_k < r_n\,. \tag{34}$$

Set

$$u_{n,m}(z) = \left[\frac{\varphi_n(z)}{z-a_n}\right]^{(m-1)},\ m = 1, \dots, q_n,\ n = 1, 2, \dots\,.$$

Estimate $u_{n,m}(z)$ for $z \in \mathbb{C}_+$, $z \notin C(a_n, \Im a_n/2)$. Note, if $|t - z| = \Im a_n/4$ then, at first,

$$|t - a_n| \geqslant \frac{\Im a_n}{4}, \quad n = 1, 2, \dots\,, \tag{35}$$

secondly,

$$|t - \bar{a}_n| \geqslant \Im a_n - \frac{\Im a_n}{4} \geqslant \frac{3\Im a_n}{4}\,,$$

$$|z - \bar{a}_n| \leqslant |z - t| + |t - \bar{a}_n| = \frac{\Im a_n}{4} + |t - \bar{a}_n| \leqslant \frac{7|t - \bar{a}_n|}{3} \quad (n = 1, 2, \dots)\,,$$

and

$$|t - \bar{a}_n| \leqslant |z - t| + |z - \bar{a}_n| = \frac{\Im a_n}{4} + |z - \bar{a}_n| \leqslant \frac{5|z - \bar{a}_n|}{4}\,,$$

i.e.,

$$\frac{3|z - \bar{a}_n|}{7} \leqslant |t - \bar{a}_n| \leqslant \frac{5|z - \bar{a}_n|}{4}\,. \tag{36}$$

In addition, if $|z-t|=\Im a_n/4$, then

$$|t+i\Im a_n-\bar{a}_n|\geqslant\frac{3\Im a_n}{4}+\Im z+\Im a_n\,. \tag{37}$$

By integration around the circle $C_{z,n}=\{t:|t-z|=\Im a_n/4$, from (27), (35), (36) and (37), we obtain

$$|u_{n,m}(z)|=\frac{(m-1)!}{2\pi}\left|\int\limits_{C_{z,n}}\frac{\varphi_n(t)\,dt}{(t-a_n)(t-z)^m}\right|\leqslant\frac{4^m(m-1)!}{(\Im a_n)^m}\max_{t\in C_{z,n}}|\varphi_n(t)|\leqslant$$

$$\leqslant\frac{4^m 49(m-1)!(\Im a_n)^2}{9(\Im a_n)^m|z-\bar{a}_n|^2}\left(\frac{\sqrt{2}(|z|+1/4)}{\sqrt{1+r_n^2}}\right)^{S_n+[\rho]+3}\max_{t\in C_{z,n}}\exp[\Re(\alpha_n(a_n)-\alpha_n(t))]\,.$$

From this we get finally, taking into account (23), (24) and (37):

$$\begin{gathered}|u_{n,m}(z)|\leqslant\frac{4^m 49(m-1)!e^M(\sqrt{2}(|z|+1/4))^{S_n+[\rho]+3}}{9(\Im a_n)^m|z-\bar{a}_n|^2(1+r_n^2)^{S_n/2}}\frac{(\Im a_n)^2}{(1+r_n^2)^{([\rho]+3)/2}}\times\\ \times\exp\left[-\sum_{k=n}^{\infty}\frac{(\Im a_k)^2}{(3\Im a_n/4+\Im z+\Im a_k)^2(1+r_k^2)^{([\rho]+3)/2}}\right],\end{gathered} \tag{38}$$

$m=1,\dots,q_n$, $n=1,2,\dots$, where $M>0$ is constant from (23).

Further from (25), (32) and (38), we get for $z\in\mathbb{C}_+$, $z\notin C(a_n,\Im a_n/2)$, inequality holds

$$\begin{gathered}|P_n(z)|\leqslant\sum_{m=1}^{q_n}|\alpha_{nm}||u_{nm}(z)|\leqslant\\ \leqslant M\exp(r_n^{\rho+\varepsilon_n})\frac{(\sqrt{2}(|z|+1/4))^{S_n+[\rho]+3}(\Im a_n)^2}{(1+r_n^2)^{S_n/2}(1+r_n^2)^{([\rho]+3)/2}|z-\bar{a}_n|^2}\sum_{m=1}^{q_n}4^m(q_n-m+1)\times\\ \times\exp\left[-\sum_{k=n}^{\infty}\frac{(\Im a_k)^2}{(3\Im a_n/4+\Im z+\Im a_k)^2(1+r_k^2)^{([\rho]+3)/2}}\right]\leqslant\\ \leqslant Mq_n(q_n+1)\exp(r_n^{\rho+\varepsilon_n}+q_n\ln 4)(\sqrt{2}(|z|+1/4))^{[\rho]+3}\left(\frac{\sqrt{2}(|z|+1/4)}{\sqrt{1+r_n^2}}\right)^{S_n}\times\\ \times\frac{(\Im a_n)^2}{|z-\bar{a}_n|^2(1+r_n^2)^{([\rho]+3)/2}}\exp\left[-\sum_{k=n}^{\infty}\frac{(\Im a_k)^2}{(3\Im a_n/4+\Im z+\Im a_k)^2(1+r_k^2)^{([\rho]+3)/2}}\right],\\ n=1,2,\dots,\end{gathered} \tag{39}$$

