Nonlinear Hilfer fractional integro-partial differential system

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Abstract

By using fractional calculus and fixed point theorems, existence of mild solution for nonlinear Hilfer fractional integro-partial differential system is studied. In addition, sufficient conditions for controllability of Hilfer fractional integro-partial differential system is established.

Keywords: Hilfer fractional derivative; integro-partial differential equation; mild solution, controllability.

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1 Introduction

Fractional differential equations have received great attention due to their applications in many important applied fields such as population dynamics, heat conduction in materials with memory, seepage flow in porous media, autonomous mobile robots, fluid dynamics, traffic models, electro magnetic, aeronautics, economics and so on, see for instance (see [1-4]). The fractional partial differential systems have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering, such as diffusion process, electrical science, electrochemistry, viscoelasticity, control science, electro magnetic theory, etc (see[5]). On the other hand, Hilfer (see[6]) proposed a generalized Riemann-Liouville fractional derivative (for short, the Hilfer fractional derivative), which includes the Riemann-Liouville and Caputo fractional derivatives. Furati et al. (see[7]) considered an initial value problem for a class of nonlinear fractional differential equations involving the Hilfer fractional derivative. Very recently, Gu and Trujillo (see[8]) investigated a class of evolution equations involving the Hilfer fractional derivatives by using Mittag-Leffler functions (see [9-11]. Mathematical control theory is one of the important concept in the study of steering the dynamical system from given initial state to any other final state or to neighborhood of the final state under some admissible control input. The controllability problem for an evolution equation is also consist of driving the solution of the system to a prescribed final target state (exactly or in some approximate way) in a finite interval of time (see [12-21]) and references therein). In Section 2, we briefly recall some basic known results and lemmas concerning fractional calculus throughout this works. In Section 3, we shall study the existence and uniqueness of mild solution of Hilfer fractional partial Integro-differential equations. In section 4 we shall investigate the sufficient conditions for controllability of Hilfer fractional partial Integro-differential equations.

2 Preliminaries

In this section, we briefly recall some basic known results.

Definition 2.1 (see[22]). The fractional integral of order $\alpha > 0$ with the lower limit zero for a function $f(t)$ is defined as

$$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s)}{(t-s)^{1-\alpha}}ds,$$

$t>0,\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 (see[23-24]). The Hilfer fractional derivative of order $0\le\alpha\le1,\ 0<\beta<1$ is defined as

$$D_{0+}^{\alpha,\beta}f(t)=I^{\beta(1-\alpha)}\frac{d}{dt}I_{0+}^{(1-\beta)(1-\alpha)}f(t)$$

Remark. When $\alpha=0,\ 0<\beta<1$ the Hilfer fractional derivative corresponds to the classical Riemann-Liouville fractional derivative:

$$D_{0+}^{0,\beta}f(t)=\frac{d}{dt}I_{0_+}^{(1-\beta)}f(t)=D_{0_+}^{\beta}f(t)$$

When $\alpha=1,0<\beta<1$ the Hilfer fractional derivative corresponds to the classical Caputo fractional derivative:

$$D_{0+}^{1,\beta}f(t)=I_{0_+}^{(1-\beta)}\frac{d}{dt}f(t)={}_t^cD_{0+}^{\beta}f(t)$$

Throughout this work, we set $J=(0,b]$ where $b>0$ is constant. The state function $w(.,t)$ takes its value in the Banach space $E=L^2([0,2\pi])$ with norm $\|\cdot\|$ and the control function $u(\cdot,t)$ takes its value in the Banach space $L^2([0,2\pi])$. from E to E.

Let $Y=C^{\alpha,\beta}(J,E)=\{w\in C(J,E),\lim_{t\to0+}t^{(1-\alpha)(1-\beta)}w(x,t)$ exist and finite$\}$ with norm $\|\cdot\|_Y$ where $\|w\|_Y=\sup_{t\in J}t^{(1-\alpha)(1-\beta)}\|w(x,t)\|$. Evidently, Y is a Banach space.

For $x\in E$ we defined the two families of operators $\{S_{\alpha,\beta}(t):t>0\}$ and $\{P_\beta(t):t>0\}$ by

$$S_{\alpha,\beta}(t)=I_{0_+}^{\alpha(1-\beta)}P_\beta(t),\ P_\beta(t)=t^{\beta-1}\int_0^\infty\beta\rho\Psi_\beta(\rho)T(t^\beta\rho)d\rho$$

Where

$$\Psi_\beta(\rho)=\sum_1^\infty\frac{(-\rho)^{n-1}}{(n-1)!\Gamma(1-n\beta)}\ \ \rho\in(0,\infty),\ \ 0<\beta<1$$

is the function of Wright type which satisfies

$$\int_0^\infty\rho^\tau\Psi_\beta(\rho)d\rho=\frac{\Gamma(1+\tau)}{\Gamma(1+\beta\tau)}\ \ for\ \rho\ge0$$

