1 Problem 2.16

Proposition 1.1.

$$(\exists x > \mathbf{0} : Ax = \mathbf{0}) \iff (y^T A \geq \mathbf{0} \implies y^T A = \mathbf{0}) \tag{1.0.1}$$

Proposition 1. Proof in direction $\implies$:

$$\text{choose } x \text{ s.t: } Ax = \mathbf{0},\ x > \mathbf{0} \tag{1.0.2}$$

$$y^T A \geq \mathbf{0} \wedge y^T Ax = \mathbf{0} \implies y^T A = \mathbf{0} \tag{1.0.3}$$

$y^T A \geq \mathbf{0}$ and by $y^T Ax = 0, x > \mathbf{0}$, we know that $y^T A = \mathbf{0}$, because you can't sum up positive number and still get zero. Geometrically it's saying that if $\mathbf{0}$ is in the affine hull of the cone made by the columns of A, and, there is a vector y such that every columns of A falls to one side of y, then it must be the case that the columns of A are on the hyperplane perpendicular to y.
For the proof in the direction $\impliedby$ we consider a direct proof starting with:

$$(y^T A \geq \mathbf{0} \implies y^T A = \mathbf{0}) \implies (\exists x \geq \mathbf{0} : Ax = \mathbf{0}) \tag{1.0.4}$$

$$\neg(\exists y^T A \geq 0 \wedge y^T A \neq \mathbf{0}) \implies (\exists x \geq \mathbf{0} : Ax = \mathbf{0}) \tag{1.0.5}$$

Take note that if $y^T A \geq \mathbf{0} \wedge y^T A \neq \mathbf{0}$, assuming it's true then it can be simplified into $y^T A \geq \mathbf{0} \wedge y^T A > \mathbf{0}$. This is saying that if there doesn't exist any hyperplane defined by y such that it separates the columns of y onto the positive side and no columns are on the hyperplane, then they have to be all lay on the strict positive side of y. Then we can consider the statement for each of the column a_i of A making it looks like one of the cases of Ferkas's Lemma:

$$\forall i \in [n] : \neg(\exists y^T A \geq \mathbf{0} \wedge y^T a_i > 0) \tag{1.0.6}$$

$$\implies \forall i \in [n] : \exists z_i \geq \mathbf{0} : Az_i = -a_i \tag{1.0.7}$$

In this case, we are setting the vector $b = -a_i$ and then applying the ferkas's lemma, attempting to crate a cone usin z_i scaling columns of A to include $-a_i$. Which then implies the fact that:

$$A\left(\sum_{j=1}^{n} z_i\right) = -A\mathbf{1} \tag{1.0.8}$$

$$A\left(\underbrace{\sum_{j=1}^{n} z_i + e_i}_{>\mathbf{0}}\right) = \mathbf{0} \tag{1.0.9}$$

Here,we summed up the results from (1.0.7) and then move then to one side of the equation, then the quantity multipled by matrix A is strictly positive because $z_i \geq \mathbf{0}$ at and the sum of e_i are all strictly positive.

□

2 Problem 2.21

Proposition 2.1. If the polytope $P := \{A|Ax \leq b\} \neq \emptyset$, prove $x^+ : x^+ = \max\{c^T x | Ax \leq b\}$ is attained by an vertex $x^+ \in P$.

Here is the approach for this problem. A polytope is closed therefore the objective value is going to be bounded. Next, if supremum of the objective exists then there is a point inside of the closed polytope p that attains it.

To show that the pint x^+ is a vertex, we assume it's not, then we show that either we can wiggle it around to improve $\langle c, x\rangle$, or we can just wiggle it so it becomes an vertex in P eventually, hence it has to be a vertex.

Proof. Let the set $\mathcal{I}$ be the indices for all the tight constraints for the given point $x^+ \in P$, then $\mathcal{I} := \{i \in [m] : a_i^T x^+ = b_i\}$ where a_i is a vector denoting the ith row of matrix A, and $|\mathcal{I}| \leq n$. Next we consider the wiggle vector $w \in \mathbb{R}^n$ such that $w \neq \mathbf{0}$ for x^+. By the definition that x^+ is not a vertex, we know that the sub matrix $A_{\mathcal{I},:}$ whose rows are indexed by $\mathcal{I}$ has $\text{rank}(A_{\mathcal{I},:}) \leq n$. Therefore:

$$
\begin{aligned}
&\exists w \neq \mathbf{0} : (A_{\mathcal{I},:})w = \mathbf{0} && (2.0.1)\\
&\implies \forall j \in [m] \setminus \mathcal{I} : a_j^T x^+ < b_j && (2.0.2)
\end{aligned}
$$

