On k-Connected Γ-Extensions of Binary Matroids

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Abstract—Slater introduced the point-addition operation on graphs to classify 4-connected graphs. The Γ-extension operation on binary matroids is a generalization of the point-addition operation. In this paper, we obtain necessary and sufficient conditions to preserve k-connectedness of a binary matroid under the Γ-extension operation. We also obtain a necessary and sufficient condition to get a connected matroid from a disconnected binary matroid using the Γ-extension operation.

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1. INTRODUCTION

We refer to [9] for standard terminology in graphs and matroids. The matroids considered here are loopless and coloopless. Slater [12] introduced the point-addition operation on graphs and used it to classify 4-connected graphs. Azanchiler [1] extended this operation to binary matroids as follows:

Definition 1. *[1] Let M be a binary matroid with ground set S and standard matrix representation A over $GF(2)$. Let $X = \{x_1, x_2, \ldots, x_m\} \subset S$ be an independent set in M and let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ be a set such that $S \cap \Gamma = \phi$. Suppose A' is the matrix obtained from the matrix A by adjoining m columns labeled by $\gamma_1, \gamma_2, ..., \gamma_m$ such that the column labeled by γ_i is same as the column labeled by x_i for $i = 1, 2, \ldots, m$. Let A^X be the matrix obtained by adjoining one extra row to A' which has entry 1 in the column labeled by γ_i for $i = 1, 2 \ldots, m$ and zero elsewhere. The vector matroid of the matrix A^X, denoted by M^X, is called as the Γ-extension of M and the transition from M to M^X is called as Γ-extension operation on M.*

An example given at the end of the paper illustrates the definition. Note that the ground set of the matroid M^X is $S \cup \Gamma$ and $M^X \setminus \Gamma = M$. Therefore M^X is an extension of M. The Γ-extension operation is related to the *splitting operation* on binary matroids, which is defined by Shikare et al. [11], as follows:

Definition 2. *[11] Let M be a binary matroid with standard matrix representation A over $GF(2)$ and let Y be a non-empty set of elements of M. Let A_Y be the matrix obtained by adjoining one extra row to the matrix A whose entries are 1 in the columns labeled by the elements of the set Y and zero otherwise. The vector matroid of the matrix A_Y, denoted by M_Y, is called as the splitting matroid of M with respect to Y, and the transition from M to M_Y is called as the splitting operation with respect to Y.*

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Let M be a binary matroid with ground set S and let $X = \{x_1, x_2, \dots, x_m\}$ be an independent set in M. Obtain the extension M' of M with ground set $S \cup \Gamma$, where $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_m\}$ is disjoint from S, such that $\{x_i, \gamma_i\}$ is a 2-circuit in M' for each i. The matroid M'_Γ obtained from M' by splitting the set Γ is the Γ-extension matroid M^X.

The splitting operation with respect to a pair of elements, which is a special case of Definition 1.2, was earlier defined by Raghunathan et al. [10] for binary matroids as an extension of the corresponding graph operation due to Fleischner [7].

Whenever we write M^X, it is assumed that X is a non-empty independent set of the matroid M.

Azanchiler [1] characterized the circuits and the bases of the Γ-extension matroid M^X in terms the circuits and bases of M, respectively. Some results on preserving graphicness of M under the Γ-extension operation are obtained in [2]. Borse and Mundhe [6] characterized the binary matroids M for which M^X is graphic for any independent set X of M.

A *k-separation* of a matroid M is a partition of its ground set S into two disjoint sets A and B such that $min\ \{|A|, |B|\} \geq k$ and $r(A) + r(B) - r(M) \leq k - 1$. A matroid M is *k-connected* if it does not have a $(k-1)$-separation. Also, M is *connected* if it is 2-connected.

In general, the splitting operation does not preserve the connectivity of a given matroid. Borse and Dhotre [4] provided a sufficient condition to preserve connectedness of a matroid while Borse [3] gave a sufficient condition to get a k-connected matroid from given the $(k+1)$-connected binary matroid, under the splitting with respect to a pair of elements. Borse and Mundhe [5], and Malwadkar et al. [8] gave two characterizations for getting a k-connected matroid from the given $(k+1)$-connected binary matroid by splitting with respect to any set of k elements.

The Γ-extension operation also does not give k-connected matroid from the given k-connected binary matroid in general. Azanchiler [1] obtained sufficient conditions to preserve 2-connectedness and 3-connectedness of a binary matroid under this operation.

In this paper, we obtain necessary and sufficient conditions to preserve k-connectedness under the Γ-extension operation for any integer $k \geq 2$. We also give necessary and sufficient conditions to get a *connected* matroid from a disconnected binary matroid in terms of the Γ-extension operation.

