Mediation: Controlled Direct Effects

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Soc 212C2

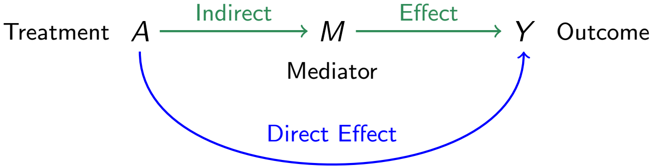
April 28, 2025

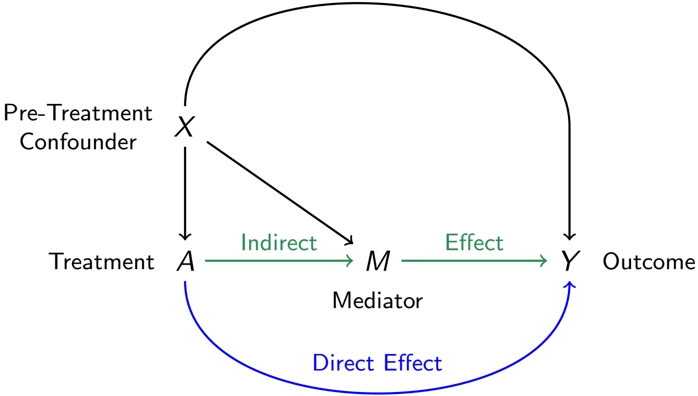
Learning goals for today

At the end of class, you will be able to:

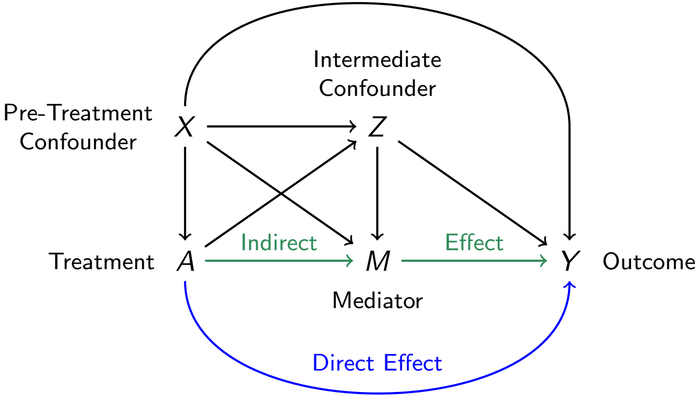
1. Define controlled direct effects
2. Connect them to longitudinal treatments
3. Built intuition for a new estimator: sequential g-estimation

$$\text{Treatment} \quad A \xrightarrow{\text{Total Effect}} Y \quad \text{Outcome}$$


Treatment
A
Indirect
M
Effect
Y
Outcome
Mediator
Direct Effect



Pre-Treatment
Confounder
X
Treatment
A
Indirect
M
Effect
Y
Outcome
Mediator
Direct Effect


Intermediate
Confounder
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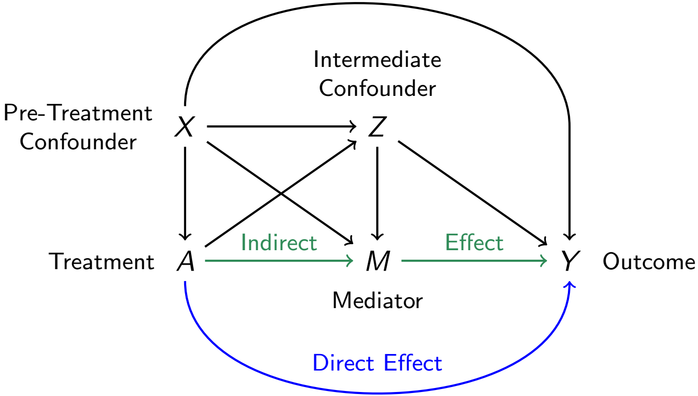


Before formally defining direct effects, we need a new tool

Single World Intervention Graphs (SWIGs)

Richardson & Robins 2013

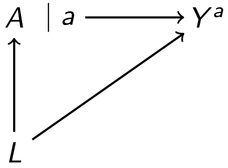
Single World Intervention Graphs (SWIGs)

Richardson & Robins 2013

$$\begin{tikzcd}
A \arrow[r] & Y \\
L \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Single World Intervention Graphs (SWIGs)

Richardson & Robins 2013



Single World Intervention Graphs (SWIGs)

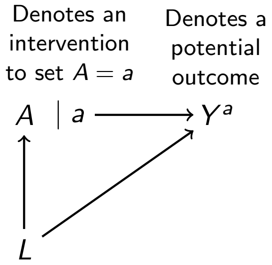
Richardson & Robins 2013

Denotes an
intervention
to set $A = a$

$$\begin{tikzcd} A \mid a \arrow[r] & Y^a \\ L \arrow[u] \arrow[ur] & \end{tikzcd}$$

