

MONOMIAL IDEALS WITH QUASI-LINEAR QUOTIENTS

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ABSTRACT. In this paper, we extend the notion of quasi-linear quotients for a pure monomial ideal (not necessarily square-free) of degree d . We introduce the notion of *quasi-linear free resolution* and show that if a pure monomial ideal $I = (u_1, u_2, \dots, u_m)$ of degree d in the polynomial ring $S = k[x_1, \dots, x_n]$ admits quasi-linear quotients then $L_q = (u_1, \dots, u_{q-1}) : u_q$ admits *quasi-linear free resolution* for all $q \leq m$. Moreover, we show that if a pure monomial ideal I of degree d admits quasi-linear quotients then $I_{<t>}$ will also have quasi-linear quotients for $t \geq d$.

Key words : shellable simplicial complex, , facet ideal, graded free resolution, graded betti numbers, betti diagram, colon of monomial ideals.

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1. INTRODUCTION

The class of square-free monomial ideals with quasi-linear quotients was introduced in [1]. A monomial ideal is said to have *quasi-linear quotients*, if there exists a minimal monomial system of generators u_1, u_2, \dots, u_m of I such that $\beta_{01}(L_k) \neq 0$ for all $1 < k \leq m$, where $L_k = (u_1, u_2, \dots, u_{k-1}) : (u_k)$. The concept of shellability has been the subject of a considerable number of geometric and algebraic investigations for instance see [2], [3], [4], and [6]. The facet ideal $I_{\mathcal{F}}(\Delta)$ (introduced in [5]) of a pure shellable simplicial complex Δ admits quasi-linear quotients [2]. Moreover, the class of monomial ideals with linear quotients (introduced by Herzog and Takayama [7]) is contained in the class of monomial ideals with quasi-linear quotients.

The potential aim of this article is to study the monomial ideals with quasi-linear quotients. We introduce the notion of *quasi-linear free resolution* and show that if a pure monomial ideal $I = (u_1, u_2, \dots, u_m)$ of degree d admits quasi-linear quotients then $L_q = (u_1, \dots, u_{q-1}) : u_q$ admits *quasi-linear free resolution* for all $q \leq m$. We also show that if a pure monomial ideal I of degree d admits quasi-linear quotients then $I_{<t>}$ will also have quasi-linear quotients for $t \geq d$.

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2. BACKGROUND

In this section, we define some basic notions that we will use in the paper along with some background theory.

Let $S = k[x_1, \dots, x_n]$ denote the polynomial ring in n variables over an infinite field k with each $\deg(x_i) = 1$ and I a monomial ideal of S . The graded Betti numbers $\beta_{i,j} = \beta_{i,j}(I)$ appear in the graded minimal free resolution

$$\mathfrak{F}: 0 \rightarrow \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{h,j}} \rightarrow \dots \rightarrow \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{0,j}} \rightarrow I$$

of I in S . Then $\beta_{i,j}(I) = \dim_k \operatorname{Tor}_i(I, k)_j$ for all i and j . There are two important invariants attached to a monomial ideal $I \subset S$ defined in terms of the minimal graded free resolution of I :

$$\operatorname{proj dim}(I) = \max\{i : \beta_{i,j}(I) \neq 0\},$$

$$\operatorname{reg}(I) = \max\{j : \beta_{i,i+j}(I) \neq 0 \text{ for some } i\}$$

namely *projective dimension* and *regularity* of I respectively. We say that I has a **linear resolution** if there exists an integer $m \geq 1$ such that $\beta_{i,i+j} = 0$ for all i and j with $j \neq m$. Thus, in particular, if I has a linear resolution, then I is generated by homogeneous polynomials of the same degree.

Definition 2.1. Let I be a monomial ideal in S and u be any monomial in S . Then the *colon ideal* of I with respect to (u) is defined as

$$I : (u) = \{f \in S \mid fu \in I\}.$$

Here, we recall an important class of monomial ideals namely *ideals with linear quotients* along with some important results.

Definition 2.2. Let $I \subset S = k[x_1, \dots, x_n]$ be a monomial ideal, we say that I has *linear quotients*, if there exists a minimal monomial system of generators u_1, u_2, \dots, u_m of I such that L_k is generated by linear forms for all $1 < k \leq m$, where

$$L_k = (u_1, u_2, \dots, u_{k-1}) : (u_k).$$

Moreover for each k , we denote the number of generators of L_k by r_k or in other words $\mu(L_k) = r_k$.

