Classification of Second Order Linear Ordinary Differential Equations with Rational Coefficients

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Abstract—In the present work we study linear ordinary differential equations of second order with rational coefficients. Such equations are very important in complex analysis (for example they include the so–called Fuchs equations, which appear in 21 Hilbert's problem) and in the theory of special functions (Bessel function, Gauss hypergeometric function, etc.). We compute the symmetry group of this class of equations, and it appears that this group includes non-rational (and even non-algebraic) transformations. Also, the field of differential invariants is described and the effective equivalence criterion is obtained. Finally, we present some examples.

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1. INTRODUCTION

The class of second order linear differential equations

$$y'' + A(z)y' + B(z)y = 0$$

is one of the most important classes of ODEs, which appears in many different areas of mathematics and physics. For example, such is the equation of the free oscillations of a spring pendulum with the friction force. Another famous example comes from complex analysis and describes the so–called special functions (see [11, 19, 25]):

- Airy function Ai is the solution of the equation $y'' - zy = 0$;
- Bessel function J_t is the solution of the equation $z^2y'' + zy' + (z^2 - t^2)y = 0$;
- hypergeometric function $F(a, b; c; z)$ is the solution of the equation

$$z(1 - z)y'' + [c - (a + b + 1)]y' - aby = 0 \dots$$

In all these examples coefficient functions A and B are rational. We may ask, when two second order linear differential equations with rational coefficients are equivalent with respect to the action of holomorphic diffeomorphisms in variables (z, y) (the corresponding group of these diffeomorphisms is called point pseudogroup)?

Let us make some historical remarks related to this question.

The problems of the classification of differential equations were studied by S. Lie, who introduced differential geometry and geometric methods in the theory of differential equations. In particular, S. Lie proved (see [1]) that each first order differential equation is point equivalent to the trivial equation $y' = 0$ in the neighborhood of regular point. Also, he proved that each second order

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differential equation is contact equivalent to the trivial one $y''=0$ in the neighborhood of regular point. But the point classification of second order differential equations is not trivial!

This problem was studied by many famous mathematicians such as S. Lie, A. Tresse, R. Liouville, E. Cartan, etc. A. Tresse in [22] calculated the algebra of differential sub-invariants (the so–called differential parameters). Further, B. Kruglikov in [14] found the algebra of absolute differential invariants and classified non-degenerated second order differential equations with respect to the action of point pseudogroup.

The most important class of equations which is not included in the Tresse–Kruglikov classification is the class of equations cubic in the first variable. This class of ODEs includes, for example, Painleve equations (see [3]) and equations associated with the projective connections (see [9]) and projective geometry (see [16, 20]). These equations were also studied by many mathematicians: R. Liouville found two tensor differential invariants and obtained a criterion of trivialization (i.e. point equivalence to the equation $y''=0$) for such ODEs (see [17]); V. Arnold found the so–called non-Dezarg differential 5/2-form and studied the canonical forms of such ODEs (see [2]); finally, V. Yumaguzhin calculated the algebra of scalar differential invariants and found the criterion of point equivalence for two non-degenerated ODEs (see [21, 24]).

Another important case of second order ODEs is case of linear ODEs. According to the result of E. Cartan (see [9]), all second order linear equations are point equivalent. The study of the linear equations of higher order was very popular at the beginning of the XX-th century, because linear equations are closely connected with differential projective geometry. The most important results in this area belong to E. Wilczynski (see [26]).

But all these results have one general flaw: they cannot be calculated with the help of the computer. On the other hand, if we consider the linear differential equations with rational coefficients, it is possible to apply different results from algebra and algebraic geometry. Such idea was used by V. Lychagin and P. Bibikov in series of works [5–7] related to the famous algebraic problem of classification of homogeneous forms. Another application of this idea to differential equations was considered by P. Bibikov and A. Malakhov in [8].

The present paper is a logical continuation of papers [4–8]. We consider the class of linear second order ODEs with the rational coefficients over the complex field $\mathbb{C}$ and study the action of the symmetry group on these equations. In work [8] the corresponding group was a subgroup of plane Cremona group $\mathrm{Cr}(2)$. It appears that our group is larger and includes non-rational holomorphic transformations. Nevertheless it is possible to give the global effective classification of linear ODEs with respect to the action of this group. First of all, we calculate the field of differential invariants for this action and then using this field we suggest an effective criterion for the equivalence of two non-degenerated linear differential equations with rational coefficients. Finally, we give some examples.

2. SYMMETRY GROUP AND ITS LIE ALGEBRA

Let $J^2\mathbb{C}$ be the k-jet space of germs of holomorphic functions with canonical coordinates (z,y,p,q). Then each linear differential equation can be represented as a surface in $J^2\mathbb{C}$ defined by equation

$$q+A(z)p+B(z)y=0. \tag{1}$$

This surface is an algebraic cone over base $\mathbb{C}$.

It is well known that point transformations which preserve the class of linear equations have the following form:

$$z\mapsto \mu(z), \quad y\mapsto \lambda(z)y.$$

But the condition of rationality of coefficients A and B imposes additional restrictions on functions μ and λ.

