# Mixed solutions of monotone iterative technique for hybrid fractional differential equations

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**Abstract:** The fractional hybrid iterative differential equations are interesting equations and widely used in mathematical modelling of biological experiments. Our study is concerned with hybrid fractional differential equations iterative type in order to model the biological experiments, and to see their influence in daily lives. The technique that used in the present study is based on the Dhage fixed point theorem. This tool describes the mixed solutions by monotone iterative technique in the nonlinear analysis. Further in this method we combine two solutions namely: lower and upper solutions. We also provide an approximate result for the hybrid fractional differential equations iterative in the closed assembly formed by the lower and upper solutions.

**Keywords:** Fractional differential equation; fractional differential operator; monotonous sequences; mixed solutions

**AMS Mathematics Subject Classification:** 26A33

## 1 Introduction

The fractional calculus is an active field of mathematical sciences and recently become famous since it is considered to extend the classical calculus and has wide range of applications almost all areas. It follows the traditional definition of derivatives and integrals of calculus in form fractional order, see ([4],[1],[2]). Thus to use fractional order differential operator in mathematical modeling has become more and more popular and important because of their common occurrence in diverse applications in economics, biology, physics and engineering. Recently, many researcher make the use of fractional differential and integral equations and contributed to the current literature by applying to solve the nonlinear differential equations of fractional order [3].

It is considered that the class of fractional order differential equations is a generalization of the class of ordinary differential equations. We discuss that the fractional order differential equations are more appropriate than the ordinary in mathematical modeling of biological studies, as well as economics and social systems, see [4, 5, 6]. Further, the fractional calculus is utilized in biology and medicine to explore the potential of fractional differential equations to describe and to develop a structure and study functional properties of population as well as try to understand the growth of biological organisms. Of course it is an hope to extend this concept to evaluate the changes which are associated with the disease that contribute to the understanding of the pathogenic processes of medicine, see [7] and to employ to the bacteria and other microbes to making something useful, such as genetically engineered human insulin [4].

Among the fractional differential equations the hybrid type are important classes and deal with special case of dynamical systems, ([8], [9]). In the related development, Dhage, Lakshmikantham and Jadhav proved some of the major outcomes of hybrid linear differential equations of the first order and second type disturbances ([10],[11],[12]). However there were many open applied problems such as to develop a mathematical model for bacteria growth by using the iterative difference equations. In this regard, Ibrahim [13] established the existence of solutions for an iterative fractional differential equation (Cauchy type) by using the technique of nonexpansive operator and similar problems were considered in the same direction as in the [14, 15, 16, 17].

In this work, we discuss a mathematical model of biological experiments and its influence in our daily life. The most prominent influence of biological organisms that effect negative or positive in our lives the bacteria. The fractional hybrid iterative differential equations are more appropriate to observe and to develop a model. Our technique is based on the Dhage fixed point theorem. This tool describes mixed solutions by monotone iterative technique in the nonlinear analysis. This method is used to combine two solutions: lower and upper. It is shown an approximate result for the hybrid fractional differential equations iterative in the closed assembly formed by the lower and upper solutions.

## 2 Preliminaries

Next we recall the following definitions that will be used in the development of the study:

**Definition 2.1** The derivative of fractional ($\gamma$) order for the function $\phi(s)$ where $0 < \gamma < 1$ is introduced by

$$D_a^\gamma \phi(s) = \frac{d}{ds}\int_a^s \frac{(s-\beta)^{-\gamma}}{\Gamma(s-\beta)}\phi(\beta)d\beta = \frac{d}{ds}I_a^{1-\gamma}\phi(s) \tag{1}$$

$$(\kappa - 1) < \gamma < \kappa,$$

in which $\kappa$ is a whole number and $\gamma$ is real number.

**Definition 2.2** The integral of fractional ($\gamma$) order for the function $\phi(s)$ where $\gamma > 0$ is introduced by

$$I_a^\gamma \phi(s) = \int_a^s \frac{(s-\beta)^{\gamma-1}}{\Gamma(\gamma)}\phi(\beta)d\beta. \tag{2}$$

In particular, if $a = 0$, then it becomes $I_a^\gamma \phi(s) = \phi(s) * \Upsilon_\gamma(s)$, where $(*)$ denotes the convolution product

$$\Upsilon_\gamma(s) = \frac{s^{\gamma-1}}{\Gamma(\gamma)}$$

and $\Upsilon_\gamma(s) = 0,\ \ s \leq 0$ and $\Upsilon_\gamma \to \delta(s)$ as $\gamma \to 0$ wherever $\delta(s)$ is the delta function

Based on the Riemann-Liouville differential operator, we have the following useful definitions:

**Definition 2.3** Assume $I = [s_0, s_0 + a]$ is a closed period bounded interval in $\Re$ (where $\Re$ the real line), for some $s_0 \in \Re$, $a \in \Re$. Then the problem of initial value of fractional iterative hybrid differential equations ($FIHDE$) can be formulated as

$$D^\alpha[v(s) - \psi(s, v(s), v(v(s))] = \aleph(s, v(s), v(v(s))), s \in I \tag{3}$$

with $v(s_0) = v_0$, where $\psi, \aleph : I \times \Re \to \Re$ are continuous. Then a solution $v \in C(I, \Re)$ of the $FIHDE$ (3) can be given as

1. the map $s \to v - \psi(s, v, v(v)))$ is a continuous function $\forall v \in \Re$, and
2. $v$ is the contented equations in (3) where $C(I, \Re)$ is the space of real-valued continuous functions defined on $I$.

The definitions of the lower and upper solutions of (3) given as follows:

**Definition 2.4** The function $\imath \in C(I, \Re)$ is said to be a lower solution for the equation which introduced by (3) on $I$ if satisfy

1. the map $s \mapsto (\imath(s) - \psi(s, \imath(s)), \imath(\imath(s))))$, is continuous, and
2. $D^{\alpha}[\imath(s) - \psi(s, v(s), v(v(s)))] \geq \aleph(s, \imath(s), \imath(\imath(s))), s \in I, \imath(s_0) \geq v_0$.

