A Note On Generalized Spectrum Approximation

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Abstract—The purpose of this paper is to solve the spectral pollution. We suggest a modern method based on generalized spectral techniques, where we show that the propriety L is hold with norm convergence. In addition, we prove that under collectively compact convergence the proprieties U and L are hold . We describe the theoretical foundations of the method in details, As well as illustrate its effectiveness by numerical results.

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1. INTRODUCTION

It is well known that the spectral pollution is the weakness of projection methods for an unbounded operator. Thus, our technic is an alternative method based on disrupting an unbounded operator by a bounded one until that the spectral properties transform on controlled case. The generalized spectrum takes its place as the desired case. We will show that every unbounded operator contains a decomposition of two bounded operators which carry all the spectral properties. Through this decomposition and basing upon its numerical approximations the phenomenon of spectral pollution will be resolved. This phenomenon is considered as a serious problem in several areas in the field of applied mathematics that has been studied in detail in [1–3]. The concept of generalized spectrum was constructed under the generalized spectrum of the matrices (see [4]) then extended for bounded operators (see [5, 6]).

The natural framework of our research is a complex separable Banach space $(X, \|\cdot\|)$, we will define an unbounded operator A with non empty resolvent set. In the first section, we will prove that each spectral problem contains a generalized spectral problem equivalent. In this respect, we will show our results for generalized spectrum. Throughout the second section we will work on sufficient condition for the generalized spectral convergence of a sequences of linear operators, which is norme convergence. Finally, we will prove the previous results under different condition, that is collectively compact convergence, while our numerical results are applied on harmonic oscillator operator, which is defined over $L^2(\mathbb{R})$ by $Au = -u'' + x^2u$.

Accordingly, the sufficient condition for the generalized spectral convergence of a sequences of linear operators is that, the generalized spectrum is eventually contained in any neighborhood of the generalized spectrum of its approach, and the corresponding generalized spectral subspace has the same dimension as its approach (this former is identical with spectral subspace of the original problem). The spectral convergence is adopted under the proprieties U and L. Although in numerical test we will use the upper and lower semi continuity of the spectrum, where the proprieties U and L are consequences of its respectively (see [7]).

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2. GENERALIZED SPECTRUM

Let T and S be two operators in $\mathrm{BL}(X)$, we define the generalized resolvent set by

$$\mathrm{re}(T,S) = \{\lambda \in \mathbb{C} : (T - \lambda S) \text{ is bijective}\}.$$

The generalized spectrum set is $\mathrm{sp}(T,S) = \mathbb{C} \setminus \mathrm{re}(T,S)$. For $z \in \mathrm{re}(T,S)$, we define $\mathrm{R}(z,T,S) = (T - zS)^{-1}$, the generalized resolvent operator.

We define $\lambda \in \mathbb{C}$ as a generalized eigenvalue when $(T - \lambda S)$ is not injective, then the set $E(\lambda) = \mathrm{Ker}(T - \lambda S)$ is the generalized spectral subspace. We say that λ has an finite algebraic multiplicity if there exists α where $\dim \mathrm{Ker}(T - \lambda S)^\alpha < \infty$. If the operator S is invertible, we have $\mathrm{sp}(T,S) = \mathrm{sp}(S^{-1}T)$. If S^{-1} does not exist, the generalized spectrum it can be bounded or the whole $\mathbb{C}$ or empty.

The next three results are a generalization of the classical case $S = I$ (see [6]).

Theorem 1. *Let $\lambda \in \mathrm{re}(T,S)$ and $\mu \in \mathbb{C}$ where $|\lambda - \mu| < \| \mathrm{R}(\lambda,T,S)S\|^{-1}$, then $\mu \in \mathrm{re}(T,S)$.*

Corollary 1. *The set $\mathrm{sp}(T,S)$ is closed in $\mathbb{C}$.*

Theorem 2. *The function $\mathrm{R}(\cdot,T,S) : \mathrm{re}(T,S) \to \mathrm{BL}(X)$ is analytic, and its derivative given by $\mathrm{R}(\cdot,T,S)\, S\, \mathrm{R}(\cdot,T,S)$.*

We consider an unbounded operator A with domain $\mathrm{D}(A) \subset X$. The following theorem shows that every unbounded operator contains a decomposition of two bounded operators which will express it in the theory of the generalized spectrum.

