9. Parametric g-formula: Continuous treatments

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Cornell Info 6751: Causal Inference in Observational Settings
Fall 2022

20 Sep 2022

Learning goals for today

At the end of class, you will be able to:

1. Define causal effects with continuous treatments
2. Understand the risks of extrapolation
3. Select causal estimands for continuous treatments that support the production of credible estimates

Why continuous treatments are hard

Why continuous treatments are hard

Person 1

Person 2

Person 3

Person 4

Why continuous treatments are hard

	Outcome under treatment value	
	Untreated	Treated
Person 1	∘	∘
Person 2	∘	∘
Person 3	∘	∘
Person 4	∘	∘

Why continuous treatments are hard

	Factual treatment	Outcome under treatment value	
		Untreated	Treated
Person 1	Untreated	$\circ$	$\circ$
Person 2	Treated	$\circ$	$\circ$
Person 3	Treated	$\circ$	$\circ$
Person 4	Untreated	$\circ$	$\circ$

Why continuous treatments are hard

	Factual treatment	Outcome under treatment value	
		Untreated	Treated
Person 1	Untreated	●	○
Person 2	Treated	○	●
Person 3	Treated	○	●
Person 4	Untreated	●	○

Why continuous treatments are hard

	Factual treatment	Outcome under treatment value 1	2	3	4	5	...
Person 1	3	○	○	●	○	○	○
Person 2	2	○	●	○	○	○	○
Person 3	5	○	○	○	○	●	○
Person 4	4	○	○	○	●	○	○

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	Factual treatment	Outcome under treatment value
		1 2 3 4 5 . . .
Person 1	3	○○○○○○○○●○○○○○○○○○○○○
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Person 3	5	○○○○○○○○○○○○○○○○●○○○○
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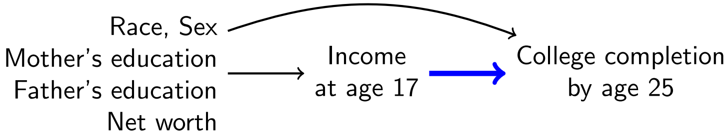
Question: Why is this hard? How would you proceed?

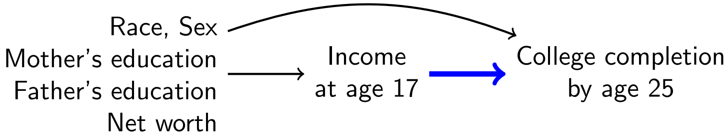
A motivating example

(current work with Jennie E. Brand)

Causal identification.



Causal identification.



Mayer 1997
Brooks-Gunn & Duncan 1997
Duncan et al. 2010
Elwert & Pfeffer 2022

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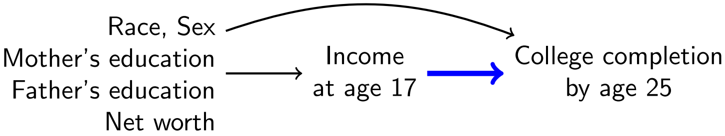

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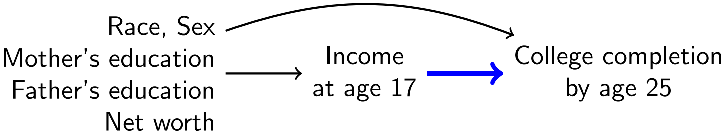

Ideal estimation. What we'd like to do in a huge sample.

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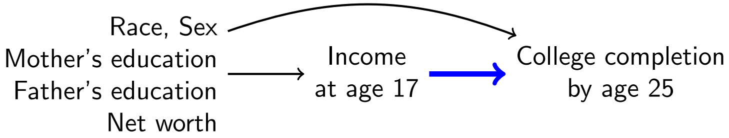

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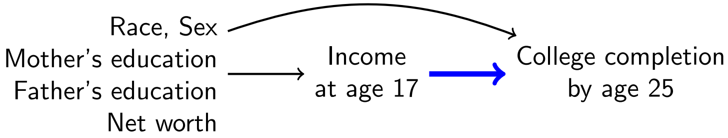