**Definition 2.5** The function $\tau \in C(I, \mathbb{R})$ is called an upper solution for the equation (3) on $I$ if satisfies

1. the $s \mapsto (\tau(s) - \psi(s, \tau(s), \tau(\tau(s)))$, is continuous, and
2. $D^{\alpha}[\tau(s) - \psi(s, v(s), v(v(s)))] \leq \aleph(s, \tau(s), \tau(\tau(s))), s \in I, \tau(s_0) \leq v_0$.

We can build the monotonous sequence of consecutive iterations to converging towards the extremes among the lower and upper solutions of the differential equation related hybrid on $I$. We treat the case that if $\psi$ is neither non-decreasing nor non-increasing in the state of the variable $v$. Then if the function $\aleph$ can be separated into two components as

$$\aleph(s, v, v(v))) = \aleph_1(s, v, v(v))) + \aleph_2(s, v, v(v))$$

where $\aleph_1(s, v, v(v)))$ is a non-decreasing component while the another component is not $\aleph_2(s, v, v(v)))$ increasing in the state variables of $v$, then we may be constructed sequences iteration converged to solutions extremal $FIHDE(3)$ on $I$.

**Definition 2.6** Currently thought to be a initial value problem $FIHDE$

$$\begin{cases} D^{\alpha}[v(s) - \psi(s, v(s), v(v(s))] = \aleph_1(s, v, v(v))) + \aleph_2(s, v, v(v))), s \in I, \\ v(s_0) = v_0 \end{cases} \tag{4}$$

where, $\psi \in C(I \times R, R)$ and $\aleph_1, \aleph_2 \in \mathfrak{L}(I \times R, R)$.

Thus the lower and upper solutions of (4) can be as defined as follows:

**Definition 2.7** The functions $\sigma, \rho \in C(I, \Re)$ fulfill the following condition: the maps $s \to \sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s)))$ and $s \to \rho(s) - \psi(s, \rho(s), \rho(\rho(s)))$ are absolute continuous on $I$. Thus the functions $(\sigma, \rho)$ are supposed to be of the kind

(a) which is mixed lower solutions and upper solutions for (4) on $I$, as following

$$\begin{cases} D^{\alpha}[\sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s))] \leq \aleph_1(s, \sigma, \sigma(\sigma(s))) + \aleph_2(s, \rho(s), \rho(\rho(s)))), s \in I, \\ \sigma(s_0) \leq v_0 \end{cases} \tag{5}$$

and

$$\begin{cases} D^{\alpha}[\rho(s) - \psi(s, \rho(s), \rho(\rho(s))] \geq \aleph_1(s, \rho, \rho(\rho(s))) + \aleph_2(s, \sigma(s), \sigma(\sigma(s))), s \in I, \\ \rho(s_0) \geq v_0 \end{cases} \tag{6}$$

if the equality is achieved in (5) and (6), then the pair of functions $(\sigma, \rho)$ is called a mixed solution of kind (a) for the $FIHDE$ (4) on $I$.

(b) which is mixed lower solutions and upper for (4) on $I$, as follows

$$\begin{cases} D^{\alpha}[\sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s))] \leq \aleph_1(s, \rho, \rho(\rho(s))) + \aleph_2(s, \sigma(s), \sigma(\sigma(s)))), s \in I, \\ \sigma(s_0) \leq v_0 \end{cases} \tag{7}$$

and
$$\begin{cases} D^{\alpha}[\rho(s) - \psi(s, \rho(s), \rho(\rho(s))] \geq \aleph_1(s, \sigma, \sigma(\sigma(s))) + \aleph_2(s, \rho(s), \rho(\rho(s))), s \in I, \\ \rho(s_0) \geq v_0 \end{cases} \tag{8}$$
if the equality is held in relations (7) and (8), then the pair of functions $(\sigma, \rho)$ is called a mixed solution of kind (b) for (4) on $I$.

### 2.1 Assumptions

In the following assumptions the function $\psi$ is very important in the study of Eq(4).

(a0) The function $v \mapsto (v - \psi(s_0, v, v(v)))$ is injective in $\Re$.

(b0) $\aleph$ is a bounded real-valued function on $I \times \Re$.

(a1) The function $v \mapsto (v - \psi(s, v, v(v)))$ is increasing in $\Re$ for all $s \in I$.

(a2) There is a constant $\ell > 0$ so that
$$|\psi(s, v, v(v)) - \psi(s, z, z(z))| \leq \frac{\ell|v - z|}{M + |v - z|}, \quad M > 0,$$
$\forall s \in I,\ v, z \in \Re$ and $\ell \leq M$.

(b1) There exists a constant $\kappa > 0$ such that $|\aleph(s, v, v(v)| \leq \kappa\ \forall s \in I$ and $\forall v \in \Re$.

(b2) $\aleph_1(s, v, v(v))$ is a non-decreasing in $v$, and $\aleph_2(s, v, v(v))$ is not increasing in $v$ for each $s \in I$

(b3) $(\sigma_0, \rho_0)$ is Functions which are mixing the lower and upper solutions for (4) kind(a) on $I$ with $\sigma_0 \leq \rho_0$ .

(b4) The pair is $(\sigma_0, \rho_0)$, the upper and lower mixing solutions for (4) kinds (b) on $I$ with $\sigma_0 \leq \rho_0$.

## 3 Main results

In this section, our purpose is to discuss the approximate solution of (4).

**Lemma 3.1**([9]) Suppose the assumptions $(a0) - (b0)$ are achieved. Then the function $v$ is a solution for Eq.(3) if and only if it must be the solution of the fractional iterative of hybrid equation integrated $FIHIE$
$$v(t) = [v_0 - \psi(s_0, v_0, v(v_0))] + \psi(s, v(s), v(v(s))) + \int_0^s \aleph(\beta, v(\beta), v(v(\beta))) \frac{(s - \beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta, \tag{9}$$
$$(s \in I,\ v(0) = v_0).$$

**Theorem 3.1** ([18]) Let $\varrho$ be a closed convex and bounded subset of the Banach space $A$. Moreover, let $Q : A \to A$ and $P : \varrho \to A$ be two operators so that

(i) $Q$ is nonlinear D-contraction,

(ii) $P$ is compact and continuous,

(iii) $v = Qv + Pz$ for all $v \in \varrho \Rightarrow z \in \varrho$.

