$*$-RICCI SOLITONS ON THREE-DIMENSIONAL NORMAL ALMOST CONTACT METRIC MANIFOLDS

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Abstract. The purpose of the paper is to study $*$-Ricci solitons and $*$-gradient Ricci solitons on three-dimensional normal almost contact metric manifolds. First, we prove that if a non-cosymplectic normal almost contact metric manifold with α, $\beta =$ constant of dimension three admits a $*$-Ricci soliton, then the manifold is $*$-Ricci flat, provided $\beta \neq 0$ and $\alpha \neq \pm\beta$. Further, we prove that if a normal almost contact metric manifold with α, $\beta =$ constant, of dimension three admits $*$-gradient Ricci soliton, then the manifold is $*$-Einstein, provided $\alpha^2 - \beta^2 \neq 0$.

1. Introduction

A Ricci soliton is a generalization of an Einstein metric. A Riemannian metric g, defined on a smooth manifold M of dimension n is said to be a Ricci soliton if there exists a vector field V and a constant λ such that

$$\pounds_V g + 2Ric + 2\lambda g = 0, \tag{1.1}$$

where $\pounds_V$ denotes the Lie-derivative in the direction of V and Ric is the Ricci tensor of g. This is considered as a generalization of Einstein metric and often arises as a fixed point of Hamiltons Ricci flow: $\frac{\partial}{\partial t}g(t) = -2Ric(g(t))$, where $g(t)$ is a one-parameter family of metrics on M.

De et. al. [7] studied Ricci soliton and gradient Ricci soliton on three-dimensional normal almost contact metric manifolds. Ricci soliton and gradient Ricci soliton have been studied by several authors such as Bejan and Crasmareanu [2], Calin and Crasmareanu [3], De and Matsuyama [4], De and Mandal [5], Ghosh [8], Sharma [15], Wang and Liu[17]) and many others.

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The notion of $*$-Ricci solitons on almost Hermitian manifolds was introduced by Tachibana [16], in 1959. Hamada [10] studied $*$-Ricci flat real hypersurfaces in non-flat complex space forms. The $*$-Ricci tensor in contact metric manifold is given by [9]

$$S^*(X,Y) = \frac{1}{2}(Trace\{\phi \circ R(X, \phi Y)\}), \tag{1.2}$$

where Q^* is the $*$-Ricci operator.

Recently, Kaimakamis and Panagiotidou [12] introduced the notion of $*$-Ricci soliton where they essentialy modified the definition of Ricci soliton by replacing the Ricci tensor Ric in (1.1) with the $*$-Ricci tensor Ric^*.

Definition 1.1. [9] *A Riemannian metric g on M is called a $*$-Ricci soliton, if*

$$(\pounds_V g)(X,Y) + 2Ric^*(X,Y) + 2\lambda g(X,Y) = 0, \tag{1.3}$$

where λ is a constant and V is a vector field.

Definition 1.2. [9] *A Riemannian metric g on M is called a $*$-gradient Ricci soliton, if*

$$\nabla\nabla f = S^* + \lambda g. \tag{1.4}$$

Definition 1.3. [9] *A normal almost contact metric manifold of dimension $n > 2$ is said to be $*$-Einstein, if the $*$-Ricci tensor S^* of type (0,2) satisfies the relation*

$$S^*(X,Y) = \mu g(X,Y), \tag{1.5}$$

where μ is a constant.

Ghosh and Patra [9] studied $*$-Ricci solition in the frame-work of Sasakian and (k, μ)-contact manifold. Recently, Majhi et al. [13] studied $*$-Ricci solitons and $*$-gradient Ricci solitons on three-dimensional Sasakian manifold. Motivated by the above studies in the present paper we consider $*$-Ricci solition and $*$-gradient Ricci soliton on three-dimensional normal almost contact manifolds.

The paper is structured as follows: After preliminary, in Sections 3 and 4 we study $*$-Ricci solitons and $*$-gradient Ricci solitons on three-dimensional normal almost contact metric manifolds respectively.