Single World Intervention Graphs (SWIGs)

Richardson & Robins 2013



Single World Intervention Graphs (SWIGs)

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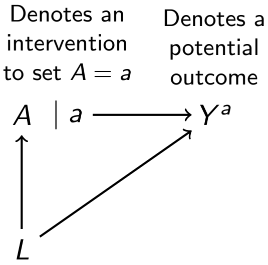

SWIGs help in at least two settings:

1. When causal assumptions differ for each potential outcome
2. When we want to focus on a particular intervention

SWIGs help (1): When causal assumptions differ for each potential outcome

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Suppose an unobserved U affects the treatment A

SWIGs help (1): When causal assumptions differ for each potential outcome

Suppose an unobserved U affects the treatment A

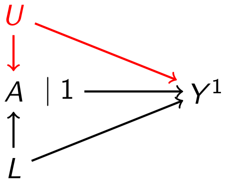
Suppose U affects Y^1

$$
\begin{tikzcd}
U \arrow[d] \arrow[dr] & \\
A \mid 1 \arrow[r] & Y^1 \\
L \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

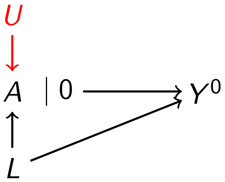
SWIGs help (1): When causal assumptions differ for each potential outcome

Suppose an unobserved U affects the treatment A

Suppose U affects Y^1



But U does not affect Y^0



SWIGs help (1): When causal assumptions differ for each potential outcome

Suppose an unobserved U affects the treatment A

Suppose U affects Y^1

$$\begin{tikzcd}
U \arrow[d] \arrow[drr] & & \\
A \mid 1 \arrow[rr] & & Y^1 \\
L \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

But U does not affect Y^0

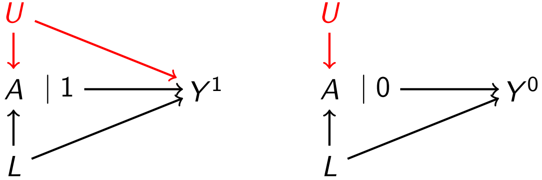
$$\begin{tikzcd}
U \arrow[d] & & \\
A \mid 0 \arrow[rr] & & Y^0 \\
L \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

In this case, $\mathrm{E}(Y^1)$ is not identified but $\mathrm{E}(Y^0)$ is identified.

SWIGs help (1): When causal assumptions differ for each potential outcome

Suppose an unobserved U affects the treatment A

Suppose U affects Y^1 But U does not affect Y^0



In this case, $\mathrm{E}(Y^1)$ is not identified but $\mathrm{E}(Y^0)$ is identified.

- The ATC $\mathrm{E}(Y^1 - Y \mid A = 0)$ is not identified

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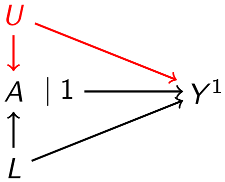
Suppose U affects Y^1



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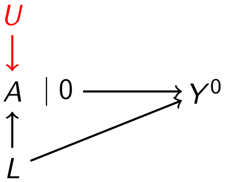
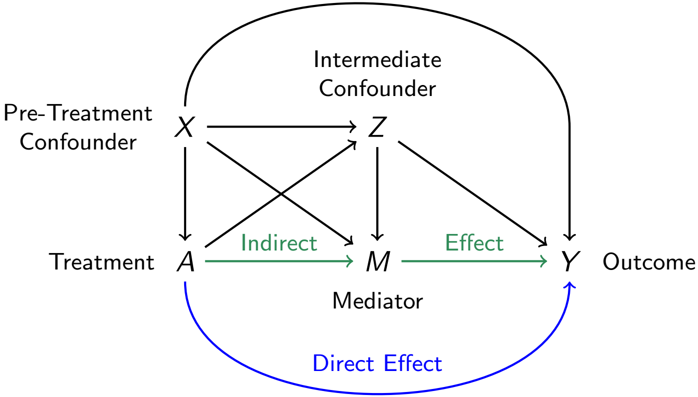


In this case, $\mathrm{E}(Y^1)$ is not identified but $\mathrm{E}(Y^0)$ is identified.

