Analytical solution of fractional Burgers–Huxley equations via residual power series method

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Abstract.

This paper is aimed at constructing fractional power series (FPS) solutions of fractional Burgers–Huxley equations using residual power series method (RPSM).RPSM is combining Taylor's formula series with residual error function. The solutions of our equation are computed in the form of rapidly convergent series with easily calculable components using Mathematica software package. Numerical simulations of the results are depicted through different graphical representations and tables showing that present scheme are reliable and powerful in finding the numerical solutions of fractional Burgers–Huxley equations. The numerical results reveal that the RPSM is very effective, convenient and quite accurate to time dependence kind of nonlinear equations. It is predicted that the RPSM can be found widely applicable in engineering.

Mathematics Subject Classification: 65M99

Keywords: Generalized Taylor series, fractional power series, fractional Burgers–Huxley equations, residual power series method, Caputo’s fractional derivative.

1. Introduction

Fractional calculus, including integrals and derivatives of arbitrary order, is a generalization of classical integer-order differentiation and integration [9]. In the past few decades, fractional calculus theory has played an important role in the fields of fluid mechanics, physics, entropy and engineering [4,7,8,12]. Fractional partial differential (FPD) equations are important tool to describe physical and natural phenomena such as: damping laws, rheology, diffusion, electrostatics, electrodynamics, fluid flows, and so on [5,6,20,22]. And in most of these applications it is too complicated to obtain exact solutions in terms of composite elementary functions, so approximation and numerical techniques are used extensively, such as the tanh method [26], the differential transform method [25,27], the Homotopy perturbation method [18,24], the Adomian decomposition method [15,23], the variational iteration method [13], The Homotopy analysis method [1,3,14], the optimal homotopy asymptotic method [21] and finite difference method [19]. In this paper, we apply the RPSM to find series solution for fractional Burgers–Huxley equations. The RPSM is an effective and easy tool to construct a power series solution for strongly linear and nonlinear equations without linearization, perturbation, or discretization. Different from the classical power series method, the RPS method does not need to compare the coefficients of the corresponding terms and a recursion relation is not required. The RPSM method does not require any conversion while switching from the low-order to the higher-order and from simple linearity to complex nonlinearity; consequently, the method can be applied directly to the given problem by choosing an appropriate initial guess approximation. Thus, through RPSM, explicit analytic solutions of nonlinear problems are possible to obtain. The RPSM was developed as an efficient method for fuzzy differential equations [17]. It has been advantageously implemented for the fractional foam drainage equation [10], for the time-fractional two-component evolutionary system of order two [11] and for other equations [16]. The remainder of the paper is organized as follows. In the next section, we review some fundamental definitions and theorems of fractional calculus theory and fractional power series. In Section 3, the procedure of the RPSM is described, and then, the residual power series to fractional Burgers–Huxley equations is derived. In Section 4 our algorithm is applied graphical and numerical results are presented. And in Section 5 conclusions are given.

2. Preliminaries

This section seeks to describe the operational properties on factional calculus theory that will help us follow through the principle with the solutions of fractional partial differential equations. There are many definitions of the fractional operators that have been constructed as Riemann Liouville, Grunwald–Letnikov, Weyl, Riesz and Caputo. In our work we will use Caputo's definition since the derivative of the constant is zero, and the initial conditions of the fractional PDE's with Caputo's derivatives take the usual form of the integer order PDE's Which reduces the chance of the occurrence of complications as in the Riemann-Liouville case.

Definition 2.1: For n to be the smallest integer that exceeds α, the Caputo fractional derivatives of order $\alpha > 0$ defined as

$$D_t^{\alpha}u(x,t)=\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t\frac{u^{(n)}(x,\tau)}{(t-\tau)^{\alpha+1-n}}\,d\tau, & n-1<\alpha<n,\\[2ex] \dfrac{\partial^n u(x,t)}{\partial t^n}, & \alpha=n\in N.\end{cases}\tag{2.1}$$

Some properties of the Caputo fractional derivatives are stated here:

$D_t^{\alpha}A=0,$ A is a constant, $D_t^{\alpha}(af(t)\mp bg(t))=aD_t^{\alpha}f(t)\mp bD_t^{\alpha}g(t),$

Next, we will collect some important definitions and theorems of fractional power series. For a more detailed discussion, the reader is referred to [2].

Definition 2.2: The fractional power series (FPS) about $t=t_0$ has the form

$$\sum_{m=0}^{\infty}a_m(t-t_0)^{m\alpha}=a_0+a_1(t-t_0)^{\alpha}+a_2(t-t_0)^{2\alpha}+\ldots,\ 0<\alpha\le 1,\ t\le t_0.$$

Theorem 2.1: Suppose that $C(t)$ has a FPS of the form $C(t)=\sum_{m=0}^{\infty}a_m(t-t_0)^{m\alpha}$ $t_0\le t<t_0.+R.$

If $D_t^{m\alpha}C(t),\ m=0,1,2,...$ are continuous on $t_0\le t<t_0.+R,$ then $a_m=\dfrac{D_t^{m\alpha}C(t_0)}{\Gamma(1+m\alpha)},$ where

$D_t^{m\alpha}=D_t^{\alpha}D_t^{\alpha}...D_t^{\alpha}$ ($m-$times) and R is the radius of convergence.

