13. Inverse Probability Weighting

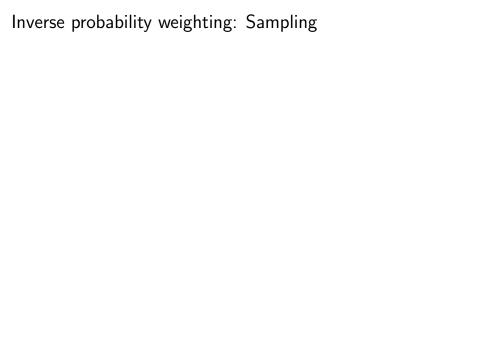
lan Lundberg Cornell Info 6751: Causal Inference in Observational Settings Fall 2022

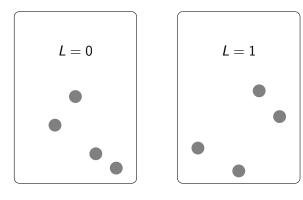
4 Oct 2022

Learning goals for today

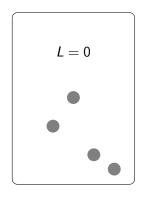
At the end of class, you will be able to:

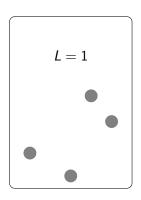
- 1. Trace inverse probability weighting to survey sampling
- 2. Apply the Horvitz-Thompson estimator for causal inference
- 3. Recognize the bias-variance tradeoff of trimmed weights
- 4. Estimate the ATE, ATT, and ATC by weighting



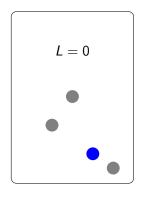


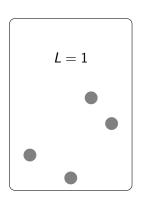
- Unsampled
- Sampled



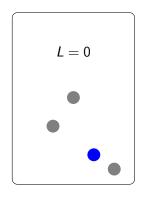


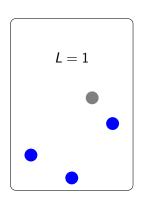
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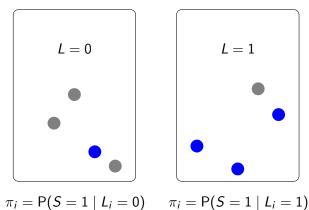


- Unsampled
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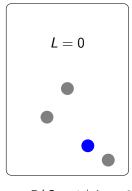
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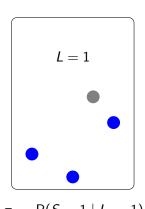
$$=\frac{1}{4}$$

$$=\frac{3}{4}$$

- Unsampled
- Sampled



Each counts for:
$$w_i = \frac{1}{\pi_i} = 4$$



$$\pi_i = P(S = 1 \mid L_i = \frac{3}{4})$$

$$= \frac{3}{4}$$

$$= \frac{3}{4}$$

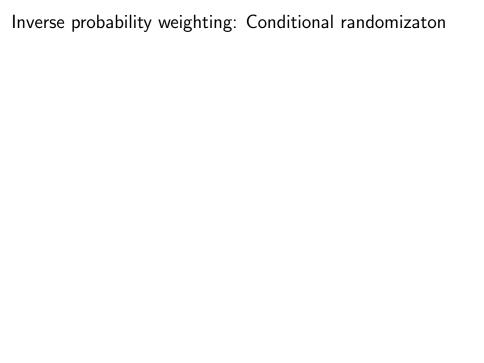
$$= \frac{3}{4}$$

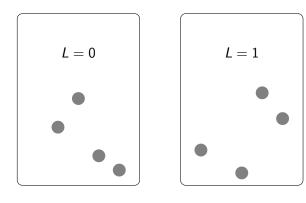
$$v_i = \frac{1}{\pi_i} = \frac{1}{2}$$

The Horvitz-Thompson estimator for a population mean:

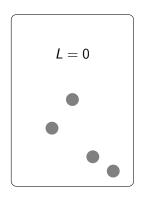
$$\hat{\mathsf{E}}(Y) = \frac{1}{N} \sum_{i:S \leftarrow 1} \frac{Y_i}{\pi_i}$$

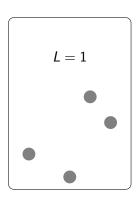
where N is the population size and π_i is the known probability of sampling



