Phys. Rev. Lett. (Published: Dec 23, 2025)

Preprint: arXiv:2508.04310 [quant-ph] (Aug 6, 2025)

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December 30, 2025

Introduction: The Quest for Quantum Advantage

Context. Quantum Information Science seeks tasks where quantum systems fundamentally outperform classical ones.

This Work's Focus. A physically natural problem involving **permutations of particles**.

- No Oracles.
- No Computational Promises.
- **Sharp, Non-asymptotic separation.**

The Problem: Parity of a Hidden Permutation

Problem Definition. Given a set of n particles subject to a hidden permutation $\sigma \in S_n$, determine the parity:

- Is σ **Even**? (e.g., identity, 3-cycles)
- Is σ **Odd**? (e.g., single swap)

Goal: Identify parity with **Certainty** $(P_s = 1)$.

The Classical Limit

Case 1: Distinguishable ($d = n$)

Unique Parity ✓

Case 2: Indistinguishable ($d < n$)

Ambiguous Parity ?
(Even or Odd?)

Full Distinguishability (n labels)

- perfectly reconstruct the permutation
- determine its parity
- $P_s = 1$

Insufficient Labels ($< n$ labels)

- perfect identification becomes impossible
- indistinguishable opposite-parity counterparts
- limited to random guessing
- $P_s = 1/2$

Theorem: The Quadratic Quantum Advantage

Theorem (Main Result). Perfect parity identification ($P_s = 1$) is achievable if

$$d \geq d_{min} := \lceil \sqrt{n} \rceil$$

Below this threshold, assuming all permutations are equally likely, parity remains indistinguishable and random guessing is the best one can do ($P_s = 1/2$).

Below $d = \lceil \sqrt{n} \rceil$, even Quantum Mechanics cannot help ($P_s = 1/2$).

Example: 4 Qubits ($n = 4, d = 2$)

Physical Setup. Consider a system of **four spin-1/2 particles**. Alice acts by reshuffling their positions *without* disturbing the spin degrees of freedom.

Schur-Weyl Decomposition.

$$(\mathbb{C}^2)^{\otimes 4} \cong (\mathcal{K}_2 \otimes \mathcal{H}_2) \oplus (\mathcal{K}_1 \otimes \mathcal{H}_1) \oplus (\mathcal{K}_0 \otimes \mathcal{H}_0)$$

- $\mathcal{H}_j$: $SU(2)$-invariant subspaces (labeled by total spin j).
- $SU(2)$ acts *only* on $\mathcal{H}_j$ and trivially on $\mathcal{K}_j$.

Multiplicity Spaces ($\mathcal{K}_j$).

- $j = 2$: $\mathcal{K}_2 \cong \mathbb{C}$ (1 copy)
- $j = 1$: $\mathcal{K}_1 \cong \mathbb{C}^3$ (3 copies)
- $j = 0$: $\mathcal{K}_0 \cong \mathbb{C}^2$ (2 copies)

Symmetry and Representation: The S_4 Perspective

Permutational Symmetry (S_4)

- Reshuffling 4 qubits $\longleftrightarrow$ "Rotation" within subspaces K_j.
- These spaces carry **irreducible representations (irreps)** of S_4.
- *Note:* No proper subspace is invariant under $\sigma|\psi\rangle$.

Schur-Weyl Duality Structure

- Agent's operations are restricted to permutations.
- **Consequence:** Action is **trivial** on spin sectors H_j.
- **Interpretation:**
  - K_j: Irrep spaces (Dynamic part).
  - H_j: Multiplicity spaces (Static part).

Restriction to Even Permutations ($A_4 \subset S_4$)

Group Restriction: Restrict agent actions to *even permutations* (A_4).

- $\mathcal{K}_1$ and $\mathcal{K}_2$: Remain irreducible.
- $\mathcal{K}_0$: **Decomposes** into two orthogonal 1D invariant subspaces:

$$\mathcal{K}_0 \xrightarrow{A_4} \mathcal{K}_{0a} \oplus \mathcal{K}_{0b}$$

Action of Odd Permutations ($\sigma_{\text{odd}} \in S_4 \setminus A_4$)

Any odd permutation swaps the subspaces:

$$\sigma_{\text{odd}} : \mathcal{K}_{0a} \longleftrightarrow \mathcal{K}_{0b}$$

Note: This explains why $\mathcal{K}_{0a/b}$ are not invariant under the full group S_4.

Parity Identification Protocol

Prepare initial state $|\psi_e\rangle \in \mathcal{K}_{0a} \otimes \mathcal{H}_0$. The odd/even sectors become orthogonal:

$$\sigma_{\text{odd}}|\psi_e\rangle \perp \sigma_{\text{even}}|\psi_e\rangle \tag{1}$$

Result: Parity can be identified with certainty via projective measurement.

