7. Positivity: The problem of empty cells

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Cornell Info 6751: Causal Inference in Observational Settings
Fall 2022

13 Sep 2022

Learning goals for today

At the end of class, you will be able to:

1. Define positivity
2. Understand how positivity relates to the adjustment set
3. Make estimates for the feasible subpopulation
4. Begin translating ideas to actual data

Positivity

Assumption that $\mathrm{P}(A = a \mid \vec{L} = \vec{\ell}) > 0$ (i.e., is positive) for all treatment values a in all population strata $\vec{\ell}$ defined by confounders

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Why this matters:

Positivity

Assumption that $\mathrm{P}(A = a \mid \vec{L} = \vec{\ell}) > 0$ (i.e., is positive) for all treatment values a in all population strata $\vec{\ell}$ defined by confounders

Why this matters:
If there is a subgroup $\vec{L} = \vec{\ell}$ where the treatment $A = a$ never happens, then we could never learn from data about the outcome in that subgroup under that treatment.

Running example for today's discussion

Government Assistance Protects Low-Income Families from Eviction

Ian Lundberg
Sarah L. Gold
Louis Donnelly
Jeanne Brooks-Gunn
Sara S. McLanahan

Journal of Policy Analysis and Management

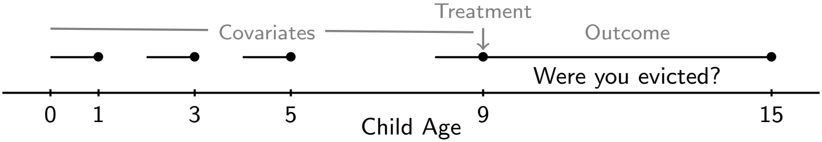
2021

Running example for today's discussion

Does public housing protect families from eviction?

Running example for today's discussion

		Treatment Condition	
		1: Public Housing	0: No Rental Assistance
Population	Person 1		
	Person 2		
	Person 3		



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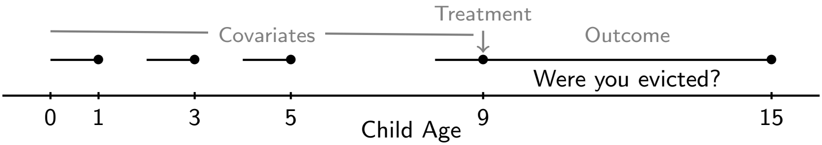

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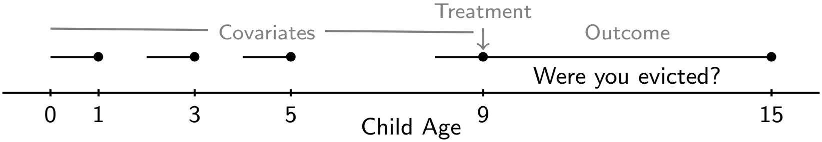

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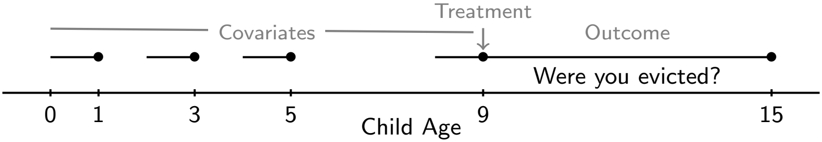

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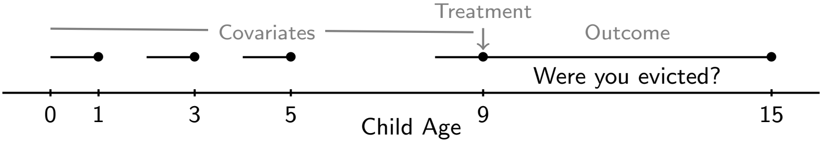

