Approximation to Constant Functions by Electrostatic Fields due to Electrons and Positrons

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Abstract—We study a uniform approximation to constant functions $f(z) = \mathrm{const}$ on compact subsets K of complex plane by logarithmic derivatives of rational functions with free poles. This problem can be treated in terms of electrostatics: we construct on K the constant electrostatic field due to electrons and positrons at points $\notin K$. If K is a disk or an interval, we get the approximation, which close to the best. Also we get the new identity for generalized Laguerre polynomials. Our results related to the classical problem of rational approximation to the exponential function.

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1. INTRODUCTION

1.1. The problem of uniform approximation to analytic functions on compact subsets K of complex plane $\mathbb{C}$ by sums

$$S_n(z) = \sum_{j=1}^{n} \frac{1}{z - z_j}, \qquad z \in K, \quad z_j \in \overline{\mathbb{C}} \setminus K, \quad n = 0, 1, 2, \ldots \quad (S_0(z) \equiv 0)$$

with free[1] poles [1–4] as well as restricted poles [5–7] is well-known. Approximation by $S_n(z)$ in other spaces studied, for example, in [8–11]. Sums S_n are also called *simple partial fractions* or *simplest fractions* (suggestion of E. P. Dolzhenko). It may be noted, that the complex conjugate of $-S_n(z)$ represents the electrostatic field at the point $z \in K$ due to electrons at points z_j (sf. [5, 9]).

A natural generalization is approximation by sums

$$S_{mn}(z) = \sum_{j=1}^{m} \frac{1}{z - z_j} - \sum_{j=1}^{n} \frac{1}{z - \tilde{z}_j}, \qquad z \in K, \quad z_j, \tilde{z}_j \in \overline{\mathbb{C}} \setminus K, \tag{1}$$

$m, n = 0, 1, 2, \ldots$. First results on approximation by S_{nn} ($m = n$) with free poles proved in [1]. Later author in [12] has showed, that the order of approximation to complex polynomials by sums S_{nn} much better, than by sums S_n, and so, sums S_{nn} much more effective. Density of the set of sums (1) with special constraints on poles in spaces of analytic functions studied in [7].

In this paper we study a uniform approximation to *constant functions* $f(z) = \mathrm{const}$ on compact sets K by sums (1) with arbitrary m and n and free poles, i.e., we construct constant electrostatic fields due

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[1] We set $(z - z_j)^{-1} \equiv 0$ as $z_j = \infty$.

to electrons and positrons at points $z_j, \tilde{z}_j \notin K$. Since $cS_{mn}(cz)$ is also a sum of type (1) for any constant c, we need only consider the case

$$f(z) = 1, \qquad z \in K.$$

The problem $S_{mn}(z) \to 1$ related to the classical problem of approximation to e^z by the set $\mathcal{R}_{mn}$ of all rationals of type (m, n), because 1 is a logarithmic derivative of e^z, and any function S_{mn} is a logarithmic derivative r'/r of $r \in \mathcal{R}_{mn}$ (while S_n is a logarithmic derivative of polynomial).

1.2. In the Section 2 for arbitrary K, m and n we construct the rational function $S(z) = S(m, n; z)$ of the form (1), such that uniformly for $z \in K$

$$S(z) - 1 = (-1)^{n+1} \frac{m!n!z^{m+n}}{(m+n)!^2} e^{z(n-m)/(m+n)}(1 + o(1)) \qquad \text{as} \quad m + n \to \infty \tag{2}$$

(Theorem 1). We prove, that in every disk $K = K_a = \{|z| \le a\}$ the order of best approximation $d_{mn} = \inf \|S_{mn} - 1\|_a$ (where S_{mn} ranges over the set of sums (1) and $\| \cdot \|_a$ is the sup-norm over K_a) very close to order of approximation (2) (Theorem 2). For example, as $m = n \to \infty$ we have

$$\frac{e^{-a}A_{nn}}{2n+1}(1 + o(1)) \le d_{nn} \le \|S(n, n; \cdot) - 1\|_a = A_{nn}(1 + o(1)),$$

where $A_{nn} = n!^2 a^{2n}/(2n)!^2$. Note, that an estimate of d_{nn} in [12] have the same order at n, but with worse constants (for example, $d_{nn} \le 2A_{nn}$). If we put $(n, 0)$ instead of (m, n) in (2), we get $\inf_{S_n} \|S_n - 1\|_a \le e^a a^n (n!)^{-1}(1 + o(1))$. Remark, that upper bound $2e^a a^n (n!)^{-1}$ (as $n \ge 5a$) proved in [1, Theorem 1].

