Matrix

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Definition

$$A = \begin{array}{c@{\quad}l} \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} & \begin{array}{l} \\ \\ \\ \longleftarrow i\text{th row} \\ \\ \\ \end{array} \\ \uparrow j\text{th column} & \end{array} \qquad (1)$$

The ***i*th row** of A is

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \qquad (1 \le i \le m);$$

the ***j*th column** of A is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \qquad (1 \le j \le n).$$

We shall say that A is $\boldsymbol{m}$ **by** $\boldsymbol{n}$ (written as $\boldsymbol{m \times n}$). If $m = n$, we say that A is a **square matrix of order** $\boldsymbol{n}$ and that the numbers $a_{11}, a_{22}, \ldots, a_{nn}$ form the **main diagonal** of A. We refer to the number a_{ij}, which is in the ith row and jth column of A, as the $\boldsymbol{i, j}$**th element** of A, or the $\boldsymbol{(i, j)}$ **entry** of A, and we often write (1) as

Example

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 4 \\ 2 & -3 \end{bmatrix}, \qquad C = \Bigg[\; -$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}, \qquad E = [3], \qquad F = [-1 \quad 0$$

2 × 3 matrix with $a_{12} = 2$, $a_{13} = 3$, $a_{22} = 0$, and a
trix with $b_{11} = 1$, $b_{12} = 4$, $b_{21} = 2$, and $b_{22} = -3$;
$c_{11} = 1$, $c_{21} = -1$, and $c_{31} = 2$; D is a 3 × 3 matrix;
F is a 1 × 3 matrix. In D, the elements $d_{11} = 1$, d
n the main diagonal.

A $1 \times n$ or an $n \times 1$ matrix is also called an ***n*-vecto**
by lowercase boldface letters. When n is understood,
merely as **vectors**. In Chapter 4 we discuss vectors at le

EXAMPLE 2 $\mathbf{u} = \begin{bmatrix} 1 & 2 & -1 & 0 \end{bmatrix}$ is a 4-vector and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ is

Example 3

ith the linear system considered in Example 5 in Section 1.1,

$$\begin{aligned} x + 2y &= 10 \\ 2x - 2y &= -4 \\ 3x + 5y &= 26, \end{aligned}$$

: can associate the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \\ 3 & 5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 10 \\ -4 \\ 26 \end{bmatrix}.$$

Section 1.3, we shall call A the coefficient matrix of the linear system.

DEFINITION

A square matrix $A = [a_{ij}]$ for which every term off the main diagor
that is, $a_{ij} = 0$ for $i \neq j$, is called a **diagonal matrix**.

EXAMPLE 7

$$G = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

are diagonal matrices.

DEFINITION

A diagonal matrix $A = [a_{ij}]$, for which all terms on the mai equal, that is, $a_{ij} = c$ for $i = j$ and $a_{ij} = 0$ for $i \neq j$, is **matrix**.

EXAMPLE 8

The following are scalar matrices:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}.$$

EXAMPLE 9 The matrices

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 4 \\ 0 & -4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & w \\ 2 & x & 4 \\ y & -4 & z \end{bmatrix}$$

are equal if $w = -1$, $x = -3$, $y = 0$, and $z = 5$.

We shall now define a number of operations that will produce new ces out of given matrices. These operations are useful in the applicat matrices.

Matrix Addition

DEFINITION

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices, then the **sum** the $m \times n$ matrix $C = [c_{ij}]$, defined by

$$c_{ij} = a_{ij} + b_{ij} \qquad (1 \le i \le m, 1 \le j \le n).$$

That is, C is obtained by adding corresponding elements of A a

EXAMPLE 10

Let

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 & -4 \\ 1 & 3 & 1 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} 1+0 & -2+2 & 4+(-4) \\ 2+1 & -1+3 & 3+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

Scalar Multiplication

DEFINITION

If $A = [a_{ij}]$ is an $m \times n$ matrix and r is a real number, then the **scalar** of A by r, rA, is the $m \times n$ matrix $B = [b_{ij}]$, where

$$b_{ij} = ra_{ij} \qquad (1 \le i \le m, 1 \le j \le n).$$

That is, B is obtained by multiplying each element of A by r.

If A and B are $m \times n$ matrices, we write $A + (-1)B$ as $A - B$ this the **difference** of A and B.

EXAMPLE 12

Let

$$A = \begin{bmatrix} 2 & 3 & -5 \\ 4 & 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 5 & -2 \end{bmatrix}.$$

Then

$$A - B = \begin{bmatrix} 2-2 & 3+1 & -5-3 \\ 4-3 & 2-5 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & -8 \\ 1 & -3 & 3 \end{bmatrix}.$$

Linear Combination

EXAMPLE 14 (a) If

$$A_1 = \begin{bmatrix} 0 & -3 & 5 \\ 2 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 5 & 2 & 3 \\ 6 & 2 & 3 \\ -1 & -2 & 3 \end{bmatrix},$$

then $C = 3A_1 - \frac{1}{2}A_2$ is a linear combination of A_1 and A_2. Using scalar multiplication and matrix addition, we can compute C:

$$C = 3\begin{bmatrix} 0 & -3 & 5 \\ 2 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 5 & 2 & 3 \\ 6 & 2 & 3 \\ -1 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{2} & -10 & \frac{27}{2} \\ 3 & 8 & \frac{21}{2} \\ \frac{7}{2} & -5 & -\frac{21}{2} \end{bmatrix}.$$

(b) $2\begin{bmatrix} 3 & -2 \end{bmatrix} - 3\begin{bmatrix} 5 & 0 \end{bmatrix} + 4\begin{bmatrix} -2 & 5 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 3 & -2 \end{bmatrix}$, $\begin{bmatrix} 5 & 0 \end{bmatrix}$, and $\begin{bmatrix} -2 & 5 \end{bmatrix}$. It can be computed (verify) as $\begin{bmatrix} -17 & 16 \end{bmatrix}$.

(c) $-0.5\begin{bmatrix} 1 \\ -4 \\ -6 \end{bmatrix} + 0.4\begin{bmatrix} 0.1 \\ -4 \\ 0.2 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ -4 \\ -6 \end{bmatrix}$ and $\begin{bmatrix} 0.1 \\ -4 \\ 0.2 \end{bmatrix}$.

It can be computed (verify) as $\begin{bmatrix} -0.46 \\ 0.4 \\ 3.08 \end{bmatrix}$. ■

Transpose of a Matrix

DEFINITION If $A = [a_{ij}]$ is an $m \times n$ matrix, then the $n \times m$ matrix A^T

$$a^T_{ij} = a_{ji} \qquad (1 \leq i \leq n, 1 \leq j \leq m)$$

is called the **transpose** of A. Thus, the entries in each r
entries in the corresponding column of A.

EXAMPLE 15 Let

$$A = \begin{bmatrix} 4 & -2 & 3 \\ 0 & 5 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 2 & -4 \\ 3 & -1 & 2 \\ 0 & 4 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 4 \\ -3 & 2 \\ 2 & -3 \end{bmatrix},$$

$$D = \begin{bmatrix} 3 & -5 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Then

$$A^T = \begin{bmatrix} 4 & 0 \\ -2 & 5 \\ 3 & -2 \end{bmatrix}, \quad B^T = \begin{bmatrix} 6 & 3 & 0 \\ 2 & -1 & 4 \\ -4 & 2 & 3 \end{bmatrix},$$

$$C^T = \begin{bmatrix} 5 & -3 & 2 \\ 4 & 2 & -3 \end{bmatrix}, \quad D^T = \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}, \quad \text{and} \quad E^T = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}.$$

■

and

$$C = \begin{bmatrix} 7 & 3 & 2 \\ -4 & 3 & 5 \\ 6 & 1 & -1 \end{bmatrix}.$$

(a) What is a_{12}, a_{22}, a_{23}?

(b) What is b_{11}, b_{31}?

(c) What is c_{13}, c_{31}, c_{33}?

2. If

$$\begin{bmatrix} a+b & c+d \\ c-d & a-b \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & 2 \end{bmatrix},$$

find a, b, c, and d.

3. If

$$\begin{bmatrix} a+2b & 2a-b \\ 2c+d & c-2d \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 4 & -3 \end{bmatrix},$$

find a, b, c, and d.

In Exercises 4 through 7, let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix},$$

$$E = \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix},$$

$$\textit{and} \quad O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) $3(2A)$ and $6A$

(c) $3A + 2A$ and $5A$

(d) $2(D + F)$ and $2D + 2F$

(e) $(2 + 3)D$ and $2D + 3D$

(f) $3(B + D)$

6. If possible, compute:

(a) A^T and $(A^T)^T$

(b) $(C + E)^T$ and $C^T + E^T$

(c) $(2D + 3F)^T$

(d) $D - D^T$

(e) $2A^T + B$

(f) $(3D - 2F)^T$

7. If possible, compute:

(a) $(2A)^T$ (b) $(A - B)^T$

(c) $(3B^T - 2A)^T$

(d) $(3A^T - 5B^T)^T$

(e) $(-A)^T$ and $-(A^T)$

(f) $(C + E + F^T)^T$

8. Is the matrix $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$? Justify your answer.

9. Is the matrix $\begin{bmatrix} 4 & 1 \\ 0 & -3 \end{bmatrix}$ a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$? Justify your answer.

10. Let

Dot product and Matrix Multiplication

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Dot ptoduct

DEFINITION

The **dot product** or **inner product** of the n-vectors **a** and **b** is the su
products of corresponding entries. Thus, if

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i.^{\dagger}$$

Similarly, if **a** or **b** (or both) are n-vectors written as a $1 \times n$ matrix,
dot product $\mathbf{a} \cdot \mathbf{b}$ is given by (1). The dot product of vectors in C^n i
in Appendix A.2.

EXAMPLE 1 The dot product of

$$\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$

is

$$\mathbf{u} \cdot \mathbf{v} = (1)(2) + (-2)(3) + (3)(-2) + (4)(1) = -$$

EXAMPLE 2 Let $\mathbf{a} = \begin{bmatrix} x & 2 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$. If $\mathbf{a} \cdot \mathbf{b} = -4$, find x.

Solution We have

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} = 4x + 2 + 6 &= -4 \\ 4x + 8 &= -4 \\ x &= -3. \end{aligned}$$

MATRIX MULTIPLICATION

DEFINITION If $A = [a_{ij}]$ is an $m \times p$ matrix and $B = [b_{ij}]$ is a $p \times n$ matrix, then the **product** of A and B, denoted AB, is the $m \times n$ matrix $C = [c_{ij}]$, defined by

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} \\ &= \sum_{k=1}^{p} a_{ik}b_{kj} \quad (1 \le i \le m, 1 \le j \le n). \end{aligned} \tag{2}$$

Equation (2) says that the i, jth element in the product matrix is the dot product of the ith row, $\mathrm{row}_i(A)$, and the jth column, $\mathrm{col}_j(B)$, of B; this is shown in Figure 1.4.

Figure 1.4 ►

$$\begin{array}{cc} & \\ & \\ & \\ \mathrm{row}_i(A) & \\ & \\ & \end{array}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{array}{c} \begin{array}{cccccc} & & & \mathrm{col}_j(B) & & \end{array} \\ \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}.$$

$$\mathrm{row}_i(A) \cdot \mathrm{col}_j(B) = \sum_{k=1}^{p} a_{ik}b_{kj}$$

EXAMPLE 4 Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 5 \\ 4 & -3 \\ 2 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned} AB &= \begin{bmatrix} (1)(-2) + (2)(4) + (-1)(2) & (1)(5) + (2)(-3) + (-1)(1) \\ (3)(-2) + (1)(4) + (4)(2) & (3)(5) + (1)(-3) + (4)(1) \end{bmatrix} \\ &= \begin{bmatrix} 4 & -2 \\ 6 & 16 \end{bmatrix}. \end{aligned}$$

■

EXAMPLE 5 Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 \\ 3 & -1 \\ -2 & 2 \end{bmatrix}.$$

Compute the (3, 2) entry of AB.

Solution If $AB = C$, then the (3, 2) entry of AB is c_{32}, which is $\mathrm{row}_3(A) \cdot \mathrm{col}_2(B)$. now have

$$\mathrm{row}_3(A) \cdot \mathrm{col}_2(B) = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = -5.$$

EXAMPLE 6 The linear system

$$\begin{aligned} x + 2y - z &= 2 \\ 3x + 4z &= 5 \end{aligned}$$

can be written (verify) using a matrix product as

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

EXAMPLE 7 Let

$$A = \begin{bmatrix} 1 & x & 3 \\ 2 & -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 \\ 4 \\ y \end{bmatrix}.$$

If $AB = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$, find x and y.

Solution We have

$$AB = \begin{bmatrix} 1 & x & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ y \end{bmatrix} = \begin{bmatrix} 2 + 4x + 3y \\ 4 - 4 + y \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}.$$

Then

$$\begin{aligned} 2 + 4x + 3y &= 12 \\ y &= 6, \end{aligned}$$

so $x = -2$ and $y = 6$.

EXAMPLE 8 If A is a 2×3 matrix and B is a 3×4 matrix, then AB is a 2
BA is undefined.