Explicit Construction: The M4 State

Construction via Projection

We construct $|\psi_e\rangle$ by projecting the basis state $|0011\rangle$ (with magnetic quantum number $m = 0$) onto the invariant subspace $\mathcal{K}_{0a} \otimes \mathcal{H}_0$:

$$|\psi_e\rangle = \mathcal{P}_{0a}|0011\rangle, \quad \text{where } \mathcal{P}_{0a} = \frac{1}{|A_4|}\sum_{\sigma \in A_4} \overline{\chi}_{0a}(\sigma)\sigma$$

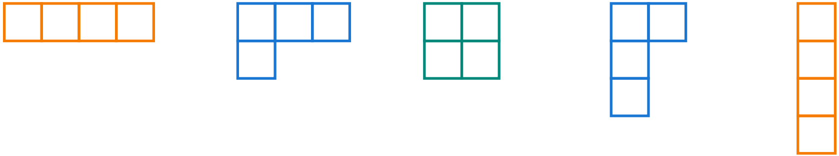
Computational Basis Expansion:

$$|\psi_e\rangle = (|0011\rangle + |1100\rangle) + \zeta_3\ (|0101\rangle + |1010\rangle) + \zeta_3^2\ (|0110\rangle + |1001\rangle)$$

- Also known as the **M4 (or D4) state**.
- A Fourier-type combination of **Dicke states**.

$\zeta_3 = e^{i2\pi/3}$ (Cubic root of unity).

Labeling Invariant Subspaces: Young Diagrams



Partitions & Young Diagrams (YDs). Subspaces are labeled by partitions $\lambda \vdash n$, visualized as YDs with n boxes.

$$\lambda = [\lambda_1, \lambda_2, \dots]$$

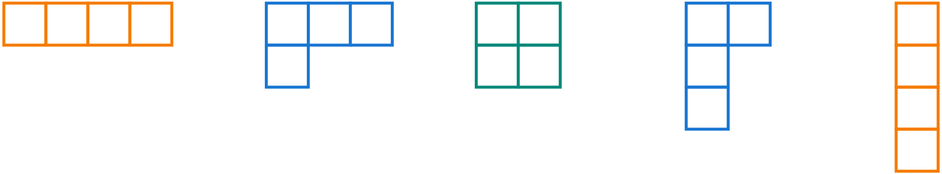
$$(\lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \ \sum \lambda_p = n)$$

- S_n **Irreps ($\mathcal{K}_\lambda$):** Correspond to *any* distinct $\lambda \vdash n$.
- $SU(d)$ **Irreps ($\mathcal{H}_\lambda$):** Restricted by dimension d:

$$\ell(\lambda) \leq d$$

(The YD cannot have more than d rows)

Labeling Subspaces: Young Diagrams (YDs)



Partitions of $n = 4$

$[4], \quad [3,1], \quad [2,2], \quad [2,1,1], \quad [1^4]$

Crucial Subspaces for $n = 4$

For $d \geq 4$, all 5 irreps appear.

- $\mathcal{K}_{[4]}$: **Fully Symmetric** (Trivial, $+1$).
- $\mathcal{K}_{[1^4]}$: **Fully Antisymmetric** (Sign, ± 1).

Mixing these enables perfect parity identification.

$$|\psi_e\rangle = |\psi_{\text{sym}}\rangle + |\psi_{\text{ant}}\rangle$$

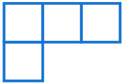
$$|\psi_{\text{sym}}\rangle \in \mathcal{K}_{[4]} \otimes \mathcal{H}_{[4]}$$
$$|\psi_{\text{ant}}\rangle \in \mathcal{K}_{[1^4]} \otimes \mathcal{H}_{[1^4]}$$

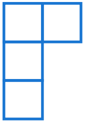
Case 2: Restricted Dimensions ($d = 3$)

The Constraint Problem($d = 3$)

Fully antisymmetric partition $[1^4]$ is **forbidden**.

$\ell([1^4]) = 4 > d = 3 \implies |\psi_{\text{ant}}\rangle$ does not exist.





Solution: Conjugate Pairs. We use conjugate partitions λ and λ^T (reflecting YD along diagonal). They satisfy a crucial representation property:

$$D_{\lambda^T}(\sigma) = \text{sign}(\sigma) D_\lambda(\sigma)$$

New Construction:. Combine partitions $\lambda = [3, 1]$ and $\lambda^T = [2, 1^2]$:

$$|\psi_e\rangle = |\psi_{[3,1]}\rangle + |\psi_{[2,1^2]}\rangle$$

Even though we lack the trivial/sign singlets, these higher-dimensional irreps mimic the sign-flipping behavior required for orthogonality.

