Functional Description of C*-algebras Associated with Group Graded Systems

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Abstract—The well known pure algebraic concept of group grading arises naturally in considering the crossed products, especially in the context of irreversible dynamical systems. In the paper some general aspects concerning group graded systems and related algebras are considered. In particular, a functional description of a C*-algebra associated with an Abelian group graded system is presented.

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1. INTRODUCTION

In recent years the attention of many specialists in operator algebras were focused on the constructions of algebras associated with irreversible dynamical systems. Since the concept of the group crossed product can not be directly transferred to the semigroup case, new methods are developed to avoid the arising difficulties.

It can be mentioned papers concerning the different aspects of algebras associated with semigroup systems. Some of them arise within the framework of algebraical structurs, namely, as corresponding to semigroups with certain properties, see a recent detailed review of Xin Li with references in [11], others describe irreversible dynamical systems: endomorphisms, polymorphisms (see e.g. [1, 3, 5, 10]), irreversible mappings (see e.g. [9]), or interactions, (see e.g. [6, 7]).

Involvment of the concept of graded algebra into consideration, seems to be most promising in this regard. As the first step a notion of group graded system arises which allows different interpretations. Group graded system is a C*-variant of Fell bundle concept, which was developed in detail by Ruy Exel and presented in the book [8]. In his turn, the mentioned work in the substantial part is a continuation of a large series of works of the author and others (see e.g. [4] as one of the recent).

It should be noted that our purposes are a little differ from the accents of the book of Exel, who is mainly interested in partial actions. On the contrary, it seems that by applying the tools of graded systems, the difficulties associated with the use of partial isometries usually accompanying the irreversible dynamical systems can be avoided.

In the present paper we prefer consider the main object as an involutive semigroup structured in a special way.

Although almost all definitions are available in general case the main result is formulated for Abelian groups. We mention in text when this restriction is necessary.

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The paper is arranged as follows.
In the first section preliminary definitions and facts are provided. The next section is devoted to the modular structure on the graded systems under consideration since their main properties are based just on it and on the natural action of the dual group in the Abelian case. In the third section we consider the modular representations and associated algebras in a suitable Hilbert module. In the last section a theorem describing the reduced C*-algebra as an algebra of continuous functions on the dual group is presented. We consider the work as an our first step in studying semigroup dynamical systems in pursuit to replace partial actions on a given C*-algebra by the actions of a suitable group on a modifying algebra equipped with complementary relations which should accumulate some problems arising in similar situations. In a certain part, we follow the contours of the paper [2], which was introductory in nature, with some abbreviations, new details, explanations and corrections. We exclude from consideration the important concept of graded C*-algebra and, accordingly, do not touch the connection between graded systems and algebras connected with semigroup systems.

2. DEFINITIONS AND ELEMENTARY PROPERTIES.

Let Γ be a discrete group (say, with the unit e), and $\mathfrak{A}$ a multiplicative semigroup with zero. We will say that $\mathfrak{A}$ is Γ-*equipped* with a system $\mathfrak{A}_\Gamma = \{\mathfrak{A}_\gamma, \gamma \in \Gamma\}$ of subsets of $\mathfrak{A}$ if

(i) $\mathfrak{A}_\gamma$ is a Banach space for each $\gamma \in \Gamma$,

(ii) $\cup_{\gamma \in \Gamma} \mathfrak{A}_\gamma = \mathfrak{A}$,

(iii) $\mathfrak{A}_\alpha \cap \mathfrak{A}_\beta = \{0\}$ for each $\alpha, \beta \in \Gamma, \alpha \neq \beta$.

Definition 1. *A Γ-equipped star semigroup $\mathfrak{A} = (\Gamma, \mathfrak{A}_\Gamma)$ is called* Γ-graded system *if the operations of multiplication and involution on the semigroup are consistent with the operations on the Banach* spaces *(components of the system), and*

(i) $ab \in \mathfrak{A}_{\alpha\beta}$, *for* $a \in \mathfrak{A}_\alpha,\ b \in \mathfrak{A}_\beta$,

(ii) $a^* \in \mathfrak{A}_{\gamma^{-1}}$, *for* $a \in \mathfrak{A}_\gamma$,

(iii) $\|ab\| \leq \|a\|\|b\|$, *for* $a \in \mathfrak{A}_\alpha,\ \ b \in \mathfrak{A}_\beta$,

(iv) $\|a^*a\| = \|a\|^2 = \|a^*\|^2,$ *for* $a \in \mathfrak{A}_\gamma$.

