Precept 6: Duration models

Soc 504: Advanced Social Statistics

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Outline

We've gotten some questions about our personal project workflows.

Rmarkdown is in some ways ideal:

- Fully reproducible
- Code and results in one place

Problem: If code is slow to run, Rmarkdown is slow to compile each time.

I more often use **R** and LaTeX:

- In RStudio, you can create a new R script. This is your code but does not produce a PDF.
- Save results (see ?save, ggsave, etc.)
- Produce final report in LaTeX
  - I use TexShop
  - You can also work in an online platform like Overleaf. They also provide great templates!

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Duration models are useful when we are interested in the

time T to an event

but some observations are **censored**: the event has not occurred at the end of data collection

Think, pair, share:
Why can't we do OLS when some observations are censored?

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but some observations are **censored**: the event has not occurred at the end of data collection

Think, pair, share:
Why can't we do OLS when some observations are censored?
Because for those observations we don't know T!

The time to death T is a random variable.
Its distribution is described by four critical functions:

1. **Density function** $f(t)$
   - Density of death at t
2. **CDF** $F(t) = P(T < t)$
   - Probability of death by t
3. **Survival function** $S(t) = P(T > t) = 1 - F(t)$
   - Probability of survival to t
4. **Hazard function** $h(t) = \frac{f(t)}{S(t)}$
   - Density of death at t given survival up to t

Question: Why isn't the hazard function a probability?

Photo credit: J Zamudio via
https://www.nps.gov/yose/planyourvisit/stargazing.htm

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We just proved the memoryless property! **How?**
$h(t)$ is not a function of t. The hazard is **constant**.

Modeling with covariates

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This changes for different families of hazard models

For the exponential, $h_0(t) = \lambda$

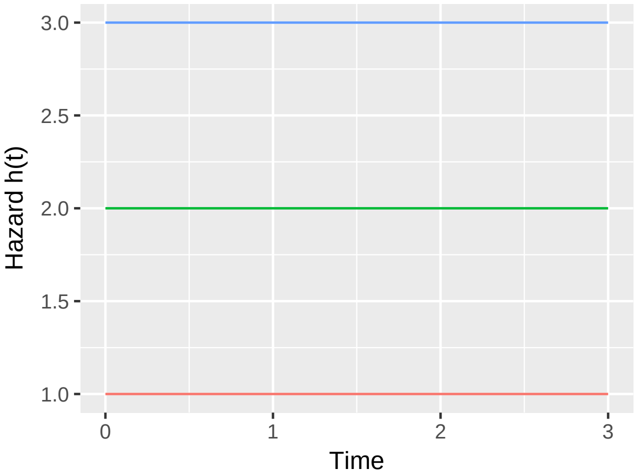
This is the same for all proportional hazard models

Why add covariates?

Why add covariates? It might be cloudy.

Photo credit: Hannah Lundberg

Exponential hazards



Question: If the green is the baseline hazard $h_0(t)$, what is the hazard ratio that produces the blue line? The red line?

Fitting an Exponential with survreg

```
> library(survival)
> fit <- survreg(Surv(time, event) ~ age + sex,
+                dist = "exponential",
+                data = lung)
> summary(fit)

Call:
survreg(formula = Surv(time, event) ~ age + sex, data = lung,
    dist = "exponential")
              Value Std. Error     z        p
(Intercept)  6.3597    0.63547 10.01 1.41e-23
age         -0.0156    0.00911 -1.72 8.63e-02
sex          0.4809    0.16709  2.88 4.00e-03

Exponential distribution
Loglik(model)= -1156.1   Loglik(intercept only)= -1162.3
Chisq= 12.48 on 2 degrees of freedom, p= 0.002
Number of Newton-Raphson Iterations: 4
n= 228
```

Interpreting hazard ratios

$$h(t \mid x) = h_0(t)e^{-x\beta}$$

```
> exp(-coef(fit))
(Intercept)         age         sex
      0.002       1.016       0.618
```

Q: How would you interpret these?

