CutFEM 有限元介绍

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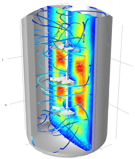


Overview

1. Part I: 一般有限元

2. Part II: CutFEM 有限元

3. Part III: 线弹性结构有限元

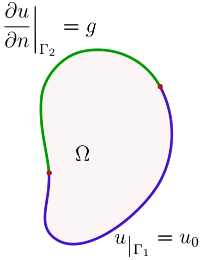
Part I: 一般有限元

一般有限元

Let Ω be a domain in R^d , d $=$ 2 or 3, with a piecewise smooth boundary $\partial\Omega$ consisting of two disjoint parts

$$\partial\Omega = \Gamma_D \cup \Gamma_N$$

where Γ_D and Γ_N are the Dirichlet and Neumann parts of the boundary, respectively.



一般有限元

Let us consider first the Poisson equation in Ω with Dirichlet boundary conditions on $\Gamma_D \subset \partial\Omega$ and Neumann boundary conditions on $\Gamma_N \subset \partial\Omega/\Gamma_D$.

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= u_0 \quad \text{on } \Gamma_D \\ \mathbf{n} \cdot \nabla u &= g_N \quad \text{in } \Gamma_N \end{aligned}$$

一般有限元

$$\mathcal{H}_D^1 := \{u \in \mathcal{H}^1(\Omega) \mid u = u_0 \text{ on } \Gamma_D\}$$
$$\mathcal{H}_N^1 := \{u \in \mathcal{H}^1(\Omega) \mid u = g_N \text{ on } \Gamma_N\}$$

The associated weak formulation is the following:
find $u \in \mathcal{H}_D^1$ such that

$$\boxed{\int_\Omega \nabla u \cdot \nabla v \mathrm{d}\Omega = \int_\Omega v f \mathrm{d}\Omega + \int_{\Gamma_N} v g_N \mathrm{d}s}$$

for all $v \in \mathcal{H}_N^1$.

Example 1: 二维泊松方程

考虑一个定义在单位圆区域上的特殊问题，在单位元内部

$$u = \frac{1 - x^2 - y^2}{2}$$

在边界 $\partial\Omega$ 上 $u = 0$。

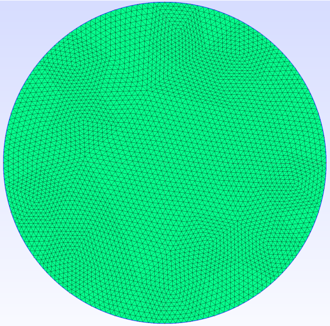


Figure 1: 网格

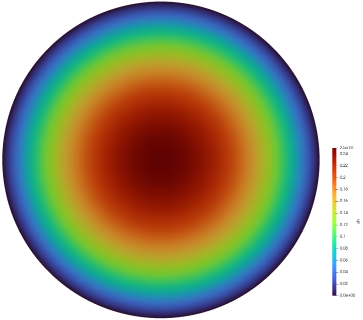


Figure 2: 分析结果

Example 1: 二维泊松方程

```julia
using Gridap
using GridapGmsh

# 读取网格
model = GmshDiscreteModel("./src/circle.msh")

Ω = Triangulation(model)
dΩ = Measure(Ω, 2)
```

Example 1: 二维泊松方程

```julia
# FE space
reffe = ReferenceFE(lagrangian, Float64, 1)
V = TestFESpace(model, reffe,
dirichlet_tags="fixed")

g(x) = 0.0
U = TrialFESpace(V, g)

f(x) = 1.0
a(u,v) = ∫( ∇(v)⋅∇(u) )dΩ
l(v) = ∫( v*f )dΩ
```

Example 1: 二维泊松方程

```
op = AffineFEOperator(a, l, U, V)

uh = solve(op)

writevtk(Ω, "results", cellfields=["uh"=>uh])
```

The Nitsche Method

We introduce a method for treating general boundary conditions in the finite element method generalizing an approach, due to Nitsche, for approximating Dirichlet boundary conditions.
Find $u_h \in V_h$ such that

$$\mathcal{A}_h(u_h, v) = l_h(v).$$

for all $v \in V_h$.