**Theorem3.2** Let the assumptions $(a1), (a2)$ and $(b1)$ be hold. Then (3) has a solution on $I$.

**Proof.** Let $A = C(I, \Re)$ be a set and $\subseteq A$, such that

$$\varrho = \{v \in A | \|A\| \leq M\} \tag{10}$$

where,

$$M = |v_0 - \psi(s_0, v_0, v(v(0))| + \ell + \Psi_0 + \frac{a^\alpha}{\Gamma(\alpha+1)} \|\xi\|_{\ell^1}.$$

and $\Psi_0 = \sup_{s \in I} \mid \psi(s, 0, 0) \mid$ . Obviously $\varrho$ is a convex, bounded and closed subset of the space $A$. By using the assumptions (a1) and (b1) together with the help of the Lemma 3.1, we conclude that the $FIHDE(3)$ is tantamount to the nonlinear $FIHIE(9)$. We define two operators $Q : A \to A$ and $P : \varrho \to A$ as follows:

$$Qy(s) = \psi(s, v(s), v(v(s))), s \in I, \tag{11}$$

and

$$Pv(s) = [v_0 - \psi(s_0, v_0, v(v_0))] + \int_0^s \aleph(\beta, v(\beta), v(v(\beta))) \frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta, s \in I. \tag{12}$$

Consequently, the $FIHIE(9)$ is equivalent to the operator equation

$$Qv(s) + Pv(s) = v(s), s \in I. \tag{13}$$

We demonstrate that the operators $Q$ and $P$ fulfill all the conditions of Theorem 3.1. Foremost, we examine that $Q$ is a nonlinear $\Upsilon$-contraction on $Q$ with a $\Upsilon$ function $\varphi$. Let $v, z \in A$. In view of assumption $(a2)$, we conclude that

$$|Qv(s) - Qz(s)| = |\psi(s, v(s)) - \psi(s, z(s))| \leq \frac{\ell |v(s) - z(s)|}{M + |v(s) - z(s)|} \leq \frac{\ell |v - z|}{M + |v - z|}$$

for all $s \in I$. Take the supremum over $s$ yields

$$\|Av - Az\| \leq \frac{\ell |v - z|}{M + |v - z|}$$

$\forall v, z \in A$. This proves that $Q$ is a nonlinear $D$-contraction $A$ with the $D$-function $\varphi$ defined by $\varphi(r) = \frac{\ell r}{M+r}$.

Next, we examine that $P$ is a continuous and compact operator on $\varrho$ into $A$. Let $\{v_t\}$ be a sequence in $\varrho$ converging to a point $v \in \varrho$, thus we have

$$\begin{aligned}
\lim_{t\to\infty} Pv_t(s) &= \lim_{t\to\infty} [v_0 - \psi(s_0, v_0, v(v_0)) + \int_0^s \aleph(\beta, v_t(\beta), v_t(v_t(\beta))) \frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta] \\
&= v_0 - \psi(s_0, y_0, v(v_0)) + \lim_{t\to\infty} \int_0^s \aleph(\beta, v_t(\beta), v_t(v_t(\beta))) \frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta \\
&= v_0 - \psi(s_0, v_0, v(v_0)) + \int_0^s \lim_{t\to\infty} [\aleph(\beta, v_t(\beta), v_t(v_t(\beta))) \frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)}] d\beta \\
&= v_0 - \psi(s_0, v_0, v(v_0)) + \int_0^s \aleph(\beta, v(\beta), v(v(\beta))) \frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta = Pv(s)
\end{aligned}$$

for all $s \in I$. Now, we proceed to prove that $\{Pv_t\}$ is equi-continuous with respect to $v$. According to [19], we attain that $P$ is a continuous operator on $\varrho$. To show that $P$ is a compact operator on $\varrho$. It suffices to examine that $\varrho$ is a regularly bounded and equi-continuous set in $A$. Let $v \in \varrho$ be arbitrary, then by the assumption (b1), we have

$$\begin{aligned} |Pv(s)| &\leq |v_0 - \psi(s_0, v_0, v(v_0))| + \int_0^s |\aleph(\beta, v(\beta), v(v(\beta)))\frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)}|d\beta \\ &\leq |v_0 - \psi(s_0, v_0, v(v_0))| + \int_0^s \xi(\beta)\frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)}d\beta \\ &\leq |v_0 - \psi(s_0, v_0, v(v_0))| + \frac{a^\alpha}{\Gamma(\alpha+1)}\|\xi\|_{\ell^1} \end{aligned}$$

for all $s \in I$. By taking the supremum over $t$, we obtain

$$|Pv(s)| \leq |v_0 - \psi(s_0, v_0, v(v_0))| + \frac{a^\alpha}{\Gamma(\alpha+1)}\|\xi\|_{\ell^1}$$