Theorem 3. *If $\mathrm{re}(A) \neq \emptyset$, then there exist $T, S \in \mathrm{BL}(X)$ such that $\mathrm{sp}(A) = \mathrm{sp}(T,S)$. In particulary, λ is an eigenvalue for A if and only if λ is a generalized eigenvalue for the couple (T,S). In addition we have $\mathrm{Ker}(A - \lambda I) = \mathrm{Ker}(T - \lambda S)$.*

Proof. Let $\alpha \in \mathrm{re}(A)$, we put $S = (A - \alpha I)^{-1}$, $T = A(A - \alpha I)^{-1}$. We prove that

$$\lambda \in \mathrm{re}(A) \Leftrightarrow (A - \lambda I)^{-1} \in \mathrm{BL}(X) \Leftrightarrow (A - \lambda I)(A - \alpha I)^{-1} \in \mathrm{BL}(X)$$

$$\Leftrightarrow (T - \lambda S) \in \mathrm{BL}(X) \Leftrightarrow (T - \lambda S)^{-1} \in \mathrm{BL}(X) \Leftrightarrow \lambda \in \mathrm{re}(T,S).$$

Let $\lambda \in \mathrm{sp}(T,S)$, there exists $u \in X \backslash \{0\}$ such that $Tu = \lambda S u$, thus

$$Tu = \lambda S u \Rightarrow A(A - \alpha I)^{-1} u = \lambda (A - \alpha I)^{-1} u \Rightarrow u = (\lambda - \alpha)(A - \alpha I)^{-1} u \Rightarrow u \in \mathrm{D}(A).$$

We apply $(A - \alpha I)$ on $Tu = \lambda S u$, we find that $Au = \lambda u$. Finally, we observe by construction that $\mathrm{Ker}(A - \lambda I) = \mathrm{Ker}(T - \lambda S)$. $\square$

We note that the choice of taking the couple (T,S) as a function of the resolvent operator of A is not only the case (see the numerical application below).

Theorem 4. *Let T and S be two operators in $\mathrm{BL}(X)$, and let λ be a generalized eigenvalue of finite type isolated in $\mathrm{sp}(T,S)$, we designate by Γ the Cauchy contour that separating λ from $\mathrm{sp}(T,S)$. Then the operator*

$$P = -\frac{1}{2i\pi} \int_\Gamma (T - zS)^{-1} S \, dz,$$

define a projection.

Proof. We note that the bounded operator P does not depend on the choice of Γ (see theorem 2). We consider now another Cauchy contour Γ' such that $\Gamma' \subset \mathrm{int}(\Gamma)$, then for any $u \in X$ we have

$$P^2 u = \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma (T - zS)^{-1} S \, dz \int_{\Gamma'} (T - z'S)^{-1} S u \, dz' = \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma \int_{\Gamma'} (T - zS)^{-1} S (T - z'S)^{-1} S u \, dz' \, dz$$

$$= \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma \int_{\Gamma'} \frac{(T-zS)^{-1}-(T-z'S)^{-1}}{(z-z')} Su\, dz'\, dz = \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma (T-zS)^{-1} \left(\int_{\Gamma'} \frac{1}{z-z'}\, dz'\right) dzSu$$

$$+ \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma'} (T-z'S)^{-1} \left(\int_\Gamma \frac{1}{(z'-z)} Su\, dz\right) dz' = \frac{-1}{2i\pi} \int_\Gamma (T-zS)^{-1} Su\, dz = Pu.$$

We used $(T-zS)^{-1}-(T-z'S)^{-1} = (z-z')(T-zS)^{-1}S(T-z'S)^{-1}$, and

$$\int_{\Gamma'} \frac{dz'}{z-z'} = 0, \qquad \int_\Gamma \frac{dz}{z-z'} = 2\pi i.$$

□

We call M as the maximal invariant subspace, where λ is generalized eigenvalue of finite type for the couple (T, S).