First of all, function μ should be rational. Then, it is known (see, for example, [10]) that rational morphism $z\mapsto\mu(z)$ is birational if and only if it is projective. So, $\mu(z)=\frac{\alpha z+\beta}{\gamma z+\delta}$, where α, β, γ and δ are complex constants and $\alpha\delta-\beta\gamma=1$.

Now let us apply transformation

$$z \mapsto \frac{\alpha z+\beta}{\gamma z+\delta}, \quad y \mapsto \lambda(z)y$$

to our equation (1). After such transformation the equation (1) transforms into the equation

$$q+\Big(2R(\widetilde{z})+\frac{2\gamma}{\gamma z+\delta}+\frac{A(\widetilde{z})}{(\gamma z+\delta)^2}\Big)p+ \\ +\Big(R'(\widetilde{z})+R(\widetilde{z})^2+\frac{2\gamma}{\gamma z+\delta}R(\widetilde{z})+\frac{A(\widetilde{z})R(\widetilde{z})}{(\gamma z+\delta)^2}+\frac{B(\widetilde{z})}{(\gamma z+\delta)^4}\Big)y=0, \quad (2)$$

where $R(z) := (\ln\lambda(z))'$ and $\widetilde{z}=\frac{\alpha z+\beta}{\gamma z+\delta}$.

Hence, our transformation preserves the class of linear ODEs with rational coefficients if and only if function $R(z)=(\ln\lambda(z))'$ is rational. Then we get that $\lambda(z)=e^{\int R(z)dz}$, where $R(z)$ is a rational function. We obtain the following result.

Theorem 1. *The symmetry group G of the linear second order differential equations with rational coefficients consists of transformations*

$$z \mapsto \frac{\alpha z+\beta}{\gamma z+\delta}, \quad y \mapsto e^{\int R(z)dz}y, \quad \text{where } R \text{ is an arbitrary rational function.}$$

The action of group G on coefficients A and B of the equation $q+Ap+By=0$ is given by the formulas

$$A(z) \mapsto 2R(\widetilde{z})+\frac{2\gamma}{\gamma z+\delta}+\frac{A(\widetilde{z})}{(\gamma z+\delta)^2}, \\ B(z) \mapsto R'(\widetilde{z})+R(\widetilde{z})^2+\frac{2\gamma}{\gamma z+\delta}R(\widetilde{z})+\frac{A(\widetilde{z})R(\widetilde{z})}{(\gamma z+\delta)^2}+\frac{B(\widetilde{z})}{(\gamma z+\delta)^4},$$

where $\widetilde{z}=\frac{\alpha z+\beta}{\gamma z+\delta}$. The corresponding Lie algebra consists of vector fields

$$X=(\alpha z^2+\beta z+\gamma)\partial_z+(-2A\alpha z+2\lambda(z)-2\alpha-A\beta)\partial_A+(-A\lambda(z)+\lambda'(z)-4\alpha zB-2\beta B)\partial_B. \quad (3)$$

Remark. It is easy to describe the integrals $\int R(z)dz$ in terms of elementary functions. Namely, such integrals have the form $U(z)+\ln V(z)$, where U and V are also rational. Hence, despite the fact that the coefficients of our equations are rational, the symmetry group includes non-rational transformations of coordinates z and y.

3. FIELD OF DIFFERENTIAL INVARIANTS

In this section we study the field of differential invariants for the action of the symmetry group G on the space of pairs of rational functions (A,B). First of all, we recall necessary definitions.

Consider the arbitrary meromorphic map $F\colon \mathbb{C}\to\mathbb{C}^2$. Denote by $\mathbf{J}^k$ the k-jet space of such functions. The canonical coordinates on this space are denoted by $(z,a,b,a_1,b_1,a_2,b_2,\dots,a_k,b_k)$.

The action of the symmetry group G on the maps $F\colon\mathbb{C}\to\mathbb{C}^2$ prolongs to the action on the k-jet space $\mathbf{J}^k$ for all k. We denote the corresponding group acting on $\mathbf{J}^k$ as $G^{(k)}$. Also, introduce the infinite jet $\mathbf{J}^\infty=\varprojlim \mathbf{J}^k$ and the corresponding group $G^{(\infty)}=\varprojlim G^{(k)}$ as projective limits.

A differential invariant of order $\leqslant k$ for the action of group G is a rational $G^{(k)}$-invariant function J on k-jet space $\mathbf{J}^k$.

An invariant derivation is a derivation ∇ with rational coefficients on infinite jet space $\mathbf{J}^\infty$, which commutes with the action of group $G^{(\infty)}$.

Using computer system `Maple` we get the following proposition.

Proposition 1. *Function*

$$J := \frac{1}{(-2a_1+a^2+4b)^3}\cdot\Big(4a_1^3+(3a^2-8b)a_1^2+((20b_1-6a_2)a+8b_2-4a_3)a_1-$$
$$-2a^3a_2+(-4b_2+2a_3)a^2-8aa_2b+(8a_3-16b_2)b+5(a_2-2b_1)^2\Big)$$

and derivation

$$\widetilde{\nabla} := \frac{1}{\sqrt{-2a_1+a^2+4b}}\frac{d}{dz}$$

are invariant.