$\forall v \in \varrho$. This proves that $P$ is uniformly bounded on $\varrho$.
Also let $s_1, s_2 \in I$ with $s_1 < s_2$. Then for any $v \in \varrho$, one has

$$\begin{aligned} |Pv(s_1) - Pv(s_2)| &= |\int_{s_0}^{s_1} |\aleph(\beta, v(\beta), v(v(\beta)))\frac{(s_1-\beta)^{\alpha-1}}{\Gamma(\alpha)}d\beta - \int_{s_0}^{s_2} |\aleph(\beta, v(\beta), v(v(\beta)))\frac{(s_2-\beta)^{\alpha-1}}{\Gamma(\alpha)}d\beta| \\ &\leq |\int_{s_0}^{s_1} |\aleph(\beta, v(\beta), v(v(\beta)))\frac{(s_1-\beta)^{\alpha-1}}{\Gamma(\alpha)}d\beta - \int_{s_0}^{s_1} |\aleph(\beta, v(\beta), v(v(\beta)))\frac{(s_2-\beta)^{\alpha-1}}{\Gamma(\alpha)}d\beta| \\ &+ |\int_{s_o}^{s_1} |\aleph(\beta, v(\beta), v(v(\beta)))\frac{(s_2-\beta)^{\alpha-1}}{\Gamma(\alpha)}d\beta - \int_{s_0}^{s_2} |\aleph(\beta, v(\beta), v(v(\beta)))\frac{(s_2-\beta)^{\alpha-1}}{\Gamma(\alpha)}d\beta| \\ &\leq \frac{\|\xi\|_{\ell^1}}{\Gamma(\alpha+1)}[|(s_2-s_2)^\alpha - (s_1-s_0)^\alpha - (s_2-s_1)^\alpha| + (s_2-s_1)^\alpha]. \end{aligned}$$

Hence, for $\delta > 0$, there exists a $\epsilon > 0$ so that

$$|s_1 - s_2| < \epsilon \Rightarrow |Pv(s_1) - Pv(s_2)| < \delta$$

$\forall s_1, s_2 \in I$ and $\forall v \in \varrho$. This examines for $P(\varrho)$ is equi-continuous in $A$. presently $P(\varrho)$ is bounded and hence it is compact by Arzel-Ascoli Theorem. Resulting, $\varrho$ is a continuous and compact operator on $\varrho$.
Then, we prove that assumptions (iii) of Theorem 3.1 is fulfilled. Let $v \in A$ be fixed and $z \in \varrho$ be arbitrary such that $v = Qv + Pz$. In view of the assumption (a2) yields

$$\begin{aligned} |v(s)| &\leq |Qv(s)| + |Pz(s)| \\ &\leq |v_0 - \psi(s_0, v_0)| + |\psi(s, v(s), v(v(s))| + \int_0^s |\aleph(\beta, v(\beta), v(v(\beta)))\frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)}|d\beta \\ &\leq |v_0 - \psi(s_0, v_0)| + |\psi(s, v(s), v(v(s))| + \int_0^s |\aleph(\beta, v(\beta), v(v(\beta)))\frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)}|d\beta \\ &\leq |v_0 - \psi(s_0, v_0, v(v_0))| + \ell + \Psi_0 + \int_0^s |\xi(\beta)\frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)}|d\beta \\ &\leq |v_0 - \psi(s_0, v_0, v(v_0))| + \ell + \Psi_0 + \frac{a^\alpha}{\Gamma(\alpha+1)}\|\xi\|_{\ell^1}. \end{aligned}$$

Taking the supremum over $s$, implies that

$$\|v\| \leq |v_0 - \psi(s_0, v_0, v(v_0))| + \ell + \Psi_0 + \frac{a^\alpha}{\Gamma(\alpha+1)}\|\xi\|_{\ell^1} = M.$$

Thus, $v \in \varrho$.

Therefore, fulfilled all conditions of the Theorem 3.1 and thus the operator equation $v = Qv + Pz$ has a solution in $\varrho$. Resulting, the $FIHDE(3)$ has a solution introduced on $I$. This completes the proof.

**Theorem 3.3** Let $\imath, \tau \in C(I, \Re)$ be lower and upper solutions of $FIHDE(3)$ fulfilling $\imath(s) \leq \tau(s), s \in I$ and let the assumptions $(a1) - (a2)$ and $(b1)$ achieved. Then, there is a solution $v(s)$ of (3), in the closed set $\overline{\mho}$, satisfying

$$\imath(s) \leq v(s) \leq \tau(s), s \in I.$$

**Proof.** Assume that $\Theta : I \times \Re \mapsto \Re$ is a function defined by

$$\Theta(s, v, v(v)) = \max\{\imath(s), \min v(s), \tau(s)\},$$

satisfying

$$\check{\aleph}(s, v, v(v))) := \aleph(s, \Theta(s, v, v(v)))).$$

Moreover, define a continuous extension of $\aleph$ on $I \times \Re$ such that

$$|\check{\aleph}(s, v, v(v)))| = |\aleph(u, \Theta(s, v, v(v))))| \leq \kappa, s \in I \ \forall v \in \Re.$$

In view of Theorem 3.2, the $FIHDE$

$$\begin{cases} D^{\alpha}[v(s) - \psi(s, v(s), v(v(s))] = \check{\aleph}(s, v, v(v))), s \in I \\ \\ v(u_0) = v_0 \in \Re \end{cases} \tag{14}$$

has a solution $v$ defined on $I$.
For any $\delta > 0$, define

$$\imath_\delta(s)\psi(s, \imath_\delta(\imath_\delta(s))) = (\imath(s) - \psi(s, \imath(s), \imath(\imath(s)))\delta(1 + s) \tag{15}$$

and

$$\tau_\delta(s)\psi(s, \tau_\delta(\tau_\delta(s))) = (\tau(s) - \psi(s, \tau(s), \tau(\tau(s)))\delta(1 + s) \tag{16}$$

for $s \in I$. In virtue of the assumptions $(a1)$, we get

$$\imath_\delta(s) < \imath(s), \quad and, \quad \tau(s) < \tau_\delta(s) \tag{17}$$

for $s \in I$. Since

$$\imath(s_0) \leq v_0 \leq \tau(s_0),$$

one has

$$\imath_\delta(s_0) < v_0 < \tau_\delta(s_0). \tag{18}$$

To show that

$$\imath_\delta(s) < v_0 < \tau_\delta(s), \quad s \in I, \tag{19}$$

we define

$$v(s) = v(s) - \psi(s, v(s), v(v(s)), s \in I.$$

Likewise, we consider

$$\hbar_\delta(s) = \imath_\delta(s) - \psi(s, \imath_\delta(\imath_\delta(s))),$$

$$\hbar(s) = \imath(u) - \psi(s, \imath(s), \imath(\imath(s))),$$

and

$$T_\delta(s) = \tau_\delta(s)\psi(s, \tau_\delta(s), \tau(\tau_\delta(s)),$$
$$T(s) = \tau(s)\psi(s, \tau(s), \tau(\tau(s))$$