Obviously, the *central* algebra $A = \mathfrak{A}_e$ is a C*-algebra as well as an involutive subsemigroup of the semigroup $\mathfrak{A}$.

Remark 1. *We suppose almost everywere that the initial semigroup is unital. The reader himself will understand when such an assuming does not work (e.g. when the considering system is an ideal).*

Definition 2. *We say that a Γ-graded system $\mathfrak{B} = (\Gamma, \mathfrak{B}_\Gamma)$ is a* subsystem *of the Γ-graded system $\mathfrak{A} = (\Gamma, \mathfrak{A}_\Gamma)$ if $\mathfrak{B}$ is a *-subsemigroup of $\mathfrak{A}$, and for each $\gamma \in \Gamma$ the Banach space $\mathfrak{B}_\gamma$ is a subspace of the Banach space $\mathfrak{A}_\gamma$.*
A Γ-graded subsystem $\mathfrak{I} = (\Gamma, \mathfrak{I}_\Gamma)$ is called an ideal *(two-sided) of the system $\mathfrak{A} = (\Gamma, \mathfrak{A}_\Gamma)$ if it is an involutive (two-sided) ideal of the semigroup $\mathfrak{A}$.*

Thus, the central algebras $B = \mathfrak{B}_e$ and $I = \mathfrak{I}_e$ are, respectively, a C*-subalgebra and an ideal of the central C*-algebra A. When $\mathfrak{I}$ is an ideal of the draded system $\mathfrak{A}$, a standard equivalence relation can be defined on $\mathfrak{A}$ in the following way: elements a and b are equivalent, $a \sim b$ if they are from the same $\mathfrak{A}_\gamma$, and $a - b \in \mathfrak{I}_\gamma$.

The equivalence relation $\sim$ is a *-congruence.

Proof. Let $a \sim b$, with $a, b \in A_\alpha$, and $u \sim v$, $u, v \in \mathfrak{A}_\beta$. Then, both au and bv are from $\mathfrak{A}_{\alpha\beta}$, so

$$au - bv = a(u - v) + (a - b)v \in \mathfrak{I}_{\alpha\beta},$$

and

$$a^* - b^* = (a - b)^* \in \mathfrak{I}_{\alpha^{-1}}$$

which means $au \sim bv$, and $a^* \sim b^*$. □

The corresponding quotient semigroup $\mathfrak{A}/\mathfrak{I}$, defined in the standard way presents a Γ-graded system composed of the quotient spaces $\mathfrak{A}_\gamma/\mathfrak{I}_\gamma$, $\gamma \in \Gamma$.

As a morphism of a graded system $\mathfrak{A} = (\Gamma, \mathfrak{A}_\Gamma)$ into another graded system $\mathfrak{B} = (\Delta, \mathfrak{A}_\Delta)$ we understand the pair $\Phi = (\rho, \varphi)$, where $\rho : \Gamma \to \Delta$ is a group homomorphism, and $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a *-morphism of the semigroup $\mathfrak{A}$ into the semigroup $\mathfrak{B}$ such that $\varphi(\mathfrak{A}_\gamma) \subset \mathfrak{A}_{\rho(\gamma)}$ for each $\gamma \in \Gamma$, and the restriction of φ on each $\mathfrak{A}_\gamma$ is linear.

Let Φ be a morphism $\Phi = (\rho, \varphi) : \mathfrak{A} \to \mathfrak{B}$ of graded systems. Then the morphism φ does not increase the norm:

$$\|\varphi(a)\| \leq \|a\|$$

for any $a \in A_\gamma$, $\gamma \in \Gamma$.

