1   Notations

1.) cvxh denotes the convex hull of some sets in Euclidean space.

2.) $\mathbb{1}_M$ denotes that activation vector defined by the set M; $(\mathbb{1}_M)_e = 1 \iff e \in M$; else $(\mathbb{1}_M)_e = 0 \iff e \notin M$.

3.) Bold face $\mathbf{1}, \mathbf{0}$ denotes a vector full of ones and zeros, the size of them depends on the context, if ambiguity arised, a subscript will be there denoting the size.

4.) We define the set of edges pointing out of a vertex v on a digraph to be $\delta^+(v)$, and the set of edges pointint into the vertex as $\delta^-(v)$. It's like flux, but on the vertex.

2   Problem 8.12

Proposition 2.1 (Non Bipartite Matching). Let $G = (V, E)$ be a non-bipartite graph. Show that the inequalities (LP formulations of the matching problem) are not enough t define the matching polytope. Mathematically Speaking:

$$P := \text{cvxh}\{\text{All Matching } \mathbb{1}_M \text{on G}\} \subsetneq Q := \{x \geq \mathbf{0} \,|\, Ax \leq \mathbb{1}\} \tag{2.0.1}$$

2.1   Proof Strategies

The inequality can be proved by choosing one element $x \in P$ such that it's not in Q. We make use of the fact that a graph is non-bipartite iff it has odd cycles in it.

Denote C to be the set of edges that describes the odd cycle on G.

2.2   The Proooooo0000oooooOWOOOf

Proof. Choose an element $x \in P$ we have:

$$\mathcal{M} := \{\text{All Integral Matching on G}\} \tag{2.2.1}$$

$$x = \sum_{M \in \mathcal{M}} \lambda_i \mathbb{1}_M; \quad \langle \mathbb{1}, \vec{\lambda} \rangle = 1, \vec{\lambda} \geq \mathbf{0} \tag{2.2.2}$$

$$\forall e \in E : x_e = \sum_{M \in \mathcal{M}} \lambda_i (\mathbb{1}_M)_e \quad \langle \mathbb{1}, \vec{\lambda} \rangle = 1, \vec{\lambda} \geq \mathbf{0} \tag{2.2.3}$$

The first line decribes that the vector x is a convex combinations of all indicator vectors of a matching on the graph. The next line highlights the fact that, for all edges e on the cycle, it's still a convex combinations of the subset of elements in $\mathbb{1}_M$ selected by M.

Now I am going to choose another element $x^+ \in P$ such that it's not in Q. define this element to be:

$$\forall e \in E : (x^+)_e := \begin{cases} \frac{1}{2} & e \in C \\ 0 & \text{else} \end{cases} \tag{2.2.4}$$

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$x^+ \in Q$ because $\forall v \in V : \sum_{v \ni e} x_e = 1 \leq 1$, and $\forall v \in V \setminus C : \sum_{e \ni v} x_e = 0 \leq 1$. Observe that if I sum up all $(x^+)_e$ for all $e \in C$, we have:

$$\sum_{e \in C} (x^+)_e = \langle \mathbb{1}_C, x^+ \rangle = \frac{(2n+1)}{2} = n + \frac{1}{2} \tag{2.2.5}$$

Here, we assume that the odd cycle have length $2n + 1$. Now, for any element $x \in P$, we can sum up edges only on cycle C too, obtaining:

$$\langle \mathbb{1}_C, x \rangle = \left\langle \mathbb{1}_C, \sum_{\forall M} \lambda_i \mathbb{1}_M \right\rangle \tag{2.2.6}$$

$$= \sum_{\forall M \in \mathcal{M}} \lambda_i \langle \mathbb{1}_M, \mathbb{1}_C \rangle = \sum_{\forall M \in \mathcal{M}} \lambda_i \mathbb{1}_{M \cap C} \leq n \tag{2.2.7}$$

The above expression denotes the fact that, if we zoom into the odd cycle and exam the matching on the odd cycle, then their convex combinations is going to be always less than n. This is true because M on C have a maximum assignment of n edges, more than that it won't be a matching. The weighted sum of λ_i preserves the bound (summation is a linear transform). Which is impossible to represent $x^+ \in Q$ (it sums up to $n + 1/2$, which is more). Therefore, we have proven the fact that $P \subsetneq Q$. $\qquad \square$

3   Problem 8.14

Proposition 3.1. The stable set polytope P_{stable} of $G := (V, E)$ is the convex hull of the incidence vectors of the stable set in G, give a description of the linear inequalities defining the stable set of the polytope of a bipartite graph G.

