A note on unbounded generalized multiplication operators

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Abstract—With the present paper, we attempt to describe the unbounded generalized multiplication operators, induced by some specific symbols, defined on the weighted Hardy spaces. We study their densely defined behaviour and closedness together with the discussion of their normality and self-adjointness.

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1. INTRODUCTION

Multiplication operators are associated closely with various significant classes of operators such as Hankel operators, Toeplitz operators and composition operators (see [1, 7] and references therein). In fact, the unilateral shift on the Hardy space H^2, one of the most interesting operators, is unitarily equivalent to the multiplication operator M_z on H^2. Because of their wide appearance and their tendency to bring in new classes of operators, multiplication operators have been studied extensively over the years and their study has been lifted to various function spaces, such as Bergman spaces, Dirichlet spaces, Lorentz spaces and Orlicz spaces, to name a few (see [3, 6, 10] and the references therein).

The weighted Hardy spaces were introduced by Kelley [9] and have the ability to yield the classical Hardy, Bergman, Dirichlet and Fischer spaces for specific choices of the weight sequences (see [16]). Thus the study over these spaces is quite fruitful and hence the multiplication operators and their generalizations have been discussed on these spaces (see [15, 16]). In recent times, the generalized multiplication operators on weighted Hardy spaces have been introduced in [13] and further extended to k^{th}-order generalized multiplication operators in [4], where $k \geq 1$ is an integer.

However, the operators under consideration in the aforementioned studies ([4, 13]) are the bounded ones. In fact, assumptions are taken on the weight sequence and the inducing symbols so that the induced generalized multiplication operators are bounded and the discussion is carried forward. We find that it is not a difficult task to ensure the existence of unbounded generalized multiplication operators. Our focus, in this paper, is to undertake the study of these unbounded operators.

Before we proceed ahead, we would like to add here that the discussion of unbounded linear operators over Hilbert spaces is of prime interest in physics. Also, unbounded operators arise in varied applications, notably in quantum mechanics and in connection with differential equations. The theory of unbounded operators was stimulated in 1920's and the systematic development of theory is credited to J. Von Neumann and M.H. Stone (see [11] and the references therein). The two most important unbounded linear operators are the multiplication operator and the differentiation operator.

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We now provide the outline of the paper. In Section 2, we provide a brief orientation about unbounded operators alongwith the setting of terminology to be used subsequently in the paper. In Section 3, we discuss the closedness and densely defined nature of some unbounded operators. We also describe some basic properties of the adjoints of these operators.

2. NOTATIONS AND PRELIMINARIES

Let $\{\beta_n\}$ be a sequence of positive real numbers with $\beta_0 = 1$. For $1 \leq p < \infty$, let $H^p(\beta)$ be the space of formal power series $\{f(z) = \sum_{n=0}^{\infty} f_n z^n : \sum_{n=0}^{\infty} |f_n|^p \beta_n^p < \infty\}$. Then $H^p(\beta)$ is a Banach space under the norm

$$\|f\|_\beta^p = \sum_{n=0}^{\infty} |f_n|^p \beta_n^p.$$

For $p = 2$, the space $H^2(\beta)$ is a Hilbert space, called weighted Hardy space, with norm $\|.\|_\beta$ induced by the inner product defined as

$$\langle f, g\rangle = \sum_{n=0}^{\infty} f_n \overline{g}_n \beta_n^2,$$

where $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $g(z) = \sum_{n=0}^{\infty} g_n z^n$ are elements of $H^2(\beta)$. Hence $\|f\|_\beta$ is given as $\|f\|_\beta^2 = \sum_{n=0}^{\infty} |f_n|^2 \beta_n^2$. The set $\{e_n\}_{n\geq 0}$, where $e_n(z) = z^n/\beta_n$, forms an orthonormal basis for the space $H^2(\beta)$. The symbol $H^\infty(\beta)$ denotes the set of formal power series ϕ such that $\phi(H^2(\beta)) \subseteq H^2(\beta)$. We refer to [16] as well as the references therein for the details and applications of these spaces.

