On n-Weak Cotorsion Modules

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Received April 29, 2016

Abstract—Let R be a ring and n a fixed non-negative integer. In this paper, n-weak cotorsion modules are introduced and studied. A right R-module N is called n-weak cotorsion module if $Ext^1_R(F,N)=0$ for any right R-module F with weak flat dimension at most n. Also some characterizations of rings with finite super finitely presented dimensions are given.

2010 Mathematical Subject Classification: 16D10, 16E30, 16D50

Keywords and phrases: *weak injective module; weak flat module; n-weak cotorsion module; super finitely presented dimension*

1. INTRODUCTION

Throughout this paper, R denotes an associative ring with identity and all modules are unitary. Denote by R-Mod the category of left R-modules and by Mod-R the category of right R-modules. As usual, $pd_R(M)$, $id_R(M)$, and $fd_R(M)$ will denote the projective, injective and flat dimensions of an R-module M, respectively. We use $\mathcal{F}_n$ to stand for the class of all right R-modules with flat dimension at most n and $w.gl.dim(R)$ to stand for the weak global dimension of a ring R. For unexplained concepts and notations, we refer the reader to [1, 5, 13, 16].

We first recall some known notions and facts needed in the sequel. Given a class $\mathcal{C}$ of right R-modules, we write

$$\mathcal{C}^{\perp}=\left\{M\in Mod\text{-}R \mid Ext^1_R(C,M)=0,\ \forall\, C\in\mathcal{C}\right\};$$
$$^{\perp}\mathcal{C}=\left\{M\in Mod\text{-}R \mid Ext^1_R(M,C)=0,\ \forall\, C\in\mathcal{C}\right\}.$$

Let $\mathcal{C}$ be a class of right R-modules and M a right R-module. Following [5], we say that a map $f\in Hom_R(C,M)$ with $C\in\mathcal{C}$ is a *$\mathcal{C}$-precover* of M, if the group homomorphism $Hom_R(C',f)$: $Hom_R(C',C)\to Hom_R(C',M)$ is surjective for each $C'\in\mathcal{C}$. A $\mathcal{C}$-precover $f\in Hom_R(C,M)$ of M is called a *$\mathcal{C}$-cover* of M if f is right minimal, that is, if $fg=f$ implies that g is an automorphism for each $g\in End_R(C)$. Dually, we have the definition of *$\mathcal{C}$-preenvelope* (*$\mathcal{C}$-envelope*).

A $\mathcal{C}$-envelope $\phi: M\to C$ is said to have the *unique mapping property* [3] if for every any homomorphism $f: M\to C'$ with $C'\in\mathcal{C}$, there is a unique homomorphism $g: C\to C'$ such that $g\phi=f$. Dually, we have the definition of $\mathcal{C}$-cover with unique mapping property.

Following [5], a monomorphism $\alpha: M\to C$ with $C\in\mathcal{C}$ is said to be a *special $\mathcal{C}$-preenvelope* of M if $coker(\alpha)\in{}^{\perp}\mathcal{C}$. Dually, we have the definition of a special $\mathcal{C}$-precover. Special $\mathcal{C}$-preenvelopes (resp., special $\mathcal{C}$-precovers) are obviously $\mathcal{C}$-preenvelopes (resp., $\mathcal{C}$-precovers). In general, $\mathcal{C}$-covers ($\mathcal{C}$-envelopes) may not exist, if exists, they are unique up to isomorphism.

A pair $(\mathcal{F},\mathcal{C})$ of classes of right R-modules is called a *cotorsion theory* [5] if $\mathcal{F}^{\perp}=\mathcal{C}$ and $^{\perp}\mathcal{C}=\mathcal{F}$. A cotorsion theory $(\mathcal{F},\mathcal{C})$ is called *complete* [15] if every right R-module has a special $\mathcal{C}$-preenvelope

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and a special $\mathcal{F}$-precover. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called *perfect* [6] if every right R-module has a $\mathcal{C}$-envelope and an $\mathcal{F}$-cover. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called *hereditary* [6] if whenever $0 \to L' \to L \to L'' \to 0$ is exact with $L, L'' \in \mathcal{F}$, then L' is also in $\mathcal{F}$. By [6, Proposition 1.2] $(\mathcal{F}, \mathcal{C})$ is hereditary if and only if whenever $0 \to C' \to C \to C'' \to 0$ is exact with $C, C' \in \mathcal{C}$, then C'' is also in $\mathcal{C}$.