Theorem 5. *Under the same hypothesis as in theorem 4, if* $M = \mathrm{Ker}(T-\lambda S)$*, then*

$$PX = \mathrm{Ker}(T-\lambda S).$$

Proof. Firstly we fix $\alpha \in \mathrm{re}(T,S)$ where for any Cauchy contour Γ associated with λ, $\alpha \notin \Gamma$. For $\mu \in \Gamma$ we have

$$\mu S - T = (\alpha S - T)[(\alpha-\mu)^{-1}I - (\alpha S - T)^{-1}S](\alpha-\mu),$$

which gives

$$(\mu S - T)^{-1} = [(\alpha-\mu)^{-1}I - (\alpha S - T)^{-1}S]^{-1}(\alpha-\mu)^{-1}(\alpha S - T)^{-1}.$$

Thus, we can see that $(\alpha-\lambda)^{-1}$ is an eigenvalue for the operator $(\alpha S - T)^{-1}S$. Indeed

$$u \in \mathrm{Ker}(T-\lambda S) \Rightarrow (T-\lambda S)u = 0 \Rightarrow (\alpha S - T)^{-1}(\alpha S - T + T - \lambda S)u = u$$

$$\Rightarrow (\alpha S - T)^{-1}Su = (\alpha-\lambda)^{-1}u \Rightarrow u \in \mathrm{Ker}((\alpha S - T)^{-1}S - (\alpha-\lambda)^{-1}I).$$

We reverse the last process, we have

$$\mathrm{Ker}(T-\lambda S) = \mathrm{Ker}((\alpha S - T)^{-1}S - (\alpha-\lambda)^{-1}I).$$

Now, under the choice of α, we can see that for all Cauchy contour Γ, $\eta(\Gamma)$ is also a Cauchy contour for the eigenvalue $(\alpha-\lambda)^{-1}$ where $\eta(\mu) = (\alpha-\mu)^{-1}$. We put $B = (\alpha S - T)^{-1}S$ and $z = (\alpha-\mu)^{-1}$ for any $\mu \in \Gamma$, we write

$$(\mu S - T)^{-1}S = z[-I + z(zI - B)^{-1}].$$

Thus, by passage to the integral over Γ we have

$$\frac{1}{2\pi i} \int_\Gamma (\mu S - T)^{-1}S\, d\mu = \frac{1}{2\pi i} \int_{\eta(\Gamma)} z[-I + z(zI-B)^{-1}]\, \frac{dz}{z^2}$$

$$= \frac{1}{2\pi i} \int_{\eta(\Gamma)} [-z^{-1}I + (zI-B)^{-1}]\, dz = -\frac{1}{2\pi i} \int_{\eta(\Gamma)} \frac{1}{z}\, dz\, I + \frac{1}{2\pi i} \int_{\eta(\Gamma)} (zI-B)^{-1}\, dz = P_{\{(\alpha-\lambda)^{-1}\}},$$

while $P_{\{(\alpha-\lambda)^{-1}\}}$ is the spectral projection associated with the operator $B = (\alpha S - T)^{-1}S$ around $(\alpha-\lambda)^{-1}$. Hence, according to the spectral decomposition theory

$$PX = P_{\{(\alpha-\lambda)^{-1}\}}X = \mathrm{Ker}((\alpha S - T)^{-1}S - (\alpha-\lambda)^{-1}I) = \mathrm{Ker}(T-\lambda S).$$

□

In the case where $\mathrm{Ker}(T-\lambda S)\subset M$, for our generalization takes sense i.e. $PX=\mathrm{Ker}(T-\lambda S)^{\alpha}$. We can assume then, T is commuting with S to obtain

$$\mathrm{Ker}(T-\lambda S)^{\alpha}=\mathrm{Ker}((\alpha S-T)^{-1}S-(\alpha-\lambda)^{-1}I)^{\alpha}.$$

We designate by $B(0,k)$ the bull with center 0 and radius $k>0$.

