Configurations On Curvilinear Three-Web Boundaries

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Abstract—On the boundaries of the first and second kind of curvilinear three-web, configurations are defined that are analogous to the known Thomsen and Bol configurations. This makes it possible to find the relative invariants defined on the boundaries corresponding to the closure of these configurations.

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1. INTRODUCTION

Let $\lambda_\alpha, \alpha, \beta, \gamma = 1, 2, 3$, be 3 smooth families of curves in the plane, D be the maximum possible domain of the plane such that each of its points has a neighborhood in which a) each of the families λ_α forms a foliation; b) families of lines have no singularities and are pairwise transversal. Then we say that the families λ_α form a three-web W in the domain D, and the domain D is called the domain of definition of the 3-web or the domain of its transversality. Points the families λ_α are defined at, but the lines of the families or families themselves have singularities, form the set of singularities of the three-web W. In particular, points the leaves of 3-web W are tangent at do not fall into the domain D, and the leaves common to two or three families λ_α do not fall. In [1] we introduced boundary curves of the first kind $\Gamma_{\alpha\beta}$, at the points of which the lines of the families λ_α and λ_β are tangent. In particular, if the lines of two families coincide, then this common line is called the boundary curve of the second kind.

The differential topological theory of three-webs studies them up to local diffeomorphisms preserving only incidence of points or lines of a three-web [2]. Therefore the main object of observation in this theory are configurations formed by three-web lines. Reidemeister, Thomsen, Blaschke, and Bol introduced various configurations (R, T, H, B) later generalized to multidimensional three-webs. A necessary and sufficient condition for the closure of all sufficiently small configurations of the same type is equivalent to the vanishing of some relative differential invariant that is a tensor. Thus the classification of three-webs by the closure conditions (and not only on them) is connected with the finding of relative differential invariants. [1]

Note that in terms of relative differential invariants, many important problems in the web theory have been solved, for example, the well-known problem of linearizability of three-webs, see [3].

The main relative invariants of a curvilinear three-web are its curvature expressed in terms of derivatives up to the third order of the web function, and the web curvature covariant derivatives [1], [2]. The geometric meaning of curvature is that it determines the main part of the obstuction to the closure of three-webs configurations. If the curvature is zero in the domain D, then all the classical configurations on the curvilinear three-web are closed. Such a web is equivalent to a web formed by three families of parallel straight lines and is called parallelizable (hexagonal, regular).

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[1] Absolute invariants are introduced in the framework of the differential topological theory of webs [1], but they do not have a geometric interpretation. Therefore the classification of webs in accordance with absolute invariants is difficult since there is no way to describe the geometric properties of the corresponding classes.

But the classical (sufficiently small) configuration of the web and its curvature are defined only in the domain D, that is, outside its boundaries. Meanwhile, the boundaries play an important role in the curvilinear web theory: as shown in [1], a three-web is regular if and only if its boundary curves are lines of this web. Therefore the study of the geometry of webs near the boundaries seems to us an urgent problem.

In the recent paper [4], absolute differential invariants of curvilinear three-webs defined on its boundaries were introduced. In this paper we introduce an analogue of the hexagonal configuration defined at the points of the boundary curve of the first kind, an analogue of the right Bol configuration for the boundary curve of the second kind, and we find the corresponding relative differential invariants, that are, like the curvature, the main part of the obstruction to closure of the introduced configurations. Here, we use a simple but effective technique of computations used by Chern in [5] to find the necessary and sufficient conditions for the closure of classical figures on a multidimensional three-web.

All functions considered in this paper are assumed to be real analytic.

2. PENTAGONAL CONFIGURATIONS ON THE BOUNDARY OF THE FIRST KIND

It is known [1] that every three-web is equivalent (locally diffeomorphic) to a three-web whose two families form a Cartesian net. We can always assume that in a neighbourhood of a point in the domain D, the third family is the level lines of some function $f(x, y)$. Then the web equation that connects the parameters of the web lines through a point has the form $z = f(x, y)$. The function f is called the web function.