Lemma 2.1 (see[25]). The operators $\{S_{\alpha,\beta}(t), t > 0\}$ and $\{P_\beta(t) : t > 0\}$ have the following properties

(i) For any fixed $t > 0$ the operators $S_{\alpha,\beta}(t)$ and $P_\beta(t)$ are bounded linear operators,

$\|P_\beta(t)x\| \leq \dfrac{Mt^{\beta-1}}{\Gamma(\beta)}\|x\| \;\; for \;\; x \in [0, 2\pi]$,

$\|S_{\alpha,\beta}(t)x\| \leq \dfrac{Mt^{(\alpha-1)(1-\beta)}}{\Gamma(\alpha(1-\beta)+\beta)}\|x\| \;\; for \;\; x \in [0, 2\pi]$.

(ii) $\{S_{\alpha,\beta}(t), t > 0\}$ and $\{P_\beta(t) : t > 0\}$ are strongly continuous.

(iii) $P_\beta(t),\; t > 0$ is continuous in the uniform operator topology.

3 Existence and uniqueness of mild solution

In this section, we prove the existence and uniqueness for mild solution of nonlinear Hilfer fractional partial integro-differential equation in the following form

$$\begin{cases} D^{\alpha,\beta}_{0+}w(x,t) = Aw(x,t) + f(t, w(x,t)) + \int_0^t g(t, s, w(x,s), \int_0^s H(s, \tau, w(x, \tau))d\tau)ds \\ , t \in J = (0, b], x \in [0, 2\pi] \\ w(0,t) = w(2\pi, t) = 0, \;\; t \in J \\ I^{(1-\gamma)(1-\beta)}_{0+}w(x, 0) = w_0(x) \;, x \in [0, 2\pi] \end{cases} \tag{3.1}$$

Where $D^{\alpha,\beta}_{0+}$ is the Hilfer fractional derivative of order $0 \leq \alpha \leq 1$, $0 < \beta < 1$,

A is a closed, linear operator and densely defined operator in E. The operator A is the infinitesimal generator of a C_0-semigroup $T(t)$ on E, and there exists a constant $M > 0$ such that $\|T(t)\| \leq M$. $w(., t)$ takes the value in Banach space E.

The nonlinear operators $f : J \times E \to E$, $H : J \times J \times E \to E$ and $g : J \times J \times E \times E \to E$ are given.

Definition 3.1. We say that $w(x, t) \in Y$ mild solution of system (3.1), if it satisfies

$$w(x,t) = S_{\alpha,\beta}(t)w_0 + \int_0^t P_\beta(t-s)f(s, w(x,s))ds + \int_0^t P_\beta(t-s)$$

$$\times \int_0^s g\bigg(s, \tau, w(x, \tau), R(\tau))\bigg)d\tau ds, \quad t \in J = (0, b], x \in [0, 2\pi]$$

where $\int_0^\tau H\Big(\tau,\delta,w(x,\delta)\Big)d\delta = R(\tau)$.

To establish the results, we assume the following hypotheses:

(H1)(i) For each $t \in J$ the function $f(t,.) : E \longrightarrow E$ is continuous, and for each $w \in E$ the function $f(.,w(.,x)) : J \longrightarrow E$ is strongly measurable.

(ii) For each positive number $r \in N$, there is positive function $z_r(.) : J \longrightarrow R^+$ such that

$$\sup_{\|w\|<r} \|f(t,w(x,t))\| \leq z_r(t),$$

the function $s \longrightarrow (t-s)^{\beta-1}z_r(s) \in L^1([0,b],R^+)$ and there exist a $\nu > 0$ such that

$$lim_{r\to\infty} \inf \frac{\int_0^t (t-s)^{\beta-1}z_r(s)ds}{r} = \nu$$

(H2) For each $(t,s) \in J \times J$, the function $H(t,s,.) : E \longrightarrow E$ is continues and for each $w \in E$, the function $H(.,.,w(x,.)) : J \times J \longrightarrow E$ is strongly measurable.

(H3) (i)For each $(t,s,v,\kappa) \in J \times J \times E \times E$, the function $g(t,s,.,.) : E \times E \longrightarrow E$ is continues,and for each $v,\kappa \in E$ the function $g(.,.,v,\kappa) : J \times J \longrightarrow E$ strongly measurable.

(ii)For each positive number $r \in N$, there is positive function $h_r(.) : J \longrightarrow R^+$ such that

$$\sup_{\|w\|<r} \| \int_0^t g\Big(t,s,w(x,t),\int_0^s H(s,\tau,w(x,\tau))d\tau\Big)ds\| \leq h_r(t),$$

$$lim_{r\to\infty} \inf \frac{\int_0^t (t-s)^{\beta-1}h_r(s)ds}{r} = \mu$$

Theorem 3.1. If hypotheses (H1)-(H3) are satisfied, then the system (3.1) has a unique mild solution in J Provided that $\frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)}\{\nu+\mu\} < 1$.