A right R-module M is called *FP-injective* [14] if $Ext^1_R(F, M) = 0$ for all finitely presented right R-modules F. Accordingly, the *FP-injective dimension* of M, denoted by $FP\text{-}id(M)$, is defined to be the smallest $n \geq 0$ such that $Ext^{n+1}_R(F, M) = 0$ for all finitely presented right R-modules F(if no such n exists, set $FP\text{-}id(M) = \infty$). We use $\mathcal{FP}_n$ to stand for the class of all right R-modules with FP-injective dimension at most n.

A left R-module M is called *super finitely presented* [8] if there exists an exact sequence $\cdots \to F_1 \to F_0 \to M \to 0$, where each F_i is finitely generated and projective. Following this, Gao and Wang in [9] gave the definitions of weak injective and weak flat modules in terms of super finitely presented modules. A left R-module M is called *weak injective* if $Ext^1_R(F, M) = 0$ for any super finitely presented left R-module F. A right R-module N is called *weak flat* if $Tor^R_1(N, F) = 0$ for any super finitely presented left R-module F.

Accordingly, the *weak injective dimension* of a left R-module M, denoted by $wid_R(M)$, is defined to be the smallest $n \geq 0$ such that $Ext^{n+1}_R(F, M) = 0$ for all super finitely presented left R-modules F. If no such n exists, set $wid_R(M) = \infty$. The *weak flat dimension* of a right R-module N, denoted by $wfd_R(N)$, is defined to be the smallest $n \geq 0$ such that $Tor^R_{n+1}(N, F) = 0$ for all super finitely presented left R-modules F. If no such n exists, set $wfd_R(N) = \infty$. The *left super finitely presented dimension*, denoted by $l.sp.gldim(R)$, of a ring R is defined as

$$l.sp.gldim(R) = sup\,\{pd_R(M) | M \text{ is a super finitely presented left } R\text{-module}\}.$$

Let n be a fixed non-negative integer. In what follows, the symbol $\mathcal{WI}_n(\mathcal{WF}_n)$ denotes the class of all left (right) R-modules with weak injective (weak flat) dimension less than or equal to n.

In [12], Mao and Ding proved that $(\mathcal{F}_n, \mathcal{F}_n^{\perp})$ is a perfect hereditary cotorsion theory and introduced the notion of n-cotorsion modules. Recently, Zhao proved $(\mathcal{WF}_n, \mathcal{WF}_n^{\perp})$ is a perfect hereditary cotorsion theory in [17, Proposition 4.18]. Inspired by [12, 17], in this paper, we will introduce and study the notion of n-weak cotorsion modules.

In Section 2, n-weak cotorsion modules are defined and studied. A right R-module N is called an n-weak cotorsion module if $N \in \mathcal{WF}_n^{\perp}$. For a ring with $wid(R) \leq n$, we prove that a right R-module M is n-weak cotorsion if and only if M is a kernel of a $\mathcal{WF}_n$-precover $f : A \to B$ with A injective if and only if M is a direct sum of an injective right R-module and a reduced n-weak cotorsion right R-module.

In Section 3, we characterize rings with finite super finitely presented dimension in terms of, among others, n-weak cotorsion modules. It is proven that $l.sp.gldim(R) \leq n$ if and only if every n-weak cotorsion right R-module is injective if and only if every n-weak cotorsion right R-module belongs to $\mathcal{WF}_n$. It is also shown that if every n-weak cotorsion right R-module has a $\mathcal{WF}_n$-envelope with the unique mapping property, then $l.sp.gldim(R) \leq n + 2$.

