On a difference equation generated by two “close” squares

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Abstract

We consider a set of two squares constructed for the primitive periods 1 and i and having four vertices on one straight line. In a neighbourhood of this set, we study a four-element difference equation with constant coefficients. The linear shifts of this equation are the generating transformations of the corresponding doubly periodic group and their inverse transformations. The solution is sought in the class of functions that are analytic outside this set and vanish at infinity. We give some applications to the moment problem for entire functions of exponential type.

Keywords: difference equations, method of regularisation, moment problem.

Introduction. Let D_1 be a square with vertices $t_1 = \lambda(1+i)$, $t_2 = t_1 + 1$, $t_3 = t_2 + i$, $t_4 = t_1 + i$, $\lambda > 0$, and sides ℓ_j, written as they are traversed along the positively oriented boundary $\Gamma_1 = \partial D_1$ ($t \in \ell_1 \Rightarrow \operatorname{Im} t = \lambda$). Consider also the square $D_2 = -D_1$. Denote its sides by ℓ'_j $\left(t \in \ell'_1 \Rightarrow \operatorname{Im} t = -\lambda - 1\right)$. The generating transformations of the corresponding doubly periodic group and their inverse transformations will be written as $\sigma_m(z) = z + i^m$, $m = \overline{1,4}$. They map the interior of any square to its exterior. Assume that $D = D_1 \cup D_2$, $\Gamma = \Gamma_1 \cup \Gamma_2$.

We will study the linear difference equation (l.d.e.)

$$(Vf)(z) \equiv \sum_{m=1}^{4} (-1)^m f\left[\sigma_m(z)\right] = g(z), \; z \in D \tag{1}$$

under the following assumptions:

1. The solution will be sought in the class of functions $f(z)$ that are holomorphic in the complement of the set D and vanish at infinity. The boundary values $f^-(t)$ should satisfy the Hölder condition on each open side of the squares. At the vertices, there may be singularities of, at most, logarithmic type.

2. The independent term $g(z) = g_k(z)$, $z \in D_k$, $k = \overline{1,2}$, is piecewise holomorphic in D (i. e. holomorphic in each D_k) and its boundary values $g_k^+(t) \in H(\Gamma_k)$.

We denote by B the class of such solutions. Under these assumptions, the non-connectivity of the set $C \setminus Q$, with $Q = \bigcup_{k=1}^{2} Q_k$, where $Q_k = \bigcup_{m=1}^{4} \sigma_m(D_k)$, is the most important factor, which completely determines the structure of the research. This set splits into three connected components, and only one of them contains the point at infinity. At the same time, Equation (1) is defined in the other two components, which makes it impossible to apply to it the powerful classical methods of research of convolution operators [1]. For this reason, here the solution and the independent term belong in the general case to different classes of holomorphic functions, which makes it different from the standard approach to the theory of analytic solutions of l.d.e. In particular, the independent term must not be analytically continuable across some interval Γ_k. Even in the case of an homogeneous equation

$$(Vf)(z) = 0, \quad z \in D, \tag{2}$$

we cannot in general conclude that it holds in a certain neighbourhood of infinity. The need for additional restrictions on the nature of the boundary values of the solution to the studied equations is determined by the essence of the matter (a detailed commentary can be found in [2, p. 551] and in the introduction to [3]).

That approach to l.d.e. was first suggested in [4] in the case of a single square. Subsequently, the task was generalised in various directions, including the case of variable coefficients that are holomorphic in D [5, 6]. Some applications arising from this approach were considered in [7, 8]. The case when those equations are defined on several squares was studied for the first time in [2], where it was assumed that the squares are so "far distant" from each other that the set $Q_1 \bigcap Q_2 = \varnothing$ for $j_1 \neq j_2$. We therefore suppose here that $\lambda \in (0, 2^{-1})$.