$\forall s \in I$. If Eq.(19) is wrong, then there exists a $s_\varepsilon \in (s_0, s_0 + a]$ such that

$$v(_\varepsilon) = \tau_\delta(s_\varepsilon)$$

and

$$\imath_\delta(s) < v(s) < \tau_\delta(s), s_0 \leq s < s_\varepsilon$$

If $v(s_\varepsilon) > \tau(s_\varepsilon)$, then $\Theta(s_\varepsilon, v(s_\varepsilon), v(v(s_\varepsilon))) = \tau(s_\varepsilon)$. Furthermore,

$$\imath(s_\varepsilon) \leq \Theta(s_\varepsilon, v(s_\varepsilon), v(v(s_\varepsilon))) \leq \tau(s_\varepsilon).$$

Now,

$$D^\alpha T(s_\varepsilon) \geq \aleph(s_\varepsilon, \tau(s_\varepsilon), \tau(\tau(s_\varepsilon))) = \check{\aleph}(s_\varepsilon, v(s_\varepsilon), v(v(s_\varepsilon)))) = D^\alpha V(s)$$

$\forall s \in I$. Since $T_\delta(us) > D^\alpha T(s)$, $\forall s \in I$ , we have

$$D^\alpha T_\delta(s_\varepsilon) > D^\alpha V(s_\varepsilon). \tag{20}$$

But,

$$V(s_\varepsilon) = T_\delta(s_\varepsilon)$$

also

$$V(s) = T_\delta(s), s_0 \leq s < s_\varepsilon,$$

means that together

$$\frac{V(s_\varepsilon + \rho) - V(s_\varepsilon)}{\rho^\alpha} > \frac{T_\delta(s_\varepsilon + \rho) - T_\delta(s_\varepsilon)}{\rho^\alpha}$$

if $\rho < 0$ a small. Take the limit $\rho \to 0$ in the up variance yields

$$D^\alpha V(s_\varepsilon) \geq D^\alpha T_\delta(s_\varepsilon)$$

that is a contradiction to (20). Hence,

$$v(s) < \tau_\delta(s)$$

$\forall s \in I$. Consequently

$$\imath_\delta(s) < v(s) < \tau_\delta(s), s \in I.$$

Letting $\delta \to 0$ in the up inequality, we get

$$\imath(s) \leq v(s) \leq \tau(s), s \in I.$$

This completes the proof. □

**Theorem 3.4** Let assumptions (a1) - (a2) and (b2) - (b3) achieved. Then there are the monotonous sequences $\{\sigma_t\}, \{\rho_t\}$ such that $\sigma_t \to \sigma$ and $\rho_t \to \rho$ uniformly on $I$ in which $(\sigma, \rho)$ are mixed extremal solutions $FIHDE(4)$ type(a) on $I$.

**Proof.** Note the following a quadratic $FIHDE$

$$\begin{cases} D^\alpha[\sigma_{t+1}(s) - \psi(s, \sigma_{t+1}(s), \sigma(\sigma_{t+1}(s))] \leq \aleph_1(s, \sigma_t(s), \sigma(\sigma_t(s))) + \aleph_2(s, \rho_t(s), \rho(\rho_t(s)))), s \in I, \\ \sigma_{t+1}(s_0) \leq v_0 \end{cases} \tag{21}$$

and

$$\begin{cases} D^{\alpha}[\rho_{t+1}(s) - \psi(s, \rho_{t+1}(s), \rho(\rho_{t+1}(s))] \geq \aleph_1(s, \rho_t(s), \rho(\rho_t(s))) + \aleph_2(s, \sigma_t(s), \sigma(\sigma_t(s))), s \in I, \\ \rho_{t+1}(s_0) \geq v_0 \end{cases} \tag{22}$$

for $t \in N$.

Obviously, the equations (21) and (22) having unique solutions $\sigma_{t+1}$ and $\rho_{t+1}$ on $I$ respectively given Banach contraction mapping principle. We now want to demonstrate that

$$\sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_t \leq \rho_t \leq \ldots \leq \rho_2 \leq \rho_1 \leq \rho_0 \tag{23}$$

on $I$ for $t = 0, 1, 2, \ldots$ Let $t = 0$ and set

$$\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\sigma_0(s) - \psi(s, \sigma_0(s), \sigma(\sigma_0(s)) -)))\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s)))$$

for $s \in I$. Next by monotonicity of $\aleph_1$ and $\aleph_2$, we get

$$\begin{aligned} D^{\alpha}[\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)))] &= D^{\alpha}[(\sigma_0(s)\psi(s, \sigma_0(s), \sigma(\sigma_0(s)))] - D^{\alpha}[\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s))))] \\ \leq \aleph_1(s_0, \sigma_0(s), \sigma(\sigma_0(s))) &+ \aleph_2(s, \rho_0(s), \rho(\rho_0(s))) - \aleph_1(s_0, \rho_0(s), \rho(\rho_0(s))) + \aleph_2(s, \sigma_0(s), \sigma(\sigma_0(s))) \\ &= 0 \end{aligned}$$

$\forall s \in I$ and $\Theta(s_0) = 0$. This implies that

$$\sigma_0(s) - \psi(s, \sigma_0(s), \sigma(\sigma_0(s))) \leq \sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s))),$$

$\forall s \in I$. In view of (a1), one can get $\sigma_0(s) \leq \sigma_1(s)$, $\forall s \in I$. Likewise it can be demonstrated which $\rho_1(s) \leq \rho_0(s)$ on $I$. Setting

$$\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s)))) - (\rho_1(s) - \psi(s, \rho_1(s), \rho(\rho_1(s))))$$