In Fig. 1 we can see the Thomsen configuration T. We are going to construct its analogue. Here the lines of the first and second families form the Cartesian net, and the lines of the third family represented by inclined lines. The vertical, horizontal and slanted lines are marked by the parameters x_α, y_α and z_α, respectively. The figure T is constructed, for example, as follows. We take two arbitrary lines x_1 and y_1 from the first and second families, then we draw the sloping lines z_1 and z_2 sufficiently close to the point $x_1 \cap y_1$. Through the resulting intersection points, we draw the vertical and horizontal lines x_2 and y_2, x_3 and y_3, as shown in Fig. 1. We obtain points A and B. The resulting configuration is called the Thomsen figure or T. If the points A and B lie on one line of the third family of web, then we say that the figure T is closed.

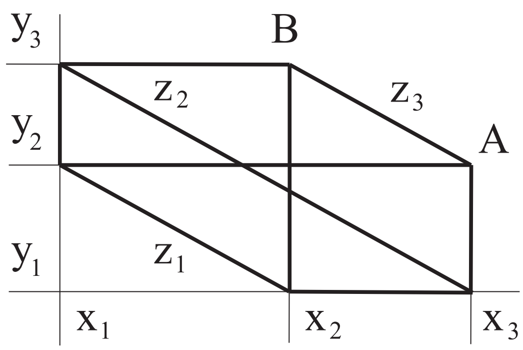

Figure 1. Thomsen configuration

In particular, if the lines x_2, y_2, and z_2 pass through a common point, then the Thomsen configuration is called a hexagonal configuration or figure H. One of the main theorems of the three-web theory is the following: the three-web W is regular if and only if all sufficiently small figures H are closed on it.

Since the third family of web lines consists of level lines of the function f, the boundary Γ_{13} (at the points the lines of the first and third families are tangent to), is given by the equation

$$\frac{\partial f}{\partial y} = 0. \tag{1}$$

Let O be an arbitrary point on Γ_{13}. We choose the coordinates so that the point O has zero coordinates. Then in a neighbourhood of this point the curve ℓ of the third family lies on one side of the vertical line $x = 0$ that is a line of the first family of web W, see Fig. 2. We parametrize the third family so that the equation of the line ℓ has the form $f(x, y) = 0$. Let $p(x, 0)$ be an arbitrary point on the line of the second family $y = 0$ that is sufficiently close to O (see Fig. 2). Then the vertical line of the web through p intersects ℓ at the points $p_1(x, y_1)$ and $p_2(x, y_2)$. The horizontal web lines through p_1 and p_2 intersect the y axis at the points p_3 and p_4, respectively. Let the curves of the third family through p_3 and p_4 intersect the axis x at the points $A(x_1, 0)$ and $B(x_2, 0)$, respectively.

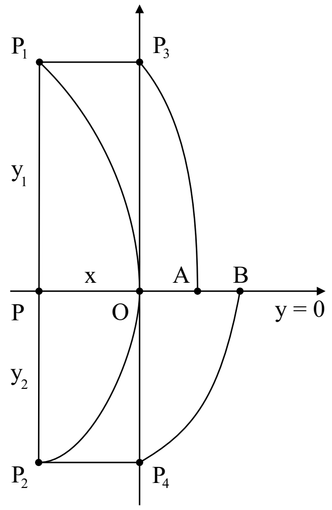

Figure 2. Pentagonal configuration P_{13}

The constructed pentagonal configuration is denoted by P_{13}. It can be regarded as an analogue of the configuration H for the boundary Γ_{13}.

We estimate the difference $x_2 - x_1$ up to the fourth order. First of all, note that, in view of (1)

$$\frac{\partial f}{\partial y}(0, 0) = 0.$$

Therefore in a neighbourhood of the point O, we have the expansion

$$\begin{aligned} f(x, y) &= f_x(0, 0)x + \frac{1}{2} f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + f_{yy}(0, 0)y^2 \\ &+ \frac{1}{6} f_{xxx}(0, 0)x^3 + \frac{1}{2} f_{xxy}(0, 0)x^2 y + \frac{1}{2} f_{xyy}(0, 0)xy^2 + \frac{1}{6} f_{yyy}(0, 0)y^3 + O_4. \end{aligned} \tag{2}$$