2. Preliminaries

Let M be an almost contact manifold and (ϕ, ξ, η) its almost contact structure. Then, M is an odd-dimensional smooth manifold and carries a (1,1)-tensor ϕ, a vector field ξ and a 1-form η satisfying [1]:

(i) $\phi^2 X = -X + \eta(X)\xi$, for all $X \in \chi(M)$,
(ii) $\eta(\xi) = 1$, $\phi(\xi) = 0$, $\eta \circ \phi = 0$.

Let t be a coordinate on $\mathbb{R}$, where $\mathbb{R}$ is the real line. Define an almost complex structure J on $M \times \mathbb{R}$ by

$$J\left(X, \lambda\frac{d}{dt}\right) = \left(\phi X - \lambda\xi, \eta(X)\frac{d}{dt}\right),$$

where the pair $(X, \frac{\lambda d}{dt})$ denotes a tangent vector to $M \times R$, X and $\frac{\lambda d}{dt}$ being tangent to M and $\mathbb{R}$ respectively. If J is integrable then M with the structure (ϕ, ξ, η) is said to be normal or, equivalently, if

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where the pair $[\phi, \phi]$ is the Nijenhuis tensor of ϕ define by

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y],$$

for $X, Y \in \chi(M)$. We say that the form η has rank $r = 2s$ if $(d\eta)^s \neq 0$, and $\eta \wedge (d\eta)^s = 0$, and has rank $r = 2s + 1$ if $\eta \wedge (d\eta)^s \neq 0$ and $(d\eta)^{s+1} = 0$. We also say that r is the rank of the structure (ϕ, ξ, η).

A Riemannian metric g on M is said to be compatible with the structure (ϕ, ξ, η) if the metric g satisfy the condition

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.1}$$

and

$$g(X, \xi) = \eta(X), \tag{2.2}$$

for X, $Y \in \chi(M)$. If the metric g is compatible with the structure (ϕ, ξ, η), then the quadruplet (ϕ, ξ, η, g) is called an almost contact metric structure on M and M is an almost contact metric manifold. For an almost contact metric manifold we can define the 2-form Φ by

$$\Phi(X, Y) = g(X, \phi Y), \tag{2.3}$$

for $X, Y \in \chi(M)$.

In a normal almost contact metric structure (ϕ, ξ, η, g) on M, we have [14]

$$\nabla_X \xi = \alpha\{X - \eta(X)\xi\} - \beta\phi X, \tag{2.4}$$

$$(\nabla_X \eta)(Y) = \alpha g(X, Y) - \alpha\eta(X)\eta(Y) - \beta g(Y, \phi X), \tag{2.5}$$

where 2α =divξ and $2\beta = tr(\phi\nabla\xi)$, div$\xi$ is the divergence of ξ defined by divξ=trace$\{X \to \nabla_X\xi\}$ and tr$(\phi\nabla\xi)$=trace$\{X \to \phi\nabla_X\xi\}$. From (2.4) we obtain

$$(\nabla_X\phi)(Y) = \alpha\{g(\phi X, Y)\xi - \eta(Y)\phi X\} + \beta\{g(X, Y)\xi - \eta(Y)X\}, \tag{2.6}$$

for all $X \in \chi(M)$, where ∇ denotes the Levi-Civita connection of the pseudo-Riemannian manifold.

The curvature tensor in a three-dimensional Riemannian manifold given by [6]

$$\begin{aligned} R(X,Y)Z &= \{\frac{r}{2} + 2(\alpha^2 - \beta^2)\}\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad + g(X,Z)\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}\eta(Y)\xi - \{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}\eta(Y)\eta(Z)X \\ &\quad - g(Y,Z)\{(\frac{r}{2} + 3(\alpha^2 - \beta^2))\eta(X)\xi\} + (\frac{r}{2} + 3(\alpha^2 - \beta^2))\eta(X)\eta(Z)Y, \end{aligned} \tag{2.7}$$

where α, β = constant.

It is known that if α, β = constant, then the manifold is either β-Sasakian, or α-Kenmotsu [11] or cosympletic [1].