Finally, we will dwell on the results of the article [9], where Equation (1) was studied in the case $D_2 = D_1 + \alpha$, $\alpha \in (2, 3)$. Note that the problem becomes meaningless when $\alpha \in (0, 2)$ since in this case some transformations of the group map the interior of one square to the interior of the other square, where the solution $f(z)$ is not defined.

We consider a problem that differs essentially from the one studied in [9]. Firstly, the positive number $\operatorname{dist}(\Gamma_1, \Gamma_2)$ may be arbitrarily small. Secondly, the set $Q_0 = Q_1 \bigcap Q_2$ in [9] is a rectangle; here we assume that this set is disconnected. It splits into two squares, thus giving a different kind of power problem of moments for entire functions of exponential type (e.f.e.t.).

This article consists of three sections. In § 1, we perform an equivalent regularisation of problem (1). In § 2, we prove that the problem is unconditionally solvable if $\lambda \in (3^{-1}, 2^{-1})$. In § 3, we consider applications to the moment problem for e.f.e.t.

§1. We seek a solution to problem (1) in the form of a Cauchy-type integral:

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} (\tau - z)^{-1} \varphi(\tau)\, d\tau, \quad z \notin \overline{D} \tag{3}$$

Consider a piecewise constant $\theta_t = (-1)^j$, $t \in \ell_j \cup \ell'_j$. Without loss of generality, we can assume that

$$\varphi(t) = -\theta_t \, \varphi[\alpha(t)]. \tag{4}$$

Here $\alpha(t) = t + i^j$, $t \in \ell_j \cup \ell'_j$, $j = \overline{1,4}$, is a Carleman shift induced on each contour Γ_k by the transformations $\sigma_m(z)$. Indeed, integral (3) does not change if one replaces its density on any contour Γ_k with $\varphi(\tau) + a_k^+(\tau)$, where $a_k(z)$ is a holomorphic function in D_k.

We consider (4) as a special case of the Carleman problem for a rectangle (see [10]) with unknown function $a_k(z)$. This problem is unconditionally solvable. Taking into account expression (3), we obtain

$$(1) \Leftrightarrow (A\varphi)(z) \equiv \frac{1}{2\pi i} \int\limits_{\Gamma} \varphi(\tau) E(\tau - z) \, d\tau = g(z), \;\; z \in D, \tag{5}$$

where the kernel function

$$E(u) = (u+1)^{-1} + (u-1)^{-1} - (u+i)^{-1} - (u-i)^{-1}. \tag{6}$$

Let us regularise Equation (1). With (4) in mind, we obtain

$$\left(A^+\varphi\right)(t) = 2^{-1}\varphi(t) + (A\varphi)(t), \tag{7}$$

where the integral operator is produced by the formal substitution of $z \in D$ with $t \in \Gamma$ and is understood in the sense of the Cauchy principal value. Replace in (7) the variable t with $\alpha(t)$, multiply it by θ_t, and subtract the obtained equality from the original one. Then, taking into account (5), we have

$$(T\varphi)(t) \equiv \varphi(t) + \frac{1}{2\pi i} \int\limits_{\Gamma} \varphi(\tau) K(t,\tau) d\tau = g^+(t) - \theta_t g^+[(\alpha(t))], \tag{8}$$

where

$$K(t,\tau) = E(\tau - t) - \theta_t \theta_\tau E\left(\alpha(\tau) - \alpha(t)\right). \tag{9}$$

Lemma 1. Kernel (9) is bounded.

Proof. The proof reduces to a direct examination of different dispositions of the points τ and t on the sides of the squares.

Theorem 1. L.d.e. (1) has at most a finite number of solvability conditions. All of them are solvability conditions of the Fredholm integral equation (8).