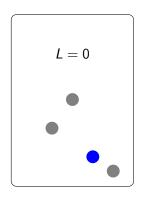


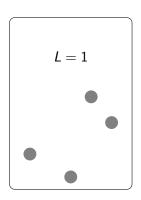
- Untreated
- Treated



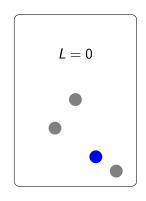


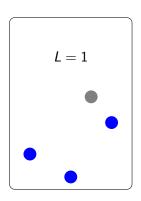
- Untreated
- Treated





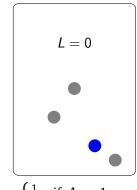
- Untreated
- Treated







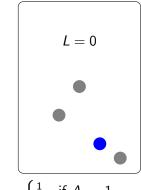




$$\pi_i = P(A_i \mid L_i) = \begin{cases} \frac{1}{4} & \text{if } A_i = 1\\ \frac{3}{4} & \text{if } A_i = 0 \end{cases}$$

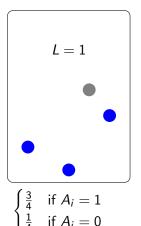


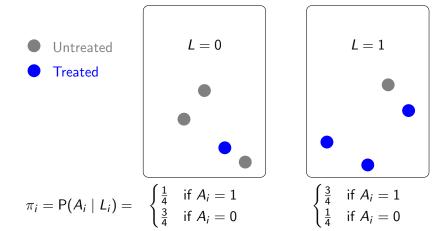
Treated



$$\pi_i = \mathsf{P}(A_i \mid L_i) =$$

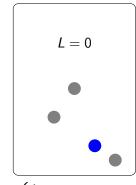
$$\begin{cases} \frac{1}{4} & \text{if } A_i = 1 \\ \frac{3}{4} & \text{if } A_i = 0 \end{cases}$$





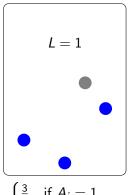
Each counts for:





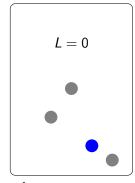
$$\pi_i = P(A_i \mid L_i) = \begin{cases} \frac{1}{4} & \text{if } A_i = 1\\ \frac{3}{4} & \text{if } A_i = 0 \end{cases}$$

Each counts for:
$$\begin{cases} \frac{4}{1} & \text{if } A_i = 1\\ \frac{4}{2} & \text{if } A_i = 0 \end{cases}$$



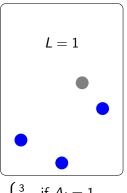
$$\begin{cases} \frac{3}{4} & \text{if } A_i = 1\\ \frac{1}{4} & \text{if } A_i = 0 \end{cases}$$





$$\pi_i = \mathsf{P}(A_i \mid L_i) = \begin{cases} \frac{1}{4} & \text{if } A_i = 1\\ \frac{3}{4} & \text{if } A_i = 0 \end{cases}$$

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Inverse probability weighted (IPW) estimator for the average treatment effect (ATE)

 $\mathsf{E}(Y^1) - \mathsf{E}(Y^0) = \frac{1}{N} \sum_{i:A:=1} \frac{Y_i}{\pi_i} - \frac{1}{N} \sum_{i:A:=0} \frac{Y_i}{\pi_i}$

where $\pi_i = P(A = a_i \mid \vec{L} = \vec{\ell_i})$ is the probability of the observed treatment given confounders

¹Hernán & Robins Technical Point 2.3

$$\mathsf{E}\left(\frac{\mathbb{I}(A=a)}{\mathsf{P}(A=a\mid\vec{L})}Y\right)$$

$$= \mathsf{E}(Y^a) \tag{6}$$

(1)

¹Hernán & Robins Technical Point 2.3

$$E\left(\frac{\mathbb{I}(A=a)}{\mathsf{P}(A=a\mid\vec{L})}Y\right)$$

$$= E\left(\frac{\mathbb{I}(A=a)}{\mathsf{P}(A=a\mid\vec{L})}Y^{a}\right)$$
consistency (2)

$$= \mathsf{E}(Y^a) \tag{6}$$

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consistency (2)
$$= E\left(E\left[\frac{\mathbb{I}(A=a)}{P(A=a\mid\vec{L})}Y^{a}\mid\vec{L}\right]\right)$$
iterated expectation (3)

$$= \mathsf{E}(Y^{\mathsf{a}}) \tag{6}$$

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iterated expectation (3)
$$= E\left(E\left[\frac{\mathbb{I}(A=a)}{P(A=a\mid\vec{L})}\mid\vec{L}\right]E\left[Y^{a}\mid\vec{L}\right]\right)$$
exchangeability (4)