Unfortunately, derivation $\widetilde{\nabla}$ has an irrational coefficient. It is not good, because the coefficients of our equations are rational, and it is natural to stay in the field of rational functions. Moreover, according to the Kruglikov–Lychagin theorem (see [15]) the field of differential invariants is generated by rational differential invariants and invariant derivations with rational coefficients. So, we will take invariant derivation $\nabla := (\widetilde{\nabla}J)\cdot\widetilde{\nabla}$.

Now we are ready to describe the field of all differential invariants.

Theorem 2. *The field of differential invariants for the action of the symmetry group $\widehat{G}$ on the infinite jet space $\mathbf{J}^\infty$ is freely generated by differential invariant J of the order 3 and invariant derivation ∇.*

Proof. We use the classical construction of isotropy algebras (see also [4, 8]).

Consider the projection $\pi_{k,k-1}\colon \mathbf{J}^k\to\mathbf{J}^{k-1}$, the jet sequence $\{\theta_k\}$ (where $\theta_k\in\mathbf{J}^k$) and $\pi_{k,k-1}(\theta_k)=\theta_{k-1}$) and the fiber $V_{\theta_{k-1}}$ of projection $\pi_{k,k-1}$ over jet θ_{k-1}.

Let us define the isotropy subalgebra $\widehat{\mathfrak{g}}_{\theta_{k-1}}\subset\widehat{\mathfrak{g}}^{(k)}$, which consists of tangent vectors $\widehat{X}^{(k)}_{\theta_{k-1}}$ vanishing in jet θ_{k-1}:

$$\widehat{\mathfrak{g}}_{\theta_{k-1}}=\{\widehat{X}^{(k)}_{\theta_{k-1}}:\widehat{X}^{(k-1)}_{\theta_{k-1}}=0\}.$$

The isotropy subalgebra $\widehat{\mathfrak{g}}_{\theta_{k-1}}$ acts on the fiber $V_{\theta_{k-1}}$. The number of independent differential invariants of pure order k equals the codimension of the regular orbit of this action.

Let us compute all isotropy subalgebras with the help of computer system `Maple`.

It follows from formula (3) that the vectors from isotropy subalgebra $\widehat{\mathfrak{g}}_{\theta_{k-1}}$ depend on variables α, β, γ, $l_i:=\lambda^{(i)}(0)$, where $i\leqslant k+1$.

$$\widehat{\mathfrak{g}}_{\theta_0}=\Big\{\beta(a^2+4b-2a_1)\partial_{a_1}+(-a^2\alpha-a^3\beta/2-2a\beta b+l_2-4\alpha b-\alpha a_1-a_1a\beta/2-3b_1\beta)\partial_{b_1}\Big\},$$
$$\widehat{\mathfrak{g}}_{\theta_1}=\Big\{2l_0(a^2+4b-2a_1)\partial_{a_2}+(-a^3l_0-4abl_0-3al_0a_1+l_3-10l_0b_1-l_0a_2)\partial_{b_2}\Big\},$$
$$\widehat{\mathfrak{g}}_{\theta_{k-1}}=\Big\{l_{k+1}\partial_{b_k}\Big\},$$

where $k\geqslant 3$.

The dimension of the fiber $V_{\theta_{k-1}}$ equals 2. Hence, there are no differential invariants of order $\leqslant 2$ and there is only one independent differential invariant of pure order $k\geqslant 3$.

According to the corollary of the Rosenlicht theorem (see [23]), differential invariant J and its invariant derivatives rationally generate the field of differential invariants. □

Remark. Using the constructions from [5] it can be proved, that the algebra of all differential invariants which are polynomials in coordinates of jet space and denominator $(-2a_1+a^2+4b)^{-1}$ is also generated by invariant J and derivation ∇.

4. CLASSIFICATION

Now we consider the following question: when are two linear second order differential equations with rational coefficients G-equivalent? We are interested in a global and effective criterion.

Consider the pair of rational functions (A, B) from the linear differential equation $y'' + A(z)y' + B(z)y = 0$ and the invariants J and $J_1 := \nabla J$. Assume that the restrictions of these invariants on the given pair are well-defined (i.e. $-2A' + A^2 + 4B \not\equiv 0$ in denominators of J and ∇). Then we can define the rational morphism

$$\pi_{(A,B)} \colon \mathbb{C}^1 \to \mathbb{C}^2, \quad \pi_{(A,B)}(z) = (J([A]^4_z, [B]^4_z), J_1([A]^4_z, [B]^4_z)).$$

The closure (in Zariski topology) of the image $\mathrm{Im}(\pi_{(A,B)})$ is an algebraic curve $\mathcal{C}_{(A,B)}$ defined by the algebraic dependence $\mathcal{D}_{(A,B)}$ between $J([A]^4_z, [B]^4_z)$ and $J_1([A]^4_z, [B]^4_z)$:

$$\mathcal{C}_{(A,B)} := \{(\xi, \eta) \in \mathbb{C}^2 : \mathcal{D}_{(A,B)}(\xi, \eta) = 0\}.$$

Theorem 3. *1. Pairs of functions (A, B) and $(\widetilde{A}, \widetilde{B})$ are G-equivalent if and only if $\mathcal{C}_{(A,B)} = \mathcal{C}_{(\widetilde{A},\widetilde{B})}$.*

2. Pairs of functions (A, B) and $(\widetilde{A}, \widetilde{B})$ are G-equivalent if and only if $\mathcal{D}_{(A,B)} = \mathcal{D}_{(\widetilde{A},\widetilde{B})}$.