$\forall s \in I$. By monotonicity of $\aleph_1$ and $\aleph_2$, we obtain

$$\begin{aligned} D^{\alpha}[\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)))] &= D^{\alpha}[\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s))))] - D^{\alpha}[(\rho_1(s)\psi(s, \rho_1(s), \rho(\rho_1(s))))] \\ \leq \aleph_1(s_0, \sigma_0(s), \sigma(\sigma_0(s))) &+ \aleph_2(s, \rho_0(s), \rho(\rho_0(s))) - \aleph_1(s_0, \rho_0(s), \rho(\rho_0(s))) + \aleph_2(s, \sigma_0(s), \sigma(\sigma_0(s))) \\ &\leq 0 \end{aligned}$$

$\forall s \in I$ and $\Theta(s_0) = 0$. This leads to

$$\sigma_1(s)\psi(s, \sigma_1(s), \sigma(\sigma_1(s))) \leq \rho_1(s) - \psi(s, \rho_1(s), \rho(\rho_1(s)))$$

$\forall s \in I$. By (a1), we attain to

$$\sigma_1(s) \leq \rho_1(s), \quad \forall s \in I.$$

Next, for $\jmath \in N$, yields

$$\sigma_{\jmath+1} \leq \sigma_{\jmath} \leq \rho_{\jmath} \leq \rho_{\jmath-1}$$

and hence

$$\sigma_{\jmath} \leq \sigma_{\jmath+1} \leq \rho_{\jmath+1} \leq \rho_{\jmath}.$$

Setting

$$\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\sigma_{\jmath}(s) - \psi(s, \sigma_{\jmath}(s), \sigma(\sigma_{\jmath}(s)))) - (\sigma_{\jmath+1}(s) - \psi(s, \sigma_{\jmath+1}(s), \sigma(\sigma_{\jmath+1}(s))))$$

Then the humdrum of $\aleph_1$ and $\aleph_2$, we receive

$$D^\alpha[\Theta(s)-\psi(s,\Theta(s),\Theta(\Theta(s)))] = D^\alpha[(\sigma_\jmath(s)-\psi(s,\sigma_\jmath(s),\sigma(\sigma_\jmath(s))))]-D^\alpha[(\sigma_{\jmath+1}(s)-\psi(s,\sigma_{\jmath+1}(s),\sigma(\sigma_{\jmath+1}(s))))]$$

$$\le \aleph_1(s,\sigma_{\jmath+1},\sigma(\sigma_{\jmath-1}(s)) + \aleph_2(s,\rho_{\jmath-1},\rho(\rho_{\jmath-1})) - \aleph_1(s,\sigma_\jmath,\sigma(\sigma_\jmath)) - \aleph_2(s,\rho_\jmath,\rho(\rho_\jmath))$$
$$\le 0$$

$\forall s \in I$ and $\Theta(s_0) = 0$. This implies that

$$\sigma_\jmath - \psi(s,\sigma_\jmath(s),\sigma(\sigma_\jmath(s))) \le \sigma_{\jmath+1}(s) - \psi(s,\sigma_{\jmath+1}(s),\sigma(\sigma_{\jmath+1}(s)))$$

for every $s \in I$. Since assumption (a1) achieved, we have $\sigma_\jmath(s) \le \sigma_{\jmath+1}(s)$, $\forall s \in I$. Likewise it can be demonstrated which $\rho_{\jmath+1}(s) \le \rho_\jmath(s)$ on $I$. The same way it is assumed that the inequality

$$\sigma_{\jmath-1} \le \sigma_\jmath \le \rho_\jmath \le \rho_{\jmath-1}$$

achieves on $I$. We are going to demonstrate that

$$\sigma_\jmath \le \sigma_{\jmath+1} \le \rho_{\jmath+1} \le \rho_\jmath$$

on $I$. Set

$$\Theta(s) - \psi(s,\Theta(s),\Theta(\Theta(s))) = (\sigma_{\jmath+1}(s) - \psi(s,\sigma_{\jmath+1}(s),\sigma(\sigma_{\jmath+1}(s)))) - (\rho_{\jmath+1}(s) - \psi(s,\rho_{\jmath+1},\rho(\rho_{\jmath+1})))$$

for $s \in I$. So by monotonicity of $\aleph_1$ and $\aleph_2$ we get

$$D^\alpha[\Theta(s)-\psi(s,\Theta(s),\Theta(\Theta(s)))] = D^\alpha[(\sigma_{\jmath+1}(s)-\psi(s,\sigma_{\jmath+1}(s),\sigma(\sigma_{\jmath+1}(s))))]-D^\alpha[(\rho_{\jmath+1}(s)-\psi(s,\rho_{\jmath+1},\rho(\rho_{\jmath+1})))]$$

$$\le \aleph_1(s,\sigma_\jmath(s),\sigma(\sigma_\jmath(s))) + \aleph_2(s,\rho_\jmath(s),\rho(\rho_\jmath(s))) - \aleph_1(s,\rho_{\jmath+1},\rho(\rho_{\jmath+1})) - \aleph_2(s,\sigma_\jmath(s),\sigma(\sigma_\jmath(s)))$$
$$\le 0$$

for the whole $s \in I$ and $\Theta(s_0) = 0$. This means that

$$\sigma_{\jmath+1}(s) - \psi(s,\sigma_{\jmath+1}(s),\sigma(\sigma_{\jmath+1}(s)))) \le \rho_{\jmath+1} - \psi(s,\rho_{\jmath+1},\rho(\rho_{\jmath+1}))$$

for every $s \in I$. Since assumption (a1) is achieved, we have $\sigma_{\jmath+1}(s) \le \rho_{\jmath+1}(s)$, $\forall s \in I$.

Up to now it is shown that the sequences $\{\sigma\}$ and $\{\rho\}$ are bounded uniformly and equi-continuous sequences and therefore converge uniformly on $I$. As they are monotonous sequences, $\{\sigma_t\}$ and $\{\rho_t\}$ the converse uniformly monotonous $\sigma$ and $\rho$ on $I$ respectively.