Proposition 2.1. *Let T and S be two operators in* $\mathrm{BL}(X)$*, then* $\mathrm{sp}(T,S)\subset B(0,k)\iff 0\notin\mathrm{sp}(S)$.

Proof. Suppose that $\mathrm{sp}(T,S)\subset B(0,k)$, then for $\alpha\in\mathrm{re}(T,S)$ we have

$$\lambda S-T=(\alpha S-T)[(\alpha S-T)^{-1}S-(\alpha-\lambda)^{-1}](\lambda-\alpha). \tag{1}$$

As $\alpha\in\mathrm{re}(T,S)$, then $\lambda\in\mathrm{sp}(T,S)\iff(\alpha-\lambda)^{-1}\in\mathrm{sp}((\alpha S-T)^{-1}S)$. So the fact that $\mathrm{sp}(T,S)\subset B(0,k)$ Implies that $0\notin\mathrm{sp}((\alpha S-T)^{-1}S)$, hence $0\notin\mathrm{sp}(S)$. □

The set of all generalized eigenvalues is denoted by $\mathrm{sp}_p(T,S)$. It's clear that when X is finite-dimensional the generalized spectrum consists only of the generalized eigenvalues, except $\{\infty\}$.

Proposition 2.2. *Let T and S be two operators in $BL(X)$. If S is compact, then*

$$\mathrm{sp}(T,S)=\mathrm{sp}_p(T,S)\cup\{\infty\}.$$

Proof. We use the expression (1), since the operator $(\alpha S-T)^{-1}S$ is compact. Thus $\mathrm{sp}(T,S)$ is a set of isolated points. Let $\lambda\in\mathrm{sp}(T,S)$, then there is $z\in\mathrm{sp}((\alpha S-T)^{-1}S)$ where $z=(\alpha-\lambda)^{-1}$, hence there exists $u\in X$ such that

$$(\alpha S-T)^{-1}Su=zu\Longrightarrow(\alpha S-T)^{-1}(\alpha S-\lambda S)u=u$$

$$\Longrightarrow u+(\alpha S-T)^{-1}(T-\lambda S)u=u\Longrightarrow Tu=\lambda Su.$$

□

3. GENERALIZED SPECTRAL APPROXIMATION UNDER NORM CONVERGENCE

Under the propriety U which is studied in [6], we will prove that the propriety L is hold, therefore the phenomenon of spectral pollution is closed. As is known, our study is around the eigenvalue of finite type.

Let T and S be two bounded operators in $\mathrm{BL}(X)$, we assume that there exist two sequences of bounded operators $(T_n)_{n\in\mathbb{N}}$ and $(S_n)_{n\in\mathbb{N}}$ in $\mathrm{BL}(X)$ such that

(A1) $\|T_n-T\|\to 0$,

(A2) $\|S_n-S\|\to 0$.

Lemma 1. *Let P_1 and P_2 be two projections in* $\mathrm{BL}(X)$ *such that* $\|(P_1-P_2)P_1\|<1$*, then* $\dim P_1X\leq\dim P_2X$.