Interpreting hazard ratios

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Q: How would you interpret these?

A year increase in age is associated with a 1.6% increase in the hazard, holding sex constant.

There are some things demographers just memorize.

We recommend just looking these up when you need them.

For instance, this fact:

The survival function is e to the minus cumulative hazard.

Hazard function → survival function

The derivative of the negative log of the survival function is

$$\begin{aligned}\frac{\partial}{\partial t}\left(-\log\left[S(t)\right]\right) &= \frac{\frac{\partial}{\partial t}\left(-S(t)\right)}{S(t)} \\ &= \frac{\frac{\partial}{\partial t}\left(-[1-F(t)]\right)}{S(t)} \\ &= \frac{f(t)}{S(t)} = h(t)\end{aligned}$$

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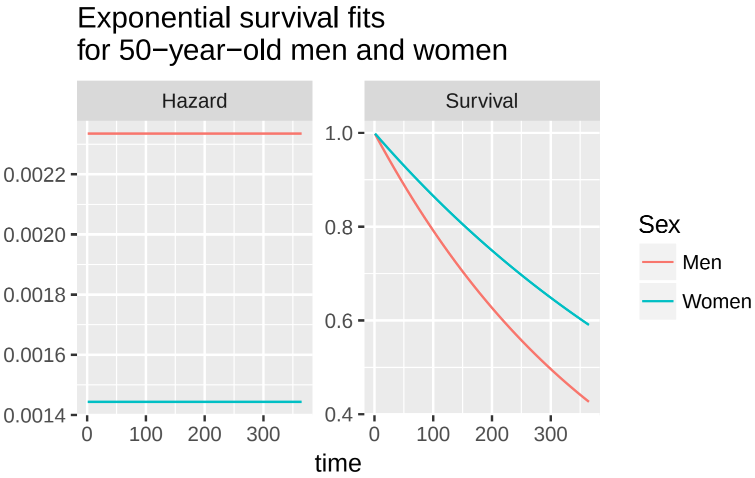
Doing the reverse, we can go from $h(t)$ to $S(t)$

$$\begin{aligned}\int_0^t \frac{\partial}{\partial t'}\left(-\log\left[S(t')\right]\right) dt &= \int_0^t h(t')dt' \\ -\log\left[S(t)\right] &= \int_0^t h(t')dt' \\ S(t) &= e^{-\int_0^t h(t')dt'}\end{aligned}$$

Plotting survival curves



Plotting survival curves

How we made the previous slide:

```
data.frame(t = seq(.5,20,.5)) %>%
  mutate(Men.Hazard = lambda[1],
         Women.Hazard = lambda[2],
         Men.Survival = exp(-lambda[1]*t),
         Women.Survival = exp(-lambda[2]*t)) %>%
  melt(id = "t") %>%
  separate(variable, into = c("Sex","QOI")) %>%
  ggplot(aes(x = t, y = value, color = Sex)) +
  geom_line() +
  facet_wrap(~QOI, scales = "free") + ylab("") + xlab("time") +
  ggtitle("Exponential survival fits, for 50-year-old men and women") +
  ggsave("ExpoFit.pdf",
         height = 3, width = 5)
```

Scales and rates

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In general, you have to be careful with the parameterization of survival distributions.

What if we want the hazard to be a function of time?

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Many options.