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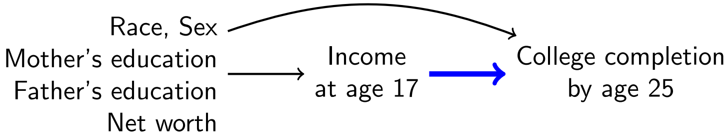

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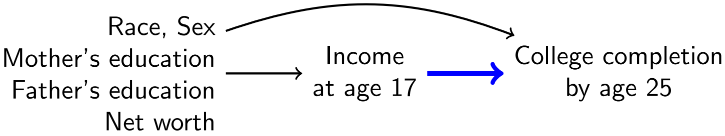

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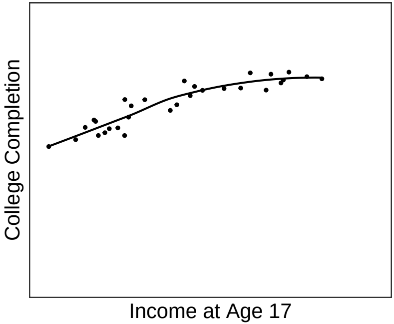

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Race, Sex
Mother's education
Father's education
Net worth

$\longrightarrow$ Income at age 17 $\longrightarrow$ College completion by age 25

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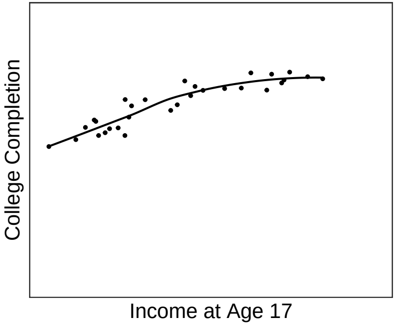

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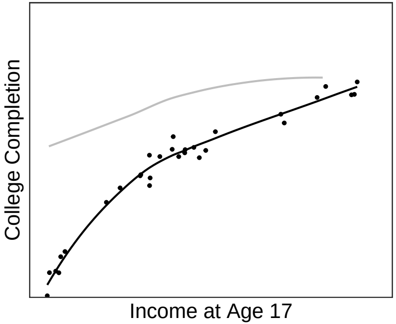

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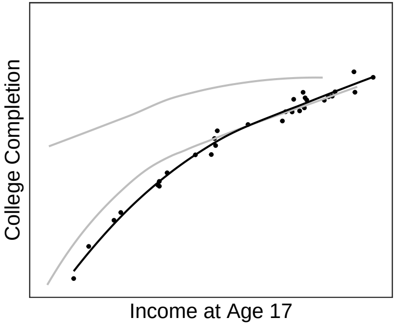

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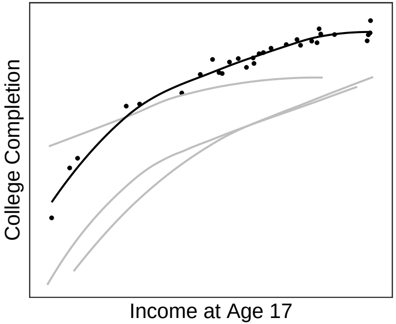

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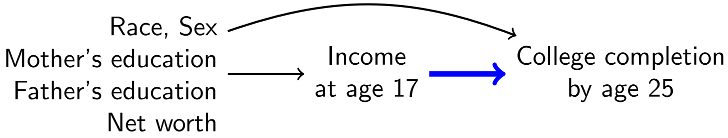

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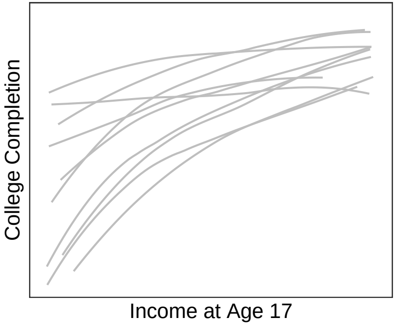

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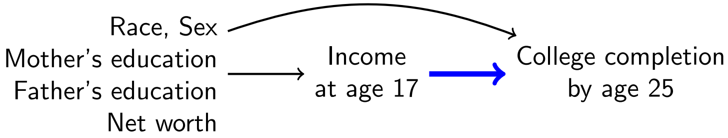