The next interesting identity for generalized Laguerre polynomials L_n^α immediately follows from Theorem 1 (see the Sec. 2.2):

$$L_m^{-m-n}(z)L_n^{-m-n}(-z) - L_{m-1}^{-m-n}(z)L_{n-1}^{-m-n}(-z) = \frac{z^n(-z)^m}{m!n!}. \tag{3}$$

1.3. In the Section 3 we study an approximation to $f(x) = 1$ in the sup-norm $\| \cdot \|$ over interval $[-1, 1]$ by sums $S_{nn}(x)$ (i.e., $m = n$). We construct the function S_* (which belongs to the set of sums S_{nn}), such that uniformly for $x \in [-1, 1]$

$$S_*(x) - 1 = (-1)^{n+1} B_n \cdot \left(\frac{U_{2n}(x)}{2n+1} + o(1) \right), \qquad B_n = (2n+1)\frac{n!^2}{4^n(2n)!^2} \tag{4}$$

as $n \to \infty$, where $U_n(x)$ is a Chebyshev polynomial of the second kind. Since $\|U_n\| = U_n(1) = n + 1$, we have $\|S_* - 1\| = B_n(1 + o(1))$. In particular, for minimal error $d_n = \inf_{S_{nn}} \|S_{nn} - 1\|$ we get the estimate $d_n \le 2n \cdot n!^2(2^n(2n)!)^{-2}(1 + o(1))$. This estimate close to sharp order of d_n (Theorem 3).

Recall, that best uniform approximation to constant functions by sums $S_n(x)$, $-1 \le x \le 1$, with free poles studied in [13, 14]; in particular, $\inf_{S_n} \|S_n - 1\| \asymp (2^n n!)^{-1}$. Thus, sums S_{nn} much more effective, than S_n, not only on a disk (see [12] and Sec. 1.2), but also on an interval.

2. APPROXIMATION ON COMPACT SETS. APPROXIMATION ON A DISK

2.1. We first recall, that the fraction

$$R = p/q, \qquad p(z) = \int_0^\infty t^n(t+z)^m e^{-t}dt, \qquad q(z) = \int_0^\infty (t-z)^n t^m e^{-t}dt, \tag{5}$$

is the (m, n)-Padé approximant to e^z at zero (O. Perron). Let

$$S = R'/R = p'/p - q'/q, \qquad S(z) = S(m, n; z). \tag{6}$$

We will show, that $S(z) - 1 = O(z^{m+n})$ as $z \to 0$ for sufficiently large $m + n$.

Theorem 1. *For the function* (6) *we have the identity*

$$S(z) - 1 = (-1)^{n+1} \frac{m!n!z^{m+n}}{p(z)q(z)} \tag{7}$$

and the uniform for z in any compact set K asymptotical representation (2).

The equality (2) proved in the Sec. 2.3, and (7) follows from (2), indeed, we have

$$S(z) - 1 = (-1)^{n+1} \frac{m!n!z^{m+n}}{(m+n)!^2} + O(z^{m+n+1}) \qquad \text{as} \quad z \to 0.$$

But $S - 1 = (p'q - q'p - pq)/pq$ is a *rational* function of degree $m + n$ and $p(0) = q(0) = (m + n)!$, therefore its numerator is equal to $(-1)^{n+1}m!n!z^{m+n}$.

