Series with integer coefficients by systems of contractions and shifts of one function

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Abstract—In the spaces $L_p(0,1)$, $1 \le p < \infty$, we investigate the systems consisting of contractions and shifts of one function. We study Fourier type series expansions with integer coefficients by such systems. The resulting decompositions have the property of image compression, that is, many their coefficients vanish. This study may also be of interest to the specialists in transmission and processing of digital information.

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The author continues the research published in [1–5]. Systems consisting of contractions and shifts of one function have become interesting for researchers with the advent of results on wavelets and frames.

In this paper, we construct decompositions in the L_p-spaces directly by a given system consisting of contractions and shifts of one function. The proposed algorithm allows one to obtain a decomposition with a large number of zero coefficients and, at the same time, to some extent is optimal when approximating with a fixed accuracy. The algorithm also admits a significant inaccuracy of intermediate calculations; these inaccuracies are corrected in further calculations.

The possibility of uniform approximation of continuous functions with any accuracy by polynomials with integer rational coefficients on a segment of the real axis has been studied since 1914, in the works of I. Pal, Kakeya, M. Fekete, I.N. Khlodovsky, S.N. Bernshtein, L.V. Kantorovich, and others. The author of the paper [6] continues these studies and formulates some results of the above authors and other researchers on this problem.

The existence of a sequence of trigonometric polynomials with integer (not necessarily positive) coefficients that converges to zero almost everywhere follows from [7].

Recently, there has been a certain interest in decompositions in a series with integer coefficients. Thus, [8] gives a result on the existence of a sequence of trigonometric polynomials with integer positive coefficients that converges to zero almost everywhere.

The papers [9, 10] also consider systems of contractions and shifts of one function, but the expansion coefficients are found almost like the Fourier coefficients (the coefficients tend to zero) and are slightly different than in this paper.

Here we consider functional systems of the form

$$\{\phi_{k,j}\} = \alpha_k\{\phi(b^k t - j)\} = \{\phi_l\},\ k = 0,1,2,...,\ j = 0,1,...,b^k - 1,\ l = b^k + j,$$
$$b \in \mathbb{N},\ b > 1,\ \alpha_k \searrow 0,\ \alpha_k > 0; \qquad (1)$$

here $\phi(t)$ is an arbitrary function from $L_p[0,1]$, $1 \le p < \infty$, continued by the value 0 outside of $[0,1]$.

In Theorem 3, we assume that $\phi_{k,0}(0) = \alpha_k$, $k \ge 0$.

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In Theorems 1 and 3, we consider the function

$$\phi(t)=\begin{cases} 1, \text{ if } t\in(0,1];\\ 0, \text{ if } t\notin(0,1].\end{cases} \tag{2}$$

Denote $\Delta_{k,j}=\left(\frac{j}{b^k},\frac{j+1}{b^k}\right)$, $k\geq 0$, $j=0,1,...,b^k-1$, $|\Delta_{k,j}|$ is the measure of the set $\Delta_{k,j}$.

For a given function g, we construct a series

$$\sum_{l=1}^{\infty} c_l^*\phi_l=\sum_{i=0}^{\infty}\sum_{j=0}^{b^i-1} c_{i,j}^*\phi_{i,j}. \tag{3}$$

Here the coefficients $c_{i,j}^*$ are defined with the help of auxiliary functions g_k and the recurrent relations:

$$g_0=g,\quad g_{k+1}=g_k-\sum_{j=0}^{b^k-1} c_{k,j}^*\phi_{k,j},\ k\geq 0, \tag{4}$$

$$c_l^*=c_{b^k+j}^*=c_{k,j}^*=\begin{cases}\left[\frac{1}{\alpha_k|\Delta_{k,j}|}\int\limits_{\Delta_{k,j}} g_k dt\right], \text{ if } \int\limits_{\Delta_{k,j}} g_k dt\geq 0;\\ \left[\frac{1}{\alpha_k|\Delta_{k,j}|}\int\limits_{\Delta_{k,j}} g_k dt\right]+1, \text{ if } \int\limits_{\Delta_{k,j}} g_k dt< 0,\end{cases} \tag{5}$$

where $[a]$ denoted the integer part of the number a.