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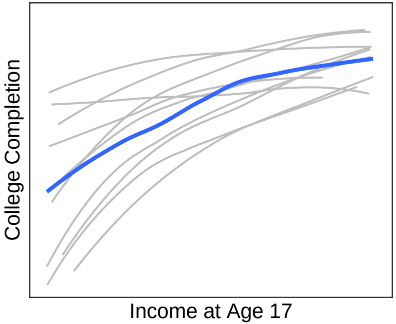

Dose-response curve

The previous slide introduced the **dose-response curve**

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Target trial:

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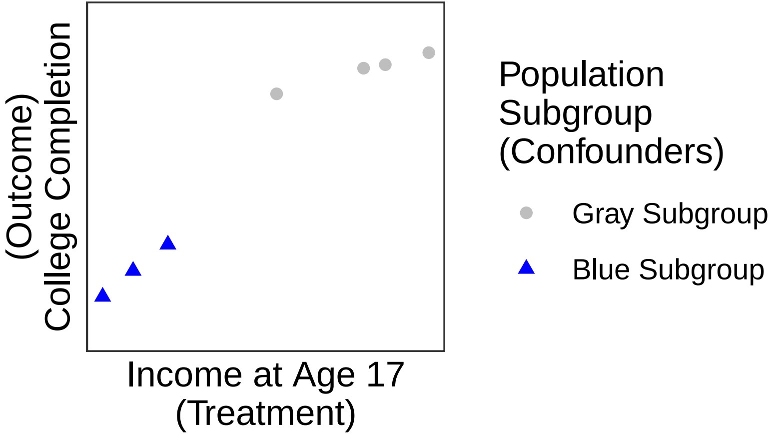
    - Model Y given A and $\vec{L}$
    - Set A to the value a
    - Make a prediction for everyone. Report the average

Warning: In strongly confounded settings, the dose-response curve is a dangerous goal. Here is why.

Sparse data can lead to **extrapolation**



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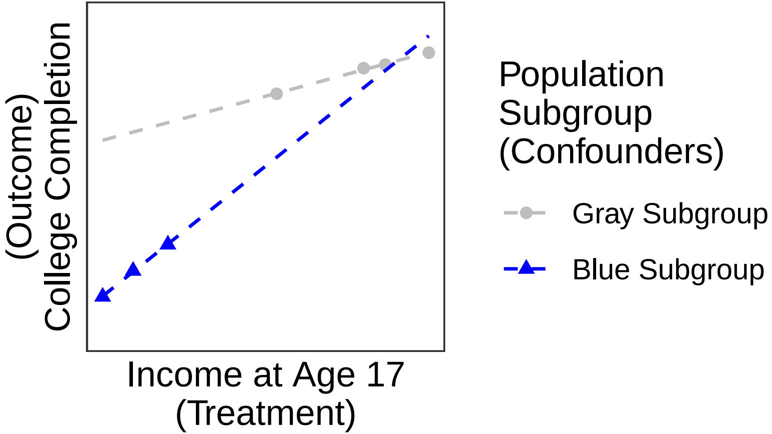

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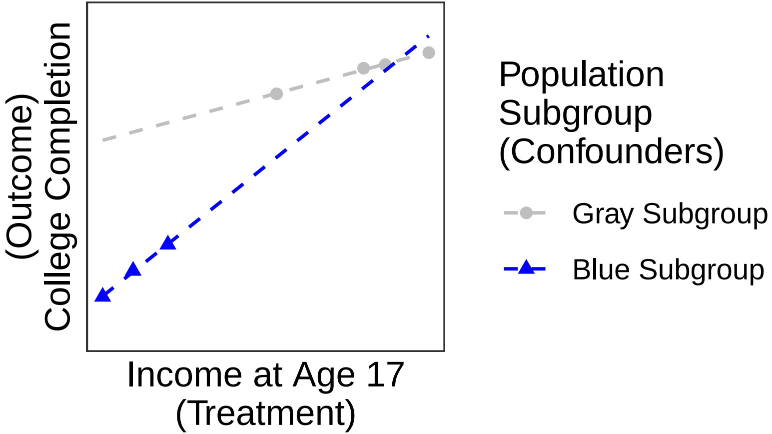
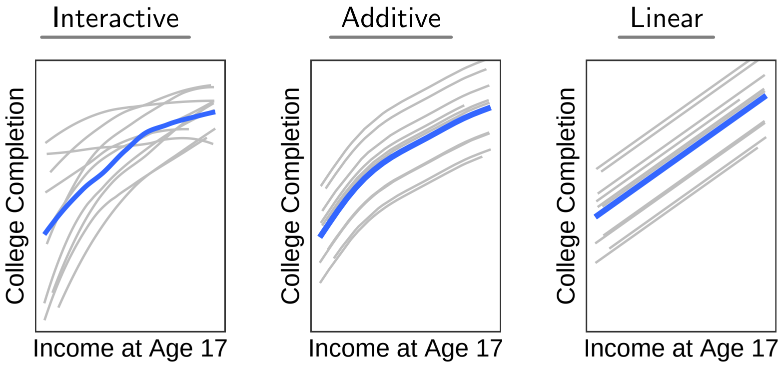