- The ATC $\mathrm{E}(Y^1 - Y \mid A = 0)$ is not identified
- The ATT $\mathrm{E}(Y - Y^0 \mid A = 1)$ is identified

SWIGs help (2): When we want to focus on a particular intervention



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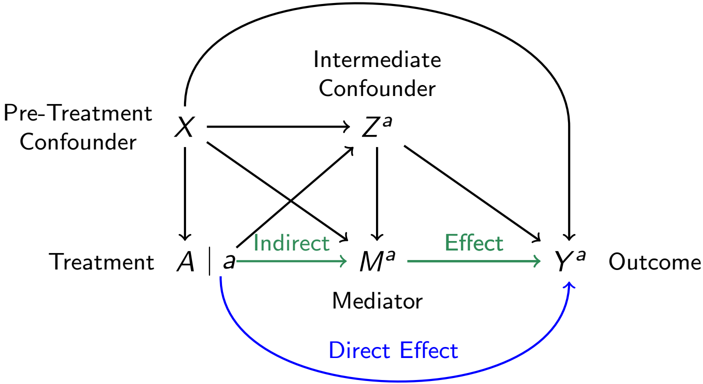

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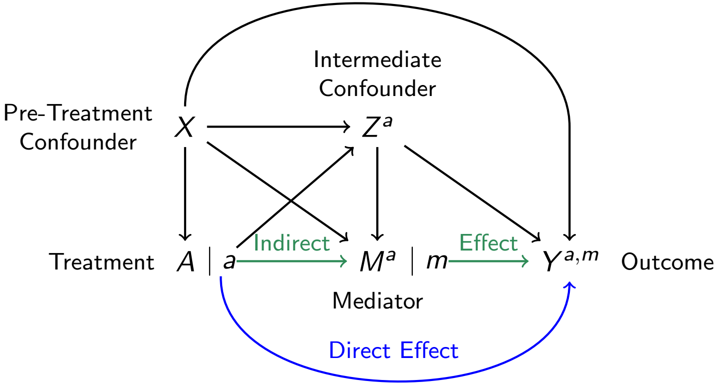
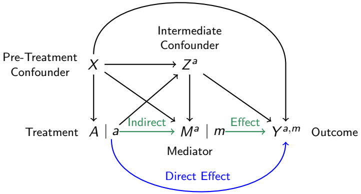


Controlled direct effect (CDE)



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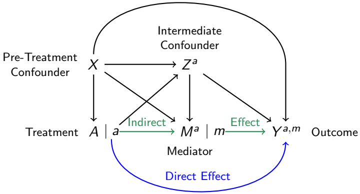

Definition: Controlled Direct Effect

$$\tau(m) = \mathrm{E}\left(Y^{1,m} - Y^{0,m}\right)$$

The effect of an intervention to set treatment $A = 1$ vs $A = 0$ while also intervening to set the mediator to $M = m$

CDE in an experiment

You are an elementary school principal

CDE in an experiment

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Y Kids Read Books

CDE in an experiment

You are an elementary school principal

Librarian Visits Class A

Y Kids Read Books

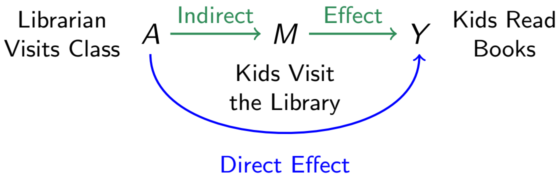
CDE in an experiment

You are an elementary school principal

$$\underset{\text{Visits Class}}{\text{Librarian}}\ A \xrightarrow{\text{Indirect}} \underset{\substack{\text{Kids Visit}\\ \text{the Library}}}{M} \xrightarrow{\text{Effect}} Y\ \underset{\text{Books}}{\text{Kids Read}}$$

CDE in an experiment

You are an elementary school principal



CDE in an experiment

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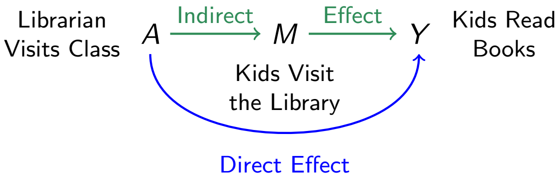

Experiment for the
Total Effect

CDE in an experiment

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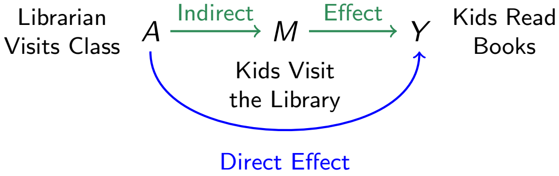

Experiment for the Total Effect

1) Librarian visits random classes

CDE in an experiment

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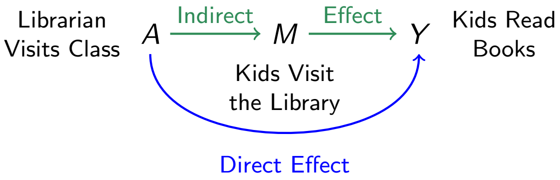