Weibull distribution

$$T \sim \text{Weibull}(\alpha, \lambda)$$

PDF [1]

$$f(t) = t^{\alpha-1}\alpha\lambda^{\alpha}e^{-(\lambda t)^{\alpha}}$$

CDF

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Survival function

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The hazard decreases with t when $\alpha < 1$
The hazard is constant over t when $\alpha = 1$
In that case, it's the exponential!

$$h(t \mid \alpha = 1) = t^{\alpha-1}\alpha\lambda^{\alpha} = t^{1-1}1\lambda^{1} = \lambda$$

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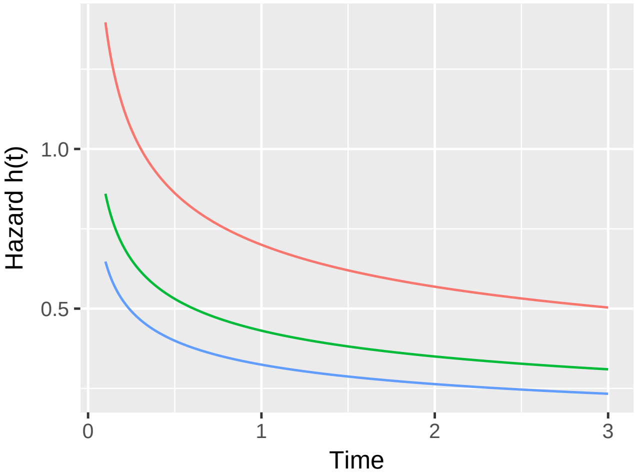
If the world is actually Exponential, we gain **efficiency** by making the assumption that the hazard is constant over time.

This is a **general theme** of statistics: Modeling assumptions buy us efficiency if they are correct.

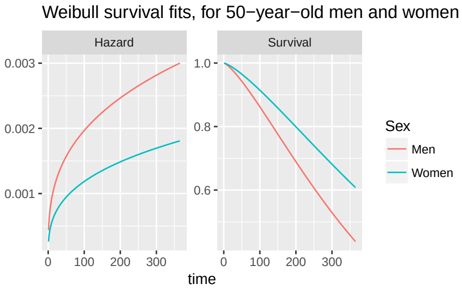
Weibull hazards



Fitting a Weibull model

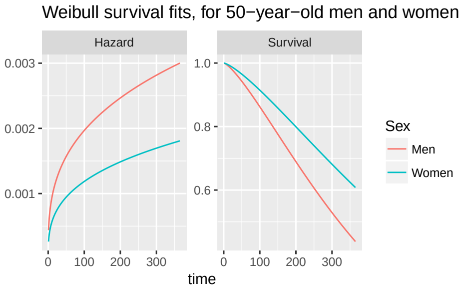
```
## Fitting a Weibull model
fit <- survreg(Surv(time, event) ~ age + sex,
               dist = "weibull",
               data = lung)
```

Weibull results



Common question: The gap between those hazards clearly changes over time! Is this a violation of a modeling assumption?

Weibull results



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A: No, they are still proportional!

Weibull results

Common question: The gap between those hazards clearly changes over time! Is this a violation of a modeling assumption?

A: No, they are still proportional!

(Also since these are fitted values, they necessarily agree with the modeling assumptions, so this was a trick question.)

You can fit a survival model using
any distribution
for which the support is
all positive numbers.

There are a huge number of options.

Lognormal distribution

$$T \sim \text{LogNormal}(\mu, \sigma^2) \sim e^Z \text{ (where } Z \sim N(\mu, \sigma^2))$$

$$f(t) = \frac{1}{Y_i \sigma \sqrt{2\pi}} \exp\left(-\frac{(\ln(Y_i) - \mu)^2}{2\sigma^2}\right)$$

CDF

$$F(t) = \int_0^t f(x)dx = \text{ugly formula}$$

Survival function

$$S(t) = P(T > t) =$$

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$$S(t) = P(T > t) = 1 - P(T < t) =$$

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Hazard function:

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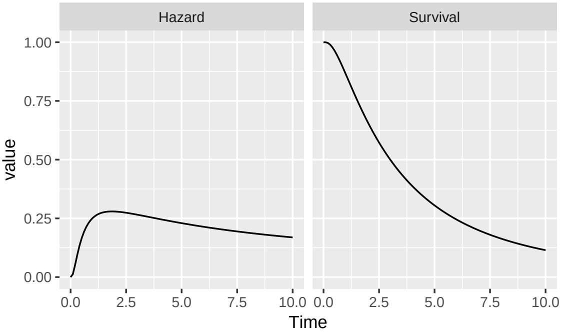
Hazard function: Risk of event at t given survival up to t

$$h(t) = \frac{f(t)}{S(t)} = \text{ugly formula}$$

Fitting a Lognormal

```
fit <- survreg(Surv(time, event) ~ age + sex,
               dist = "lognormal",
               data = lung)
```

Note: This figure doesn't correspond to the model above - just an example of a LogNormal



Gompertz distribution

$$f(t) = b\eta e^{bt} e^{\eta} \exp(-\eta e^{bt})$$

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The log of the hazard function is linear in time!

This is why people like the Gompertz.

Gompertz distribution

Gompertz hazard with $\alpha = -7, \beta = .09$

$$\log[h(t)] = \alpha + \beta t, \quad h(t) = \exp(\alpha + \beta t)$$

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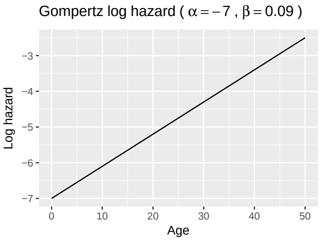

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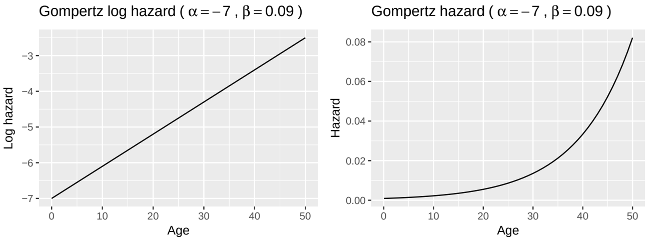

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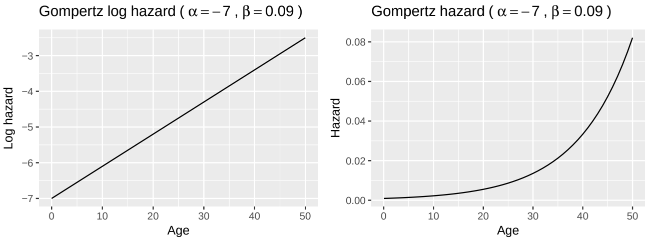

Q: For what questions would this be a good choice?

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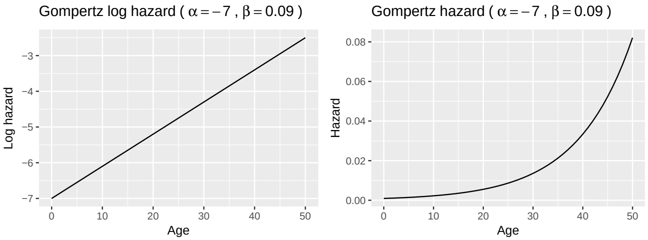

Q: For what questions would this be a good choice? Mortality

Note: Example motivated by U.S. mortality; see German Rodriguez's example here.

Time between breaks while hiking out of this valley.
You don't need a rest right away...

Donahue Pass, Yosemite. Photo credit: Riley Brian

...but after going for a while your hazard of resting increases.
Gompertz.

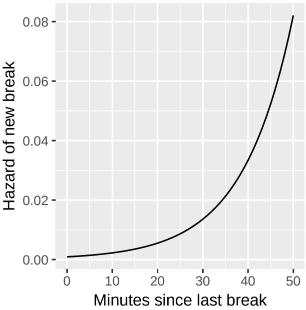

Photo credit: Riley Brian

As I said at the beginning, all of the survival models above have the form:

Hazard function given covariate set x

Baseline hazard

Hazard ratio

$$h(t \mid x) = h_0(t)e^{x\beta}$$

This changes for different families of hazard models

This is the same for all proportional hazard models

Different models allow different kinds of flexibility in the baseline hazard $h_0(t)$.