2. n-WEAK COTORSION MODULES

For any ring R and a fixed non-negative integer n, it is known that $(\mathcal{WF}_n, \mathcal{WF}_n^{\perp})$ is a perfect hereditary cotorsion theory by [17, Proposition 4.18]. In this section, n-weak cotorsion modules are defined to be the modules in the class $\mathcal{WF}_n^{\perp}$. We start with the following

Definition 1. *Let R be a ring and n a fixed non-negative integer. A right R-module N is called n-weak cotorsion module if $Ext^1_R(F, N) = 0$ for any right R-module $F \in \mathcal{WF}_n$.*

In what follows, $\mathcal{WC}_n$ stands for the class of all n-weak cotorsion right R-modules.

By Definition 1, we have the following proposition.

Proposition 1. *The following assertions hold:*

1. *Let C_i be a family of right R-modules. Then $\prod_i C_i$ is n-weak cotorsion if and only if each C_i is n-weak cotorsion.*
2. *$\mathcal{WC}_n$ is closed under extensions and direct summands.*
3. *If $m \geq n$, then every m-weak cotorsion modules is n-weak cotorsion.*

Recall that, a right R-module C is called *cotorsion* [4] provided that $Ext^1_R(F, C) = 0$ for any flat right R module F. For a fixed non-negative integer n, a right R-module M is called *n-cotorsion* [12] if $Ext^1_R(N, M) = 0$ for any $N \in \mathcal{F}_n$. 0-cotorsion modules are preciously cotorsion modules.

Remark 1.

1. *For any non-negative integer n, we have the following implications: injective modules $\Rightarrow$ n-weak cotorsion modules $\Rightarrow$ n-cotorsion modules $\Rightarrow$ cotorsion modules;*
2. *If R is a coherent ring, then n-weak cotorsion modules coincide with n-cotorsion modules since $l.sp.gldim(R) = w.gl.dim(R)$.*

The following theorem is due to Zhao [17, Proposition 4.17 and Proposition 4.18].

Theorem 1. *Let n be a fixed non-negative integer. Then following hold:*

1. *For a ring R with $wid_R({}_RR) \leq n$, $(\mathcal{WI}_n, \mathcal{WI}_n^{\perp})$ is a perfect cotorsion theory.*
2. *For any ring R, $(\mathcal{WF}_n, \mathcal{WC}_n)$ is a perfect hereditary cotorsion theory.*

Proposition 2. *Let R be a ring, m and n two non-negative integers.*

1. *If C is an n-weak cotorsion right R-module, then $Ext^{i+1}_R(C, M) = 0$ for any integer $i \geq m$ and any $M \in \mathcal{WF}_{m+n}$.*
2. *The mth cosyzygy of any n-weak cotorsion right R-module is $(m + n)$-weak cotorsion.*

Proof. (1) For any $M \in \mathcal{WF}_{m+n}$, we have an exact sequence

$$0 \to K_m \to P_{m-1} \to P_{m-2} \to \cdots \to P_1 \to P_0 \to M \to 0$$

with each P_i is projective. It is clear that $K_m \in \mathcal{WF}_n$. Therefore, $Ext^{m+1}_R(M, C) \cong Ext^1_R(K_m, C) = 0$ since C is n-weak cotorsion, and the result follows by induction.

(2) Let C be any n-weak cotorsion right R-module and L^m the mth cosyzygy of C. Note that $Ext^1_R(F, L^m) \cong Ext^{m+1}_R(F, C) = 0$ for any $F \in \mathcal{WF}_{m+n}$ by (1). Thus L^m is $(m + n)$-weak cotorsion. □

Proposition 3. *Let R be a ring and N a right R-module. Then, the following are equivalent:*

1. *N is n-weak cotorsion;*
2. *N is injective with respect to every exact sequence $0 \to K \to M \to L \to 0$, where $L \in \mathcal{WF}_n$;*

   Moreover if $wid_R({}_RR) \leq n$ then, the above conditions are also equivalent to:

3. *For every exact sequence $0 \to N \to E \to L \to 0$, where E is injective, $E \to L$ is a $\mathcal{WF}_n$-precover of L;*
4. *N is a kernel of a $\mathcal{WF}_n$-precover, $E \to L$ with E injective.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). For every right R-module $L \in \mathcal{WF}_n$, there is a short exact sequence $0 \to K \to P \to L \to 0$ with P projective, which induces an exact sequence

$$Hom(P, N) \to Hom(K, N) \to Ext^1_R(L, N) \to 0.$$

Since $Hom(F, N) \to Hom(K, N) \to 0$ is exact by (2), $Ext^1_R(L, N) = 0$. So (1) follows.