Remark. Note that this theorem is similar to the theorem in work [5].

Proof. The direct statements of the theorem are obvious. Let us prove the inverse ones.

Note that $\mathcal{C}_{(A,B)} = \mathcal{C}_{(\widetilde{A},\widetilde{B})}$ if and only if $\mathcal{D}_{(A,B)} = \mathcal{D}_{(\widetilde{A},\widetilde{B})}$. If $\mathcal{D}_{(A,B)} = \mathcal{D}_{(\widetilde{A},\widetilde{B})} =: \mathcal{D}$, then there exist two points z_1, $z_2 \in \mathbb{C}$ such that $\pi_{(A,B)}(z_1) = \pi_{(\widetilde{A},\widetilde{B})}(z_2)$. Hence, the values of differential invariants J and J_1 in 4-jets $([A]^4_{z_1}, [B]^4_{z_1})$ and $([\widetilde{A}]^4_{z_2}, [\widetilde{B}]^4_{z_2})$ coincide.

Let us differentiate the polynomial $\mathcal{D}$ by invariant derivation ∇. We get $\mathcal{D}_J \cdot J_1 + \mathcal{D}_{J_1} \cdot J_2 = 0$, where $J_2 := \nabla^2 J$. As the degree of $\mathcal{D}$ is minimal, then $\mathcal{D}_{J_1} \not\equiv 0$ and $J_2 = -\frac{\mathcal{D}_J}{\mathcal{D}_{J_1}} J_1$. Then the values of the differential invariant J_2 in 5-jets $([A]^4_{z_1}, [B]^4_{z_1})$ and $([\widetilde{A}]^4_{z_2}, [\widetilde{B}]^4_{z_2})$ also coincide.

In the same way we obtain that for each point z_1 from the open (in Zariski topology) set from the base $\mathbb{C}^2$ there exists a point $z_2 \in \mathbb{C}^2$ such that the values of all basic differential invariants $J_k := \nabla^k J$ in the infinite jets $([A]^\infty_{z_1}, [B]^\infty_{z_1})$ and $([\widetilde{A}]^\infty_{z_2}, [\widetilde{B}]^\infty_{z_2})$ coincide. Hence, according to theorem 2 there exists an infinite jet $g^\infty_{z_1,z_2} \in G^{(\infty)}$ such that $g^\infty_{z_1,z_2} \circ ([A]^\infty_{z_1}, [B]^\infty_{z_1}) = ([\widetilde{A}]^\infty_{z_2}, [\widetilde{B}]^\infty_{z_2})$.

Not let us prove that there exists an element $g \in G$ such that $g \circ (A, B) = (\widetilde{A}, \widetilde{B})$. This relation can be considered as a system of equations on the unknown constants α, β, γ, δ and function λ. Note that the constants α, β, γ, δ can be found from the element $g^\infty_{z_1,z_2}$. To find function λ one should use the formula (2):

$$2(\ln \lambda(\widetilde{z}))' + \frac{2\gamma}{\gamma z + \delta} + \frac{A(\widetilde{z})}{(\gamma z + \delta)^2} = \widetilde{A}(z).$$

Above we prove that the system of equations $g \circ (A, B) = (\widetilde{A}, \widetilde{B})$ is formally integrable in some open in the Zariski topology subset of $\mathbb{C}^2$. Hence, this system is integrable and has the solution $g \in G$. $\square$

5. EXAMPLES

In this section we provide some examples of the curves $\mathcal{C}_{(A,B)}$ and dependencies $\mathcal{D}_{(A,B)}$ for different pairs (A, B).

Let us describe the procedure for computing the polynomial $\mathcal{D} := \mathcal{D}_{(A,B)}$. We consider the restrictions $j := J(A, B)$ and $j_1 := J_1(A, B)$ of basic differential invariants J and J_1 on the pair of rational functions (A, B) of the differential equation $y'' + A(z)y' + B(z)y = 0$. Then we consider the polynomials

$$pol := \mathrm{numer}(j) - \xi \cdot \mathrm{denom}(j) \quad \text{and} \quad pol_1 := \mathrm{numer}(j_1) - \eta \cdot \mathrm{denom}(j_1),$$

where numer$(\cdot)$ and denom$(\cdot)$ are the numerator and denominator of a given fraction. Then the irreducible factor of the resultant of polynomials pol and pol_1 with respect to variable z is our polynomial $\mathcal{D}$.