We recall from [4] that for a natural number k and an element f of $H^2(\beta)$ with expression $f(z) = \sum_{n=0}^{\infty} f_n z^n$, $f^{(k)}$ is defined as

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \alpha_n f_n z^{n-k},$$

where $\alpha_n = n!/(n-k)!,\ n \geq k$ and is called the k^{th}-derivative of f.

For a fixed natural number $k \geq 1$, if ϕ is a formal power series, then the mapping $f \mapsto \phi.f^{(k)}$ for each $f \in \mathfrak{D} \subseteq H^2(\beta)$, where $\mathfrak{D}$ denotes the domain of this mapping, is denoted by $M_{\phi,k}$. In general, this mapping need not be bounded. However, under suitable assumptions on the sequence β, we can see that this mapping becomes bounded (continuous) on $H^2(\beta)$ and in this case, we call these operators as k^{th}-order generalized multiplication operators on $H^2(\beta)$. We refer to [4] for the bounded case of $M_{\phi,k}$.

It is worth recalling here that the differentiation operator ($x \mapsto x'$, the derivative of x) is an unbounded linear operator ([11], Example 4.13-4). The similarity of this operator with the mapping $f \mapsto f^{(k)}$ intuitively leads us to discuss the unbounded behaviour of this mapping. In the present paper, we describe the structure and behaviour of the unbounded operator $M_{z^p,k}$, where p is a fixed natural number. In this pursuit, we begin with describing the properties of the operator $f \mapsto f^{(k)}$, denoted by D_k (which is same as the operator $M_{1,k}$).

Throughout the paper, the symbol $\mathbb{N}$ denotes the set of all natural numbers and k refers to a fixed natural number. By an operator on a Hilbert space H, we mean a linear mapping $T : \mathfrak{D}(T) \subseteq H \to H$ defined on a linear subspace $\mathfrak{D}(T)$, known as domain of T, of H. We use the symbols $\mathcal{K}er(T)$ and $\mathcal{R}an(T)$ to denote respectively the kernel of T and its range space. We reserve the symbol $\mathcal{G}_T$ to denote the graph of an operator T. By definition, $\mathcal{G}_T = \{(x, Tx) : x \in \mathfrak{D}(T)\}$. For a closable operator T, the smallest closed extension of T is denoted by $\overline{T}$ and called the closure of T. The closure of a subset S of H is denoted by $\overline{S}$ and the orthogonal complement by $S^\perp$. We refer to [5, 11, 14] for the definitions and details of these notions.

3. UNBOUNDED GENERALIZED MULTIPLICATION OPERATORS

For a natural number k, consider the linear operator $D_k : \mathfrak{D}(D_k) \subseteq H^2(\beta) \to H^2(\beta)$ given by $f \mapsto f^{(k)}$ for each $f \in \mathfrak{D}(D_k) = \{f(z) = \sum_{n=0}^{\infty} f_n z^n \in H^2(\beta) : f^{(k)}(z) = \sum_{n=k}^{\infty} \alpha_n f_n z^{n-k} \in H^2(\beta)\}$. We claim that for the sequence β satisfying $\beta_{n-k}/\beta_n \geq K$ for each $n \geq k$ (or $\lim_{n\to\infty} \beta_{n-k}/\beta_n \geq K$) for some $K > 0$, the mapping D_k is unbounded. For, let $f_n(z) = z^n/\beta_n$, where $n \in \mathbb{N}$. Then, $\|f_n\|_\beta = 1$ and $(D_k f_n)(z) = \alpha_n z^{n-k}/\beta_n$ so that $\|D_k f_n\|_\beta = \alpha_n \beta_{n-k}/\beta_n \geq K\alpha_n$ for $n \geq k$, which ensures that D_k is unbounded.

In the same vein, we obtain that the restriction $\lim_{n\to\infty} \beta_{n+p-k}/\beta_n \geq K$ for some $K > 0$, leads to unbounded generalized multiplication operator $M_{z^p,k}$, where $p \in \mathbb{N}$. Thus the existence of unbounded operators D_k and $M_{z^p,k}$ is ensured on specific sequence spaces $H^2(\beta)$. Henceforth in the paper, whenever we discuss the operators D_k and $M_{z^p,k}$ on $H^2(\beta)$, the weight sequence β is assumed to be such that these operators are unbounded.