Since the points O and p_1 lie on the line ℓ, we have $f(0, 0) = f(x, y_1)$ or

$$\begin{aligned} 0 &\sim f_x(0, 0)x + \frac{1}{2} f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy_1 + f_{yy}(0, 0)(y_1)^2 \\ &+ \frac{1}{6} f_{xxx}(0, 0)x^3 + \frac{1}{2} f_{xxy}(0, 0)x^2 y_1 + \frac{1}{2} f_{xyy}(0, 0)x(y_1)^2 + \frac{1}{6} f_{yyy}(0, 0)(y_1)^3. \end{aligned} \tag{3}$$

The symbol $\sim$ means that the calculations are carried out up to the fourth order terms. In what follows, we write f_x instead of $f_x(0, 0)$ and so on.

It is clear from (3) that we can put

$$x \sim -\frac{f_{yy}}{2f_x}(y_1)^2 + v(y_1)^3, \tag{4}$$

Substituting into (3) we find:

$$v \sim \frac{f_{xy}f_{yy}}{2(f_x)^2} - \frac{f_{yyy}}{6f_x}. \tag{5}$$

Similar conclusions are obtained for the point p_2:

$$x \sim -\frac{f_{yy}}{2f_x}(y_2)^2 + v(y_2)^3. \tag{6}$$

Let $f(x, y) = c$ be the line of the third family through the points $p_3(0, y_1)$ and $A(x_1, 0)$, then $f(0, y_1) = f(x_1, 0)$, or, by virtue of (2),

$$\frac{1}{2}f_{yy}(y_1)^2 + \frac{1}{6}f_{yyy}(y_1)^3 \sim f_x x_1 + \frac{1}{2}f_{xx}(x_1)^2 + \frac{1}{6}f_{xxx}(x_1)^3.$$

Hence we get $x_1 \sim \frac{1}{f_x}(\frac{1}{2}f_{yy}(y_1)^2 + \frac{1}{6}f_{yyy}(y_1)^3)$. Similarly we find $x_2 \sim \frac{1}{f_x}(\frac{1}{2}f_{yy}(y_2)^2 + \frac{1}{6}f_{yyy}(y_2)^3)$. From the last two equalities we have:

$$x_2 - x_1 \sim \frac{f_{yy}}{2f_x}((y_2)^2 - (y_1)^2) + \frac{f_{yyy}}{6f_x}((y_2)^3 - (y_1)^3). \tag{7}$$

On the other hand, from (4) and (6) we find $\frac{f_{yy}}{2f_x}((y_2)^2 - (y_1)^2) \sim v((y_2)^3 - (y_1)^3)$. Substituting into (7), taking into account (5) we obtain:

$$x_2 - x_1 \sim (v + \frac{f_{yyy}}{6f_x})((y_2)^3 - (y_1)^3) = \frac{f_{xy}f_{yy}}{2(f_x)^2}((y_2)^3 - (y_1)^3).$$

Thus the main part of the difference $x_2 - x_1$ is of order 3 and is determined by the quantity

$$\pi_{13} = \frac{f_{xy}f_{yy}}{(f_x)^2}, \tag{8}$$

where the values of the derivatives are calculated at an arbitrary point of the boundary Γ_{13}. It follows that π_{13} is a relative invariant of the three-web W on the boundary of the first kind Γ_{13}.

We proved

Theorem 1. *Let the three-web W be formed by the Cartesian net and the level lines of the function f. Then the main part of the obstruction to the closure of the figure P_{13} constructed on the boundary of the first kind Γ_{13} is determined by the relative invariant π_{13} computed at the points of this boundary by the formula (8).*