From (2.7), we have

$$\begin{aligned} R(X,Y)\phi Z &= \{\frac{r}{2} + 2(\alpha^2 - \beta^2)\}\{g(Y,\phi Z)X - g(X,\phi Z)Y\} \\ &\quad + g(X,\phi Z)\{(\frac{r}{2} + 3(\alpha^2 - \beta^2))\eta(Y)\xi\} \\ &\quad - g(Y,\phi Z)[\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}\eta(X)\xi]. \end{aligned} \tag{2.8}$$

Using (2.8), we obtain

$$\begin{aligned} g(R(X,Y)\phi Z, \phi W) = \{\tfrac{r}{2} + 2(\alpha^2 - \beta^2)\}\{g(Y,\phi Z)g(X,\phi W) \\ - g(X,\phi Z)g(Y,\phi W)\}. \end{aligned} \tag{2.9}$$

Let $\{e_i\}$, $i = 1, 2, 3$ be a local orthogonal basis of vector fields in M. Substituting $X = W = e_i$ in (2.9) and summing over $i = 1$ to 3, we infer that

$$S^*(Y,Z) = \{\frac{r}{2} + 2(\alpha^2 - \beta^2)\}\{g(Y,Z) - \eta(Y)\eta(Z)\}. \tag{2.10}$$

From (2.10), we get

$$Q^*Y = \{\frac{r}{2} + 2(\alpha^2 - \beta^2)\}\{Y - \eta(Y)\xi\} \tag{2.11}$$

From (2.10) we can state the following:

Proposition 2.1. *A three-dimensional normal almost contact metric manifold $(M^3, \phi, \xi, \eta, g)$ is $*$-Ricci flat if and only if $r = -4(\alpha^2 - \beta^2)$.*

The following lemma is very crucial for the next results.

Lemma 2.1. *On a three-dimensional normal almost contact metric manifold* $(M^3, \phi, \xi, \eta, g)$ *we have*

$$(\xi r) = -4\alpha\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}. \tag{2.12}$$

Proof. From (2.7), we obtain

$$S(X,Y) = \{\frac{r}{2} + (\alpha^2 - \beta^2)\}g(X,Y) - \{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}\eta(X)\eta(Y). \tag{2.13}$$

Contracting Y from the above equation we have

$$QX = \{\frac{r}{2} + (\alpha^2 - \beta^2)\}X - \{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}\eta(X)\xi. \tag{2.14}$$

Using (2.14) in the well known formula on Riemannian manifolds

$$trace\{Y \to (\nabla_Y Q)X\} = \frac{1}{2}\nabla_X r,$$

we infer

$$(\xi r)\eta(X) = -4\alpha\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}\eta(X). \tag{2.15}$$

Substituting X with ξ in the above equation we get

$$(\xi r) = -4\alpha\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}.$$

This completes the proof. □

3. $*$-Ricci solitons on normal almost contact metric manifolds

In this section we study $*$-Ricci solitons on normal almost contact metric manifolds. Applying (2.10) in (1.3), we get

$$(\pounds_V g)(X,Y) = -2\{\frac{r}{2} + 2(\alpha^2 - \beta^2) + \lambda\}g(X,Y) + (\frac{r}{2} + 2(\alpha^2 - \beta^2))\eta(X)\eta(Y). \tag{3.1}$$

Taking covariant differentiation of (3.1) with respect to any vector field Z, we have

$$\begin{aligned}(\nabla_Z \pounds_V g)(X,Y) &= -(Zr)\{g(X,Y) - \eta(X)\eta(Y)\}\\ &\quad +2\{\frac{r}{2} + 2(\alpha^2 - \beta^2) + \lambda\}\{\alpha g(X,Z)\eta(Y) + \alpha g(Y,Z)\eta(X)\\ &\quad -2\alpha\eta(X)\eta(Y)\eta(Z) - \beta g(X,\phi Z)\eta(Y) - \beta g(Y,\phi Z)\eta(X)\}.\end{aligned} \tag{3.2}$$

In [18], Yano proved that

$$\begin{aligned}(\pounds_V \nabla_X g - \pounds_X \nabla_V g - \nabla_{[V,X]} g)(Y,Z) &= -\ g((\pounds_V \nabla)(X,Y)Z)\\ &\quad -g((\pounds_V \nabla)(X,Z)Y),\end{aligned} \tag{3.3}$$

for any vector fields X, Y, Z on M. Since g is parallel with respect to the Levi-Civita connection ∇, then the above formula becomes

$$(\nabla_X \pounds_V g)(Y,Z) = g(\pounds_V \nabla)(X,Y),Z) + g(\pounds_V \nabla)(X,Z),Y). \tag{3.4}$$

