A New Generalization of Ostrowski Type Inequalities for Mappings of Bounded Variation

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Abstract—In this paper, a new generalization of Ostrowski type integral inequality for mappings of bounded variation is obtained and the quadrature formula is also provided.

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1. INTRODUCTION

Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b) , i.e. $\|f'\|_\infty := \sup\limits_{t\in(a,b)} |f'(t)| < \infty$. Then, we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int\limits_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_\infty , \tag{1}$$

for all $x \in [a,b]$[16]. The constant 1/4 is the best possible. This inequality is well known in the literature as the *Ostrowski inequality*.

Definition 1. *Let $P : a = x_0 < x_1 < ... < x_n = b$ be any partition of $[a,b]$ and let $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$. Then $f(x)$ is said to be of bounded variation if the sum $\sum\limits_{i=1}^{m} |\Delta f(x_i)|$ is bounded for all such partitions. Let f be of bounded variation on $[a,b]$, and $\sum(P)$ denotes the sum $\sum\limits_{i=1}^{n} |\Delta f(x_i)|$ corresponding to the partition P of $[a,b]$. The number*

$$\bigvee_a^b (f) := \sup \left\{ \sum (P) : P \in P([a,b]) \right\}$$

is called the total variation of f on $[a,b]$. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [9], Dragomir proved following Ostrowski type inequalities for functions of bounded variation:

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Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt-(b-a)f(x)\right|\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f) \tag{2}$$

holds for all $x\in[a,b]$. The constant $1/2$ is the best possible.

We introduce the notation $I_n: a=x_0<x_1<...<x_n=b$ for a division of the interval $[a,b]$ with $h_i:=x_{i+1}-x_i$ and $v(h)=\max\{h_i: i=0,1,...,n-1\}$ and let intermediate points $\xi_i\in[x_i,x_{i+1}]$ $(i=0,1,...,n-1)$. Then we have

$$\int_a^b f(t)dt=A(f,I_n,\xi)+R(f,I_n,\xi), \tag{3}$$

where

$$A(f,I_n,\xi):=\sum_{i=0}^{n} f(\xi_i)h_i \tag{4}$$

and the remainder term satisfies

$$|R(f,I_n,\xi)|\leq\left[\frac{1}{2}v(h)+\max_{i\in\{0,1,...,n-1\}}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]\bigvee_a^b(f)\leq v(h)\bigvee_a^b(f). \tag{5}$$

In [7], Dragomir obtained following Ostrowski type inequality for functions of bounded variation:

Theorem 2. *Let $I_k: a=x_0<x_1<...<x_k=b$ be a division of the interval $[a,b]$ and α_i $(i=0,1,...,k+1)$ be $k+2$ points so that $\alpha_0=a$, $\alpha_i\in[x_{i-1},x_i]$ $(i=1,...,k)$, $\alpha_{k+1}=b$. If $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$, then we have the inequality:*

$$\left|\int_a^b f(x)dx-\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)\right| \tag{6}$$

$$\leq\left[\frac{1}{2}v(h)+\max\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|, i=0,1,...,k-1\right]\bigvee_a^b(f)\leq v(h)\bigvee_a^b(f),$$

where $v(h):=\max\{h_i|\ i=0,...,n-1\}$, $h_i:=x_{i+1}-x_i$ $(i=0,1,...,k-1)$ and $\bigvee_a^b(f)$ is the total variation of f on the interval $[a,b]$.

For recent results concerning the above Ostrowski's inequality and other related results see [1], [21]. The aim of this paper is to obtain a new generalization of Ostrowski type integral inequalities for functions of bounded variation. And we give some applications for our results.