EXAMPLE 9 Let A be 2×3 and let B be 3×2. Then AB is 2×2 while

EXAMPLE 10 Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 2 & 3 \\ -2 & 2 \end{bmatrix} \quad \text{while} \quad BA = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus $AB \neq BA$.

EXAMPLE 12 Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix}.$$

Then the second column of AB is

$$A\text{col}_2(B) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \\ 7 \end{bmatrix}.$$

■

Remark If $\mathbf{u}$ and $\mathbf{v}$ are n-vectors, it can be shown (Exercise T.14) that if we view them as $n \times 1$ matrices, then

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}.$$

This observation will be used in Chapter 3. Similarly, if $\mathbf{u}$ and $\mathbf{v}$ are viewed as $1 \times n$ matrices, then

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}\mathbf{v}^T.$$

Finally, if $\mathbf{u}$ is a $1 \times n$ matrix and $\mathbf{v}$ is an $n \times 1$ matrix, then $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}\mathbf{v}$.

EXAMPLE 13 Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. Then

$$\mathbf{u} \cdot \mathbf{v} = 1(2) + 2(-1) + (-3)(1) = -3.$$

Moreover,

$$\mathbf{u}^T\mathbf{v} = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 1(2) + 2(-1) + (-3)(1) = -3.$$

EXAMPLE 14 Let

$$A = \begin{bmatrix} 2 & -1 & -3 \\ 4 & 2 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}.$$

Then the product $A\mathbf{c}$ written as a linear combination of the column

$$A\mathbf{c} = \begin{bmatrix} 2 & -1 & -3 \\ 4 & 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} - 3 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} -3 \\ -2 \end{bmatrix} =$$

Augmented matrix

EXAMPLE 16 Consider the linear system

$$\begin{aligned} -2x \quad\quad + \ z &= 5 \\ 2x + 3y - 4z &= 7 \\ 3x + 2y + 2z &= 3. \end{aligned}$$

Letting

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 2 & 3 & -4 \\ 3 & 2 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix},$$

we can write the given linear system in matrix form as

$$A\mathbf{x} = \mathbf{b}.$$

The coefficient matrix is A and the augmented matrix is

$$\left[\begin{array}{rrr:r} -2 & 0 & 1 & 5 \\ 2 & 3 & -4 & 7 \\ 3 & 2 & 2 & 3 \end{array}\right].$$

■

EXAMPLE 17 The matrix

$$\left[\begin{array}{rrr:r} 2 & -1 & 3 & 4 \\ 3 & 0 & 2 & 5 \end{array}\right]$$

is the augmented matrix of the linear system

$$2x - y + 3z = 4$$

PARTITIONED MATRICES (OPTIONAL)

If we start out with an $m \times n$ matrix $A = [a_{ij}]$ and cross out s
of its rows or columns, we obtain a **submatrix** of A.

EXAMPLE 18

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 4 & -3 & 5 \\ 3 & 0 & 5 & -3 \end{bmatrix}.$$

If we cross out the second row and third column, we obtain t

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & -3 \end{bmatrix}.$$

EXAMPLE 19 The matrix

$$A = \left[\begin{array}{ccc:cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \hdashline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{array}\right]$$

is partitioned as

$$A = \left[\begin{array}{c:c} A_{11} & A_{12} \\ \hdashline A_{21} & A_{22} \end{array}\right].$$

We could also write

$$A = \left[\begin{array}{cc:cc:c} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \hdashline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{array}\right] = \left[\begin{array}{c:c:c} \widehat{A}_{11} & \widehat{A}_{12} & \widehat{A}_{13} \\ \hdashline \widehat{A}_{21} & \widehat{A}_{22} & \widehat{A}_{23} \end{array}\right],$$

which gives another partitioning of A. We thus speak of **partitioned m**

EXAMPLE 22 Let

$$B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 9 & 8 & -4 \\ 6 & 7 & 5 \end{bmatrix}$$

Then we have

$$\left[B \;\vdots\; D\right] = \left[\begin{array}{c:ccc} 2 & 9 & 8 & -4 \\ 3 & 6 & 7 & 5 \end{array}\right], \quad \left[\begin{array}{c} D \\ \hdashline C \end{array}\right] = \left[\begin{array}{ccc} 9 & 8 & -4 \\ 6 & 7 & 5 \\ \hdashline 1 & -1 & 0 \end{array}\right],$$

and

$$\left[\left[\begin{array}{c} D \\ \hdashline C \end{array}\right] \;\vdots\; C^T\right] = \left[\begin{array}{ccc:c} 9 & 8 & -4 & 1 \\ 6 & 7 & 5 & -1 \\ \hdashline 1 & -1 & 0 & 0 \end{array}\right].$$

SUMMATION NOTATION (OPTIONAL)

We shall occasionally use the **summation notation** and we now revi useful and compact notation, which is widely used in mathematics.

By $\sum_{i=1}^{n} a_i$ we mean

$$a_1 + a_2 + \cdots + a_n.$$

The letter i is called the **index of summation**; it is a dummy variable t be replaced by another letter. Hence we can write

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j = \sum_{k=1}^{n} a_k.$$

EXAMPLE 23 If

$$a_1 = 3, \quad a_2 = 4, \quad a_3 = 5, \quad \text{and} \quad a_4 = 8,$$

then

$$\sum_{i=1}^{4} a_i = 3 + 4 + 5 + 8 = 20.$$

EXAMPLE 24

By $\sum_{i=1}^{n} r_i a_i$ we mean

$$r_1 a_1 + r_2 a_2 + \cdots + r_n a_n .$$

It is not difficult to show (Exercise T.11) that the summation notation sati the following properties:

(i) $$\sum_{i=1}^{n} (r_i + s_i) a_i = \sum_{i=1}^{n} r_i a_i + \sum_{i=1}^{n} s_i a_i .$$

(ii) $$\sum_{i=1}^{n} c(r_i a_i) = c \left(\sum_{i=1}^{n} r_i a_i \right) .$$

.3 Exercises

Exercises 1 and 2, compute $\mathbf{a} \cdot \mathbf{b}$.

(a) $\mathbf{a} = \begin{bmatrix} 1 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

(b) $\mathbf{a} = \begin{bmatrix} -3 & -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(c) $\mathbf{a} = \begin{bmatrix} 4 & 2 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$

(d) $\mathbf{a} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(a) $\mathbf{a} = \begin{bmatrix} 2 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(b) $\mathbf{a} = \begin{bmatrix} 1 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) $\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

(d) $\mathbf{a} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

3. Let $\mathbf{a} = \begin{bmatrix} -3 & 2 & x \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -3 \\ 2 \\ x \end{bmatrix}$. If $\mathbf{a} \cdot \mathbf{b} = 17$, find x.

4. Let $\mathbf{w} = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$. Compute $\mathbf{w} \cdot \mathbf{w}$.

5. Find all values of x so that $\mathbf{v} \cdot \mathbf{v} = 1$, where $\mathbf{v} = \Big[-$

6. Let $A = \begin{bmatrix} 1 & 2 & x \\ 3 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} y \\ x \\ 1 \end{bmatrix}$. If $AB = \Big[$

find x and y.

$$C = \begin{bmatrix} 2 & 3 & 1 \\ 3 & -4 & 5 \\ 1 & -1 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & 0 & -3 \\ -2 & 1 & 5 \\ 3 & 4 & 2 \end{bmatrix}, \quad \textit{and} \quad F = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}.$$

7. If possible, compute:

(a) AB (b) BA (c) $CB + D$

(d) $AB + DF$ (e) $BA + FD$

8. If possible, compute:

(a) $A(BD)$ (b) $(AB)D$ (c) $A(C + E)$

(d) $AC + AE$ (e) $(D + F)A$

9. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$.

Compute the following entries of AB:

(a) The $(1, 2)$ entry (b) The $(2, 3)$ entry

(c) The $(3, 1)$ entry (d) The $(3, 3)$ entry

10. If $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$, compute DI_2 and I_2D.

11. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}.$$

Show that $AB \neq BA$.

12. If A is the matrix in Example 4 and O is the 3×2

Express $A\mathbf{c}$ as a linear combination of the columns of A.

16. Let

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 4 & 3 \\ 3 & 0 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 4 \end{bmatrix}.$$

Express the columns of AB as linear combinations of the columns of A.

17. Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$.

(a) Verify that $AB = 3\mathbf{a}_1 + 5\mathbf{a}_2 + 2\mathbf{a}_3$, where $\mathbf{a}_j$ is the jth column of A for $j = 1, 2, 3$.

(b) Verify that $AB = \begin{bmatrix} (\text{row}_1(A))B \\ (\text{row}_2(A))B \end{bmatrix}$.

18. Write the linear combination

$$3\begin{bmatrix} -2 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

as a product of a 2×3 matrix and a 3-vector.

19. Consider the following linear system:

$$\begin{aligned} 2x + w &= 7 \\ 3x + 2y + 3z &= -2 \\ 2x + 3y - 4z &= 3 \\ x + 3z &= 5. \end{aligned}$$

(a) Find the coefficient matrix.

(b) Write the linear system in matrix form.

(c) Find the augmented matrix.

20. Write the linear system with augmented matrix

Determinant

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The determinant of a Matrix

ion of the
inant of a
2 Matrix

The **determinant** of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by

$$\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}.$$

Example

Compute $\det\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

lution. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. Since A is obtained from I_2 by multiplying the second v by the constant 3, we have

$$\det(A) = 3\det(I_2) = 3 \cdot 1 = 3.$$

Note that our answer agrees with this definition of the determinant.

Example

Find the determinant of each matrix.

(a) $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ (c) $C = \left[\begin{matrix} 0 \\ 2 \end{matrix}\right.$

UTION (a) $|A| = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$

(b) $|B| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) = 4 - 4 = 0$

(c) $|C| = \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 0(4) - 2(3) = 0 - 6 = -6$

Minors and cofactors of a Matrix

If A is a square matrix, then the **minor** M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The **cofactor** C_{ij} is given by

$$C_{ij} = (-1)^{i+j}M_{ij}.$$

For example, if A is a 3×3 matrix, then the minors and cofactors of a_{21} and a_{22} are as shown in the diagram below.

Minor of a_{21}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Delete row 2 and column 1.

Cofactor of a_{21}

$$C_{21} = (-1)^{2+1}M_{21} = -M_{21}$$

Minor of a_{22}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Delete row 2 and column 2.

Cofactor of a_{22}

$$C_{22} = (-1)^{2+2}M_{22} = M_{22}$$

you can see, the minors and cofactors of a matrix can differ only in sign. To obtain t
actors of a matrix, first find the minors and then apply the checkerboard pattern of +
. −'s shown below.

Sign Pattern for Cofactors

$$\begin{matrix} - & + \\ + & - \\ - & + \end{matrix}\Bigg]$$

3 × 3 *matrix*

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

4 × 4 *matrix*

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$n \times n$ *matrix*

e that *odd* positions (where $i + j$ is odd) have negative signs, and even positions (whe
j is even) have positive signs.

Definition of the determinant of a Matrix

natrix (of order 2 or greater), then the determinant of A i
st row of A multiplied by their cofactors. That is,

$$A| = \sum_{j=1}^{n} a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdot \cdot \cdot + a_{1n}C_{1n}.$$

The Determinant of a Matrix of Order 3

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

SOLUTION This matrix is the same as the one in Example 2. There you found the cofactors of the entries in the first row to be

$$C_{11} = -1, \qquad C_{12} = 5, \qquad C_{13} = 4.$$

By the definition of a determinant, you have

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} && \textbf{First row expansion} \\ &= 0(-1) + 2(5) + 1(4) = 14. \end{aligned}$$

Although the determinant is defined as an expansion by the cofactors in the first row, it can be shown that the determinant can be evaluated by expanding by *any* row or column. For instance, you could expand the 3×3 matrix in Example 3 by the second row to obtain

$$\begin{aligned} |A| &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} && \textbf{Second row expansion} \\ &= 3(-2) + (-1)(-4) + 2(8) = 14 \end{aligned}$$

or by the first column to obtain

$$|A| = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \qquad \textbf{First column expansion}$$

Theorem.
Expansion by cofactors

A be a square matrix of order n. Then the determinant of A is given by

$$\det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdot \cdot \cdot + a_{in}C_{in}$$

$$\det(A) = |A| = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdot \cdot \cdot + a_{nj}C_{nj}.$$

The Determinant of a Matrix of Order 4

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix}.$$

By inspecting this matrix, you can see that three of the entries in the third column are zeros. You can eliminate some of the work in the expansion by using the third column.

