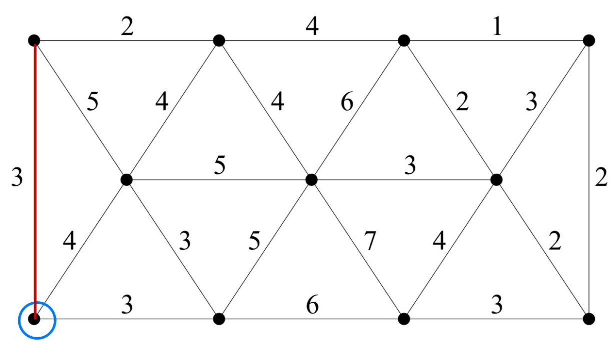
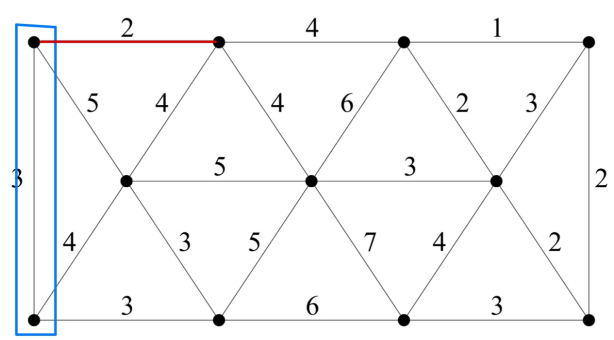
1 1.7

Below are my hand wrriten notes.

§1.7

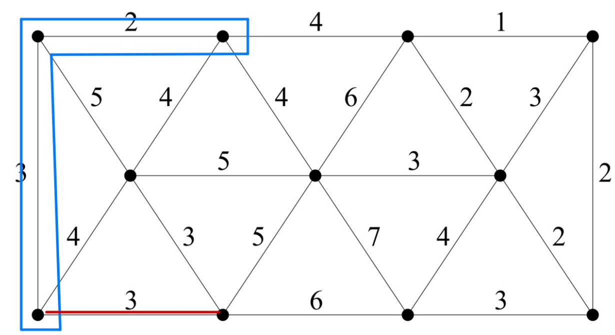
* Dijkstra Prim's Algorithm is like:





— : $\min\limits_{e \in \delta(T_i)} l(T_i)$

□ : T_i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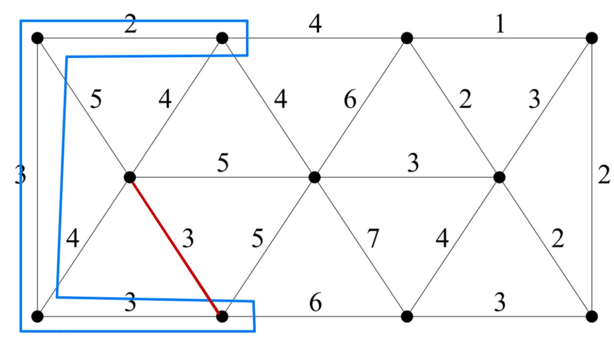
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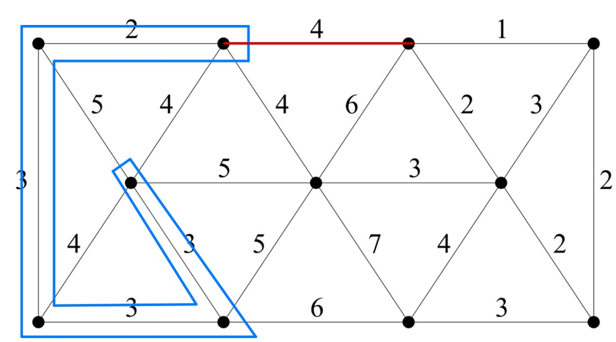
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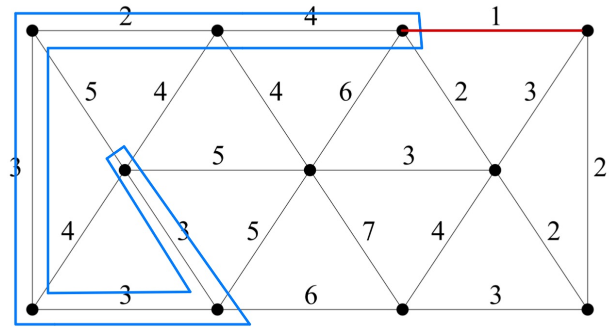
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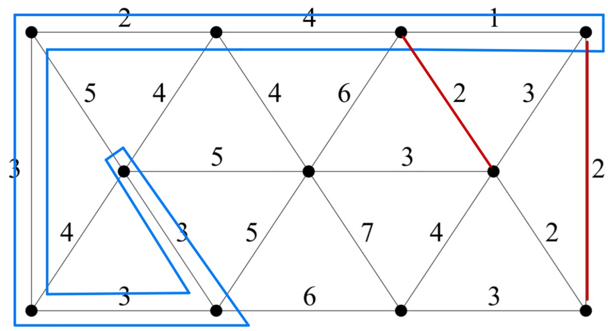
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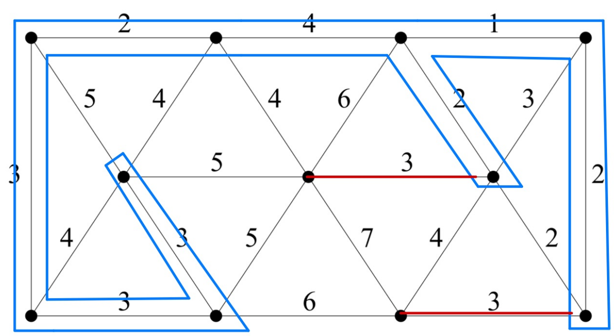
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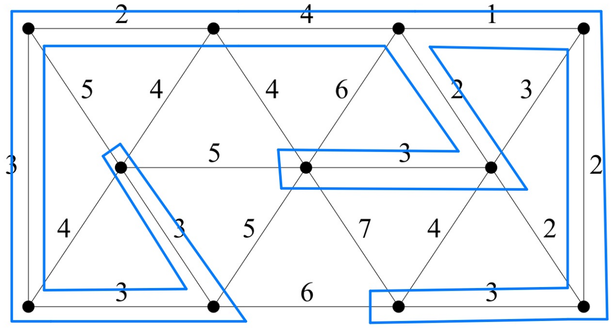


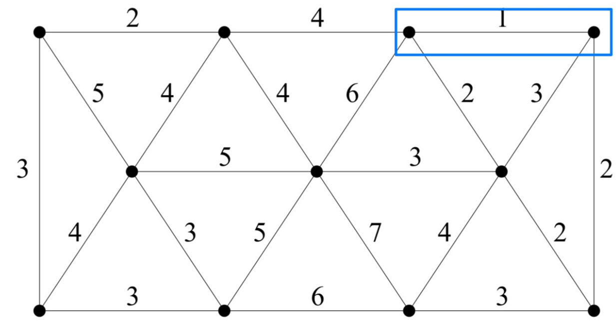
Done Score:

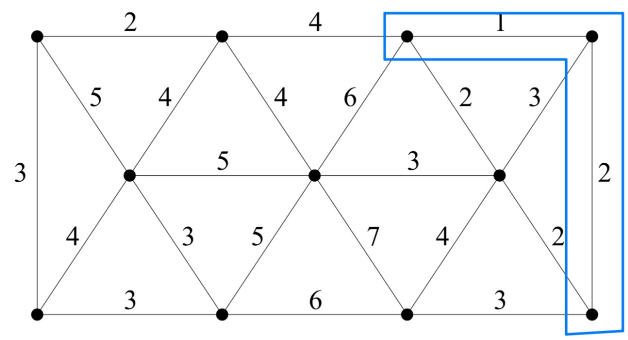
$1+2+2+2+3+3+3+3+3+4$

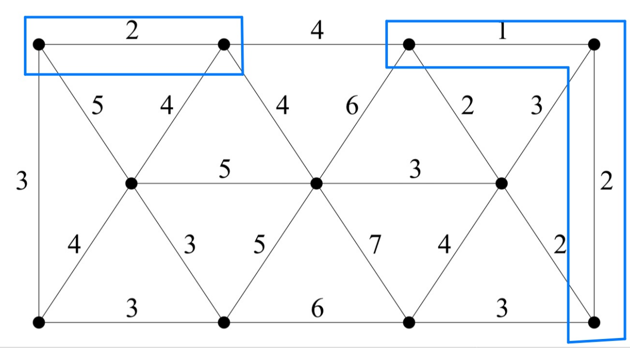
$1+6+15+4$

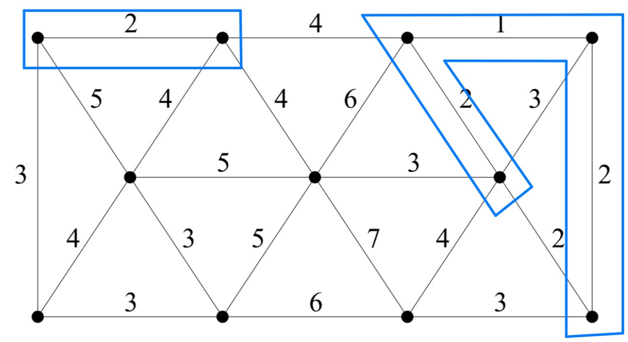
$=10+1+15=26$

*Kruskal Algorithm is like





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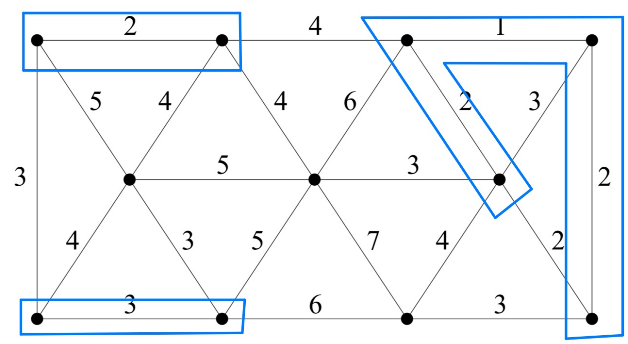
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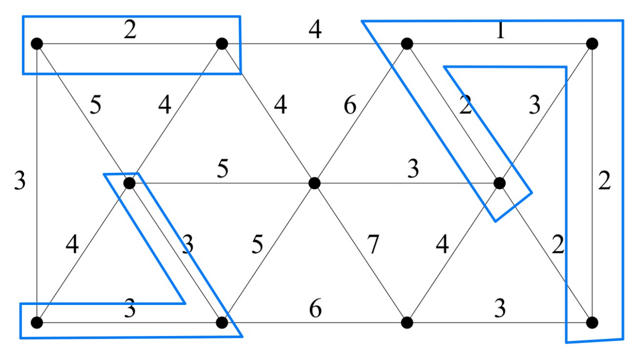
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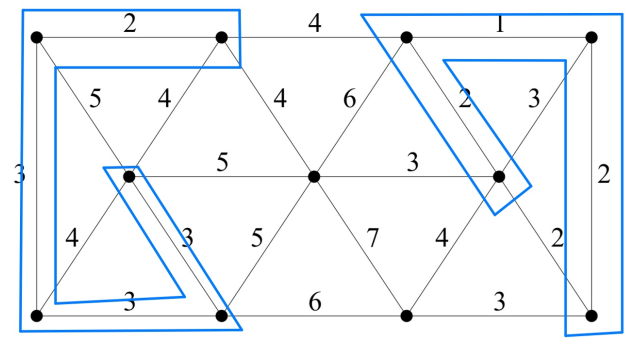
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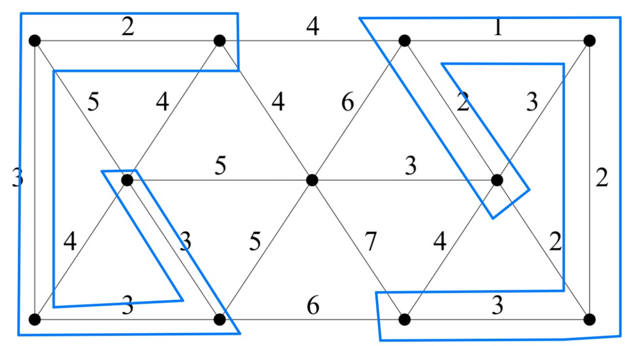
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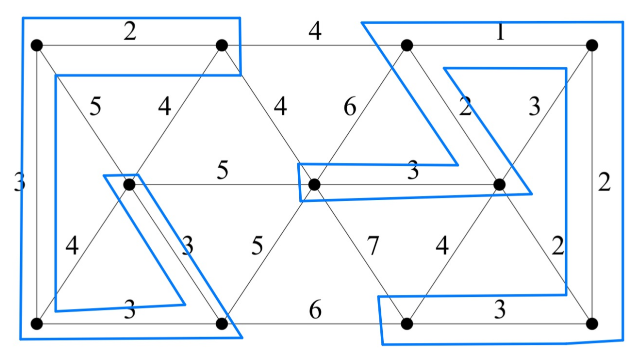
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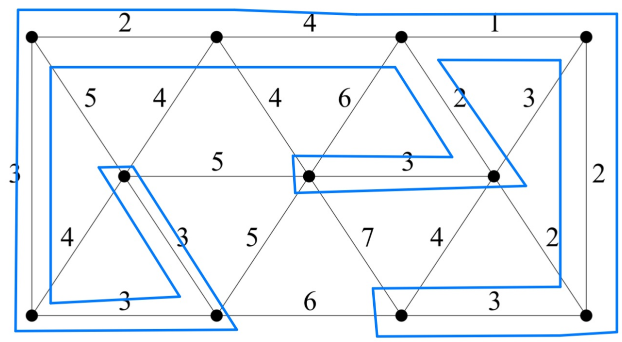