Proof. See [7]. □

Theorem 6. *Let λ be a generalized eigenvalue of finite type isolated in* $\mathrm{sp}(T,S)$*. Under (A1) and (A2) there is a positive integer n_0 such that for $n\geq n_0$* $\dim PX=\dim P_nX$*, where*

$$P_n=-\frac{1}{2\pi i}\int_{\Gamma}(T_n-zS_n)^{-1}S_n\,dz.$$

Proof. We put $E_n = T - T_n$, $F_n = S - S_n$. Let z be in Γ, then

$$T_n - zS_n = [I - (E_n - zF_n)R(z,T,S)](T - zS).$$

It proves that under (A1) and (A2), $T_n - zS_n$ has a bounded inverse which is uniformly bounded for $n \in \mathbb{N}$, where

$$(T_n - zS_n)^{-1} = R(z,T,S)\sum_{k=0}^{\infty}[E_n - zF_n]^k R^k(z,T,S).$$

We put $\lambda_n = \operatorname{int}(\Gamma) \cap \operatorname{sp}(T_n, S_n)$. We define the bounded operator

$$P_n = -\frac{1}{2\pi i}\int_{\Gamma}(T_n - zS_n)^{-1}S_n\, dz,$$

for n larger enough. We mention that λ_n and P_n do not depend on Γ (see[4]). Hence, $\|P_n - P\|$ tends to 0. Indeed, it suffices that to see,

$$\|(T_n - zS_n)^{-1} - (T - zS)^{-1}\| = \|(T_n - zS_n)^{-1}[E_n - zF_n](T - zS)^{-1}\| =\leq c\|E_n - zF_n\|,$$

and

$$(T_n - zS_n)^{-1}S_n - (T - zS)^{-1}S = (T_n - zS_n)^{-1}F_n + [(T_n - zS_n)^{-1} - (T - zS)^{-1}]S.$$

Finally we call the lemma 1 to finish the result. □

We remark that $\{\lambda_n\} \neq \emptyset$ for n large enough, otherwise $P_n = 0$ which implies that $P = 0$.

Corollary 2. *Let λ be a generalized eigenvalue of finite type isolated in $\operatorname{sp}(T,S)$. Under (A1) and (A2), there exists a sequence $\lambda_n \in \operatorname{sp}(T_n, S_n)$ such that $\lambda_n \to \lambda$.*

Proof. We fix $\epsilon > 0$ such that the sequence $\lambda_n = \operatorname{int}(\Gamma) \cap \operatorname{sp}(T_n, S_n)$ belongs to the bull $B(\lambda, \epsilon)$. Let $(\lambda_{n'})_{n' \in \mathbb{N}}$ a subsequence converges to $\tilde{\lambda}$ where $\tilde{\lambda} \neq \lambda$. So by according to propriety U proved in [6], we see that $\tilde{\lambda} \in \operatorname{sp}(T,S)$. But $\tilde{\lambda} \in B(\lambda, \epsilon)$ and $\operatorname{sp}(T,S) \cap B(\lambda,\epsilon) = \{\lambda\}$, hence $\lambda = \tilde{\lambda}$, thus $\lambda_n \to \lambda$. □

4. GENERALIZED SPECTRAL APPROXIMATION UNDER COLLECTIVELY COMPACT CONVERGENCE

We assume that

(B1) $T_n \xrightarrow{cc} T$,

(B2) $S_n \xrightarrow{cc} S$.

We state in this section a set of lemmas which will be needed in the proof of our main theorems.

Lemma 2. *If $T_n \xrightarrow{p} T$ and $S_n \xrightarrow{cc} S$, then for any bounded operator H in $\mathrm{BL}(X)$,*

$$\|(T_n - T)H(S_n - S)\| \to 0.$$

Proof. Since $T_n \xrightarrow{p} T$, and the set $H(\bigcup_{n \geq n_0}\{Sx - S_n x : \|x\| = 1\})$, has compact closure, then $\|(T_n - T)H(S_n - S)\| \to 0$. □

Lemma 3. *Let $T, \widetilde{T}, S$ and $\widetilde{S}$ belong to $\mathrm{BL}(X)$, and let $z \in \operatorname{re}(T,S)$ such that*

$$\|\left[\left((T - \widetilde{T}) - z(S - \widetilde{S})\right)\mathrm{R}(z,T,S)\right]^2\| < 1.$$