$$|A| = 3(C_{13}) + 0(C_{23}) + 0(C_{33}) + 0(C_{43})$$

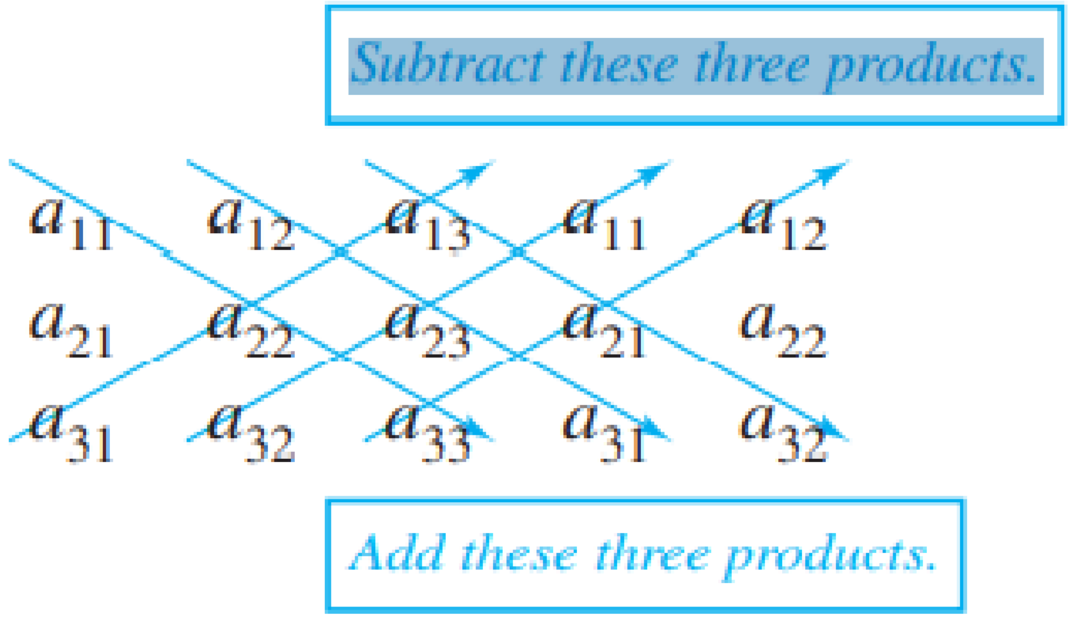
Because C_{23}, C_{33}, and C_{43} have zero coefficients, you need only find the cofactor C_{13}. To do this, delete the first row and third column of A and evaluate the determinant of the resulting matrix.

$$C_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

Expanding by cofactors in the second row yields

$$\begin{aligned} C_{13} &= (0)(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2}\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^{2+3}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} \\ &= 0 + 2(1)(-4) + 3(-1)(-7) = 13. \end{aligned}$$

There is an alternative method commonly used for evaluating the determinant of a 3×3 matrix A. To apply this method, copy the first and second columns of A to form fourth and fifth columns. The determinant of A is then obtained by adding (or subtracting) the products of the six diagonals, as shown in the following diagram.



confirming that the determinant of A is

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

Example

the determinant of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix}.$$

n by recopying the first two columns and then computing the six diagonal product:
ws.

$$\begin{array}{ccccccc} & & -4 & 0 & 6 & & \leftarrow \text{Subtract these products.} \\ 0 & 2 & 1 & 0 & 2 & & \\ 3 & -1 & 2 & 3 & -1 & & \\ 4 & -4 & 1 & 4 & -4 & & \\ & & 0 & 16 & -12 & & \leftarrow \text{Add these products.} \end{array}$$

, by adding the lower three products and subtracting the upper three products, you
the determinant of A to be $|A| = 0 + 16 + (-12) - (-4) - 0 - 6 = 2.$

REMARK :

- The diagonal process illustrated in Example 5 is valid only for matrices of order 3. For matrices of higher orders, another method must be used.

Triangular Matrices

Evaluating determinants of matrices of order 4 or higher can be tedious. There is, however, an important exception: the determinant of a *triangular* matrix. Recall from Section 2.4 that a square matrix is called *upper triangular* if it has all zero entries below its main diagonal, and *lower triangular* if it has all zero entries above its main diagonal. A matrix that is both upper and lower triangular is called **diagonal.** That is, a diagonal matrix is one in which all entries above and below the main diagonal are zero.

Upper Triangular Matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Lower Triangular Matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Determinant of a triangular matrix

ılar matrix of order n, then its determinant is the product of
ıal. That is,

$$|A| = a_{11}a_{22}a_{33} \cdot \cdot \cdot a_{nn}.$$

Example

Find the determinant of each matrix.

(a) $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix}$ (b) $B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$

'ION (a) The determinant of this lower triangular matrix is giver

$$|A| = (2)(-2)(1)(3) = -12.$$

(b) The determinant of this *diagonal* matrix is given by

$$|B| = (-1)(3)(2)(4)(-2) = 48.$$

In Exercises 1–12, find the determinant of the matrix.

1. $[1]$

2. $[-3]$

3. $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

4. $\begin{bmatrix} -3 & 1 \\ 5 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 5 & 2 \\ -6 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

7. $\begin{bmatrix} -7 & 6 \\ \frac{1}{2} & 3 \end{bmatrix}$

8. $\begin{bmatrix} \frac{1}{3} & 5 \\ 4 & -9 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$

10. $\begin{bmatrix} 2 & -3 \\ -6 & 9 \end{bmatrix}$

11. $\begin{bmatrix} \lambda - 3 & 2 \\ 4 & \lambda - 1 \end{bmatrix}$

12. $\begin{bmatrix} \lambda - 2 & 0 \\ 4 & \lambda - 4 \end{bmatrix}$

In Exercises 13–16, find (a) the minors and (b) the cofactors of the matrix.

13. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

14. $\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$

15. $\begin{bmatrix} -3 & 2 & 1 \\ 4 & 5 & 6 \\ 2 & -3 & 1 \end{bmatrix}$

16. $\begin{bmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{bmatrix}$

17. Find the determinant of the matrix in Exercise 15 using the method of expansion by cofactors. Use (a) the second row and (b) the second column.

In Exercises 19–34, use expansion by cofactors to find the determinant of the matrix.

19. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{bmatrix}$

20. $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 4 \\ 1 & 0 & 2 \end{bmatrix}$

21. $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{bmatrix}$

22. $\begin{bmatrix} -3 & 0 & 0 \\ 7 & 11 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

23. $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$

24. $\begin{bmatrix} -0.4 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.2 \\ 0.3 & 0.2 & 0.2 \end{bmatrix}$

25. $\begin{bmatrix} x & y & 1 \\ 2 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

26. $\begin{bmatrix} x & y & 1 \\ -2 & -2 & 1 \\ 1 & 5 & 1 \end{bmatrix}$

27. $\begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 7 & 3 & 6 \\ 1 & 5 & 0 & 1 \\ 3 & 7 & 0 & 7 \end{bmatrix}$

28. $\begin{bmatrix} 1 & 4 & 3 & 2 \\ -5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \end{bmatrix}$

29. $\begin{bmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{bmatrix}$

30. $\begin{bmatrix} 3 & 0 & 7 & 0 \\ 2 & 6 & 11 & 12 \\ 4 & 1 & -1 & 2 \\ 1 & 5 & 2 & 10 \end{bmatrix}$

31. $\begin{bmatrix} w & x & y & z \\ 21 & -15 & 24 & 30 \\ \ldots \end{bmatrix}$

32. $\begin{bmatrix} w & x & y & z \\ 10 & 15 & -25 & 30 \\ \ldots \end{bmatrix}$

Inverse of a Matrix

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introduction

Section 2.2 discussed some of the similarities between the algebra of real numbers and the algebra of matrices. This section further develops the algebra of matrices to include the solutions of matrix equations involving matrix multiplication. To begin, consider the real number equation $ax = b$. To solve this equation for x, multiply both sides of the equation by a^{-1} (provided $a \neq 0$).

$$\begin{aligned} ax &= b \\ (a^{-1}a)x &= a^{-1}b \\ (1)x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

The number a^{-1} is called the *multiplicative inverse* of a because $a^{-1}a$ yields 1 (the identity element for multiplication). The definition of a multiplicative inverse of a matrix is similar.

Definition of the Inverse of a Matrix

An $n \times n$ matrix A is **invertible** (or **nonsingular**) if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n$$

where I_n is the identity matrix of order n. The matrix B is called the (multiplicative) **inverse** of A. A matrix that does not have an inverse is called **noninvertible** (or **singular**).

Nonsquare matrices do not have inverses. To see this, note that if A is of size $m \times n$ and B is of size $n \times m$ (where $m \neq n$), then the products AB and BA are of different sizes and cannot be equal to each other. Indeed, not all square matrices possess inverses. (See Example 4.) The next theorem, however, tells you that if a matrix *does* possess an inverse, then that inverse is unique.

EXAMPLE 1 The Inverse of a Matrix

Show that B is the inverse of A, where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

SOLUTION Using the definition of an inverse matrix, you can show that B is the inverse of A by showing that $AB = I = BA$, as follows.

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

REMARK: Recall that it is not always true that $AB = BA$, even if both products are defined. If A and B are both square matrices and $AB = I_n$, however, then it can be shown that $BA = I_n$. Although the proof of this fact is omitted, it implies that in Example 1 you needed only to check that $AB = I_2$.

The next example shows how to use a system of equations to find the inverse of a matrix.

EXAMPLE 2 Finding the Inverse of a Matrix

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}.$$

SOLUTION To find the inverse of A, try to solve the matrix equation $AX = I$ for X.

$$\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, by equating corresponding entries, you obtain the two systems of linear equations shown below.

$$\begin{aligned} x_{11} + 4x_{21} &= 1 \\ -x_{11} - 3x_{21} &= 0 \end{aligned} \qquad \begin{aligned} x_{12} + 4x_{22} &= 0 \\ -x_{12} - 3x_{22} &= 1 \end{aligned}$$

Solving the first system, you find that the first column of X is $x_{11} = -3$ and $x_{21} = 1$. Similarly, solving the second system, you find that the second column of X is $x_{12} = -4$ and $x_{22} = 1$. The inverse of A is

$$X = A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}.$$

Try using matrix multiplication to check this result.

Solution 2

Generalizing the method used to solve Example 2 provides a convenient method for finding an inverse. Notice first that the two systems of linear equations

$$\begin{aligned} x_{11} + 4x_{21} &= 1 \\ -x_{11} - 3x_{21} &= 0 \end{aligned} \qquad\qquad \begin{aligned} x_{12} + 4x_{22} &= 0 \\ -x_{12} - 3x_{22} &= 1 \end{aligned}$$

have the *same coefficient matrix*. Rather than solve the two systems represented by

$$\left[\begin{array}{rr|r} 1 & 4 & 1 \\ -1 & -3 & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{rr|r} 1 & 4 & 0 \\ -1 & -3 & 1 \end{array}\right]$$

separately, you can solve them simultaneously. You can do this by **adjoining** the identity matrix to the coefficient matrix to obtain

$$\left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right].$$

By applying Gauss-Jordan elimination to this matrix, you can solve *both* systems with a single elimination process, as follows.

$$\left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \qquad R_2 + R_1 \to R_2$$

$$\left[\begin{array}{rr|rr} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right] \qquad R_1 + (-4)R_2 \to R_1$$

Applying Gauss-Jordan elimination to the "doubly augmented" matrix $[A \vdots I]$, you obtain the matrix $[I \vdots A^{-1}]$.

Finding the Inverse of a Matrix by Gauss-Jordan Elimination

Let A be a square matrix of order n.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain $[A \vdots I]$. Note that you separate the matrices A and I by a dotted line. This process is called **adjoining** matrix I to matrix A.
2. If possible, row reduce A to I using elementary row operations on the *entire* matrix $[A \vdots I]$. The result will be the matrix $[I \vdots A^{-1}]$. If this is not possible, then A is noninvertible (or singular).
3. Check your work by multiplying AA^{-1} and $A^{-1}A$ to see that $AA^{-1} = I = A^{-1}A$.

EXAMPLE 3 **Finding the Inverse of a Matrix**

Find the inverse of the matrix.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

SOLUTION Begin by adjoining the identity matrix to A to form the matrix

$$[A \ \vdots \ I] = \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right].$$

Now, using elementary row operations, rewrite this matrix in the form $[I \ \vdots \ A^{-1}]$, as follows.

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \quad R_2 + (-1)R_1 \rightarrow R_2$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right] \quad R_3 + (6)R_1 \rightarrow R_3$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right] \quad R_3 + (4)R_2 \rightarrow R_3$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \quad (-1)R_3 \rightarrow R_3$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \quad R_2 + R_3 \rightarrow R_2$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \quad R_1 + R_2 \rightarrow R_1$$

The matrix A is invertible, and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}.$$

Try confirming this by showing that $AA^{-1} = I = A^{-1}A$.

EXAMPLE 4 A Singular Matrix

Show that the matrix has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

SOLUTION Adjoin the identity matrix to A to form

$$[A \ \vdots \ I] = \left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right]$$

and apply Gauss-Jordan elimination as follows.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right] \qquad R_2 + (-3)R_1 \rightarrow R_2$$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 7 & -2 & 2 & 0 & 1 \end{array}\right] \qquad R_3 + (2)R_1 \rightarrow R_3$$

Now, notice that adding the second row to the third row produces a row of zeros on the left side of the matrix.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \qquad R_3 + R_2 \rightarrow R_3$$

Because the "A portion" of the matrix has a row of zeros, you can conclude that it is not possible to rewrite the matrix $[A \ \vdots \ I]$ in the form $[I \ \vdots \ A^{-1}]$. This means that A has no inverse, or is noninvertible (or singular).

EXAMPLE 5 Finding the Inverse of a 2 x 2 Matrix

If possible, find the inverse of each matrix.

(a) $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ (b) $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

SOLUTION (a) For the matrix A, apply the formula for the inverse of a 2×2 matrix to obtain $ad - bc = (3)(2) - (-1)(-2) = 4$. Because this quantity is not zero, the inverse is formed by interchanging the entries on the main diagonal and changing the signs of the other two entries, as follows.

