Notations

1. $P_G(u,v)$ a path, which is a list of vertices, or edges, or both, that starts with the vertex u and ends with vertex v in the graph G.
2. $\mathrm{cc}(v)$ Denotes the connected component, is the set of all rechable vertices from a vertex V in G. G can be directed or undirected. It can also be applied to a set of vertices: S, which is just $\mathrm{cc}(S) := \bigcup_{v\in S}\mathrm{cc}(v)$

1 Problem 3.19

Proposition 1.1 (Minimal Bipartite Vertex Cover from Maximum Matching)**.** Given a maximum matching on bipartite graph: $G = (U\dot{\cup}V, E)$ let M^+ be a matching of maximum size.

Suppose that solution of a maximum is given after the execution of the matching algorithm and $e \in M$ goes from U to V, and $\notin M$ goes from V to U. To get the minimum vertex cover:

We choose every reachable vertices from L that is in V (Name that set S). Which are going to be covered by M. For the remaining vertices that is covered by M and not sharing the same edge in the matching with vertices in S, choose then as well, and they will form a vertex cover F with $|F| = |M|$.

Define the sets and directed edges in the following way:

$$M :: \text{The maximum Matching!} \tag{1.0.1}$$

$$L := \left\{ v \in U : v \notin \bigcup_{e\in M} e \right\} \tag{1.0.2}$$

$$S := \mathrm{cc}(L) \cap V \tag{1.0.3}$$

$$e \in M, e = (v_1, v_2) \implies v_1 \in V, v_2 \in U \tag{1.0.4}$$

$$e \notin M, e = (v_1, v_2) \implies v_1 \in U, v_2 \in V \tag{1.0.5}$$

(1.0.2) : L is the set of vertices in U that are not covered by the matching.

(1.0.3) : S is the set of reachable vertices from all vertices in L.

(1.0.4) : An edge in matching goes from V to U.

(1.0.5) : an edge not in matching goes from U to V.

Lemma 1.0.1 (Lemma 2)**.** It's impossible to have a path going from L to S to $U \setminus L$ to $V \setminus S$.

Proof. This is true because S by definition is set of all vertices reachable from L in V. And if we reached some vertices in $V \setminus S$, then it's not in S, which violate the definition of S. □

Lemma 1.0.2 (Lemma 3)**.** All vertices in S are covered by M.

Proof. If not, there exists a path going from $u \in L$ to $v \in S$ such that v not covered by M, since u not covered by M by definition of L; an augmented path is found, therefore M is not maximum. $\square$

Lemma 1.0.3 (Lemma 4)**.** No edges, in any directions exists between the set $V \setminus S$, L.

Proof. For contradiction, suppose there is such an edge and denote that edge as e^+. Then the contradiction is:

$$e^+ \notin M \wedge e^+ \in M \tag{1.0.6}$$

Because $V \setminus S$ is the set of vertices in V that can't be reached by L, therefore there are no direct edges going from $L \subseteq U$ to $(V \setminus S) \subseteq V$, therefore, $e^+ \notin M$; which also means e^+ will go from $(V \setminus S) \subseteq V$ to $L \subseteq U$, therefore $e^+ \in M$. Which is impossible because by definition L is not covered by M. $\square$

Proposition 1.1. Let $\overline{F} := U \setminus L$. The claim is the I can keep the $|\overline{F}|$ fixed and exchange vertices to make this into a vertex cover.

If $L = \emptyset$, then $\overline{F}$ is a vertex cover because $\overline{F} = U$. Using the fact that G is bipartite, $\overline{F}$ covers all edges. And that means M covers all U because $L = \emptyset$; implying $|\overline{F}| = |M|$

If $L \neq \emptyset$, then for all $e \in E, e = \{u, v\}$ (direction doesn't matter). Then there are 3 cases:

(1.) e goes from $u \in L$ to $v \in S$, let $e = (u, v)$. $e \notin M$ because $u \in L$ by def of L, u not covered by M. However, v is covered by M because $v \in S$ and we use lemma 1.0.2. Therefore $\exists! u' \in U \setminus L : \{u', v\} \in M$.

I can then construct $\overline{F} := (F \setminus \{u'\}) \cup \{v\}$ to be a minimum vertex cover, without losing edges. u' can be removed from $\overline{F}$ by lemma 1.0.1. To convince you further, assuing it's not the case, suppose that removing u' expose an edge $e' = (u', v')$ that I am unbale to cover. Observe that v' must be in $V \setminus S$ because S are already all covered by M. Then the path is possible:

$$u \to v \to u' \to v' \tag{1.0.7}$$
$$u \in L \tag{1.0.8}$$
$$v \in S \tag{1.0.9}$$
$$u' \in U \setminus L \tag{1.0.10}$$
$$v' \in V \setminus S \tag{1.0.11}$$

Which contradicts lemma 1.0.1. Therefore $(F \setminus \{u'\}) \cup \{v\}$ now covers the additional edge: e without exposing any other edges.