Question: What should one do?

Sparse data can lead to **extrapolation**: Three solutions

1. Extrapolate transparently: Additive model
2. Reduce extrapolation: Small shift
3. Avoid extrapolation: Incremental causal effects

1) Extrapolate transparently: Additive model



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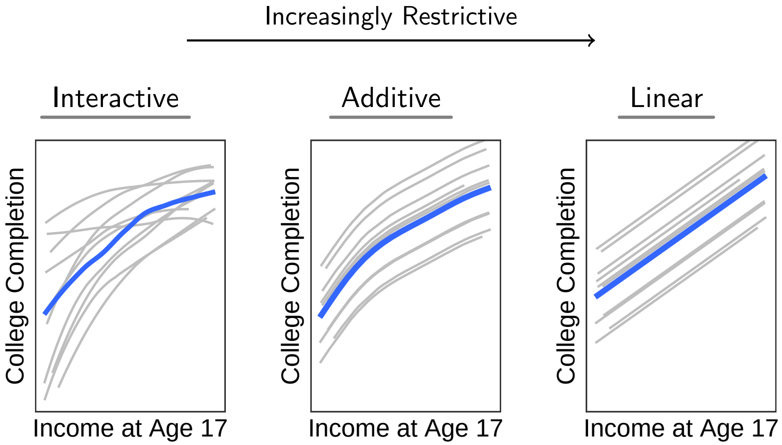

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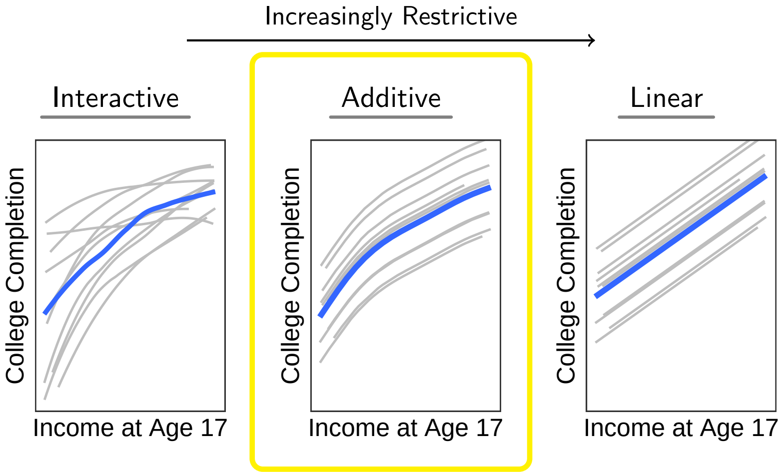

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- For each person, do not consider all treatment values a

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  - $A + \delta$ (additive shift)

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Example: The effect of adding 1: $\mathrm{E}(Y^{A+1} - Y^{A})$

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Example: The effect of adding 1: $\mathrm{E}(Y^{A+1} - Y^{A})$

Advantages:

- Only two potential outcomes for each person
- One of them is observed
- The other is not far away

3) Avoid extrapolation: Incremental causal effects[1]

[1] Rothenhäusler, D., & Yu, B. (2019). Incremental causal effects. arXiv preprint arXiv:1907.13258.