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

(b) For the matrix B, you have $ad - bc = (3)(2) - (-1)(-6) = 0$, which means that B is noninvertible.

Properties of Inverses

Some important properties of inverse matrices are listed below.

THEOREM 2.8 Properties of Inverse Matrices

If A is an invertible matrix, k is a positive integer, and c is a scalar not equal to zero, then A^{-1}, A^k, cA, and A^T are invertible and the following are true.

1. $(A^{-1})^{-1} = A$
2. $(A^k)^{-1} = A^{-1}A^{-1} \cdot \cdot \cdot A^{-1} = (A^{-1})^k$
3. $(cA)^{-1} = \frac{1}{c}A^{-1}, \ c \neq 0$
4. $(A^T)^{-1} = (A^{-1})^T$

EXAMPLE 6 The Inverse of the Square of a Matrix

Compute A^{-2} in two different ways and show that the results are equal.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$$

SOLUTION One way to find A^{-2} is to find $(A^2)^{-1}$ by squaring the matrix A to obtain

$$A^2 = \begin{bmatrix} 3 & 5 \\ 10 & 18 \end{bmatrix}$$

and using the formula for the inverse of a 2×2 matrix to obtain

$$(A^2)^{-1} = \frac{1}{4}\begin{bmatrix} 18 & -5 \\ -10 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{2} & -\frac{5}{4} \\ -\frac{5}{2} & \frac{3}{4} \end{bmatrix}.$$

Another way to find A^{-2} is to find $(A^{-1})^2$ by finding A^{-1}

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$$

and then squaring this matrix to obtain

$$(A^{-1})^2 = \begin{bmatrix} \frac{9}{2} & -\frac{5}{4} \\ -\frac{5}{2} & \frac{3}{4} \end{bmatrix}.$$

Note that each method produces the same result.

THEOREM 2.9
The Inverse of a Product

If A and B are invertible matrices of size n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

EXAMPLE 7 **Finding the Inverse of a Matrix Product**

Find $(AB)^{-1}$ for the matrices

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{bmatrix}$$

using the fact that A^{-1} and B^{-1} are represented by

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix}.$$

SOLUTION Using Theorem 2.9 produces

$$(AB)^{-1} = B^{-1}A^{-1} = \overset{B^{-1}}{\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix}} \overset{A^{-1}}{\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 8 & -5 & -2 \\ -8 & 4 & 3 \\ 5 & -2 & -\frac{7}{3} \end{bmatrix}.$$

THEOREM 2.11
Systems of Equations with Unique Solutions

If A is an invertible matrix, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution given by

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

PROOF Because A is nonsingular, the steps shown below are valid.

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

This solution is unique because if $\mathbf{x}_1$ and $\mathbf{x}_2$ were two solutions, you could apply the cancellation property to the equation $A\mathbf{x}_1 = \mathbf{b} = A\mathbf{x}_2$ to conclude that $\mathbf{x}_1 = \mathbf{x}_2$.

Theorem 2.11 is theoretically important, but it is not very practical for solving a system of linear equations. It would require more work to find A^{-1} and then multiply by $\mathbf{b}$ than simply to solve the system using Gaussian elimination with back-substitution. A situation in which you might consider using Theorem 2.11 as a computational technique would be one in which you have *several* systems of linear equations, all of which have the same coefficient matrix A. In such a case, you could find the inverse matrix once and then solve each system by computing the product $A^{-1}\mathbf{b}$. This is demonstrated in Example 8.

EXAMPLE 8 Solving a System of Equations Using an Inverse Matrix

Use an inverse matrix to solve each system.

(a) $\begin{aligned} 2x + 3y + z &= -1 \\ 3x + 3y + z &= 1 \\ 2x + 4y + z &= -2 \end{aligned}$

(b) $\begin{aligned} 2x + 3y + z &= 4 \\ 3x + 3y + z &= 8 \\ 2x + 4y + z &= 5 \end{aligned}$

SOLUTION First note that the coefficient matrix for each system is

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}.$$

Using Gauss-Jordan elimination, you can find A^{-1} to be

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}.$$

To solve each system, use matrix multiplication, as follows.

(a) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

The solution is $x = 2$, $y = -1$, and $z = -2$.

(b) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}$

The solution is $x = 4$, $y = 1$, and $z = -7$.

(c) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The solution is trivial: $x = 0$, $y = 0$, and $z = 0$.

Inverse of a Matrix
using Minors, Cofactors and Adjugate

(Note: also check out [Matrix Inverse by Row Operations] and the [Matrix Calculator].)

We can calculate the Inverse of a Matrix by:

- Step 1: calculating the Matrix of Minors,
- Step 2: then turn that into the Matrix of Cofactors,
- Step 3: then the Adjugate, and
- Step 4: multiply that by 1/Determinant.

But it is best explained by working through an example!

Example: find the Inverse of A:

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

It needs 4 steps. It is all simple arithmetic but there is a lot of it, so try not to make a mistake!

Step 1: Matrix of Minors

The first step is to create a "Matrix of Minors". This step has the most calculations.

For each element of the matrix:

- ignore the values on the current row and column
- calculate the determinant of the remaining values

The Calculations

Here are the first two, and last two, calculations of the "**Matrix of Minors**" (notice how I ignore the values in the current row and columns, and calculate the determinant using the remaining values):

$$\begin{bmatrix} \bullet & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \quad 0\times1 - (-2)\times1 = 2$$

$$\begin{bmatrix} 3 & \bullet & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \quad 2\times1 - (-2)\times0 = 2$$

$$\cdots$$

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & \bullet & 1 \end{bmatrix} \quad 3\times-2 - 2\times2 = -10$$

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & \bullet \end{bmatrix} \quad 3\times0 - 0\times2 = 0$$

And here is the calculation for the whole matrix:

$$\begin{bmatrix} 0\times1 - (-2)\times1 & 2\times1 - (-2)\times0 & 2\times1 - 0\times0 \\ 0\times1 - 2\times1 & 3\times1 - 2\times0 & 3\times1 - 0\times0 \\ 0\times(-2) - 2\times0 & 3\times(-2) - 2\times2 & 3\times0 - 0\times2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix}}_{\text{Matrix of Minors}}$$

Step 2: Matrix of Cofactors

This is easy! Just apply a "checkerboard" of minuses to the "Matrix of Minors".
In other words, we need to change the sign of alternate cells, like this:

$$\begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -2 & 2 \\ +2 & 3 & -3 \\ 0 & +10 & 0 \end{bmatrix}$$

Matrix of Minors *Matrix of CoFactors*

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

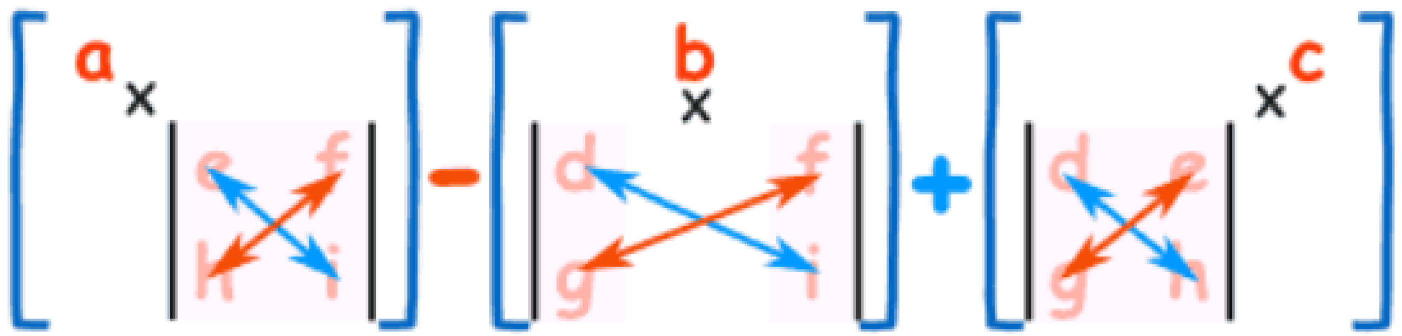
Step 3: Adjugate (also called Adjoint)

Now "Transpose" all elements of the previous matrix... in other words swap their positions over the diagonal (the diagonal stays the same):

$$\begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix}$$

Step 4: Multiply by 1/Determinant

Now find the determinant of the original matrix. This isn't too hard, because we already calculated the determinants of the smaller parts when we did "Matrix of Minors".



In practice we can just multiply each of the top row elements by the cofactor for the same location:

Elements of top row: 3, 0, 2
Cofactors for top row: 2, −2, 2

$$\text{Determinant} = 3\times2 + 0\times(-2) + 2\times2 = \mathbf{10}$$

(Just for fun: try this for any other row or column, they should also get 10.)

And now multiply the Adjugate by 1/Determinant:

$$A^{-1} = \frac{1}{10}\underbrace{\begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix}}_{\textit{Adjugate}} = \underbrace{\begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{bmatrix}}_{\textit{Inverse}}$$

And we are done!

Conclusion

- For each element, calculate the **determinant of the values not on the row or column**, to make the Matrix of Minors
- Apply a **checkerboard** of minuses to make the Matrix of Cofactors
- **Transpose** to make the Adjugate
- Multiply by **1/Determinant** to make the Inverse

System of linear equations

Gulush Nabadova

Linear Equations in n Variables

Recall from analytic geometry that the equation of a line in two-dimensional space has the form

$$a_1x + a_2y = b, \quad a_1, a_2, \text{ and } b \text{ are constants.}$$

This is a **linear equation in two variables** x and y. Similarly, the equation of a plane in three-dimensional space has the form

$$a_1x + a_2y + a_3z = b, \quad a_1, a_2, a_3, \text{ and } b \text{ are constants.}$$

Such an equation is called a **linear equation in three variables** x, y, and z. In general, a linear equation in n variables is defined as follows.

A **linear equation in n variables** $x_1, x_2, x_3, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b.$$

The **coefficients** $a_1, a_2, a_3, \ldots, a_n$ are real numbers, and the **constant term** b is a real number. The number a_1 is the **leading coefficient,** and x_1 is the **leading variable.**

EXAMPLE 1 Examples of Linear Equations and Nonlinear Equations

Each equation is linear.

(a) $3x + 2y = 7$

(b) $\frac{1}{2}x + y - \pi z = \sqrt{2}$

(c) $x_1 - 2x_2 + 10x_3 + x_4 = 0$

(d) $\left(\sin \frac{\pi}{2}\right)x_1 - 4x_2 = e^2$

Each equation is not linear.

(a) $xy + z = 2$

(b) $e^x - 2y = 4$

(c) $\sin x_1 + 2x_2 - 3x_3 = 0$

(d) $\frac{1}{x} + \frac{1}{y} = 4$

A **solution** of a linear equation in n variables is a sequence of n real numbers $s_1, s_2, s_3, \ldots, s_n$ arranged so the equation is satisfied when the values

$$x_1 = s_1, \quad x_2 = s_2, \quad x_3 = s_3, \quad \ldots, \quad x_n = s_n$$

are substituted into the equation. For example, the equation

$$x_1 + 2x_2 = 4$$

is satisfied when $x_1 = 2$ and $x_2 = 1$. Some other solutions are $x_1 = -4$ and $x_2 = 4$, $x_1 = 0$ and $x_2 = 2$, and $x_1 = -2$ and $x_2 = 3$.

The set of *all* solutions of a linear equation is called its **solution set,** and when this set is found, the equation is said to have been **solved.** To describe the entire solution set of a linear equation, a **parametric representation** is often used, as illustrated in Examples 2 and 3.

EXAMPLE 2 Parametric Representation of a Solution Set

Solve the linear equation $x_1 + 2x_2 = 4$.

SOLUTION To find the solution set of an equation involving two variables, solve for one of the variables in terms of the other variable. If you solve for x_1 in terms of x_2, you obtain

$$x_1 = 4 - 2x_2.$$

In this form, the variable x_2 is **free,** which means that it can take on any real value. The variable x_1 is not free because its value depends on the value assigned to x_2. To represent the infinite number of solutions of this equation, it is convenient to introduce a third variable t called a **parameter.** By letting $x_2 = t$, you can represent the solution set as

$$x_1 = 4 - 2t, \qquad x_2 = t, \quad t \text{ is any real number.}$$

Particular solutions can be obtained by assigning values to the parameter t. For instance, $t = 1$ yields the solution $x_1 = 2$ and $x_2 = 1$, and $t = 4$ yields the solution $x_1 = -4$ and $x_2 = 4$.

The solution set of a linear equation can be represented parametrically in more than one way. In Example 2 you could have chosen x_1 to be the free variable. The parametric representation of the solution set would then have taken the form

$$x_1 = s, \qquad x_2 = 2 - \tfrac{1}{2}s, \quad s \text{ is any real number.}$$

For convenience, choose the variables that occur last in a given equation to be free variables.

Systems of Linear Equations

A **system of m linear equations in n variables** is a set of m equations, each of which is linear in the same n variables:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}$$

REMARK: The double-subscript notation indicates a_{ij} is the coefficient of x_j in the ith equation.