Then $z \in \operatorname{re}(\widetilde{T}, \widetilde{S})$, and

$$\|(\widetilde{T} - z\widetilde{S})^{-1}\| \leq \frac{\|\mathrm{R}(z,T,S)\|\left[1 + \|\left((T - \widetilde{T}) - z(S - \widetilde{S})\right)\mathrm{R}(z,T,S)\|\right]}{1 - \|\left[\left((T - \widetilde{T}) - z(S - \widetilde{S})\right)\mathrm{R}(z,T,S)\right]^2\|}.$$

Proof. We put $\widetilde{E} = (T - \widetilde{T})\,\mathrm{R}(z,T,S)$ and $\widetilde{F} = (S - \widetilde{S})\,\mathrm{R}(z,T,S)$, we can see then

$$\widetilde{T} - z\widetilde{S} = [I - (\widetilde{E} - z\widetilde{F})](T - zS).$$

So, by using the second Neumann expansion (see[7]), we obtain that

$$(\widetilde{T} - z\widetilde{S})^{-1} = \mathrm{R}(z,T,S)\sum_{k=0}^{\infty}(\widetilde{E} - z\widetilde{F})^{2k} + \mathrm{R}(z,T,S)\sum_{k=0}^{\infty}(\widetilde{E} - z\widetilde{F})^{2k+1}$$

$$= \mathrm{R}(z,T,S)\left[I + (\widetilde{E} - z\widetilde{F})\right]\sum_{k=0}^{\infty}\left[(\widetilde{E} - z\widetilde{F})^{2}\right]^{k}, \|(\widetilde{T} - z\widetilde{S})^{-1}\| \le \frac{\|R(z,T,S)\|\left(1 + \|\widetilde{E} - z\widetilde{F}\|\right)}{1 - \|(\widetilde{E} - z\widetilde{F})^2\|}.$$

□

Proposition 4.1. *If (B1) and (B2) are obtained, then for $z \in \mathrm{re}(T,S)$, $z \in \mathrm{re}(T_n,S_n)$ for n larger enough.*

Proof. Let $z \in \mathrm{re}(T,S)$, and for n larger enough,

$$T_n - zS_n = [I - (\widetilde{E}_n - z\widetilde{F}_n)](T - zS),$$

where $\widetilde{E}_n = (T - T_n)\,\mathrm{R}(z,T,S)$ and $\widetilde{F}_n = (S - S_n)\,\mathrm{R}(z,T,S)$. Firstly, we have $(\widetilde{E}_n - z\widetilde{F}_n)^2 = (\widetilde{E}_n)^2 + (z\widetilde{F}_n)^2 - \widetilde{E}_n\widetilde{F}_n - \widetilde{F}_n\widetilde{E}_n$. So, according to the lemma 2, $\|(\widetilde{E}_n - z\widetilde{F}_n)^2\| \to 0$. Thus, by applying the last lemma, we obtain that $z \in \mathrm{re}(T_n,S_n)$. □

The following theorem shows that property U is obtained under the collectively compact convergence.

Theorem 7. *Under (B1) and (B2), if $\lambda_n \in \mathrm{sp}(T_n,S_n)$ and $\lambda_n \to \lambda$ then $\lambda \in \mathrm{sp}(T,S)$.*

Proof. Assume that $\lambda \notin \mathrm{sp}(T,S)$, then according to the corollary 1, there exists $r > 0$ such that the ball $B(\lambda,r)$ is contained in $\mathrm{re}(T,S)$. Hence, according to the proposition 4.1, $B(\lambda,r)$ is contained also in $\mathrm{re}(T_n,S_n)$ for n larger enough. In the other hand, $\lambda_n \to \lambda$. Thus, there exists n_0 such that for $n \ge n_0$, $\lambda_n \in B(\lambda,r) \subset \mathrm{re}(T_n,S_n)$ which forms the contradiction. □

To show that the property L is obtained we need the following lemmas.