Proof. If Equation (8) is solvable, then it has a solution that satisfies (4) (see [11]). Thus

$$(8) \;\Rightarrow \left(A^+\varphi\right)(t) - \theta_t \left(A^+\varphi\right)(\alpha(t)) = g^+(t) - \theta_t g^+\left[\alpha(t)\right].$$

And since the homogeneous Carleman problem $a^+(t) = \theta_t a^+\left[\alpha(t)\right]$ has only the trivial solution, we can conclude that $(8) \Rrightarrow (1)$. This completes the proof.

§2. We assume now that $\lambda \in (3^{-1}, 2^{-1})$. Consider the homogeneous equation

$$T\varphi = 0\ . \tag{10}$$

It is obvious that (10) $\Leftrightarrow (T\varphi)(-t) = 0$. To make this conclusion, it is enough to replace in (10) the variables τ and t respectively with $-\tau$ and $-t$, taking into account that $\alpha(-t) = -\alpha(t)$. Kernel (9) is antisymmetric. The adjoint equation has therefore the form

$$(T'\psi)(t) \equiv \psi(t) - \frac{1}{2\pi i}\int\limits_{\Gamma} \varphi(\tau) K(t,\tau)\, d\tau = 0. \tag{11}$$

The fundamental system of solutions (f.s.s.) of Equation (10) or (11) can be taken in such a way that each function belonging to it satisfies either condition (4) or the opposite condition

$$\psi(t) = \theta_t \psi\left[\alpha(t)\right]. \tag{12}$$

The solvability conditions of the inhomogeneous Fredholm equation (8) can be obtained by taking into account only the solutions of the adjoint equation that have the property (12), since the solutions with property (4) are orthogonal to the right member of Equation (8).

Lemma 1. The f.s.s. of the adjoint equation (11) does not contain any function with property (12).

Proof. Assume that $M = \max\{|\psi(t)|, t \in \Gamma\}$. By virtue of condition (12), the symmetry, and the parity (evenness or oddness) of the function $\psi(t)$, we can assume without loss of generality that $t \in \ell_1$. We also assume that $\lambda = 3^{-1}$.

1) $\tau \in \ell_1 \cup \ell'_1 \ \Rightarrow\ K(t,\tau) = 0.$

2) $\tau \in \ell_3$. Then $\tau - t = i + \gamma$, $\alpha(\tau) - \alpha(t) = \gamma - i$, with $\gamma \in [-1, 1]$ $\Rightarrow K(t,\tau) = 4i\varphi(x)$, where $\varphi(x) = (x+4)^{-1} - (x+2)(x^2+4)^{-1}$ and $x = \gamma^2$. Therefore $|K(t,\tau)| \leq 1{,}6$.

3) $\tau \in \ell'_3 \ \Rightarrow |K(t,\tau)| = \big|(u+i)^{-1} - (u-1)^{-1} - (u+1)^{-1} - (u-3i)^{-1} + {}+ (u-1-2i)^{-1} + (u+1-2i)^{-1}\big| \leq 2,13$. Here $u = \tau - t$.

4) $\tau \in \ell_2 \cup \ell_4$. Since $\sum\limits_{k=1}^{2} \int\limits_{\ell_{2k}} \varphi(\tau) K(t,\tau)\, d\tau = \int\limits_{\ell_4} \varphi(\tau) K_1(t,\tau)\, d\tau$, we will find an upper bound for the absolute value of the kernel $K_1(t,\tau) = K(t,\tau) - K(t,\alpha(\tau))$. Thus, $|K_1| = \big|(u+i)^{-1} - (u-i)^{-1} + + (u+1-2i)^{-1} - (u+2-i)^{-1} - (u+i+1)^{-1} + (u+2)^{-1} - (u-2i)^{-1} + + (u-1-i)^{-1}\big| \leq \dfrac{\sqrt{3}}{2}$.

5) $\tau \in \ell'_2 \cup \ell'_4$. If $\tau \in \ell'_4$, then we have $|K_1(t,\tau)| \leq 1{,}8$.