Case 3: Self-Conjugate Irreps & The General Rule

Definition: Self-Conjugate Partitions

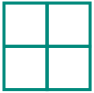
Its Young Diagram is symmetric along the main diagonal ($\lambda = \lambda^T$).

- **Example ($n = 4$):** $\lambda = [2, 2]$ (The square diagram).

Mechanism: Splitting under A_n

Unlike generic irreps, self-conjugate ones split when restricted to even permutations (A_n):

$$\mathcal{K}_\lambda \xrightarrow{A_n} \mathcal{K}_{\lambda a} \oplus \mathcal{K}_{\lambda b}$$



Action of Odd Permutations:

$$\mathcal{K}_{\lambda a} \xrightarrow{\sigma_{\text{odd}}} \mathcal{K}_{\lambda b}$$

Example: Five Qutrits ($d = 3$) & Non-Trivial Coefficients

Setup: Self-Conjugate Irrep $\mathcal{K}_{[3,1^2]}$

Consider $(\mathbb{C}^3)^{\otimes 5}$. The partition $\lambda = [3, 1^2]$ is self-conjugate with dimension $d_{[3,1^2]} = 6$.

- Parity can be identified with certainty using any state $|\psi_e\rangle$ in the invariant subspace $\mathcal{K}_{[3,1^2]a} \otimes \mathcal{H}_{[3,1^2]}$.

Mechanism: Projector Construction

The projector $\mathcal{P}_{[3,1^2]a}$ onto the subspace (analogous to $\mathcal{P}_{0a}$) is given by:

$$\mathcal{P}_{\lambda a} = \frac{d_\lambda/2}{|A_n|} \sum_{\sigma \in A_n} \bar{\chi}_{\lambda a}(\sigma)\sigma$$

$$\begin{aligned}
|\psi_e\rangle = 3\big(|00012\rangle - |00021\rangle\big) - |00102\rangle + |00120\rangle + |00201\rangle - |00210\rangle \\
- |01002\rangle + |01020\rangle - |10002\rangle + |10020\rangle + \ldots \\
+ \sqrt{5}\big(|01200\rangle - |02100\rangle - |10200\rangle + |12000\rangle + \ldots\big)
\end{aligned}$$

Proof (I): Hypothesis Testing & Schur-Weyl Decomposition

Binary Discrimination Problem. Distinguish between even (ρ_0) and odd (ρ_1) mixtures of permutations.

$$\rho_0 = \frac{1}{|A_n|} \sum_{\sigma \in A_n} \sigma \left|\psi_e\right\rangle \left\langle\psi_e\right| \sigma^{-1}$$

$$\rho_1 = (12)\rho_0(12)$$

- Perfect discrimination requires orthogonal supports: $\mathrm{Tr}(\rho_0\rho_1) = 0$.
- If $\rho_0 = \rho_1$, then $P_s = 1/2$ (random guessing).

Schur-Weyl Decomposition. The Hilbert space decomposes into sectors labeled by partitions $\lambda \vdash n$:

$$(\mathbb{C}^d)^{\otimes n} \cong \bigoplus_{\lambda \vdash n, \ell(\lambda) \leq d} \mathcal{K}_\lambda \otimes \mathcal{H}_\lambda$$

Partition λ Into 3 Sectors.

1. $\lambda = \lambda^T$ (Self-conjugate).
2. $\lambda \neq \lambda^T$ and $\ell(\lambda^T) \leq d$ (Conjugate pairs).
3. $\ell(\lambda^T) > d$ (Restricted sector).

Proof (II): Action of A_n and Block Structure

Lemma (Block-diagonal Form). Under the restriction to the Alternating Group A_n, the irreps split differently in each sector. By Schur's lemma, ρ_0 takes the block-diagonal form:

$$\rho_0 = \rho_0^{(i)} + \rho_0^{(ii)} + \rho_0^{(iii)}$$

1 **$[\lambda = \lambda^T]$:** Irreps split $\mathcal{K}_\lambda \downarrow_{A_n} = \mathcal{K}_{\lambda a} \oplus \mathcal{K}_{\lambda b}$.

$$\rho_0^{(\mathrm{i})} = \sum_{\lambda \in R_{\mathrm{i}}} \left(\frac{\mathbb{1}_{\lambda a}}{d_\lambda/2} \otimes \Phi^{\lambda a} + \frac{\mathbb{1}_{\lambda b}}{d_\lambda/2} \otimes \Phi^{\lambda b} \right)$$

If either $\Phi^{\lambda a} = 0$ or $\Phi^{\lambda b} = 0$, then $\rho_0^{(1)}$ and $\rho_1^{(1)}$ have orthogonal supports;