2. PROOFS

We need some lemmas.

Lemma 1. *[1] Let M be a binary matroid with ground set S and let X be an independent set in M. Suppose M^X is the Γ-extension of M with ground set $S \cup \Gamma$. Let r and r' be the rank functions of M and M^X, respectively. Then*
(i) Γ is independent in M^X;
(ii) $r'(A) = r(A)$ if $A \subset S$;
(iii) $r'(A) \geq r(S \cap A) + 1$ if A intersects Γ;
(iv) $r'(M^X) = r(M) + 1$.

Lemma 2. *[1] Let M be a binary matroid with ground set S and let X be an independent set in M. Then $Z \subset S \cup \Gamma$ is a circuit of M^X if and only if one of the following conditions holds:*
(i) Z is a circuit of M;
(ii) $Z = \{x_i, x_j, \gamma_i, \gamma_j\}$ for some distinct elements x_i, x_j of X and the corresponding elements γ_i, γ_j of Γ;
(iii) $Z = J \cup (D - X_J)$, where $J \subset \Gamma$ with $|J|$ even and D is a circuit of M containing the set $X_J = \{x_i \in X : \gamma_i \in J\}$.

Lemma 3 ([9], pp 273). *Let M be a k-connected matroid with at least $2(k-1)$ elements. Then every circuit and every cocircuit of M contains at least k elements.*

The next lemma is a consequence of [9, Proposition 2.1.6].

Lemma 4. *[3] Let M be a matroid with ground set S and let $Y \subset S$ such that $r(M \setminus Y) = r(M) - 1$. Then Y contains a cocircuit of M.*

The following result follows immediately from Lemma 3 and Lemma 4.

Corollary 1. *Let M be a k-connected matroid with ground set S such that $|S| \geq 2(k-1)$. Then $r(M \backslash Y) = r(M)$ for any $Y \subset S$ with $|Y| < k$.*

We now give necessary and sufficient conditions to obtain a k-connected matroid from the given k-connected binary matroid as follows.

Theorem 1. *Let $k \geq 2$ be an integer and M be a k-connected binary matroid with at least $2(k-1)$ elements and X be an independent set in M. Then the Γ-extension matroid M^X is k-connected if and only if $|X| \geq k$ and $2 \leq k \leq 4$.*

Proof. Suppose $|X| \geq k$ and $2 \leq k \leq 4$. We prove that M^X is k-connected. The ground set of M^X is $S \cup \Gamma$, where Γ is disjoint from the ground set S of M. Since $|\Gamma| = |X|$, $|\Gamma| \geq k$. By Lemma 1(i), Γ is independent in M^X. Suppose r and r' denote the rank functions of M and M^X, respectively. Assume that M^X is not k-connected. Then M^X has a $(k-1)$-separation (A, B). Therefore A and B are non-empty disjoint subsets of $S \cup \Gamma$ such that $S \cup \Gamma = A \cup B$ and further,

$$\min\{|A|, |B|\} \geq k-1 \quad \text{and} \quad r'(A) + r'(B) - r'(M^X) \leq k-2. \tag{1}$$

As A and B are non-empty, each of them intersects S or Γ or both. We consider the three cases depending on whether A intersect only S or only Γ or both and obtain a contradiction in each of these cases.

Case (i). A intersects only Γ.

As $A \subset \Gamma$, $B = S \cup (\Gamma - A)$. Since Γ is independent, A is independent in M^X. Consequently, $r'(A) = |A| \geq k-1$. Suppose $A \neq \Gamma$. Then, by Lemma 1(iii) and (iv), $r'(B) \geq r(S) + 1 = r(M) + 1 = r'(M^X)$. Therefore $r'(B) = r'(M^X)$. Hence $r'(A) + r'(B) - r'(M^X) \geq k-1$, which contradicts (1). Therefore $A = \Gamma$. Hence $B = S$ and $r'(A) = |\Gamma| \geq k$. By Lemma 1(ii) and (iv), $r'(B) = r'(S) = r(S) = r(M) = r'(M^X) - 1$. Therefore $r'(A) + r'(B) - r'(M^X) \geq k-1$, which is a contradiction to (1).

Case (ii). A intersects only S.