 $= \mathsf{E}(Y^a) \tag{6}$

¹Hernán & Robins Technical Point 2.3

$$E\left(\frac{\mathbb{I}(A=a)}{P(A=a\mid\vec{L})}Y\right) \tag{1}$$

$$= E\left(\frac{\mathbb{I}(A=a)}{P(A=a\mid\vec{L})}Y^a\right) \qquad \text{consistency} \tag{2}$$

$$= E\left(E\left[\frac{\mathbb{I}(A=a)}{P(A=a\mid\vec{L})}Y^a\mid\vec{L}\right]\right) \qquad \text{iterated expectation} \tag{3}$$

$$= E\left(E\left[\frac{\mathbb{I}(A=a)}{P(A=a\mid\vec{L})}\mid\vec{L}\right]E\left[Y^a\mid\vec{L}\right]\right) \qquad \text{exchangeability} \tag{4}$$

$$= E\left(E\left[Y^a\mid\vec{L}\right]\right) \qquad \text{since left term was 1} \tag{5}$$

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In an observational study, we do not control assignment. We assume the dashed edge does not exist.

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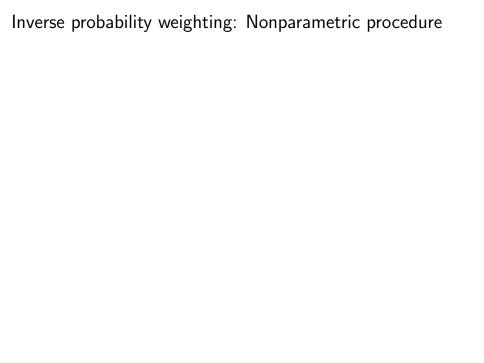
In an observational study, we do not control assignment. We assume the dashed edge does not exist. We estimate the propensity score.

In an experiment, we control treatment assignment. We know the dashed edge does not exist. We know the propensity score π .



In an observational study, we do not control assignment. We assume the dashed edge does not exist. We estimate the propensity score.

Otherwise all estimators (including IPW) are the same.



Inverse probability weighting:	Nonparametric procedure
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1. Assume a DAG where \vec{L} blocks backdoor paths

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- 4. Apply the IPW estimator

$$\hat{\mathsf{E}}(Y^a) = \frac{1}{N} \sum_{i: A = a} \frac{Y_i}{\hat{\pi}_i}$$

²On the Hajek estimator, see Hernán & Robins Technical Point 12.1

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$$\begin{split} \hat{\mathsf{P}}(A = 1 \mid \vec{L}) &= \mathsf{logit}^{-1} \left(\hat{\alpha} + \hat{\vec{\gamma}} \vec{L} \right) \\ \hat{\pi}_i &= \begin{cases} \mathsf{logit}^{-1} \left(\hat{\alpha} + \hat{\vec{\gamma}} \vec{L} \right) & \text{if } A_i = 1 \\ 1 - \mathsf{logit}^{-1} \left(\hat{\alpha} + \hat{\vec{\gamma}} \vec{L} \right) & \text{if } A_i = 0 \end{cases} \end{split}$$

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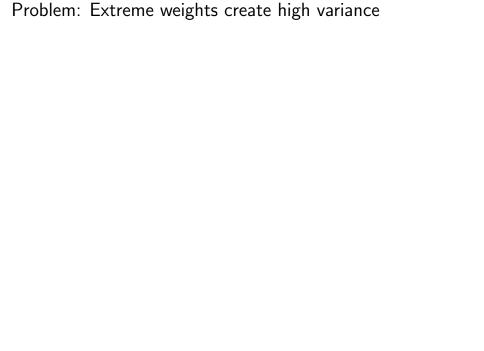
$$\hat{\pi}_i = \begin{cases} \mathsf{logit}^{-1} \left(\hat{\alpha} + \hat{\vec{\gamma}} \vec{L} \right) & \text{if } A_i = 1 \\ 1 - \mathsf{logit}^{-1} \left(\hat{\alpha} + \hat{\vec{\gamma}} \vec{L} \right) & \text{if } A_i = 0 \end{cases}$$