Proof: We defined the operator $K : Y \longrightarrow Y$ by

$$Kw(x,t) = S_{\alpha,\beta}(t)w_0(x) + \int_0^t P_\beta(t-s)F(s,w(x,s))ds$$

$$+\int_0^t P_\beta(t-s)\int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau ds$$

Let

$$B_r = \{w(x,t) \in Y, w(x,0) = w_0(x), \|w(x,t)\| \leq r\}$$

which is bounded and closed subset of Y

Step1. We will show that there exists $r > 0$ such that $Kw(x,t) \subset B_r$.

We claim that there exists a positive number r such that $KB_r \subset B_r$. If this is not true, then for each positive number r, there exists a function $w(x,t) \in B_r$ and $\|Kw(x,t)\|_Y \geq r$ then $1 < r^{-1}\|Kw_r(x,t)\|$, and so $1 < \lim_{r\to\infty} r^{-1}\|Kw_r(x,t)\|_Y$. However,

$$
\begin{aligned}
&\lim_{r\to\infty} r^{-1}\|Kw_r(x,t)\|_Y \\
&\leq \lim_{r\to\infty} r^{-1}\Bigg\{\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\|S_{\alpha,\beta}(t)w_0(x)\| \\
&+\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\|\int_0^t P_\beta(t-s)f(s,w(x,s))ds\| \\
&+\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\|\int_0^t P_\beta(t-s)\int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau ds\|\Bigg\} \\
&\leq \lim_{r\to\infty} r^{-1}\Bigg\{\frac{M\|w_0(x)\|}{\Gamma(\alpha(1-\beta)+\beta)}+\frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\|f(s,w(x,s))\|ds \\
&+\frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\|\int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau\|ds\Bigg\} \\
&\leq \lim_{r\to\infty} r^{-1}\Bigg\{\frac{M\|w_0(x)\|}{\Gamma(\alpha(1-\beta)+\beta)}+\frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}z_r(s)ds \\
&+\frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h_r(s)ds\Bigg\} \\
&\leq \frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)}\{\nu+\mu\} < 1,
\end{aligned}
$$

a contradiction. Hence, $KB_r \subset B_r$ for some positive number r.

Step2. We prove that the operator K maps B_r into a compact subset of B_r.

Let the set $D_r(t) = \{(Kw)(t) : w \in B_r\}$ is precompact in E for all $t \in J$. This is trivial for $t = 0$, since $D_r(0) = \{w_0(x)\}$. Let $0 < t \leq b$ be fixed, for $0 < \epsilon < t$ and arbitrary $\varrho > 0$;

$$
\begin{aligned}
(K^{\epsilon,\delta}w)(t) &= \int_0^{t-\epsilon}\frac{(t-s)^{\alpha(1-\beta)-1}}{\Gamma(\alpha(1-\beta))}\int_\varrho^\infty \beta\rho s^{\beta-1}\Psi_\beta(\rho)T(s^\beta\rho)w_0(x)d\rho ds \\
&+\beta\int_0^{t-\epsilon}\int_\varrho^\infty \rho(t-s)^{\beta-1}\Psi_\beta(\rho)T((t-s)^\beta\rho)f(s,w(x,s))d\rho ds
\end{aligned}
$$

$$+\beta \int_0^{t-\epsilon}\int_{\varrho}^{\infty}\rho(t-s)^{\beta-1}\Psi_\beta \rho T((t-s)^\beta\rho)\int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau d\rho ds$$

$$= T(\epsilon^\alpha\varrho)\int_0^{t-\epsilon}\frac{(t-s)^{\alpha(1-\beta)-1}}{\Gamma(\alpha(1-\beta))}\int_{\varrho}^{\infty}\beta\rho s^{\beta-1}\Psi_\beta(\rho)T(s^\beta\rho-\epsilon^\alpha\varrho)w_0(x)d\rho ds$$

$$+\beta T(\epsilon^\alpha\varrho)\int_0^{t-\epsilon}\int_{\varrho}^{\infty}\rho(t-s)^{\beta-1}\Psi_\beta\rho T((t-s)^\beta\rho-\epsilon^\alpha\varrho)f(s,w(x,s))d\rho ds$$

$$+\beta T(\epsilon^\alpha\varrho)\int_0^{t-\epsilon}\int_{\varrho}^{\infty}\rho(t-s)^{\beta-1}\Psi_\beta\rho T((t-s)^\beta\rho-\epsilon^\alpha\varrho)\int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau d\rho ds$$

