Notations

1. $\mathbb{1}_C$ to be an indicator set, where $C \subseteq E$, and it's indexed by element $e \in E$ such that $(\mathbb{1}_C)_e = 1$ when $e \in C$ and 0 when $e \notin C$

2. Define $\delta^+(v) := \{(v,u) \in A | u \in V\}$ to be the set of arcs coming out of the vertex v on the direction graph $D := (V, A)$. Follows a similar manner, $\delta^-(v) := \{(u,v) | u \in V\}$ be the set of arcs that are coming into the vertex v on the digraph. Similarly, one can define it for a set of vertices as well, which will be a indicator vector representing the set of arcs cutting into or out of a set of vertices on the digraph.

3. Define $\mathbb{1}_{\delta^\pm(v)} = \mathbb{1}_{\delta^+(v)} - \mathbb{1}_{\delta^-(v)}$, which is a vector of $\pm$ denoting arcs that are coming into or out of the vertex $v \in V$.

1 Problem 1

Proposition 1.1. Let $D := (V, A)$ be a digraph with $|V| = n$ and $|A| = m$, and define $M_D \in \mathbb{R}^{n\times m}$ be an incidence matrix of D. Then the determinant of any $(n-1)\times(n-1)$ sub matrix M' of M_D hs a determinant of ± 1 when the chosen columns of M' from M_D forms a tree on the digraph, disregard the directions of the chosen edges.

1.1 Proof Strategies

For the proof of sufficiency ($\Longleftarrow$), we assume that the submatrix M' has columns of M_D where it corresponds to a cycle: C on the original graph, regardless of directions of the edges. Then, I will show that the absolute values of $\det(M')$ is preseved when I make the directions of edges of C so they aligns; which means that now I can send through a circulation on the cycle, which give me a vector on the null space of M'.

For the proof of neccessity ($\Longrightarrow$), we assume that the sub graph represented by M' is a tree, which implies that each arc must introduce us to a new vertex in the graph, which in the end actually gives us a matrix that is bi-diagonal with nonzeros on the diagonal.

1.2 Proof Direction $\Longleftarrow$

WOLG Let M' be an $(n-1)\times(n-1)$ sub matrix of M_D that takes $\mathcal{C} \subset [m]$ columns and $[n-1]$ rows of of M_D (v_n is not chosen to be a row of M') such that they doesn't form a tree on D, disregarding the directions of the arcs. Not a tree means columns of M' can contain a cycle if we treat the arcs as edges, for example:

$$\underbrace{\text{WOLG let}}_{\text{Read Remark!}} C := v_0 \xrightarrow[a_{k_1}]{} v_1 \xleftarrow[a_{k_2}]{} v_2 \xrightarrow[a_{k_3}]{} v_3 \cdots v_{l-1} \xrightarrow[a_{k_l}]{} v_l, \quad l \le n-1 \tag{1.2.1}$$

We want to send a flow to it, because the cycle is subset of arcs represented by M', and if we can send a flow: $\mathbb{1}_C$, then $M'\mathbb{1}_C = \mathbf{0}$. The good news is, swapping the direction of any arcs a_{k_i} on C a subgraph of D corresponds to multiplying the k_i column of M' by -1, which perserves the absolute value of the determinant.

Consider doing this for all the arcs in C to aling all of them to form a directed cycle for a circulations and we obtained M'' as the new matrix, then:

$$
\begin{aligned}
& |\det(M'')| = |\det(M')| && (1.2.2)\\
& M''\mathbb{1}_C = \mathbf{0} \implies |\det(M'')| = 0 && (1.2.3)\\
\implies & |\det(M)| = 0 && (1.2.4)
\end{aligned}
$$

Remark 1.2.1 (A tiny Subtlety here)**.** We made the assumption that all the vertices in the cycle C indeed corresponds to the first $(n-1)$ vertices. This is a legit assumption because if any of the vertices v_i is not in the cycle, them that tth row is going to be all zeros! Which trivially makes the matrix having a null space, hence a determinant of zero.

1.3 Proof Direction $\implies$

WLOG, let M' be $(M_D)_{1:n-1,1:n-1}$, a sub-matrix that takes all vertices except v_n, and it takes the first $(n-1)$ columns of M_D too.[1]

M''s columns represents arcs that connects all the vertices because they form a spanning tree. Therefore, for all rows of M', it has at least one ± 1 on it because they are the first $n-1$ vertices in the graph, connected by all first $n-1$ arcs in the graph, therefore it's possible to permute the matrix such that all it's diagonal elements are $\in \{\pm 1\}$. Let's reorder the columns of M' allows for the diagonal to be all $\in \{\pm 1\}$

Moreover, tree has the proprty that we can remove one edge from the tree and it will always connect to a new vertex in the original graph.[2] Let's order the vertices the same way as how they are removed! That implies the following structure about the matrix M':

$$
(M_D)_{1:k,1:k+1} = \begin{bmatrix}
\underbrace{\begin{bmatrix}
\{\pm 1\} & & & & \\
\{\pm 1\} & \{\pm 1\} & & & \\
 & \{\pm 1\} & \{\pm 1\} & & \\
 & & \ddots & \ddots & \\
 & & & \{\pm 1\} & \{\pm 1\}
\end{bmatrix}}_{(M_D)_{1:k,1:k}} \\
\pm \mathbf{e}_k^T
\end{bmatrix} \tag{1.3.1}
$$

Each time, We introduce a new edge to the sub tree represented by $(M_D)_{1:k,1:k}$ by connecting the last column, a_k to a new unvisted vertex, indutively giving that structure. The base case is a 2×1 matrix filled with ones, represnting that one arc connects 2 vertices together.