Definition 2.3: The multiple FPS about $t=t_0$ has the form $\sum_{m=0}^{\infty}C_m(x)(t-t_0)^{m\alpha}$

Theorem 2.2: Suppose that $u(x,t)$ has a multiple FPS representation at $t=t_0$ of the form

$$u(x,t)=\sum_{m=0}^{\infty}C_m(x)(t-t_0)^{m\alpha},\ x\in I,\ t_0\le t<t_0.+R.$$

If $D_t^{m\alpha}u(x,t),\ m=0,1,2,...$ are continuous on $I\times(t_0,t_0.+R),$ then $C_m(x)=\dfrac{D_t^{m\alpha}u(x,t_0)}{\Gamma(1+m\alpha)}.$

Corollary 2.1: Suppose that $u(x,y,t)$ has a multiple FPS representation at $t=t_0$ of the form

$$u(x,y,t)=\sum_{m=0}^{\infty}h_m(x,y)(t-t_0)^{m\alpha},\ (x,y)\in I_1\times I_2,\ t_0\le t<t_0.+R.$$

If $D_t^{m\alpha}u(x,y,t),\ m=0,1,2,...$ are continuous on $I_1 \times I_2 \times (t_0, t_0.+R),$ then

$$h_m(x,y)=\frac{D_t^{m\alpha}u(x,y,t_0)}{\Gamma(1+m\alpha)}.$$

3. RPS algorithm for solving generalized fractional Burgers-Huxley equation

The aim of this section is to construct power series solution to the generalized fractional Burgers-Huxley equation by substituting its power series (PS) expansion among its truncated residual function. From the resulting equation, a recursion formula for the computation of the coefficients is derived, while the coefficients in the fractional PS expansion can be computed recursively by recurrent fractional differentiation of the truncated residual function. The fractional Burgers-Huxley equation has the form

$$D_t^{\alpha}u(x,t)=\kappa\frac{\partial^2}{\partial x^2}u(x,t)-\omega u^{\delta}(x,t)\frac{\partial}{\partial x}u(x,t)+\beta u(x,t)(1-u^{\delta}(x,t))(\eta u^{\delta}(x,t)-\gamma), \tag{3.1}$$
$$0<\alpha\le 1.\ t\ge 0,\ x\ge 0.$$

Subject to the initial condition

$$u(x,0)=C(x), \tag{3.2}$$

where $\kappa,\ \omega,\ \beta,\eta,\gamma$ are real constants and δ is a positive integer. The RPS method assumes the solution for the equation (3.1) as fractional power series about the initial point $t=0,$ as follows

$$u(x,t)=\sum_{m=0}^{\infty}C_n(x)\frac{t^{m\alpha}}{\Gamma(1+m\alpha)},\quad 0<\alpha\le 1,\quad x\in I,\ 0\le t<R. \tag{3.3}$$

Next, we let $u_k(x,t)$ to denote the k-th truncated series of $u(x,t)$ i.e.,

$$u_k(x,t)=\sum_{m=0}^{k}C_m(x)\frac{t^{m\alpha}}{\Gamma(1+m\alpha)},\quad 0<\alpha\le 1,\ x\in I,\ 0\le t<R. \tag{3.4}$$

By using the initial condition (3.2), the 0-th RPS approximate solution of $u(x,t)$ is

$$u_0(x,t)=C_0(x)=u(x,0)=C(x). \tag{3.5}$$

Also, in general the k-th RPS approximate solution of $u(x,t)$ can be written in the form

$$u_k(x,t)=C(x)+\sum_{m=1}^{k}C_m(x)\frac{t^{m\alpha}}{\Gamma(1+m\alpha)},\quad 0<\alpha\le 1,\ x\in I,\ 0\le t<R.,\ k=1,2,3,... \tag{3.6}$$

Now, we define the residual function for Eq. (3.1) as:

$$\begin{aligned}\operatorname{Re} s_u(x,t) &= D_t^{\alpha} u(x,t) - \kappa \frac{\partial^2}{\partial x^2} u(x,t) + \omega u^{\delta}(x,t) \frac{\partial}{\partial x} u(x,t) \\ &\quad - \beta u(x,t)(1-u^{\delta}(x,t))(\eta u^{\delta}(x,t)-\gamma),\end{aligned} \tag{3.7}$$

And the k-th residual function has the form

$$\begin{aligned}\operatorname{Re} s_{u,k}(x,t) &= D_t^{\alpha} u_k(x,t) - \kappa \frac{\partial^2}{\partial x^2} u_k(x,t) + \omega u_k^{\delta}(x,t) \frac{\partial}{\partial x} u_k(x,t) \\ &\quad - \beta u_k(x,t)(1-u_k^{\delta}(x,t))(\eta u_k^{\delta}(x,t)-\gamma).\end{aligned} \tag{3.8}$$

As in [19,20], $\operatorname{Re} s(x,t) = 0$ and $\lim_{k \longrightarrow \infty} \operatorname{Re} s_k(x,t) = \operatorname{Re} s(x,t)$ for all $x \in I$ and $t \geq 0$. Therefore, $D_t^{i\alpha} \operatorname{Re} s(x,t) = 0$ since the fractional derivative of a constant in the Caputo's sense is 0. Also, the fractional derivative $D_t^{i\alpha}$ of $\operatorname{Re} s(x,t)$ and $\operatorname{Re} s_k(x,t)$ is matching at $t = 0$ for each $i = 0,1,2,...,k.$ Now to clarify the RPS technique, we substitute (3.6) in Eq. (3.8) to get