(1) $\Rightarrow$ (3). Let $0 \to N \to E \to L \to 0$ be an exact sequence with E injective. Then $E \in \mathcal{WF}_n$ is by [17, Proposition 4.11]. For any right R-module $F \in \mathcal{WF}_n$, the exact sequence $0 \to N \to E \to L \to 0$ induces the exact sequence

$$0 \to Hom(F, N) \to Hom(F, E) \to Hom(F, L) \to Ext^1_R(F, N) = 0.$$

So $E \to L$ is a $\mathcal{WF}_n$-precover of L.

(3) $\Rightarrow$ (4). It follows from the exact sequence $0 \to N \to E(N) \to L \to 0$ and (3).

(4) $\Rightarrow$ (1). Let N be a kernel of a $\mathcal{WF}_n$-precover $E \to L$ with E injective. Then we have an exact sequence $0 \to N \to E \to L \to 0$. So, we have the exact sequence $Hom(M, E) \to Hom(M, L) \to Ext^1_R(M, N) \to 0$ for each right R-module $M \in \mathcal{WF}_n$. Note that $Hom(M, E) \to Hom(M, L) \to 0$ is exact by (4). Hence, $Ext^1_R(M, N) = 0$, as desired. □

The following example shows that $\mathcal{WF}_0^{\perp}$(the class of all 0-weak cotorsion modules) is a proper subclass of $\mathcal{F}_0^{\perp}$(class of all cotorsion modules) and $\mathcal{WF}_1^{\perp}$(the class of all 1-weak cotorsion modules) is a proper subclass of $\mathcal{F}_1^{\perp}$(class of all 1-cotorsion modules).

Example 1. *As Gao mentioned in [9, Remark 3.11(2)], we have a ring R with $l.sp.gldim(R) = 0$ but $w.gl.dim(R) \neq 0$ by [11, Theorem 3.4] and from [2] we have a ring R with $l.sp.gldim(R) = 1$ but $w.gl.dim(R) \neq 1$. Then $\mathcal{F}_0$ (resp., $\mathcal{F}_1$) is a proper subclass of $\mathcal{WF}_0$ (resp., $\mathcal{WF}_1$). Note that for any non-negative integer n, $(\mathcal{F}_n, \mathcal{F}_n^{\perp})$ and $(\mathcal{WF}_n, \mathcal{WF}_n^{\perp})$ are cotorsion theories by [12, Theorem 3.4(2)] and [17, Proposition 4.18] respectively, so $\mathcal{WF}_0^{\perp}$ (resp., $\mathcal{WF}_1^{\perp}$) is a proper subclass of $\mathcal{F}_0^{\perp}$ (resp., $\mathcal{F}_1^{\perp}$).*

Recall that an R-module M is said to be *reduced* [5] if M has no non zero injective submodules.

Proposition 4. *Let R be a ring with $wid_R({}_RR) \leq n$. Then the following are equivalent for a right R-module M:*

1. *M is a reduced n-weak cotorsion right R-module;*
2. *M is a kernel of a $\mathcal{WF}_n$-cover $f\colon A \to B$ with A injective.*

Proof. (1) $\Rightarrow$ (2). Consider an exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$. By Proposition 3, the natural map $\alpha\colon E(M) \to E(M)/M$ is a $\mathcal{WF}_n$-precover. Thus $E(M)$ has no non zero direct summand K contained in M since M is reduced. Note that $E(M)/M$ has a $\mathcal{WF}_n$-cover by Theorem 1(2). It follows that $\alpha\colon E(M) \to E(M)/M$ is a $\mathcal{WF}_n$-cover by [16, Corollary 1.2.8] and hence (2) follows.