To ensure the existence of the adjoint T^* of an unbounded linear operator T, it must be densely defined (i.e. $\mathfrak{D}(T)$ must be dense in H) (see [11]). We know that the closed subspace generated by $\{e_n : n \geq 0\}$ is dense in $H^2(\beta)$. We find that for each $n \geq 0$, $e_n(z) = z^n/\beta_n \in \mathfrak{D}(D_k)$ because $e_n(z) \in H^2(\beta)$ and

$$e_n^{(k)}(z) = \begin{cases} 0 & \text{if } n < k \\ \frac{\alpha_n}{\beta_n} z^{n-k} & \text{if } n \geq k, \end{cases}$$

so that $e_n^{(k)}(z) \in H^2(\beta)$ for each $n \geq 0$. As a consequence, the operator D_k is densely defined.

Theorem 1. *D_k is a densely defined unbounded operator.*

Theorem 3.1 ensures the existence of the adjoint of D_k, so we now proceed to compute D_k^*. Consider the collection of all elements $g \in H^2(\beta)$ for which there exists a $h \in H^2(\beta)$ such that $\langle D_k f, g\rangle = \langle f, h\rangle$ for each $f \in \mathfrak{D}(D_k)$. It is known from the theory of unbounded operators [5, 11] that for each g in this collection, we obtain a unique $h \in H^2(\beta)$ satisfying

$$\langle D_k f, g\rangle = \langle f, h\rangle.$$

The collection of all such g is represented as $\mathfrak{D}(D_k^*)$ (the domain of D_k^*) and in this situation, we define $D_k^* g = h$ for each $g \in \mathfrak{D}(D_k^*)$. We obtain the following.

Theorem 2. *The adjoint of D_k is the operator $D_k^* : \mathfrak{D}(D_k^*) \subseteq H^2(\beta) \to H^2(\beta)$ given by $D_k^*(g(z)) = \sum_{n=0}^{\infty} \gamma_n z^n$, for each $g(z) = \sum_{n=0}^{\infty} g_n z^n \in \mathfrak{D}(D_k^*)$, where γ_n's are given as*

$$\gamma_n = \begin{cases} 0 & \text{if } n < k \\ \frac{\alpha_n g_{n-k} \beta_{n-k}^2}{\beta_n^2} & \text{if } n \geq k. \end{cases}$$

Proof. The proof follows using straightforward computations as follows. Let $f(z) = \sum_{n=0}^{\infty} f_n z^n \in \mathfrak{D}(D_k)$ and $g(z) = \sum_{n=0}^{\infty} g_n z^n \in \mathfrak{D}(D_k^*)$. Then, $\langle D_k f, g\rangle = \langle f, h\rangle$ provides that for each $f \in \mathfrak{D}(D_k)$,

$$\begin{aligned} \langle f, h\rangle = \langle \sum_{n=0}^{\infty} \alpha_{n+k} f_{n+k} z^n, \sum_{n=0}^{\infty} g_n z^n\rangle &= \sum_{n=0}^{\infty} \alpha_{n+k} f_{n+k} \overline{g_n} {\beta_n}^2 \\ &= \sum_{n=0}^{\infty} (f_{n+k})(\overline{\alpha_{n+k} g_n}) {\beta_n}^2 \\ &= \langle \sum_{n=0}^{\infty} f_n z^n, \sum_{n=0}^{\infty} \gamma_n z^n\rangle = \langle f, \sum_{n=0}^{\infty} \gamma_n z^n\rangle, \end{aligned}$$

where $\gamma_n = \alpha_n g_{n-k} \beta_{n-k}^2/\beta_n^2$ if $n \geq k$ and zero otherwise. Since $f \in \mathfrak{D}(D_k)$ is arbitrary and $\overline{\mathfrak{D}(D_k)} = H^2(\beta)$, we obtain the desired result. □