Thus, the pair $(\sigma,\rho)$ is a mixed solution of these equations (4) on $I$. Lastly, we can establish which $(\sigma,\rho)$ is a mixed solution of minimum and maximum for the equations (4) on $I$. Let $v$ whatever solution of the equations (4) on $I$ as $\sigma_0(s) \le v(s) \le \rho(s)$ on$I$. Assume that for $\jmath \in N$, $\sigma_\jmath(s) \le v(s) \le \rho_\jmath(s)$, $s \in I$. We will demonstrate which $\sigma_{\jmath+1}(s) \le v(s) \le \rho_{\jmath+1}(s)$, $s \in I$. adjustment

$$\Theta(s) - \psi(s,\Theta(s),\Theta(\Theta(s))) = (\sigma_{\jmath+1}(s) - \psi(s,\sigma_{\jmath+1}(s),\sigma(\sigma_{\jmath+1}(s)))) - (v(s) - \psi(s,v(s),v(v(s))))$$

for every $s \in I$. After, for the monotony of $\aleph_1$ and $\aleph_2$ we get

$$D^\alpha[\Theta(s)-\psi(s,\Theta(s),\Theta(\Theta(s)))] = D^\alpha[(\sigma_{\jmath+1}(s)-\psi(s,\sigma_{\jmath+1}(s),\sigma(\sigma_{\jmath+1}(s))))]-D^\alpha[(v(s)-\psi(s,v(s),v(v(s))))]$$

$$\le \aleph_1(s,\sigma_\jmath(s),\sigma(\sigma_\jmath(s))) + \aleph_2(s,\rho_\jmath(s),\rho(\rho_\jmath(s))) - \aleph_1(s,v(s),v(v(s))) - \aleph_2(s,v(s),v(v(s)))$$
$$\le 0$$

for the whole $s \in I$ and $\Theta(s_0) = 0$. This yields

$$\sigma_{\jmath+1}(s) - \psi(s,\sigma_{\jmath+1}(s),\sigma(\sigma_{\jmath+1}(s))) \le v(s) - \psi(s,v(s),v(v(s)))$$

for every $s \in I$. Since assumption (a1) is valid, we get $\sigma_{\jmath+1}(s) \leq v(s)$, $\forall s \in I$. Likewise it can be demonstrated which $v(s) \leq \rho_{\jmath+1}(s)$ on $I$. In principle, the method of induction, $\sigma_t \leq v \leq \rho_t$ for every $s \in I$. By taking $t \to \infty$ limit, we get $\sigma \leq v \leq \rho$ on $I$. So $(\sigma, \rho)$ they are mixed type (a) extreme solutions for the equations (4) on $I$., i.e,

$$\begin{cases} D^{\alpha}[\sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s))] \leq \aleph_1(s, \sigma(s), \sigma(\sigma(s))) + \aleph_1(s, \rho(s), \rho(\rho(s)))), s \in I, \\ \sigma(s_0) = v_0 \end{cases} \tag{24}$$

and

$$\begin{cases} D^{\alpha}[\rho(s) - \psi(s, \rho(s), \rho(\rho(s))] \geq \aleph_1(s, \rho(s), \rho(\rho(s))) + \aleph_1(s, \sigma(s), \sigma(\sigma(s))), s \in I, \\ \rho(s_0) = v_0 \end{cases} \tag{25}$$

the proof is completed. $\square$

**Corollary 3.1** Suppose the hypothesis of Theorem 3.4 are fulfilled. Assume that for $\imath_1 \geq \imath_2$, $\imath_1, \imath_2 \in \overline{\mho}$, then

$$\aleph_1(s, \imath_1(s), \imath(\imath_1(s))) - \aleph_1(s, \imath_2(s), \imath(\imath_2(s))) \leq N_1[\imath_1(s) - \psi(s, \imath_1(s), \imath(\imath_1(s))) - (\imath_2(s) - \psi(s, \imath_2(s), \imath(\imath_2(s))), N_1 > 0,$$

and

$$\aleph_2(s, \imath_1(s), \imath(\imath_1(s))) - \aleph_2(s, \imath_2(s), \imath(\imath_2(s))) \leq N_2[\imath_1(s) - \psi(s, \imath_1(s), \imath(\imath_1(s))) - (\imath_2(s) - \psi(s, \imath_2(s), \imath(\imath_2(s))), N_2 > 0,$$

thus $\sigma(s) = v(s) = \rho(s)$ on $I$.

**Proof.** For $\sigma \leq \rho$ on $I$, it suffices to demonstrate that $\rho \leq \sigma$ on $I$. Introduce a function $\Theta \in C(I, \Re)$

$$\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\rho(s) - \psi(s, \rho(s), \rho(\rho(s)))) - (\sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s)))).$$

Next, $\Theta(s_0) = 0$ and

$$\begin{aligned} D^{\alpha}[\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)))] &= D^{\alpha}[(\rho(s) - \psi(s, \rho(s), \rho(\rho(s))))] - D^{\alpha}[(\sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s))))] \\ &= \aleph_1(s, \rho(s), \rho(\rho(s))) - \aleph_1(s, \sigma(s), \sigma(\sigma(s))) + \aleph_2(s, \sigma(s), \sigma(\sigma(s))) - \aleph_2(s, \rho(s), \rho(\rho(s))) \\ &\leq N_1[(\rho(s) - \psi(s, \rho(s), \rho(\rho(s))) - (\sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s))))] \\ &+N_2[(\sigma(s) - \sigma(s, \sigma(s), \sigma(\sigma(s)))) - (\rho(s) - \psi(s, \rho(s), \rho(\rho(s))))] \\ &= (N_1 + N_2)[\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)))]. \end{aligned}$$

This demonstrates that $\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) \leq 0$ on $I$, demonstrating that $\rho \leq \sigma$ on $I$. Therefore $\sigma = \rho = v$ $I$, the proof is completed. $\square$