(2) $\Rightarrow$ (1). Let M be a kernel of a $\mathcal{WF}_n$-cover $f\colon A \to B$ with A injective. By Proposition 3, M is n-weak cotorsion. Now let K be an injective submodule of M. Suppose $A = K \oplus L$, $p\colon A \to L$ is the projection and $i\colon L \to A$ is the inclusion. It is easy to see that $f(K) = 0$, and $f(ip) = f$. This implies that ip is an isomorphism. Thus i is an epimorphism, and hence $A = L$, $K = 0$. So M is reduced. □

Theorem 2. *Let R be a ring with $wid({}_RR) \leq n$. Then a right R-module M is n-weak cotorsion if and only if M is a direct sum of an injective right R-module and a reduced n-weak cotorsion right R-module.*

Proof. $\Leftarrow$ is clear.

$\Rightarrow$. let M be an n-weak cotorsion right R-module. Consider an exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$. By Proposition 3, $E(M) \to E(M)/M$ is a $\mathcal{WF}_n$-precover of $E(M)/M$. But $E(M)/M$ has a $\mathcal{WF}_n$-cover $L \to E(M)/M$ by Theorem 1(2), so we have the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "\phi"] & L \arrow[r] \arrow[d, "\gamma"] & E(M)/M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, "\sigma"] & E(M) \arrow[r] \arrow[d, "\beta"] & E(M)/M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r, "f"] & L \arrow[r] & E(M)/M \arrow[r] & 0
\end{tikzcd}$$

Note that $\beta\gamma$ is an isomorphism, and so $E(M) = \ker\beta \oplus im\gamma$. Since $im\gamma \cong L$, thus L and $\ker\beta$ are injective. Therefore K is a reduced n-weak cotorsion module by Proposition 4. By the Five lemma, $\sigma\phi$ is an isomorphism. Hence we have $M = im\phi \oplus \ker\sigma$, where $im\phi \cong K$. In addition, we get the following commutative diagram:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker\sigma \arrow[r] \arrow[d] & \ker\beta \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, "\sigma"] & E(M) \arrow[r] \arrow[d, "\beta"] & E(M)/M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r, "f"] \arrow[d] & L \arrow[r] \arrow[d] & E(M)/M \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Hence $\ker\sigma \cong \ker\beta$. This completes the proof. □

Theorem 3. *Let R be a ring. Then the following are equivalent:*

1. *Every right R-module is n-weak cotorsion;*
2. *Every right R-module in $\mathcal{WF}_n$ is projective;*
3. *Every right R-module in $\mathcal{WF}_n$ is n-weak cotorsion;*
4. *$Ext^1_R(M,N) = 0$ for all right R-modules $M, N \in \mathcal{WF}_n$;*
5. *$Ext^i_R(M,N) = 0$ for all $i \geq 1$ and all right R-modules $M, N \in \mathcal{WF}_n$;*
6. *Every right R-module M has a $\mathcal{WC}_n$-envelope with the unique mapping property.*

Proof. (1) $\Leftrightarrow$ (2). It follows from Theorem 1(2).

(1) $\Rightarrow$ (4) $\Rightarrow$ (3), (4) $\Leftrightarrow$ (5) and (1) $\Rightarrow$ (6) are trivial.

(6) $\Rightarrow$ (3). Let $M \in \mathcal{MF}_n$. Then we have the following commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "\alpha_M"] \arrow[drr, "0"'] & \mathcal{WC}_n(M) \arrow[r, "\gamma"] \arrow[dr, "\alpha_L\gamma"] & L \arrow[r] \arrow[d, "\alpha_L"] & 0 \\
 & & & \mathcal{WC}_n(L). &
\end{tikzcd}$$

where $L \in \mathcal{MF}_n$ by Wakamatsu's Lemma [16, Lemma 2.1.2]. Note that $\sigma_L\gamma\sigma_M = 0 = 0\sigma_M$, so $\sigma_L\gamma = 0$ by (6). Therefore, $L = im(\gamma) \subseteq ker(\sigma_L) = 0$, and hence M is n-weak cotorsion. Thus (3) follows.