The Nitsche Method

$$\mathcal{A}_h(u_h, v) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}\Omega + \int_{\Gamma_D} (\lambda u v - v(\mathbf{n} \cdot \nabla u) - u(\mathbf{n} \cdot \nabla v))\mathrm{d}s$$

$$l_h(v) = \int_\Omega v f \mathrm{d}\Omega + \int_{\Gamma_D} \left(\lambda v g_D - \frac{\partial v}{\partial \mathbf{n}} g_D\right)\mathrm{d}s$$

Example 2: Nitsche 法求解

```julia
# 读取网格
Ω = Triangulation(model)
dΩ = Measure(Ω, 2)

Γd = BoundaryTriangulation(model,tags="fixed")
dΓd = Measure(Γd, 2)
n_Γd = get_normal_vector(Γd)

# FE space
reffe = ReferenceFE(lagrangian, Float64, 1)
V = TestFESpace(model, reffe, conformity=:H1)

U = TrialFESpace(V)
```

Example 2: Nitsche 法求解

```
γd = 10.0
h = 1.0 / 400

ud(x) = 0.0
f(x) = 1.0

a(u,v) =∫( ∇(v)⋅∇(u) )dΩ +
∫( (γd/h)*v*u  - v*(n_Γd⋅∇(u)) - (n_Γd⋅∇(v))*u )dΓd

l(v) =  ∫( v*f )dΩ  +
∫( (γd/h)*v*ud - (n_Γd⋅∇(v))*ud ) * dΓd
```

Example 2: Nitsche 法求解

分析结果对比:

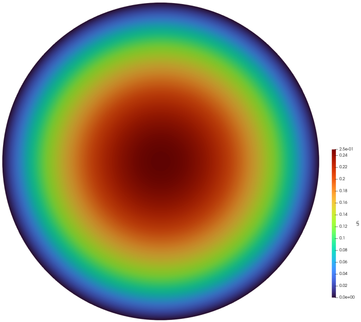


Figure 3: 一般方法

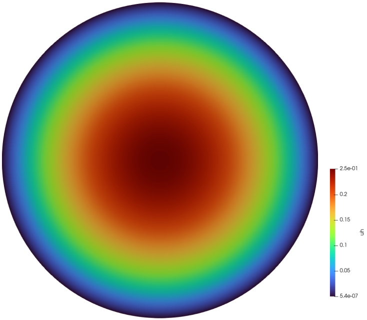


Figure 4: The Nitsche Method

Part II: CutFEM 有限元

CutFEM 有限元

Like in any other *embedded boundary method*, we build the computational mesh by introducing an **artificial domain** Ω_{art} such that it has a simple geometry that is easy to mesh using Cartesian grids and it includes the **physical domain** $\Omega \subset \Omega_{\text{art}}$.

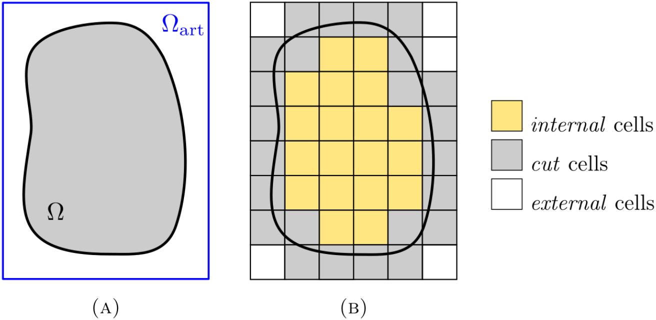


Figure 5: Embedded boundary setup.