3. Apply an IPW estimator

$$\hat{\mathsf{E}}(Y^a) = \frac{1}{N} \sum_{i:A_i = a} \frac{Y_i}{\hat{\pi}_i} \qquad \qquad \mathsf{(Horvitz\text{-Thompson})}$$
 or
$$\hat{\mathsf{E}}(Y^a) = \frac{1}{\sum_{i:A_i = a} \frac{1}{\hat{\pi}_i}} \sum_{i:A_i = a} \frac{Y_i}{\hat{\pi}_i} \qquad \qquad \mathsf{(Hajek)}$$

(Hajek)

²On the Hajek estimator, see Hernán & Robins Technical Point 12.1



Suppose a stratum $\vec{L} = \vec{\ell}$ contains

- ▶ 100 untreated units
- ▶ 1 treated unit

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The estimate depends heavily on which treated unit happens to be included in the sample \rightarrow high-variance estimator

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Two solutions

- 1. Trim the weights
- 2. Truncate the weights

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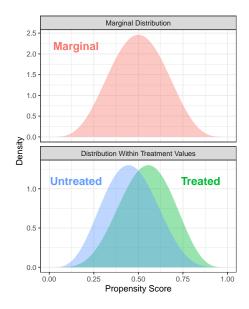
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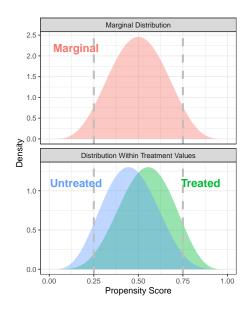
The estimate depends heavily on which treated unit happens to be included in the sample \rightarrow high-variance estimator

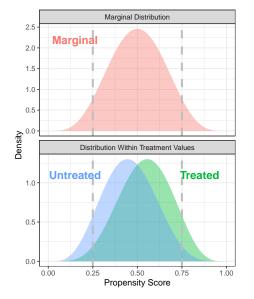
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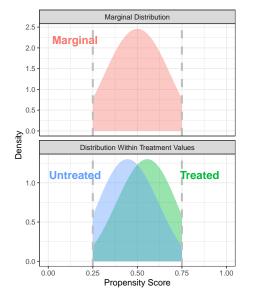
Both solutions accept bias in order to reduce variance