1. Consider the equation $y'' - zy = 0$, whose solution is Airy function Ai. The polynomial $\mathcal{D}$ in this case looks as follows:

$$\mathcal{D}(\xi, \eta) = 5\eta - 36\xi^3.$$

The corresponding curve $\mathcal{C}$ is a simple cubic parabola (see Fig. 1).

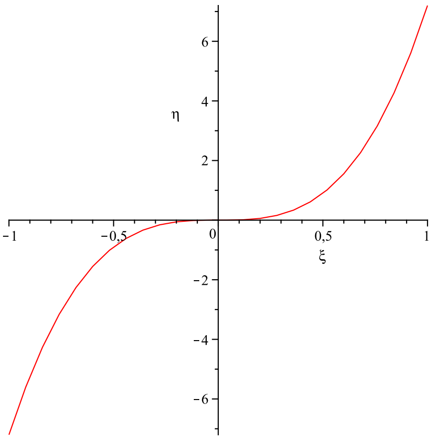

Figure 1.

2. Consider the Bessel equation $y'' + \frac{1}{z}y' + (1 - \frac{t^2}{z^2})y = 0$.

In case $t = 0$ we get

$$\mathcal{D}_0(\xi, \eta) = 78125\eta^3 + (-1687500\xi^3 + 2565000\xi^2 - 2980800\xi + 1119744)\eta^2 + (12150000\xi^6 - 36936000\xi^5 + 38646720\xi^4 - 14432256\xi^3)\eta + 132969600\xi^8 - 29160000\xi^9 - 47775744\xi^5 + 170201088\xi^6 - 226234944\xi^7.$$

In case $t = 1$ we get

$$\mathcal{D}_1(\xi, \eta) = 78125\eta^3 + (-41472 - 855000\xi^2 - 331200\xi - 1687500\xi^3)\eta^2 + (12312000\xi^5 + 4294080\xi^4 + 534528\xi^3 + 12150000\xi^6)\eta - 29160000\xi^9 - 589824\xi^5 - 6303744\xi^6 - 25137216\xi^7 - 44323200\xi^8.$$

The corresponding curves $\mathcal{C}_0$ and $\mathcal{C}_1$ are presented in Fig. 2 and Fig. 3.

In the arbitrary case we get the following polynomial:

$$\begin{aligned}
\mathcal{D}_t = {} & (-33750000t^2 + 67500000t^4 - 60000000t^6 + 6328125 + 20000000t^8)\eta^3 + \\
& + ((729000000t^2 + 1296000000t^6 - 136687500 - 1458000000t^4 - 432000000t^8)\xi^3 + \\
& + (277020000t^2 - 69255000 - 369360000t^4 + 164160000t^6)\xi^2 + \\
& + (71539200t^2 - 26827200 - 47692800t^4)\xi + 4478976t^2 - 3359232)\eta^2 + \\
& + ((984150000 + 10497600000t^4 - 5248800000t^2 + 3110400000t^8 - 9331200000t^6)\xi^6 + \\
& + (5318784000t^4 + 997272000 - 2363904000t^6 - 3989088000t^2)\xi^5 + \\
& + (347820480 + 618347520t^4 - 927521280t^2)\xi^4 + (-57729024t^2 + 43296768)\xi^3)\eta + \\
& + (22394880000t^6 + 12597120000t^2 - 2361960000 - 7464960000t^8 - 25194240000t^4)\xi^9 + \\
& + (8510054400t^6 - 3590179200 + 14360716800t^2 - 19147622400t^4)\xi^8 + \\
& + (-2036114496 + 5429638656t^2 - 3619759104t^4)\xi^7 + (-510603264 + 680804352t^2)\xi^6 - 47775744\xi^5.
\end{aligned}$$

The set of curves $\{\mathcal{C}_t\}$ forms two-dimensional algebraic variety in three-dimensional complex space $\mathbb{C}^3$. From different angles of view it looks as follows (see Figures 4, 5).