Since $1{,}6 + 2{,}13 + \sqrt{3}/2 + 1{,}8 < 2\pi$, we obtain $\psi \equiv 0$. This finishes the proof of the lemma.

Note 1. If $\lambda > 3^{-1}$, then estimates 3) and 5) only become better, which means that the lemma remains valid in this case. Inequalities 2) and 3) do not depend on λ.

Note 2. It is possible to show that the f.s.s. of Equations (11) and (12) are empty if $\lambda \leq 3^{-1}$. We do not give the proof here since the estimates involved are quite cumbersome, and the result is not used in the rest of this study.

Theorem 2. Problem (1) is solvable and has a unique solution. The homogeneous problem (2) has only the trivial solution.

§3. One of the squares the set Q_0 splits into is a square Δ_1 with vertices $\lambda(i-1)$, $i(1-\lambda)-\lambda$, $(1-\lambda)i-1+\lambda$, $\lambda-1+\lambda i$. The second one is the square $\Delta_2 = -\Delta_1 \;\Leftrightarrow\; \Delta_2 = \Delta_1 + 1 - i$. Assume that $z \in \Delta_1$ and $b(z) = f(z) - f(z+1-i)$. Then, taking into account (1), this expression can be transformed in two ways. On the one hand, $b(z) = g(z+1) - f(z+2) + f(z+1+i)$. On the other hand, $b(z) = f(z-1-i) - f(z-2i) - g_2(z-i)$. Equating both expressions, we obtain $f(z+2) - f(z+1+i) - f(z-1-i) + f(z-2i) = g_0(z)$, $z \in \Delta_1$, where $g_0(z) = g_1(z+1) + g_2(z-i)$. Take the point $z_0 = 2^{-1}(i-1)$, the centre of the square Δ_1. Assume that

$$g_0(z) = \sum_{k=0}^{\infty} \frac{\beta_k (z-z_0)^k}{k!},$$

and the convergence radius of the series satisfies the inequality $R \leq (1-2\lambda)\sqrt{2}/2$. Using the Borel transform (see [12, ch. 1]), we get

$$\int_0^{\infty} F(x)\exp\left(-x(z+2)\right)\,dx - \int_{-\infty}^{0} F(x)\exp\left(-x(z-1-i)\right)\,dx +$$

$$+ \int_{\theta_2} F(\tau)\exp\left(-\tau(z-z_0)\right)\,d\tau = g_0(z),\;\; z \in \Delta_1.$$

Here the ray $\theta_k = \dfrac{\pi}{2}(-1)^k$, $k = 1,2$. Note that the conjugate indicator diagram of the e.f.e.t. $F(z)$, which is Borel-associated with the lower function $f(z) \in B$, is, generally speaking, a hexagon D with vertices $\pm t_2$, $\pm t_3$, $\pm t_4$. Equating the Taylor coefficients in the last equality at the point z_0, we get

$$\int_0^{\infty} F(x)x^k\exp\left(-x(z_0+2)\right)\,dx - \int_{\theta_1} F(\tau)\tau^k\exp\left(\tau(z_0+1+i)\right)\,d\tau -$$

$$- \int_{-\infty}^{0} F(x)x^k\exp\left(-x(z_0-1-i)\right)\,dx + \int_{\theta_2} F(\tau)\tau^k\exp\left(-\tau(z-2i)\right)\,d\tau = \beta_k(-1)^k. \tag{13}$$

Theorem 3. Moment problem (13) is solvable in the class of e.f.e.t. $F(z)$ that are Borel-associated with the lower function $f(z) \in B$ and its solution is unique.

Note 3. It is possible (albeit of little interest) the case when the conjugate indicator diagram is a "smaller" convex set $D'' \subset D'$. For this, it is necessary (but not sufficient) that at least one of the functions $g_k(z)$ could be analytically continued from D''_k to some neighbourhood of the point at infinity, and $g_k(\infty) = 0$. See Note 2 in [9] for more details.

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