Since $T(\epsilon^\alpha\varrho),\ \epsilon^\alpha\varrho>0$ is compact operator, then the set $D_r^{\epsilon,\varrho}=\{(Kw)(t): w\in B_r\}$ is a precompact set in Y for every $\epsilon,\ 0<\epsilon<t$ and for all $\varrho>0$, also for $w(x,t)\in B_r$

$$\|Kw(x,t)-K^{\epsilon,\varrho}w(x,t)\|_Y =$$

$$\leq \sup_{t\in J} t^{(1-\alpha)(1-\beta)}\|\int_0^t\frac{(t-s)^{\alpha(1-\beta)-1}}{\Gamma(\alpha(1-\beta))}\int_0^{\varrho}\beta\rho s^{\beta-1}\Psi_\beta(\rho)T(s^\beta\rho)w_0(x)d\rho ds\|$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\|\int_{t-\epsilon}^t\frac{(t-s)^{\alpha(1-\beta)-1}}{\Gamma(\alpha(1-\beta))}\int_{\varrho}^{\infty}\beta\rho s^{\beta-1}\Psi_\beta(\rho)T(s^\beta\rho)w_0(x)d\rho ds\|$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\beta\|\int_{t-\epsilon}^t\int_{\varrho}^{\infty}\rho(t-s)^{\beta-1}\Psi_\beta\rho T((t-s)^\beta\rho)f(s,w(x,s))d\rho ds\|$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\beta\|\int_{t-\epsilon}^t\int_{\varrho}^{\infty}\rho(t-s)^{\beta-1}\Psi_\beta\rho T((t-s)^\beta\rho)\int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau d\rho ds\|$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\beta\|\int_0^t\int_0^{\varrho}\rho(t-s)^{\beta-1}\Psi_\beta\rho T((t-s)^\beta\rho)f(s,w(x,s))d\rho ds\|$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\beta\|\int_0^t\int_0^{\varrho}\rho(t-s)^{\beta-1}\Psi_\beta\rho T((t-s)^\beta\rho)\int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau d\rho ds\|$$

$$\leq \frac{\beta M\|w_0(x)\|}{\Gamma(\alpha(1-\beta))}\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\int_0^t\Big((t-s)^{\alpha(1-\beta)-1}s^{\beta-1}\Big)\Big(\int_0^{\varrho}\rho\Psi_\beta(\rho)d\rho\Big)ds$$

$$+\frac{M\|w_0(x)\|\beta}{\Gamma(\alpha(1-\beta))}\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\int_{t-\epsilon}^t\Big((t-s)^{\alpha(1-\beta)-1}s^{\beta-1}\Big)\Big(\int_{\varrho}^{\infty}\rho\Psi_\beta(\rho)d\rho\Big)ds$$

$$+M\beta\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\Big(\int_{t-\epsilon}^t(t-s)^{\beta-1}z_r(t)ds\Big)\Big(\int_{\varrho}^{\infty}\rho\Psi_\beta(\rho)d\rho\Big)$$

$$+M\beta\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\Big(\int_{t-\epsilon}^t(t-s)^{\beta-1}h_r(t)ds\Big)\Big(\int_{\varrho}^{\infty}\rho\Psi_\beta(\rho)d\rho\Big)$$

$$+M\beta\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\Big(\int_0^t(t-s)^{\beta-1}z_r(t)ds\Big)\Big(\int_0^{\varrho}\rho\Psi_\beta(\rho)d\rho\Big)$$

$$+M\beta \sup_{t\in J} t^{(1-\alpha)(1-\beta)}\Big(\int_0^t (t-s)^{\beta-1} h_r(t)ds\Big)\Big(\int_0^{\varrho} \rho\Psi_\beta(\rho)d\rho\Big)$$

As $\epsilon \to 0+$ and $\varrho \to 0+$ there are precompact sets arbitrary close to the set $D_r(t)$ and so $D_r(t)$ is precompact in Y

Step3. We will show that the $Kw_r = \{Kw :, w \in B_r\}$ is an equicontinuous family of functions.

Let $w \in B_r$ and $t, \ell \in J$ such that $0 < t < \ell < b$

$$\|(Kw)(t) - (Kw)(\ell)\|_Y$$

$$\leq \sup_{t\in J} t^{(1-\alpha)(1-\beta)} \|\Big(S_{\alpha,\beta}(\ell) - S_{\alpha,\beta}(t)\Big) w_0(x)\|$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)} \|\int_0^t \Big(P_\beta(\ell-s) - P_\beta(t-s)\Big) f(s, w(x,s))ds\|$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)} \|\int_t^\ell P_\beta(\ell-s) f(s, w(x,s))ds\|$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)} \|\int_0^t \Big(P_\beta(\ell-s) - P_\beta(t-s)\Big)\int_0^s g\Big(s,\tau, w(x,\tau), R(\tau)\Big)d\tau ds\|$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)} \|\int_t^\ell P_\beta(\ell-s)\int_0^s g\Big(s,\tau, w(x,\tau), R(\tau)\Big)d\tau ds\|$$

Its clearly that, the right-hand side of the above inequality tends to zero as $t \to \ell$. The compactness of $P_\beta(t)$ for $t > 0$ implies the continuity in the uniform operator topology , thus KB_r is both equicontinuous and bounded .By using Arzela-Ascoli theorem, KB_r is precompact in Y. Hence K is a completely continuous operator on Y, from the Schauder fixed-point theorem, K has a fixed point in B_k. Any fixed point of K is a mild solution of (3.1) on J satisfying $Kw(x,t) = w(x,t)$. The proof is completed.