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Different models allow different kinds of flexibility in the baseline hazard $h_0(t)$.

Can we model hazard ratios without any assumptions about $h_0(t)$?

Cox proportional hazards model

Then we can fit a Cox proportional hazards model!

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To save time, I won't cover this here, but it's important and in lecture slides.

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The Cox model is fit based on the order at which people die, rather than the times, so it does not assume a baseline hazard.

You can fit one with `coxph()`

Outline

Using distributions

Most common question we are asked:
How do I know when to use a given distribution for a given problem?

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How do I know when to use a given distribution for a given problem?

When you know the **story of the distributions**, you can find one that **maps onto** your current problem.

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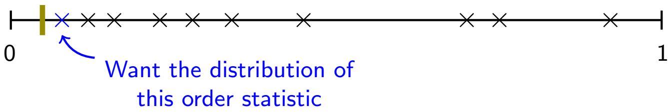

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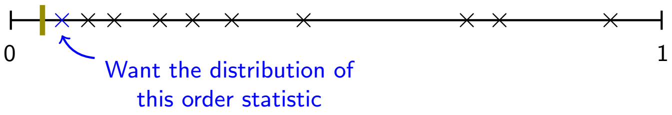

You reply:
"You want to know the distribution of the order statistic $U_{(3)}$.

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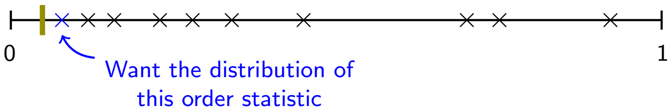
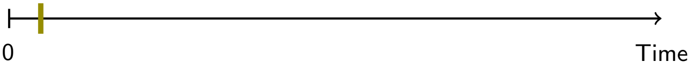


You reply:
“You want to know the distribution of the order statistic $U_{(3)}$.
Let me take you to the wilderness. We will count shooting stars.”

PC: http://wilderness.org/30-prettiest-lakes-wildlands

Imagine laying out on your pad on the granite, looking up at the sky.

We will count shooting stars and record the times we see them.[2]



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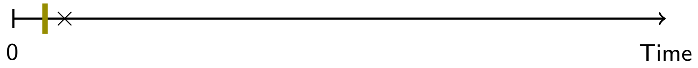

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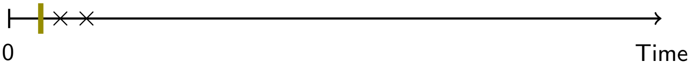

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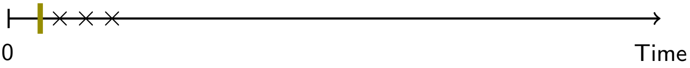

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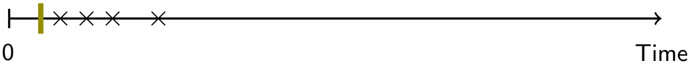

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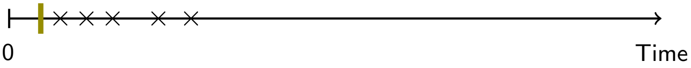

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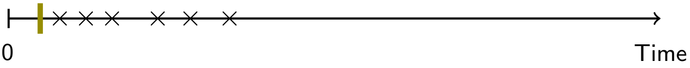

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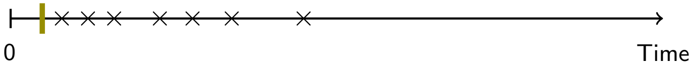

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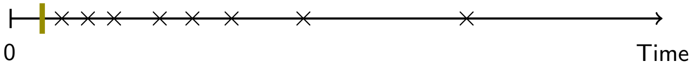

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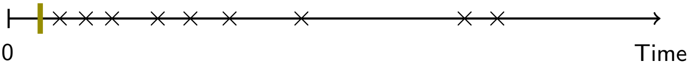