[1] We can do this because we can get the tree first and then permute columns and rows of M_D so that the first $(n-1)$ arcs are in the tree.

[2] The arc connecting to v_n is removed as the last arc! This is absolutely doable.

Therefore, the full tree $(M_D)_{1:n,1:n-1}$ will look like:

$$(M_D)_{1:n,1:n-1} = \begin{bmatrix} \{\pm1\} & & & & & \\ \{\pm1\} & \{\pm1\} & & & & \\ & \{\pm1\} & \{\pm1\} & & & \\ & & \ddots & \ddots & & \\ & & & \{\pm1\} & \{\pm1\} \\ & & & & \{\pm1\} \end{bmatrix} \tag{1.3.2}$$

$$\implies M' = \begin{bmatrix} \{\pm1\} & & & & \\ \{\pm1\} & \{\pm1\} & & & \\ & \{\pm1\} & \{\pm1\} & & \\ & & \ddots & \ddots & \\ & & & \{\pm1\} & \{\pm1\} \end{bmatrix} \tag{1.3.3}$$

$$\implies \det(M') \in \{\pm1\} \tag{1.3.4}$$

Because the diagonal of M' after some permutations are all nonzero, the determinant of M' is nonzero.

2 Problem 2

2.1 Problem Statement

LP for (50) in the textbook won't work if the objective vector C contains some negative numbers to it.

2.2 Show Strategies

I aim to reduce the system of LP to another form that is easier too analyze and show that if any $c_{i,j} < 0, (i,j) \in A$, then the dual problem will become unbounded.

Let $D = (V, A)$ be a digraph with a set of vertices V and a set of arcs A. Let's define M be the incidence matrix of the directed graph G. Denotes M' to be the incidence matrix of the digraph. Let $c \in \mathbb{R}^{|A|}$ be a capacity vector.

2.3 Proof

The primal formulation of the max cpacity flow is:

$$\max\left\{ \langle \mathbb{1}_{\delta^{\pm}(s)}, x\rangle \middle| \mathbf{0} \le x \le c, M'x = \mathbf{0}, x \in \mathbb{R}^{|A|} \right\} \tag{2.3.1}$$

And after applying duality, we obtain the following dual problem:

$$\min\left\{ \langle c, y\rangle \middle| y \ge \mathbf{0}, y^T + z^T M' \ge \mathbb{1}_{\delta^{\pm}(s)}, z \in \mathbb{R}^{|V|-2}, y \in \mathbb{R}_+^{|A|} \right\} \tag{2.3.2}$$

Let me expand the system out and get:

$$\min \sum_{(i,j)\in A} c_{i,j} y_{i,j} \tag{2.3.3}$$
$$y_{i,j} + z_i - z_j \geq 0 \quad \forall\ (i,j) \in A : i \neq 0 \wedge j \neq 0 \tag{2.3.4}$$
$$y_{s,j} - z_j \geq \pm 1 \quad \forall\ (i,j) \in \delta^+(s) \cup \delta^-(s) \tag{2.3.5}$$
$$y_{i,t} + z_i \geq 0 \quad \forall\ j = t \wedge i \neq s \tag{2.3.6}$$

Here, the variable $y \geq \mathbf{0}$, z is free and I can apply the following tricks:

$$\forall (i,j) \in A \delta_{i,j} \geq 0 \tag{2.3.7}$$
$$y_{i,j} = \delta_{i,j} + \max(z_j - z_i, 0) \quad \forall\ (i,j) \in A : i \neq 0 \wedge j \neq 0 \tag{2.3.8}$$
$$y_{s,j} = \delta_{s,j} + \max(z_j \pm 1, 0) \quad \forall\ (i,j) \in \delta^+(s) \cup \delta^-(s) \tag{2.3.9}$$
$$y_{i,t} = \delta_{i,t} + \max(-z_i, 0) \quad \forall\ j = t \wedge i \neq s \tag{2.3.10}$$

Now, we may consider splitting the objective expression for the miniizations:

$$\sum_{(i,j)\in A} c_{i,j} y_{i,j} = \sum_{(i,j)\in A, i\neq s \wedge j \neq t} c_{i,j} y_{i,j} + \sum_{(s,j)\in A} c_{s,j} y_{s,j} + \sum_{(i\neq s,t)\in A} c_{i,t} y_{i,t} \tag{2.3.11}$$
$$= \sum_{(i,j)\in A, i\neq s \wedge j \neq t} c_{i,j}(\delta_{i,j} + \max(z_j - z_i, 0)) \cdots \tag{2.3.12}$$
$$+ \sum_{(s,j)\in A} c_{s,j}(\delta_{s,j} + \max(z_j \pm 1, 0)) \cdots \tag{2.3.13}$$
$$+ \sum_{(i\neq s,t)\in A} c_{i,t}(\delta_{i,t} + \max(-z_i, 0)) \tag{2.3.14}$$

Notice that, we can factor out the term $\sum_{(i,j)\in A} c_{i,j}\delta_{i,j}$, in which case if any of the $c_{i,j} \leq 0$, we can make it unbounded for any feasible solution of y, z.

3 Problem 3

4 Problem 4