3.1   Proof Strategies

The polytope would need the help of a incidence matrix A for the graph G. The indicence matrix is a $|V| \times |E|$ defined as:

$$A_{v,e} := \begin{cases} 1 & v \in e \\ 0 & \text{else} \end{cases} \tag{3.1.1}$$

Then, for all independent sets on a graph, it will fall into the following linear program:

$$x \in \mathbb{R}^{|V|} \tag{3.1.2}$$

$$A^T x \leq \mathbf{1} \tag{3.1.3}$$

Observe that for all v, $(A^T x)_v = \sum_{v \ni e} 1$, which counts the total number of edges incident to the vertex v. All independent set F has incidator $\mathbb{1}_F$ will satisfies this condition.

3.2 The ProooOooOOooF

Proof. Recall: Theorem 8.3 that the incidence matrix A_G for a bipartite graph is total unimodular. Then A^T is T.U as well therefore, the polytope $P := \{x \geq \mathbf{0} : A^T x \leq \mathbf{1}\}$ is integral. All stable set vector $\mathbb{1}_F \in P$ and this is direct from its definition. Then any convex combinations of $\mathbb{1}_F$ will be $\in P$. Therefore $P_{\text{stable}} \subseteq P$.

For all integral vectors inside P, it has to be $\mathbb{1}_F$ where F is an independent set. A polytope is defined as the convex hull of all its vertices(In this case, P has all integral vertices). Therefore all its vertices are describing an independet set as well, therefore $P = P_{\text{stable}}$. $\square$

4   Problem 4.5

Proposition 4.1. Let $D : (V, A)$ be a digraph, set $s, t \in V$ to be the source and sink for a networkflow, then define flow $f : A \mapsto \mathbb{R}_+$ as the $s-t$ flow of value β. How that $\exists f' : A \mapsto \mathbb{Z}_+$ of value $\lceil \beta \rceil$ such that $\lfloor f(a) \rfloor \leq f'(a) \leq \lceil f(a) \rceil$.

4.1   Proof Strategies

The problem can be viewed as looking for a flow that is $\lfloor f \rfloor \leq x \leq \lceil f \rceil$ (f is the flow vector $f \in \mathbb{R}^{|A|}$, $f_a = f(a)$) such that it sends exactly a flow of $\lceil \beta \rceil$. This is always possible because rounding up f to attain something $\geq \lceil f \rceil$. Therefore, we always have a solution for x, but the problem is, is it neccessarily integral?

To do that we make use of total unimodularity of matrices. Notation wise we use $\mathbf{e}_k$ to denote the standard basis vectors.

4.2   The proof

Proof. Define the flow digraph matrix A:

$$
A = \begin{bmatrix}
\mathbb{1}_{\delta^+(s)} - \mathbb{1}_{\delta^-(s)} \\
\mathbb{1}_{\delta^+(t)} - \mathbb{1}_{\delta^-(t)} \\
\mathbb{1}_{\delta^+(v_1)} - \mathbb{1}_{\delta^-(v_1)} \\
\vdots \\
\mathbb{1}_{\delta^+(v_n)} - \mathbb{1}_{\delta^-(v_n)}
\end{bmatrix}
\tag{4.2.1}
$$

The first row and the second row are edges that are coming in and out of the source and the sink vertex. I want the flow coming in and out to be $\lceil \beta \rceil$, then we consider:

$$
Ax = \lceil \beta \rceil (\mathbf{e}_1 - \mathbf{e}_2)
\tag{4.2.2}
$$

To assert the conditions that $\lfloor f \rfloor \leq x \leq \lceil f \rceil$ we have:

$$
0 \leq x - \lfloor f \rfloor \leq \lceil f \rceil
\tag{4.2.3}
$$
$$
x - \lfloor f \rfloor + s = \lceil f \rceil \quad s \geq \mathbf{0}, x \geq \mathbf{0}
\tag{4.2.4}
$$
$$
x + s = \lceil f \rceil - \lfloor f \rfloor \quad s, x \geq \mathbf{0}
\tag{4.2.5}
$$