The inequality is loose, therefore we can try inserting the quantity $\alpha_j a_j^T w$ into the inequality like:

$$
\begin{aligned}
&\exists\, \alpha_j > 0 : a_j^T x^+ \pm \alpha_j a_j^T w \leq b_j && (2.0.3)\\
&\implies \pm \alpha_j a_j^T w \leq b_j - a_j^T x^+ && (2.0.4)\\
&\implies -(b_j - a_j^T x^+) \leq \alpha_j a_j^T w \leq b_j - a_j^T x^+ && (2.0.5)\\
&\implies \alpha_j |a_j^T w| \leq |b_j - a_j^T x^+| && (2.0.6)\\
&\implies \alpha_j \leq \left| \frac{b_j - a_j^T x^+}{a_j^T w} \right| && (2.0.7)
\end{aligned}
$$

Now, we can choose the minimal α_j to determine how much wiggle room we have for x^+ such that it stills remains in P, name that variable α and it would be given as:

$$
\begin{aligned}
&\alpha := \min_{j \in [m] \setminus \mathcal{I}} \left\{ \left| \frac{b_j - a_j}{a_j^T w} \right| \right\} && (2.0.8)\\
&\implies A(x^+ \pm \alpha w) \leq b \implies x^+ \pm \alpha w \in P && (2.0.9)
\end{aligned}
$$

Next, we may consider the objective value achieve by this wiggled point:

$$
\begin{aligned}
&\langle c, x^+ \pm \alpha w\rangle && (2.0.10)\\
&\langle c, \alpha w\rangle \neq 0 && (2.0.11)\\
&\implies (\langle c, x^+ + \alpha w\rangle > \langle c, x^+\rangle) \vee (\langle c, x^+ - \alpha w\rangle > \langle c, x^+\rangle) && (2.0.12)
\end{aligned}
$$

Therefore, in this case, we x^+ cannot be an optimal yet, which is a contradiction. Otherwise, it has to be the case that $\langle c, \alpha w\rangle = 0$, which tells use that we can make a new point x^{++} be either $x^+ + \alpha w$ or $x^+ - \alpha w$, and then we will have one more tight constraint for the set $\mathcal{I}$. More specifically, choose:

$$\forall j^+ \in \arg\min_{j\in[m]\setminus\mathcal{I}} \left\{\left|\frac{b_j - a_j}{a_j^T w}\right|\right\} \tag{2.0.13}$$

$$\implies (\alpha_{j^+}x^+ + \alpha_{j^+}a_{j^+}^T w = b_i) \vee (\alpha_{j^+}x^+ - \alpha_{j^+}a_{j^+}^T w = b_i) \tag{2.0.14}$$

Choose the right sign for α_{j^+} such that it makes the constraints j^+ tight, then we have one or more more tight constraints for our point x^+, which means that: $A_{\mathcal{I}\cup\{j^+\},:}$ must have a higher rank than $A_{\mathcal{I},:}$. To this regard, we can repeat this process by redefining $x^+ = x^{++}, \mathcal{I} := \mathcal{I} \cup \{j^+\}$, and repeat the proof. Eventually, we will have x^+ becoming a vartex because $\text{rank}(A_{\mathcal{I},:}) = n$ eventually x^+ will be a vertex in P. This is true because polytope is closed, it's impossible that x^+ gives us some constraints that is unbounded.

□

2.26

Find a case where $\{x|Ax \geq b\}$ and $\{y|y \geq \mathbf{0} : y^T A = c^T\}$ are empty.

Consider 2d where we have x_1, x_2 as coordinates for primal and y_1, y_2 for dual:

$$\underbrace{\begin{bmatrix}-1 & 0\\ 1 & 0\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \leq \begin{bmatrix}-1\\ -1\end{bmatrix} \tag{2.0.15}$$

$$c = \begin{bmatrix}0\\ 1\end{bmatrix} \tag{2.0.16}$$

Observe that c is in the null space of A, therefore it's impossible that $y^T A = c^T$, at the same time the polytope is empty too.

2.27

Proposition 2.2. define the primal and dual problem pairs for LP to be the following:

$$\max\{\langle c, x\rangle : Ax \leq b\} \leq \min\{y^T b : y^T A = c^T, y \geq \mathbf{0}\} \tag{2.0.17}$$

Both x, y are optimal iff $\forall i \in [m] : y_i = 0 \implies a_i^T x = b_i$. The weights on the loose constraints will have to be zero.

Proof.

$$Ax \leq b \tag{2.0.18}$$

$$\implies Ax + s = b \quad s \geq \mathbf{0} \tag{2.0.19}$$

$$y^T(Ax + s) = y^T b \tag{2.0.20}$$

$$\underbrace{y^T A}_{=c^T} x + \underbrace{y^T s}_{\geq \mathbf{0}} = y^T b \tag{2.0.21}$$

$$c^T x + y^T s = y^T b \tag{2.0.22}$$

Take note that $y^T b = c^T x$ by strong duality (we proved that part in class), therefore $y^T s = 0$. However, take note that whenever index j are of a loose constraints for the optimal $x \in P$, $a_j^T x \leq b_j \implies s_j > 0$, by the fact that $y \geq \mathbf{0}, s \geq \mathbf{0}$, it must be the case that $y_j = 0$ for all such j. Proof done. $\square$