Consider approximation on a disk $K = K_a$. Set

$$A_{mn} = \frac{m!n!a^{m+n}}{(m+n)!^2}, \qquad \mu_{mn} = \frac{|n-m|}{m+n}.$$

Theorem 2.

$$\frac{e^{-a}A_{mn}}{m+n+1}(1 + o(1)) \le d_{mn} \le e^{a\mu_{mn}} A_{mn}(1 + o(1)) \qquad \text{as} \quad m + n \to \infty.$$

Proof. An upper bound follows from (2). To prove a lower estimate we take any function $s = r'/r$, $r \in \mathcal{R}_{mn}$, of the form (1), such that $r(0) = 1$, $\delta := \|s - 1\|_a = d_{mn}(1 + o(1))$, and set $I(z) = \int_0^z (s(t) - 1)\, dt$. We have $r(z) - e^z = e^z(e^{I(z)} - 1)$ and $\|I\|_a < a\delta$, therefore

$$E_{mn} \le \|r - e^z\|_a < e^a(e^{a\delta} - 1) = ae^a d_{mn}(1 + o(1)) \qquad \text{as} \quad m + n \to \infty,$$

where E_{mn} is the error in best Chebyshev approximation to e^z by the class $\mathcal{R}_{mn}$ on a disk K_a. Theorem follows, because $E_{mn} \sim A_{mn}a(m + n + 1)^{-1}$, see [16]. □

2.2. We need to make a few remarks. First of all, Theorems 1 and 2 show that $\|S - 1\|_a = e^{a\mu_{mn}} A_{mn}(1 + o(1)) = d_{mn}O(m + n)$, i.e., the order of approximation to $f(z) = 1$ by the function $S(z)$ very close to the order of best approximation by all sums (1) on an every disk K_a.

Further, let $N = m + n$. Well-known, that $p(z) = (-1)^m m!n!L_m^{-N-1}(z)$, $q(z) = (-1)^n m!n!L_n^{-N-1}(-z)$ with generalized Laguerre polynomials L_n^α. Since $(L_n^\alpha(z))' = -L_{n-1}^{\alpha+1}(z)$, and (7) is equivalent to $p'q - q'p - pq = (-1)^{n+1}m!n!z^N$, we get the identity

$$L_{m-1}^{-N}(z)L_n^{-N-1}(-z) + L_m^{-N-1}(z)\left[L_{n-1}^{-N}(-z) + L_n^{-N-1}(-z)\right] = \frac{(-1)^m z^N}{m!n!},$$

and (3) follows after applying the formula $L_n^{\alpha-1}(z) = L_n^\alpha(z) - L_{n-1}^\alpha(z)$.

Also we can see, that choice $m = n$ is optimal in approximation to $f(z) = 1$ by $S(z)$ (it's natural, because in this case $q(z) = p(-z)$ and so $S(n, n; z)$ is even as well as $f(z)$).

As $m = n$, the fraction $\tilde{S} = S(n, n; \cdot)$ was introduced in [12] at once in the form $\tilde{S} = \tilde{R}'/\tilde{R}$, $\tilde{R}(z) = L_n^{-2n-1}(z)/L_n^{-2n-1}(-z)$, and identities (7), (3) with the estimate $2A_{nn}/3 \le \|\tilde{S} - 1\|_a \le 2A_{nn}$ were proved. Proofs in [12] based only on Laguerre polynomials' properties, but not on results about approximation to e^z, as here.

2.3. Now we prove the formula (2). It is known [15, Eq. (8)], that

$$R(z) - e^z = (-1)^{n+1} \frac{m!n!z^{m+n+1}e^{2Jz}}{(m+n)!(m+n+1)!} \cdot \gamma(z), \qquad J := \frac{n}{m+n}, \tag{8}$$

$\gamma(z) = \gamma(m, n; z) = 1 + o(1)$ as $m + n \to \infty$ uniformly for $z \in K$. Hence

$$(S(z) - 1)R(z) \equiv R'(z) - R(z) = (-1)^{n+1}\frac{m!n!z^{m+n}}{(m+n)!^2}e^{2Jz} \cdot \gamma_1(z), \tag{9}$$

$$\gamma_1(z) := \gamma(z) + z\frac{(2J-1)\gamma(z) + \gamma'(z)}{m+n+1},$$

and (2) follows from (9), because $R(z) = e^z + o(1)$ and true

Lemma 1. $\gamma'(z) = o(1)$, $\gamma''(z) = o(1)$ *as* $m + n \to \infty$ *uniformly for* $z \in K$.