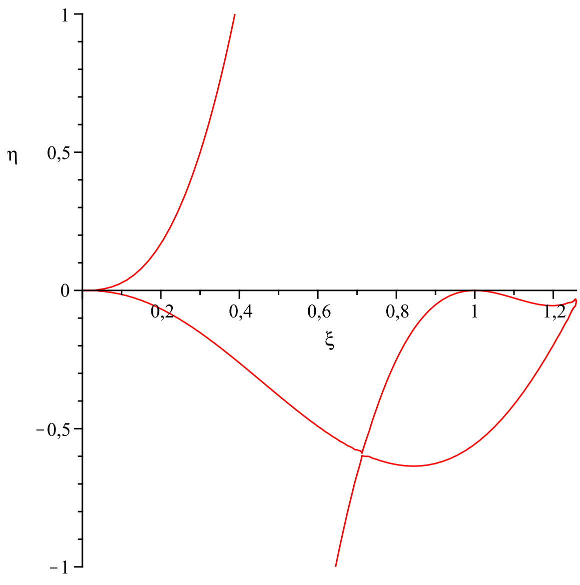

Figure 2. Curve $\mathcal{C}_0$

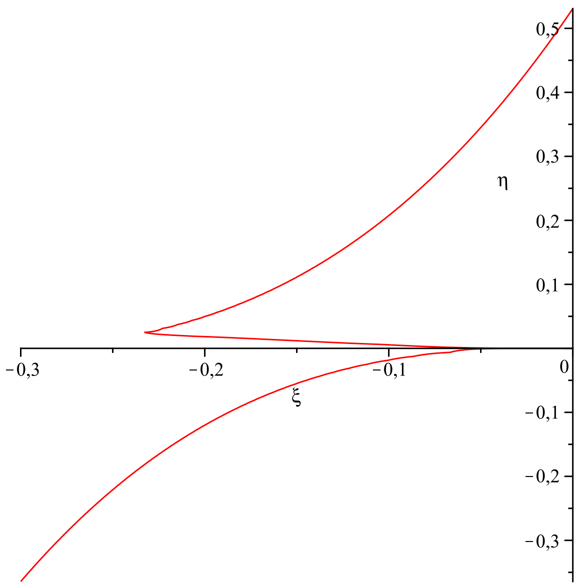

Figure 3. Curve $\mathcal{C}_1$

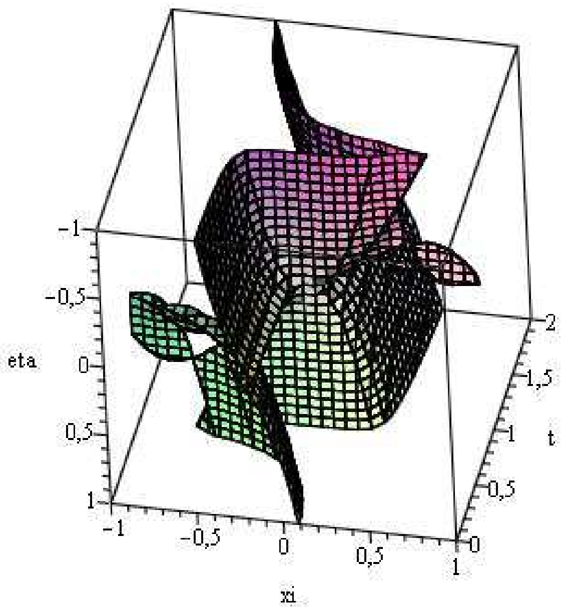

Figure 4.

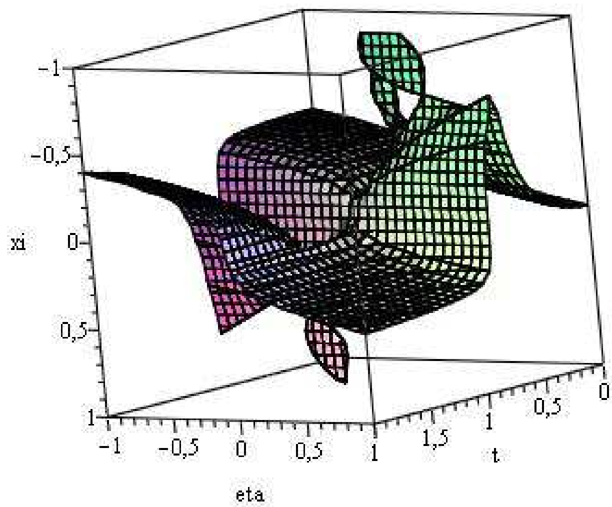

Figure 5.

3. For hypergeometric equation $y'' + \frac{c-(a+b+1)}{z(1-z)}y' - \frac{ab}{z(1-z)}y = 0$ polynomial $\mathcal{D}$ is very big, so we just show the corresponding curves $\mathcal{C}$ for parameters $a = b = c = 1$ (Fig. 6) and parameters $a = 1/4$, $b = 3/4$, $c = 1$ (Fig. 7).