The similar relative invariant on the boundary Γ_{23} will be

$$\pi_{23} = \frac{f_{xy}f_{xx}}{(f_y)^2}.$$

3. ANALOGUE OF BOL CONFIGURATION ON THE BOUNDARY OF THE SECOND KIND

Let the line $x = 0$ be the common line of the first and third families, that is, the line is the boundary Γ_{13} of the second kind of the 3-web W, given by the Cartesian net and some family of lines $F(x, y, c) = 0$. Let the line $x = 0$ have the parameter $c = 0$, then

$$F(x, y, c) = x + a(x, y)c + b(x, y)c^2 + \dots . \tag{9}$$

We place the origin in an arbitrary point of the axis y and construct Fig. 3. We choose an arbitrary point $p(x, 0)$, draw a line of the third family through it, take an arbitrary point $p_1(x_1, y)$ on the latter, and take x and y close to zero. Next, on axis x we take the point $\bar{p}(-x, 0)$, draw the line of the third family through it, take the point $p_2(x_2, y)$ on it. Through the points p_1 and p_2 we draw the vertical lines of the web W, then we obtain the points $p_3(x_1, 0)$ and $p_4(x_2, 0)$, respectively. Through the points p and $\bar{p}$ we draw vertical lines of the web W, through the points p_3 and p_4 draw inclined lines. We obtain the points $p_5(x, y_1)$ and $p_6(-x, y_2)$, respectively.

The constructed configuration coincides with the well-known right Bol figure B_r [1]. The fundamental difference between this figure and the classical one is that in the latter the left and right parts are not separated by the boundary, that is, they lie in one connected part of the domain D of the 3-web W. The configuration in Fig. 3 we denote by $\mathcal{B}_{13}$. If the constructed configuration is closed, then $y_1 = y_2$. Therefore the main part of the difference $y_1 - y_2$ is a relative invariant of the 3-web W.

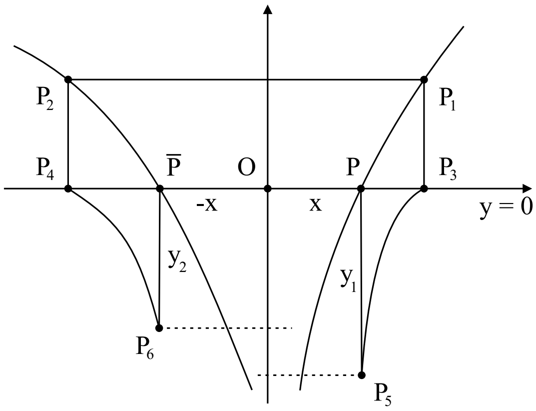

Figure 3. Analogue of the Bol configuration $\mathcal{B}_{13}$

We compute y_1 and y_2, assuming the figure $\mathcal{B}_{13}$ to be sufficiently small. This means that the quantities x, y defining this figure, and the parameters of the lines of the third family entering into it, should be assumed to be close to zero.

We write down the conditions for the pairs of points p and p_1, p_3 and p_5, $\bar{p}$ and p_2, p_4 and p_6 to belong to the same curve of the third family:

$$\begin{aligned} &(1) \quad F(x, 0, c_1) = 0, \quad F(x_1, y, c_1) = 0; \\ &(2) \quad F(-x, 0, c_2) = 0, \quad F(x_2, y, c_2) = 0; \\ &(3) \quad F(x_1, 0, c_3) = 0, \quad F(x, y_1, c_3) = 0; \\ &(4) \quad F(x_2, 0, c_4) = 0, \quad F(-x, y_2, c_4) = 0. \end{aligned} \tag{10}$$

It is clear from (9) that

$$F_c(x, y, 0) = a(x, y), \; F_x(x, y, 0) = 1,$$

all the other derivatives of F with respect to the variables x and y for $c = 0$ (that is, on the boundary Γ_{13}) being equal to zero.

Let the series expansion of $a(x, y)$ be written as

$$a(x, y) = a_0 + a_1 x + a_2 y + a_{11} x^2 + a_{12} xy + a_{22} y^2 + a_{111} x^3 + a_{112} x^2 y + a_{122} xy^2 + a_{222} y^3 + \dots. \tag{11}$$

We assume that $a_0 \neq 0$, that is, the quantities x and y are small relative to $a(x, y)$. Then, up to the second order members, we obtain from (9) and (10, 1): $x + a(x, 0)c_1 \sim 0, \quad x_1 + a(x_1, y)c_1 \sim 0$, and

$$x a(x_1, y) \sim x_1 a(x, 0). \tag{12}$$

We look for x_1 in the form $x_1 = u_0 + u_1 x + u_2 y + u_{11} x^2 + \dots$ Substituting in the previous equation and using (11), after some calculations we get the following formula.