Consider the space $H^2(\beta) \times H^2(\beta)$ with the inner product defined as

$$\langle (f_1, g_1), (f_2, g_2) \rangle = \langle f_1, f_2 \rangle + \langle g_1, g_2 \rangle$$

for each $(f_1, g_1), (f_2, g_2) \in H^2(\beta) \times H^2(\beta)$. Then, $H^2(\beta) \times H^2(\beta)$ is a Hilbert space and $U : H^2(\beta) \times H^2(\beta) \to H^2(\beta) \times H^2(\beta)$ defined as $U(f, g) = (g, -f)$ for each $(f, g) \in H^2(\beta) \times H^2(\beta)$ is a unitary operator. Now, utilizing [11, Problem 7, Pg-541] and the fact that U is a unitary operator, we can relate the graphs of D_k and D_k^*.

Theorem 3. *We have the following for the operator D_k:*

1. $\mathcal{G}_{D_k^*} = U(\mathcal{G}_{D_k}^{\perp}) = (U(\mathcal{G}_{D_k}))^{\perp}$.
2. *D_k is closable i.e. D_k permits closure.*

Proof. We prove only 2. Let $\{f_n\}, \{h_n\}$ be two sequences in $\mathfrak{D}(D_k)$ such that $\lim_{n\to\infty} f_n = \lim_{n\to\infty} h_n = f$, for some $f(z) = \sum_{m=0}^{\infty} f_m z^m \in H^2(\beta)$, where $f_n(z) = \sum_{m=0}^{\infty} f_{n,m} z^m$ and $h_n(z) = \sum_{m=0}^{\infty} h_{n,m} z^m$. Let $\epsilon > 0$ be an arbitrary real number. Since $\|f_n - f\|_\beta \to 0$, there exists $n_0 \geq 0$ such that

$$|f_{n,l} - f_l|^2 \beta_l{}^2 \leq \sum_{m=0}^{\infty} |f_{n,m} - f_m|^2 \beta_m{}^2 = \|f_n - f\|_\beta^2 < \epsilon \beta_l{}^2,$$

$\forall\, n \geq n_o$, where $l \geq 0$ is fixed. Utilizing the fact that $\|h_n - f\|_\beta \to 0$, we obtain that for each $l \geq 0$,

$$\lim_{n\to\infty} f_{n,l} = f_l = \lim_{n\to\infty} h_{n,l}. \tag{1}$$

Further assuming that $\lim_{n\to\infty} D_k f_n = g$ and $\lim_{n\to\infty} D_k h_n = h$, where $g, h \in H^2(\beta)$ are given as $g(z) = \sum_{m=0}^{\infty} g_m z^m$ and $h(z) = \sum_{m=0}^{\infty} h_m z^m$ and using the structure of D_k, we obtain that for each $l \geq 0$,

$$g_l = \alpha_{l+k} \lim_{n\to\infty} f_{n,l+k} \quad \text{and} \quad h_l = \alpha_{l+k} \lim_{n\to\infty} h_{n,l+k}.$$

This together with equation (1) yields that $g_l = \alpha_{l+k} f_{l+k} = h_l$ for each $l \geq 0$. Thus $g = h$ and the proof is complete. □

For we have obtained that D_k is closable, we now compute the closure $\overline{D_k}$ of D_k. In this pursuance, consider a sequence $\{f_n\}$ in M such that $f_n \to f$ and $D_k f_n \to g$ for some $f, g \in H^2(\beta)$. Working along parallel lines as in Theorem 3, we obtain that for each $l \geq 0$,

$$f_l = \lim_{n\to\infty} f_{n,l} \text{ and } g_l = \alpha_{l+k} \lim_{n\to\infty} f_{n,l+k},$$

where $f_n(z) = \sum_{m=0}^{\infty} f_{n,m} z^m$, $f(z) = \sum_{m=0}^{\infty} f_m z^m$ and $g(z) = \sum_{m=0}^{\infty} g_m z^m$ respectively are the formal power series expansions of f_n, f and g. Thus $g_{l-k} = \alpha_l f_l$ for each $l \geq k$ and consequently, $f^{(k)}(z) = \sum_{m=0}^{\infty} \alpha_{m+k} f_{m+k} z^m = \sum_{m=0}^{\infty} g_m z^m = g(z) \in H^2(\beta)$. This helps to provide the following.

