1 Problem 2.16

Proposition 1.

$$(\exists x \geq \mathbf{0} : Ax = \mathbf{0}) \iff (y^T A \geq \mathbf{0} \implies y^T A = \mathbf{0}) \tag{1.0.1}$$

Introduce the lemma:

Lemma 1. if $x > \mathbf{0}$, and $y \geq \mathbf{0}$, then $\langle x, y \rangle \geq \mathbf{0}$.

Lemma 1. This is true because $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \geq \mathbf{0}$, we are just multiplying each of the inequality by a non-negative number and then sum then all up. □

Proposition 1. Proof of sufficiency $\implies$:

$$\text{choose } x \text{ s.t: } Ax = \mathbf{0},\ x > \mathbf{0} \tag{1.0.2}$$

$$y^T A \geq \mathbf{0} \wedge y^T Ax = \mathbf{0} \implies y^T A = \mathbf{0} \tag{1.0.3}$$

$y^T A \geq \mathbf{0}$ and by $y^T Ax = 0, x \geq 0$, we know that $y^T A = \mathbf{0}$, because you can't sum up positive number and still get zero.

Proof of neccessity $\impliedby$: we will use prove by contradiction, we assume that $y^T A \geq \mathbf{0}$ and $y^T A = \mathbf{0}$, and for contradiction we assume $\nexists x > \mathbf{0} : Ax = \mathbf{0}$.

$$y^T A = \mathbf{0} \implies y^T Ax = \mathbf{0} \quad \forall x \tag{1.0.4}$$

$$\exists x > \mathbf{0} : \underbrace{y^T A}_{\geq \mathbf{0}} \underbrace{x}_{x > 0} = \mathbf{0} \tag{1.0.5}$$

$$\text{Contradicts Lemma 1} \tag{1.0.6}$$

□

2 Problem 2.21

Proposition 2. If the polytope $P := \{A | Ax \leq b\} \neq \emptyset$, prove $x^+ : x^+ = \max\{c^T x | Ax \leq b\}$ is attained by an vertex $x^+ \in P$.

Here is the approach for this problem. A polytope is closed therefore the objective value is going to be bounded. Next, if supremum of the objective exists then there is a point inside of the closed polytope p that attains it.

To show that the pint x^+ is a vertex, we assume it's not, then we show that either we can wiggle it around to improve $\langle c, x \rangle$, or we can just wiggle it so it becomes an vertex in P eventually, hence it has to be a vertex.

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