(3) $\Rightarrow$ (1). Let M be a right R-module. By Theorem 1(2), M has a special $\mathcal{WF}_n$-precover, and hence there exists a short exact sequence $0 \to K \to N \to M \to 0$, where $K \in \mathcal{WF}_n$ and $N \in \mathcal{WC}_n$. Since N is n-weak cotorsion by (3), M is n-weak cotorsion by Theorem 1(2). So (1) follows. $\square$

In general $l.sp.gldim(R) \neq w.gl.dim(R)$ (see [9, Remark 3.7(3)]). Here we have the following

Theorem 4. *The following are equivalent for a ring R:*

1. $l.sp.gldim(R) = w.gl.dim(R)$*;*
2. $\mathcal{WF}_n = \mathcal{F}_n$ *for any* $n \geq 0$*;*
3. *Every n-cotorsion module is n-weak cotorsion for any $n \geq 0$.*

Proof. (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (2). Since $(\mathcal{WF}_n, \mathcal{WF}_n^{\perp})$ and $(\mathcal{F}_n, \mathcal{F}_n^{\perp})$ are cotorsion theories for any integer $n \geq 0$ so the assertion hold. $\square$

3. APPLICATIONS

In this section, we characterize rings with finite super finitely presented dimension in terms, among others, of n-weak cotorsion modules. We start with the following

Lemma 1. *Let R be a ring with $wid(_RR) \leq n$ and $n \geq 1$. If $M \in \mathcal{WI}_{n-1}^{\perp}$, then there is an exact sequence $0 \to K \to E \to M \to 0$ such that E is injective and $K \in \mathcal{WI}_n^{\perp}$.*

Proof. Let $M \in \mathcal{WI}_{n-1}^{\perp}$ and $0 \to N \to P \to M \to 0$ an exact sequence of left R-modules, where P is projective. Consider the following pushout diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] & E(P) \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
 & & L \arrow[r, equal] \arrow[d] & L \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where P is projective and $P \to E(P)$ is an injective envelope. Note that $wid_R(P) \leq n$ by [17, Proposition 4.11], it follows that $wid_R(L) \leq n-1$ by [9, Proposition 3.3]. Then $Ext^1_R(L, M) = 0$ since $M \in \mathcal{WI}^{\perp}_{n-1}$. Thus the exact sequence $0 \to M \to D \to L \to 0$ is split, and so M is a quotient of $E(P)$.

Now let $f : E \to M$ be a weak injective cover of M with E injective, then f is epic. So we have the exact sequence $0 \to K \to E \to M \to 0$. Note that $K \in \mathcal{WI}^{\perp}_0$. We claim that $K \in \mathcal{WI}^{\perp}_n$. Let $F \in \mathcal{WI}_n$ and consider the exact sequence $0 \to E(F) \to G \to 0$. Then $G \in \mathcal{WI}_{n-1}$ by [9, Proposition 3.3]. So we have the exact sequence

$$0 = Ext^1_R(G, M) \to Ext^2_R(G, K) \to Ext^2_R(G, E) = 0.$$

Thus $Ext^2_R(G, K) = 0$. On the other hand, the exactness of the sequence $0 \to F \to E(F) \to G \to 0$ induces the exact sequence

$$0 = Ext^1_R(E(F), K) \to Ext^1_R(F, K) \to Ext^2_R(G, K) = 0.$$

Therefore, $Ext^1_R(F, K) = 0$ and so $K \in \mathcal{WI}^{\perp}_n$. □

The following theorem extends the result of Zhao [17, Proposition 4.12].

Theorem 5. *The following are equivalent for a ring R and a fixed non-negative integer n:*

1. *$l.sp.gldim(R) \leq n$;*
2. *Every n-weak cotorsion right R module is injective;*
3. *Every n-weak cotorsion right R-module is in $\mathcal{WF}_n$;*
4. *$Ext^1_R(M, N) = 0$ for all n-weak cotorsion right R-modules M, N;*
5. *$Ext^i_R(M, N) = 0$ for all $i \geq 1$ and all n-weak cotorsion right R-modules M, N;*
6. *Every right R-module M has a $\mathcal{WF}_n$-cover with the unique mapping property.*

   If $n \geq 1$, then the above conditions are also equivalent to:

7. *Every left R-module has weak injective dimension at most n;*
8. *Every right R-module has weak flat dimension at most n;*
9. *Every left R-module has a monic $\mathcal{WI}_{n-1}$-cover;*
10. *Every right R-module has an epic $\mathcal{WF}_{n-1}$-envelope;*
11. *Every quotient of any weak injective left R-module is in $\mathcal{WI}_{n-1}$;*
12. *Every submodule of any weak flat right R-module is in $\mathcal{WF}_{n-1}$;*
13. *The kernel of any $\mathcal{WI}_{n-1}$-precover of any left R-module is in $\mathcal{WI}_{n-1}$;*
14. *The cokernel of any $\mathcal{WF}_{n-1}$-preenvelope of any right R-module is in $\mathcal{WF}_{n-1}$.*

Proof. The equivalence of (7)−(14) with (1) follows from [17, Proposition 4.12].

(1) ⇔ (2). It follows from Theorem 1(2).

(1) ⇒ (3) ⇔ (4) ⇔ (5), and (1) ⇒ (6) are trivial.

(6) ⇒ (3). Let M be any n-weak cotorsion right R-module. Then we have the following commutative diagram

$$\begin{tikzcd}
 & \mathcal{WF}_n(K) \arrow[d, "\epsilon_K"] \arrow[dr, "\alpha\epsilon_K"] \arrow[drr, "0"] & & & \\
0 \arrow[r] & K \arrow[r, "\alpha"'] \arrow[d] & \mathcal{WF}_n(M) \arrow[r, "\epsilon_M"'] & M \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

Note that $\epsilon_M \alpha \epsilon_K = 0 = \epsilon_M 0$, so $\alpha\epsilon_K = 0$ by (8). Therefore $K = im(\epsilon_K) \subseteq ker(\alpha) = 0$, and so $M \in \mathcal{WF}_n$, as required.

(3) $\Rightarrow$ (1). Let M be any right R-module. By Theorem 1(2), there exists a shot exact sequence $0 \to M \to C \to L \to 0$ with $C \in \mathcal{WC}_n$ and $L \in \mathcal{WF}_n$. Then $C \in \mathcal{WF}_n$ by (3), and hence $M \in \mathcal{WF}_n$. Thus $l.sp.gldim(R) \leq n$. $\square$

Remark 2. *Note that if R is a coherent ring, then Theorem 5 gives some of the equivalent conditions proved in [12, Theorem* 6.4*].*

Theorem 6. *Let R be a ring with $wid_R({}_RR) \leq n$ for a fixed $n \geq 1$, then the following are equivalent:*

1. *$l.sp.gldim(R) < \infty$;*
2. *$l.sp.gldim(R) \leq n$;*
3. *Every left R-module in $\mathcal{WI}_{n-1}^{\perp}$ is injective;*
4. *Every left R-module in $\mathcal{WI}_{n-1}^{\perp}$ is weak injective;*
5. *Every left R-module in $\mathcal{WI}_{n}^{\perp}$ is weak injective;*
6. *Every left R-module in $\mathcal{WI}_{n}^{\perp}$ is injective;*
7. *Every left R-module in $\mathcal{WI}_{n}^{\perp}$ belongs to $\mathcal{WI}_n$.*

Proof. (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (4) $\Rightarrow$ (5), and (6) $\Rightarrow$ (5) are trivial.

(1) $\Rightarrow$ (2). By [10, Proposition 4.2], $l.sp.gldim(R) = wid({}_RR) \leq n$.

(1) $\Rightarrow$ (7) follows from Theorem 1(1).

(7) $\Leftarrow$ (1). Let M be a left R-module. Then by Theorem 1(1), there is a short exact sequence $0 \to K \to F \to M \to 0$ with $K \in \mathcal{WI}_n^{\perp}$ and $F \in \mathcal{WI}_n$. Then $K \in \mathcal{WI}_n$ by (8), and hence $M \in \mathcal{WI}_n$ as desired.

(4) $\Rightarrow$ (3). Let M be any left R-module in $\mathcal{WI}_{n-1}^{\perp}$. There exists an exact sequence $0 \to M \to E \to L \to 0$ with E injective. Note that L is weak injective by (4) and [9, Proposition 3.3], and so the sequence is split since $Ext_R^1(L, M) = 0$. Therefore, M is injective.