Done, score:

$$1+2+2+2+3+3+3+3+3+4$$
$$=1+6+15+4$$
$$=10+15+1=26$$

2 1.8

I wrote code to do the job (I tried by hands but it took me several hours and produced the wrong results I give up) and here is my Julia Code:

```julia
function Cities()
return ["ame", "ams","ape", "arn", "ass", "boz", "bre", "ein", "ens",
    "s-g", "gro", "Haa", "dh", "s-h", "hil", "lee", "maa", "mid", "nij", "roe",
    "rot,", "utr", "win", "zut","zwo"
] end

function GetNameListCrossProduct()
    NameList = Cities
    n = NameList |> length
    N = fill(("", ""), n, n)
    for i = 1:n, j = 1:n
        N[i, j] = (NameList[i], NameList[j])
    end
return N end

function GetDistances()
    M =
    [0.0 47 47 46 139 123 86 111 114 81 164 67 126 73 18 147 190 176 63 141 78 20 109 65 70;
    47 0 89 92 162 134 100 125 156 57 184 20 79 87 30 132 207 175 109 168 77 40 151 107 103;
    47 89 0 25 108 167 130 103 71 128 133 109 154 88 65 129 176 222 42 127 125 67 66 22 41;
    46 92 25 0 132 145 108 78 85 116 157 112 171 63 64 154 151 200 17 102 113 59 64 31 66;
    139 162 108 132 0 262 225 210 110 214 25 182 149 195 156 68 283 315 149 234 217 159 143 108 69;
    123 134 167 145 262 0 37 94 230 83 287 124 197 82 119 265 183 59 128 144 57 103 209 176 193;
    86 100 130 108 225 37 0 57 193 75 250 111 179 45 82 228 147 96 91 107 49 66 172 139 156;
    111 125 103 78 210 94 57 0 163 127 235 141 204 38 107 232 100 153 61 50 101 91 142 109 144;
    114 156 71 85 110 230 193 163 0 195 135 176 215 148 132 155 236 285 102 187 192 134 40 54 71;
    81 57 128 116 214 83 75 127 195 0 236 41 114 104 72 182 162 124 133 177 26 61 180 146 151;
    164 184 133 157 25 287 250 235 135 236 0 199 147 220 178 58 308 340 174 259 242 184 168 133 94;
    67 20 109 112 182 124 111 141 176 41 199 0 73 103 49 141 203 165 129 184 67 56 171 127 123;
    126 79 154 171 149 197 179 204 215 114 147 73 0 166 109 89 276 238 188 247 140 119 220 176 144;
    73 87 88 63 195 82 45 38 148 104 220 103 166 0 69 215 123 141 46 81 79 53 127 94 129;
    18 30 65 64 156 119 82 107 132 72 178 49 109 69 0 146 192 172 81 150 74 16 127 83 88;
    147 132 129 154 68 265 228 232 155 182 58 141 89 215 146 0 306 306 171 256 208 162 183 139 91;
    190 207 176 151 283 183 147 100 236 162 308 203 276 123 192 305 0 242 135 50 188 176 213 182 217;
    176 175 222 200 315 59 96 153 285 124 340 165 238 141 172 306 242 0 187 203 98 156 264 231 246;
    63 109 42 17 149 128 91 61 102 133 174 129 188 46 81 171 135 187 0 85 111 76 81 48 83;
    141 168 127 102 234 144 107 50 187 177 259 184 247 81 150 256 50 203 85 0 151 134 166 133 168;
    78 77 125 113 217 57 49 101 192 26 242 67 140 79 74 208 188 98 111 151 0 58 177 143 148;
    20 40 67 59 159 103 66 91 134 61 184 56 119 53 16 162 176 156 76 134 58 0 123 85 90;
    109 151 66 64 143 209 172 142 40 180 168 171 220 127 127 183 213 264 81 166 177 123 0 44 92;
    65 107 22 31 108 176 139 109 54 146 133 127 176 94 83 139 182 231 48 133 143 85 44 0 48;
    70 103 41 66 69 193 156 144 71 151 94 123 144 129 88 91 217 246 83 168 148 90 92 48 0]
    M .+= diagm(fill(Inf, size(M, 1)))
return copy(M) end

using LinearAlgebra

function RunThis()
    v = [1]
    M = GetDistances()
    M[v, :] .= +Inf
    EdgeList = Vector{Tuple}()
    for _ in 1: 24
        e = argmin(M[:, v])
        push!(EdgeList, (e[1], v[e[2]]))
        M[e[1], :] .= +Inf
        push!(v, e[1])
    end

    C = Cities()
    CityPairs = Vector{Tuple}()
    for (II, III) in EdgeList
        push!(CityPairs, (C[II], C[III]))
    end

return EdgeList, CityPairs end

function ProduceResults()
    E, Pairs = RunThis()
    D = GetDistances()
    WeightSum = 0
    for (e, f) in zip(E, Pairs)
        println("$(e), $(D[e[1], e[2]]), $(f)")
        WeightSum += D[e[1], e[2]]
    end
    print("Total: $(WeightSum)")

end

ProduceResults()
```