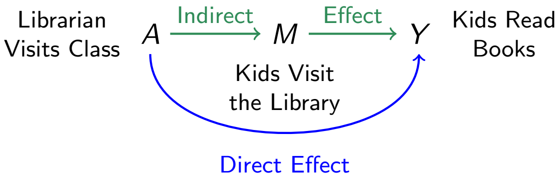
Experiment for the
Total Effect

1) Librarian visits random classes

2) Measure the outcome

CDE in an experiment

You are an elementary school principal

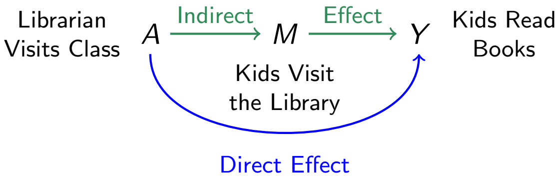


Experiment for the
Direct Effect
$\tau(0) = E\left(Y^{10} - Y^{00}\right)$

Experiment for the
Direct Effect
$\tau(1) = E\left(Y^{11} - Y^{01}\right)$

CDE in an experiment

You are an elementary school principal



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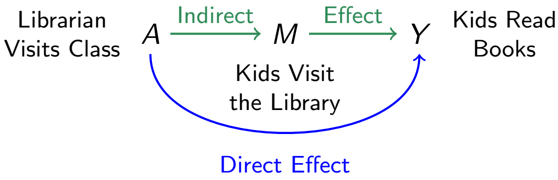

Experiment for the Direct Effect
$\tau(0) = E\left(Y^{10} - Y^{00}\right)$

1) Librarian visits random classes

2) You close the school library

Experiment for the Direct Effect
$\tau(1) = E\left(Y^{11} - Y^{01}\right)$

1) Librarian visits random classes

CDE in an experiment

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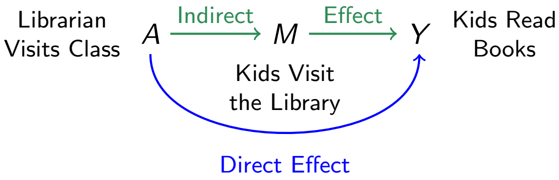

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1) Librarian visits random classes

2) You close the school library

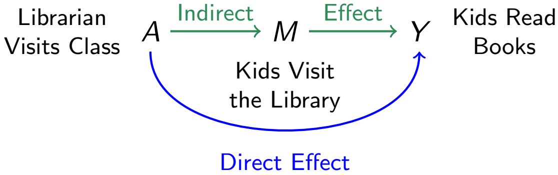
Experiment for the
Direct Effect
$\tau(1) = E\left(Y^{11} - Y^{01}\right)$

1) Librarian visits random classes

2) You make every kid visit the library

CDE in an experiment

You are an elementary school principal



Experiment for the
Direct Effect
$\tau(0) = E\left(Y^{10} - Y^{00}\right)$

1) Librarian visits random classes
2) You close the school library
3) Measure the outcome

Experiment for the
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$\tau(1) = E\left(Y^{11} - Y^{01}\right)$

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3) Measure the outcome

CDE in an experiment

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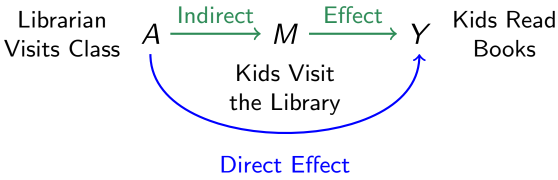

Note

These two estimands are **not** the same.

There are **two** direct effects.

Experiment for the Direct Effect
$\tau(0) = E\left(Y^{10} - Y^{00}\right)$

1) Librarian visits random classes
2) You close the school library
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Experiment for the Direct Effect
$\tau(1) = E\left(Y^{11} - Y^{01}\right)$

1) Librarian visits random classes
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CDE warning: Mediators that are not manipulated

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It is hard to study mediators that occur inside a person's head

CDE warning: Mediators that are not manipulated

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- Psychological stimulus $\rightarrow$ Stress $\rightarrow$ Test performance

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- Exposure to racial outgroup $\rightarrow$ Racial resentment $\rightarrow$ Voting

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- Psychological stimulus $\rightarrow$ Stress $\rightarrow$ Test performance
- Exposure to racial outgroup $\rightarrow$ Racial resentment $\rightarrow$ Voting
- Father incarcerated $\rightarrow$ Mother depressed $\rightarrow$ Child behavior

CDE warning: Mediators that are not manipulated

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- Psychological stimulus $\rightarrow$ Stress $\rightarrow$ Test performance
- Exposure to racial outgroup $\rightarrow$ Racial resentment $\rightarrow$ Voting
- Father incarcerated $\rightarrow$ Mother depressed $\rightarrow$ Child behavior