$$\begin{aligned}\operatorname{Re} s_{u,k}(x,t) &= D_t^{\alpha}\Big(C(x) + \sum_{m=1}^{k} C_m(x) \frac{t^{m\alpha}}{\Gamma(1+m\alpha)}\Big) - \kappa \frac{\partial^2}{\partial x^2}\Big(C(x) + \sum_{m=1}^{k} C_m(x) \frac{t^{m\alpha}}{\Gamma(1+m\alpha)}\Big) \\ &\quad + \omega\Big(C(x) + \sum_{m=1}^{k} C_m(x) \frac{t^{m\alpha}}{\Gamma(1+m\alpha)}\Big)^{\delta} \frac{\partial}{\partial x}\Big(C(x) + \sum_{m=1}^{k} C_m(x) \frac{t^{m\alpha}}{\Gamma(1+m\alpha)}\Big) \\ &\quad - \beta\Big(C(x) + \sum_{m=1}^{k} C_m(x) \frac{t^{m\alpha}}{\Gamma(1+m\alpha)}\Big)\Big(1-\Big(C(x) + \sum_{m=1}^{k} C_m(x) \frac{t^{m\alpha}}{\Gamma(1+m\alpha)}\Big)^{\delta}\Big) \\ &\quad \times\Big(\eta\Big(C(x) + \sum_{m=1}^{k} C_m(x) \frac{t^{m\alpha}}{\Gamma(1+m\alpha)}\Big)^{\delta} - \gamma\Big).\end{aligned} \tag{3.9}$$

To get the required coefficients $C_m(x),\ m = 1,2,3,...,k,$ take the fractional derivative formula when $i = k-1$ (i.e.: $D_t^{(k-1)\alpha}$) of both $\operatorname{Re} s_{u,k}(x,t),\ k = 1,2,3,...,$ and then solve the obtained algebraic system

$$D_t^{(k-1)\alpha} \operatorname{Re} s_{u,k}(x,0) = 0, \quad 0 < \alpha \leq 1,\ x \in I, \quad k = 1,2,3,... \tag{3.10}$$

After solving algebraic System (3.10), we have the coefficients $C_1(x), C_2(x),...,C_k(x).$ Therefore; the k-th RPS approximate solution is derived. Next, we will deduce the first approximate solution in detail. In fact, it is very convenient to perform computations by using the Mathematica software package.

4. Application and numerical results

In this section we will generalize a classical test problem from Burgers-Huxley equation into a fractional one by replacing the first time derivative by a fractional derivative of order $0<\alpha\le 1$ then we will apply the RPS method demonstrated above on this problem, later, graphics and numerical results will be discussed.

Application 1. Consider the following time fractional Burgers equation:

$$D_t^{\alpha}u(x,t)=u_{xx}(x,t)+u(x,t)u_x(x,t)+u(x,t)(1-u(x,t))(u(x,t)-1),\qquad 0<\alpha\le 1.\ t\ge 0,\ x\ge 0. \tag{4.1}$$

Subject to the initial conditions:

$$u(x,0)=C(x), \tag{4.2}$$

When $\alpha=1$ and $C(x)=\frac{1}{2}-\frac{1}{2}\tanh(\frac{x}{4})$, the exact solutions of equation (4.1) is

$$u(x,t)=\frac{1}{2}-\frac{1}{2}\tanh(\frac{x}{4}+\frac{3}{8}t) \tag{4.3}$$

According to the process of the RPSM described in Section 3, we get the first few RPS approximate solutions

$$\begin{aligned}
u_0(x,t)&=C_0(x)=C(x),\\
u_1(x,t)&=C(x)+C_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)},\\
u_2(x,t)&=C(x)+C_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+C_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)},\\
u_3(x,t)&=C(x)+C_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+C_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+C_3(x)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)},\\
&\vdots
\end{aligned} \tag{4.4}$$

The k-th residual function for Eq. (4.1) has the form

$$\operatorname{Re}s_{u,k}(x,t)=D_t^{\alpha}u_k(x,t)-\frac{\partial^2}{\partial x^2}u_k(x,t)-u_k(x,t)\frac{\partial}{\partial x}u_k(x,t)-u_k(x,t)(1-u_k(x,t))(u_k(x,t)-1).$$

To get the coefficient $C_1(x)$ substitute $u_1(x,t)=C(x)+C_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}$ in the 1-th residual function

$$\begin{aligned}
\operatorname{Re}s_{u,1}(x,t)=&C_1(x)-C''(x)+C(x)((C(x)-1)^2-C'(x))\\
&-\frac{1}{\Gamma(1+\alpha)}(C(x)C_1'(x)+C_1''(x)-C_1(x)(1-C'(x)+C(x)(3C(x)-4)))\,t^{\alpha}\\
&-\frac{C_1(x)}{\Gamma(1+\alpha)^2}(C_1'(x)-C_1(x)(3C(x)-2))\,t^{2\alpha}+\frac{C_1^3(x)}{\Gamma(1+\alpha)^3}\,t^{3\alpha}.
\end{aligned}$$

From Eq. (3.10), we deduce that $\operatorname{Re} s_{u,1}(x,0)=0$ and thus,

$C_1(x)=C''(x)+C(x)(C'(x)-(C(x)-1)^2)$.