Lemma 4. *Under (B1) and (B2), then*

$$\forall z \in \mathrm{re}(T,S) \cap \mathrm{re}(T_n,S_n), \quad (T_n - zS_n)^{-1} \xrightarrow{p} (T - zS)^{-1}.$$

Proof. For $z \in \mathrm{re}(T,S) \cap \mathrm{re}(T_n,S_n)$, to find the desired result, we use

$$(T_n - zS_n)^{-1} - (T - zS)^{-1} = (T_n - zS_n)^{-1}[\widetilde{E}_n - z\widetilde{F}_n],$$

where

$$\widetilde{E}_n = (T - T_n)\,\mathrm{R}(z,T,S), \quad \widetilde{F}_n = (S - S_n)\,\mathrm{R}(z,T,S).$$

□

Lemma 5. *Under (B1) and (B2), and for $z \in \mathrm{re}(T,S) \cap \mathrm{re}(T_n,S_n)$, we have*

$$\|\left((T_n - zS_n)^{-1} - (T - zS)^{-1}\right)S\| \to 0.$$

Proof. We put that $\widetilde{E}_n = (T - T_n)(T - zS)^{-1}$, $\widetilde{F}_n = (S - S_n)(T - zS)^{-1}$. Since

$$\left((T - zS)^{-1} - (T_n - zS_n)^{-1}\right)S = [(T_n - zS_n)^{-1} - (T - zS)^{-1}](\widetilde{E}_n - z\widetilde{F}_n)S + (T - zS)^{-1}(\widetilde{E}_n - z\widetilde{F}_n)S.$$

Then, by applying the Lemma 2, we find the result. □

Lemma 6. *Under (B1) and (B2), we have* $\|\left[(T_n - zS_n)^{-1}S_n - (T - zS)^{-1}S\right]^2\| \to 0$, *for all* $z \in \mathrm{re}(T,S) \cap \mathrm{re}(T_n, S_n)$.

Proof. For $z \in \mathrm{re}(T,S) \cap \mathrm{re}(T_n, S_n)$ where n is larger enough,

$$(T_n - zS_n)^{-1}S_n - (T - zS)^{-1}S = [(T_n - zS_n)^{-1} - (T - zS)^{-1}](S_n - S)$$

$$+[(T_n - zS_n)^{-1} - (T - zS)^{-1}]S + (T - zS)^{-1}(S_n - S) = G_n + H_n + K_n.$$

So, according to the Lemma 2 and the Lemma 4 and the Lemma 5, we find H_n^2, K_n^2, H_nK_n and K_nH_n tend to zeros. □

Theorem 8. *Let* λ *be a generalized eigenvalue of finite type isolated in* $\mathrm{sp}(T,S)$, *under (B1) and (B2) there exists a positive integer* n_0 *such that for* $n \geq n_0$ $\dim PX = \dim P_nX$, *where* $P_n = -\frac{1}{2\pi i}\int_\Gamma (T_n - zS_n)^{-1}S_n\,dz$.

Proof. Let $z \in \Gamma$, we apply the proposition 4.1, we find that $(T_n - zS_n)^{-1} \in \mathrm{BL}(X)$ which is uniformly bounded for $n \in \mathbb{N}$ and $z \in \Gamma$. We define $\lambda_n = \mathrm{int}(\Gamma) \cap \mathrm{sp}(T_n, S_n)$ for n larger enough. The bounded operator P_n is well define by

$$P_n = -\frac{1}{2\pi i}\int_\Gamma (T_n - zS_n)^{-1}S_n\,dz.$$

Hence, according to the theorem 2, there exists $z_0 \in \mathrm{re}(T,S)$ and $c > 0$ such that

$$\|(P_n - P)^2\| \leq c\,\|(T_n - z_0S_n)^{-1}S_n - (T - z_0S)^{-1}S\|.$$

So, by applying the lemma 6, we find that $\|(P_n - P)^2\| \to 0$.

