Power Filtration On Morphisms of Formal Group Law

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Abstract—The height filtration on the stack of formal groups $\mathcal{M}_{FG}$ is well known. We explore analogous filtration on a set of morphisms of formal group laws, which extends to the stack $\mathcal{M}_{FG}$. It is correctly defined colimit object for this filtration which can be identified with the colimit $\mathcal{M}_{FG,\infty}$. As a corollary we prove explicitly density of additive formal group in any group law.

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1. INTRODUCTION

Formal group laws is a powerful tool for local problems in number theory. It is classically known that kernels of isogenies are the main reason for all arithmetic fenomenas concerning formal groups. For instance, p-divisible groups were invented exactly as a generalization of the Tate module (kernel of ∞-isogeny) of formal groups, not formal groups itself. Also the definition of the stack $\mathcal{M}_{FG}$ [1] incarnates the idea of high importance only of the isogenies among all homomorphisms of formal group laws.

It was shown that additive group isomorphic modulo a Tate module to a subgroup of any formal group by highly abstract reasonings [1]. We give absolutely direct method of constructing such groups.

Section 2 contains preliminaries for further sections. In section 3 we following J.Lubin develop lattice—finite subgroups correspondence theory on the level of full functors. In section 4 we develop power filtration on morphisms of formal group laws, which in case of endomorphism ring give us an explicit subgroup isomorphic to the additive group law.

2. PRELIMINARIES

We fix a prime number p and a local field K, which we assume to be a finite extension of the field of p-adic numbers $\mathbb{Q}_p$. Ring of integers and maximal ideal of the field K we denote by $\mathcal{O}_K$ and $\mathfrak{m}_K$ respectively. It is known that we can continue discrete valuation $v_{\mathbb{Q}_p}$ from $\mathbb{Q}_p$ to K in the unique way. Then by a uniformizer π_K of maximal ideal $\mathfrak{m}_K$ we mean any element with valuation equals 1.

By FGL we denote the category of all one dimensional formal group laws [2]. Its quotient (in stack terminology) by a group of all isomorphisms we get $\mathcal{M}_{FG}$. We assume that all formal group laws are defined over the ring $\mathcal{O}_K$. For any such group law F and maximal ideal $\mathfrak{m}_L$ of any extension L/K we form an actual group — group of points $F(\mathfrak{m}_L)$ with addition given by $a +_F b := F(a, b)$.

We refer to [3] for structural results on ring of endomorphisms of formal group law. For any extension L/K abelian group $F(\mathfrak{m}_L)$ has a natural $End_{\mathcal{O}_K}(F)[Gal(L/K)]$—module structure.

Also we remind that the notion of height [4] is well-defined for formal group laws. Moreover, the height induces a natural filtration not only on the category FGL, but on a stack $\mathcal{M}_{FG}$

$$FGL_{\geq 1} \supset FGL_{\geq 2} \supset \cdots \supset FGL_{\infty},$$

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$$\mathcal{M}_{FG,\geq 1} \supset \mathcal{M}_{FG,\geq 2} \supset \cdots \supset \mathcal{M}_{FG,\infty}.$$

By FGL_{fin} we denote a full subcategory of formal group laws of **finite** height. It is well known that all groups of infinite height are isomorphic to the additive group law $F_a(X,Y)$.

Set of homomorphisms between any two formal groups $Hom_{FGL}(F,G)$ is not empty if and only if the corresponding formal groups F and G have the same height.

3. LATTICE–SUBGROUPS CORRESPONDENCE FOR FORMAL GROUP LAWS

The most complete exposition on the results in classical lattice–subgroups correspondence can be found in the paper [5] of J.Lubin. We notice that results of this section is just an incarnation of the general idea of correspondence between sets and corresponding annihilators in a context of formal group laws.

The only endomorphisms of any formal group law $F(X,Y)$ over $\mathcal{O}_K$ without inverse are of the form $[\pi^t]_F(X)$ for $t \in \mathbb{N}$, where π is a prime element of an endomorphisms ring[1] $End_{\mathcal{O}_K}(F) = End(F) \cap \mathcal{O}_K$. Kernels of this endomorphisms $W_F^m(K) = \{x \in \mathfrak{m}_K : [\pi^m]_F(x) = 0\}$ actually are $End_{\mathcal{O}_K}(F)$ and $Gal(L/K)$–modules. Union of all this kernels we denote by $W_F^\infty(K)$.

Let $V(F)$ be a space of all sequences $(a_0, a_1, \dots)$ such that $[\pi^k]_F(a_0) = 0$ for some $k \geq 1$ and $a_i = [p]_F(a_{i+1})$. The Tate module $T(F)$ of formal group law is a subspace of $V(F)$ with a_0 equals 0.

First form of the following theorem which describes a correspondence between finite subgroups in F and sublattices of $V(F)$ appeared in [5]. We present a natural generalization of the theorem to a level of equivalence of the corresponding functors. The striking difference of the statement from the original one is that sets of morphisms are expanded to a maximal one.