Lemma 1.

$$x_1 \sim x + \frac{a_2}{a_0} xy + \frac{a_{12}}{a_0} x^2 y + \frac{a_{22}}{a_0} xy^2 + (\frac{a_{112}}{a_0} + \frac{a_2 a_{11}}{a_0^2}) x^3 y + (\frac{a_{122}}{a_0} + \frac{a_2 a_{12}}{a_0^2}) x^2 y^2 + \frac{a_{222}}{a_0} xy^3. \tag{13}$$

From (10.2) we similarly find $-xa(x_2, y) \sim x_2a(-x, 0)$. This equality is obtained from (17) by replacing the variable x by $-x$, so the following is true.

Lemma 2.

$$x_2 \sim -x - \frac{a_2}{a_0}xy + \frac{a_{12}}{a_0}x^2y - \frac{a_{22}}{a_0}xy^2 - (\frac{a_{112}}{a_0} + \frac{a_2a_{11}}{a_0^2})x^3y + (\frac{a_{122}}{a_0} + \frac{a_2a_{12}}{a_0^2})x^2y^2 - \frac{a_{222}}{a_0}xy^3.$$

The third equality (10) gives $xa(x_1, 0) \sim x_1a(x, y_1)$. Substituting here x_1 from (18) and solving with respect to y_1, we obtain the following result.

Lemma 3.

$$y_1 \sim -y + u_{22}y^2 + u_{122}xy^2 - u_{22}^2y^3,$$

where $u_{22} = \frac{a_2}{a_0} - 2\frac{a_{22}}{a_2}$, $\quad u_{122} = -2\frac{a_{122}}{a_2} + 2\frac{a_{12}a_{22}}{a_2^2}$.

Consider the fourth equality (10). It is obtained from the second one by replacing x by $-x$ in the right-hand side and by changing x_1 to x_2 in the left-hand side. But x_2 is obtained from x_1 also by replacing x by $-x$. Therefore y_2 is obtained from y_1 also by replacing x by $-x$, that is, the following is true.

Lemma 4. $y_2 \sim -y + u_{22}y^2 - u_{122}xy^2 - u_{22}^2y^3$.

From the above we get $y_1 - y_2 \sim 2u_{122}xy^2 \equiv 2\beta_{13}xy^2$. So, the quantity

$$\beta_{13} = -2\frac{a_{122}}{a_2} + 2\frac{a_{12}a_{22}}{a_2^2} \tag{14}$$

is a relative invariant of the three-web on the 2-kind boundary Γ_{13}. We have proved

Theorem 2. *Let the three-web be formed by the Cartesian net and family of curves (9), so that the y axis is the boundary curve of the second kind Γ_{13}. Then the main part of the obstruction to the closure of the figure $\mathcal{B}_{13}$ constructed on the boundary Γ_{13} is determined by the relative invariant β_{13} calculated at every point of this boundary by formula (14).*

4. CALCULATION OF THE INVARIANT β_{13}

We express β_{13} in terms of partial derivatives of the function a. From (11) we obtain:

$$a_2 = a_y(0,0), a_{12} = a_{xy}(0,0), 2a_{22} = a_{yy}(0,0), 2a_{122} = a_{xyy}(0,0).$$

Substituting in the second formula (14), we obtain:

$$\beta_{13} = -\frac{a_y a_{xyy} - a_{xy}a_{yy}}{a_y^2} = -(\frac{a_{xy}}{a_y})_y = -(\ln a_y)_{xy},$$

all derivatives being computed at point $(0,0)$, and the last equality is possible in a domain, where $a_y > 0$. But since $(0,0)$ is an arbitrary point of axis y, we can write

$$\beta_{13} = -(\ln a_y)_{xy}|_{c=0}. \tag{15}$$

On the other hand, it follows from (9) that for $c = 0$ $F_c = a(x, y)$,

$$F_{cy} = a(x, y)_y,\ F_{cxy} = a(x, y)_{xy},\ F_{cyy} = a(x, y)_{yy},\ F_{cxyy} = a(x, y)_{xyy}.$$