Let $\chi_\Delta(t)$ be the characteristic function of the set Δ.

Lemma 1. *For any stepwise function of the form $R(t)=\sum_{j=0}^{b^k-1} c_{k,j}\cdot\chi_{\Delta_{k,j}}(t)$, where $c_{k,j}\in\mathbb{R}$, $k=0,1,2,...$, $j=0,1,...,b^k-1$, consider the sum $P(t)=\sum_{j=0}^{b^k-1} c_{k,j}^*\phi_{k,j}$ where $\phi_{k,j}$ are defined by (1) with the same ϕ as in (2), and*

$$c_{k,j}^*=\begin{cases}\left[\frac{1}{\alpha_k}c_{k,j}\right], \text{ if } c_{k,j}\geq 0,\\ \left[\frac{1}{\alpha_k}c_{k,j}\right]+1, \text{ if } c_{k,j}<0,\end{cases}$$

Then

$$\|R(t)-P(t)\|_p\leq\alpha_k,\ 1\leq p<\infty,\qquad \left\|\sum_{j=0}^{m} c_{k,j}^*\phi_{k,j}(t)\right\|_p\leq\|R(t)\|_p+\alpha_k,\ 0\leq m\leq b^k-1.$$

Proof. Indeed,

$$\left\|\sum_{j=0}^{b^k-1} c_{k,j}\chi_{\Delta_{k,j}}(t)-\sum_{j=0}^{b^k-1} c_{k,j}^*\phi_{k,j}(t)\right\|_p\leq\left\|\sum_{j=0}^{b^k-1}\left(c_{k,j}-\alpha_k c_{k,j}^*\right)\chi_{\Delta_{k,j}}(t)\right\|_p$$

$$=\left\|\sum_{j=0}^{b^k-1}\alpha_k\left(\frac{1}{\alpha_k}c_{k,j}-c_{k,j}^*\right)\chi_{\Delta_{k,j}}(t)\right\|_p\leq\alpha_k.$$

Then

$$\left\|\sum_{j=0}^{m} c_{k,j}^{*}\phi_{k,j}(t)\right\|_p \leq \left\|\sum_{j=0}^{b^k-1} c_{k,j}^{*}\phi_{k,j}(t)\right\|_p \leq \|R(t)-P(t)\|_p + \|R(t)\|_p$$

$$\leq \|R(t)\|_p + \alpha_k, \quad 0 \leq m \leq b^k-1.$$

□

Theorem 1. *Let an arbitrary function $g \in L_p(0,1)$, $1 \leq p < \infty$. Then series (3) with respect to system (1) with the generating function ϕ as in formula (2) converges in the norm of the space $L_p(0,1)$, $1 \leq p < \infty$, to $g(t)$, i.e. $\|g - \sum_{l=0}^{m} c_l^{*}\phi_l\|_p \to 0, \quad m \to \infty$.*

Proof. Let $g_0 = g$. Then, by induction, we construct sequences of stepwise functions $S_k(t)$, $k \geq 0$, functions g_k, $k \geq 0$, and linear combinations $\sum_{j=0}^{b^k-1} c_{k,j}^{*}\phi_{k,j}(t)$, $k \geq 0$, so that takes place (4) - (5).