Therefore, the first RPS approximate solution is

$$u_1(x,t)=C(x)+(C''(x)+C(x)(C'(x)-(C(x)-1)^2))\frac{t^{\alpha}}{\Gamma(1+\alpha)}. \tag{4.5}$$

To get the coefficient $C_2(x)$ substitute $u_2(x,t)$ in the second residual function

$$\begin{aligned}
\operatorname{Re} s_{u,2}(x,t)&=C(x)-2C^2(x)+C^3(x)+C_1(x)-C(x)C'(x)-C''(x)\\
&+\frac{1}{\Gamma(1+\alpha)}(C_2(x).-C(x)C_1'(x)-C_1''(x)+C_1(x)(1-C'(x)+C(x)(3C(x)-4)))t^{\alpha}\\
&+[\frac{1}{\Gamma(1+2\alpha)}(C_2(x)(1-C'(x)+C(x)(3C(x)-4))-C(x)C_2'(x)-C_2''(x))\\
&+\frac{C_1(x)}{\Gamma(1+\alpha)^2}(C_1(x)(3C(x)-2)-C_1'(x))]t^{2\alpha}-[\frac{1}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}(C_2(x)C_1'(x)\\
&+C_1(x)(C_2'(x)+C_2(x)(4-6C(x))))-\frac{C_1^3(x)}{\Gamma(1+\alpha)^3}]t^{3\alpha}+[\frac{C_2(x)}{\Gamma(1+2\alpha)^2}(-C_2'(x)\\
&+C_2(x)(3C(x)-2))+\frac{3C_2(x)C_1^2(x)}{\Gamma(1+\alpha)^2\Gamma(1+2\alpha)}]t^{4\alpha}+\frac{3C_1(x)C_2^2(x)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)^2}t^{5\alpha}\\
&+\frac{C_2^3(x)}{\Gamma(1+2\alpha)^3}t^{6\alpha}.
\end{aligned} \tag{4.6}$$

Applying D_t^{α} on both sides of Eq. (4.6) gives

$$\begin{aligned}
D_t^{\alpha}\operatorname{Re} s_{u,2}(x,t)&=C_2(x).-C(x)C_1'(x)-C_1''(x)+C_1(x)(1-C'(x)+C(x)(3C(x)-4))\\
&+[\frac{1}{\Gamma(1+\alpha)}(C_2(x)(1-C'(x)+C(x)(3C(x)-4))-C(x)C_2'(x)-C_2''(x))\\
&+\frac{\Gamma(1+2\alpha)C_1(x)}{\Gamma(1+\alpha)^3}(C_1(x)(3C(x)-2)-C_1'(x))]t^{\alpha}-[\frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)^2}(C_2(x)C_1'(x)\\
&+C_1(x)(C_2'(x)+C_2(x)(4-6C(x))))-\frac{\Gamma(1+3\alpha)C_1^3(x)}{\Gamma(1+\alpha)^3\Gamma(1+2\alpha)}]t^{2\alpha}\\
&+[\frac{\Gamma(1+4\alpha)C_2(x)}{\Gamma(1+3\alpha)\Gamma(1+2\alpha)^2}(-C_2'(x)+C_2(x)(3C(x)-2))+\frac{3\Gamma(1+4\alpha)C_2(x)C_1^2(x)}{\Gamma(1+\alpha)^2\Gamma(1+2\alpha)\Gamma(1+3\alpha)}]t^{3\alpha}\\
&+\frac{3\Gamma(1+5\alpha)C_1(x)C_2^2(x)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)^2\Gamma(1+4\alpha)}t^{4\alpha}+\frac{\Gamma(1+6\alpha)C_2^3(x)}{\Gamma(1+2\alpha)^3\Gamma(1+5\alpha)}t^{5\alpha}.
\end{aligned}$$

By the fact that ($D_t^{\alpha}\,\mathrm{Re}\,s_{u,2}(x,0)=0$) and solving the above resulting system for the unknown coefficient function $C_2(x),$ we get

$$C_2(x)=C(x)C_1^{'}(x)+C_1^{''}(x)+C_1(x)(C^{'}(x)+C(x)(4-3C(x))-1))$$

and the second RPS approximate solution is

$$\begin{aligned}u_2(x,t)&=C(x)+(C^{''}(x)+C(x)(C^{'}(x)-(C(x)-1)^2))\frac{t^{\alpha}}{\Gamma(1+\alpha)}\\&\quad+(C(x)C_1^{'}(x)+C_1^{''}(x)+C_1(x)(C^{'}(x)+C(x)(4-3C(x))-1)))\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}.\end{aligned}\tag{4.7}$$

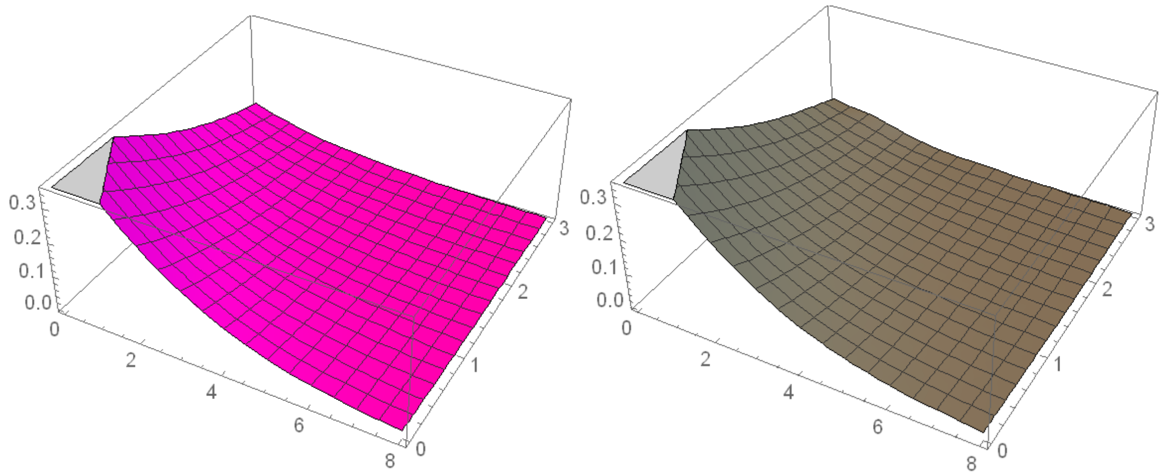
Thus, by using the same manner as above and from Eq. (3.10) solving the equation $D_t^{2\alpha}\,\mathrm{Re}\,s_{u,3}(x,0)=0$ results in the following formula