One of the reasons that makes ideals with linear quotient an important subject for considerations is due to the following result [6, Proposition 8.2.1].

Theorem 2.3. Suppose that $I \subset S$ is a graded ideal generated in degree d and that I has linear quotients. Then I has a d -linear resolution.

3. PURE MONOMIAL IDEALS WITH QUASI-LINEAR QUOTIENTS

In this section, we will consider only monomial ideals with quasi-linear quotients in the polynomial ring $S = k[x_1, \dots, x_n]$. We say that a monomial ideal $I \subset S$ is pure of degree d , if and only if $\beta_{0,j}(I) = 0$ for all $j \neq d$.

Definition 3.1. Let I be a monomial ideal in S . We define the $\text{indeg}(I)$ as follows

$$\text{indeg}(I) = \min\{j : \beta_{0,j}(I) \neq 0\}.$$

Definition 3.2. Let $I \subset S$ be a monomial ideal, we say that I has *quasi-linear quotients*, if there exists a minimal monomial system of generators u_1, u_2, \dots, u_m of I such that $\text{indeg}(L_i) = 1$ for all $1 < i \leq m$, where

$$L_i = (u_1, u_2, \dots, u_{i-1}) : (u_i).$$

Remark 3.3. Let $I \subset S$ be a monomial ideal having *linear quotients*, then I has *quasi-linear quotients* as well. Therefore, the class of ideals with quasi-linear quotients is a generalization of the class of monomial ideals with linear quotients.

Definition 3.4. We say that I has a *quasi-linear resolution* if for each integer i with $0 \leq i < \text{projdim}(I)$, there exists an integer j such that, $\beta_{i,j}(I) \neq 0$ implies $\beta_{i+1,j+1}(I) \neq 0$. Thus, in particular, if I will have a quasi-linear resolution, then there is at least one linear shift in each step of the minimal free resolution of I .

The following example is given to show the various aspects of quasi-linear resolution of an ideal I .

Example 3.5. Let us consider the polynomial ring $S = k[x_1, \dots, x_7]$ and monomial ideal $I = (x_1x_2x_3, x_1x_3x_4, x_3x_4x_5, x_4x_5x_6, x_5x_6x_7)$. Its minimal free resolution is as follows:

$$0 \rightarrow S(-7) \rightarrow S(-4)^4 \oplus S(-6) \rightarrow S(-3)^5 \rightarrow I$$

with Betti diagram:

$$\begin{array}{cccc} - & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 5 & 4 & 0 \\ 4 & 0 & 0 & 0 \\ 5 & 0 & 1 & 1 \end{array}$$

One can see the linear shift in each step clearly.

Theorem 3.6. Let $I \subset S = k[x_1, \dots, x_n]$ be a pure monomial ideal of degree d , minimally generated by $\{u_1, u_2, \dots, u_m\}$. If I has quasi-linear quotients, then $L_q = (u_1, u_2, \dots, u_{q-1}) : u_q$ has a quasi-linear free resolution for all $1 < q \leq m$.

Proof. Since I has quasi-linear quotients, so L_q has a variable as a generator, for all $1 < q \leq m$. To show L_q has a quasi-linear free resolution, it is enough to show that a monomial ideal containing a variable has a quasi-linear free resolution.

Let L be an ideal containing a variable. We may assume that $L = (J, x_1)$ with $x_1 \notin \text{supp}(J)$. By using the definition of minimal graded free resolution, one may obtain a minimal graded free resolution of L from a minimal graded free resolution of J , as

$$\text{Tor}_i^S(L, k)_j = \text{Tor}_i^S(J, k)_j \oplus \text{Tor}_{i-1}^S(J(-1), k)_j, \quad \text{for all } i \geq 1 \text{ and}$$

$$\text{Tor}_0^S(L, k)_j = \text{Tor}_0^S((x_1, k)_j \oplus \text{Tor}_0^S(J, k)_j).$$

Now suppose that $\beta_{i+1}(L) \neq 0$. Then $\beta_i(J) \neq 0$. (In fact, if $\beta_i(J) = 0$, then $\beta_{i+1}(J) = 0$, which implies $\beta_{i+1}(L) = 0$, a contradiction.) In particular, there exists j such that $\beta_{i,j}(J) \neq 0$. The presence of $\text{Tor}_i^S(J, k)_j \oplus \text{Tor}_{i-1}^S(J(-1), k)_j$ ensures that $\beta_{i,j}(L) \neq 0$ and $\beta_{i+1,j+1}(L) \neq 0$. Therefore, L has a quasi-linear free resolution for all $k \leq m$.