As $A \cap \Gamma = \phi$, $A \subset S$ and $B = (S - A) \cup \Gamma$. Therefore, by Lemma 1(i) and (ii), $r'(A) = r(A)$ and $r'(B) \geq r'(\Gamma) = |\Gamma| \geq k$. Suppose $|S - A| \leq k-2$. Then, by Corollary 1, $r(A) = r(M)$. Consequently, by Lemma 1(iv),

$$r'(A) + r'(B) - r'(M^X) = r(A) + r'(B) - (r(M) + 1) \geq r'(B) - 1 \geq k-1,$$

which is a contradiction to (1). Hence $|S - A| \geq k-1$. By Lemma 1 (ii) and (iii), $r(S - A) = r'(S - A) \leq r'(B) - 1$. Therefore, by Inequality (1),

$$r(A) + r(S - A) - r(M) \leq r'(A) + r'(B) - 1 - r(M^X) + 1 \leq k-2.$$

This shows that A and $S - A$ gives a $(k-1)$-separation of M, which is a contradiction to fact that M is k-connected.

Case (iii). A intersects both S and Γ.

Let $S_1 = A \cap S$ and $\Gamma_1 = A \cap \Gamma$. Since $B \neq \phi$, it intersects S or Γ. If B intersects only S or only Γ, then we get a contradiction by interchanging roles of A and B in Case (i) and Case (ii). Therefore B intersects both S and Γ. Let $S_2 = B \cap S$ and $\Gamma_2 = B \cap \Gamma$. Then $S_i \neq \phi$ and $\Gamma_i \neq \phi$ for $i = 1, 2$. By Lemma 1(ii) and (iii), $r(S_1) = r'(S_1) \leq r'(A) - 1$ and $r(S_2) = r'(S_2) \leq r'(B) - 1$. By (1),

$$r(S_1) + r(S_2) - r(M) \leq r'(A) - 1 + r'(B) - 1 - r'(M^X) + 1 \leq k-3.$$

Hence, if $|S_1| \geq k-2$ and $|S_2| \geq k-2$, then (S_1, S_2) gives a $(k-2)$-separation of M, a contradiction to fact that M is k-connected. Consequently, $|S_1| \leq k-3$ or $|S_2| \leq k-3$.

Suppose $|S_1| \leq k-3$. As $k \leq 4$ and $1 \leq |S_1|$, $k=4$ and $|S_1| = k-3 = 1$. Thus A contains exactly one element, say x, of M. Further, $|A| \geq k-1 = 4-1 = 3$. We claim that $r'(A) \geq 3$. Suppose $r'(A) \leq 2$. Then A contains a circuit C of M^X such that $|C| \leq 3$. Since Γ is independent in M^X, C is not a subset of Γ. Therefore C contains x and $C - \{x\} \subset A - \{x\} \subset \Gamma$. In the last row of the matrix A^X which represents the matroid M^X, the columns corresponding to the elements of Γ have entries 1 and rest of the entries in that row are zero. As C is a circuit, the sum of the columns of A^X corresponding to the elements of C is zero over GF(2). This implies that C contains at least two elements of Γ. Hence $C = \{x, \gamma_1, \gamma_2\}$ for some $\gamma_1, \gamma_2 \in \Gamma$. Let x_1 and x_2 be elements of the matroid M corresponding to γ_1 and γ_2, respectively. By Lemma 2(ii), $C_1 = \{x_1, x_2, \gamma_1, \gamma_2\}$ is a circuit in M^X. Since M^X is a binary matroid, the symmetric difference $C \Delta C_1 = \{x, x_1, x_2\}$ of the circuits C and C_1 contains a circuit, say C_2, of M^X. Hence C_2 is a circuit in $M^X \setminus \Gamma = M$ such that $|C_2| \leq 3 = 4-1 = k-1$, a contradiction by Lemma 3. Hence $r'(A) \geq 3$. Since $|S_1| \leq k-3$, by Corollary 1, $r(S_2) = r(S - S_1) = r(M)$. Therefore, by Lemma 1(iii), $r'(B) \geq r'(S_2) + 1 = r(S_2) + 1 = r(M) + 1 = r'(M^X)$. Therefore $r'(B) = r'(M^X)$. Hence $r'(A) + r'(B) - r'(M^X) = r'(A) \geq 3 = k-1$, a contradiction to (1).

Suppose $|S_2| \leq k-3$. Then, as in the above paragraph, we see that $r'(B) \geq 3 = k-1$ and $r'(A) = r'(M^X)$ and so $r'(A) + r'(B) - r'(M^X) = r'(B) \geq k-1$, a contradiction to (1).

Thus we get contradictions in Cases (i), (ii) and (iii). Therefore M^X is k-connected.

Conversely, suppose M^X is k-connected. The last row of the matrix A^X, which represents M^X, has 1's in the columns corresponding to the set Γ and zero elsewhere. Hence Γ contains a cocircuit of M^X. By Lemma 3, $|\Gamma| \geq k$ and so $|X| = |\Gamma| \geq k$. By Lemma 2(ii), M^X contains a 4-circuit. Therefore, by Lemma 3, $k \leq 4$. This completes the proof. □

We now give a necessary and sufficient condition to get a connected matroid M^X from the disconnected matorid M. If X is disjoint from a component D of M, then it follows from Lemma 2 that D is a component of M^X also. Therefore to get a connected matroid M^X from the disconnected matroid M, it is necessary that X intersects every component of M. In the following theorem, we prove that this obvious necessary condition is also suffcient.