A **solution** of a system of linear equations is a sequence of numbers $s_1, s_2, s_3, \ldots, s_n$ that is a solution of each of the linear equations in the system. For example, the system

$$\begin{aligned}
3x_1 + 2x_2 &= 3\\
-x_1 + x_2 &= 4
\end{aligned}$$

has $x_1 = -1$ and $x_2 = 3$ as a solution because *both* equations are satisfied when $x_1 = -1$ and $x_2 = 3$. On the other hand, $x_1 = 1$ and $x_2 = 0$ is not a solution of the system because these values satisfy only the first equation in the system.

EXAMPLE 4 **Systems of Two Equations in Two Variables**

Solve each system of linear equations, and graph each system as a pair of straight lines.

(a) $x + y = 3$
$x - y = -1$

(b) $x + y = 3$
$2x + 2y = 6$

(c) $x + y = 3$
$x + y = 1$

SOLUTION

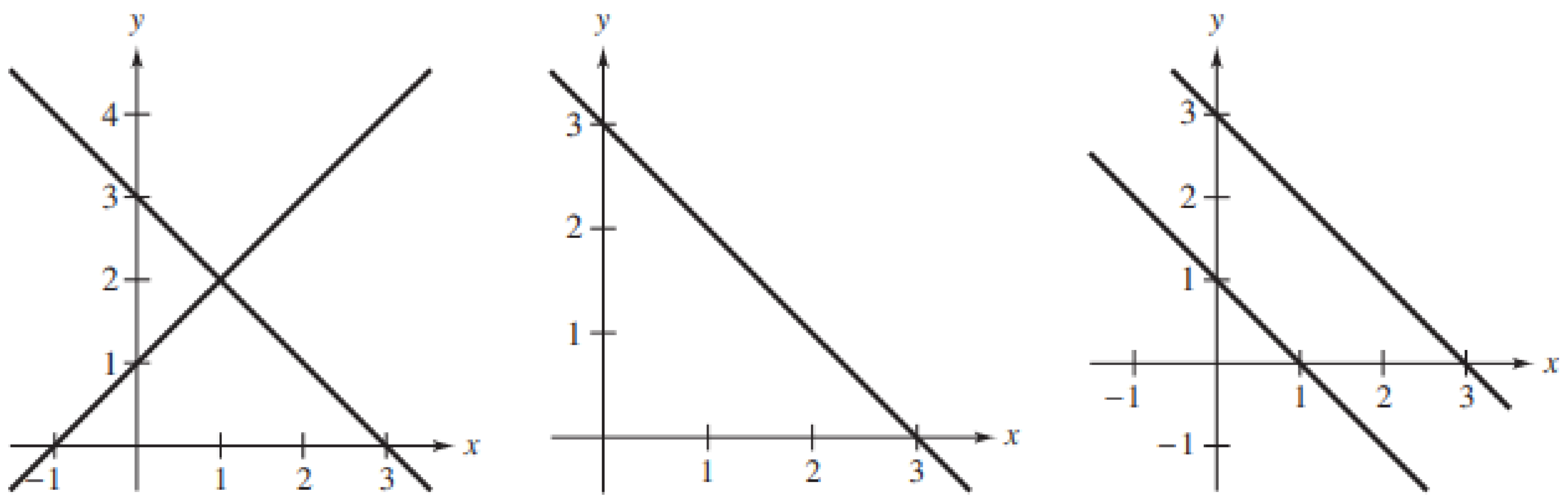
(a) This system has exactly one solution, $x = 1$ and $y = 2$. The solution can be obtained by adding the two equations to give $2x = 2$, which implies $x = 1$ and so $y = 2$. The graph of this system is represented by two *intersecting* lines, as shown in Figure 1.1(a).

(b) This system has an infinite number of solutions because the second equation is the result of multiplying both sides of the first equation by 2. A parametric representation of the solution set is shown as

$$x = 3 - t, \qquad y = t, \qquad t \text{ is any real number.}$$

The graph of this system is represented by two *coincident* lines, as shown in Figure 1.1(b).

(c) This system has no solution because it is impossible for the sum of two numbers to be 3 and 1 simultaneously. The graph of this system is represented by two *parallel* lines, as shown in Figure 1.1(c).



(a) Two intersecting lines:
$x + y = 3$
$x - y = -1$

(b) Two coincident lines:
$x + y = 3$
$2x + 2y = 6$

(c) Two parallel lines:
$x + y = 3$
$x + y = 1$

Number of Solutions of a System of Linear Equations

For a system of linear equations in n variables, precisely one of the following is true.

1. The system has exactly one solution (consistent system).
2. The system has an infinite number of solutions (consistent system).
3. The system has no solution (inconsistent system).

Solving a System of Linear Equations

Which system is easier to solve algebraically?

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned} \qquad\qquad \begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

The system on the right is clearly easier to solve. This system is in **row-echelon form,** which means that it follows a stair-step pattern and has leading coefficients of 1. To solve such a system, use a procedure called **back-substitution.**

EXAMPLE 5 Using Back-Substitution to Solve a System in Row-Echelon Form

Use back-substitution to solve the system.

$$\begin{aligned} x - 2y &= 5 &\quad& \text{Equation 1} \\ y &= -2 && \text{Equation 2} \end{aligned}$$

SOLUTION From Equation 2 you know that $y = -2$. By substituting this value of y into Equation 1, you obtain

$$\begin{aligned} x - 2(-2) &= 5 &\quad& \text{Substitute } y = -2. \\ x &= 1. && \text{Solve for } x. \end{aligned}$$

The system has exactly one solution: $x = 1$ and $y = -2$.

EXAMPLE 6 Using Back-Substitution to Solve a System in Row-Echelon Form

Solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 && \text{Equation 1} \\ y + 3z &= 5 && \text{Equation 2} \\ z &= 2 && \text{Equation 3} \end{aligned}$$

SOLUTION From Equation 3 you already know the value of z. To solve for y, substitute $z = 2$ into Equation 2 to obtain

$$\begin{aligned} y + 3(2) &= 5 && \text{Substitute } z = 2. \\ y &= -1. && \text{Solve for } y. \end{aligned}$$

Finally, substitute $y = -1$ and $z = 2$ in Equation 1 to obtain

$$\begin{aligned} x - 2(-1) + 3(2) &= 9 && \text{Substitute } y = -1, z = 2. \\ x &= 1. && \text{Solve for } x. \end{aligned}$$

The solution is $x = 1$, $y = -1$, and $z = 2$.

Operations That Lead to Equivalent Systems of Equations

Each of the following operations on a system of linear equations produces an *equivalent* system.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

Rewriting a system of linear equations in row-echelon form usually involves a *chain* of equivalent systems, each of which is obtained by using one of the three basic operations. This process is called **Gaussian elimination,** after the German mathematician Carl Friedrich Gauss (1777–1855).

EXAMPLE 7 Using Elimination to Rewrite a System in Row-Echelon Form

Solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \phantom{{}+5z} &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

SOLUTION Although there are several ways to begin, you want to use a systematic procedure that can be applied easily to large systems. Work from the upper left corner of the system, saving the x in the upper left position and eliminating the other x's from the first column.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Adding the first equation to the second equation produces a new second equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$$

Adding -2 times the first equation to the third equation produces a new third equation.

Now that everything but the first x has been eliminated from the first column, work on the second column.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$$

Adding the second equation to the third equation produces a new third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

Multiplying the third equation by $\frac{1}{2}$ produces a new third equation.

This is the same system you solved in Example 6, and, as in that example, the solution is

$$x = 1, \qquad y = -1, \qquad z = 2.$$

EXAMPLE 8 An Inconsistent System

Solve the system.

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 2x_1 - x_2 - 2x_3 &= 2 \\ x_1 + 2x_2 - 3x_3 &= -1 \end{aligned}$$

SOLUTION

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ x_1 + 2x_2 - 3x_3 &= -1 \end{aligned}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ 5x_2 - 4x_3 &= -2 \end{aligned}$$

Adding -1 times the first equation to the third equation produces a new third equation.

(Another way of describing this operation is to say that you *subtracted* the first equation from the third equation to produce a new third equation.) Now, continuing the elimination process, add -1 times the second equation to the third equation to produce a new third equation.

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ 0 &= -2 \end{aligned}$$

Adding -1 times the second equation to the third equation produces a new third equation.

Because the third "equation" is a false statement, this system has no solution. Moreover, because this system is equivalent to the original system, you can conclude that the original system also has no solution.

EXAMPLE 9 A System with an Infinite Number of Solutions

Solve the system.

$$\begin{aligned} x_2 - x_3 &= 0 \\ x_1 - 3x_3 &= -1 \\ -x_1 + 3x_2 &= 1 \end{aligned}$$

SOLUTION Begin by rewriting the system in row-echelon form as follows.

$$\begin{aligned} x_1 - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ -x_1 + 3x_2 &= 1 \end{aligned}$$

The first two equations are interchanged.

$$\begin{aligned} x_1 - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ 3x_2 - 3x_3 &= 0 \end{aligned}$$

Adding the first equation to the third equation produces a new third equation.

$$\begin{aligned} x_1 - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

Adding -3 times the second equation to the third equation eliminates the third equation.

Because the third equation is unnecessary, omit it to obtain the system shown below.

$$\begin{aligned} x_1 - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \end{aligned}$$

To represent the solutions, choose x_3 to be the free variable and represent it by the parameter t. Because $x_2 = x_3$ and $x_1 = 3x_3 - 1$, you can describe the solution set as

$$x_1 = 3t - 1, \qquad x_2 = t, \qquad x_3 = t, \quad t \text{ is any real number.}$$

Homework

In Exercises 11–16, use back-substitution to solve the system.

11. $$\begin{aligned} x_1 - x_2 &= 2 \\ x_2 &= 3 \end{aligned}$$

12. $$\begin{aligned} 2x_1 - 4x_2 &= 6 \\ 3x_2 &= 9 \end{aligned}$$

13. $$\begin{aligned} -x + y - z &= 0 \\ 2y + z &= 3 \\ \tfrac{1}{2}z &= 0 \end{aligned}$$

14. $$\begin{aligned} x - y \quad &= 4 \\ 2y + z &= 6 \\ 3z &= 6 \end{aligned}$$

In Exercises 79–84, determine the value(s) of k such that the system of linear equations has the indicated number of solutions.

79. An infinite number of solutions

$$\begin{aligned} 4x + ky &= 6 \\ kx + y &= -3 \end{aligned}$$

80. An infinite number of solutions

$$\begin{aligned} kx + y &= 4 \\ 2x - 3y &= -12 \end{aligned}$$

In Exercises 37–56, solve the system of linear equations.

37. $$\begin{aligned} x_1 - x_2 &= 0 \\ 3x_1 - 2x_2 &= -1 \end{aligned}$$

38. $$\begin{aligned} 3x + 2y &= 2 \\ 6x + 4y &= 14 \end{aligned}$$

39. $$\begin{aligned} 2u + v &= 120 \\ u + 2v &= 120 \end{aligned}$$

40. $$\begin{aligned} x_1 - 2x_2 &= 0 \\ 6x_1 + 2x_2 &= 0 \end{aligned}$$

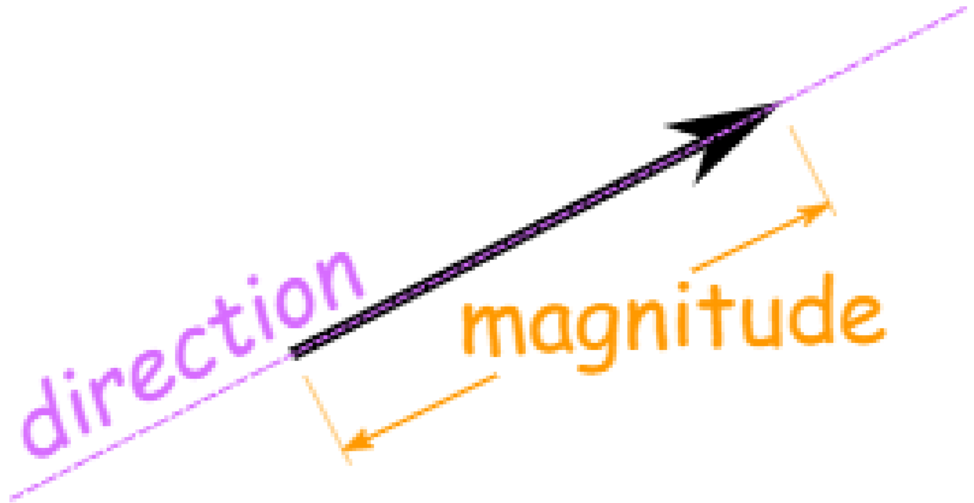
41. $$\begin{aligned} 9x - 3y &= -1 \\ \tfrac{1}{5}x + \tfrac{2}{5}y &= -\tfrac{1}{3} \end{aligned}$$

42. $$\begin{aligned} \tfrac{2}{3}x_1 + \tfrac{1}{6}x_2 &= 0 \\ 4x_1 + x_2 &= 0 \end{aligned}$$