Since $\pounds_V \nabla$ is $(1,2)$ type symmetric tensor, then it follows from (3.4) that

$$\begin{aligned} 2g((\pounds_V \nabla)(X,Y),Z) = (\nabla_X \pounds_V g)(Y,Z) + (\nabla_Y \pounds_V g)(X,Z) \\ -(\nabla_Z \pounds_V g)(X,Y). \end{aligned} \tag{3.5}$$

Applying (3.2) in (3.5) yields

$$\begin{aligned} 2g((\pounds_V \nabla)(X,Y),Z) &= -(Xr)\{g(Y,Z) - \eta(Y)\eta(Z)\} + 2\{\frac{r}{2} + 2(\alpha^2 - \beta^2)\} \\ &\quad \{\alpha g(X,Y)\eta(Z) + \alpha g(Z,X)\eta(Y) - 2\alpha\eta(X)\eta(Y)\eta(Z) \\ &\quad -\beta g(Y,\phi X)\eta(Z) - \beta g(Z,\phi X)\eta(Y)\} - (Yr)\{g(X,Z) \\ &\quad -\eta(X)\eta(Z)\} + 2\{\frac{r}{2} + 2(\alpha^2 - \beta^2)\}\{\alpha g(X,Y)\eta(Z) \\ &\quad +\alpha g(Z,Y)\eta(X) - \beta g(X,\phi Y)\eta(Z) - \beta g(Z,\phi Y)\eta(X)\} \\ &\quad +(Zr)\{g(X,Y) - \eta(X)\eta(Y)\} - 2\{\frac{r}{2} - 2(\alpha^2 - \beta^2)\} \\ &\quad \{\alpha g(X,Z)\eta(Y) + \alpha g(Y,Z)\eta(X) \\ &\quad -\beta g(X,\phi Z)\eta(Y) - \beta g(Y,\phi Z)\eta(X)\}. \end{aligned} \tag{3.6}$$

Removing Z from (3.6), we get

$$\begin{aligned} 2(\pounds_V \nabla)(X,Y) &= -(Xr)\{Y - \eta(Y)\xi\} + 2\{\frac{r}{2} + 2(\alpha^2 - \beta^2)\} \\ &\quad \{\alpha g(X,Y)\xi - 2\alpha\eta(X)\eta(Y)\xi) - \beta g(Y,\phi X)\xi \\ &\quad -\beta\phi X)\eta(Y)\} - (Yr)\{X - \eta(X)\xi\} \\ &\quad +2\{\frac{r}{2} + 2(\alpha^2 - \beta^2)\}\{\alpha g(X,Y)\xi - \beta g(X,\phi Y)\xi \\ &\quad -\beta\phi Y)\eta(X)\} + (Dr)\{g(X,Y) - \eta(X)\eta(Y)\} \\ &\quad -2\{\frac{r}{2} + 2(\alpha^2 - \beta^2)\}\{\beta\eta(Y)\phi X + \beta\eta(X)\phi Y\}. \end{aligned} \tag{3.7}$$

Substituting $Y = \xi$ in (3.7), we obtain

$$(\pounds_V \nabla)(X,\xi) = -\{r + 4(\alpha^2 - \beta^2)\}\beta\phi X - \frac{(\xi r)}{2}(X - \eta(X)\xi. \tag{3.8}$$

With the help of (2.12) and (3.8), we infer that

$$(\pounds_V \nabla)(X,\xi) = -\{r + 4(\alpha^2 - \beta^2)\}\beta\phi X + 2\alpha\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}(X - \eta(X)\xi. \tag{3.9}$$