4 Controllability of system

In this chapter, we will establish a set of sufficient conditions for controllability of nonlinear Hilfer fractional partial integro-differential equation in the following form:-

$$\begin{cases} D_{0+}^{\alpha,\beta} w(x,t) = Aw(x,t) + f(t, w(x,t)) + Bu(x,t) + \int_0^t g(t,s,w(x,s), \int_0^s H(s,\tau,w(x,\tau))d\tau)ds \\ , t \in J = (0,b], x \in [0, 2\pi] \\ w(0,t) = w(2\pi,t) = 0, \ t \in J \\ I_{0+}^{(1-\gamma)(1-\beta)} w(x,0) = w_0(x) \ , x \in [0, 2\pi] \end{cases} \tag{4.2}$$

where $D_0^{\alpha,\beta} w(x,t)$, $w(x,t)$, $f(t,w(x,t))$ and $g\Big(t,s,w(x,s), \int_0^s h\Big(s,\tau,w(x,\tau)\Big)d\tau\Big)$ are defined in the previous sections and the control function $u(\cdot,t)$ is in $L^2(J,U)$, a Banach space of admissible control functions where $U = L^2([0,2\pi])$, with U as a Banach space. The symbol B stands for a bounded linear from U into E.

Definition 4.1. We say $w(x,t) \in Y$ is a mild solution of system (4.2) if it satisfies:

$$w(x,t) = S_{\alpha,\beta}(t)w_0 + \int_0^t P_\beta(t-s)f(s,w(x,s))ds + \int_0^t P_\beta(t-s)Bu(x,s)ds + \int_0^t P_\beta(t-s)$$
$$\times \int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau ds, \quad x \in [0,2\pi], \ t \in J$$

Definition 4.2. The system (4.2) is said to be controllable on interval J if, for every w_0, $w_1 \in Y$, there exists a control $u(\cdot,t) \in L^2(J,U)$ such that the mild solution $w(x,t)$ of system(4.2) satisfies $w(.,b) = w_1(x)$, where w_1 and b are the preassigned terminal state and time, respectively. To prove the result let us assume the following hypotheses:

(H4)The linear operator Q from U into E defined by

$$Qw = \int_0^t P_\beta(b-s)Bu(x,s)ds$$

has an inverse operator Q^{-1} which takes values in $L^2(J,U)/Ker\ Q$ where the kernel space of Q is defined by:-

Ker $Q = \{w \in Y : Qw = 0\}$ and B is bounded liner operator.

Theorem 4.1. If the hypotheses (H1)-(H4) are satisfied, then the system (4.1) is controllable on interval J provided that

$$\frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)}\{\mu+\nu\}+\frac{Mb^{\beta}\|Q^{-1}\|\|B\|}{\Gamma(\beta+1)}\left\{\frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)}\{\nu+\mu\}\right\}<1$$

Proof. Using assumption (H4), define the control

$$u(x,t)=Q^{-1}\Bigg\{w_1-S_{\alpha,\beta}(b)w_0(x)-\int_0^b P_\beta(b-s)\Bigg\{f(s,w(x,s))$$

$$+\int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau\Bigg\}ds\Bigg\}(t)$$

We defined the operator $\Omega: Y\longrightarrow Y$ by:-

$$\Omega w(x,t)=S_{\alpha,\beta}(t)w_0(x)+\int_0^t P_\beta(t-s)F(s,w(x,s))ds$$

$$+\int_0^t P_\beta(t-s)\int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau ds$$

$$+\int_0^t P_\beta(t-\eta)Q^{-1}B\Bigg\{w_1(x)-S_{\alpha,\beta}(b)w_0(x)-\int_0^b P_\beta(b-s)\Bigg\{f(s,w(x,s))$$

$$+\int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau\Bigg\}ds\Bigg\}d\eta$$

which is bounded and closed subset of Y Then:-

$$\Omega w(x,b)=S_{\alpha,\beta}(b)w_0(x)+\int_0^b P_\beta(b-s)f(s,w(x,s))ds$$

$$+\int_0^b P_\beta(b-s)\int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau ds$$

$$+\int_0^b P_\beta(b-\eta)Q^{-1}B\Bigg\{w_1(x)-S_{\alpha,\beta}(b)w_0(x)-\int_0^b P_\beta(b-s)\Bigg\{f(s,w(x,s))$$

$$+\int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau\Bigg\}ds\Bigg\}d\eta=w_1(x)$$

Step1. We will show that there exist $r>0$ such that $\Omega w(x,t)\subset B_r$.