2. MAIN RESULTS

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then, for all $x\in[a,b]$, we have*

$$\left|(b-a)\left(1-\frac{\lambda}{2}\right)f(x)+\lambda\frac{(x-a)f(a)+(b-x)f(b)}{2}-\int_a^b f(t)dt\right|$$

$$\leq\left(1-\frac{\lambda}{2}\right)\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f),$$

where $\lambda \in [0,1]$ *and* $\bigvee_c^d (f)$ *denotes the total variation of* f *on* $[c,d]$ *.*

Proof. Define the mapping $K_\lambda(x,t)$ by

$$K_\lambda(x,t) = \begin{cases} t - \left(a + \lambda\frac{x-a}{2}\right), & a \le t \le x, \\ \\ t - \left(b - \lambda\frac{b-x}{2}\right) & x < t \le b. \end{cases}$$

Integrating by parts, we get

$$\int_a^b K_\lambda(x,t)df(t) = \int_a^x \left(t - \left(a + \lambda\frac{x-a}{2}\right)\right) df(t) + \int_x^b \left(t - \left(b - \lambda\frac{b-x}{2}\right)\right) df(t)$$

$$= \left(t - a - \lambda\frac{x-a}{2}\right) f(t)\Big|_a^x - \int_a^x f(t)dt + \left(t - b + \lambda\frac{b-x}{2}\right) f(t)\Big|_x^b - \int_x^b f(t)dt$$

$$= (x-a)\left(1 - \frac{\lambda}{2}\right) f(x) + \lambda\frac{x-a}{2} f(a) + \lambda\frac{b-x}{2} f(b) + (b-x)\left(1 - \frac{\lambda}{2}\right) f(x) - \int_a^b f(t)dt$$

$$= (b-a)\left(1 - \frac{\lambda}{2}\right) f(x) + \lambda\frac{(x-a) f(a) + (b-x) f(b)}{2} - \int_a^b f(t)dt.$$

It is well known that if $g, f : [a,b] \to \mathbb{R}$ are such that g is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$, then $\int_a^b g(t)df(t)$ exist and

$$\left|\int_a^b g(t)df(t)\right| \le \sup_{t\in[a,b]} |g(t)| \bigvee_a^b (f). \tag{7}$$

On the other hand, using (7), we get

$$\left|\int_a^b K_\lambda(x,t)df(t)\right| \le \left|\int_a^x \left(t - \left(a + \lambda\frac{x-a}{2}\right)\right) df(t)\right| + \left|\int_x^b \left(t - \left(b - \lambda\frac{b-x}{2}\right)\right) df(t)\right|$$

$$\le \sup_{t\in[a,x]} \left|t - a - \lambda\frac{x-a}{2}\right| \bigvee_a^x (f) + \sup_{t\in[x,b]} \left|t - b + \lambda\frac{b-x}{2}\right| \bigvee_x^b (f)$$

$$= (x-a)\left(1 - \frac{\lambda}{2}\right) \bigvee_a^x (f) + (b-x)\left(1 - \frac{\lambda}{2}\right) \bigvee_x^b (f) \le \left(1 - \frac{\lambda}{2}\right) \max\{x-a, b-x\} \bigvee_a^b (f)$$

$$= \left(1 - \frac{\lambda}{2}\right) \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f).$$

This completes the proof. □

Remark 1. *If we choose $\lambda = 0$ in Theorem 3, then the inequality (7) reduces the inequality (2).*

Corollary 1. *Under the assumption of Theorem 3 with $\lambda = 1$, then we have the following inequality*

$$\left|\frac{1}{2}(b-a)f(x)+\frac{(x-a)f(a)+(b-x)f(b)}{2}-\int_a^b f(t)dt\right| \leq \frac{1}{2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f). \quad (8)$$

Remark 2. *If we take $x = (a+b)/2$ in Corollary 1, then we have the inequality*

$$\left|\frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\int_a^b f(t)dt\right| \leq \frac{1}{4}(b-a)\bigvee_a^b(f)$$

which was given by Alomari in [3]. The constant $1/4$ is the best possible.

Corollary 2. *Under the assumption of Theorem 3 with $\lambda = 2/3$, then we get the inequality*

$$\left|\frac{2}{3}(b-a)f(x)+\frac{(x-a)f(a)+(b-x)f(b)}{3}-\int_a^b f(t)dt\right| \leq \frac{2}{3}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f). \quad (9)$$

Remark 3. *If we take $x = (a+b)/2$ in Corollary 2, then we have the Simpson's inequality*

$$\left|\frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)dt\right| \leq \frac{1}{3}(b-a)\bigvee_a^b(f)$$

which was given by Dragomir in [7].