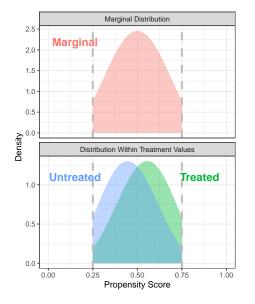




Drop units with extreme weights

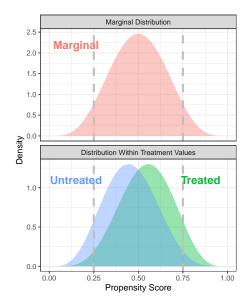


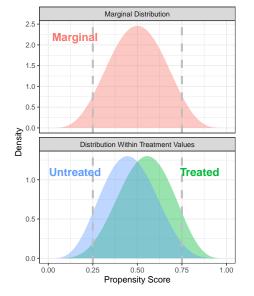
Drop units with extreme weights



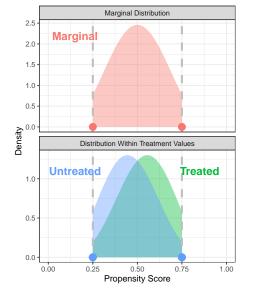
Drop units with extreme weights

Changes target population
— Biased for full population

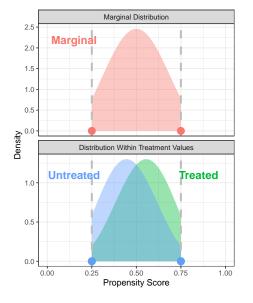




Truncate values of extreme weights

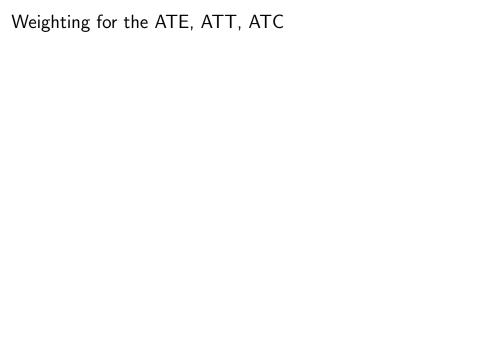


Truncate values of extreme weights

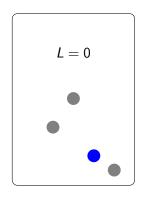


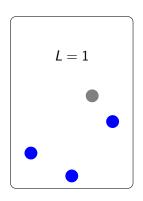
Truncate values of extreme weights

Biased: Ignores some confounding

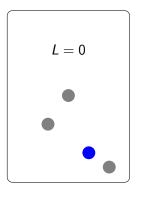


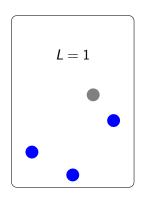
- Untreated
- Treated





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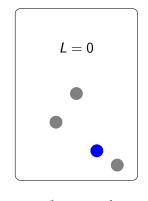


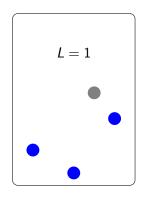


For ATE:

$$w_i = \frac{1}{P(A=a_i|L=\ell_i)}$$

- Untreated
- Treated

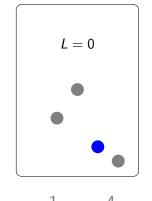


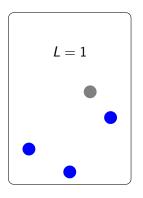


$$w_i = \frac{1}{P(A=a_i|L=\ell_i)}$$

$$\frac{1}{3/4} = \frac{2}{3}$$



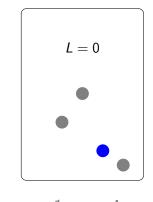


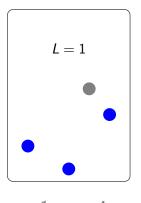


For ATE:
$$w_i = \frac{1}{P(A)}$$

$$\frac{1}{1/4} = \frac{4}{1}$$







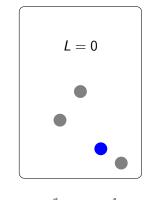
For ATE:
$$w_i = \frac{1}{2}$$

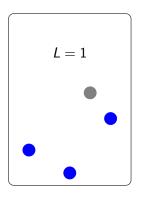
$$\frac{3/4}{\overline{\ell_{i}}} - \frac{3}{3}$$

$$1 \quad \underline{4}$$

$$\frac{1}{1/4} = \frac{4}{1}$$



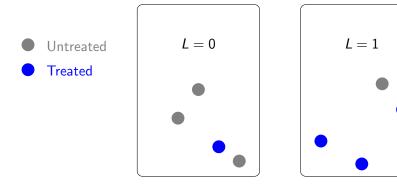




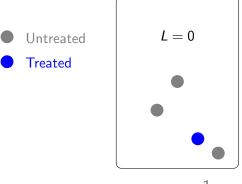
For ATE:
$$w_i = \frac{1}{P(A_i)}$$

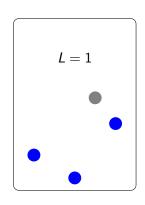
$$\frac{3}{4} = \frac{4}{1}$$

$$\frac{1}{3/4} = \frac{4}{3}$$



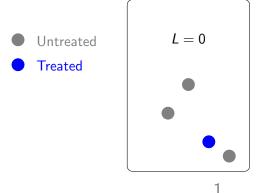
For ATT?

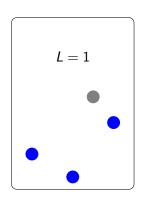




For ATT?

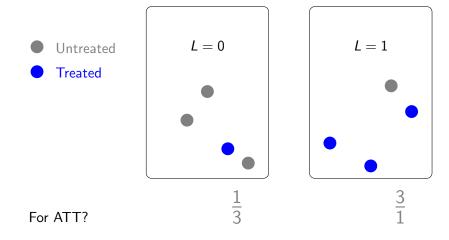
 $\frac{1}{3}$



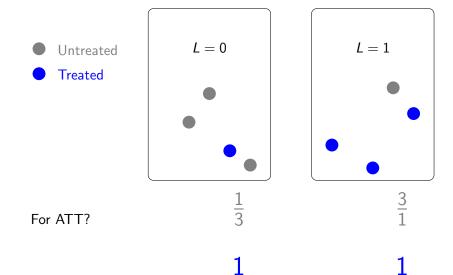


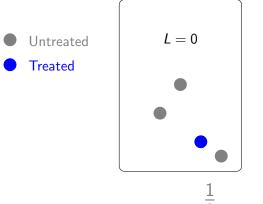
For ATT?