In the other hand, for any $u \in X$ (see the Lemma 4)

$$(T_n - zS_n)^{-1}S_nu \to (T - zS)^{-1}Su.$$

Thus, $P_n \xrightarrow{p} P$. But $\dim PX < \infty$, so we obtain $\|(P_n - P)P\| \to 0$. Finally, we apply the Lemma 3, we have $\dim PX = \dim P_nX$. □

Corollary 3. *Let* λ *be a generalized eigenvalue of finite type isolated in* $\mathrm{sp}(T,S)$. *Under (B1) and (B2), there exists a sequence* $\lambda_n \in \mathrm{sp}(T_n, S_n)$ *such that* $\lambda_n \to \lambda$.

We mention that the two conditions (A1), (A2) and (B1), (B2) do not have any connection between them in natural ways. However, under (A1), (B2) (or (A2), (B1)) the proprieties U and L are hold too.

Theorem 9. *Under (A1) and (B2) (or (A2) and (B1)) the proprieties U and L are hold.*

As before, the difficult is how to reverse $(T_n - zS_n)^{-1}$ for $z \in \mathrm{re}(T,S)$, and how to show $\|(P_n - P)^2\| \to 0$. Thus, the two next lemmas illustrate these points.

Lemma 7. *Under (A1) and (B2) (or (A2) and (B1)), we have*

$$\|[((T_n - T) - z(S_n - S))R(z,T,S)^{-1}]^2\| \to 0,$$

for $z \in \mathrm{re}(T,S) \cap \mathrm{re}(T_n, S_n)$.

Proof. See the Lemma 2) □

Lemma 8. *Under (A1) and (B2) (or (A2) and (B1)), we have*

$$\|[(T_n - zS_n)^{-1}S_n - (T - zS)^{-1}S]^2\| \to 0,$$

for $z \in re(T,S) \cap \mathrm{re}(T_n, S_n)$.

Proof. The result is a consequence through

$$(T_n - zS_n)^{-1}S_n - (T - zS)^{-1}S = [(T_n - zS_n)^{-1} - (T - zS)^{-1}](S_n - S)$$

$$+[(T_n - zS_n)^{-1} - (T - zS)^{-1}]S + (T_n - zS_n)^{-1}(S_n - S).$$

□

5. NUMERICAL APPLICATION

Our numerics test are doing under the harmonic oscillation $Au = -u'' + x^2u$, which is defined in $L^2(\mathbb{R})$. As is known, in theory its spectrum is given by (see [6]) $\mathrm{Sp}(A) = \{2n+1\}_{n\in\mathbb{N}}$. We use the same technics introduced in [6], we transform our spectral problem to $Tu - zSu = 0$, where

$$Tu(x) = u(x) + \int_{-b}^{b} G_{-b,b}(x,y)y^2u(y)\,dy, \quad Su(x) = \int_{-b}^{b} G_{-b,b}(x,y)u(y)\,dy.$$

such that $b > 0$, and

$$G_{a,b}(x,y) = \begin{cases} \frac{1}{b-a}(x-a)(b-y), & \text{if } a \le y \le x \le b, \\ \frac{1}{b-a}(y-a)(b-x), & \text{if } a \le x < y \le b. \end{cases}$$

We define T_n and S_n as the Nyström approach (see [8]), the collectively compact convergence is hold. errGS represents the error of the method described in [6] and errCC represents the error of the Nyström method.

N	errGS	time	errCC	time
15	9.8E-1	1	8.2E-1	0.05
30	1.4E-1	5	1.9E-1	0.08
60	2.3E-2	49	4.3E-2	0.17
120	4.6E-3	457	8.1E-3	0.7

Table 1. Comparison between the method described in[6] and the Nyström method

Numerical tests show that the method described in [6] is slightly more accurate than the Nyström method. But the latter is much faster.

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