Proof. Let $a \in A_\gamma$. Then, by the condition (iv) of Definition 1

$$\|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\| \leq \|a^*a\| = \|a\|^2.$$

□

Two graded systems are called *isomorphic* if the mappings ρ and φ are bijective. The group of automorphisms of a graded system $\mathfrak{A}$ is denoted by $\mathrm{Aut}(\mathfrak{A})$. The standard action of the dual group of the group Γ on the Γ-graded system can be characterized as follows. Let $\mathfrak{A}$ be a Γ-graded system with Abelian Γ, and $G = \widehat{\Gamma}$ be the dual group of the group Γ. Then for any $g \in G$ the pair $\Phi_g = (\mathbf{id}, \tau(g))$, where $\mathbf{id}$ is the identity map, and $\tau(g)(a) = g(\gamma)a$ (for $a \in \mathfrak{A}_\gamma$) is an automorphism of the system $\mathfrak{A}$. Moreover, the map $\tau : G \to \mathrm{Aut}(\mathfrak{A})$ is a faithful representation of the group G into the automorphism group of of $\mathfrak{A}$.

Proof. For $g \in G, a \in \mathfrak{A}_\alpha$, and $b \in \mathfrak{A}_\beta$ we have

$$\tau(g)(ab) = g(\alpha\beta)ab = g(\alpha)g(\beta)ab = \tau(g)(a)\tau(g)(b).$$

For $g \in G, a \in \mathfrak{A}_\alpha$,

$$\tau(g)(a^*) = g(\alpha^{-1})a^* = \overline{g(\alpha)}a^* = (\tau(g)(a))^*.$$

Evidently, $\tau(g)$ is linear on each $\mathfrak{A}_\gamma$. Thus, $\Phi_g \in \mathrm{Aut}(\mathfrak{A})$.
Now, let $g, h \in G$ and $a \in \mathfrak{A}_\gamma$. Then

$$\tau(gh)(a) = (gh)(\gamma)a = g(\gamma)h(\gamma)a = \tau(g)\tau(h)(a).$$

Let $\tau(g) = \tau(h)$. Then $g(\gamma)a = h(\gamma)a$ for all $a \in \mathfrak{A}_\gamma$ and $\gamma \in \Gamma$, which means that $g = h$. □

Remark 2. *Each ideal of a Γ-graded system is invariant under the standard action of the dual group.*

3. MODULES.

Let $\mathfrak{A}$ be a graded system, and B be a C*-algebra. We consider a notion of B-module graded system limiting ourselves to bimodules.

Definition 3. *Let $\mathfrak{A} = (\Gamma, \mathfrak{A}_\Gamma)$ be a Γ-graded system, B be a C^*-algebra. We say the system $\mathfrak{A}$ is a B-module, if each $\mathfrak{A}_\gamma$ is a B-module and the following consistency conditions satisfy (modular multiplication is denoted as $\cdot$)*

(i) $(b \cdot \xi)\eta = b \cdot (\xi\eta),\ \ (\xi \cdot b)\eta = \xi(b \cdot \eta)$

(ii) $(b \cdot \xi) \cdot a = b \cdot (\xi \cdot a),\ \ (\xi \cdot b) \cdot a = \xi \cdot (ba)$

(iii) $(b \cdot \xi)^* = \xi^* \cdot b^*,\ \ (\xi \cdot b)^* = b^* \cdot \xi^*$

for $a, b \in B,\ \xi \in \mathfrak{A}_\alpha,\ \eta \in A_\beta$.

The following four conditions follow immediately from the previous ones and complete all the possibilities with them.

(iv) $\xi(\eta \cdot b) = (\xi\eta) \cdot b, \quad \xi(b \cdot \eta) = (\xi \cdot b)\eta$

(v) $a \cdot (\xi \cdot b) = (b \cdot \xi) \cdot a, \quad a \cdot (b \cdot \xi) = (ab) \cdot \xi$.

Graded systems have the standard modular structure.

Let $\mathfrak{A} = (\Gamma, \mathfrak{A}_\Gamma)$ be a Γ-graded system with the central algebra A. Then each $\mathfrak{A}_\gamma,\ \gamma \in \Gamma$O“ is an A-module (more precisely, bimodule) with respect the operations

$$a \cdot \xi = a\xi,\ \ \xi \cdot a = \xi a,$$

for $a \in A,\ \ \xi \in A_\gamma$. Moreover, the system $\mathfrak{A}$ is also an A-module.

Proof. We omit routine calculations. □

The following proposition introduces the structure of Hilbert A-module on each $A_\gamma,\ \gamma \in \Gamma$.