for some constant $M>0$ and for some sequence $\{\varepsilon_n\}$ of positive numbers satisfying conditions (33) and (34).

Using (10), we obtain from (39) for $z\in\mathbb{C}_+$, $z\notin C(a_n,\Im a_n/2)$:

$$\begin{gathered}|P_n(z)|\leqslant M\exp(r_n^{\rho+\varepsilon_n})(\sqrt{2}(|z|+1/4))^{[\rho]+3}\left(\frac{\sqrt{2}(|z|+1/4)}{\sqrt{1+r_n^2}}\right)^{S_n}\times\\ \times\frac{(\Im a_n)^2}{|z-\bar{a}_n|^2(1+r_n^2)^{([\rho]+3)/2}}\exp\left[-\sum_{k=n}^{\infty}\frac{(\Im a_k)^2}{(3\Im a_n/4+\Im z+\Im a_k)^2(1+r_k^2)^{([\rho]+3)/2}}\right],\\ n=1,2,\dots,\end{gathered} \tag{40}$$

for some constant $M>0$ and for some sequence $\{\varepsilon_n\}$ of positive numbers satisfying conditions (33) and (34).

Further, note that if $|t-a_n| \leqslant \Im a_n/2$, and $|z-a_n| = \Im a_n/2$ then

$$|z| \leqslant |t|+1, \quad 3|t-\bar{a}_n|/5 \leqslant |z-\bar{a}_n| \leqslant 5|t-\bar{a}_n|/3\,. \tag{41}$$

Applying the principle of maximum modulus to analytic in $\mathbb{C}_+$ the functions $\Phi_n(z) = E(z)P_n(z)$, using inequalities (40) and (41) we obtain for $t \in C(a_n, \Im a_n/2)$, considering that $\Im t \geqslant \Im z/4$,

$$\begin{gathered} |\Phi_n(t)| \leqslant \max_{|z-a_n|=\Im a_n/2}\{|E(z)||P_n(z)|\} \leqslant \\ \leqslant \max_{|z-a_n|=\Im a_n/2} |E(z)| M \exp(r_n^{\rho+\varepsilon_n})(\sqrt{2}(|t|+5/4))^{[\rho]+3}\left(\frac{\sqrt{2}(|t|+5/4)}{\sqrt{1+r_n^2}}\right)^{S_n} \times \\ \times \frac{(\Im a_n)^2}{|t-\bar{a}_n|^2(1+r_n^2)^{([\rho]+3)/2}} \exp\left[-\sum_{k=n}^{\infty}\frac{(\Im a_k)^2}{(3\Im a_n/4+4\Im t+\Im a_k)^2(1+r_k^2)^{([\rho]+3)/2}}\right], \\ n=1,2,\dots, \end{gathered} \tag{42}$$

for some constant $M>0$ and for some sequence $\{\varepsilon_n\}$ of positive numbers satisfying conditions (33) and (34).

By (40), inequality (42) valid for all $t \in \mathbb{C}_+$.

We denote

$$\lambda_n(z) = \sum_{k=n}^{\infty}\frac{(\Im a)_k^2}{(3\Im a_n/4+4\Im z+\Im a_k)^2(1+r_k^2)^{([\rho]+3)/2}}\,, \quad n=1,2,\dots$$

so that

$$\lambda_n(z)-\lambda_{n+1}(z) = \frac{(\Im a)^2}{(3\Im a_n/4+4\Im z+\Im a_n)^2(1+r_n^2)^{([\rho]+3)/2}}\,, \quad n=1,2,\dots.$$

It's clear that $\lambda_n(z) \downarrow 0$ as $n\to\infty$, $z\in\mathbb{C}_+$. Noting that if $z \in \mathbb{C}_+$ then $3\Im a_n/4+4\Im z+\Im a_n \leqslant 4\Im z+7\Im a_n/4 \leqslant 4(\Im z+\Im a_n) \leqslant 4|z-\bar{a}_n|$, we obtain from (42):

$$\begin{gathered} |\Phi_n(z)| \leqslant \max_{|z-a_n|=\Im a_n/2} |E(z)| M \exp(r_n^{\rho+\varepsilon_n})(\sqrt{2}(|z|+5/4))^{[\rho]+3}\left(\frac{\sqrt{2}(|z|+5/4)}{\sqrt{1+r_n^2}}\right)^{S_n} \times \\ \times[\lambda_n(z)-\lambda_{n+1}(z)]\exp\left[-\lambda_n(z)\right], \quad n=1,2,\dots, \end{gathered}$$

for some constant $M>0$ and for some sequence $\{\varepsilon_n\}$ of positive numbers satisfying conditions (33) and (34).