Theorem 1. *We define two functors $\mathcal{L}.$ and $\mathcal{F}.$ from the category FGL_{fin}, which on formal group F equals to*

- *a category $\mathcal{L}_F$ as a category of sublattice L of $V(F)$, which is contained and contains in p-homotheties of $T(F)$ with morphisms induced by $W_F^\infty(K)$–linear maps of $V(F)$;*
- *a category $\mathcal{F}_F$ as a category of all finite subgroups of a formal group law F with morphisms induced by endomorphisms of formal group F.*

Then functors $\mathcal{L}.$ and $\mathcal{F}.$ are naturally equivalent.

Natural equivalence is given on objects by taking any lattice L to values set of the first non-zero coordinates of all points from L.

Proof. For any morphisms between lattices we just set corresponding morphisms on values groups of first non-zero coordinate.

But for homomorphisms of finite subgroups we should use the expanded form of the Theorem 4 from [2], i.e. any such homomorphism we can present as an "analytic" homomorphism (coming from the formal group structure on $F(X,Y)$) and then pull it to a morphism of lattices. □

[1] Here and below $End(F)$ is an absolute endomorphisms ring [3] for formal group law $F(X,Y)$.

4. HEIGHT FILTRATION ON HOMOMORPHISMS

In this section we consider all formal group law over algebraically closed field K^{alg} for simplicity. However, for any fixed formal group law all reasonings are the same for the fraction field $End(F) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Also we notice that in this section categories FGL and $\mathcal{M}_{FG}$ are replaceable, because we work with isogeny structures only.

For any fixed formal group law $F(X,Y)$ and any natural number m we consider a category $\mathcal{C}_F(m)$ which consists of formal group law $G(X,Y)$ together with a morphism $g : F(X,Y) \to G(X,Y)$ with a crucial condition

$$Ker[\pi^m]_F \subset Ker(g).$$

Morphisms in $\mathcal{C}_F(m)$ just coincide with morphisms in a coslice category $F \downarrow FGL$.

Then from the Theorem 1 we know that the object $[\pi^m]_F : F(X,Y) \to F(X,Y)$ is initial for $\mathcal{C}_F(m)$. So, for instance, for any $(G(X,Y), g)$ from $\mathcal{C}_F(m)$ there exists a map $\bar{g} : F(X,Y) \to G(X,Y)$ such that $g = \bar{g} \circ [\pi^m]_F$.

Definition 1. *We define an endo-analog of $\mathcal{C}_F(m)$ as a subcategory*[2] *$\mathcal{E}_F(m)$ with objects $(F(X,Y), f : F(X,Y) \to F(X,Y))$. Then a natural filtration*

$$\mathcal{E}_F(0) \subset \mathcal{E}_F(1) \subset \mathcal{E}_F(2) \subset \dots$$

coming from the one on $\{\mathcal{C}_F(m)\}_m$define ***height filtration*** *on the ring $End(F)$.*

Now we can formulate and prove the main theorem.

Theorem 2. *The colimit $\mathcal{E}_F(\infty)$ is correctly defined and equivalent to $End(F_a)$.*

Proof. The derivation homomorphism $c : End(F) \to K^{alg}$ is injective, so we can define a map

$$c^{-1} : Frac(\, c(End(F))\,) \to End(F) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

by $c^{-1}(a) := \frac{[a\cdot\pi^w]_F(x)}{\pi^w}$ for $w + v_L(a) = 0$. But natural inclusion into $\mathcal{E}_F(m)$ sends such a into $\frac{[a\cdot\pi^m]_F(x)}{\pi^m}$. So after an elementary calculation[3]

$$\lim_{m\to\infty} \frac{F(X,Y) - ([\pi^m]_F(X) + [\pi^m]_F(Y))}{\pi^m} = X + Y,$$

we have that all this maps can be continued by $\lim\limits_{m\to\infty} \frac{[\pi^m\cdot -]_F}{\pi^m}$ – operation to maps into the additive group $F_a(X,Y)$. Then by the very construction we see that $F_a(X,Y)$ is isomorphic to $\mathcal{E}_F(\infty)$.

Finally, we remark that all morphisms are commute with any isomorphisms of formal group $F(X,Y)$, so it is true not only in FGL, but also in $\mathcal{M}_{FG}$. □

The following proposition is a corollary of the proof.

Corollary 1. *The additive group $F_a(X,Y)$ is isomorphic to a dense subgroup in any formal group law $F(X,Y)$.*

Proof. From the previous it can be seen that

$$F(X,Y)|_{\mathfrak{m}_{K^{alg}} \setminus W_F^\infty(K^{alg})} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is isomorphic to the additive group $F_a(X,Y)$ restricted to the same set by the map $\lim\limits_{m\to} \frac{[\pi^m]_F(X)}{\pi^m}$. □

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[2] We identify it with its projection on the second coordinate, thereby identify with subcategory of $End(F)$.

[3] The limit is taken in π-adic topology.

So powers of isogenies $[\pi^m]_F$ can be considered as an analytic approximation sequence for logarithm log_F on the punctured set $\mathfrak{m}_{K^{alg}} \backslash W_F^{\infty}(K^{alg})$.

Also the last proposition is an incornation of an abstract idea that after localization in the set of all isogenies of any formal group law $F(X,Y)$ we get a representer of $F(X,Y)$ in the ∞-component $\mathcal{M}_{FG,\infty}$.

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