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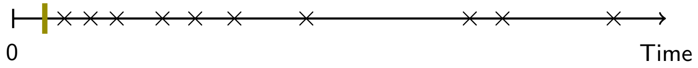

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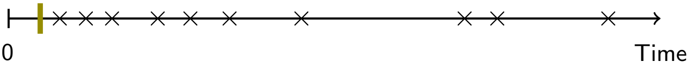

Shooting stars come at a **constant rate**.
The times between the arrivals are $X_1, X_2, \ldots \overset{\text{iid}}{\sim} \text{Exponential}(\lambda)$.

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Suppose we saw the second star at time $X_1 + X_2 = 1$.

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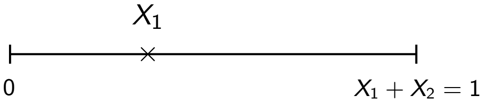
Q: What is the distribution of X_1 given this information?



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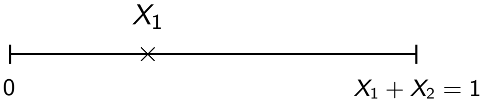

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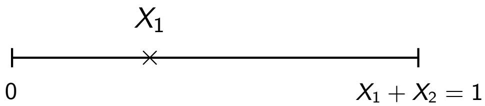

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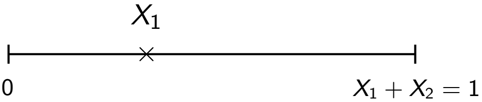

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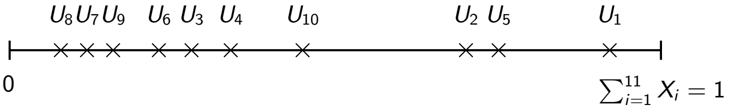
A: $P(U < .05) = P\left(\frac{X_1}{X_1+X_2} < .05\right) = 0.05$

Now suppose we observe $X_1, \ldots, X_{11}$ and we rescale so their sum is 1.



Let's re-label the $\times$ marks with U values with arbitrary indexes.

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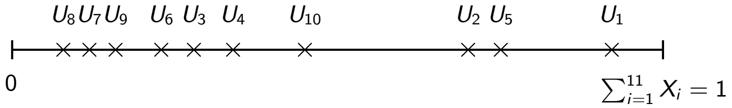

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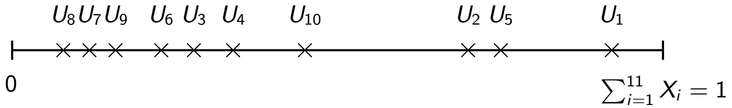

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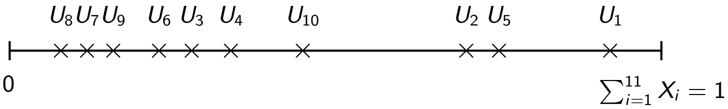
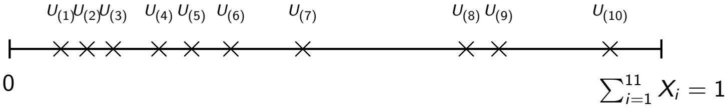
There is a connection between p-values and shooting stars.

Order statistics



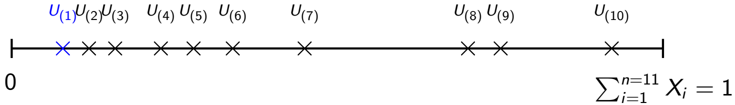
Let's denote the k-th order statistic by $U_{(k)}$.



$$U_{(k)} = \frac{\sum_{i=1}^{k} X_i}{\sum_{i=1}^{11} X_i}$$

A new distribution: The **Beta**

If $X_1, \ldots, X_n \sim$ Exponential, then $\frac{\sum_{i=1}^{k} X_i}{\sum_{i=1}^{n} X_i} \sim$ Beta(k,n - k - 1)



So $U_{(1)} \sim \text{Beta}(1, 10)$.