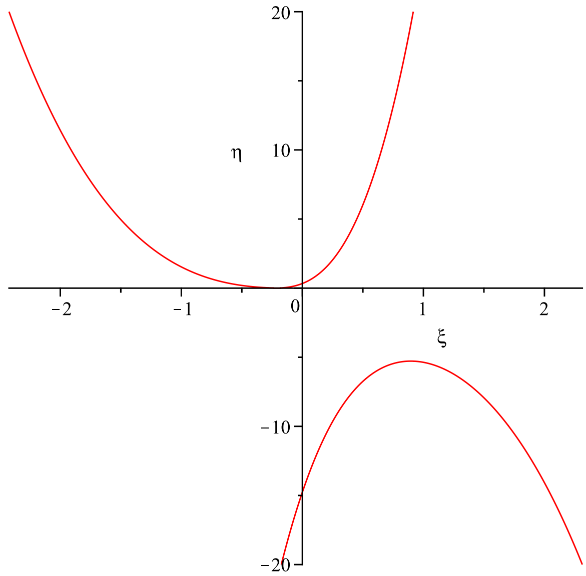

Figure 6. $a = b = c = 1$

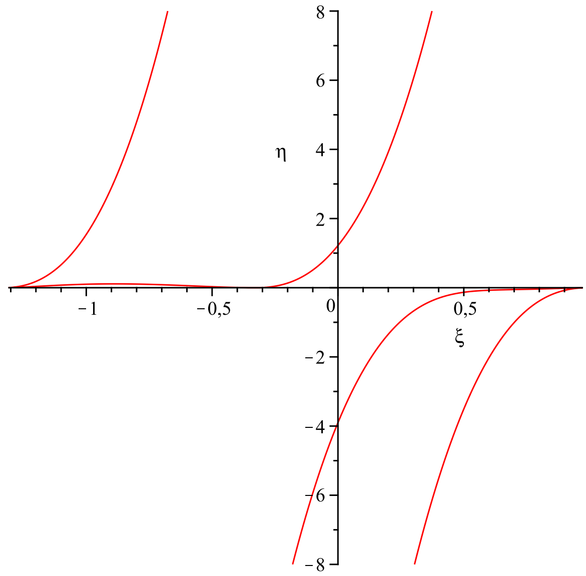

Figure 7. $a = 1/4$, $b = 3/4$, $c = 1$

4. Let $j = \frac{g_2^3}{g_2^3 - 27g_3^2}$ be the j-invariant with the modular invariants g_2 and g_3 of the elliptic curve in Weierstrass form:

$$y^2 = 4x^3 - g_2x - g_3.$$

Note that the j-invariant is an isomorphism from the Riemann surface $\mathbb{H}/\Gamma$ to the Riemann sphere $\mathbb{C} \cup \{\infty\}$, where $\mathbb{H}$ is the upper half-plane and Γ is the modular group. The Picard–Fuchs equation (see [13]) is

$$\frac{d^2y}{dj^2} + \frac{1}{j}\frac{dy}{dj} + \frac{31j - 4}{144j^2(1-j)^2}y = 0.$$

This equation can be cast into the form of the hypergeometric differential equation. It has two linearly independent solutions, called the periods of elliptic functions. The ratio of the two periods is equal to the period ratio τ, which is the standard coordinate on the upper-half plane.

Unfortunately, the coefficients of the polynomial $\mathcal{D}$ are very big (some of them are more than 10^{200}), so we just show the corresponding curve $\mathcal{C}$ (Fig. 8).

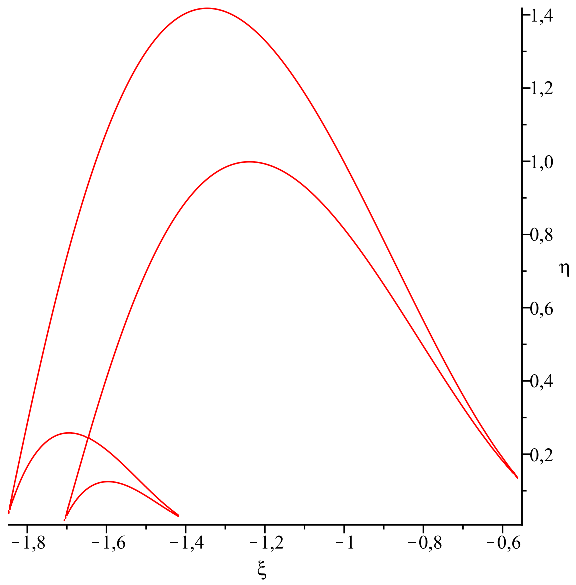

Figure 8.

5. The Gauss–Manin connection is a connection on a certain vector bundle over a base space S of a family of algebraic varieties V_s. The fibers of the vector bundle are the de Rham cohomology groups $H^k_{DR}(V_s)$ of the fibers V_s of the family. It was introduced by Manin (see [18]) for curves S and by Grothendieck (see [12]) in higher dimensions. We consider just one example of this connection.

Let $V_\lambda(x, y, z)$ be the elliptic curve

$$x^3 + y^3 + z^3 - \lambda xyz = 0.$$

Here λ is a free parameter describing the curve. It is an element of the complex projective line. Thus, the base space of the bundle is taken to be the projective line. For a fixed λ in the base space consider an element ω_λ of the associated de Rham cohomology group: $\omega_\lambda \in H^1_{dR}(V_\lambda)$. Each such element corresponds to a period of the elliptic curve. The cohomology is two–dimensional. The Gauss–Manin connection corresponds to the second-order differential equation

$$(\lambda^3 - 27)\frac{\partial^2 \omega_\lambda}{\partial \lambda^2} + 3\lambda^2 \frac{\partial \omega_\lambda}{\partial \lambda} + \lambda \omega_\lambda = 0.$$