(5) $\Rightarrow$ (4) follows from Lemma 1.

(5) $\Rightarrow$ (6). Similar to the proof (4) $\Rightarrow$ (3). $\square$

Recall that a ring R is called n-FC ring if it is a left and right coherent ring with left and right self FP-injective dimension is n. From Theorem 6 we get the following equivalent conditions proved by Gao in [7, Theorem 3.6].

Corollary 1. *Let R be a n-FC ring with $n \geq 1$. Then the following are equivalent:*

1. *$w.gl.dim(R) < \infty$;*

2. *$w.gl.dim(R) \leq n$;*
3. *Every left R-module in $\mathcal{FP}_{n-1}^{\perp}$ is injective;*
4. *Every left R-module in $\mathcal{FP}_{n-1}^{\perp}$ is FP-injective;*
5. *Every left R-module in $\mathcal{FP}_{n}^{\perp}$ is FP-injective;*
6. *Every left R-module in $\mathcal{FP}_{n}^{\perp}$ is injective;*
7. *Every left R-module in $\mathcal{FP}_{n}^{\perp}$ belongs to $\mathcal{FP}_n$.*

Proposition 5. *The following are equivalent for a ring R and a fixed integer $n \geq 0$.*

1. *R is a semisimple Artinian ring;*
2. *Every n-weak cotorsion right R-module is projective.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Since every injective right R-module is projective so R is a QF ring. Then every n-cotorsion right R-module is n-weak cotorsion, by Remark 1(2) and hence R is semisimple Artinian by [12, Corollary 6.5]. □

Theorem 7. *Assume a ring R satisfies one of the following conditions:*

1. *Every n-weak cotorsion right R-module has a $\mathcal{WF}_n$-envelope with the unique mapping property.*
2. *Every finitely presented right R-module has a $\mathcal{WF}_n$-envelope with the unique mapping property.*

Then $l.sp.gldim(R) \leq n+2$.

Proof. Assume (1). Let M be any right R-module. Then we have the exact sequences

$$0 \longrightarrow C \xrightarrow{i} F_0 \xrightarrow{\alpha} M \longrightarrow 0 \text{ and } 0 \to F_2 \xrightarrow{\psi} F_1 \xrightarrow{\beta} C \longrightarrow 0$$

by Theorem 1(2), where $\alpha : F_0 \to M$ and $\beta : F_1 \to C$ are $\mathcal{WF}_n$-covers, C and F_2 are n-weak cotorsion. Thus we get an exact sequence

$$0 \longrightarrow F_2 \xrightarrow{\psi} F_1 \xrightarrow{\phi = i\beta} F_0 \xrightarrow{\alpha} M \longrightarrow 0.$$

Let $\theta : F_2 \to H$ be an $\mathcal{WF}_n$-envelope with the unique mapping property. Then there exists $\delta : H \to F_1$ such that $\psi = \delta\theta$. Thus $\phi\delta\theta = \phi\psi = 0$, and hence $\phi\delta = 0$, which implies that $im(\delta) \subseteq ker(\phi) = im(\psi)$. So there exists $\gamma : H \to F_2$ such that $\psi\gamma = \delta$, and hence we get the following commutative diagram:

$$\begin{tikzcd} & H \arrow[d, shift right, "\gamma"'] \arrow[dr, "\delta"] & & & \\ 0 \arrow[r] & F_2 \arrow[u, shift right, "\theta"'] \arrow[r, "\psi"] & F_1 \arrow[r, "\phi"] & F_0 \arrow[r, "\alpha"] & M \arrow[r] & 0 \end{tikzcd}$$

Note that $\psi\gamma\theta = \psi$, and so $\gamma\theta = 1_{F_2}$ since ψ is monic. Thus F_2 is isomorphic to a direct summand of H, and hence $F_2 \in \mathcal{WF}_n$. Therefore, $wfdM \leq n+2$, and so $l.sp.gldim(R) \leq n+2$.

Assume (2). By [3, Lemma 3.2], every right R-module has a $\mathcal{WF}_n$-envelope with the unique mapping property since $\mathcal{WF}_n$ is closed under direct limits. So the result follows since the condition (1) is satisfied. □

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