Therefore for $c = 0$ $(\ln a_y)_{xy} = (\ln F_{cy})_{xy}$ and (15) takes the form

$$\beta_{13} = -(\ln F_{cy})_{xy}|_{c=0}. \tag{16}$$

Let us find the form of the invariant β_{13} for the case when the third family is given by the level lines of the function $f(x, y)$. Then the identity $F(x, y, f(x, y)) \equiv 0$ is fulfilled. Differentiating it we obtain

$F_x + F_c f_x = 0$. Since $F_x = 1$, $F_c = a$ for $c = 0$, from the previous formulas we have $1 + af_x = 0$ for $c = 0$. As a result formula (15) takes the form:

$$\beta_{13} = (\ln(\frac{1}{f_x})_y)_{xy}|_{c=0}.$$

For the boundary Γ_{23} of the second kind, we can construct an analogous figure $\mathcal{B}_{23}$ and find analogous formulas for the corresponding relative invariant.

5. THE NECESSARY CONDITION FOR THE P_{13} CLOSURE

The necessary condition for the closure of the figure P_{13} is that the relative invariant π_{13} vanishes, which gives the equality $f_{xy}f_{yy} = 0$ that must be satisfied at the points of the boundary of the first kind Γ_{13}.

1) Consider the case

$$f_{xy} = 0. \tag{17}$$

If condition (17) is satisfied at every point of the web domain, then $f = \alpha(x) + \beta(y)$ and the web is regular [2].

If condition (17) is satisfied on the boundary Γ_{13} (given by the equation $f_y = 0$), then the condition

$$f_{xy} = \Theta(f_y, x, y), \quad \Theta(0, x, y) = 0 \tag{18}$$

must be fulfilled.

Solutions of equation (18) can be obtained as follows. We set $f_y = (p(x, y))^k$, $\quad k \neq 0, 1$. Then $f_{xy} = kp^{k-1}p_x = k(f_y)^{\frac{k-1}{k}}p_x$, and condition (18) is satisfied. In this case, the boundary Γ_{13} is given by the equation $p = 0$, and the function f can be found by integration. The three-web W in this case will not, in general, be regular.

2) The case

$$f_{yy} = 0. \tag{19}$$

If condition (19) is satisfied at every point of the web domain D, then the web equation has the form $z = a(x)y + b(x)$, and in the plane of variables y, z it defines a rectilinear three-web $\bar{W}$ obtained from the web W by renumbering foliations and formed by the Cartesian net $y = const$, $z = const$ and the family of straight lines $z = a(x)y + b(x)$ with parameter x.

If (19) is satisfied on the boundary Γ_{13}, then the condition $f_{yy} = \Theta(f_y, x, y)$, $\quad \Theta(0, x, y) = 0$ must be fulfilled. Solutions of this equation can be found similarly to 1).

6. THE NECESSARY CONDITION FOR THE $\mathcal{B}_{13}$ CLOSURE

The necessary condition for the closure of the figure $\mathcal{B}_{13}$ is that the relative invariant β_{13} on the boundary Γ_{13} (its equation is $x = 0$) is equal to zero. Suppose that the web is given by equation (9), then we obtain from (16)

$$(\ln F_{cy})_{xy}|_{c=0} = 0. \tag{20}$$

If the equality $(\ln F_{cy})_{xy} = 0$ is satisfied at every point of domain D, then after integration we obtain

$$F = x + c(\alpha(x)\beta(y) + \gamma(x)), \tag{21}$$

where $\alpha(x)\beta(y) \neq 0$ in D (otherwise the web does not exist). The three-web, defined in this case by the equation $F = 0$, has not only the boundary Γ_{13} of the second kind ($x = 0$) but also the boundary Γ_{13} of the first kind defined by the equation $\beta(y) = 0$. Outside the boundary $x = 0$, the web equation $F = 0$ can be written in the form $c^{-1} + x^{-1}(\alpha(x)\beta(y) + \gamma(x)) = 0$, or, after admissible substitutions of variables $c^{-1} \to z$, $x^{-1}\alpha(x) \to x$, $\beta(y) \to -y$, in the form $z = xy + \varphi(x)$. Generally speaking this web is not regular and will be such if $\varphi(x)$ is a linear function. Note that the last equation determines a rectilinear three-web in the plane of the variables y and z.