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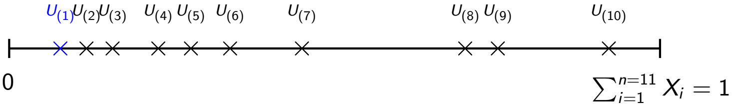
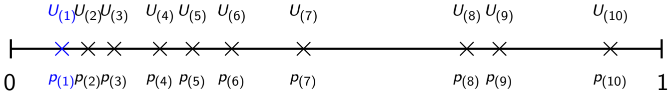


So $U_{(1)} \sim \text{Beta}(1, 10)$.

Q: Can you reason about the expected value of a Beta(1,10)?

A: There are 11 white space that we would expect to be of equal size, so we might expect that $\text{E}(U_{(1)}) = \frac{1}{11}$. **This is right!**

Q: Given what we know about shooting stars, what distribution do you think the smallest p-value takes?



$$p_{(1)} \sim \text{Beta}(1, 9)$$

Q: What is the probability that the smallest p-value is less than 0.05?

$$P(p_{(1)} < .05) = P(\text{Beta}(1, 10) < .5) = F_{\text{Beta}(1,9)}(.05) = 0.37$$

It is very easy to get a false positive by running 10 hypothesis tests!

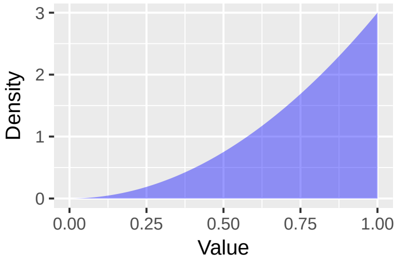
Key takeaways

We've taught you the stories of many distributions.

To use them, try to fit your problem into one of these **known stories**!

Photo credit: Hannah Lundberg

In my own research, I wanted to choose a prior distribution on a correlation that I expected to be near 1. I chose Beta(3,1).



I chose that by thinking:

- I want the distribution of the highest of 3 uniform draws.
- I want the distribution of the proportion of time spent waiting for 3 shooting stars, out of a total time spend waiting for 4.

Plugging your problem into a **known story** can help you find a solution.

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Suppose someone says to you, "I ran 100 hypothesis tests. What's the probability that the 7th-smallest p-value is less than 0.05 if the null hypotheses are true?"

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That is the proportion of time spent waiting for the 7th shooting star:

$$U_{(7)} \sim \text{Beta}(7, 93)$$

$$P(U_{(7)} < .05) = F_{\text{Beta}(7,93)}(.05) = 0.23$$

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So, it's not that strange to see 7 p-values less than 0.05. And we learned this all from shooting stars!

One other story you might use

What if we wanted a distribution for the time until the kth star comes?

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What if we wanted a distribution for the time until the kth star comes?

$$X_1, \ldots, X_k \overset{\text{iid}}{\sim} \text{Exponential}(\lambda)$$
$$G_k \sim X_1 + \cdots + X_k$$

Then we say

$$G_k \sim \text{Gamma}(k, \lambda)$$

The **Gamma distribution** characterizes the wait time until the kth star.

Outline

Side note: Zelig

Zelig is an R package designed to make everything we do in class easier.

Note the Zelig workflow overview.

We will use the Zelig-Exponential.