No experiment could manipulate these mediators

CDE warning: Mediators that are not manipulated

It is hard to study mediators that occur inside a person's head

- Psychological stimulus $\rightarrow$ Stress $\rightarrow$ Test performance
- Exposure to racial outgroup $\rightarrow$ Racial resentment $\rightarrow$ Voting
- Father incarcerated $\rightarrow$ Mother depressed $\rightarrow$ Child behavior

No experiment could manipulate these mediators

Mediators outside a person's head are easier to study

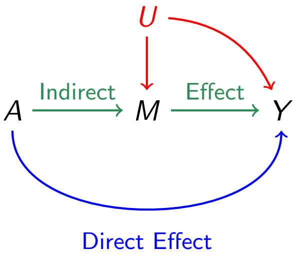
- Example: Require every kid to visit the school library

CDE warning: Mediators that are not manipulated

$$\begin{tikzcd} A \arrow[r, "\text{Indirect}"] \arrow[rr, bend right, "\text{Direct Effect}"'] & M \arrow[r, "\text{Effect}"] & Y \end{tikzcd}$$

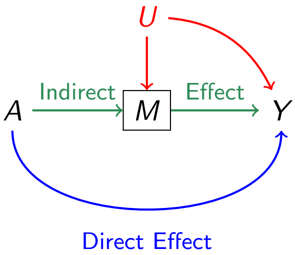
An experiment might randomize the treatment A

CDE warning: Mediators that are not manipulated



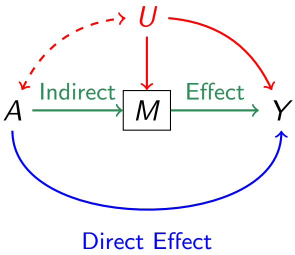
But the mediator M is not randomized

CDE warning: Mediators that are not manipulated



By adjusting for the collider M,
researchers open a backdoor path $A \rightarrow \boxed{M} \leftarrow U \rightarrow Y$

CDE warning: Mediators that are not manipulated



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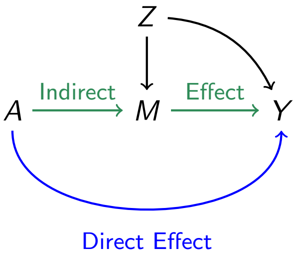


We can solve this problem by measuring the confounders Z

CDE identification and estimation in observational studies

A visual summary: Nonparametric sequential g-estimation

Estimating $\tau(0) = \mathrm{E}(Y^{10} - Y^{00})$

Text here will tell the story for those reading these slides online.

A visual summary: Nonparametric sequential g-estimation

Estimating $\tau(0) = \mathrm{E}(Y^{10} - Y^{00})$

Treatment variable A.
You can think of this as randomized, or you can take this entire story to take place within subgroups of $\vec{X}$ sufficient to yield exchangeability.

Librarian does not visit class

$A = 0$

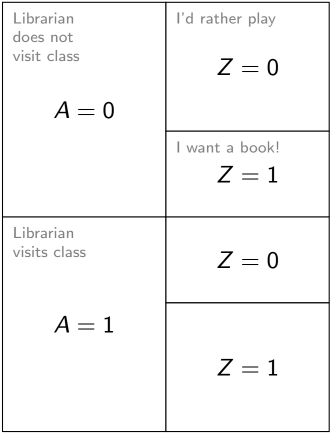
Librarian visits class

$A = 1$

A visual summary: Nonparametric sequential g-estimation

Estimating $\tau(0) = \mathrm{E}(Y^{10} - Y^{00})$

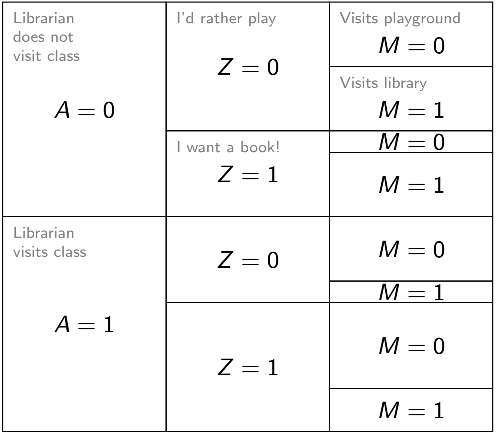
A affects an intermediate confounder Z



A visual summary: Nonparametric sequential g-estimation

Estimating $\tau(0) = E(Y^{10} - Y^{00})$

Z affects the mediator M



A visual summary: Nonparametric sequential g-estimation

Estimating $\tau(0) = \mathrm{E}(Y^{10} - Y^{00})$

We observe outcome means $\bar{Y}$ in each subgroup.
We can now impute the outcome Y^{A0} under $M = 0$
in each stratum of $\{A, Z\}$.