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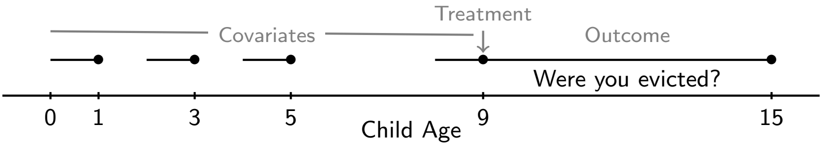

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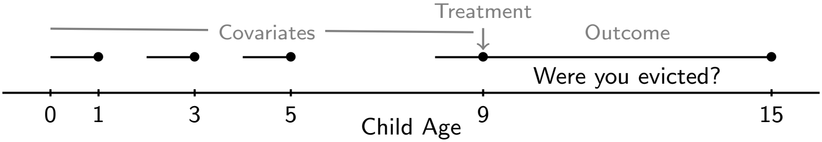

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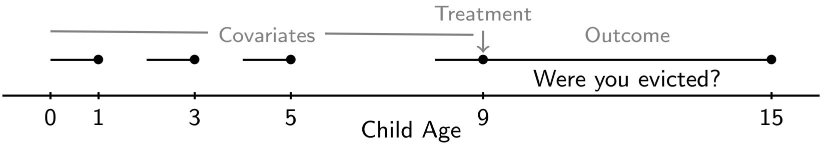
Robins 1986
Hahn 1998



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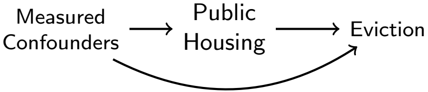

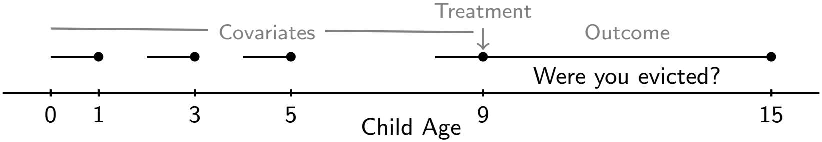

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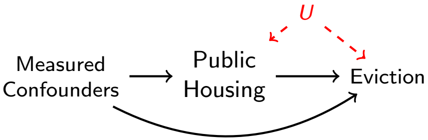

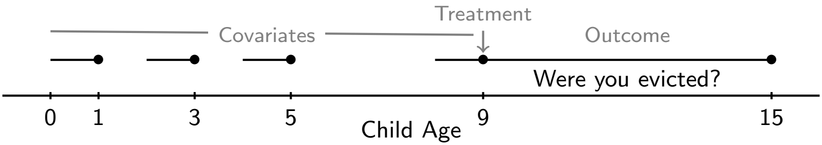

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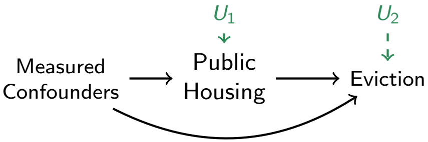

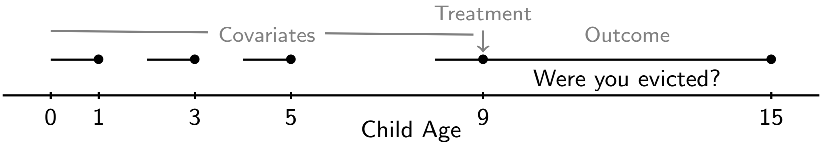

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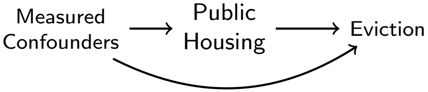

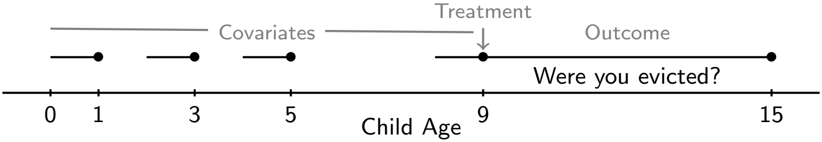

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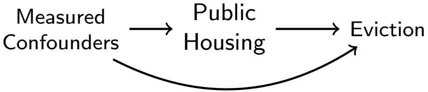

