A Non-Asymptotic Separation Based on Particle Statistics

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Introduction: The Quest for Quantum Advantage

Context. Quantum Information Science seeks tasks where quantum systems fundamentally outperform classical ones.

Common Types of Advantage.

- **Computational:** Shor's factoring, Grover's search.
- **Communication:** Superdense coding, complexity reduction.
- **Query Complexity:** Simon's problem, Bernstein-Vazirani.

This Work's Focus. A physically natural problem involving **permutations of particles**.

- No Oracles.
- No Computational Promises.
- **Sharp, Non-asymptotic separation.**

The Problem: Parity of a Hidden Permutation

[Diagram Placeholder: Alice permutes n particles inside a black box. Bob must guess Parity.

The Setup

Bob must determine:

- Is σ **Even**? (e.g., identity, 3-cycles)
- Is σ **Odd**? (e.g., single swap)

Goal: Identify parity with **Certainty** ($P_s = 1$).

The Classical Limit

Classical Hard Barrier. To identify the permutation (and thus its parity) perfectly, distinguishable labels are required.

- If particles have n distinct labels (colors), σ is readable.
- If we have fewer than n labels ($d < n$):

$$\mathrm{P}_{success} = \frac{1}{2} \quad \text{(Random Guessing)}$$

Reason: Every permutation has an opposite-parity counterpart producing the same label arrangement.

[Image: Classical Failure]
Labels: Red, Blue ($d = 2$)
Particles: 3 ($n = 3$)
$\sigma(R, B, B) \to (B, R, B)$
Ambiguous Parity!

Theorem I: The Quadratic Quantum Advantage

Theorem (Main Result). Perfect parity identification ($P_s = 1$) is achievable using quantum resources if the local dimension d (number of orthogonal states per particle) satisfies:

$$d \geq d_{min} := \lceil \sqrt{n} \rceil$$

Comparison

Quantum Requirement

$$d_{qm} \approx \sqrt{n}$$

Quadratic Reduction

Below $d = \lceil \sqrt{n} \rceil$, even Quantum Mechanics cannot help ($P_s = 1/2$).

The Mechanism: Symmetry Representation Theory

Schur-Weyl Duality. The Hilbert space decomposes into irreducible representations (irreps) of the Symmetric Group S_n and unitary group $SU(d)$:

$$(\mathbb{C}^d)^{\otimes n} \cong \bigoplus_{\lambda} \mathcal{K}_\lambda \otimes \mathcal{H}_\lambda$$

Key Insight:

- We restrict actions to the **Alternating Group** A_n (Even permutations).
- Certain subspaces $\mathcal{K}_\lambda$ split or behave differently under even vs. odd permutations.
- Specifically, we look for **Self-Conjugate** Young Diagrams or **Conjugate Pairs**.

[Image: Young Diagrams]
Partition $\lambda \vdash n$
vs λ^T (Transpose)

Explicit Example: 4 Qubits ($n = 4, d = 2$)

Here $n = 4$, so $\lceil \sqrt{4} \rceil = 2$. We can distinguish parity with Qubits!

The Hilbert space decomposes into spin sectors $j = 2, 1, 0$:

$$(\mathbb{C}^2)^{\otimes 4} \cong (\mathcal{K}_2 \otimes \mathcal{H}_2) \oplus (\mathcal{K}_1 \otimes \mathcal{H}_1) \oplus (\mathcal{K}_0 \otimes \mathcal{H}_0)$$

- The $j = 0$ sector ($\mathcal{K}_0$) corresponds to the partition $[2, 2]$ (Self-conjugate).
- Under A_4 (even perms), $\mathcal{K}_0$ splits into orthogonal subspaces $\mathcal{K}_{0a}$ and $\mathcal{K}_{0b}$.
- **Odd permutations swap these subspaces.**

$$\sigma_{odd} \left|\psi_e\right\rangle \perp \sigma_{even} \left|\psi_e\right\rangle$$

The Parity-Detecting State ($n = 4$)

Using the projector onto the invariant subspace, we derive the explicit state (Fourier-type combination of Dicke states):

$$|\psi_e\rangle = |0011\rangle + |1100\rangle + \zeta_3(|0101\rangle + |1010\rangle) + \zeta_3^2(|0110\rangle + |1001\rangle)$$

where $\zeta_3 = e^{2\pi i/3}$.

Protocol

If Outcome $= 1 \to$ **Even**.
If Outcome $= 0 \to$ **Odd**.

Higher Dimensions: 5 Qutrits

For $n = 5$, we need $d \geq \lceil \sqrt{5} \rceil = 3$ (Qutrits).

Irrep Construction:

- We use the partition $[3, 1, 1]$ (Self-conjugate).
- Dimension of irrep $= 6$.
- State is a complex superposition of permutations of $|00012\rangle$.

[Image: State Representation]
Basis: $|00012\rangle$
Coefficients involve
60 permutations (A_5)

$$|\psi_e\rangle \propto \sum_{\sigma \in A_5} \overline{\chi}(\sigma)\sigma\, |00012\rangle$$

This requires no ancillas, just the 5 particles.

The Cost: Entanglement

Does this require high entanglement? Yes.

Geometric Measure of Entanglement (GME).

$$E(|\psi\rangle) = 1 - \max_{|\phi\rangle \in SEP} |\langle\phi\rangle\, \psi|^2$$

Particles (n)	Local Dim (d)	Required GME	Max Possible GME
3	2	5/9	5/9 (Max)
4	2	7/9	7/9 (Max)
5	3	17/20	≈ 0.96 (Near Max)

Conclusion: Parity identification requires entanglement that is maximal or close to maximal.

Summary Outlook

Key Achievements.

- **Quantum Advantage:** $d \sim \sqrt{n}$ vs Classical $d \sim n$.
- **Certainty:** Deterministic identification, not probabilistic.
- **Constructive:** Explicit states provided for small n.

Future Directions

- Experimental realization (State Prep).
- Robustness against noise.
- Extension to other symmetry groups.

"A simple, rigorous example of genuine quantum advantage requiring no oracles."