**Theorem 3.5** Let us suppose that the assumption $(a1) - (a2)$ and $(b2) - (b4)$ achieved. Therefore, for any solution $v(s)$ of (4) with $\sigma_0 \leq v \leq \rho_0$, and we are an iteration $\sigma_t, \rho_t$ satisfactory for $s \in I$,

$$\begin{cases} \sigma_0 \leq \sigma_2 \leq \ldots \leq \sigma_{2t} \leq v \leq \sigma_{2t+1} \leq \ldots \leq \sigma_3 \leq \sigma_1, \\ \rho_1 \leq \rho_3 \leq \ldots \leq \rho_{2t+1} \leq v \leq \rho_{2t} \leq \ldots \leq \rho_2 \leq \rho_0, \end{cases} \tag{26}$$

as long as $\sigma_0 \leq \sigma_2$ and $\rho_2 \leq \rho_0$ on $I$, in which iterating is given by

$$\begin{cases} D^{\alpha}[\sigma_{2t+1}(s) - \psi(s, \sigma_{2t+1}(s), \sigma(\sigma_{2t+1}(s))] = \aleph_1(s, \rho_t(s), \rho(\rho_t(s))) + \aleph_2(s, \sigma_t(s), \sigma(\sigma_t(s))), \quad s \in I, \\ \sigma_{2t+1}(s_0) = v_o, \end{cases} \tag{27}$$

and

$$\begin{cases} D^{\alpha}[\rho_{2t+1}(s) - \psi(s, \rho_{2t+1}(s), \rho(\rho_{2t+1}(s))] = \aleph_1(s, \sigma_t(s), \sigma(\sigma_t(s))) + \aleph_2(s, \rho_t(s), \rho\rho_t(s))), \quad s \in I, \\ \rho_{2t+1}(s_0) = v_o, \end{cases} \tag{28}$$

of $t \in N$. Furthermore, the monotonous sequences $\{\sigma_{2t}\}, \{\sigma_{2t+1}\}, \{\rho_{2t}\}, \{\rho_{2t+1}\}$ converge uniformly to $\sigma, \rho, \sigma^{\diamond}, \rho^{\diamond}$, respectively, and fulfilling this assumptions:

$$(1)\ D^{\alpha}[\sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s)))] = \aleph_1(s, \rho(s), \rho(\rho(s))) + \aleph_2(s, \sigma(s), \sigma(\sigma(s)))$$

$$(2)\ D^{\alpha}[\rho(s) - \psi(s, \rho(s), \rho(\rho(s))] = \aleph_1(s, \sigma_t(s), \sigma(\sigma(s))) + \aleph_2(s, \rho(s), \rho\rho(s)))$$

$$(3)\ D^{\alpha}[\sigma^{\diamond}(s) - \psi(s, \sigma^{\diamond}(s), \sigma(\sigma^{\diamond}(s)))] = \aleph_1(s, \rho^{\diamond}(s), \rho(\rho^{\diamond}(s))) + \aleph_2(s, \sigma^{\diamond}(s), \sigma(\sigma^{\diamond}(s)))$$

$$(4)\ D^{\alpha}[\rho^{\diamond}(s) - \psi(s, \rho^{\diamond}(s), \rho(\rho^{\diamond}(s))] = \aleph_1(s, \sigma^{\diamond}(s), \sigma(\sigma^{\diamond}(s))) + \aleph_2(s, \rho^{\diamond}(s), \rho\rho^{\diamond}(s)))$$

for $s \in I$ and $\sigma \le v \le \rho, \sigma^{\diamond} \le v \le \rho^{\diamond}, s \in I, \sigma(0) = \sigma(0) = \sigma^{\diamond}(0) = \rho^{\diamond}(0) = v_0$.

**Proof.** By the assumptions of the theorem, we suppose that $\sigma_0 \le \sigma_2$ and $\rho_2 \le \rho_0$, on $I$. We demonstrate that

$$\begin{cases} \sigma_0 \le \sigma_2 \le v \le \sigma_3 \le \sigma_1, \\ \rho_1 \le \rho_3 \le v \le \rho_2 \le \rho_0 \end{cases} \tag{29}$$

on $I$. Set

$$\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (v(s) - \psi(s, v(s), v(v(s)))) - (\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s))))$$

utilization that $\sigma_0 \le v \le \rho_0$ on $I$, as $v$ is any solution of (4) and the monotonous the nature of functions $\aleph_1$ and $\aleph_2$, this yields

$$\begin{aligned} D^{\alpha}[\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)))] &= D^{\alpha}[(v(s) - \psi(s, v(s), v(v(s))))] - D^{\alpha}[(\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s))))] \\ &= \aleph_1(s, v(s), v(v(s))))) + \aleph_2(s, v(s), v(v(s)))) - \aleph_1(s, \rho_0(s), \rho(\rho_0(s))) - \aleph_2(s, \sigma_0(s), \sigma(\sigma_0(s)))) \\ &\le 0 \end{aligned}$$

for every $s \in I$ and $\Theta(s_0) = 0$. Thus, we reached the conclusion

$$v(s) - \psi(s, v(s), v(v(s))) \le \sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s)))$$

or

$$v(s) \le \sigma_1(s)$$

for every $s \in I$. In the same way, we can show that $\sigma_3 \le \sigma_1, \rho_1 \le v$ and $\sigma_2 \le v$, taking into account differences

$$\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\sigma_3(s) - \psi(s, \sigma_3(s), \sigma(\sigma_3(s)))) - (\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s)))),$$

$$\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\rho_1(s) - \psi(s, \rho_1(s), \rho(\rho_1(s)))) - (v(s) - \psi(s, v(s), v(v(s))))$$

and

$$\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\sigma_2(s) - \psi(s, \sigma_2(s), \sigma(\sigma_2(s)))) - (v(s) - \psi(s, v(s), v(v(s))))$$

respectively. At each of these cases, we get $\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)) \le 0$, for all $s \in I$ and representation (29) is established. This completed prove.

**Competing interests** The authors declare that they have no competing interests.

**Authors′ contributions** All the authors jointly worked on deriving the results and approved the final manuscript.

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