Vectors

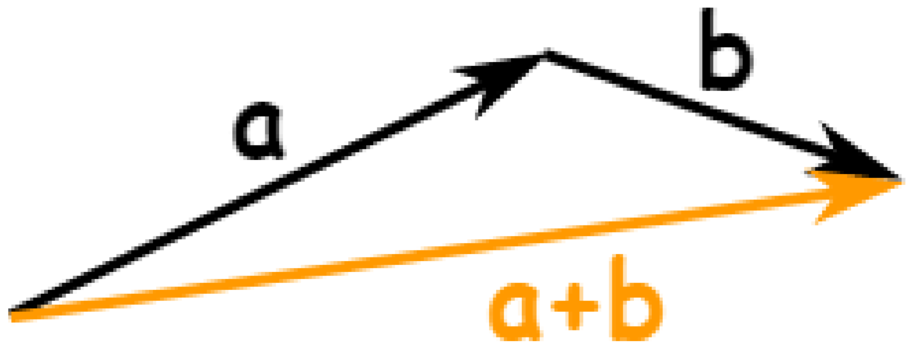
Gulush Nbabadova

Definition

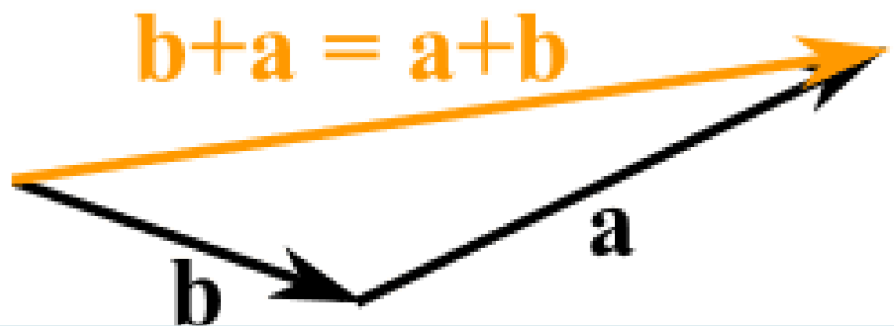


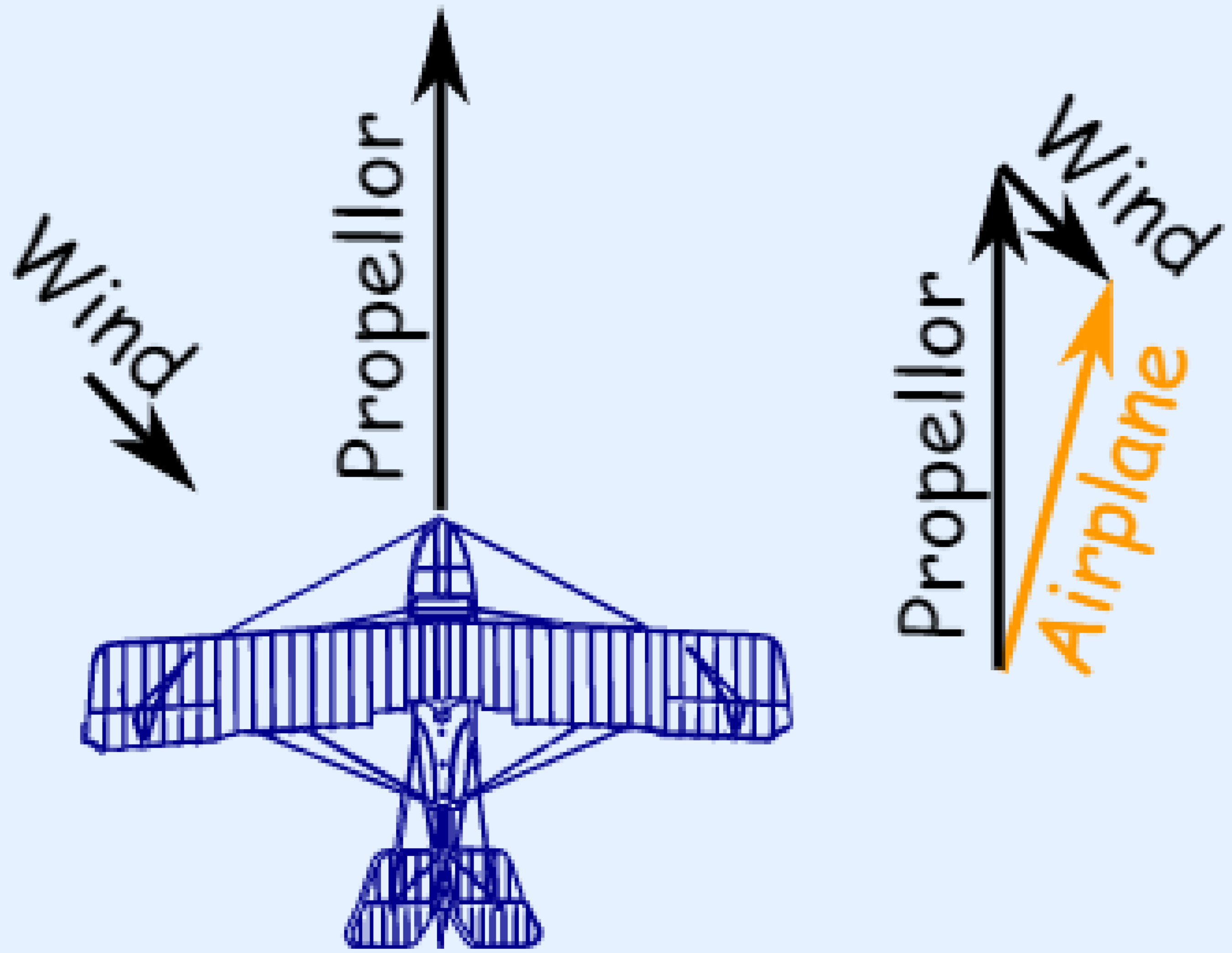
The length of the line shows its magnitude and the arrowhead points in the direction.

We can add two vectors by joining them head-to-tail:



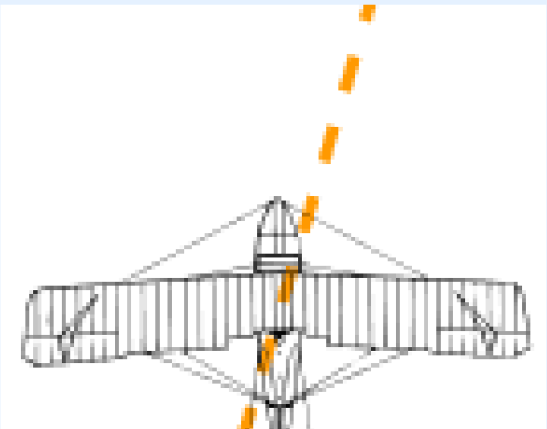
And it doesn't matter which order we add them, we get the same result:





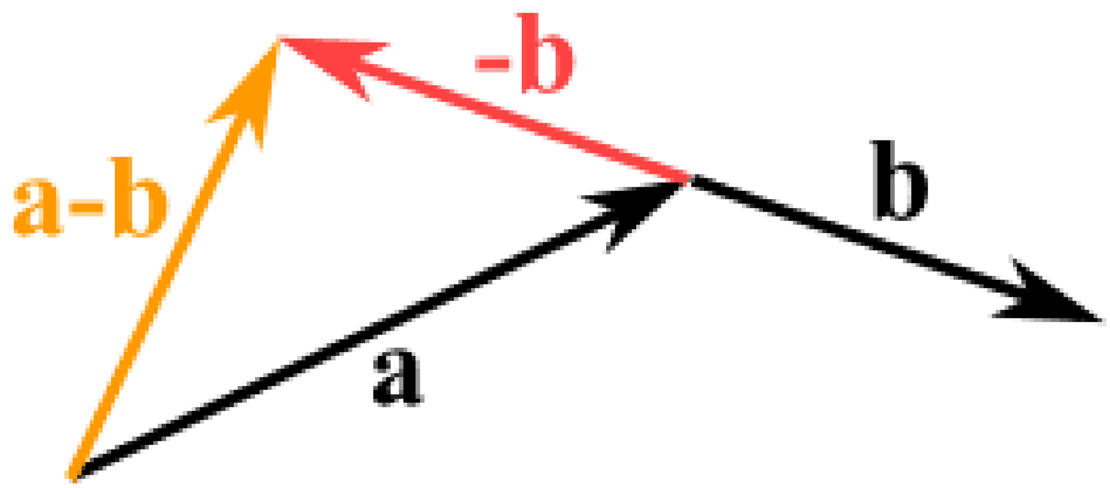
The two vectors (the velocity caused by the propeller, and the velocity of the wind) result in a slightly slower ground speed heading a little East of North.

If you watched the plane from the ground it would seem to be slipping sideways a little.



We can also subtract one vector from another:

- first we reverse the direction of the vector we want to subtract,
- then add them as usual:

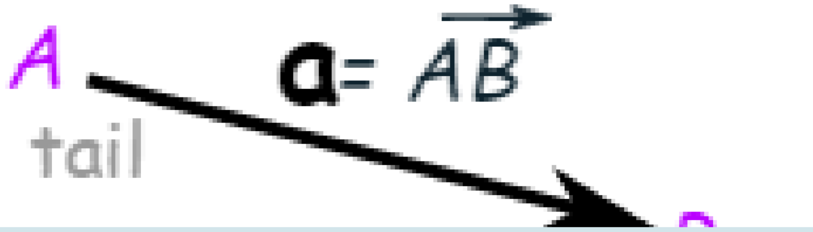


a − **b**

Notation

A vector is often written in **bold**, like **a** or **b**.

A vector can also be written as the letters of its head and tail with an arrow above it, like this:



gnitude of a Vector

iagnitude of a vector is shown by two vertical bars on either side of the vector:

$$|\mathbf{a}|$$

can be written with double vertical bars (so as not to confuse it with absolute value):

$$||\mathbf{a}||$$

se Pythagoras' theorem to calculate it:

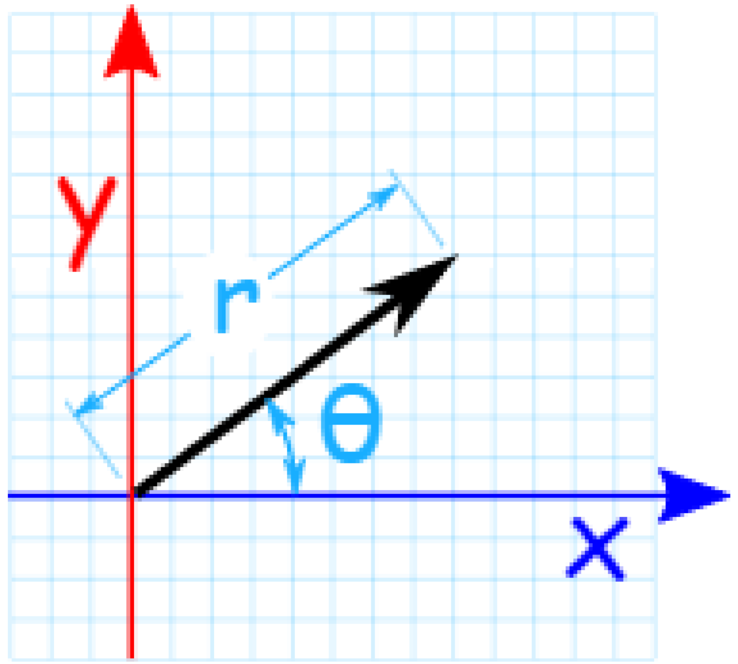
$$|\mathbf{a}| = \sqrt{(x^2 + y^2)}$$

Example: what is the magnitude of the vector **b** = (6, 8) ?