Theorem 2. *Let M be a disconnected binary matroid and let X be an independent set in M. Then M^X is connected if and only if every component of M intersects X.*

Proof. Let $M_1, M_2, \ldots, M_r$ be the components of M. Suppose each M_i intersects X. Let S be the ground set of M. Then the ground set of M^X is $S \cup \Gamma$, where $S \cap \Gamma = \phi$. Since each M_i is connected in M and $M^X \setminus \Gamma = M$, each M_i is connected in M^X too. Therefore each M_i is contained in a component of M^X. We show that all M_i are contained in a single component of M^X. Since M is disconnected, it has at least two components and so $r \geq 2$. Let D be a component of M^X containing M_1 and let $j \in \{2, 3, \ldots, r\}$. Suppose X contains an element x_1 of M_1 and an element x_j of M_j. Suppose γ_1 and γ_j are elements of Γ corresponding to x_1 and x_j, respectively. Then, by Lemma 2.2(ii), $C = \{x_1, x_j, \gamma_1, \gamma_j\}$ is a 4-circuit in M^X. As C contains an element of the component D of M^X, C is contained in D. Therefore D contains the element x_j of M_j. Consequently, M_j is contained in D. Thus all components of M are contained in D. Therefore $S \subset D$. Let γ be an arbitrary member of Γ and let x be the member of X corresponding to γ. Then, by Lemma 2.2(ii), γ and x belong to a 4-circuit, say Z, of M^X. As $x \in Z \cap D$, $Z \subset D$ and so $\gamma \in D$. Therefore $\Gamma \subset D$. Consequently, D is the only component of M^X. Hence M^X is connected.

The converse readily follows from the discussion prior to the statement of the theorem. □

Example 2.8. We illustrate Theorem 1 by using the Fano matroid F_7. The ground set of F_7 is $\{1, 2, 3, 4, 5, 6, 7\}$ and the standard matrix representation of F_7 over $GF(2)$ is as follows:

$$A = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix} \end{array}.$$

Let $X=\{1,2\}$ and $Y=\{1,2,3\}$. Then X and Y are independent in F_7. Further,

$$A^X = \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \gamma_1 & \gamma_2 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

and

$$A^Y = \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \gamma_1 & \gamma_2 & \gamma_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

Let F_7^X and F_7^Y be the vector matroids of A^X and A^Y, respectively. It is well known that F_7 is 3-connected. One can check that F_7^Y is 3-connected while F_7^X is 2-connected but not 3-connected.

REFERENCES

1. H. Azanchiler, "Γ-extension of binary matroids," ISRN Discrete Mathematics **2011**, Article 629707 (8 pages) (2011).
2. H. Azanchiler, "On extension of graphic matroids," Lobachevskii J. Math. **36**, 38–47 (2015).
3. Y. M. Borse, "A note on n-connected splitting-off matroids," Ars Combin. **128**, 279–286 (2016).
4. Y. M. Borse and S. B. Dhotre, "On connected splitting matroids," Southeast Asian Bull. Math. **36** (1), 17–21 (2012).
5. Y. M. Borse and G. Mundhe, "On n-connected splitting matroids," AKCE Int. J. Graphs Comb. (2017). doi 10.1016/j.akcej.2017.12.001
6. Y. M. Borse and G. Mundhe, "Graphic and cographic Γ-extensions of binary matroids," Discuss. Math. Graph Theory **38**, 889-898 (2018).
7. H. Fleischner, *Eulerian Graphs and Related Topics Part 1, Vol. 1* (North Holland, Amsterdam, 1990).
8. P. P. Malavadkar, M. M. Shikare and S. B. Dhotre, "A characterization of n-connected splitting matroids," Asian-European J. Combin. **7** (4), Article 14500600 (7 pages) (2014).
9. J. G. Oxley, *Matroid Theory* (Oxford University Press, Oxford, 1992).
10. T. T. Raghunathan, M. M. Shikare and B. N. Waphare, "Splitting in a binary matroid," Discrete Math. **184**, 267–271 (1998).
11. M. M. Shikare, G. Azadi and B. N. Waphare, "Generalized splitting operation for binary matroids and its applications," J. Indian Math. Soc. New Ser. **78** (1–4), 145–154 (2011).
12. P. J. Slater, "A Classification of 4-connected graphs," J. Combin. Theory Series B **17**, 281–298 (1974).