The idea is to use Dijkstra Prim's Algorithm on the Adjacency Matrix. I set the diagonal of the Adjacency matrix to be zero. Each time I discover a new vertex, I remove a row by setting them all to infinity (So all the edges that points back to the discovered vertices are not gonna be visited again). Next I search for the smallest element in all the columns of the visited vertices. Then add a new column after a new edge that points to an unvisited vertex with the smallest weight is discovered. And then I get the vertex (index for the city) and then accumulate the edges. The results produced by the algorithm are:

```
(15, 1), 18.0, ("hil", "ame")
(22, 15), 16.0, ("utr", "hil")
(2, 15), 30.0, ("ams", "hil")
(12, 2), 20.0, ("Haa", "ams")
(10, 12), 41.0, ("s-g", "Haa")
(21, 10), 26.0, ("rot,", "s-g")
(4, 1), 46.0, ("arn", "ame")
(19, 4), 17.0, ("nij", "arn")
(3, 4), 25.0, ("ape", "arn")
(24, 3), 22.0, ("zut", "ape")
(25, 3), 41.0, ("zwo", "ape")
(23, 24), 44.0, ("win", "zut")
(9, 23), 40.0, ("ens", "win")
(14, 19), 46.0, ("s-h", "nij")
(8, 14), 38.0, ("ein", "s-h")
(7, 14), 45.0, ("bre", "s-h")
(6, 7), 37.0, ("boz", "bre")
(20, 8), 50.0, ("roe", "ein")
(17, 20), 50.0, ("maa", "roe")
(18, 6), 59.0, ("mid", "boz")
(5, 25), 69.0, ("ass", "zwo")
(11, 5), 25.0, ("gro", "ass")
(16, 11), 58.0, ("lee", "gro")
(13, 12), 73.0, ("dh", "Haa")
Total: 936.0
```

3 Notes:

Expect verbose proofs and illustrations because I lack of the mathematical notations for graph theory related stuff.

4 HW: 1.9

For notations:

$$\delta(F) := \{e \in E(F) : e = \{u, v\} \not\exists \text{ Path between} u, v \text{ in F}\} \tag{4.0.1}$$

$$\gamma(F) := \{e \in E(F) : e = \{u, v\} \exists \text{ Path between} u, v \text{ in F}\} \tag{4.0.2}$$

When I said edge $e \in F$, I am saying this as in picture, not as if F is a set, because graph is not a set. Similiar logic applies to other mathematical entities like Path and tree and stuff.