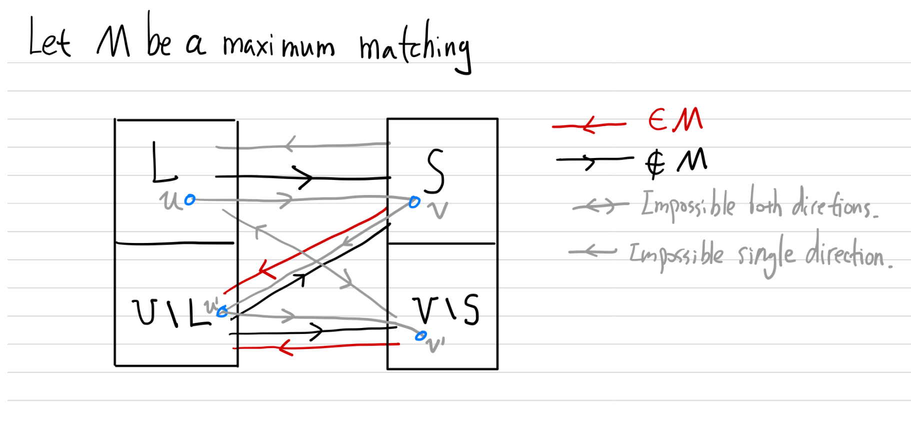
(2.) $e = \{u, v\}$, direction doesn't matter, it goes between S and $U \setminus L$. Then $\overline{F}$ covers the edge: e because $v \in U \setminus L$, and $\overline{F} = U \setminus L$ at the start, and in case (1.), we move vertices to S, therefore, such an edge is always gonna be covered by $\overline{F}$.

(3.) e goes between $U \setminus L$ and $V \setminus S$. This is covered by $\overline{F}$ because $U \setminus L$ originally covers all edges incident to $U \setminus L$, and in case (1.) above, when we remove u', we never expose any edges going between $U \setminus L$ and $V \setminus S$.

(4.) e goes between the set L and $V \setminus S$. This is impossible by 1.0.3.

For all cases, I can re-arrange $\overline{F}$ such that its cardinality remains unchanged and all the edges are covered. I started with $|\overline{F}| = |M|$, therefore, we have a vertex cover $|\overline{F}| = |M|$ in the end.

HEEEEEEY! Here is picture to get my point across fig: 1: □



2 Problem 3.25

Theorem 1. Let $G = (V, E)$ be a bipartite graph. Then the perfect matching polytope P is equal to the set of verctors $x \in \mathbb{R}^{|E|}$ satisfying:

$$Q := \begin{cases} x_e \geq 0 \quad \forall e \in E \\ \sum_{e \ni v} x_e = 1 \quad \forall v \in V \end{cases} \tag{2.0.1}$$

Here let $G = (V \dot\cup U, E)$ be a bipartite graph and we denote P as the polytope that is the convex hull of all possible perfect matching solution vectors.

Assuming that $|V| = |U| = n$ so that perfect matching is at least possible, notice that the perfect matching vector is a solution to the perfect matching problem we denote it as $\chi_M \in \{0, 1\}^{|E|}$:

$$\forall e \in E : (\chi_M)_e := \begin{cases} 1 & e \in M \\ 0 & \text{else} \end{cases} \tag{2.0.2}$$

$$P = \text{conv}\{\chi_M : M \text{ is a matching on G}\} \tag{2.0.3}$$

The theorem states that $P = Q$ **when** G is bipartite.

Observe that if $P = \emptyset$ then $Q = \emptyset$. This is obvious because if there is no perfect matching, then it's impossible to sum up all the edges incident to a vertex and sum up to 1. The trivial case holds up.

2.1 Settings Things up

Let $G := (U\dot{\cup}V, E)$ be bipartite, $w : \mathbb{R}^{|E|} \mapsto \mathbb{R}_+$ be a weight functions on all the edges of G. We further assum $|U| = |V| = n$, and the vertices are enumerable:

$$U := \{u_i\}_{i=1}^n \tag{2.1.1}$$

$$V := \{v_i\}_{i=1}^n \tag{2.1.2}$$

Then we define another matrix $A \in \mathbb{R}^{n\times n}$, which represent the weight function w for each edges on the bipartite graph:

$$a_{i,j} := \begin{cases} w(\{u_i, v_j\}) & \{u_i, v_j\} \in E \\ 0 & \text{else} \end{cases} \tag{2.1.3}$$

2.2 Show $Q \subseteq P$

Take any $x \in Q$, then $x \in \mathbb{R}^{|E|}$, and it can define a weight function for all the edges on E:

$$w(e) := x_e \tag{2.2.1}$$

Which induces a matrix because assuming that G is bipartite:

$$a_{i,j} := \begin{cases} x_e & e = \{u_i, v_j\}, e \in E \\ 0 & \text{else} \end{cases} \tag{2.2.2}$$