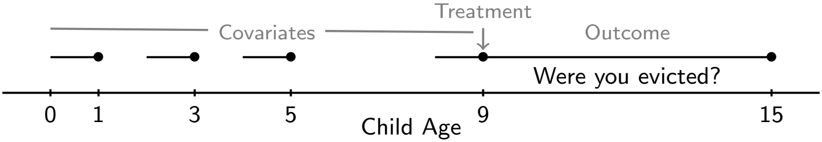

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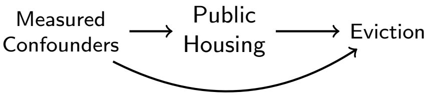
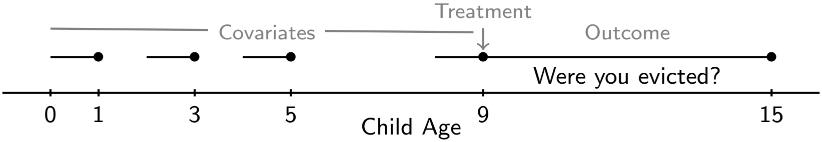
Average	**3%**	**11%**





Setting 1: A highly simplified DAG

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Question: How would you estimate in this setting?

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Setting 1: A highly simplified DAG

$$\begin{tikzcd} \text{Lagged Eviction} \arrow[r] \arrow[rr, bend right] & \text{Public Housing} \arrow[r] & \text{Eviction} \end{tikzcd}$$

Question: How would you estimate in this setting?

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1. Among those with (lagged eviction = TRUE), estimate the effect
2. Among those with (lagged eviction = FALSE), estimate the effect
3. Take a weighted average of the two estimates, weighted by the number of cases in each group

Nonparametric estimation: In R code

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```
# Load packages
library(tidyverse)

# Simulate some data
sim_data <- data.frame(x1 = rbinom(n,1,.5),
                       x2 = rbinom(n,1,.5),
                       x3 = rbinom(n,1,.5)) %>%
  # Generate the treatment
  mutate(a = rbinom(n,1,plogis(x1 + x2 + x3))) %>%
  # Generate the outcome
  mutate(y = rnorm(n, x1 + x2 + x3 + a))
```

Nonparametric estimation: In R code

```
# Define the confounders
confounders <- c("x1","x2","x3")

# Count cases in each stratum
strata_counts <- sim_data %>%
  # Group by the confounders
  group_by(across(all_of(confounders))) %>%
  # Count the number of cases
  summarize(cases = n(),
            .groups = "drop")
```

Nonparametric estimation: In R code

```
# Estimate effect in each stratum
strata_effects <- sim_data %>%
  # Group by the confounders and treatment
  group_by(a, across(all_of(confounders))) %>%
  # Estimate the mean outcome
  summarize(ybar = mean(y),
            .groups = "drop") %>%
  # Prepare to make the data wider by re-valuing the treatment
  mutate(a = paste0("ybar_",a)) %>%
  # Make the data wide
  pivot_wider(names_from = "a", values_from = "ybar") %>%
  # Estimate the effect
  mutate(conditional_effect = ybar_1 - ybar_0)
```

The `pivot_wider` step makes a conversion like this:

Stratum	a	ybar
1	ybar_1	3.6
1	ybar_0	3.2
2	ybar_1	3
2	ybar_0	2.8
⋮	⋮	⋮

$\rightarrow$

Stratum	ybar_1	ybar_0
1	3.6	3.2
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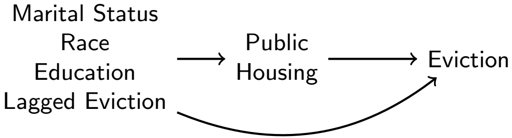
⋮	⋮	⋮

Nonparametric estimation: In R code