Let now $f(t) \in L_p(0,1)$, $1 \leq p < \infty$, and

$$S_k(f) = \sum_{l=0}^{b^k-1} p_{k,l}\cdot\phi_{k,l}(t), \quad p_{k,l} = \frac{1}{\alpha_k\,|\Delta_{k.l}|}\int\limits_{\Delta_{k,l}} f(t)dt, \quad k \geq 0,\ l = 0,1,...,b^k-1.$$

Then

$$S_k(g_k) = \sum_{l=0}^{b^k-1} c_{k,l}\cdot\chi_{\Delta_{k,l}}(t) = \sum_{l=0}^{b^k-1} \frac{1}{\alpha_k} c_{k,l}\cdot\phi_{k,l}(t), \quad S_k^{*}(g_k) = \sum_{l=0}^{b^k-1} c_{k,l}^{*}\cdot\phi_{k,l}(t).$$

Lemma 1 implies

$$\|S_k(g_k) - S_k^{*}(g_k)\|_p = \left\|\sum_{l=0}^{b^k-1}\left(\frac{1}{\alpha_k}c_{k,l} - c_{k,l}^{*}\right)\phi_{k,l}(t)\right\|_p \leq \alpha_k.$$

Consequently,

$$\|g_{k+1}\|_p \leq \|g_k - S_k(g_k)\|_p + \left\|S_k(g_k) - \sum_{l=0}^{b^k-1} c_{k,l}^{*}\phi_{k,l}(t)\right\|_p \leq \|g_k - S_k(g_k)\|_p + \alpha_k, \tag{6}$$

$$S_k(g_k) = S_k\left(g - \sum_{i=0}^{k-1}\sum_{j=0}^{b^i-1} c_{i,j}^{*}\phi_{i,j}(t)\right) = S_k(g) - S_k\left(\sum_{i=0}^{k-1}\sum_{j=0}^{b^i-1} c_{i,j}^{*}\phi_{i,j}(t)\right).$$

By the definition of system (1)–(2), we have

$$\sum_{i=0}^{k-1}\sum_{j=0}^{b^i-1} c_{i,j}^{*}\phi_{i,j}(t) = \sum_{l=0}^{b^k-1} d_{k,l}\phi_{k,l}(t).$$

Then

$$S_k\left(\sum_{i=0}^{k-1}\sum_{j=0}^{b^i-1} c_{i,j}^{*}\psi_{i,j}(t)\right) = S_k\left(\sum_{l=0}^{b^k-1} d_{k,l}\phi_{k,l}(t)\right) = \sum_{l=0}^{b^k-1} d_{k,l}\phi_{k,l}(t) = \sum_{i=0}^{k-1}\sum_{j=0}^{b^i-1} c_{i,j}^{*}\phi_{i,j}(t).$$

Therefore, using ([11], pp. 74–75), we obtain

$$\|g_{k-1}-S_{k-1}(g_{k-1})\|_p=\left\|g-\sum_{i=0}^{k-2}\sum_{j=0}^{b^i-1}c^*_{i,j}\phi_{i,j}(t)-S_{k-1}\left(g-\sum_{i=0}^{k-2}\sum_{j=0}^{b^i-1}c^*_{i,j}\phi_{i,j}(t)\right)\right\|_p$$
$$\leq\left\|g-S_{k-1}(g)-S_{k-1}\left(\sum_{i=0}^{k-2}\sum_{j=0}^{b^i-1}c^*_{i,j}\phi_{i,j}(t)\right)-\sum_{i=0}^{k-2}\sum_{j=0}^{b^i-1}c^*_{i,j}\phi_{i,j}(t)\right\|_p$$
$$=\|g-S_{k-1}(g)\|_p\leq c_p\omega_p\left(\frac{1}{b^{k-1}},g\right) \qquad (7)$$

where $\omega_p\left(\frac{1}{b^{k-1}},g\right)$ is the integral continuity module of the function $g\in L_p(0,1)$, $1\leq p<\infty$, with the step $\frac{1}{b^{k-1}}$.