The space A_γ with the inner product

$$< \xi,\ \eta > = \ \eta^* \xi, \quad \xi, \eta \in A_\gamma, \tag{1}$$

is a (right) Hilbert A-module (denoted as $\mathcal{H}_\gamma$).

Proof. We give only a few obvious relations

$< \xi \cdot a,\ \eta > = < \xi,\ \eta > a,$

$< \xi,\ \eta \cdot a > = \ a^* < \xi, \eta >,$

$< \xi,\ \xi > = \ \|\xi\|^2.$ □

Then, a right Hilbert A-module $\mathcal{H}(\mathfrak{A})$ can be associated to the Γ-graded system $\mathfrak{A}$ as a direct sum of the Hilbert modules on the fibers.
The inner product in $\mathcal{H}(\mathfrak{A})$ is determined as

$$< \xi,\ \eta > = \sum_{\gamma \in \Gamma} \eta_\gamma^* \xi_\gamma, \tag{2}$$

for $\xi,\ \eta \in \mathfrak{A}_\Gamma,\ \xi = \{\xi_\gamma\},\ \eta = \{\eta_\gamma\}$.

Remark 3. *Remind that a system $\xi \in \mathfrak{A}_\Gamma$ belongs to the Hilbert module $\mathcal{H}(\mathfrak{A})$ if the series $\sum \xi_\gamma^* \xi_\gamma$ is convergent in A. It certainly holds if the series $\sum \|\xi_\gamma\|^2$ converges.*
Denote $|\xi_\gamma| = (\xi_\gamma^* \xi_\gamma)^{\frac{1}{2}}$ for $\xi = \{\xi_\gamma\} \in \mathfrak{A}_\Gamma$. Then $\xi \in \mathcal{H}(\mathfrak{A})$ means that the series $\sum |\,\xi_\gamma|^2$ is convergent, and so the Hilbert module $\mathcal{H}(\mathfrak{A})$ associated to a Γ-graded system $\mathfrak{A}$ may be considered as a non-commutative $l^2(\mathfrak{A})$.

4. REPRESENTATIONS.

We consider group graded systems as something like a covariant system. From this viewpoint, the following concept is a continuation of this analogy.

Definition 4. *Let $\mathfrak{A} = (\Gamma, \mathfrak{A}_\Gamma)$ be a Γ-graded system, and H be a Hilbert space. We call a *-representation π of the semigroup $\mathfrak{A}$ in H a representation of the Γ-graded system $\mathfrak{A}$ if it is linear on each $\mathfrak{A}_\gamma, \gamma \in \Gamma$.*

Obviously, the restriction of the representation onto the central algebra is a representation of the central C*-algebra A. It is easy to verify that the kernel of any representation of a graded system is an ideal. The following fact is an immediate consequence of the Proposition 2.

Each representation of a Γ-graded system $\mathfrak{A}$ satisfies the inequality

$$\|\pi(a)\| \leq \|a\| \tag{3}$$

for $a \in \mathfrak{A}$. The representation π is faithful if and only if $\|\pi(a)\| = \|a\|$.
The uniformly closed involutive algebra $C^*(\mathfrak{A}, \pi)$ generated by $\pi(\mathfrak{A})$ in $\mathcal{B}(H)$ we call the C*-algebra π-associated to the graded system $\mathcal{A}$.

For any Hilbert B-module $\mathcal{H}$ we denote by $\mathcal{L}(\mathcal{H})$ the algebra of all adjointable bounded B-linear operators on $\mathcal{H}$.

Definition 5. *Let $\mathfrak{A} = (\Gamma, \mathfrak{A}_\gamma)$ be a Γ-graded B-module, and $\mathcal{H}$ be a (right) Hilbert B-module. A *-representation Π of the semigroup $\mathfrak{A}$ in the algebra $\mathcal{L}(\mathcal{H})$ is called B-representation of the system $\mathfrak{A}$ if it is a B-modular mapping on each space $\mathfrak{A}_\gamma,\ \gamma \in \Gamma$O".*

The uniformly closed involutive algebra $C^*(\mathfrak{A},\ \Pi)$ generated by $\Pi(\mathfrak{A})$ in $\mathcal{L}(\mathcal{H})$ is called the C*-algebra Π-associated to the graded system $\mathfrak{A}$.
Now we introduce a canonical A-representation of a graded system $\mathfrak{A}$ in the associated Hilbert module $\mathcal{H}(\mathfrak{A})$.