Proof (II): Action of A_n and Block Structure

2 $[\lambda \neq \lambda^T]$: Irreps appear in conjugate pairs, $(\mathcal{K}_\lambda \oplus \mathcal{K}_{\lambda^T}) \downarrow_{A_n} = \mathcal{K}_\lambda \otimes \mathbb{C}^2$.

$$\rho_0^{(\text{ii})} = \sum_{\lambda \in R_{\text{ii}}} \frac{\mathbb{1}_\lambda}{d_\lambda} \otimes \sum_{k=1}^{d_\lambda} |\phi_k^\lambda\rangle \langle \phi_k^\lambda| , \quad |\phi_k^\lambda\rangle \in \mathbb{C}^2 \otimes \mathcal{H}_\lambda$$

$|\phi_k^\lambda\rangle = \sum_{p=\pm} |s_p\rangle |\phi_{k,p}^\lambda\rangle \in \mathbb{C}^2 \otimes \mathcal{H}_\lambda.$

$|s_\pm\rangle \xrightarrow{\sigma} \pm |s_\pm\rangle$ for $\sigma \in S_n \setminus A_n$

$$\text{Tr}\left(\rho_0^{(\text{ii})} \rho_1^{(\text{ii})}\right) = \sum_{\lambda \in R_{\text{ii}}} \sum_{k,l=1}^{d_\lambda} \frac{\left| \langle \phi_{k,+}^\lambda | \phi_{l,+}^\lambda \rangle - \langle \phi_{k,-}^\lambda | \phi_{l,-}^\lambda \rangle \right|^2}{d_\lambda}$$

$\implies$ Supports are orthogonal if $\langle \phi_{k,+}^\lambda | \phi_{l,+}^\lambda \rangle = \langle \phi_{k,-}^\lambda | \phi_{l,-}^\lambda \rangle$ for all λ, k, l.

Proof (II): Action of A_n and Block Structure

3 Restriction has no effect ($\mathcal{K}_\lambda \downarrow_{A_n} = \mathcal{K}_\lambda$).

$$\rho_0^{(\text{iii})} = \sum_{\lambda \in R_{\text{iii}}} \frac{\mathbb{1}_\lambda}{d_\lambda} \otimes \Phi^\lambda$$

$\rho_0^{(\text{iii})}$ is invariant under all of S_n, so $\rho_0^{(\text{iii})} = \rho_1^{(\text{iii})}$.

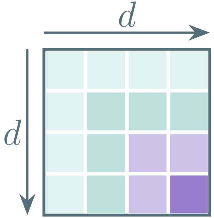
Proof (III): Distinguishability Conditions

$d \geq d_{\min} := \lceil \sqrt{n} \rceil$. 1 and/or 2 hold, enabling perfect distinguishability.

$d \leq d_{\min} := \lceil \sqrt{n} \rceil$. 1 and 2 fail,only 3 remains

$\rho_0 = \rho_1$ and $P_s = 1/2$.

This geometrically corresponds to the impossibility of fitting certain Young Diagrams within a $d \times d$ box when d is too small.



$d^2 \geq n$

Entanglement Requirement for Parity Identification

Question. How entangled must a parity-detecting state be?

Metric:geometric measure of entanglement(GME)

$$E(|\psi\rangle) = 1 - \max_{|\phi\rangle \in \mathsf{SEP}} |\langle\phi|\psi\rangle|^2$$

Minimum Required Entanglement:

$$E_{\lambda a} = 1 - \max_{|\phi\rangle \in \mathsf{SEP}} \langle\phi| \mathcal{P}_{\lambda a} |\phi\rangle$$

This lower bounds the entanglement of any parity-detecting state supported on this subspace.

Table: Minimum required ($E_{\lambda a}$) vs. known maximal ($E_{\max}$) entanglement.

n	$E_{\lambda a}$	$E_{\max}$
3	$5/9$	$5/9^{\dagger}$
4	$7/9$	$7/9^{\dagger}$
5	$17/20$	≈ 0.96

† indicates proven maxima.

Observation:
maximal for $n = 3, 4$, and close to the maximum for $n = 5$.

Summary and Outlook

A Sharp Quantum Advantage
Achievable with $d = \lceil \sqrt{n} \rceil$.

Relies purely on permutation symmetry without invoking specific dynamics.

Requires **no ancillas** (increasing multiplicities does not help).

Demands entanglement close to the theoretical maximum.

Practical Implementations:

- State preparation via experimentally accessible Hamiltonians.
- Designing specific quantum circuits and measurements.

Theoretical Extensions:

- Assessing robustness of the $\sqrt{n}$ advantage under noise.
- Exploring other symmetry groups.