3) Avoid extrapolation: Incremental causal effects[1]

Take the limit of an extremely small additive shift:

$$\frac{\mathsf{E}\left(Y^{A+\delta} - Y^{A}\right)}{\delta}$$

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Take the limit of an extremely small additive shift:

$$\frac{\mathsf{E}\left(Y^{A+\delta} - Y^{A}\right)}{\delta}$$

This is the **local slope**
of the **local dose-response curve**,
averaged over the population
(see next slide).

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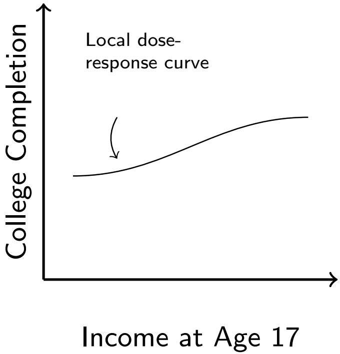
Among a subgroup with covariate values $\vec{L} = \vec{\ell}$



Rothenhausler & Yu 2019

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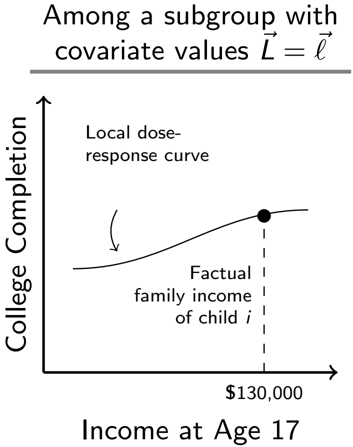

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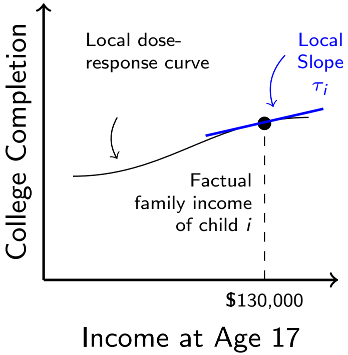

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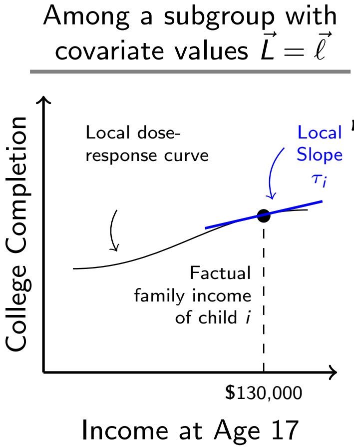

Report the
population
average
of these

Rothenhausler & Yu 2019

Example from a current working paper

[2]This estimation idea comes from: Friedberg, R., Tibshirani, J., Athey, S., & Wager, S. (2020). Local linear forests. Journal of Computational and Graphical Statistics, 30(2), 503-517.

Example from a current working paper

Locally linear estimation within coarsened subgroups[2]

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Locally linear estimation within coarsened subgroups[2]

1. Learn a random forest
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4. Coefficient on the treatment $\approx$ incremental causal effect

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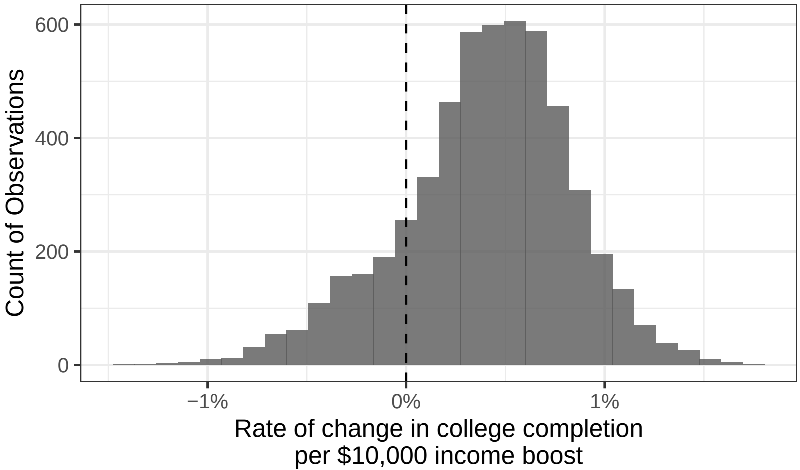
600
400
200
0
Count of Observations
−1%
0%
1%
Rate of change in college completion
per $10,000 income boost