IF this is not true. Then there exists $w_r(x,t)\in B_r$ such that $\|\Omega w_r(x,t)\|_Y>r$ for some $t\in J, x\in[0,2\pi]$.then we have:-

$$r<\|\Omega w_r(x,t)\|_Y$$

$$\le \| \sup_{t\in J} t^{(1-\alpha)(1-\beta)} S_{\alpha,\beta}(t) w_0(x)\| + \| \sup_{t\in J} t^{(1-\alpha)(1-\beta)} \int_0^t P_\beta(t-s) f(s, w(x,s)) ds\|$$

$$+ \| \sup_{t\in J} t^{(1-\alpha)(1-\beta)} \int_0^t P_\beta(t-s) \int_0^s g\Big(s, \tau, w(x,\tau), R(\tau)\Big) d\tau ds\|$$

$$+ \| \sup_{t\in J} t^{(1-\alpha)(1-\beta)} \int_0^t P_\beta(t-\eta) Q^{-1} B \Big\{ w_1(x) - S_{\alpha,\beta}(b) w_0(x) - \int_0^b P_\beta(b-s) \Big\{ f(s, w(x,s))$$

$$+ \int_0^s g\Big(s, \tau, w(x,\tau), R(\tau)\Big) d\tau \Big\} ds \Big\} d\eta \|$$

$$\le \frac{M\|w_0(x)\|}{\Gamma(\alpha(1-\beta)+\beta)} + \frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)} \int_0^t (t - s^{\beta-1}) z_r(s) ds$$

$$+ \frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h_r(s) ds$$

$$+ \int_0^t \|P_\beta(t-\eta)\| \|Q^{-1}\| \|B\| \Big\{ b^{(1-\alpha)(1-\beta)} \|w_1(x)\| + \frac{M\|w_0(x)\|}{\Gamma(\alpha(1-\beta)+\beta)}$$

$$+ \frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)} \int_0^t (t - s^{\beta-1}) z_r(s) ds$$

$$+ \frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h_r(s) ds \Big\} d\eta$$

$$\le \frac{M\|w_0(x)\|}{\Gamma(\alpha(1-\beta)+\beta)} + \frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)} \int_0^t (t - s^{\beta-1}) z_r(s) ds$$

$$+ \frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h_r(s) ds$$

$$+ \frac{Mb^{\beta}\|Q^{-1}\|\|B\|}{\beta\Gamma(\beta)} \Big\{ b^{(1-\alpha)(1-\beta)} \|w_1(x)\| + \frac{M\|w_0(x)\|}{\Gamma(\alpha(1-\beta)+\beta)}$$

$$+ \frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)} \int_0^t (t - s^{\beta-1}) z_r(s) ds$$

$$+ \frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h_r(s) ds \Big\}$$

Dividing both sides of the previous inequality by r and taking the lower limit as $r \to +\infty$, we get

$$1 \le \frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)} \{\mu + \nu\} + \frac{Mb^{\beta}\|Q^{-1}\|\|B\|}{\Gamma(\beta+1)} \Big\{ \frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)} \{\nu + \mu\} \Big\}$$

which is contradiction. Therefore $\Omega w(x,t) \subset B_r$ for some $r > 0$

Step2. To prove that the operator Ω maps B_r into a compact subset of B_r.

Let the set $D_r(t) = \{(\Omega w)(t) : w \in B_r\}$ is precompact in Y for all $t \in J$, since $D_r(0) = \{w_0(x)\}$, let $0 < t \leq b$ be fixed, for $0 < \epsilon < t$ and arbitrary $\varrho > 0$;

$$(\Omega^{\epsilon,\varrho} w)(t) = \int_0^{t-\epsilon} \frac{(t-s)^{\alpha(1-\beta)-1}}{\Gamma(\alpha(1-\beta))} \int_\varrho^\infty \beta\rho s^{\beta-1}\Psi_\beta(\rho)T(s^\beta\rho)w_0(x)d\rho ds$$

$$+\beta \int_0^{t-\epsilon}\int_\varrho^\infty \rho(t-s)^{\beta-1}\Psi_\beta(\rho)T((t-s)^\beta\rho)f(s,w(x,s))d\rho ds$$

$$+\beta \int_0^{t-\epsilon}\int_\varrho^\infty \rho(t-s)^{\beta-1}\Psi_\beta\rho T((t-s)^\beta\rho)\int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau d\rho ds$$

$$+\beta \int_0^{t-\epsilon}\int_\varrho^\infty \rho(t-\eta)^{\beta-1}\Psi_\beta(\rho)T((t-\eta)^\beta\rho)Q^{-1}B\Big\{w_1(x)-S_{\alpha,\beta}(b)w_0(x)-\int_0^b P_\beta(b-s)\Big\{f(s,w(x,s))$$

$$+\int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau\Big\}ds\Big\}(\eta)d\rho d\eta$$

$$= T(\epsilon^\alpha\varrho)\int_0^{t-\epsilon} \frac{(t-s)^{\alpha(1-\beta)-1}}{\Gamma(\alpha(1-\beta))}\int_\varrho^\infty \beta\rho s^{\beta-1}\Psi_\beta(\rho)T(s^\beta\rho-\epsilon^\alpha\varrho)w_0(x)d\rho ds$$