Taking covariant differentiation of (3.9) with respect to any vector field Y yields

$$\begin{aligned}(\nabla_Y \pounds_V \nabla)(X,\xi) &= -\{r+4(\alpha^2-\beta^2)\}\beta(\nabla_Y\phi)X \\ &\quad -2\alpha\{\frac{r}{2}+3(\alpha^2-\beta^2)\}(\nabla_Y\eta)(X)\xi \\ &\quad -\alpha(\pounds_V\nabla)(X,Y)+\beta(\pounds_V\nabla)(X,\phi Y). \end{aligned} \tag{3.10}$$

Again we know that

$$(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z). \tag{3.11}$$

Using (3.10) and (3.11), we have

$$\begin{aligned}(\pounds_V R)(X,\xi)\xi &= 2\beta^2\{r+4(\alpha^2-\beta^2)\}[X-\eta(X)\xi] \\ &\quad +2\alpha\beta\{r+5(\alpha^2-\beta^2)\phi X\}. \end{aligned} \tag{3.12}$$

Setting $Y=\xi$ in (1.3), we obtain

$$(\pounds_V g)(X,\xi) = -2\lambda\eta(X). \tag{3.13}$$

Now Lie-differentiating the equation (2.2) along V and using the equation (3.13), we have

$$\eta(\pounds_V \xi) = 0. \tag{3.14}$$

Now from (2.7), we get

$$R(X,\xi)\xi = -(\alpha^2-\beta^2)(X-\eta(X)\xi). \tag{3.15}$$

Lie-differentiating the equation (3.15) and applying (3.13), (3.14), we infer

$$(\pounds_V R)(X,\xi)\xi = 6\lambda(\alpha^2-\beta^2)\eta(X)\xi. \tag{3.16}$$

Equating (3.16) and (3.10), we have

$$\begin{aligned}6\lambda(\alpha^2-\beta^2)\eta(X)\xi &= 2\beta^2\{r+4(\alpha^2-\beta^2)\}[X-\eta(X)\xi] \\ &\quad +2\alpha\beta\{r+5(\alpha^2-\beta^2)\phi X\}. \end{aligned} \tag{3.17}$$

Substituting $X=\xi$ in (3.17), we get

$$\lambda(\alpha^2-\beta^2)=0. \tag{3.18}$$

Then either $\lambda=0$ or, $\alpha=\pm\beta$. Let us consider $\lambda=0$.
Now, taking the inner product with Y of (3.17), we obtain

$$\begin{aligned}&2\beta^2\{r+4(\alpha^2-\beta^2)\}[g(X,Y)-\eta(X)\eta(Y)]+2\alpha\beta\{r+5(\alpha^2-\beta^2)\}g(\phi X,Y) \\ &\quad -6\lambda(\alpha^2-\beta^2)\eta(X)\eta(Y)=0. \end{aligned} \tag{3.19}$$

Substituting $X = Y = e_i$ in (3.19) and summing over $i = 1$ to 3 and using $\lambda = 0$, we infer that

$$\{r + 4(\alpha^2 - \beta^2)\}\beta^2 = 0 \tag{3.20}$$

which implies that $r = -4(\alpha^2 - \beta^2)\}$, provided $\beta \neq 0$.

From the above discussions we can state the following:

Theorem 3.1. *If a non-cosymplectic normal almost contact metric manifold with α, $\beta =$ constant of dimension three admits a $*$-Ricci soliton, then the manifold is $*$-Ricci flat, provided $\beta \neq 0$ and $\alpha \neq \pm\beta$.*

4. $*$-gradient Ricci solitons on normal almost contact metric manifolds

Let (M, g) be a three-dimensional normal almost contact metric manifold and g a $*$-gradient Ricci soliton. Then (1.4) reduces to

$$\nabla_Y Df = Q^*Y + \lambda Y, \tag{4.1}$$

for any $Y \in \chi(M)$, where D denotes the gradient operator of g. From (4.1) it follows that

$$R(X, Y)Df = (\nabla_X Q^*)Y - (\nabla_Y Q^*)X. \tag{4.2}$$

Taking covariant differentiation of (2.11) along arbitrary vector field X and using (2.5) we have

$$\begin{aligned}(\nabla_X Q^*)Y &= \frac{(Xr)}{2}\{Y - \eta(Y)\xi\} - \{\frac{r}{2} + 2(\alpha^2 - \beta^2)\}\{\alpha g(X, Y) - \beta g(Y, \phi X) \\ &\quad -2\alpha\eta(X)\eta(Y)\xi - \beta\eta(Y)\phi X\}.\end{aligned} \tag{4.3}$$