$$\begin{aligned}C_3(x)&=C(x)C_2^{'}(x)+C_2^{''}(x)+C_2(x)(C^{'}(x)+C(x)(4-3C(x))-1))\\&\quad+\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}C_1(x)(C_1^{'}(x)+C_1(x)(2-3C(x))),\end{aligned}$$

and the third RPS approximate solution is

$$\begin{aligned}u_3(x,t)&=C(x)+(C^{''}(x)+C(x)(C^{'}(x)-(C(x)-1)^2))\frac{t^{\alpha}}{\Gamma(1+\alpha)}\\&\quad+(C(x)C_1^{'}(x)+C_1^{''}(x)+C_1(x)(C^{'}(x)+C(x)(4-3C(x))-1)))\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\\&\quad+(C(x)C_2^{'}(x)+C_2^{''}(x)+C_2(x)(C^{'}(x)+C(x)(4-3C(x))-1))\\&\quad+\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}C_1(x)(C_1^{'}(x)+C_1(x)(2-3C(x))))\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}.\end{aligned}\tag{4.8}$$

If we repeat the same procedures for $k=4,5,6,\ldots,$ we will get the RPS approximate solutions of our time-fractional problem. In this application, we study the solutions of the time fractional Burgers-Huxley equation numerically. In order to validate the efficiency and accuracy of the RPS method, we will compare between the exact solution and the 4th approximate solutions. Figure 1 explores the fourth RPS approximate solutions of $u(x,t)$ when $\alpha=1$ and for different values of x and t.



. exact solution 4th RPS solution

Fig. 1 the 4th RPS approx. sol. for application 1 when $C(x)=\frac{1}{2}-\frac{1}{2}\tanh(\frac{x}{4})$, $\alpha=1$, $0<x<8$ and $0<t<3$.

It is clear from the figure 1 that the 4-th order RPS approximate solutions (when $\alpha=1$) are nearly identical and in excellent agreement with the exact solution. Figure 2 shows the 4-th order RPS approximate solutions for various values of α and we observe that each of the subfigures is nearly coinciding and similar in their behavior. The graphical results in provide a numerical estimate for the convergence of the RPS method in predict the solitary pattern solution. Anyhow, the accuracy is in advanced by using only few terms of approximations. Indeed, we can conclude that higher accuracy can be achieved by computing further terms. To show the accuracy of the method, numerical results at $x=4$ with some selected grid points t for $K=10$ are given in Table 1. From the table, it can be seen that the present method provides us with an accurate approximate solution to Burgers equation (4.1). Indeed, the results reported in this table confirm the effectiveness of the RPS method.

Table 1. Numerical and error results of the RPS approximate solution for application 1 at $\alpha = 1$ and $x = 4$

t	$u_{exact}(x,t)$	$u_{10}(x,t)$	Absolute error	Relative error
$\mathbf{0.1}$	$\mathbf{0.11155054}$	$\mathbf{0.11155054}$	$\mathbf{5.5511151 \times 10^{-17}}$	$\mathbf{4.976322 \times 10^{-16}}$
$\mathbf{0.2}$	$\mathbf{0.104331223}$	$\mathbf{0.104331223}$	$\mathbf{2.220446 \times 10^{-16}}$	$\mathbf{2.128266 \times 10^{-15}}$
$\mathbf{0.3}$	$\mathbf{0.097527837}$	$\mathbf{0.097527837}$	$\mathbf{1.7152945 \times 10^{-14}}$	$\mathbf{1.7587743 \times 10^{-13}}$
$\mathbf{0.4}$	$\mathbf{0.091122961}$	$\mathbf{0.091122961}$	$\mathbf{3.9696024 \times 10^{-13}}$	$\mathbf{4.3563141 \times 10^{-12}}$
$\mathbf{0.5}$	$\mathbf{0.085099045}$	$\mathbf{0.085099045}$	$\mathbf{4.5216053 \times 10^{-12}}$	$\mathbf{5.3133443 \times 10^{-11}}$
$\mathbf{0.6}$	$\mathbf{0.079438549}$	$\mathbf{0.079438549}$	$\mathbf{3.2865377 \times 10^{-11}}$	$\mathbf{4.1372076 \times 10^{-10}}$
$\mathbf{0.7}$	$\mathbf{0.074124065}$	$\mathbf{0.074124065}$	$\mathbf{1.7519996 \times 10^{-10}}$	$\mathbf{2.3636043 \times 10^{-9}}$
$\mathbf{0.8}$	$\mathbf{0.069138420}$	$\mathbf{0.06913841}$	$\mathbf{7.4432132 \times 10^{-10}}$	$\mathbf{1.0765668 \times 10^{-8}}$
$\mathbf{0.9}$	$\mathbf{0.064464765}$	$\mathbf{0.064464763}$	$\mathbf{2.6588732 \times 10^{-9}}$	$\mathbf{4.1245372 \times 10^{-8}}$
$\mathbf{1}$	$\mathbf{0.0600866501}$	$\mathbf{0.060086641}$	$\mathbf{8.2840590 \times 10^{-9}}$	$\mathbf{1.3786854 \times 10^{-7}}$

