Notations

1. $\mathbb{1}_C$ to be an indicator set, where $C \subseteq E$, and it's indexed by element $e \in E$ such that $(\mathbb{1}_C)_e = 1$ when $e \in C$ and 0 when $e \notin C$

2. Define $\delta^+(v) := \{(v, u) \in A | u \in V\}$ to be the set of arcs coming out of the vertex v on the direction graph $D := (V, A)$. Follows a similar manner, $\delta^-(v) := \{(u, v) | u \in V\}$ be the set of arcs that are coming into the vertex v on the digraph. Similarly, one can define it for a set of vertices as well, which will be a indicator vector representing the set of arcs cutting into or out of a set of vertices on the digraph.

3. Define $\mathbb{1}_{\delta^\pm(v)} = \mathbb{1}_{\delta^+(v)} - \mathbb{1}_{\delta^-(v)}$, which is a vector of $\pm$ denoting arcs that are coming into or out of the vertex $v \in V$.

1 Problem 1

Proposition 1.1. Let $D := (V, A)$ be a digraph with $|V| = n$ and $|A| = m$, and define $M_D \in \mathbb{R}^{n\times m}$ be an incidence matrix of D. Then the determinant of any $(n-1) \times (n-1)$ sub matrix M' of M_D hs a determinant of ± 1 when the chosen columns of M' from M_D forms a tree on the digraph, disregard the directions of the chosen edges.

1.1 Proof Strategies

For the proof of sufficiency ($\Longleftarrow$), we assume that the submatrix M' has columns of M_D where it corresponds to a cycle: C on the original graph, regardless of directions of the edges. Then, I will show that the absolute values of $\det(M')$ is preseved when I make the directions of edges of C so they aligns; which means that now I can send through a circulation on the cycle, which give me a vector on the null space of M'.

For the proof of neccessity ($\Longrightarrow$), we assume that the sub graph represented by M' is a tree, which implies that each arc must introduce us to a new vertex in the graph, which in the end actually gives us a matrix that is bi-diagonal with nonzeros on the diagonal.

1.2 Proof Direction $\Longleftarrow$

WOLG Let M' be an $(n-1) \times (n-1)$ sub matrix of M_D that takes $\mathcal{C} \subset [m]$ columns and $[n-1]$ rows of of M_D (v_n is not chosen to be a row of M') such that they doesn't form a tree on D, disregarding the directions of the arcs. Not a tree means columns of M' can contain a cycle if we treat the arcs as edges, for example:

$$\underbrace{\text{WOLG let}}_{\text{Read Remark!}} C := v_0 \underset{a_{k_1}}{\longrightarrow} v_1 \underset{a_{k_2}}{\longleftarrow} v_2 \underset{a_{k_3}}{\longrightarrow} v_3 \cdots v_{l-1} \underset{a_{k_l}}{\longrightarrow} v_l, \quad l \le n-1 \tag{1.2.1}$$

We want to send a flow to it, because the cycle is subset of arcs represented by M', and if we can send a flow: $\mathbb{1}_C$, then $M'\mathbb{1}_C = \mathbf{0}$. The good news is, swapping the direction of any arcs a_{k_i} on C a subgraph of D corresponds to multiplying the k_i column of M' by -1, which perserves the absolute value of the determinant.

Consider doing this for all the arcs in C to aling all of them to form a directed cycle for a circulations and we obtained M'' as the new matrix, then:

$$
\begin{aligned}
&|\det(M'')| = |\det(M')| && (1.2.2)\\
&M''\mathbb{1}_C = \mathbf{0} \implies |\det(M'')| = 0 && (1.2.3)\\
\implies &|\det(M)| = 0 && (1.2.4)
\end{aligned}
$$

Remark 1.2.1 (A tiny Subtlety here)**.** We made the assumption that all the vertices in the cycle C indeed corresponds to the first $(n-1)$ vertices. This is a legit assumption because if any of the vertices v_i is not in the cycle, them that tth row is going to be all zeros! Which trivially makes the matrix having a null space, hence a determinant of zero.

1.3 Proof Direction $\implies$

WLOG suppose that $(M_D)_{:,1:(n-1)}$ is an incidence matrix of a spanning tree so that the first $(n-1)$ arcs spans a spanning tree in D. Further assuming that $M' = (M_D)_{2:n,1:(n-1)}$ which is M_D but without the first row. (v_1 is not a vertex in our tree... and it's arbitrary.)