Let us construct a partition of Ω_{art} into cells, represented by T^h_{art} , with characteristic cell size h. Cells in T^h_{art} can be classified as follows: a cell $K \in T^h_{\text{art}}$ such that $K \subset \Omega$ is an **internal cell**; if $K \cap \Omega = \emptyset$, K is an **external cell**; otherwise, K is a **cut cell** (see Fig. 5). Furthermore, we define the set of **active cells** as $T_h^{\text{act}} = T_h^{\text{in}} \cup T_h^{\text{cut}}$ and its union Ω_{act}.

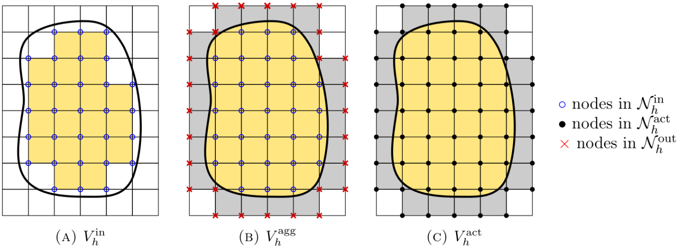


Figure 6: Finite Element spaces.

CutFEM 有限元

We define the FE-wise operators:

$$
\begin{aligned}
\mathcal{A}_K(u,v) \doteq \int_{K\cap\Omega} \nabla u \cdot \nabla v \mathrm{d}V \\
+ \int_{\partial K \cap \Gamma_D} \left(h^{-1}\lambda u v - v(\mathbf{n}\cdot\nabla u) - u(\mathbf{n}\cdot\nabla v)\right)\mathrm{d}S \\
+ \int_{\partial K\cap\Gamma_{\text{ghost}}} h^{-1}\mu [\![\partial_n u]\!][\![\partial_n v]\!]\mathrm{d}S
\end{aligned}
$$

$$
l_K(u,v) \doteq \int_{K\cap\Omega} v f \mathrm{d}V + \int_{\partial K\cap\Gamma_D} \left(h^{-1}\lambda v g_D - (\mathbf{n}\cdot\nabla v) g_D\right)\mathrm{d}S
$$

CutFEM 有限元

Given a face $F \in \mathcal{F}_h^{\text{ghost}}$ and the two cells K and K' sharing this face, we define the jump operator

$$[\![\partial_n u]\!] \doteq \mathbf{n}_K \cdot \nabla u|_K + \mathbf{n}_{K'} \cdot \nabla u|_{K'},$$

h_F is some average of h_K and $h_{K'}$.
We define the GP stabilisation term:

$$s_h(u, v) = \sum_{F \in \mathcal{F}_h^{\text{ghost}}} \left(\lambda_G h_F [\![\partial_n u]\!], [\![\partial_n v]\!]\right)_F .$$

Example 3: CutFEM 求解

```
# 1. Build background mesh
nn = 40
partition = (nn,nn)
pmin = 1.2*Point(-1,-1)
pmax = 1.2*Point(1,1)
bgmodel = CartesianDiscreteModel(pmin,pmax,partitio

# 2. Build CSG geometry
R = 1.0
geo = disk(R, name="csg")

# 3. Cut the background model
cutgeo = cut(bgmodel,geo)
```

Example 3: CutFEM 求解

```
# 生成计算域
Ω = Triangulation(cutgeo,PHYSICAL,"csg")
Ω_act = Triangulation(cutgeo,ACTIVE,"csg")
```