4.1 Forest Exchange Edges

Lemma 1 (Forest Edge Exchange)**.** Let $F' \neq F^*$ be 2 forests on the same graph, then we can exchange some edges between them. We can do it for every forets that contain some edges.

Proof. $\forall e^* \in F^* \setminus F'$, there are 2 cases:

1. if $e \in \delta(F')$, then $\exists e' \in F' : F' \setminus \{e'\} \cup \{e^*\}$ is still a forest. Because $F' \cup \{e^*\}$ is a forest since no cycle is created; and removing e' is keeps the forest a forest.

2. if $e^* \in \gamma(F')$, then adding e^* creates a cycle created in F'. Choose any $e' \neq e^*$ on the cycle, then $F' \setminus \{e'\} \cup \{e^*\}$ is still a forest.

□

As a remark, we can do this until $F^* \setminus F'$ becomes empty, meaning that $F^* \subseteq F'$, or F^* becomes empty.

4.2 Minimal Forest Edge Exchange

Definition 1 (Good Forest)**.** Forest F' of $G = (V, E)$ associated with $l : V \mapsto \mathbb{R}$ edge length function. If input is a sub graph, the the function sum up the length for each edges in th subgraph. Then, forest F' is good if and only if for all F $|F'| = |F| \implies l(F') \leq l(F)$.

Proposition 1 (Good Forest Best Edge Accumulates)**.** Let F' be a good forest, and $e_{\min}$ to be the edge with minimal length such that $F' \cup \{e_{\min}\}$ is still a forest, then it's has to be a bigger good forest.

Proof. consider F' be good and choose $e_{\min} \in \arg\min\{l(e) : e \in \delta(F')\}$. Consider ANY F^* such that $|F^*| = |F'| + 1$. Define $F = F' \cup \{e_{\min}\}$.

Choose any $e^* \in F^* \setminus F$, at least one such an edge exists by definition. Then there are 2 cases:

1) $e^* \in \delta(F')$, then $l(e^*) \geq l(e')\ \forall e' \in F'$. For contradiction if this is not true, then $\exists e' \in F'$ with $l(e') > l(e^*)$, then perform exchange $F' \setminus \{e'\} \cup \{e^*\}$, giving us a forest with the same cardinality as F' but with less length. Contradicting F' is good. Therefore let me reiterate:

$$e^* \in \delta(F') \implies l(e') \leq l(e_{min}) \leq l(e^*) \tag{4.2.1}$$

$e_{\min}$ is by definition is an element that is smallest in $\delta(F')$. As a results, any exchange between F' and e^* will not make $l(F' \cup \{e^*\} \setminus \{e'\}) \geq l(F')$ because:

$$l(F' \cup \{e^*\} \setminus \{e'\}) = l(F') + \underbrace{l(e^*) - l(e')}_{\geq 0} \geq l(F') \tag{4.2.2}$$

$$l(F' \cup \{e^*\} \setminus \{e'\} \cup \{e_{\min}\}) \geq l(F' \cup \{e_{\min}\}) \tag{4.2.3}$$

$$l(F \cup \{e^*\} \setminus \{e'\}\}) \geq l(F' \cup \{e_{\min}\}) \tag{4.2.4}$$

So, we can exchange e^* with e' already in F', but it won't improve F.

2) $e^* \in \gamma(F')$, then e^* creates a cycle in the forest F', then it's impossible that there exists $e' \in F'$ as part of cycle created by e^* such that $l(e') > l(e^*)$. If this happened, simply exchange and get $F' \setminus \{e'\} \cup \{e^*\}$ to get a forest with the same cardinality with F', contradicting F' is good. Therefore, let $P(T', e^*)$ be the path connecting the endpoints of e^* in T' then:

$$e^* \in \gamma(F') \implies \forall e' \in P(T', e^*) : l(e^*) \geq l(e') \tag{4.2.5}$$

Therefore, $F \cup \{e^*\} \setminus \{e'\}$ won't improve the forest F, everything that can be exchanged can't make it better.