Theorem 4. *The operator D_k is a closed linear transformation.*

By the definition of the adjoint of a densely defined operator, we obtain that D_k^* is a linear and closed operator. Thus, the closure of D_k^* is the operator D_k^* itself. Also, the adjoint of a densely defined operator is densely defined, so D_k^* is a densely defined operator.

Using the densely defined nature and closedness of operators, we obtain the following.

Theorem 5. *For the densely defined and closed operators D_k and D_k^*, we have:*

1. *$\mathcal{K}er(D_k)$ and $\mathcal{K}er(D_k^*)$ are closed linear subspaces of $H^2(\beta)$.*
2. *$\mathcal{R}an(D_k)$ is closed if and only if $\mathcal{R}an(D_k^*)$ is closed.*

Next, we attempt to discuss the sums and products of the operator D_k with some known operators defined on $H^2(\beta)$. It is trivial to obtain that the operator $D_k + \alpha I$ is a closed operator with domain $\mathfrak{D}(D_k)$, for each complex number α, where I denotes the identity operator on $H^2(\beta)$. If we consider the bounded operator T_ϕ^β on $H^2(\beta)$, called weighted Toeplitz operator (see [12]) induced by the symbol ϕ, we observe that $(T_\phi^\beta D_k)^* = D_k^* T_\phi^{\beta^*}$ and $(T_\phi^\beta + D_k)^* = T_\phi^{\beta^*} + D_k^*$.

It is worth pointing out here that an additional condition that $r \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$ and $r \leq \frac{\beta_n}{\beta_{n-1}} \leq 1$ for $n \leq 0$, for some $r > 0$, is imposed on the weight sequence β, so as to ensure the existence of some non-trivial symbols $\phi \in L^\infty(\beta)$ inducing bounded weighted Toeplitz operators on $H^2(\beta)$. It is imperative that we ensure the existence of bounded T_ϕ^β and unbounded D_k on the same sequence space $H^2(\beta)$ for the above discussion to be meaningful. We find that the co-existence is ensured, consider $H^2(\beta)$ with β given by $\beta_n = 2^n$ as an example.

Next, we investigate the self-adjointness of the operator D_k. Consider the following example.

Example 1. *Let $f(z) = 1$ and $g(z) = z^k$. Then, f and g both belong to $H^2(\beta)$ such that $f^{(k)}(= 0)$ and $g^{(k)}(= k!)$ also are in $H^2(\beta)$. Thus $f, g \in \mathfrak{D}(D_k)$ and using the structure of D_k, it is easy to observe that $\langle D_k f, g\rangle = 0$, while $\langle f, D_k g\rangle = k!$. This provides that D_k cannot be self-adjoint, since $k \geq 1$ is fixed.*

However, we find that $D_k^* D_k$ is a self-adjoint operator with domain $\mathfrak{D}(D_k^* D_k)$ given by $\{f \in \mathfrak{D}(D_k) : D_k f \in \mathfrak{D}(D_k^*)\}$. Further, $\mathfrak{D}(D_k^* D_k)$ can be expressed as $\{f \in H^2(\beta) : \sum_{n=0}^\infty \alpha_{n+k}^2 |f_{n+k}|^2 \beta_n^2 < \infty$ and $\sum_{n=0}^\infty \frac{\alpha_{n+k}^4 |f_{n+k}|^2 \beta_n^4}{\beta_{n+k}^2} < \infty\}$. If the sequence β is such that $\frac{\alpha_{n+k}^2 \beta_n^2}{\beta_{n+k}^2} < M$ for some $M > 0$ (for instance, the sequence given by $\beta_n = n!$), then $\mathfrak{D}(D_k^* D_k) = \{f \in H^2(\beta) : \sum_{n=0}^\infty \alpha_{n+k}^2 |f_{n+k}|^2 \beta_n^2 < \infty\}$.