Librarian does not visit class $A = 0$	I'd rather play $Z = 0$	Visits playground $M = 0$	Reads books $\bar{Y}$
		Visits library $M = 1$	$\bar{Y}$
	I want a book! $Z = 1$	$M = 0$	$\bar{Y}$
		$M = 1$	$\bar{Y}$
Librarian visits class $A = 1$	$Z = 0$	$M = 0$	$\bar{Y}$
		$M = 1$	$\bar{Y}$
	$Z = 1$	$M = 0$	$\bar{Y}$
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A visual summary: Nonparametric sequential g-estimation

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	I want a book! $Z = 1$	$M = 0$	$\bar{Y}$
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$\mathrm{E}(Y^{00} \mid A = 0, Z = 0)$</td></tr>
<tr><td>I want a book!
$Z = 1$</td><td colspan="2">$\mathrm{E}(Y^{00} \mid A = 0, Z = 1)$</td></tr>
<tr><td rowspan="3">Librarian visits class
$A = 1$</td><td>$Z = 0$</td><td colspan="2">$\mathrm{E}(Y^{10} \mid A = 1, Z = 0)$</td></tr>
<tr><td rowspan="2">$Z = 1$</td><td>$M = 0$</td><td>$\bar{Y}$</td></tr>
<tr><td>$M = 1$</td><td>$\bar{Y}$</td></tr>
</table>

A visual summary: Nonparametric sequential g-estimation

Estimating $\tau(0) = \text{E}(Y^{10} - Y^{00})$

We observe outcome means $\bar{Y}$ in each subgroup.
We can now impute the outcome Y^{A0} under $M = 0$
in each stratum of $\{A, Z\}$.

Librarian does not visit class $A = 0$	I'd rather play $Z = 0$	Proportion reading books if we prevent anyone from visiting the library ($M = 0$) $\text{E}(Y^{00} \mid A = 0, Z = 0)$
	I want a book! $Z = 1$	$\text{E}(Y^{00} \mid A = 0, Z = 1)$
Librarian visits class $A = 1$	$Z = 0$	$\text{E}(Y^{10} \mid A = 1, Z = 0)$
	$Z = 1$	$\text{E}(Y^{10} \mid A = 1, Z = 1)$

A visual summary: Nonparametric sequential g-estimation

Estimating $\tau(0) = \mathrm{E}(Y^{10} - Y^{00})$

To focus on the effect of A, we now ignore Z.

Librarian does not visit class

$A = 0$

Librarian visits class

$A = 1$

Proportion reading books if we prevent anyone from visiting the library ($M = 0$)

$\mathrm{E}(Y^{00} \mid A = 0, Z = 0)$

$\mathrm{E}(Y^{00} \mid A = 0, Z = 1)$

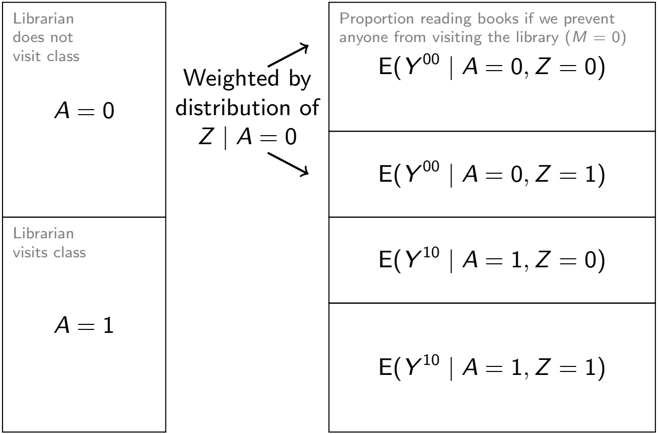
$\mathrm{E}(Y^{10} \mid A = 1, Z = 0)$

$\mathrm{E}(Y^{10} \mid A = 1, Z = 1)$

A visual summary: Nonparametric sequential g-estimation

Estimating $\tau(0) = \mathrm{E}(Y^{10} - Y^{00})$

To focus on the effect of A, we now ignore Z.
We have a weighted average over $Z \mid A = a$ for each a.
Because the effect of A is identified, $\underbrace{(Z \mid A = a)}_{\text{Observational}} \quad \sim \quad \underbrace{(Z^a)}_{\text{Interventional}}$



A visual summary: Nonparametric sequential g-estimation

Estimating $\tau(0) = \mathrm{E}(Y^{10} - Y^{00})$

To focus on the effect of A, we now ignore Z.
We have a weighted average over $Z \mid A = a$ for each a.
Because the effect of A is identified, $\underbrace{(Z \mid A = a)}_{\text{Observational}} \sim \underbrace{(Z^a)}_{\text{Interventional}}$

Librarian does not visit class

$A = 0$

Weighted by distribution of $Z \mid A = 0$

Proportion reading books if we prevent anyone from visiting the library ($M = 0$)