Proof. Recall (see in [15] Eq. (5) and above), that

$$p(z) - e^z q(z) = (-1)^{n+1}\frac{m!n!z^{m+n+1}}{(m+n+1)!}\sum_{k=0}^{\infty}\frac{c_k z^k}{k!}, \qquad c_k = \frac{(n+1)_k}{(m+n+2)_k},$$

$(u)_0 := 1$, $(u)_k := u(u+1)\dots(u+k-1)$, therefore

$$\gamma(z) = e^{-2Jz}\frac{(m+n)!}{q(z)}(e^{Jz} + v(z)), \qquad v(z) = \sum_{k=0}^{\infty}(c_k - J^k)\frac{z^k}{k!}.$$

It's easy to check the equality

$$\gamma'(z) = -\gamma(z)\left(J + \frac{q'(z)}{q(z)}\right) + e^{-2Jz}\frac{(m+n)!}{q(z)}\left(v'(z) - Jv(z)\right).$$

Since $c_k - J^k \to 0$ as $m + n \to \infty$ for every k, then $v(z), v'(z), v''(z), \dots$ tend to 0 as $m + n \to \infty$ uniformly for $z \in K$ (see [15, p. 377]). Further, $q(z) = (m+n)!e^{-Jz}(1 + o(1))$ as $m + n \to \infty$ uniformly for $z \in K$ [15, Eq. (7)]. But $q'(z)$ can be represented as an integral, analogous to $q(z)$ in (5), and

$$\frac{q'(z)}{-n} = \int_0^{\infty}(t-z)^{n-1}t^m e^{-t}dt = (m+n-1)!e^{-(n-1)z/(m+n-1)}(1 + o(1)).$$

Hence $J + q'(z)/q(z) = o(1)$. Thus, $\gamma'(z) = o(1)$. Equality $\gamma''(z) = o(1)$ can be proved analogously. $\square$

3. APPROXIMATION TO $f(x) = 1$ ON THE INTERVAL $[-1, 1]$

Now $m = n$ and $K = [-1, 1]$. An upper bound for d_n (see Sec. 1.3) follows from (4). A lower bound can be established as in the Theorem 2, but E_{mn} must be replaced on $\inf_{r \in \mathcal{R}_{nn}} \|r - e^x\| \sim (2n+1)^{-2}B_n$ (see [15]). Thus, we have

Theorem 3. $e^{-1}(2n+1)^{-2}B_n(1 + o(1)) \le d_n \le B_n(1 + o(1))$ *as* $n \to \infty$.

Describe a construction of the function S_* (Sec. 1.3). According to Newman [17, 18], if $R(z)$ is (n, n)-Padé approximant to e^z (see (5) with $m = n$) and

$$z = (x + iy)/2, \quad x \in [-1, 1], \quad x^2 + y^2 = 1, \tag{10}$$

then $r_*(x) = R(z)R(\overline{z})$ is a rational function of type (n, n) in x. We set

$$S_*(x) = r_*'(x)/r_*(x).$$

The function $S_*(x)$ is even. Indeed, $q(z) = p(-z)$ as $m = n$, and so, $r_*(x) = P_n(x)/P_n(-x)$, where $P_n(x) = p(z)p(\overline{z})$, $P_n(-x) = p(-z)p(-\overline{z})$.

Prove the formula (4). We will take $x \in [0, 1]$, since $S_*(x)$ is even.

When putting $m = n$ in (8), we get the representation

$$e^{-z}R(z) = 1 + b_n z^{2n+1}(1+\varepsilon(z)), \qquad b_n := (-1)^{n+1}\frac{n!^2}{(2n)!(2n+1)!}, \tag{11}$$

where $1+\varepsilon(z) \equiv \gamma(n,n;z)$, $\varepsilon = o(1)$ and $\varepsilon', \varepsilon'' = o(1)$ as $n \to \infty$ (Lemma 1).