Thus, from (6) and (7) we obtain

$$\|g_k\|_p\leq\alpha_{k-1}+c_p\omega_p\left(\frac{1}{b^{k-1}},g\right)\to 0,\quad k\to\infty. \qquad (8)$$

Now we make sure that the constructed series $\sum_{l=0}^{\infty}c^*_l\cdot\phi_l(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{b^i-1}c^*_{i,j}\cdot\phi_{i,j}(t)$ converges to g in L_p. Let $n>0$ be some large number. Fix $k\geq 2$ such that $0\leq j_0\leq b^k-1$, $k\geq 2$, and

$$\sum_{l=0}^{n}c^*_l\cdot\phi_l(t)=\sum_{i=0}^{k-1}\sum_{j=0}^{b^i-1}c^*_{i,j}\cdot\phi_{i,j}(t)+\sum_{j=0}^{j_0}c^*_{k,j}\cdot\phi_{k,j}(t).$$

Using Lemma 1, (7) and (8), we have

$$\left\|g-\left(\sum_{i=0}^{k-1}\sum_{j=0}^{b^i-1}c^*_{i,j}\cdot\phi_{i,j}(t)+\sum_{j=0}^{j_0}c^*_{k,j}\cdot\phi_{k,j}(t)\right)\right\|_p\leq\|g_k\|_p+\left\|\sum_{j=0}^{j_0}c^*_{k,j}\cdot\phi_{k,j}(t)\right\|_p\leq$$
$$\|g_k\|_p+\alpha_k+\|S_k(g_k)\|_p\leq 2\|g_k\|_p+\alpha_k+\|g_k-S_k(g_k)\|_p\leq$$
$$2c_p\omega_p\left(\frac{1}{b^{k-1}},g\right)+c_p\omega_p\left(\frac{1}{b^k},g\right)+\alpha_k+2\alpha_{k-1}\to 0,\ n\to\infty.$$

The theorem is proved. □

Now assume that the function ϕ is of a more general type. We give two lemmas that will be needed below.

Lemma 2. *([1]). Let $\phi\in L_p(0,1)$, $1\leq p<\infty$, $\int_0^1\phi(t)dt=\delta\neq 0$. Then there is a constant $\lambda_0\neq 0$ such that*

$$\|1-\lambda_0\phi(t)\|_p=\sigma_0<1. \qquad (9)$$

Lemma 3. *Let a function $\phi\in L_p(0,1)$, $1\leq p<\infty$, satisfy (9). Fix $\sigma\in(\max\{\frac{1}{2},\sigma_0\};1)$. Then, for any stepwise function $S(t)=\sum_{l=0}^{b^k-1}c_{k,l}\cdot\chi_{\Delta_{k,l}}(t)$, the finite sum $h(t)=\sum_{l=0}^{b^k-1}c_{k,l}\cdot\frac{1}{\alpha_k}\lambda_0\cdot\phi_{k,l}(t)$ (here $\phi_{k,l}(t)$ is the same as in (1)) satisfies the conditions*

$$\|S(t)-h(t)\|_p\leq\sigma\|S(t)\|_p,\quad\left\|\sum_{l=0}^{m}c_{k,l}\cdot\frac{1}{\alpha_k}\lambda_0\cdot\phi_{k,l}(t)\right\|_p\leq(1+\sigma)\|S(t)\|_p,\ 0\leq m\leq b^k-1.$$

Proof. Indeed,

$$\|S(t)-h(t)\|_p^p \le \left\| \sum_{l=0}^{b^k-1} c_{k,l}\cdot \chi_{\Delta_{k,l}}(t) - \sum_{l=0}^{b^k-1} c_{k,l}\cdot \frac{1}{\alpha_k}\lambda_0\cdot \phi_{k,l}(t)\right\|_p^p$$

$$\le \int_0^1 |1-\lambda_0\phi(t)|^p dt \sum_{l=0}^{b^k-1} |c_{k,l}|^p |\Delta_{k,l}| \le \sigma^p \|S(t)\|_p^p ,$$

$$\left\|\sum_{l=0}^{m} c_{k,l}\cdot \frac{1}{\alpha_k}\lambda_0\cdot \phi_{k,l}(t)\right\|_p \le \left\|\sum_{l=0}^{b^k-1} c_{k,l}\cdot \frac{1}{\alpha_k}\lambda_0\cdot \phi_{k,l}(t) - S(t) + S(t)\right\|_p$$

$$\le (1+\sigma)\|S(t)\|_p , \quad 0\le m\le b^k-1.$$

Lemma 3 is proved. □

Let now $\phi \in L_p(0,1)$, $1\le p<\infty$, $\int_0^1 \phi(t)dt = \delta \ne 0$,

$$\{\phi_{n,k}\} = \lambda_0\cdot\alpha_n\{\phi(b^n t-k)\} = \{\phi_l\}, \quad n=0,1,2,..., \ k=0,1,...,b^n-1, \ l=b^n+k. \tag{10}$$