$|\mathbf{b}| = \sqrt{(6^2 + 8^2)} = \sqrt{(36+64)} = \sqrt{100} = 10$

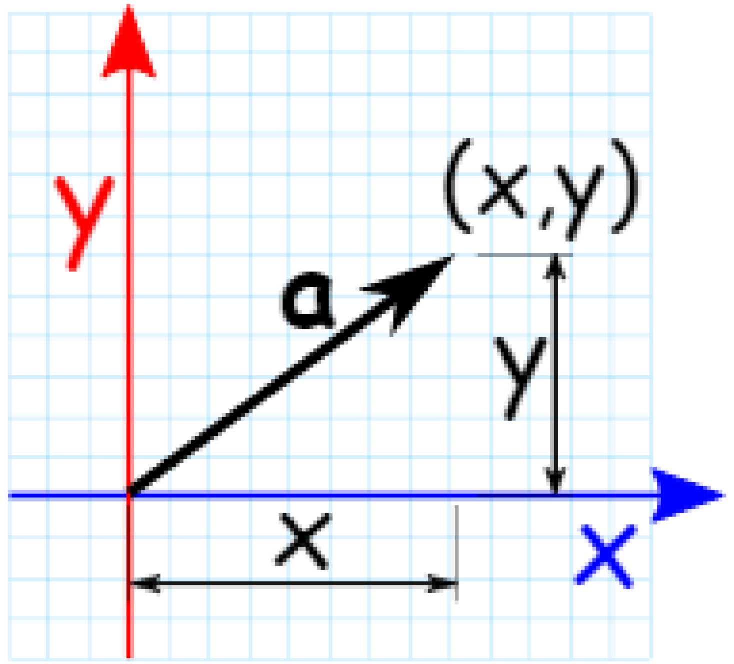
Magnitude and Direction

We may know a vector's magnitude and direction, but want its x and y lengths (or vice versa):



Vector **a** in Polar Coordinates

<=>



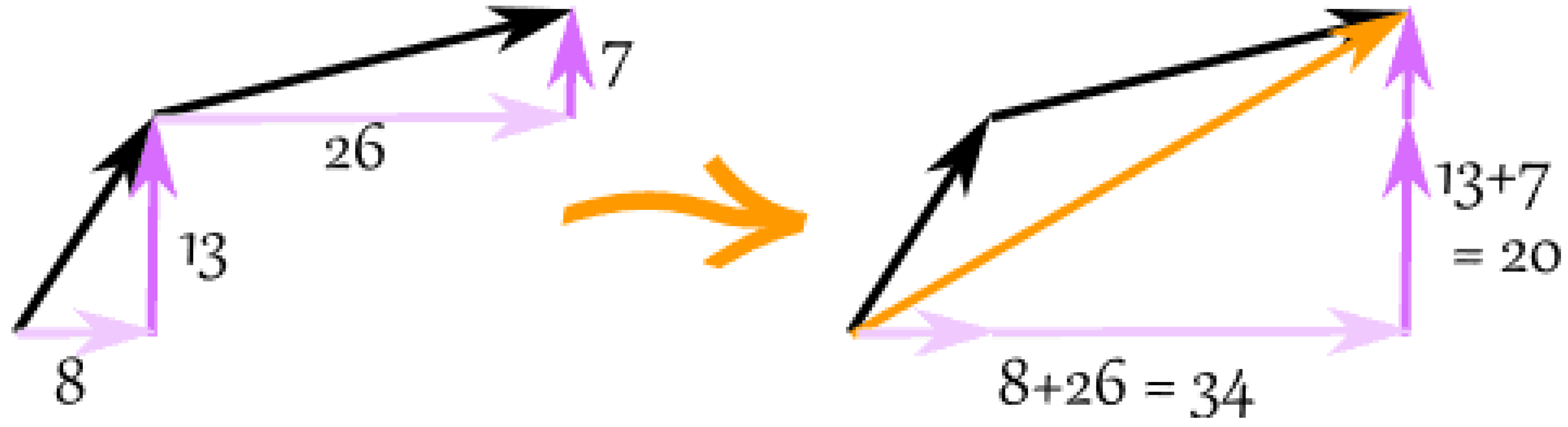
Vector **a** in Cartesian Coordinates

You can read how to convert them at [Polar and Cartesian Coordinates], but here is a quick summary:

From Polar Coordinates (r,θ) to Cartesian Coordinates (x,y)	From Cartesian Coordinates (x,y) to Polar Coordinates (r,θ)
• $x = r \times \cos(\theta)$ • $y = r \times \sin(\theta)$	• $r = \sqrt{(x^2 + y^2)}$ • $\theta = \tan^{-1}(y / x)$

g Vectors

en add vectors by **adding the x parts** and **adding the y parts**:



The vector (8, 13) and the vector (26, 7) add up to the vector (34, 20)

xample: add the vectors **a** = (8, 13) and **b** = (26, 7)

= **a + b**

= (8, 13) + (26, 7) = (8+26, 13+7) = (34, 20)

vs Scalar

s **magnitude** (size) **only**.

just a number (like 7 or −0.32) ... definitely not a vector.

as **magnitude and direction**, and is often written in **bold**, so we know it is

a vector, it has magnitude and direction

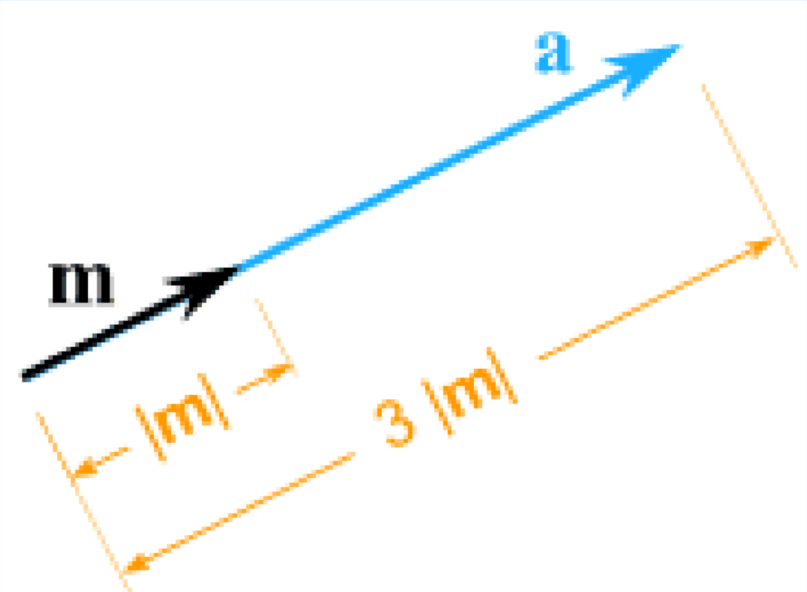
s just a value, like 3 or 12.4

ample: k**b** is actually the scalar k times the vector **b**.

ultiplying a Vector by a Scalar

en we multiply a vector by a scalar it is called "scaling" a vector, because we change how big all the vector is.

Example: multiply the vector **m** = (7, 3) by the scalar 3

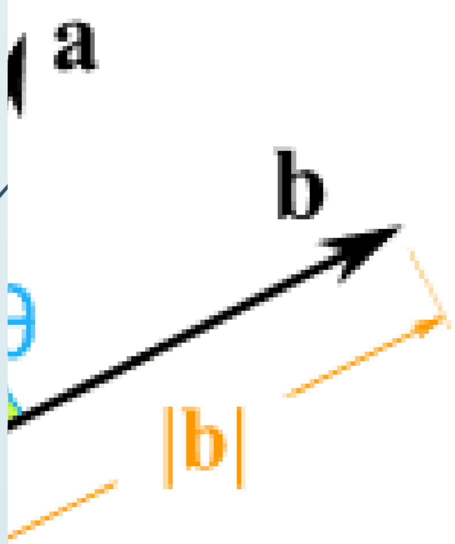


$\mathbf{a} = 3\mathbf{m} = (3\times7, 3\times3) = (21, 9)$

It still points in the same direction, but is 3 times longer

d now you know why numbers are called "scalars", because they "scale" the vector up or do

lying a Vector by a Vector (Dot Product and Cro ct)

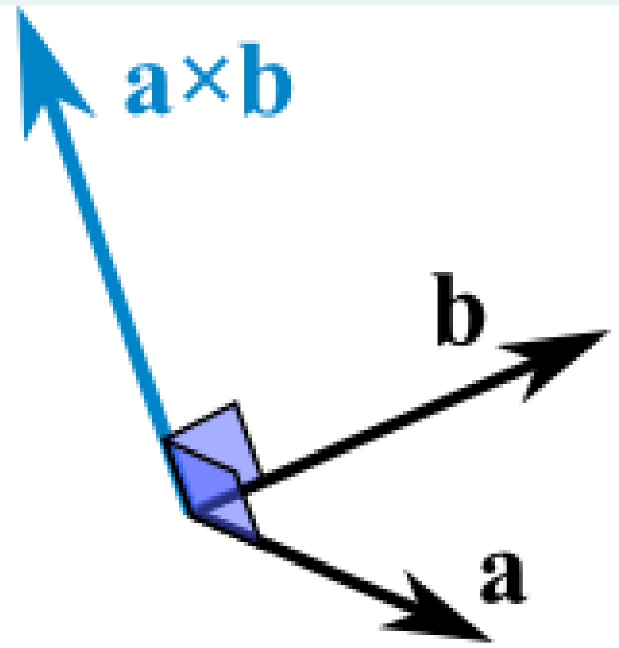


How do we **multiply two vectors** together? There is more than

- The scalar or Dot Product (the result is a scalar).
- The vector or Cross Product (the result is a vector).

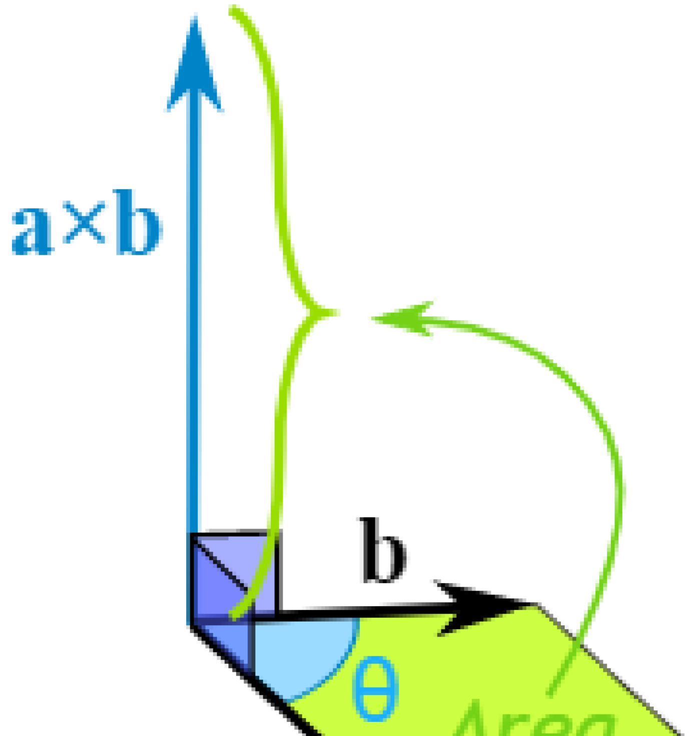
(Read those pages for more details.)

Cross product



And it all happens in 3 dimensions!

The magnitude (length) of the cross product equals the area of a parallelogram with vectors **a** and **b** for sides:



ating

calculate the Cross Product this way:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \sin(\theta)\ \mathbf{n}$$

: the magnitude (length) of vector **a**

: the magnitude (length) of vector **b**

he angle between **a** and **b**

he unit vector at right angles to both **a** and **b**

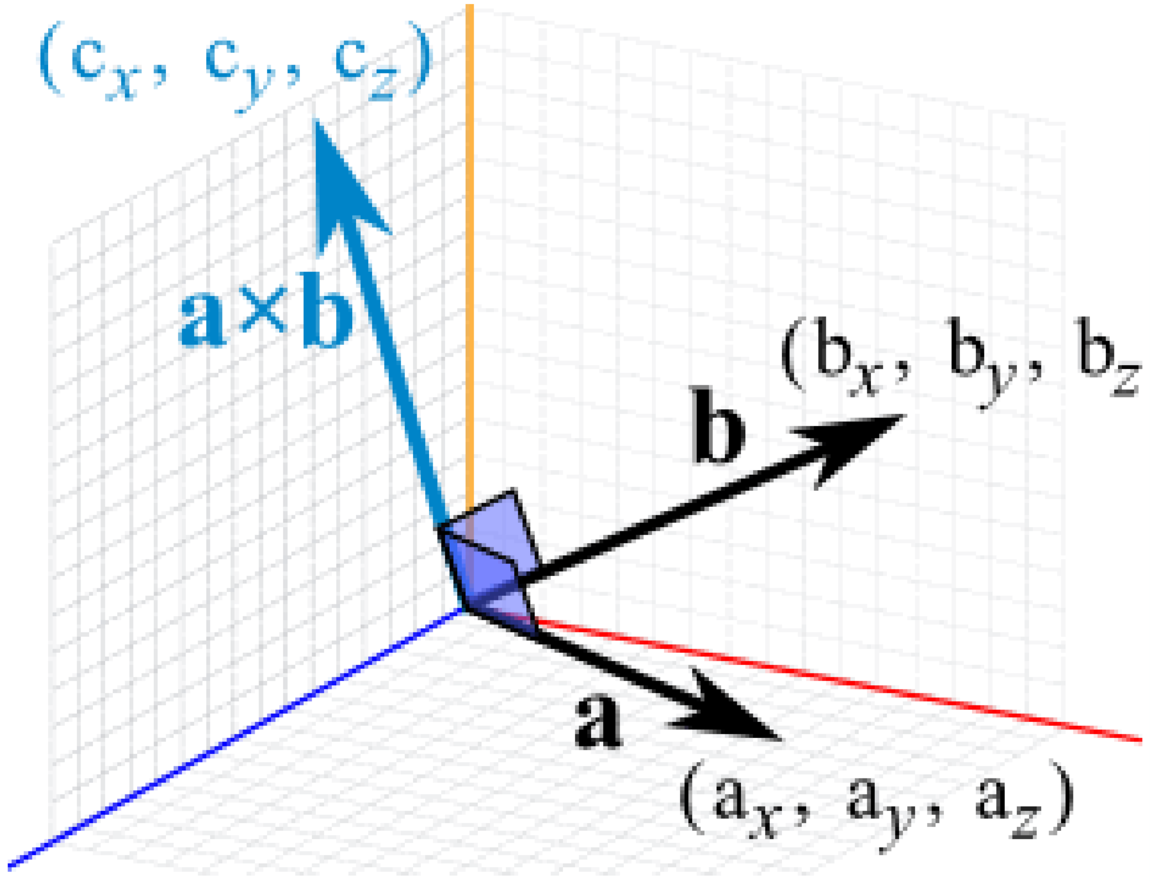
gth is: the length of **a** times the length of **b** times the sine of the
een **a** and **b**,

ultiply by the vector **n** so it heads in the correct **direction** (at right angles to