Zelig example: Lung cancer survival

We will walk through the example using data on lung cancer survival

```
> library(survival)
> data(lung)
> head(lung)
  inst time status age sex ph.ecog ph.karno pat.karno meal.cal wt.loss
1    3  306      2  74   1       1       90       100     1175      NA
2    3  455      2  68   1       0       90        90     1225      15
3    3 1010      1  56   1       0       90        90       NA      15
4    5  210      2  57   1       1       90        60     1150      11
5    1  883      2  60   1       0      100        90       NA       0
6   12 1022      1  74   1       1       50        80      513       0
lung <- mutate(lung, event = as.numeric(status == 2))
```

Variable definitions: Lung cancer survival

```
?lung
inst: Institution code
time: Survival time in days
status: censoring status 1=censored, 2=dead
age: Age in years
sex: Male=1 Female=2
ph.ecog: ECOG performance score (0=good 5=dead)
ph.karno: Karnofsky performance score (bad=0-good=100) rated by physician
pat.karno: Karnofsky performance score as rated by patient
meal.cal: Calories consumed at meals
wt.loss: Weight loss in last six months
```

Zelig step 1: Fit a model

```
fit <- zelig(Surv(time, event) ~ age + sex,
             model = "exp",
             data = lung)
```

Zelig step 1: Fit a model

```
> summary(fit)
Model:

Call:
z5$zelig(formula = Surv(time, event) ~ age + sex, data = lung)
               Value Std. Error     z        p
(Intercept)  6.3597    0.63547 10.01 1.41e-23
age         -0.0156    0.00911 -1.72 8.63e-02
sex          0.4809    0.16709  2.88 4.00e-03

Scale fixed at 1

Exponential distribution
Loglik(model)= -1156.1   Loglik(intercept only)= -1162.3
Chisq= 12.48 on 2 degrees of freedom, p= 0.002
Number of Newton-Raphson Iterations: 4
n= 228

Next step: Use 'setx' method
```

Zelig step 2: Use `setx` to set covariates of interest

```
men <- setx(fit, age = 50, sex = 1)
women <- setx(fit, age = 50, sex = 2)
```

Zelig step 2: Use `setx` to set covariates of interest

```
> men
setx:
  (Intercept) age sex
1           1  50   1

Next step: Use 'sim' method
> women
setx:
  (Intercept) age sex
1           1  50   2

Next step: Use 'sim' method
```

Zelig step 3: Use `sim` to simulate quantities of interest

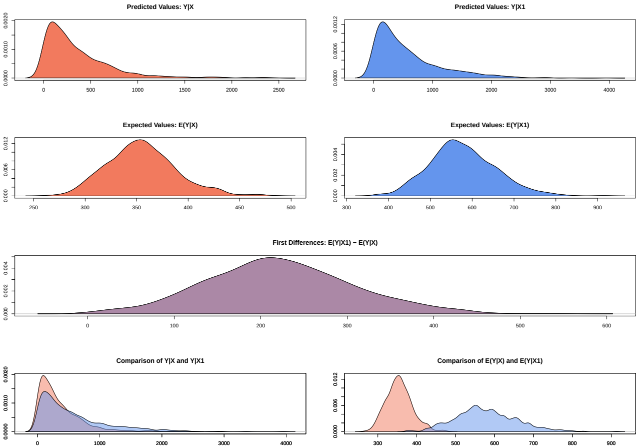
```
> sims <- sim(obj = fit, x = men, x1 = women)
> summary(sims)

 sim x :
 -----
ev
     mean       sd      50%     2.5%   97.5%
1 355.086 33.63733 353.5258 296.6169 428.758
pv
        mean       sd     50%     2.5%    97.5%
[1,] 351.414 361.6174 242.511 7.082744 1357.005

 sim x1 :
 -----
ev
      mean      sd     50%     2.5%    97.5%
1 577.5684 78.5113 571.178 438.4341 743.9957
pv
         mean       sd      50%    2.5%   97.5%
[1,] 562.8317 550.6102 382.9658 11.5627 2016.61
fd
      mean      sd      50%     2.5%    97.5%
1 222.4824 85.0493 217.0278 61.08082 396.5632
```

Zelig step 4: Use graph to plot simulation results

```
pdf("ZeligFigures.pdf",
    height = 5, width = 7)
plot(sims)
dev.off()
```



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plot(sims)
```

After break: expectation maximization, missing data

Cards! Questions?

Photo credit: Riley Brian