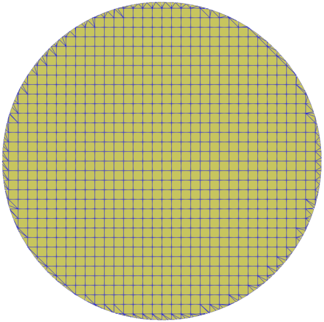


Figure 7: 物理域 Ω

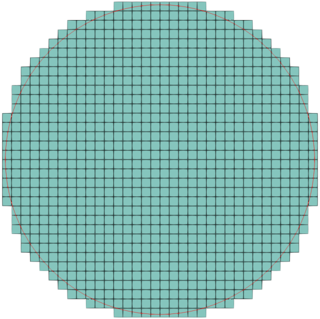


Figure 8: 求解域 Ω_{act}

Example 3: CutFEM 求解

```
Γd = EmbeddedBoundary(cutgeo,"csg")
Γg = GhostSkeleton(cutgeo,"csg")
```

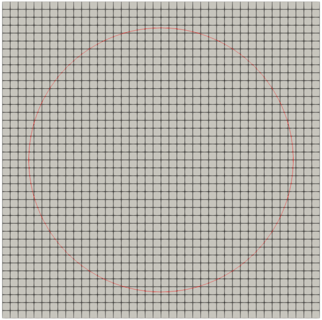


Figure 9: 背景网格 Ω_{bg}

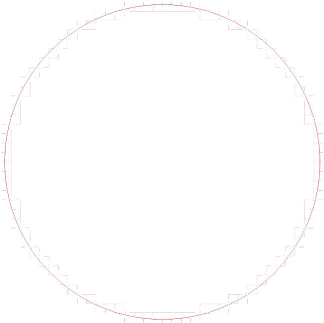


Figure 10: 几何

Example 3: CutFEM 求解

```
# 方程弱形式
a(u,v) = ∫( ∇(v)⋅∇(u) ) dΩ +
∫( (γd/h)*v*u  - v*(n_Γd⋅∇(u)) - (n_Γd⋅∇(v))*u ) dΓd
∫( (γg*h)*jump(n_Γg⋅∇(v))*jump(n_Γg⋅∇(u)) ) dΓg

l(v) = ∫( v*f ) dΩ +
∫( (γd/h)*v*g - (n_Γd⋅∇(v))*g ) dΓd
```

Example 4: 一个更复杂的例子

```
R = 0.5
geo1 = cylinder(R,v=VectorValue(1,0,0))
geo2 = cylinder(R,v=VectorValue(0,1,0))
geo3 = cylinder(R,v=VectorValue(0,0,1))
geo4 = union(union(geo1,geo2),geo3,name="source")

geo5 = sphere(1)
geo6 = cube(L=1.5)
geo7 = intersect(geo6,geo5)
geo8 = setdiff(geo7,geo4,name="csg")
```

Example 4: 一个更复杂的例子

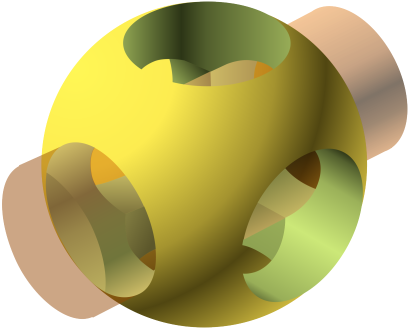


Figure 11: 几何定义

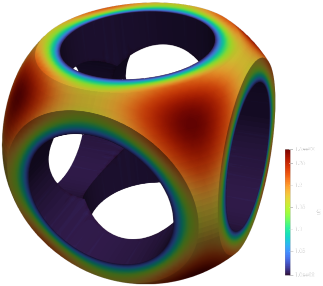


Figure 12: 分析结果

Example 4: 一个更复杂的例子

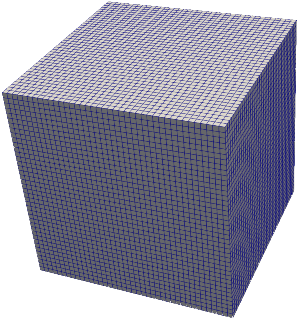


Figure 13: 背景网格

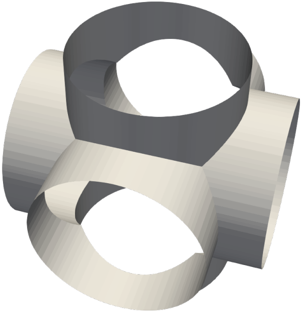


Figure 14: 边界条件

Part III: 线弹性结构有限元

线弹性有限元

Let Ω be a domain in $\mathbb{R}^d$, $d = 2$ or 3, with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$, and exterior unit normal $\mathbf{n}$. We consider the following problems:
Find the displacement $\mathbf{u} : \Omega \to \mathbb{R}^d$ such that