Describing the covariates of those with

Small Effects	Big Effects
-0.1% to 0.3% increase in P(College) per $10,000 income boost	0.6% to 0.9% increase in P(College) per $10,000 income boost

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% Black	19%	34%

Describing the covariates of those with

	Small Effects -0.1% to 0.3% increase in P(College) per $10,000 income boost	Big Effects 0.6% to 0.9% increase in P(College) per $10,000 income boost
% Black	19%	34%
% Hispanic	18%	

Describing the covariates of those with

	Small Effects -0.1% to 0.3% increase in P(College) per $10,000 income boost	Big Effects 0.6% to 0.9% increase in P(College) per $10,000 income boost
% Black	19%	34%
% Hispanic	18%	25%

Describing the covariates of those with

	Small Effects -0.1% to 0.3% increase in P(College) per $10,000 income boost	Big Effects 0.6% to 0.9% increase in P(College) per $10,000 income boost
% Black	19%	34%
% Hispanic	18%	25%
% Father Absent	21%	

Describing the covariates of those with

	Small Effects -0.1% to 0.3% increase in P(College) per $10,000 income boost	Big Effects 0.6% to 0.9% increase in P(College) per $10,000 income boost
% Black	19%	34%
% Hispanic	18%	25%
% Father Absent	21%	50%

Describing the covariates of those with

	Small Effects -0.1% to 0.3% increase in P(College) per $10,000 income boost	Big Effects 0.6% to 0.9% increase in P(College) per $10,000 income boost
% Black	19%	34%
% Hispanic	18%	25%
% Father Absent	21%	50%
% Neither Parent Finished College	74%	

Describing the covariates of those with

	Small Effects -0.1% to 0.3% increase in P(College) per \$10,000 income boost	Big Effects 0.6% to 0.9% increase in P(College) per \$10,000 income boost
% Black	19%	34%
% Hispanic	18%	25%
% Father Absent	21%	50%
% Neither Parent Finished College	74%	90%

Describing the covariates of those with

	Small Effects -0.1% to 0.3% increase in P(College) per $10,000 income boost	Big Effects 0.6% to 0.9% increase in P(College) per $10,000 income boost
% Black	19%	34%
% Hispanic	18%	25%
% Father Absent	21%	50%
% Neither Parent Finished College	74%	90%
Median Income	$77,751	

Describing the covariates of those with

	Small Effects -0.1% to 0.3% increase in P(College) per $10,000 income boost	Big Effects 0.6% to 0.9% increase in P(College) per $10,000 income boost
% Black	19%	34%
% Hispanic	18%	25%
% Father Absent	21%	50%
% Neither Parent Finished College	74%	90%
Median Income	$77,751	$31,433

Continuous treatments: Concluding thoughts

There exists a common strategy:

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$$\begin{aligned}\text{Outcome} = \beta_0 + \beta_1 \times \text{Income} &+ \beta_2 \times (\text{Confounder 1}) \\ &+ \beta_3 \times (\text{Confounder 2}) \\ &+ \ldots + \epsilon\end{aligned}$$

Continuous treatments: Concluding thoughts

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But the effect actually

— varies across people **heterogeneity**

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Continuous treatments: Concluding thoughts

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But the effect actually

— varies across people **heterogeneity**

— varies across income values **nonlinearity**

Nonlinear and heterogeneous patterns
offer opportunities for new discoveries

Learning goals for today

At the end of class, you will be able to:

1. Define causal effects with continuous treatments
2. Understand the risks of extrapolation
3. Select causal estimands for continuous treatments that support the production of credible estimates

Let me know what you are thinking

tinyurl.com/CausalQuestions

Office hours TTh 11am-12pm and at
calendly.com/ianlundberg/office-hours
Come say hi!

If time allows: Discuss

How would you define the estimand?
Pick one or invent your own example.

1. More time exercising causes lower blood pressure.
2. More time sleeping improves focus in class.
3. Reading more pages per day promotes vocabulary development.
4. Smoking more cigarettes causes higher risk of lung cancer.

Things to consider:

- What treatment values are compared?
- Over whom will you average those counterfactual outcomes?
- Are there dangers of extrapolation? How would you mitigate them?