3

Then, we have the following polytope to deal with:

$$\underbrace{\begin{bmatrix} A & \mathbf{0} \\ I & I \end{bmatrix}}_{=:B} \begin{bmatrix} x \\ s \end{bmatrix} = \underbrace{\begin{bmatrix} \lceil \beta \rceil (\mathbf{e}_1 - \mathbf{e}_2) \\ \lceil f \rceil - \lfloor f \rfloor \end{bmatrix}}_{=:b} \tag{4.2.6}$$

The claim is that, the 2 by 2 block matrix above is totally unimodular, and because its totally unimodular, it's unimodular, in additoin, A the incidence matrix will be full rank when G is connected(It will be, if not, we can remove all the edges that has $f = 0$, then we are left with a graph connected by the defined real flow f). By theorem 8.2 we proved in class, we have the fact that the polytope $\{x \geq \mathbf{0} | Bx = b\}$ is integral. We know it's definitely non-empty, therefore, an integral flow x exists and it is in between $\lfloor f \rfloor, \lceil f \rceil$ for all the arcs in the graph.

The 2 by 2 is unimodular, we denote any submatrices using the $\tilde{\cdot}$ decorator. Then any square sub matrix of B can be written in the form of:

$$\det \begin{bmatrix} \tilde{A} & \mathbf{0} \\ \tilde{I} & \tilde{I} \end{bmatrix} = \det(\tilde{A})\det(\tilde{I}) \in \{\pm 1, 0\} \tag{4.2.7}$$

We use the theorem that, for any block diagonal matrix, its determiannt is the product of the determinant of all its block matrices on its diagonal (I learned this in a numerical linear algebra class); any submatrix of I is going to be either 0 or 1, therefore, the product of this square sub matrix is still, $\pm 1, 0$. It's T.U. $\qquad \square$
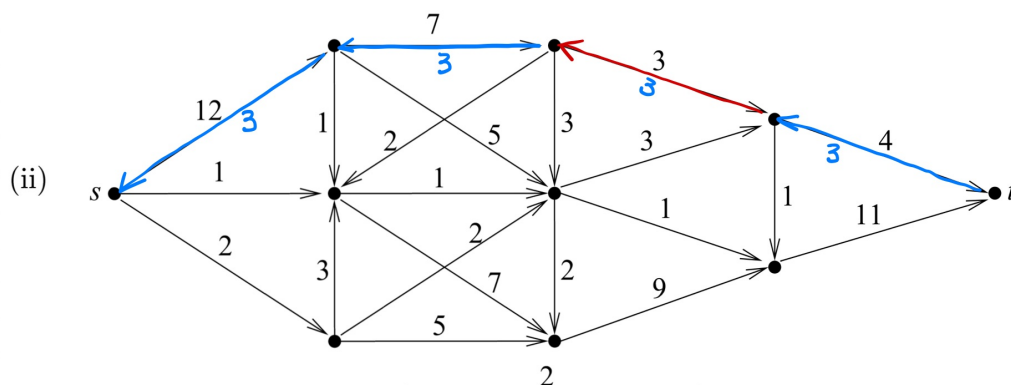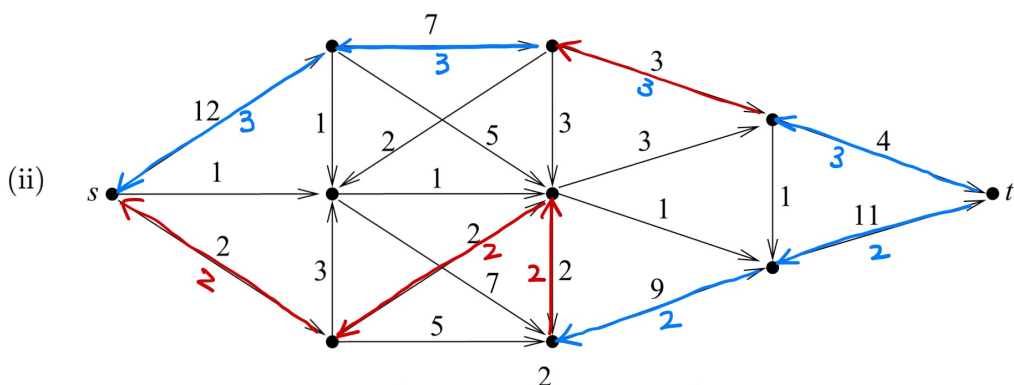
5   Problem 4.7 (ii)
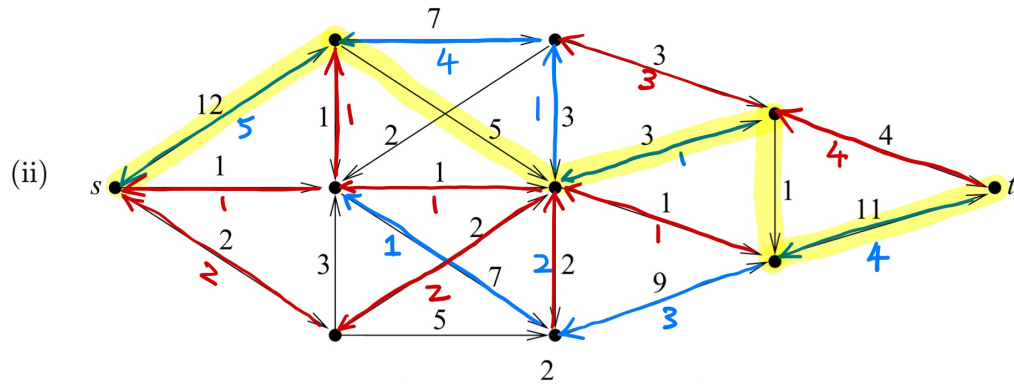
§4.7 (ii)