Note (see (10)), that $e^z e^{\overline{z}} = e^x$, $|z| = \frac{1}{2}$, $z^{2n+1} + \overline{z}^{2n+1} = 4^{-n}T_{2n+1}(x)$, where $T_n(x)$ is a Chebyshev polynomial of the first kind, therefore (11) implies

$$e^{-x}r_*(x) = 1 + b_n 4^{-n}T_{2n+1}(x) + b_n\alpha(x) + b_n^2 4^{-2n-1}\beta(x)$$

with

$$\alpha(x) = z^{2n+1}\varepsilon(z) + \overline{z}^{2n+1}\varepsilon(\overline{z}), \qquad \beta(x) = \gamma(z)\gamma(\overline{z});$$

here $\gamma(z) \equiv \gamma(n,n;z)$, $\alpha = o(4^{-n})$, $\beta = 1 + o(1)$ as $n \to \infty$ (sf. representation $e^{-x}r_*(x) - 1 = b_n 4^{-n}(T_{2n+1}(x) + o(1))$ in [18]). Hence

$$\frac{(S_*(x)-1)r_*(x)}{e^x b_n} \equiv \frac{r_*'(x) - r_*(x)}{e^x b_n} = \frac{T_{2n+1}'(x)}{4^n} + \alpha'(x) + \frac{b_n\beta'(x)}{4^{2n+1}},$$

and (4) follows, since $T_{n+1}'(x) = (n+1)U_n(x)$, $r_*(x) = e^x + o(1)$ and true

Lemma 2. $\alpha'(x) = o(n^2 4^{-n})$, $\beta'(x) = o(1)$ *as* $n \to \infty$ *uniformly for* $x \in [0,1]$.

Proof. Without loss of generality, $y \geq 0$. Thus, $z = (x + i\sqrt{1-x^2})/2$, $\overline{z} = (x - i\sqrt{1-x^2})/2$ and

$$\alpha'(x) = \frac{h(z)+h(\overline{z})}{2} - \frac{ix(h(z)-h(\overline{z}))}{2\sqrt{1-x^2}}, \qquad h(z) = (2n+1)z^{2n}\varepsilon(z) + z^{2n+1}\varepsilon'(z).$$

We have $h(z) + h(\overline{z}) = o(n4^{-n})$, since $\varepsilon, \varepsilon' = o(1)$. Set $z = e^{it}/2$. The parameter $t \in [0, \pi/2]$, because $y \geq 0$ and $0 \leq x \leq 1$. So, $x = \cos t$,

$$\frac{ix(h(z)-h(\overline{z}))}{\sqrt{1-x^2}} = \phi(t)\cot t, \qquad \phi(t) := i(h(e^{it}/2) - h(e^{-it}/2)).$$

ϕ is real-valued and $\phi(0) = 0$, therefore for any t there is a τ such that $\phi(t) = \phi'(\tau)\cdot t = -(h'(e^{i\tau}/2)e^{i\tau} + h'(e^{-i\tau}/2)e^{-i\tau})\cdot t/2$. Thus,

$$|\phi(t)\cot t| \leq t\cot t \cdot \max\nolimits_{|z|=\frac{1}{2}} |h'(z)|, \qquad 0 \leq t \leq \pi/2.$$

But $t\cot t \leq 1$ for such t. Consequently, $\alpha'(x) = o(n^2 4^{-n})$ is true, because

$$\max_{|z|=\frac{1}{2}} |h'(z)| = \frac{4n+2}{2^{2n-1}} \max_{|z|=\frac{1}{2}} \left| n\varepsilon(z) + z\varepsilon'(z) + \frac{z^2\varepsilon''(z)}{4n+2} \right| = o(n^2 4^{-n})$$

($\varepsilon, \varepsilon', \varepsilon'' = o(1)$ as $n \to \infty$). Analogously,

$$\beta'(x) = \frac{g(z)+g(\overline{z})}{2} + \frac{ix(g(z)-g(\overline{z}))}{2\sqrt{1-x^2}}, \qquad g(z) = \gamma(z)\gamma'(\overline{z}) = o(1)$$

and $\psi'(t) = o(1)$ for the function $\psi(t) = i(g(z) - g(\overline{z})) = i\gamma'(e^{-it}/2)\gamma(e^{it}/2) - i\gamma'(e^{it}/2)\gamma(e^{-it}/2)$, therefore $\beta'(x) = o(1)$. □

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