$$\begin{aligned} -\nabla \cdot \sigma(\mathbf{u}) &= \mathbf{f} \quad \text{in } \Omega \\ \sigma(\mathbf{u}) \cdot \mathbf{n} &= \mathbf{g}_N \quad \text{on } \Gamma_N \\ \mathbf{u} &= \mathbf{g}_D \quad \text{on } \Gamma_D \end{aligned}$$

where the strain and stress tensors are defined by

$$\begin{aligned} \varepsilon(\mathbf{u}) &\doteq \frac{1}{2}\left(\nabla \cdot \mathbf{u} + (\nabla \cdot \mathbf{u})^{\mathrm{T}}\right) \\ \sigma(\mathbf{u}) &\doteq 2\mu\varepsilon(\mathbf{u}) + \lambda \mathrm{tr}(\varepsilon(\mathbf{u})) \end{aligned}$$

with Lamé parameters λ and μ.

线弹性有限元

Let $\mathbf{V}_g = \left\{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u} = \mathbf{g} \text{ on } \Gamma_N\right\}$, and define the bilinear form

$$a(\mathbf{u}, \mathbf{v}) = 2\mu \left(\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v})\right)_\Omega + \lambda \left(\mathrm{tr}(\varepsilon(\mathbf{u})), \mathrm{tr}(\varepsilon(\mathbf{v}))\right)_\Omega$$

Find such that

$$a(\mathbf{u}, \mathbf{v}) = l(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_{\mathbf{o}}$$

where the linear form on right hand side is defined by

$$l(\mathbf{v}) = (\mathbf{f}, \mathbf{v})_\Omega + (\mathbf{g}_N, \mathbf{v})_{\Gamma_N}$$

线弹性有限元

Define the stabilized Nitsche form

$$\mathcal{A}_h(\mathbf{v},\mathbf{w}) = a_h(\mathbf{v},\mathbf{w}) - (\sigma(\mathbf{v})\cdot\mathbf{n},\mathbf{w})_{\Gamma_D} - (\mathbf{v},\sigma(\mathbf{w})\cdot\mathbf{n})_{\Gamma_D} + \beta h^{-1} b_h(\mathbf{v},\mathbf{w})$$

where $\beta > 0$ is a parameter and

$$b_h(\mathbf{v},\mathbf{w}) = 2\mu\,(\mathbf{v},\mathbf{w})_{\Gamma_D} + \lambda\,(\mathbf{v}\cdot\mathbf{n},\mathbf{w}\cdot\mathbf{n})_{\Gamma_D}$$

线弹性有限元

Find $\mathbf{u}_h \in \mathbf{V}_h$ such that

$$\mathcal{A}_h(\mathbf{u}_h, \mathbf{v}) = \mathcal{L}_h(\mathbf{v}), \quad \forall \mathbf{v}_h \in \mathbf{V}_h$$

where the right hand side is given by

$$\mathcal{L}_h(\mathbf{v}) = (\mathbf{f}, \mathbf{v})_\Omega + (\mathbf{g}_N, \mathbf{v})_{\Gamma_N} - (\mathbf{g}_D, \sigma(\mathbf{v}) \cdot \mathbf{n})_{\Gamma_D} + \beta h^{-1} b_h\,(\mathbf{g}_D, \mathbf{v})_{\Gamma_D}.$$

Part IV: 杂项

Part V: 项目规划