That vertex in the first row must be connected to a series of arcs: $\{a_{k_1}, a_{k_2}, \cdots, a_{k_l}\}$. Each of those must connect to a different vertex: $v_1 \notin \{v_{j_1}, v_{j_2}, \cdots, v_{j_l}\}$, so it looks like this:

$$
(M_D)_{\{v_1\}\cup\{v_{j_1}, v_{j_2}, \cdots, v_{j_l}\}, \{k_1, k_2, \cdots, k_l\}}
\begin{bmatrix}
\{\pm1\}_1 & \{\pm1\}_2 & \cdots & \{\pm1\}_l \\
\{\mp1\}_1 & & & \\
& \{\mp1\}_2 & & \\
& & \ddots & \\
& & & \{\mp1\}_l
\end{bmatrix}
\tag{1.3.1}
$$

$$
\text{let: } T_l := (M_D)_{\{v_{j_1}, v_{j_2}, \cdots, v_{j_l}\}, \{k_1, k_2, \cdots, k_l\}} =
\begin{bmatrix}
\{\mp1\}_1 & & & \\
& \{\mp1\}_2 & & \\
& & \ddots & \\
& & & \{\mp1\}_l
\end{bmatrix}
\tag{1.3.2}
$$

Observe that, T_l IS A SUBMATRIX OF M' if $T_l = M'$ then we are DONE because it's a diagonal matrix with nonzeros on its diagonal, because M' is a tree and T_l is a sub tree of M', both with v_1 missing, I can include another arc and a new vertex incidence to any existing vertex and this new vertex to get T_{l+1}:

$$
T_{l+1} =
\begin{bmatrix}
 & & & 0 \\
 & & & \vdots \\
 & T_l & & \pm1 \\
 & & & \vdots \\
 & & & 0 \\
0 & \cdots & 0 & \mp1
\end{bmatrix}
=
\begin{bmatrix}
T_k & \pm\mathbf{e}_i^{(l)} \\
\mathbf{0} & \mp1
\end{bmatrix}
\tag{1.3.3}
$$

Where, we introduce a new ∓1 at the bottom right corner, and a new column to T_1 that has exactly one nonzero element in it. This is true because if we haven includes all the arcs yet, then there exists some new arcs that is incidence to existing vertices and it connects to a new vertex that is not in the tree.

Notice that this argument can be applied inductively, we assume that T_k is upper triangular (Base case T_l already is), then the induction holds for all $k < n-1$, giving us:

$$T_{k+1} = \begin{bmatrix} & & & 0 \\ & & & \vdots \\ & T_k & & \pm 1 \\ & & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \mp 1 \end{bmatrix} \tag{1.3.4}$$

The induction completes with $k = n-2$, which gives us $T_k = M'$, and T_k is upper triangular with only ± 1 on the diagonal, therefore $\det(T_k) = \pm 1$.

2 Problem 2

2.1 Problem Statement

LP for (50) in the textbook won't work if the objective vector C contains some negative numbers to it.

2.2 Show Strategies

I claim to reduce the system of LP for the Dual of Maxflow to another form that is easier too analyze and show that if any $c_{i,j} < 0, (i,j) \in A$, then the dual problem will become unbounded.

Let $D = (V, A)$ be a digraph with a set of vertices V and a set of arcs A. Let's define M be the incidence matrix of the directed graph G. Denotes M' to be the incidence matrix of the digraph. Let $c \in \mathbb{R}^{|A|}$ be a capacity vector.

2.3 Proof

The primal formulation of the max cpacity flow is:

$$\max \left\{ \langle \mathbb{1}_{\delta^{\pm}(s)}, x \rangle \middle| \mathbf{0} \leq x \leq c, M'x = \mathbf{0}, x \in \mathbb{R}^{|A|} \right\} \tag{2.3.1}$$

And after applying duality, we obtain the following dual problem:

$$\min \left\{ \langle c, y \rangle \middle| y \geq \mathbf{0}, y^T + z^T M' \geq \mathbb{1}_{\delta^{\pm}(s)}, z \in \mathbb{R}^{|V|-2}, y \in \mathbb{R}_+^{|A|} \right\} \tag{2.3.2}$$

Let me expand the system out and get:

$$\min \sum_{(i,j)\in A} c_{i,j} y_{i,j} \tag{2.3.3}$$
$$y_{i,j} + z_i - z_j \geq 0 \quad \forall\ (i,j) \in A : i \neq 0 \wedge j \neq 0 \tag{2.3.4}$$
$$y_{s,j} - z_j \geq \pm 1 \quad \forall\ (i,j) \in \delta^+(s) \cup \delta^-(s) \tag{2.3.5}$$
$$y_{i,t} + z_i \geq 0 \quad \forall\ j = t \wedge i \neq s \tag{2.3.6}$$

Here, the variable $y \geq \mathbf{0}$, z is free and I can apply the trick of introducing a new decision variable and a max function.

$$\forall (i,j) \in A : \delta_{i,j} \geq 0 \tag{2.3.7}$$
$$y_{i,j} = \delta_{i,j} + \max(z_j - z_i, 0) \quad \forall\ (i,j) \in A : i \neq 0 \wedge j \neq 0 \tag{2.3.8}$$
$$y_{s,j} = \delta_{s,j} + \max(z_j \pm 1, 0) \quad \forall\ (i,j) \in \delta^+(s) \cup \delta^-(s) \tag{2.3.9}$$
$$y_{i,t} = \delta_{i,t} + \max(-z_i, 0) \quad \forall\ j = t \wedge i \neq s \tag{2.3.10}$$