Use of the elementary inequality $t \leqslant e^t-1$, $t \geq 0$, for $t = \lambda_n(z)-\lambda_{n+1}(z)$ gives us

$$\begin{gathered} |\Phi_n(z)| \leqslant \max_{|z-a_n|=\Im a_n/2} |E(z)| M \exp(r_n^{\rho+\varepsilon_n})(\sqrt{2}(|z|+5/4))^{[\rho]+3}\left(\frac{\sqrt{2}(|z|+5/4)}{\sqrt{1+r_n^2}}\right)^{S_n} \times \\ \times[\exp[-\lambda_{n+1}(z)]-\exp[-\lambda_n(z)]], \quad n=1,2,\dots, \end{gathered} \tag{43}$$

for some constant $M>0$ and for some sequence $\{\varepsilon_n\}$ of positive numbers satisfying conditions (33) and (34).

We now choose a sequence of numbers S_n such that the function $F(z)$ defined by the series (28) belongs to the space $[\rho,\infty]^+$. Set $S_n = 1+[r_n^{\rho+\varepsilon_n}]$, where sequence $\{\varepsilon_n\}$ from (43). Let's fix $\varepsilon>0$. Let r_n be such that $\varepsilon_n<\varepsilon$. If $\sqrt{1+r_n^2} \geq \sqrt{2}(|z|+5/4)$ then

$$\exp(r_n^{\rho+\varepsilon_n})\left(\frac{\sqrt{2}(|z|+5/4)}{\sqrt{1+r_n^2}}\right)^{S_n} \leqslant 1\,. \tag{44}$$

If $\sqrt{1+r_n^2} \le \sqrt{2}(|z|+5/4)$ then, by (34), for all $r_k \le r_n$

$$\exp(r_k^{\rho+\varepsilon_k}) \leqslant \exp(r_n^{\rho+\varepsilon_n}) \leqslant \exp(M|z|^{\rho+\varepsilon})\,, \tag{45}$$

$$\left(\frac{\sqrt{2}(|z|+5/4)}{\sqrt{1+r_k^2}}\right)^{S_k} \leqslant (\sqrt{2}(|z|+5/4))^{S_k} \leqslant (\sqrt{2})^{S_k}\exp(|z|+5/4))^{S_k} \leqslant \exp[M|z|^{\rho+\varepsilon}]\,. \tag{46}$$

Thus, with this choice of the numbers S_n, from (43) – (46) the inequality follows:

$$|\Phi_n(z)| \leqslant \exp[M|z|^{\rho+\varepsilon}]|E(z)|[\exp[-\lambda_{n+1}(z)] - \exp[-\lambda_n(z)]]\,, \tag{47}$$

for some constant $M > 0$.

From (47), we obtain for a sufficiently large natural number N:

$$\left|E(z)\sum_{n=1}^{N} P_n(z)\right| \leqslant \sum_{n=1}^{N}|\Phi_n(z)| \leqslant \exp[M|z|^{\rho+\varepsilon}]|E(z)|\times$$
$$\times[\exp[-\lambda_{N+1}(z)] - \exp[-\lambda_1(z)]] \leqslant \exp[M|z|^{\rho+\varepsilon}]|E(z)|\,.$$

Whence the convergence of the series (28) on compact sets in $\mathbb{C}_+$ follows and its belonging to the space $[\rho,\infty]^+$ for $\rho > 1$.

The theorem is proved.

Remark 1. In this paper, we consider the interpolation problem in the space $[\rho,\infty]^+$, $\rho > 1$. There are various definitions of the order of functions analytic in the half-plane [2, 13–15]. These definitions coincide for $\rho > 1$ and differ for $0 \leqslant \rho \leqslant 1$. In our opinion, each case requires an independent study.

Remark 2. In 1994, K. G. Malyutin [16] considered the problem of multiple interpolation in the space $[\rho(r),\infty)^+$ of functions of at most normal type for the proximate order $\rho(r)$, $\lim\limits_{r\to\infty}\rho(r) = \rho > 1$, in the upper half-plane $\mathbb{C}_+$.

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