Corollary 3. *Under the assumption of Theorem 3. Suppose that $f \in C^1[a,b]$, then we have*

$$\left|(b-a)\left(1-\frac{\lambda}{2}\right)f(x)+\lambda\frac{(x-a)f(a)+(b-x)f(b)}{2}-\int_a^b f(t)dt\right|$$
$$\leq \left(1-\frac{\lambda}{2}\right)\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\|f'\|_1$$

for all $x \in [a,b]$. Here as subsequently $\|.\|_1$ is the L_1-norm: $\|f'\|_1 := \int_a^b f'(t)dt$.

Corollary 4. *Under the assumption of Theorem 3, let $f : [a,b] \to \mathbb{R}$ be a Lipschitzian with the constant $L > 0$. Then for all $x \in [a,b]$*

$$\left|(b-a)\left(1-\frac{\lambda}{2}\right)f(x)+\lambda\frac{(x-a)f(a)+(b-x)f(b)}{2}-\int_a^b f(t)dt\right|$$
$$\leq \left(1-\frac{\lambda}{2}\right)\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right](b-a)L.$$

Corollary 5. *Under the assumption of Theorem 3, let $f:[a,b]\to\mathbb{R}$ be a monotone mapping on $[a,b]$. Then for all $x\in[a,b]$*

$$\left|(b-a)\left(1-\frac{\lambda}{2}\right)f(x)+\lambda\frac{(x-a)f(a)+(b-x)f(b)}{2}-\int_a^b f(t)dt\right|$$
$$\leq\left(1-\frac{\lambda}{2}\right)\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]|f(b)-f(a)|.$$

3. APPLICATION TO QUADRATURE FORMULA

We now introduce the intermediate points $\xi_i\in[x_i,x_{i+1}]$ $(i=0,1,...,n-1)$ in the division $I_n: a=x_0<x_1<...<x_n=b$. Let $h_i:=x_{i+1}-x_i$ and $v(h)=\max\{h_i: i=0,1,...,n-1\}$ and define the sum

$$A(f,I_n,\xi):=\sum_{i=0}^{n}\left[\left(1-\frac{\lambda}{2}\right)f(\xi_i)h_i+\lambda\frac{(\xi_i-x_i)f(x_i)+(x_{i+1}-\xi_i)f(x_{i+1})}{2}\right]. \tag{10}$$

Then the following Theorem holds:

Theorem 4. *Let f be as Theorem 3. Then*

$$\int_a^b f(t)dt=A(f,I_n,\xi)+R(f,I_n,\xi), \tag{11}$$

where $A(f,I_n,\xi)$ is defined as above and the remainder term $R(f,I_n,\xi)$ satisfies

$$|R(f,I_n,\xi)|\leq\left(1-\frac{\lambda}{2}\right)\left[\frac{1}{2}v(h)+\max_{i\in\{0,1,...,n-1\}}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]\bigvee_a^b(f)\leq\left(1-\frac{\lambda}{2}\right)v(h)\bigvee_a^b(f).$$

Proof. Application of Theorem 3 to the interval $[x_i,x_{i+1}]$ $(i=0,1,...,n-1)$ gives

$$\left|\left(1-\frac{\lambda}{2}\right)f(\xi_i)h_i+\lambda\frac{(\xi_i-x_i)f(x_i)+(x_{i+1}-\xi_i)f(x_{i+1})}{2}-\int_{x_i}^{x_{i+1}}f(t)dt\right|$$
$$\leq\left(1-\frac{\lambda}{2}\right)\left[\frac{h_i}{2}+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]\bigvee_{x_i}^{x_{i+1}}(f) \tag{12}$$

for all $i\in\{0,1,...,n-1\}$. Summing the inequality (12) over i from 0 to $n-1$ and using the generalized triangle inequality, we have