Using the theory of unbounded operators, we can prove the following.

Theorem 6. *The operators $(I + D_k^* D_k)^{-1}$ and $D_k(I + D_k^* D_k)^{-1}$ on $H^2(\beta)$ are bounded with norms atmost 1. In addition, the operator $(I + D_k^* D_k)^{-1}$ is a self-adjoint and positive operator.*

Next, we discuss normality and quasinormality conditions for the operator D_k. We recollect that a densely defined operator $T : \mathfrak{D}(T) \subseteq H \to H$ is normal if either of the following (equivalent) conditions hold:

1. T is closed and $T^* T = T T^*$.
2. $\mathfrak{D}(T) = \mathfrak{D}(T^*)$ and $\|Tx\| = \|T^* x\|$ for each $x \in \mathfrak{D}(T) = \mathfrak{D}(T^*)$.

For a fixed integer $m \geq 0$, we compute the values $(D_k D_k^*)e_m$ and $(D_k^* D_k)e_m$. The structure of D_k provides that

$$(D_k^* D_k)e_m = \begin{cases} \frac{\alpha_m^2 \, \beta_{m-k}^2}{\beta_m^2} e_m & \text{if } m \geq k \\ 0 & \text{otherwise} \end{cases}$$

and

$$(D_k D_k^*)e_m = \frac{\alpha_{m+k}^2 \, \beta_m^2}{\beta_{m+k}^2} e_m, \text{ for each } m \geq 0.$$

We put to use condition (2) stated above to assert that the operator D_k can not be normal, for if $\mathfrak{D}(D_k) = \mathfrak{D}(D_k^*)$, we obtain that for each $0 \leq m < k$, $\|D_k e_m\|_\beta = 0 \neq \|D_k^* e_m\|_\beta$.

In fact, D_k fails to become quasinormal (see [8], Theorem 3.6), for we compute and find that for each $m \geq 0$,

$$(D_k^{*2} D_k^2)e_m = \begin{cases} \frac{\alpha_m^2 \, \alpha_{m-k}^2 \, \beta_{m-2k}^2}{\beta_m^2} e_m & \text{if } m \geq 2k \\ 0 & \text{otherwise} \end{cases}$$

and

$$(D_k^* D_k)^2 e_m = \begin{cases} \frac{\alpha_m^4 \ \beta_{m-k}^4}{\beta_m^4} e_m & \text{if } m \geq k \\ 0 & \text{otherwise.} \end{cases}$$

Let $p \in \mathbb{N}$. We shall now focus our attention towards the unbounded generalized multiplication operator $M_{z^p,k} : \mathfrak{D}(M_{z^p,k}) \subseteq H^2(\beta) \to H^2(\beta)$, induced by the symbol z^p and given as $M_{z^p,k}(f) = z^p.f^{(k)}$ for each $f \in \mathfrak{D}(M_{z^p,k}) = \{f = \sum_{m=0}^{\infty} f_m z^m \in H^2(\beta) : z^p.f^{(k)} = \sum_{m=p}^{\infty} \alpha_{m+k-p} f_{m+k-p} \ z^m \in H^2(\beta)\}$. Adopting similar approach as in Theorem 1 and utilizing that

$$z^p.e_n^{(k)}(z) = \begin{cases} 0 & \text{if } n < k \\ \frac{\alpha_n}{\beta_n} z^{n-k+p} & \text{if } n \geq k, \end{cases}$$

we obtain the following.

Theorem 7. *The unbounded operator $M_{z^p,k}$ is densely defined.*

In our pursuance to discuss the closedness of the operator $M_{z^p,k}$, we adopt similar methods and techniques as adopted for D_k and achieve the following.