In accordance with (21), the solution of equation (20) can be written in the form

$$F = x + c(\alpha(x)\beta(y) + \gamma(x)) + \varphi_2(x, y)c^2 + \varphi_3(x, y)c^3 + \dots.$$

7. ON THE SUFFICIENT CONDITION FOR THE $\mathcal{B}_{13}$ CLOSURE

The necessary closure conditions for the figures P_{13} and $\mathcal{B}_{13}$ are not, generally speaking, sufficient. Let us find, for example, a sufficient condition for the closure of the figures $\mathcal{B}_{13}$ for the web (21). First we make an admissible substitution $\beta(y) \to y$, then the web equation takes the form

$$x + c(\alpha(x)y + \gamma(x)) = 0, \tag{21$'$}$$

and the closure conditions (10) are written in the form:

$$\begin{aligned}
&(1) \quad x + \gamma(x)c_1 = 0, \quad x_1 + (\alpha(x_1)y + \gamma(x_1))c_1 = 0;\\
&(2) \quad -x + \gamma(-x)c_2 = 0, \quad x_2 + (\alpha(x_2)y + \gamma(x_2))c_2 = 0;\\
&(3) \quad x_1 + \gamma(x_1)c_3 = 0, \quad x + (\alpha(x)y_1 + \gamma(x))c_3 = 0;\\
&(4) \quad x_2 + \gamma(x_2)c_4 = 0, \quad -x + (\alpha(-x)y_2 + \gamma(-x))c_4 = 0.
\end{aligned}$$

Since the configuration lines are outside the domain $c = 0$ $(x = 0)$, we find from these relations:

$$\begin{aligned}
&(1) \quad \frac{\gamma(x)}{x} = \frac{\alpha(x_1)}{x_1}y + \frac{\gamma(x_1)}{x_1};\\
&(2) \quad \frac{\gamma(-x)}{-x} = \frac{\alpha(x_2)}{x_2}y + \frac{\gamma(x_2)}{x_2};\\
&(3) \quad \frac{\gamma(x_1)}{x_1} = \frac{\alpha(x)}{x}y_1 + \frac{\gamma(x)}{x};\\
&(4) \quad \frac{\gamma(x_2)}{x_2} = \frac{\alpha(-x)}{-x}y_2 + \frac{\gamma(-x)}{-x}.
\end{aligned} \tag{22}$$

Adding the first and the third equalities, then the second to the fourth ones we obtain equalities

$$\frac{\alpha(x_1)}{x_1}y + \frac{\alpha(x)}{x}y_1 = 0; \quad \frac{\alpha(x_2)}{x_2}y + \frac{\alpha(-x)}{-x}y_2 = 0.$$

A necessary and sufficient condition for the closure $y_1 = y_2$ is satisfied in the case when

$$\frac{\alpha(x_1)}{x_1} : \frac{\alpha(x)}{x} = \frac{\alpha(x_2)}{x_2} : \frac{\alpha(-x)}{-x}, \quad \text{or} \quad \frac{\alpha(x_1)}{x_1\alpha(x)} = -\frac{\alpha(x_2)}{x_2\alpha(-x)}.$$

The values of x_1 and x_2 can be found from the first two equations (22): $x_1 = \chi(x, y)$, $x_2 = \chi(-x, y)$, then the previous equality takes the form:

$$\frac{\alpha(\chi(x, y))}{\chi(x, y)\alpha(x)} = -\frac{\alpha(\chi(-x, y))}{\chi(-x, y)\alpha(-x)}.$$

So *the necessary and sufficient closure condition is that the function* $\alpha \circ \chi/\chi \cdot \alpha$ *be skew-symmetric.*

Remark. The construction of figures P_{13} and $\mathcal{B}_{13}$ is impossible if the web exists only on one side of the boundary, for example, if three-web is given by the equation $(z - x)^2 + y^2 = 1$.

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