```
# Aggregate over strata
strata_counts %>%
  # Merge the effects into the counts data frame
  full_join(strata_effects, by = confounders) %>%
  # Stop working within strata. Average the effect
  ungroup() %>%
  summarize(average_effect = weighted.mean(conditional_effect,
                                           w = cases))
```

Setting 2: A slightly more complex DAG



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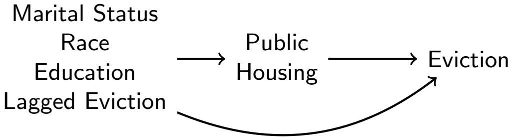

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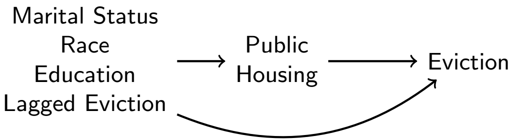

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PROBLEM: In some subgroups, the treatment does not vary

To discuss

Situation: In some subgroups, the treatment does not vary

Example: (oversimplified for concreteness)

When a mother has a college degree,
her family is never seen in public housing

Discuss: Does causal inference make sense for this subgroup

1. if this happens in the sample but not the population?
2. if this happens in the sample and in the population?

If it makes sense, how might you go about it?

Positivity assumption (in the population)

$$\mathrm{P}(A = a \mid \vec{L} = \vec{\ell}) > 0$$

for all treatment values a in the causal estimand
and
for every covariate stratum $\vec{\ell}$ in the population of interest

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Guarantees that in an infinite sample you will eventually see the needed treatment values

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But they will never happen—in an infinite sample, you'd never learn the answer.

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Note:

- Empirical positivity implies theoretical positivity
  - If we saw all treatments in this subgroup in our sample, they must exist in the population
- A lack of empirical positivity does not imply a lack of theoretical positivity
  - All the treatment values may exist in this subgroup, and our sample just happened to miss them

Feasible subsample (and subpopulation)

(No empirical positivity) $\rightarrow$ (No nonparametric SATE estimator)

where SATE is the Sample Average Treatment Effect.

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We will go do this in R.

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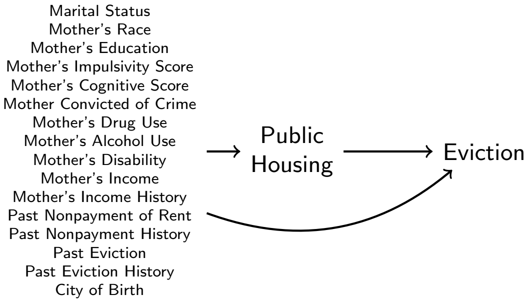

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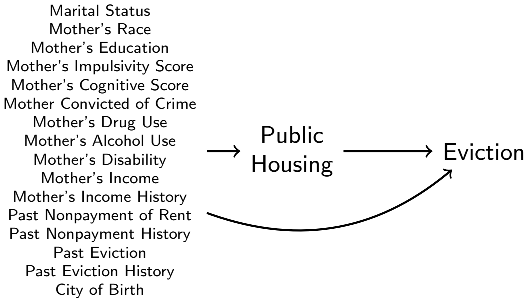

Let's see how many strata have both

- Cases in public housing and
- Cases with no assistance

Conclusion: Positivity is deceptively hard to satisfy

[1] D'Amour, A., Ding, P., Feller, A., Lei, L., & Sekhon, J. (2021). Overlap in observational studies with high-dimensional covariates. Journal of Econometrics, 221(2), 644-654.

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- Confounding may seem to require many covariates

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Conclusion: Positivity is deceptively hard to satisfy

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- You end up leaning on a model to extrapolate (next class)

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Learning goals for today

At the end of class, you will be able to:

1. Define positivity
2. Understand how positivity relates to the adjustment set
3. Make estimates for the feasible subpopulation
4. Begin translating ideas to actual data

Let me know what you are thinking

tinyurl.com/CausalQuestions

Office hours TTh 11am-12pm and at
calendly.com/ianlundberg/office-hours
Come say hi!