$\mathrm{E}(Y^{00} \mid A = 0, Z = 0)$

$\mathrm{E}(Y^{00} \mid A = 0, Z = 1)$

Librarian visits class

$A = 1$

Weighted by distribution of $Z \mid A = 1$

$\mathrm{E}(Y^{10} \mid A = 1, Z = 0)$

$\mathrm{E}(Y^{10} \mid A = 1, Z = 1)$

A visual summary: Nonparametric sequential g-estimation

Estimating $\tau(0) = \mathrm{E}(Y^{10} - Y^{00})$

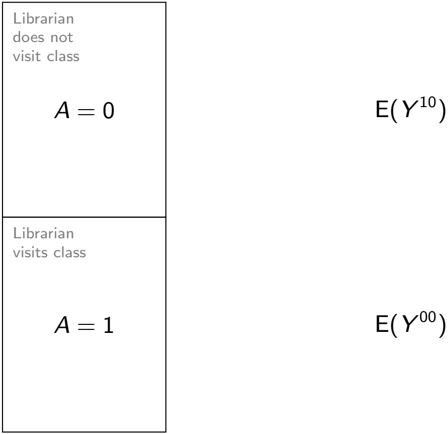
To focus on the effect of A, we now ignore Z.
We have a weighted average over $Z \mid A = a$ for each a.
Because the effect of A is identified, $\underbrace{(Z \mid A = a)}_{\text{Observational}} \quad \sim \quad \underbrace{(Z^a)}_{\text{Interventional}}$

Librarian does not visit class $A = 0$	$\mathrm{E}(Y^{10})$
Librarian visits class $A = 1$	$\mathrm{E}(Y^{00})$

A visual summary: Nonparametric sequential g-estimation

Estimating $\tau(0) = \mathrm{E}(Y^{10} - Y^{00})$

The difference is the CDE $\tau(0)$!



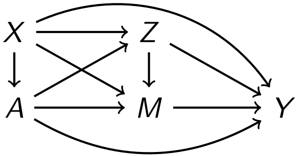
Sometimes we want to estimate with a model

Parametric sequential g-estimation (see Acharya, Blackwell, & Sen 2016)

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr] \arrow[drr, bend left] & Z \arrow[d] \arrow[dr] & \\
A \arrow[r] \arrow[ur] \arrow[rr, bend right] & M \arrow[r] & Y
\end{tikzcd}
$$

High-level overview:

Parametric sequential g-estimation (see Acharya, Blackwell, & Sen 2016)



High-level overview:

1. Estimate the effect of the mediator

Parametric sequential g-estimation (see Acharya, Blackwell, & Sen 2016)

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr] \arrow[drr, bend left] & Z \arrow[d] \arrow[dr] & \\
A \arrow[r] \arrow[ur] \arrow[rr, bend right] & M \arrow[r] & Y
\end{tikzcd}
$$

High-level overview:

1. Estimate the effect of the mediator
    - Model Y given X, A, Z, M

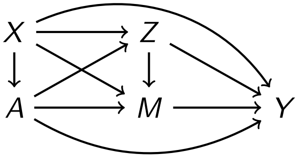
Parametric sequential g-estimation (see Acharya, Blackwell, & Sen 2016)

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr] \arrow[drr, bend left] & Z \arrow[d] \arrow[dr] & \\
A \arrow[ur] \arrow[r] \arrow[rr, bend right] & M \arrow[r] & Y
\end{tikzcd}$$

High-level overview:

1. Estimate the effect of the mediator
   - Model Y given X, A, Z, M
2. Construct $\tilde{Y}$ with the effect of the mediator removed

Parametric sequential g-estimation (see Acharya, Blackwell, & Sen 2016)



High-level overview:

1. Estimate the effect of the mediator
   - Model Y given X, A, Z, M
2. Construct $\tilde{Y}$ with the effect of the mediator removed
   - $\tilde{Y} = Y - \left[\mathrm{E}(Y^M \mid X, A, Z) - \mathrm{E}(Y^0 \mid X, A, Z)\right]$

Parametric sequential g-estimation (see Acharya, Blackwell, & Sen 2016)

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr] \arrow[drr, bend left] & Z \arrow[d] \arrow[dr] & \\
A \arrow[ur] \arrow[r] \arrow[rr, bend right] & M \arrow[r] & Y
\end{tikzcd}
$$

High-level overview:

1. Estimate the effect of the mediator
   - Model Y given X, A, Z, M
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   - $\tilde{Y} = Y - \left[\mathrm{E}(Y^M \mid X, A, Z) - \mathrm{E}(Y^0 \mid X, A, Z)\right]$
3. Estimate treatment effect on the de-mediated outcome