→ unsaturated flow (going reverse is fine)
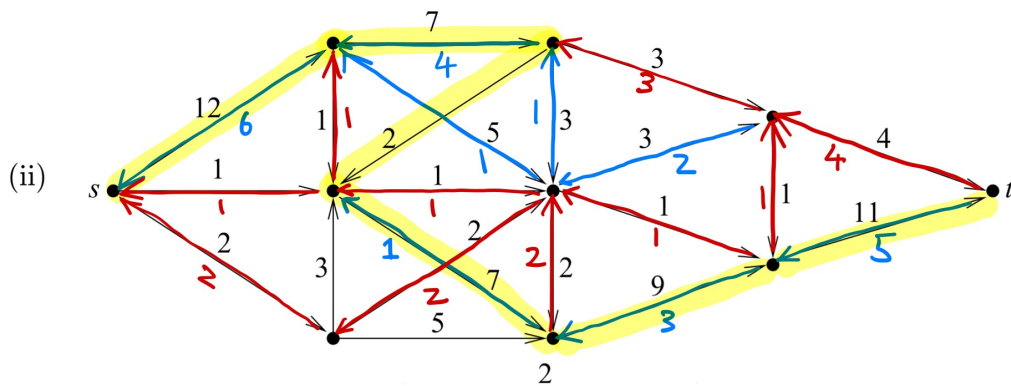
→ flow saturated (can't go in reverse anymore)

⬤ min cut set
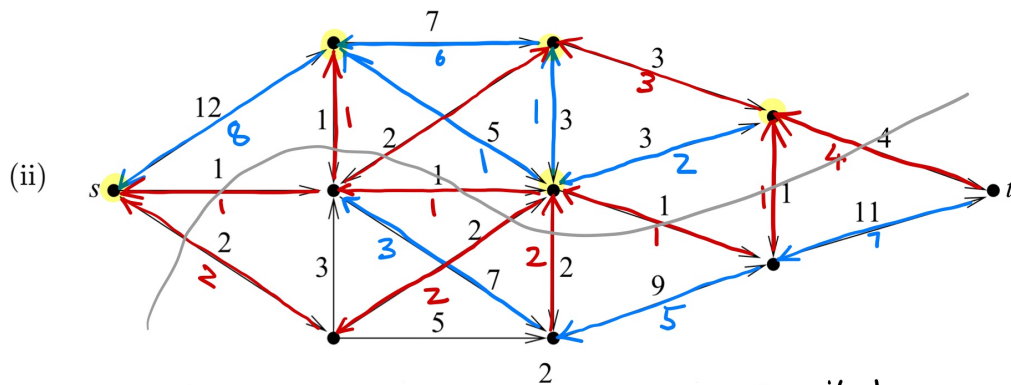
(ii)

*Found by eyeing it, that's my first flow.

(ii)

*Added another flow by looking at it.

(ii)

flow with capacity 1

(ii)

flow 2 possible

(ii)

4+1+1+2 + 1+1+1 = 8+3 = 11 ← max capacity!

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