$$+\beta T(\epsilon^\alpha\varrho)\int_0^{t-\epsilon}\int_\varrho^\infty \rho(t-s)^{\beta-1}\Psi_\beta\rho T((t-s)^\beta\rho-\epsilon^\alpha\varrho)f(s,w(x,s))d\rho ds$$

$$+\beta T(\epsilon^\alpha\varrho)\int_0^{t-\epsilon}\int_\varrho^\infty \rho(t-s)^{\beta-1}\Psi_\beta\rho T((t-s)^\beta\rho-\epsilon^\alpha\varrho)\int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau d\rho ds$$

$$-\beta T(\epsilon^\alpha\varrho)\int_0^{t-\epsilon}\int_\varrho^\infty \rho(t-\eta)^{\beta-1}\Psi_\beta\rho T((t-\eta)^\beta\rho-\epsilon^\alpha\varrho)Q^{-1}B\Big\{w_1(x)+S_{\alpha,\beta}(b)w_0(x)$$

$$+\int_0^b P_\beta(b-s)f(s,w(x,s))+\int_0^b P_\beta(b-s)\int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau ds\Big\}(\eta)d\rho d\eta.$$

Since $T(\epsilon^\alpha\varrho),\ \epsilon^\alpha\varrho > 0$ is compact operator,then the set $D_k^{\epsilon,\varrho} = \{(\Omega w)(t) : w \in B_r\}$ is a precompact set in E for every $\epsilon,\ 0 < \epsilon < t$ and for all $\varrho > 0$, also for $w(x,t) \in B_r$ Moreover, for $w \in B_r$, we have

$$\|\Omega w(x,t) - \Omega^{\epsilon,\varrho} w(x,t)\|_Y =$$

$$\leq \frac{M\|w_0(x)\|\beta}{\Gamma(\alpha(1-\beta))}\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\int_0^t \Big\{\Big((t-s)^{\alpha(1-\beta)-1}s^{\beta-1}\Big)\Big(\int_0^\varrho \rho\Psi_\beta(\rho)d\rho\Big)\Big\}ds$$

$$+\frac{M\|w_0(x)\|\beta}{\Gamma(\alpha(1-\beta))}\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\int_{t-\epsilon}^t \Big\{\Big((t-s)^{\alpha(1-\beta)-1}s^{\beta-1}\Big)\Big(\int_\varrho^\infty \rho\Psi_\beta(\rho)d\rho\Big)\Big\}ds$$

$$+M\beta\sup_{t\in J}t^{(1-\alpha)(1-\beta)}\bigg(\int_{t-\epsilon}^{t}(t-s)^{\beta-1}z_r(t)ds\bigg)\bigg(\int_{\varrho}^{\infty}\rho\Psi_\beta(\rho)d\rho\bigg)$$

$$+M\beta\sup_{t\in J}t^{(1-\alpha)(1-\beta)}\bigg(\int_{t-\epsilon}^{t}(t-s)^{\beta-1}h_r(t)ds\bigg)\bigg(\int_{\varrho}^{\infty}\rho\Psi_\beta(\rho)d\rho\bigg)$$

$$+M\beta\sup_{t\in J}t^{(1-\alpha)(1-\beta)}\bigg(\int_{0}^{t}(t-s)^{\beta-1}z_r(t)ds\bigg)\bigg(\int_{0}^{\varrho}\rho\Psi_\beta(\rho)d\rho\bigg)$$

$$+M\beta\sup_{t\in J}t^{(1-\alpha)(1-\beta)}\bigg(\int_{0}^{t}(t-s)^{\beta-1}h_r(t)ds\bigg)\bigg(\int_{0}^{\varrho}\rho\Psi_\beta(\rho)d\rho\bigg)$$

$$+M\beta\sup_{t\in J}t^{(1-\alpha)(1-\beta)}\int_{t-\epsilon}^{t}(t-\eta)^{\beta-1}\bigg(\int_{\varrho}^{\infty}\rho\Psi_\beta(\rho)d\rho\bigg)d\eta\|B\|\|Q^{-1}\bigg\{\|w_1(x)\|+\frac{M\|w_0(x)\|}{\Gamma(\alpha(1-\beta)+\beta)}$$

$$+\frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)}\int_0^b(b-s)^{\beta-1}z_r(s)ds$$

$$+\frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)}\int_0^b(b-s)^{\beta-1}h_r(s)ds\bigg\}$$

$$+M\beta\sup_{t\in J}t^{(1-\alpha)(1-\beta)}\int_{0}^{t}(t-\eta)^{\beta-1}\bigg(\int_{0}^{\varrho}\rho\Psi_\beta(\rho)d\rho\bigg)d\eta\|B\|\|Q^{-1}\bigg\{\|w_1(x)\|+\frac{M\|w_0(x)\|}{\Gamma(\alpha(1-\beta)+\beta)}$$

$$+\frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)}\int_0^b(b-s^{\beta-1})z_r(s)ds$$

$$+\frac{Mb^{(1-\alpha)(1-\beta)}}{\Gamma(\beta)}\int_0^b(b-s)^{\beta-1}h_r(s)ds\bigg\}$$

As $\epsilon\to 0+$ and $\varrho\to 0+$ there are precompact sets arbitrary close to the set $D_r(t)$ and so $D_r(t)$ is precompact in Y

Step3.