$$\begin{aligned}|R(f,I_n,\xi)| &\leq \left(1-\frac{\lambda}{2}\right)\sum_{i=0}^{n}\left[\frac{h_i}{2}+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]\bigvee_{x_i}^{x_{i+1}}(f)\\ &\leq \left(1-\frac{\lambda}{2}\right)\max_{i\in\{0,1,...,n-1\}}\left[\frac{h_i}{2}+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]\sum_{i=0}^{n}\bigvee_{x_i}^{x_{i+1}}(f)\\ &\leq \left(1-\frac{\lambda}{2}\right)\left[\frac{1}{2}v(h)+\max_{i\in\{0,1,...,n-1\}}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]\bigvee_a^b(f)\end{aligned}$$

which completes the proof of the first inequality in (4)

For the second inequality in (4), we show that

$$\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \leq \frac{h_i}{2}\ i \in \{0, 1, ..., n-1\} \quad \text{and} \quad \max_{i\in\{0,1,...,n-1\}} \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \leq \frac{1}{2}v(h)$$

which completes the proof. □

Remark 4. *If we choose $\lambda = 0$, we get (3) with (4) and (5).*

Remark 5. *If we choose $\lambda = 2/3$ and $\xi_i = (x_i + x_{i+1)/2}$, then we have $\int_a^b f(t)dt = A_S(f, I_n)$ $+R_S(f, I_n)$, where*

$$A_S(f, I_n) = \frac{1}{6}\sum_{i=0}^{n} [f(x_i) + f(x_{i+1})]\, h_i + \frac{2}{3}\sum_{i=0}^{n} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i,$$

and the remainder term $R_S(f, I_n)$ satisfies$|R_S(f, I_n)| \leq (1/3)v(h)\bigvee_a^b(f)$ which were given by Dragomir in [7].

Corollary 6. *Choosing $\lambda = 1$ gives$\int_a^b f(t)dt = A(f, I_n, \xi) + R(f, I_n, \xi)$, where*

$$A(f, I_n, \xi) = \sum_{i=0}^{n} \left[\frac{1}{2}f(\xi_i)h_i + \frac{(\xi_i - x_i)\, f(x_i) + (x_{i+1} - \xi_i)\, f(x_{i+1})}{2}\right]$$

and the remainder term $R(f, I_n, \xi)$ satisfies

$$|R(f, I_n, \xi)| \leq \frac{1}{2}\left[\frac{1}{2}v(h) + \max_{i\in\{0,1,...,n-1\}} \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_a^b(f) \leq \frac{1}{2}v(h)\bigvee_a^b(f).$$

Particularly, if we take $\xi_i = (x_i + x_{i+1})/2$, then we have

$$A(f, I_n) = \frac{1}{2}\sum_{i=0}^{n} \left[f\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{f(x_i) + f(x_{i+1})}{2}\right] h_i \quad \text{and} \quad |R(f, I_n, \xi)| \leq \frac{1}{4}v(h)\bigvee_a^b(f).$$

Corollary 7. *Let $f : [a, b] \to \mathbb{R}$ be a Lipschitzian with the constant $L > 0$. Then we have (10) and (11) and the remainder term satisfies*

$$|R(f, I_n, \xi)| \leq L\left(1 - \frac{\lambda}{2}\right)\left[\frac{1}{2}v(h) + \max_{i\in\{0,1,...,n-1\}} \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right](b - a) \leq L\left(1 - \frac{\lambda}{2}\right)v(h)\,(b - a)\,.$$

Corollary 8. *Let $f : [a, b] \to \mathbb{R}$ be a monotone mapping on $[a, b]$. Then we get (10) and (11) and the remainder term satisfies*

$$|R(f, I_n, \xi)| \leq \left(1 - \frac{\lambda}{2}\right)\left[\frac{1}{2}v(h) + \max_{i\in\{0,1,...,n-1\}} \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] |f(b) - f(a)|$$

$$\leq \left(1 - \frac{\lambda}{2}\right) v(h)\, |f(b) - f(a)|\,.$$

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