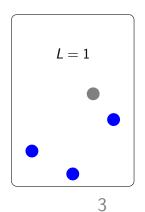
)



1

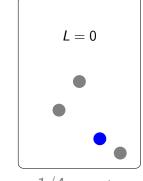






For A	TT:
$w_i = $	$P(A=1 L=\ell_i)$
	$P(A=a_i L=\ell_i)$





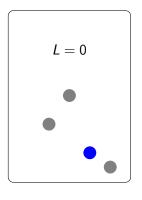
For ATT:

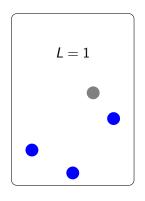
$$w_i = \frac{P(A=z)}{P(A=z)}$$

$$\frac{1/4}{3/4} = \frac{1}{3}$$

$$\frac{8/4}{8/4} = 1$$

- Untreated
- Treated

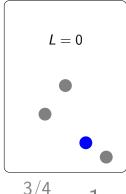


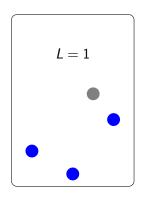


For ATC:
$$P(A=0)$$

$$w_i = \frac{P(A=0|L=\ell_i)}{P(A=a_i|L=\ell_i)}$$

- Untreated
- Treated



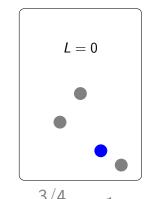


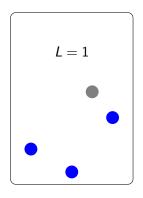
For ATC:

$$w_i = \frac{P(A=0|L=\ell_i)}{P(A=a_i|L=\ell_i)}$$

$$\frac{3/4}{3/4} = 1$$

- Untreated
- Treated





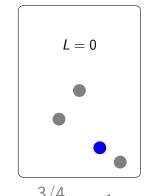
For ATC:

$$w_i = \frac{P(A=0|L=\ell_i)}{P(A=a_i|L=\ell_i)}$$

$$\frac{3}{4} = \frac{3}{4}$$

$$\frac{3/4}{1/4} = 3$$

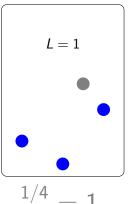
- Untreated
- Treated



For ATC:

$$w_i = \frac{P(A=0|L=\ell_i)}{P(A=a;|I=\ell_i)}$$

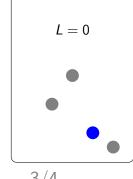
$$v_i = \frac{\frac{1}{P(A=a_i|L=\ell_i)}}{\frac{3/4}{1/4}} = 3$$



$$\frac{1/4}{1/4} = 1$$

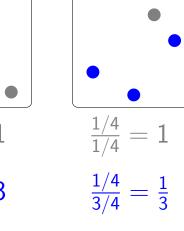


Treated



For ATC:

$$w_i = \frac{P(A=0|L=\ell_i)}{P(A=a_i|L=\ell_i)}$$



L = 1

General formula:

$$w_i = rac{ ext{Size of target population in } \vec{L} = \vec{\ell_i}}{ ext{Size of factual population in } \vec{L} = \vec{\ell_i}}$$

Intuition:

- ► Normalize the factual population across strata (denominator)
- ► Upweight to the counterfactual population (numerator)

Inverse probability weighting: Reading

Hernán & Robins

- **▶** 2.4
- ► 12.1–12.3

Learning goals for today

At the end of class, you will be able to:

- 1. Trace inverse probability weighting to survey sampling
- 2. Apply the Horvitz-Thompson estimator for causal inference
- 3. Recognize the bias-variance tradeoff of trimmed weights
- 4. Estimate the ATE, ATT, and ATC by weighting

Let me know what you are thinking

tinyurl.com/CausalQuestions

Office hours TTh 11am-12pm and at calendly.com/ianlundberg/office-hours Come say hi!