We will show that the $\Omega w_r=\{\Omega w, w\in B_r\}$ is an equicontinuous family of functions. Let $w\in B_r$ and $t,\ell\in J$ such that $0<t<\ell<b$

$$\|(\Omega w)(t)-(\Omega w)(\ell)\|_Y$$

$$\leq\sup_{t\in J}t^{(1-\alpha)(1-\beta)}\|S_{\alpha,\beta}(\ell)-S_{\alpha,\beta}(t)\|\|w_0(x)\|$$

$$+\sup_{t\in J}t^{(1-\alpha)(1-\beta)}\|\int_0^t\Big\{P_\beta(\ell-s)-P_\beta(t-s)\Big\}f(s,w(x,s))ds\|$$

$$+\sup_{t\in J}t^{(1-\alpha)(1-\beta)}\|\int_t^{\ell}P_\beta(\ell-s)f(s,w(x,s))ds\|$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\| \int_0^t \Big\{P_\beta(\ell-s)-P_\beta(t-s)\Big\} \int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau ds\|$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\| \int_t^\ell P_\beta(\ell-s) \int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau ds\|$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\| \int_0^t \Big\{P_\beta(\ell-\eta)-P_\beta(t-\eta)\Big\}\|$$

$$\times\|Q^{-1}B\Big\{w_1(x)-S_{\alpha,\beta}(b)w_0(x)-\int_0^b P_\beta(b-s)\Big\{f(\tau,w(x,\tau))$$

$$+\int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau\Big\}ds\Big\}d\eta\|$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)}\| \int_t^\ell P_\beta(\ell-\eta)\|$$

$$\times\|Q^{-1}B\Big\{w_1(x)-S_{\alpha,\beta}(b)w_0(x)-\int_0^b P_\beta(b-s)\Big\{f(\tau,w(x,\tau))$$

$$+\int_0^s g\Big(s,\tau,w(x,\tau),R(\tau)\Big)d\tau\Big\}ds\Big\}d\eta\|$$

$$\leq \frac{2M\|w_0(x)\|}{\Gamma(\alpha(1-\beta)+\beta)}$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)} \int_0^t \|P_\beta(\ell-s)-P_\beta(t-s)\|z_r(s)ds$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)} \int_t^\ell \|P_\beta(\ell-s)\|z_r(s)ds$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)} \int_0^t \|P_\beta(\ell-s)-P_\beta(t-s)\|h_r(s)ds$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)} \int_t^\ell \|P_\beta(\ell-s)\|h_r(s)ds$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)} \int_0^t \|P_\beta(\ell-s)-P_\beta(t-s)\|\|Q^{-1}\|\|B\|$$

$$\times\Big\{\|w_1(x)\|+\frac{M\|w_0(x)\|}{\Gamma(\alpha(1-\beta)+\beta)}+\frac{M}{\Gamma(\beta)}\int_0^b (b-\eta)^{(\beta-1)}z_r(\eta)d\eta$$

$$+\frac{M}{\Gamma(\beta)}\int_0^\eta (t-\eta)^{\beta-1}h_r(\eta)d\eta\Big\}ds$$

$$+\sup_{t\in J} t^{(1-\alpha)(1-\beta)} \int_t^\ell \|P_\beta(\ell-s)\|\|Q^{-1}\|\|B\|$$

$$\times\left\{\|w_1(x)\|+\frac{M\|w_0(x)\|}{\Gamma(\alpha(1-\beta)+\beta)}+\frac{M}{\Gamma(\beta)}\int_0^b(b-\eta)^{(\beta-1)}z_r(\eta)d\eta\right.$$
$$\left.+\frac{M}{\Gamma(\beta)}\int_0^\eta(t-\eta)^{\beta-1}h_r(\eta)d\eta\right\}ds$$

its clearly that and the right-hand side of the above inequality tends to zero as $t \to \ell$ the compactness of $P_\beta(t)$ for $t > 0$ implies the continuity in the uniform operator topology , thus ΩB_r is both equicontinuous and bounded .By using Arzela-Ascoli theorem, ΩB_r is precompact in Y. Hence Ω is a completely continuous operator on Y, from the Schauder fixed-point theorem, Ω has a fixed point in B_r. Any fixed point of Ω is a mild solution of (4.2) on J satisfying $\Omega w(x,t) = w(x,t)$. Hence the system (4.2) is controllable on interval J.

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