Theorem 2. *For any function $g\in L_p(0,1)$, $1\le p<\infty$, the series $\sum_{l=1}^{\infty} c_l^*\cdot\phi_l(t)$ with respect to system (10), where $c_l^*\in Z$, converges in the space $L_p(0,1)$, $1\le p<\infty$, to $g(t)$, i.e.,*

$$\left\| g - \sum_{l=0}^{m} c_l^*\cdot \phi_l(t)\right\|_p \to 0, \quad m\to\infty.$$

Proof. Let $g_0 = g$. Using Lemma 3, we construct, by induction, a sequence of stepwise functions $R_k(t)$, $k\ge 1$, numbers $n_k\,(n_1\ge 4)$, functions g_k, $k\ge 1$, and linear combinations $\sum_{j=0}^{b^{n_k}-1} c^*_{n_k,j}\cdot \phi_{n_k,j}(t)$, $k\ge 1$, such that

$$g_k = g_{k-1} - \sum_{j=0}^{b^{n_k}-1} c^*_{n_k,j}\cdot\phi_{n_k,j}(t),$$

$$\|g_{k-1}-R_k(t)\|_p < \frac{1}{2^{k+2}}, \ \alpha_{n_k} \le \frac{1}{2^{k+2}}, \tag{11}$$

where

$$R_k(t) = \sum_{l=0}^{b^{n_k}-1} c_{n_k,l}\cdot \chi_{\Delta_{n_k,l}}(t), \quad k\ge 1, \quad c_{n_k,l} = \frac{1}{|\Delta_{n_k,l}|}\int_{\Delta_{n_k,l}} g_{k-1}dt, \ l=0,1,...,b^{n_k}-1,$$

$$c^*_{n_k,l} = \begin{cases} \left[\frac{1}{\alpha_{n_k}} c_{n_k,l}\right], & \text{if } c_{n_k,l}\ge 0,\\ \left[\frac{1}{\alpha_{n_k}} c_{n_k,l}\right]+1, & \text{if } c_{n_k,l}<0.\end{cases}$$

Note that we choose n_k such that relations (11) hold.

Let

$$S_k(g_{k-1}) = R_k(t) = \sum_{l=0}^{b^{n_k}-1} c_{n_k,l}\cdot\chi_{\Delta_{n_k,l}}(t), \quad S_k^*(g_{k-1}) = \sum_{l=0}^{b^{n_k}-1} c^*_{n_k,l}\cdot \phi_{n_k,l}(t),$$

$$R_k^*(t) = \sum_{l=0}^{b^{n_k}-1} \alpha_{n_k} \cdot c_{n_k,l}^* \cdot \chi_{\Delta_{n_k,l}}(t).$$

Using Lemma 3, we have

$$\|R_k^*(t) - S_k^*(g_{k-1})\|_p < \sigma \|R_k^*(t)\|_p , \tag{12}$$

$$\left\| \sum_{l=0}^{m} c_{n_k,l}^* \cdot \phi_{n_k,l}(t) \right\|_p \le (1+\sigma) \|R_k^*(t)\|_p , \; 0 \le m \le b^{n_k} - 1.$$