The polynomial $\mathcal{D}$ for the Gauss–Manin equation looks as follows:

$$\begin{aligned}\mathcal{D}(\xi,\eta) = {} & 18377297265625\eta^4 + (-202848531451740 - 529266161250000\xi^3 - 2427103191037500\xi^2 - \\ & - 1355889788151000\xi)\eta^3 + (583666949048790 + 5716074541500000\xi^6 + 5242542892641000\xi^5 + \\ & + 4391394621155562\xi^2 + 8157094179740280\xi + 12979651106125935\xi^4 + 1117377563842172 88\xi^3)\eta^2 + \\ & + (-545844047521500 - 1594134359097842160\xi^7 - 493991273915290208\xi^3 - 100318927174209540\xi^2 - \\ & - 11316158315270280\xi - 2832686258160931452\xi^6 - 27437157799200000\xi^9 - 377463088270152000\xi^8 - \\ & - 2659271645796921000\xi^5 - 1466602233487439076\xi^4)\eta + 1126678742412395688 9\xi^8 + 58913206108200\xi + \\ & + 931446963225 + 7273779536279101752\xi^7 + 49386884038560000\xi^12 + 1064232109561209681 6\xi^9 + \\ & + 5255199403736563296\xi^10 + 905911411848364800\xi^11 + 163529198665795926\xi^4 + \\ & + 865836692744285160\xi^5 + 14904123705970 20\xi^2 + 20042767004672520\xi^3 + 3065484711035804796\xi^6.\end{aligned}$$

The corresponding curve $\mathcal{C}$ is presented in Fig. 9.

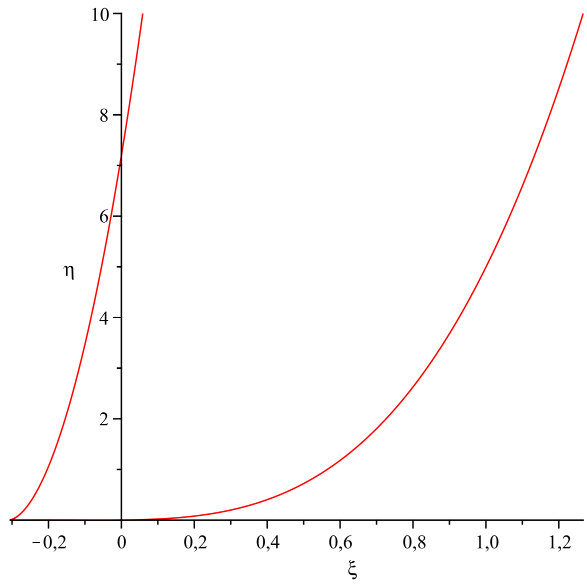

Figure 9.

6. Finally, let us take two differential equations

$$y'' + zy' + \frac{1}{z}y = 0 \quad \text{and} \quad y'' + (z+1)y' + \frac{1}{z}y = 0.$$

The corresponding curves $\mathcal{C}$ for these equations are presented in Figure 10 and Figure 11. As these curves do not coincide, these equations are not G-equivalent.

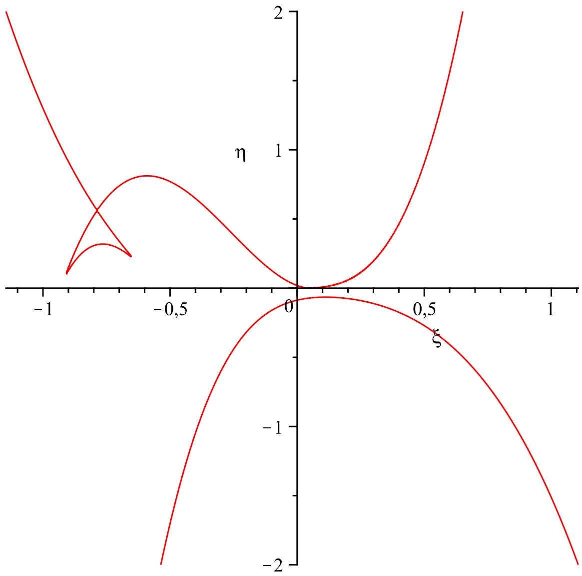

Figure 10. Curve $\mathcal{C}$ for the equation $y'' + zy' + \frac{1}{z}y = 0$

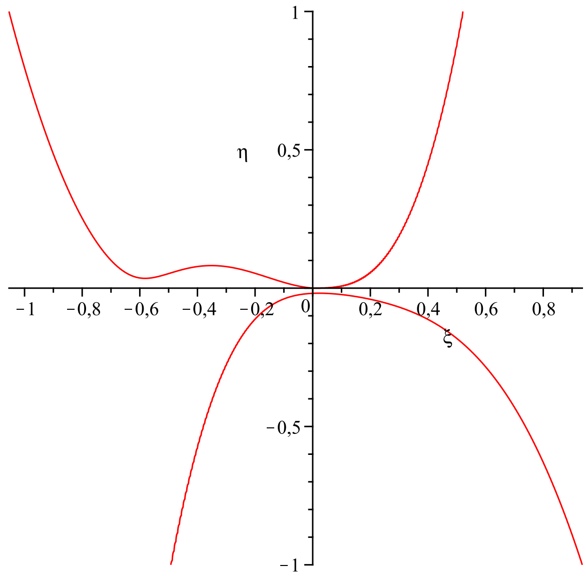

Figure 11. Curve $\mathcal{C}$ for the equation $y'' + (z+1)y' + \frac{1}{z}y = 0$

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