Theorem 8. *$M_{z^p,k}$ is a closed linear transformation.*

Proof. Let $\{F_n\}$ be a sequence in $\mathfrak{D}(M_{z^p,k})$ such that $F_n \to f$ and $M_{z^p,k} F_n \to g$ for some $f, g \in H^2(\beta)$. We obtain that for each $l \geq 0$, $f_l = \lim_{n\to\infty} F_{n,l}$ and

$$g_l = \begin{cases} \alpha_{l+k-p} f_{l+k-p} & \text{if } l \geq p \\ 0 & \text{otherwise,} \end{cases}$$

where $F_n(z) = \sum_{m=0}^{\infty} F_{n,m} z^m$, $f(z) = \sum_{m=0}^{\infty} f_m z^m$ and $g(z) = \sum_{m=0}^{\infty} g_m z^m$ respectively. This provides that $z^p f^{(k)} = \sum_{m=p}^{\infty} \alpha_{m+k-p} f_{m+k-p} z^m = g(z)$. Hence $f \in \mathfrak{D}(M_{z^p,k})$ and $M_{z^p,k} f = g$ and the operator $M_{z^p,k}$ is a closed linear transformation. □

It is easy to compute the adjoint $M_{z^p,k}^*$ of $M_{z^p,k}$.

Theorem 9. *The adjoint $M_{z^p,k}^*$ of $M_{z^p,k}$ is the operator*

$$M_{z^p,k}^* : \mathfrak{D}(M_{z^p,k}^*) \subseteq H^2(\beta) \to H^2(\beta)$$

given by

$$M_{z^p,k}^*(g(z)) = \sum_{n=k}^{\infty} \frac{\alpha_n g_{n+p-k} \beta_{n+p-k}^2}{\beta_n^2} \ z^n,$$

for each $g(z) = \sum_{n=0}^{\infty} g_n z^n \in \mathfrak{D}(M_{z^p,k}^*)$.

It is known that a linear transformation B is an extension of a linear transformation A, written in symbols as $A \subset B$, if $\mathfrak{D}(A) \subseteq \mathfrak{D}(B)$ and $Af = Bf$ for each $f \in \mathfrak{D}(A) = \mathfrak{D}(B)$ (see [14]). Since D_k and $M_{z^p,k}$ are both densely defined unbounded operators, we obtain that $D_k^* M_{z^p,k}^* \subset (M_{z^p,k} D_k)^*$.

It is interesting to observe that for $p \neq k$, the operator $M_{z^p,k}$ is not self-adjoint. For, if we consider $f(z) = z^p$ and $g(z) = z^k$, then $f, g \in \mathfrak{D}(M_{z^p,k})$ and $\langle M_{z^p,k} f, g \rangle = 0$ while $\langle f, M_{z^p,k} g \rangle = k! {\beta_p}^2$. However, for $p = k$, $M_{z^p,k}$ turns out to be a self-adjoint operator.

Theorem 10. *The unbounded operator $M_{z^k,k}$ is self-adjoint.*

Proof. Let $f, g \in \mathfrak{D}(M_{z^k,k})$ such that $f(z) = \sum_{m=0}^{\infty} f_m z^m$ and $g(z) = \sum_{m=0}^{\infty} g_m z^m$. Then, it is a matter of routine computations to see that

$$\begin{aligned}\langle M_{z^k,k} f, g\rangle = \langle \sum_{m=k}^{\infty} \alpha_m f_m z^m, \sum_{m=0}^{\infty} g_m z^m\rangle &= \sum_{m=k}^{\infty} \alpha_m f_m \overline{g_m} {\beta_m}^2 \\ &= \sum_{m=k}^{\infty} f_m (\overline{\alpha_m g_m}) {\beta_m}^2 \\ &= \langle f, M_{z^k,k} g\rangle.\end{aligned}$$

Next, we observe that the operators $M_{z^k,k} \pm iI$ are onto. For, let $h(z) = \sum_{m=0}^{\infty} h_m z^m \in H^2(\beta)$. Define $F(z) = \sum_{m=0}^{\infty} F_m z^m$, where $F_m = h_m/(\alpha_m \pm i)$ for each $m \geq 0$, where

$$\alpha_m = \begin{cases} 0 & \text{if } m < k \\ \frac{m!}{(m-k)!} & \text{if } m \geq k. \end{cases}$$