Now, we may consider splitting the objective expression for the miniizations:

$$\sum_{(i,j)\in A} c_{i,j} y_{i,j} = \sum_{(i,j)\in A, i\neq s \wedge j \neq t} c_{i,j} y_{i,j} + \sum_{(s,j)\in A} c_{s,j} y_{s,j} + \sum_{(i\neq s,t)\in A} c_{i,t} y_{i,t} \tag{2.3.11}$$
$$= \sum_{(i,j)\in A, i\neq s \wedge j \neq t} c_{i,j}(\delta_{i,j} + \max(z_j - z_i, 0)) \cdots \tag{2.3.12}$$
$$+ \sum_{(s,j)\in A} c_{s,j}(\delta_{s,j} + \max(z_j \pm 1, 0)) \cdots \tag{2.3.13}$$
$$+ \sum_{(i\neq s,t)\in A} c_{i,t}(\delta_{i,t} + \max(-z_i, 0)) \tag{2.3.14}$$

Notice that, we can factor out the term $\sum_{(i,j)\in A} c_{i,j}\delta_{i,j}$, in which case if any of the $c_{i,j} \leq 0$, we can make it unbounded for any feasible solution of y, z by increasing values of δ indefinitely.

2.4 An Example

There is no example for Maxflow, only the Mincut, because the primal is infeasible due to the constraints $x \geq \mathbf{0}$. The polytope is empty.

Consider a graph that has only $s \underset{c<0}{\longrightarrow} t$ to it, then the dual is: $\min\{cy : y \geq \mathbf{0}, y \geq 1\}$, decision variable z is gone because the graph only has $\{s, t\}$ as the vertex set. Obivously it's unbounded when $c < 0$.

Remark 2.4.1. If people want to add negative capacity to model flow in reverse direction, please consider adding parallel arcs in opposite direction with positive capacity between those 2 vertices instead.

3 Problem 3

3.1 Problem Statement

Explain why the model in Application 4.4 on pp 73 works.

3.2 The Setting up of The Graph

Putting the citites having surplus, deficit into a bipartite directed graph $D = (U\dot{\cup}W, A)$. Let U be the cities of curplus and V be the cities of deficit in freighters. Connects every $u \in U$ to every $w \in V$ by an arc with infinite capacity, going from U to W, associate cost with each arc: (u_i, w_j) by the distance between these 2 cities of surplus and deficit. Let $k : A \mapsto \mathbb{R}_+$ be our cost functions, and $c : A \mapsto \mathbb{R}_+$ be our capacity function. Mathematically:

$$U :: \text{FreightersSurplus Cities} \tag{3.2.1}$$
$$W :: \text{Freighers Deficit Cities} \tag{3.2.2}$$
$$\forall u \in U, w \in W \tag{3.2.3}$$
$$c((u, w)) := +\infty \tag{3.2.4}$$
$$k((u, w)) := \text{min\_distance(city: u, city v)} \tag{3.2.5}$$

Next, we introduce auxilary vertex s, t as the source and the sink vertex for the flow. Construct arcs from s to all vertices in U, cities with surplus, with capacity equals to the empty freighters in that each city. $(s, u) \in E, u \in U$, $c((s, v)) = \sigma$, connecting each vertex $w \in W$, cities with deficit in freighters to t, with an edge (w, t) with a capacity equals to the deficit of freighters for that city: ν.

$$c((s, u)) := \sigma_{s,u}, k((s, u)) := 0 \quad \forall u \in U \tag{3.2.6}$$
$$c((w, t)) := \nu_{w,t}, k((w, t)) := 0 \quad \forall w \in W \tag{3.2.7}$$
$$\forall w^* \in W : \sum_{u \in U} \sigma_{u,w} = \sigma_u \tag{3.2.8}$$

Take note that from the original problem, the total number of deficit among all cities and the surplus adds up to zero. A maximum flow will have a flow value equals to the total number of surplus/deficit among all cities: $\text{value}(f) = \sum_{i=1}^{|U|} u_i = \sum_{i=1}^{|W|} \nu_i$. The Mincut won't include any arcs going between U, W because they have infinite capacity. This implies that for all $u \in U$, the flow cutting out of it equals the total surplus of that city.

A maximum flow from the cities of surplus to deficit is a fixed flow problem, and to minimize the costs will minimizes the distance travelled perunit freighters. Let's just stick with real numbers for now which (Cause in the real world, we can't cut a frieghters and send it to different cities).

3.3 Why Bipartite Graph, Why Only Transport Between Cities of Freighters Surplus and Deficit?

Triangle inequality, sending some freighters from city u_i to u_j then to w_k is always longer than sending directly from u_i to w_k.[1]

3.4 Why is Maxflow Best Routing?

4 Problem 4

[1] Triangule Inequality holds for noneuclidean geometry: The earth surface.