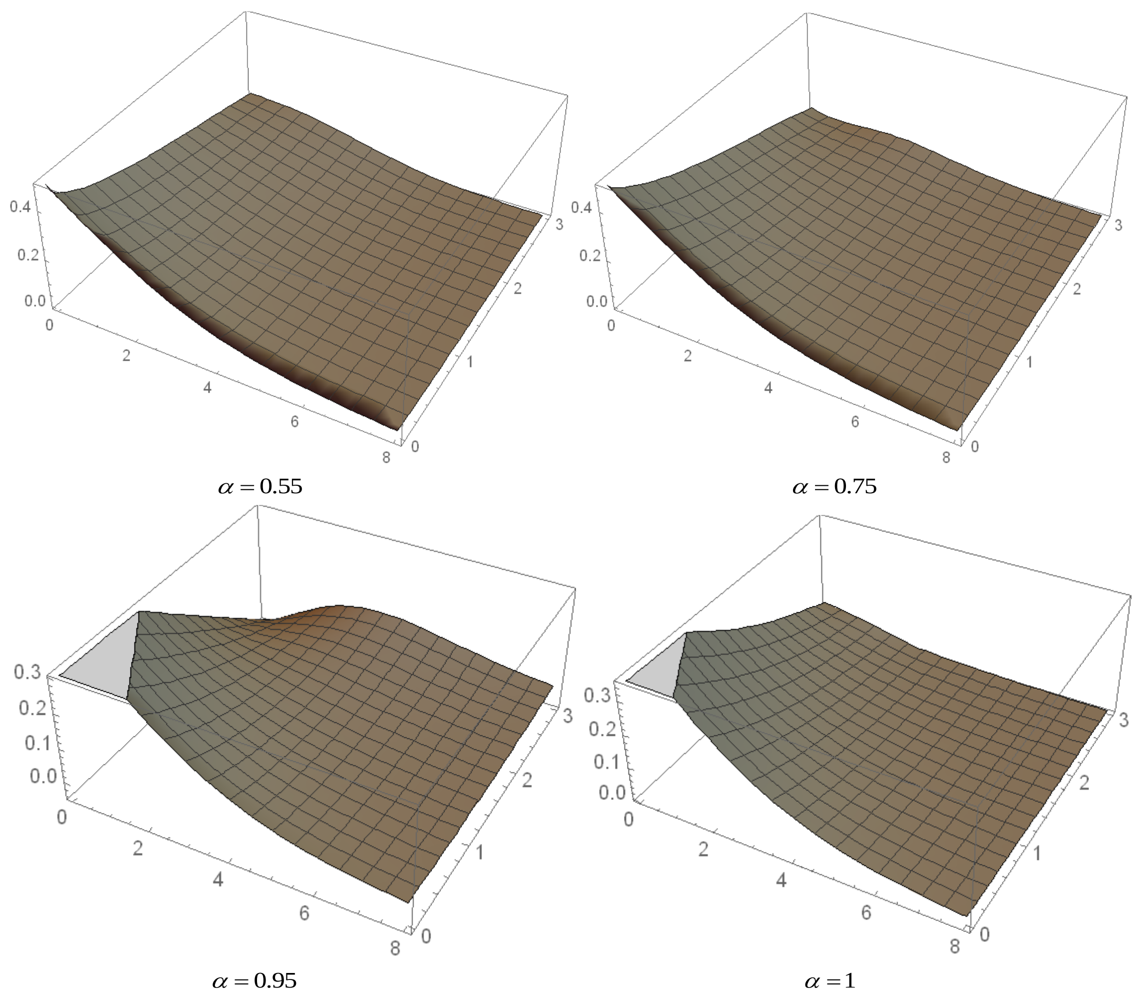


Fig. 2 the 4th RPS approx. sol. for application 1 when $C(x) = \frac{1}{2} - \frac{1}{2}\tanh(\frac{x}{4})$.

Application 2. Consider the following time fractional Burgers equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} - u^2 \frac{\partial u}{\partial x} + u(1-u^2),, \quad 0 < \alpha \le 1.\ t \ge 0,\ x \ge 0. \tag{4.9}$$

Subject to the initial conditions:

$$u(x,0) = C(x), \tag{4.10}$$

When $\alpha = 1$ and $C(x) = \sqrt{\frac{1}{2} - \frac{1}{2}\tanh(\frac{x}{3})}$, the exact solutions of equation (4.9) is

$$u(x,t) = \sqrt{\frac{1}{2} - \frac{1}{2}\tanh(\frac{x}{3} - \frac{10}{9}t)} \tag{4.11}$$

According to the process of the RPSM described in Section 3, we get the same results as in equation (4.4). The k-th residual function for Eq. (4.9) has the form

$$\operatorname{Re} s_{u,k}(x,t) = D_t^\alpha u_k(x,t) - \frac{\partial^2}{\partial x^2} u_k(x,t) + u_k^2(x,t)\frac{\partial}{\partial x} u_k(x,t) - u_k(x,t)(1-u_k^2(x,t)). \tag{4.12}$$