Now we make sure that the constructed series $\sum_{l=0}^{\infty} c_l^* \cdot \phi_l(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{b^{n_i}-1} c_{n_i,j}^* \cdot \phi_{n_i,j}(t)$ converges to the function g in the metric of the space L_p. Let $n > 0$ be a large number. Define $k \ge 2$ such that $0 \le j_0 \le b^{n_k} - 1$, $k \ge 2$, and

$$\sum_{l=0}^{n} c_l^* \cdot \phi_l(t) = \sum_{i=0}^{k-1} \sum_{j=0}^{b^{n_i}-1} c_{n_i,j}^* \cdot \phi_{n_i,j}(t) + \sum_{j=0}^{j_0} c_{n_k,j}^* \cdot \phi_{n_k,j}(t).$$

Then from Lemma 1 and (11) we obtain

$$\begin{aligned} \left\| g - \left(\sum_{i=0}^{k-1} \sum_{j=0}^{b^{n_i}-1} c_{n_i,j}^* \cdot \phi_{n_i,j}(t) + \sum_{j=0}^{j_0} c_{n_k,j}^* \cdot \phi_{n_k,j}(t) \right) \right\|_p &\le \|g_{k-1}\|_p + \left\| \sum_{j=0}^{j_0} c_{n_k,j}^* \cdot \phi_{n_k,j}(t) \right\|_p \\ \le \|g_{k-1}\|_p + 2\|R_k^*(t)\|_p &\le \|g_{k-1}\|_p + 2\|R_k^*(t) - R_k(t)\|_p + 2\|R_k(t)\|_p \\ \le \|g_{k-1}\|_p + 2\alpha_{n_k} + 2\|g_{k-1} - R_k(t)\|_p + 2\|g_{k-1}\|_p &\le 3\|g_{k-1}\|_p + 2\alpha_{n_k} + \frac{1}{2^{k+1}}. \end{aligned}$$

Using Lemmas 1, 3 and formulas (11), (12), we obtain

$$\begin{aligned} \|g_{k-1}\|_p &\le \left\| g_{k-2} - R_{k-1}^*(t) \right\|_p + \left\| R_{k-1}^*(t) - S_{k-1}^*(g_{k-2}) \right\|_p \le \|g_{k-2} - R_{k-1}(t)\|_p \\ &+ \left\| R_{k-1}^*(t) - R_{k-1}(t) \right\|_p + \sigma \left\| R_{k-1}^*(t) \right\|_p \le \frac{1}{2^k} + \sigma \left\| R_{k-1}^*(t) - R_{k-1}(t) \right\|_p \\ &+ \sigma \|R_{k-1}(t)\|_p \le \frac{1}{2^k} + \sigma \cdot \alpha_{n_{k-1}} + \sigma \left(\frac{1}{2^{k+1}} + \|g_{k-2}\|_p \right) \le \frac{1}{2^{k-1}} + \sigma \|g_{k-2}\|_p . \end{aligned}$$

Consequently,

$$\|g_k\|_p \le \frac{1}{2^k} + \frac{\sigma}{2^{k-1}} + \ldots + \frac{1}{2}\sigma^{k-1} + \sigma^k \|g\|_p \le k\sigma^k + \sigma^k \|g\|_p \to 0, \quad k \to \infty.$$

Thus,

$$\left\| g - \sum_{l=0}^{n} c_l^* \phi_l \right\|_p \to 0, \quad n \to \infty.$$

□

Theorem 3. *Let $g \in C[0,1]$. The series (3) by the system (1) with generating function ϕ as in (2) converges in the space $C[0,1]$ to $g(t)$, i.e., $\|g - \sum_{l=0}^{m} c_l^* \phi_l\|_{C[0,1]} \to 0, \quad m \to \infty$.*

Proof. Repeating the considerations of Lemma 1 and Theorem 1, we obtain the proof of Theorem 3 with the estimate $3\omega\left(\frac{1}{b^k}, g\right)$ ([11], p. 74) in place of $c_p\omega_p\left(\frac{1}{b^k}, g\right)$ in formulas (7) and (8). □

Remark 1. *We call the expansions of functions in Theorems 1–3 the integer expansions of functions in the corresponding spaces.*

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