If we keep replacing e^* that is from $F^* \setminus F$, doing exchange with some $e' \in F'$ or $e_{\min} \in \delta(F')$, and then redefining F' if it's in the first case, and then redefined F if it's the second case; keep doing this until $F = F^*$, then regardless of what we do, $l(F^*)$ only incrases compare to F, therefore $F := F' \cup \{e_{\min}\}$ must be good too because $|F| = |F'| + 1$ and that is the definition of good forest. $\square$

5 1.11

Skipping the problem statement, Firstly Notice that:

$$r_G(u,v) \geq r_T(u,v) \tag{5.0.1}$$

Because all path in the Max Reliability tree T is a path in G, all path from T is a subset of all paths in G. threfore the Reliability between 2 vertices in G is always better than T
Next, we wish to prove the statement that:

$$r(P_G(u,v)) \leq r_T(u,v) \tag{5.0.2}$$

If there exists a path from u to v in G such that the reliability is better, then I can find a better reliability tree that improves the strength and has the better reliability path between u, v. WLOG suppose that:

$$r(P_G^+(v_0,v_n)) > r_T(v_0,v_n) = r(P_T(v_0,v_n)) \tag{5.0.3}$$
$$P_G^+(v_0,v_n) = v_0, e_1^+ v_1, e_2^+, v_2, \cdots, e_n^+, v_n \tag{5.0.4}$$
$$P_T(v_0,v_n) = v_0, e_1 v_1, e_2, v_2, \cdots, e_n, v_n \tag{5.0.5}$$

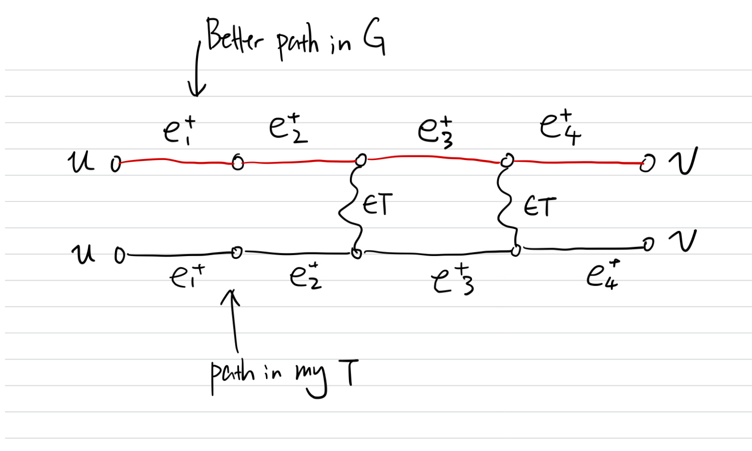
If reliability of $P_G^+(v_0,v_n)$ is better than $P_T(v_0,v_n)$, then the minimum strength edge in $P_G^+(v_0,v_n)$ is better than the minimum strength edge $P_T(v_0,v_n)$. again WLOG let it be the case that:

$$\{v_{j-1}^+, v_j^+\} = e_j^+ \in \arg\min\{l(e) : e \in P_G^+(v_0,v_n)\} \tag{5.0.6}$$
$$\{v_{i-1}, v_i\} = e_i \in \arg\min\{l(e) : e \in P_T(v_0,v_n)\} \tag{5.0.7}$$
$$l(e_j^+) \geq l(e_i) \tag{5.0.8}$$

Then we can improve T by includeing e_j^+ trading off e_i on the way. By the definition that T is a tree, then there exists a unique path between v_{i-1}, v_{j-1}^+ and v_i^+, v_j^+, therefore the inclusion of e_j^+ creates the cycle:

$$P_T(v_{i-1}, v_{j-1}^+), e_j^+, P_T(v_j^+, v_i), e_i, v_{i-1} \tag{5.0.9}$$

For an simple illustration with concrete numbers for the concept:



Both e_j^+ and e_i are in the cycle after the inclusion, all other edges already exists in the tree. Therefore: $T^+ := T \setminus \{e_i\} \cup \{e_j^+\}$ will improve $r_{T^+}(u, v)$ and T^+ is still a tree. Worth noting is that, the total weights of all the egdes in T^+ is larger: $l(T^+) > l(T)$ because $l(e_j^+) > l(e_i)$. And the path now has the best reliability giving us: $r_G(v_0, v_n) = r_{T^+}(v_0, v_n)$.
Repeat this process, redefining $T := T^+$, do this for all paths in G that has better reliability, then the weight of the tree T^+ improves monotonically. There is an upper bound to the total weight of the edges of the tree because E is finite, therefore it has to be the case that there exists T that has the best reliability, and when that happens, it includes all the best path in G, giving us $r_G(u, v) = r_T^+(u, v)\ \forall u, v \in V$.