For how Q is defined in (2.0.1) we have this:

$$\forall v \in U\dot{\cup}V : \sum_{v\ni e} x_e = 1 \quad x_e \geq 0\ \forall e \in E \tag{2.2.3}$$

$$\implies \forall u_i \in U : \sum_{\{u_i,v_j\}\in E} x_{\{u_i,v_j\}} = 1 = \sum_{j=1}^{n} a_{i,j} \tag{2.2.4}$$

$$\implies \forall v_j \in V : \sum_{\{u_i,v_j\}\in E} x_{\{u_i,v_j\}} = 1 = \sum_{i=1}^{n} a_{i,j} \tag{2.2.5}$$

For any vertices in G, we can break it into 2 cases, in U or V. For both cases we can invoke the definition of Q, and sume up all x_e where e is incident to the vertex: u_i, v_j. Then by the definition of $a_{i,j}$ back in (2.2.2), both of them has to be equal to 1. Therefore, the matrix A whose (i, j) entry are $a_{i,j}$ are going to be a doubly stochastic matrix. The non-negativity constraint of x_e translate to corresponding $a_{i,j}$ by (2.2.2) as well.

From 3.11 of the last HW, we know that the matrix A is a convex combinations of permutations matrices. And most importantly, a permutation matrix is a perfect matching on a bipartite graph of the same size. Let $\mathbf{P}$ denotes any $n \times n$ permutations matrix, and let $\pi : [n] \mapsto [n]$ be bijective, then π induces a permutations matrix:

$$\mathbf{P} := \begin{bmatrix} e_{\pi(1)} & e_{\pi(2)} & \cdots & e_{\pi(n)} \end{bmatrix} \tag{2.2.6}$$

$$\text{define } \mathcal{M}_{\mathbf{P}} \text{ s.t: } \mathbf{P}_{i,j} \neq 0 \iff \{u_i, u_j\} \in \mathcal{M}_{\mathbf{P}} \tag{2.2.7}$$

$$\implies \mathcal{M}_{\mathbf{P}} \text{ is a matching on G} \tag{2.2.8}$$

This is true because for each $v_j \in V$, $\exists!\{u_{\pi(j)}, v_j\} \in \mathcal{M}_{\mathbf{P}}$. Therefore all vertices in U are covered, and since $|U| = |V| = n$, the matching is perfect. Further more, the matching introduced by $\mathbf{P}$ can be converted into a binary vector in $\chi_{\mathcal{M}_{\mathbf{P}}} \in \{1,0\}^{|E|}$:

$$(\chi_{\mathcal{M}_{\mathbf{P}}})_e = 1 \iff e \in \mathcal{M}_{\mathbf{P}} \tag{2.2.9}$$

For each non zero element $a_{i,j}$, they are a convex combinations of all possible $\mathbf{P}_{i,j}$ where $\mathbf{P}$ is a permutation matrix, by (2.2.2) we know that x_e is a convex combinations of all possible $(\chi_{\mathcal{M}_{\mathbf{P}}})_e$. Therefore the whole vector x is a convex combinations of $\chi_{\mathcal{M}_{\mathbf{P}}}$ for all possible $\mathbf{P}$; x now fits the definition of P, therefore, $Q \subseteq P$.

2.3 Show $P \subseteq Q$

This is true because each χ_M is also an element of Q, this is direct from the definition of a perfect matching (we don't even need G to be bipartite in this direction). Using the fact that the set Q is convex, and it contains all χ_M, therefore, it contains the convex hull of all χ_M.

Theorem 1. Since $P \subseteq Q$ and $Q \subseteq P$ from the previous 2 sections, $P = Q$. □

3 Problem 8.4

Let $G = (V, E)$ be a graph. Describe the problem of finding a clique (= complete subgraph) of maximum cardinality as an integer linear programming problem.

We consider decision variables of both vertices and edges. Let $x \in [0,1]^{|V|}$, then:

$$P := \forall u, v \notin E : x_u + x_v \leq 1 \tag{3.0.1}$$

$$P_I := P \cap \mathbb{Z}^{|E|+|V|} \leftarrow \text{This is what we want} \tag{3.0.2}$$

For every vertices chosen, there must exist an edge $e \in E$ between them, then it will be a clique on the graph. To assert it we prevent the case where u, v are chosen and there is no edges betwee them (which is (2.0.1)). The second line (2.0.2) asserts the conditions that we want the integral solutions.

4 Problem 8.7

Proposition 4.1. Give integer matrix A and an integer vector b s.t: polyhedron $P := \{x|Ax \leq b\}$ is integer and A is not T.U (totally Unimodular).

For the trivial case consider $A \in \mathbb{R}^{1\times 1}$, $A = 2$, and the polytope: $P := \{x|Ax \leq 4\}$, then:

$$2x \leq 4 \implies x = 2 \tag{4.0.1}$$

The vertex is an integer and $\det(A) = 2$, not T.U. For a bigger example consider:

$$A := \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} b := \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \tag{4.0.2}$$

Then determinant of upper 2 by 2 of A is not 4 which means A is not T.U. Next the polytope define by such A, b will heve vertex: $[0\ 1]^T, [1\ 0]^T$ which are integral.