Then,

$$\sum_{m=0}^{\infty} |F_m|^2 {\beta_m}^2 = \sum_{m=0}^{\infty} \frac{|h_m|^2}{\alpha_m^2 + 1} {\beta_m}^2 \leq \sum_{m=0}^{\infty} |h_m|^2 {\beta_m}^2 < \infty$$

and

$$\sum_{m=k}^{\infty} |\alpha_m F_m|^2 {\beta_m}^2 = \sum_{m=k}^{\infty} \frac{\alpha_m^2}{\alpha_m^2 + 1} |h_m|^2 {\beta_m}^2 \leq \sum_{m=0}^{\infty} |h_m|^2 {\beta_m}^2 < \infty.$$

Thus $F \in \mathfrak{D}(M_{z^k,k})$ and $(M_{z^k,k} \pm iI)F(z) = \sum_{m=0}^{\infty} h_m z^m = h(z)$. Thus the range of both the operators $M_{z^k,k} + iI$ and $M_{z^k,k} - iI$ is the space $H^2(\beta)$ itself. These observations together with the fact that $M_{z^p,k}$ is densely defined provide the desired result. □

Theorem 10 helps to provide that $\mathcal{K}er(M^*_{z^k,k} + iI)$ and $\mathcal{K}er(M^*_{z^k,k} - iI)$ are both equal to the singleton set $\{0\}$.

In our attempt to discuss the normality of the operator $M_{z^p,k}$, we observe that for each $m \geq 0$,

$$(M^*_{z^p,k} M_{z^p,k}) e_m = \begin{cases} \frac{\alpha_m^2 \ \beta_{m-k+p}^2}{\beta_m^2} e_m & \text{if } m \geq k \\ 0 & \text{otherwise} \end{cases}$$

and

$$(M_{z^p,k} M^*_{z^p,k}) e_m = \begin{cases} \frac{\alpha_{m+k-p}^2 \ \beta_m^2}{\beta_{m+k-p}^2} e_m & \text{if } m \geq p \\ 0 & \text{otherwise.} \end{cases}$$

This helps to conclude that

1. For $p = k$, we obtain that for each $m \geq 0$,

   $$(M^*_{z^k,k} M_{z^k,k}) e_m = (M_{z^k,k} M^*_{z^k,k}) e_m = \begin{cases} \alpha_m^2 e_m & \text{if } m \geq k \\ 0 & \text{otherwise.} \end{cases}$$

   By linearity of $M_{z^k,k}$ and that of its adjoint, we find that $(M^*_{z^k,k} M_{z^k,k}) f = (M_{z^k,k} M^*_{z^k,k}) f$ for each $f \in \mathfrak{D}(M^*_{z^k,k} M_{z^k,k}) = \mathfrak{D}(M_{z^k,k} M^*_{z^k,k})$ provided the domains are equal. Thus the operator $M_{z^k,k}$ is a normal operator if $\mathfrak{D}(M^*_{z^k,k} M_{z^k,k}) = \mathfrak{D}(M_{z^k,k} M^*_{z^k,k})$.

2. For natural numbers $p \neq k$, the operator $M_{z^p,k}$ can not be normal.

3. For $p \neq k$, we compute and obtain that for each $m \geq 0$,

$$(M^{*}_{z^p,k}{}^2 M^2_{z^p,k})e_m = \begin{cases} \frac{\alpha_m^2\ \alpha_{m-k+p}^2\ \beta_{m-2k+2p}^2}{\beta_m^2} e_m & \text{if } m \geq 2k-p \\ 0 & \text{otherwise} \end{cases}$$

and

$$(M^{*}_{z^p,k} M_{z^p,k})^2 e_m = \begin{cases} \frac{\alpha_m^4\ \beta_{m-k+p}^4}{\beta_m^4} e_m & \text{if } m \geq k \\ 0 & \text{otherwise.} \end{cases}$$

This provides that the operator $M_{z^p,k}$, for $p \neq k$, can not be quasinormal.

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