Parametric sequential g-estimation (see Acharya, Blackwell, & Sen 2016)

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr] \arrow[drr, bend left] & Z \arrow[d] \arrow[dr] & \\
A \arrow[r] \arrow[ur] \arrow[rr, bend right] & M \arrow[r] & Y
\end{tikzcd}$$

High-level overview:

1. Estimate the effect of the mediator
   - Model Y given X, A, Z, M
2. Construct $\tilde{Y}$ with the effect of the mediator removed
   - $\tilde{Y} = Y - \left[\mathrm{E}(Y^M \mid X, A, Z) - \mathrm{E}(Y^0 \mid X, A, Z)\right]$
3. Estimate treatment effect on the de-mediated outcome
   - Model $\tilde{Y}$ given X, A

Parametric sequential g-estimation (see Acharya, Blackwell, & Sen 2016)

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr] \arrow[drr, bend left] & Z \arrow[d] \arrow[dr] & \\
A \arrow[r] \arrow[ur] \arrow[rr, bend right] & M \arrow[r] & Y
\end{tikzcd}
$$

Step 1: What outcome would have been realized at each $M = m$?

Parametric sequential g-estimation (see Acharya, Blackwell, & Sen 2016)

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr] \arrow[drr, bend left] & Z \arrow[d] \arrow[dr] & \\
A \arrow[r] \arrow[ur] \arrow[rr, bend right] & M \arrow[r] & Y
\end{tikzcd}$$

Step 1: What outcome would have been realized at each $M = m$?

$$\text{E}(Y^m \mid X, A, Z) = \text{E}(Y \mid X, A, Z, M = m)$$

because $M \to Y$ is identifiied given $\{X, A, Z\}$

Parametric sequential g-estimation (see Acharya, Blackwell, & Sen 2016)

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr] \arrow[drr, bend left] & Z \arrow[d] \arrow[dr] & \\
A \arrow[r] \arrow[ur] \arrow[rr, bend right] & M \arrow[r] & Y
\end{tikzcd}$$

Step 2: Construct a **de-mediated outcome**

Parametric sequential g-estimation (see Acharya, Blackwell, & Sen 2016)

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr] \arrow[drr, bend left] & Z \arrow[d] \arrow[dr] & \\
A \arrow[ur] \arrow[r] \arrow[rr, bend right] & M \arrow[r] & Y
\end{tikzcd}
$$

Step 2: Construct a **de-mediated outcome**

$$\tilde{Y} = Y - \gamma(X, A, M)$$

where the de-mediation function γ is

$$\underbrace{\gamma(X, A, M)}_{\substack{\text{Not a function of } Z \\ \text{See below}}} = \underbrace{\mathrm{E}(Y \mid X, A, Z, M) - \mathrm{E}(Y \mid X, A, Z, M = 0)}_{\text{Causal effect of the factual mediator value } M \text{ vs } 0}$$

Parametric sequential g-estimation (see Acharya, Blackwell, & Sen 2016)

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr] \arrow[drr, bend left] & Z \arrow[d] \arrow[dr] & \\
A \arrow[r] \arrow[ur] \arrow[rr, bend right] & M \arrow[r] & Y
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New assumption: No $Z \times M$ interactions (simplifies estimation)

- The effect $M \rightarrow Y$ does not depend on Z
- By this assumption, γ is not a function of Z

Parametric sequential g-estimation (see Acharya, Blackwell, & Sen 2016)

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr] \arrow[drr, bend left] & Z \arrow[d] \arrow[dr] & \\
A \arrow[r] \arrow[ur] \arrow[rr, bend right] & M \arrow[r] & Y
\end{tikzcd}$$

Step 3: Estimate the treatment effect on the de-mediated outcome

$$\mathrm{E}(Y^{a,0} \mid X) = \mathrm{E}(\tilde{Y} \mid X, A = a)$$

Parametric sequential g-estimation (see Acharya, Blackwell, & Sen 2016)

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr] \arrow[drr, bend left] & Z \arrow[d] \arrow[dr] & \\
A \arrow[ur] \arrow[r] \arrow[rr, bend right] & M \arrow[r] & Y
\end{tikzcd}
$$

High-level overview:

1. Estimate the effect of the mediator
   - Model Y given X, A, Z, M
2. Construct $\tilde{Y}$ with the effect of the mediator removed
   - $\tilde{Y} = Y - \left[\mathrm{E}(Y^M \mid X, A, Z) - \mathrm{E}(Y^0 \mid X, A, Z)\right]$
3. Estimate treatment effect on the de-mediated outcome
   - Model $\tilde{Y}$ given X, A

Learning goals for today

At the end of class, you will be able to:

1. Define controlled direct effects
2. Connect them to longitudinal treatments
3. Built intuition for a new estimator: sequential g-estimation