Using (4.2) and (4.3) infer

$$\begin{aligned}R(X, Y)Df &= \frac{(Xr)}{2}\{Y - \eta(Y)\xi\} - \frac{(Yr)}{2}\{X - \eta(X)\xi\} \\ &\quad -\beta\{\frac{r}{2} + 2(\alpha^2 - \beta^2)\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(Y, \phi X)\}.\end{aligned} \tag{4.4}$$

From (4.4), we have

$$g(R(\xi, Y)Df, \xi) = 0. \tag{4.5}$$

Also from (2.7) it follows that

$$R(\xi, Y)Df = -(\alpha^2 - \beta^2)\{g(Y, Df)\xi - g(\xi, Df)Y\}. \tag{4.6}$$

Taking inner product of (4.6) with ξ gives

$$g(R(\xi, Y)Df, \xi) = -(\alpha^2 - \beta^2)\{g(Y, Df) - g(\xi, Df)\eta(Y)\}. \tag{4.7}$$

In view of (4.5) and (4.7) we have

$$(\alpha^2 - \beta^2)\{g(Y, Df)\xi - g(\xi, Df)Y\} = 0. \tag{4.8}$$

From (4.8), we have

$$Df = (\xi f)\xi, \tag{4.9}$$

provided $\alpha^2 - \beta^2 \neq 0$.

Taking differentiation of (4.9) along any arbitrary vector field X, we have $\nabla_X Df = X(\xi f)\xi + (\xi f)\nabla_X \xi$. Replacing X by ϕX and taking inner product with ϕY we have

$$g(\nabla_{\phi X} Df, \phi Y) = (\xi f)\{\alpha g(X, Y) + \beta g(X, \phi Y) - \alpha\eta(X)\eta(Y)\}. \tag{4.10}$$

Interchanging X and Y in the above equation yields

$$g(\nabla_{\phi Y} Df, \phi X) = (\xi f)\{\alpha g(X, Y) + g(Y, \phi X) - \alpha\eta(X)\eta(Y)\}. \tag{4.11}$$

Applying Poincaré's lemma: On a contractible manifold, all closed forms are exact. Therefore $d^2 f(X, Y) = 0$, for all $X, Y \in \chi(M)$. From which we have

$$XY(f) - YX(f) - [X, Y]f = 0,$$

that is,

$$Xg(\mathrm{grad}f, Y) - Yg(\mathrm{grad}f, X) - g(\mathrm{grad}f, [X, Y]) = 0.$$

This is equivalent to

$$\nabla_X g(\mathrm{grad}f, Y) - g(\mathrm{grad}f, \nabla_X Y) - \nabla_Y g(\mathrm{grad}f, X) + g(\mathrm{grad}f, \nabla_Y X) = 0.$$

Since $\nabla g = 0$, the above equation yields

$$g(\nabla_X \mathrm{grad}f, Y) - g(\nabla_Y \mathrm{grad}f, X) = 0,$$

that is, $g(\nabla_X Df, Y) = g(\nabla_Y Df, X)$. Replacing X by ϕX and Y by ϕY in the foregoing equation we obtain $g(\nabla_{\phi X} Df, \phi Y) = g(\nabla_{\phi Y} Df, \phi X)$. Applying this in (4.10) and (4.11) we have $\beta(\xi f)g(X, \phi Y) = 0$, that is, $(\xi f)d\eta(X, Y) = 0$. Since $d\eta \neq 0$, it follows that $\xi f = 0$. Consequently from (4.9) we obtain $Df = 0$, this implies f is constant. Therefore from (4.1) we have

$$S^*(X, Y) = -\lambda g(X, Y),$$

for all vector field X and Y. This shows the manifold is an $*$-Einstein manifold.

Theorem 4.1. *If a normal almost contact metric manifold with α, β = constant, of dimension three admits $*$-gradient Ricci soliton, then the manifold is $*$-Einstein, provided $\alpha^2 - \beta^2 \neq 0$.*

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