To determine $C_1(x)$, we consider $(k=1)$ in equation (4.12) and substitute $u_1(x,t) = C(x) + C_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)}$ in the 1-th residual function ($\operatorname{Re} s_{u,1}(x,t)$) to get

$$\begin{aligned}\operatorname{Re} s_{u,1}(x,t) =& C_1(x) - C''(x) + C(x)(-1 + C(x)(C(x) + C'(x))) \\ &+ \frac{1}{\Gamma(1+\alpha)}(.C^2(x)C_1'(x) - C_1''(x) + C_1(x)(3C^2(x) + 2C(x)C'(x) - 1))t^\alpha \\ &+ \frac{C_1(x)}{\Gamma(1+\alpha)^2}(2C(x)C_1'(x) + C_1(x)(3C(x) + C'(x)))t^{2\alpha} + \frac{C_1^2(x)}{\Gamma(1+\alpha)^3}(C_1(x) + C_1'(x))\, t^{3\alpha}.\end{aligned}$$

From Eq. (3.10), we deduce that $\operatorname{Re} s_{u,1}(x,0) = 0$ and thus,

$$C_1(x) = C''(x) - C(x)(-1 + C(x)(C(x) + C'(x))).$$

Therefore, the first RPS approximate solution is

$$u_1(x,t) = C(x) + (C''(x) - C(x)(-1 + C(x)(C(x) + C'(x))))\frac{t^\alpha}{\Gamma(1+\alpha)}. \tag{4.13}$$

To obtain the coefficient $C_2(x)$, we substitute the 2nd truncated series $u_2(x,t) = C(x) + C_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + C_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$ into the second residual function $\operatorname{Re} s_{u,2}(x,t)$

and by using the same manner as above, we get $C_2(x)=0.$ So, the second RPS approximate solution is

$$u_2(x,t)=C(x)+(C^{''}(x)-C(x)(-1+C(x)(C(x)+C^{'}(x))))\frac{t^{\alpha}}{\Gamma(1+\alpha)}. \tag{4.14}$$

Thus, solving the equation $D_t^{2\alpha}\operatorname{Re}s_{u,3}(x,0)=0$ results in the following recurrence formula

$$C_3(x)=-\frac{\Gamma(1+2\alpha)C_1(x)}{\Gamma(1+\alpha)^2}(2C(x)C_1^{'}(x)+C_1(x)(3C(x)+C^{'}(x))),$$

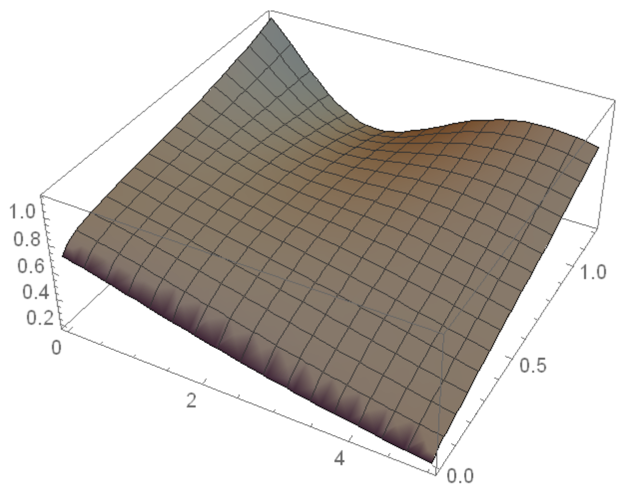
and the third RPS approximate solution is

$$\begin{aligned}u_3(x,t)&=C(x)+(C^{''}(x)-C(x)(-1+C(x)(C(x)+C^{'}(x))))\frac{t^{\alpha}}{\Gamma(1+\alpha)}\\&\quad-\frac{\Gamma(1+2\alpha)C_1(x)}{\Gamma(1+\alpha)^2}(2C(x)C_1^{'}(x)+C_1(x)(3C(x)+C^{'}(x)))\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\end{aligned} \tag{4.15}$$

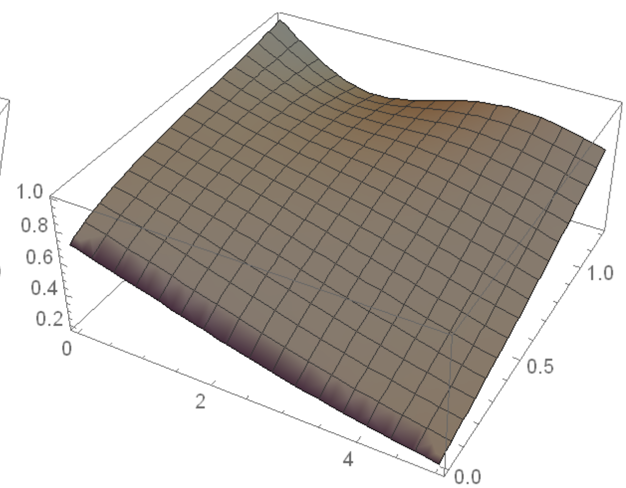
Repeat the same procedures for $k=4,5,6,\ldots$, to get the RPS approximate solutions of our time-fractional problem. The geometric behavior of the solutions of equation (4.9) and (4.10)(when the initial condition $u(x,0)=C(x)=\sqrt{\frac{1}{2}-\frac{1}{2}\tanh(\frac{x}{3})}$) are studied next by drawing the 3-dimensoinal space figures of the 5-th order RPS approximate solution together with the exact solution ($u(x,t)=\sqrt{\frac{1}{2}-\frac{1}{2}\tanh(\frac{x}{3}-\frac{10}{9}t)}$) when $\alpha=1$. Anyhow, the scenario of subfigures is to plot $u_5(x,t)$ when $\alpha=0.55$, $\alpha=0.75$, $\alpha=0.95$, and $\alpha=1$ respectively, on the domain $[0,\ 5]\times[0,\ 1,2]$. It is clear from the figure 3 that each of the subfigures are nearly coinciding and similar in their behavior, while for the special case the subfigures ($\alpha=1$, exact solution) are nearly identical and in excellent agreement to each other in terms of the accuracy. The performance errors analysis is obtained by the RPS at x = 3 with some selected grid points t for K = 20 are summarized in Table 2. Numerically, it is showed that the proposed approach is effective, accurate and convenient method.