Here we give an example to demonstrate the step recursively mentioned in the proof of above theorem.

Example 3.7. Let $I = (a, b, cd, ce, de)$ be a monomial ideal in the polynomial ring $S = k[a, b, c, d, e]$. Therefore, we have the following chain of ideals

$$I_0 = (cd, ce, de) \subset I_1 = (a, I_0) \subset I_2 = I$$

The Betti diagram of I_0 is given as:

$$\begin{array}{ccc} - & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 2 \end{array}$$

The Betti diagram of I_1 can be obtained from I_0 (changes in shift are shown in bold entries):

$$\begin{array}{cccc} - & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & \mathbf{1} & 0 & 0 \\ 2 & 3 & 5 = \mathbf{3} + 2 & 2 = \mathbf{2} + 0 \end{array}$$

The Betti diagram of $I_2 = I$ can be obtained from I_1 (changes in shift are shown in bold entries):

$$\begin{array}{ccccc} - & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & \mathbf{1} + 1 & 1 = 1 + 0 & 0 & 0 \\ 2 & 3 & 8 = \mathbf{3} + 5 & 7 = \mathbf{5} + 2 & 2 = \mathbf{2} + 0 \end{array}$$

One can notice the changes in the shifts by comparing the above Betti diagrams. It is clear from above that

$$\text{Tor}_i^S(I_1, k)_j = \text{Tor}_i^S(I_0, k)_j \oplus \text{Tor}_{i-1}^S(I_0(-1), k)_j \text{ for } i > 0$$

and

$$\text{Tor}_i^S(I, k)_j = \text{Tor}_i^S(I_1, k)_j \oplus \text{Tor}_{i-1}^S(I_1(-1), k)_j \text{ for } i > 0.$$

Moreover,

$$\text{Tor}_0^S(I_1, k)_j = \text{Tor}_0^S((a), k)_j \oplus \text{Tor}_0^S(I_0, k)_j$$

and

$$\text{Tor}_0^S(I, k)_j = \text{Tor}_0^S((b), k)_j \oplus \text{Tor}_0^S(I_1, k)_j.$$

For a pure monomial ideal $I \subset S$ of degree d , we have $I_{<t>}$ is a monomial ideal generated by t degree monomials in I . In particular, $I_{<t>} = I.(x_1, x_2, \dots, x_n)^{t-d}$.

Theorem 3.8. Let $I \subset S$ be a pure monomial ideal of degree d having quasi-linear quotients with respect to the generating set $G(I) = \{u_1, u_2, \dots, u_m\}$. Then $I_{<t>}$ will also have quasi-linear quotients for $t \geq d$.

Proof. We prove our claim by giving the an ordered generating set of $I_{<t>} = I.(x_1, x_2, \dots, x_n)^{t-d}$ that admits quasi-linear quotients. We claim that $I_{<t>}$ have quasi-linear quotients with respect to the following generating system.

$$\begin{aligned} &v_1u_1, v_1u_2, \dots, v_1u_m, \\ &v_2u_1, v_2u_2, \dots, v_2u_m, \\ &\vdots \\ &v_ru_1, v_ru_2, \dots, v_ru_m, \end{aligned}$$

such that $r = \binom{n+t-d-1}{n-1}$. We order the above list of generators in lexicographic order with respect to v_i by ignoring the monomial u_j . One may observe that all the monomials in the above list are of type $v_i u_j$ where v_i is a monomial of degree $t - d - 1$. Firstly, observe that if u_1, \dots, u_m admits quasi-linear quotients then $x_1^{t-d}u_1, x_1^{t-d}u_2, \dots, x_1^{t-d}u_m$ also admits the quasi-linear quotients, therefore all the colon $L_{v_1u_k}$ will have a linear term for all $1 \leq k \leq m$. Now take any term from the list of monomials $v_i u_k$ for $2 \leq i$ from the above list with the assumption that $v_i = x_i^p \mathbf{x}$ (with $\deg(\mathbf{x}) = t - d - p$ and $\text{supp}(\mathbf{x}) = \{x_j \mid j > i\}$) then there exists a term $v_j = x_1 x_i^{p-1} \mathbf{x}$ such that $v_i <_{\text{lex}} v_j$ (or equivalently $i > j$). Therefore, the colon ideal $L_{v_i u_k}$ will have a linear term $\frac{v_j u_k}{\text{GCD}(v_i u_k, v_j u_k)} = x_1$, as required. \square

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