)R WE CAN CALCULATE IT THIS WAY:

/hen **a** and **b** start at the origin point (0,0,0), the Cross roduct will end at:

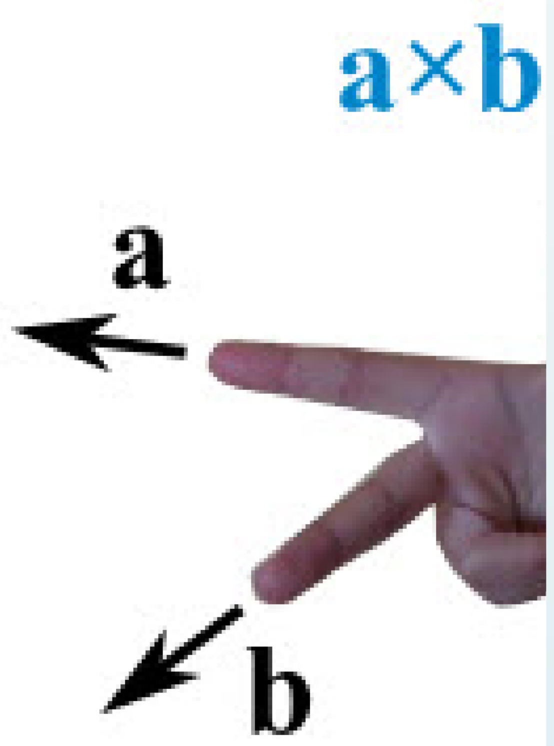
- $c_x = a_y b_z - a_z b_y$
- $c_y = a_z b_x - a_x b_z$
- $c_z = a_x b_y - a_y b_x$



Example: The cross product of **a** = (2,3,4) and **b** = (5,6,7)

- $c_x = a_y b_z - a_z b_y = 3 \times 7 - 4 \times 6 = -3$
- $c_y = a_z b_x - a_x b_z = 4 \times 5 - 2 \times 7 = 6$
- $c_z = a_x b_y - a_y b_x = 2 \times 6 - 3 \times 5 = -3$

Answer: $\mathbf{a} \times \mathbf{b} = (-3, 6, -3)$



ıich Direction?

cross product could point in the completely opposite direction and still be ght angles to the two other vectors, so we have the:

"Right Hand Rule"

ı your right-hand, point your index finger along vector **a**, and point your dle finger along vector **b**: the cross product goes in the direction of your thumb.

ɔt Product

Cross Product gives a **vector** answer, and is sometimes called the **vector product**.

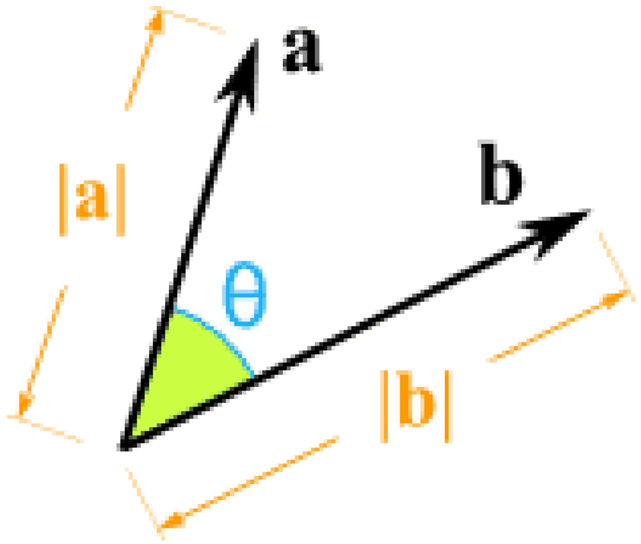
there is also the Dot Product which gives a **scalar** (ordinary number) answer, and is ıetimes called the **scalar product**.

DOT PRODUCT

$$\mathbf{a} \cdot \mathbf{b}$$

This means the Dot Product of **a** and **b**

We can calculate the Dot Product of two vectors this way:

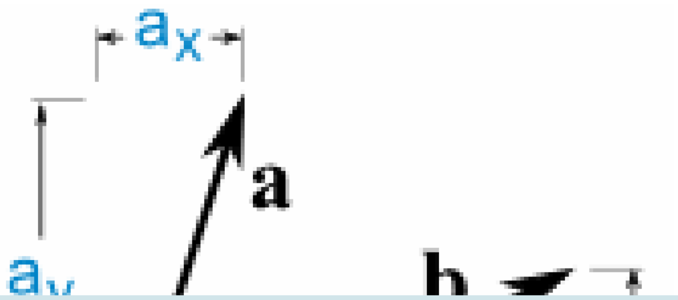


$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos(\theta)$$

Where:
$|\mathbf{a}|$ is the magnitude (length) of vector **a**
$|\mathbf{b}|$ is the magnitude (length) of vector **b**
θ is the angle between **a** and **b**

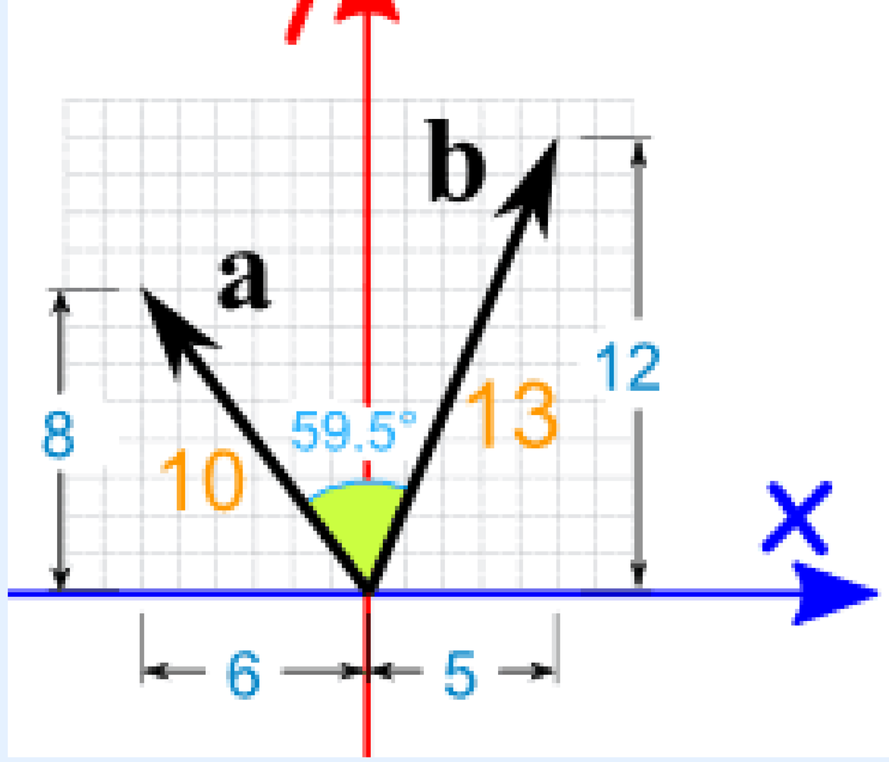
So we multiply the length of **a** times the length of **b**, then multiply by the cosine of the angle between **a** and **b**

OR we can calculate it this way:



$$\mathbf{a} \cdot \mathbf{b} = a_x \times b_x + a_y \times b_y$$

So we multiply the x's, multiply the y's, then add.



$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos(\theta)$

$\mathbf{a} \cdot \mathbf{b} = 10 \times 13 \times \cos(59.5°)$

$\mathbf{a} \cdot \mathbf{b} = 10 \times 13 \times 0.5075...$

$\mathbf{a} \cdot \mathbf{b} = 65.98... = 66$ (rounded)

OR we can calculate it this way:

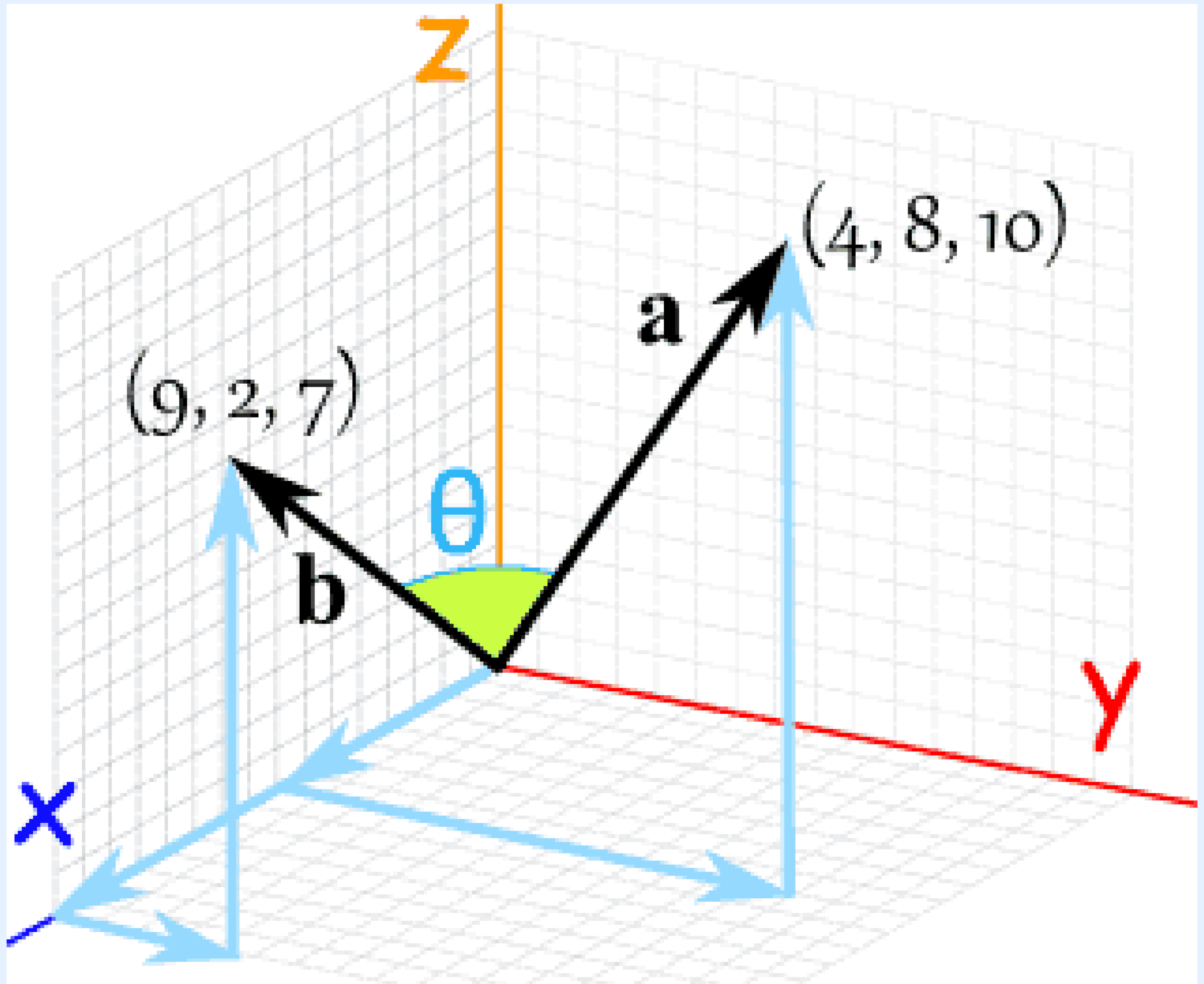
$\mathbf{a} \cdot \mathbf{b} = a_x \times b_x + a_y \times b_y$

$\mathbf{a} \cdot \mathbf{b} = -6 \times 5 + 8 \times 12$

$\mathbf{a} \cdot \mathbf{b} = -30 + 96$

And can actually be very useful!

Example: Sam has measured the end-points of two poles, and wants to know **the angle between them**:



We have 3 dimensions, so don't forget the z-components:

$\mathbf{a} \cdot \mathbf{b} = a_x \times b_x + a_y \times b_y + a_z \times b_z$

But what is |**a**| ? It is the magnitude, or length, of the vector **a**. We can use Pythagoras:

- $|\mathbf{a}| = \sqrt{(4^2 + 8^2 + 10^2)}$
- $|\mathbf{a}| = \sqrt{(16 + 64 + 100)}$
- $|\mathbf{a}| = \sqrt{180}$

Likewise for |**b**|:

- $|\mathbf{b}| = \sqrt{(9^2 + 2^2 + 7^2)}$
- $|\mathbf{b}| = \sqrt{(81 + 4 + 49)}$
- $|\mathbf{b}| = \sqrt{134}$

And we know from the calculation above that $\mathbf{a} \cdot \mathbf{b} = 122$, so:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos(\theta)$$

$$122 = \sqrt{180} \times \sqrt{134} \times \cos(\theta)$$

$$\cos(\theta) = 122 / (\sqrt{180} \times \sqrt{134})$$

$$\cos(\theta) = 0.7855...$$

$$\theta = \cos^{-1}(0.7855...) = 38.2...°$$