Table 2. Numerical and error results of the RPS approximate solution for application 2 at $\alpha = 1$ and $x = 3$

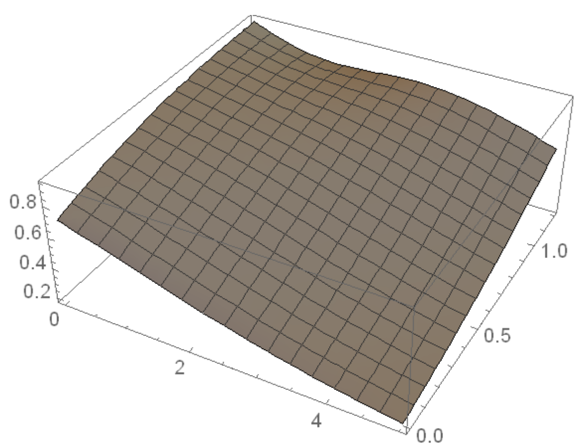
t	$u_{exact}(x,t)$	$u_{20}(x,t)$	Absolute error	Relative error
0.1	0.38023381	0.38023381	0	0
0.2	0.417474934	0.417474934	0	0
0.3	0.45673682	0.456736825	0	0
0.4	0.497658317	0.497658317	$2.49800180 \times 10^{-15}$	$5.01951181 \times 10^{-15}$
0.5	0.539758441	0.539758441	$1.93955962 \times 10^{-13}$	$3.59338451 \times 10^{-13}$
0.6	0.582446247	0.582446247	$4.97579755 \times 10^{-12}$	$8.54293005 \times 10^{-12}$
0.7	0.625045964	0.625045964	$1.89318560 \times 10^{-11}$	$3.02887421 \times 10^{-11}$
0.8	0.666837270	0.666837271	$1.543232985 \times 10^{-9}$	$2.31425724 \times 10^{-9}$
0.9	0.707106781	0.707106821	$4.064332081 \times 10^{-8}$	$5.74783355 \times 10^{-8}$
1	0.745203365	0.745203937	$5.727171323 \times 10^{-7}$	$7.685380384 \times 10^{-7}$



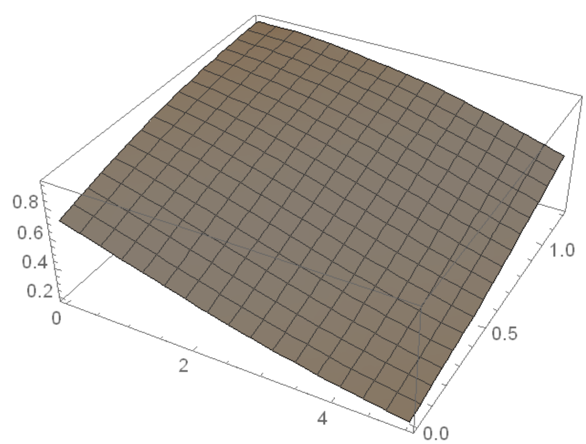
$\alpha = 0.55$



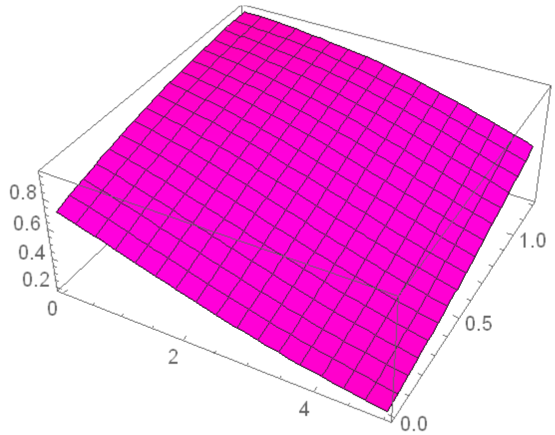
$\alpha = 0.75$



$\alpha = 0.95$



$\alpha = 1$



Exact solution

Fig. 3 The surface graph of the exact solution $u(x,t)$ and the 5th RPS approximate solution $u_5(x,t)$ for application 2 when $C(x)=\sqrt{\frac{1}{2}-\frac{1}{2}\tanh(\frac{x}{3})}$

5. Conclusions

In this work, the RPS method is successfully employed to solve the time-fractional Burgers–Huxley equations with variable pressure in two dimensions. The given examples reveal that the RPS method can be used as an alternative to obtain analytical solutions of time fractional nonlinear differential equations. The proposed technique provides solutions in terms of rapidly convergent series with easily computable components, which are in excellent agreement with the exact solutions (when $\alpha=1$) as revealed by the numerical results. The algorithm for this method is direct and easy because it is based on the recursive differentiation of time- fractional dispersive and the application of a given initial constraints conditions so that we can compute the coefficient of the multiplicity FPS solution with less computations. The simulation results obtained shows that the technique is simple and reveal the validity and reliability of RPS method.

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