Theorem 1. *The mapping specified on generators as*

$$\Pi_r(a)\xi = a \cdot \xi \tag{4}$$

for $a, \xi \in \mathfrak{A}$, determines a faithful A-representation of the system $\mathfrak{A}$ in the associated Hilbert A-module $\mathcal{H}(\mathfrak{A})$.

Proof. It is easy to verify that Π_r is a representation of the system $\mathfrak{A}$, and then $\|\Pi_r(a)\| \leq \|a\|$ by Proposition 4. Let us show that it is faithful. For any $a \in A_\gamma,\ a \neq 0$ we have by (iv) of Definition 1

$$\|\Pi_r\| \geq \|a^*\|^{-1}\|\Pi_r(a)(a^*)\| = \|a\|^{-1}\|aa^*\| = \|a\|^{-1}\|a\|^2 = \|a\|.$$

□

Definition 6. *The representation Π_r introduced via* Theorem 1 *is called (left) regular representation of the graded system $\mathfrak{A}$.*

The last result shows that a graded system can be realized as an operator system. Let us denote by $C_r^*(\mathfrak{A})$ the uniformly closed subalgebra in $\mathcal{L}(\mathcal{H}(\mathfrak{A}))$ generated by the image of the regular representation of a graded system $\mathfrak{A}$. Thus, a covariant functor could be defined from the category of graded systems (with above mentioned morphisms) to the category of C*-algebras with *-homomorphisms as morphisms.

5. FUNCTIONAL DESCRIPTION.

Now we are ready to present a description of the C*-algebra associated with a graded system as the algebra of $\mathfrak{A}$-valued mappings on the compactum $G = \widehat{\Gamma}$, the dual group to the group Γ.
For each $a \in \mathfrak{A}_\gamma,\ \gamma \in \Gamma$, let a continuous function $\hat{a}$ on G be defined as

$$\hat{a}(g) = \gamma(g)a, \tag{5}$$

and let $\widehat{\mathfrak{A}}_\gamma = \{\hat{a} : a \in \mathfrak{A}_\gamma\},\ \gamma \in \Gamma$O". Denote also $\widehat{\mathfrak{A}} = \bigcup_{\gamma\in\Gamma} \widehat{\mathfrak{A}}_\gamma$.

Theorem 2. *The system $\widehat{\mathfrak{A}}$ is a Γ-graded system isomorphic to the system $\mathfrak{A}$, and then the associated reduced C^*-algebras $C^*_r(\widehat{\mathfrak{A}})$ and $C^*_r(\mathfrak{A})$ are isomorphic.*

Proof. Obviously, $\widehat{\mathfrak{A}}$ is an involutive subsemigroup of $C(G, \mathfrak{A})$. Each $\widehat{\mathfrak{A}}_\gamma$ is a Banach space in sup-norm, a subspace of $C(G, \mathfrak{A})$. By virtue of Proposition 3 the only common element of these spaces is the zero function. Then $\widehat{\mathfrak{A}}$ is a Γ-equipped system.
For $a \in \mathfrak{A}_\alpha$ and $b \in \mathfrak{A}_\beta$ we have

$$\hat{a}\hat{b}(g) = \hat{a}(g)\hat{b}(g) = \alpha(g)a\beta(g)b = (\alpha\beta)(g)ab = \widehat{ab}(g)$$

and

$$(\hat{a})^*(g) = (\alpha(g)a)^* = \overline{\alpha(g)}a^* = \widehat{a^*}(g).$$

Thus, $\widehat{\mathfrak{A}}_\alpha\widehat{\mathfrak{A}}_\beta \subset \widehat{\mathfrak{A}}_{\alpha\beta}$, and $\widehat{\mathfrak{A}^*}_\alpha \subset \widehat{\mathfrak{A}}_{\alpha^{-1}}$ which means that the first two conditions of Definition 1 are satisfied. The conditions (iii) and (iv) of the definition are evident since $C(G, \mathfrak{A})$ is a C*-algebra.
Again by Proposition 3 we obtain that the mapping $a \to \hat{a}$ is a bijection which means that the Γ-graded systems $\mathfrak{A}$ and $\widehat{\mathfrak{A}}$ are isomorphic. ☐

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