Notations

1. $P_G(u, v)$ a path, which is a list of vertices, or edges, or both, that starts with the vertex u and ends with vertex v in the graph G.

2. $\text{cc}(v)$ Denotes the connected component, is the set of all rechable vertices from a vertex V in G. G can be directed or undirected. It can also be applied to a set of vertices: S, which is just $\text{cc}(S) := \bigcup_{v\in S} \text{cc}(v)$

1 Problem 3.19

Proposition 1.1 (Minimal Bipartite Vertex Cover from Maximum Matching)**.** Given a maximum matching on bipartite graph: $G = (U\dot{\cup}V, E)$ let M^+ be a matching of maximum size.

Suppose that solution of a maximum is given after the execution of the matching algorithm and $e \in M$ goes from U to V, and $\notin M$ goes from V to U. To get the minimum vertex cover:

We choose every reachable vertices from L that is in V (Name that set S). Which are going to be covered by M. For the remaining vertices that is covered by M and not sharing the same edge in the matching with vertices in S, choose then as well, and they will form a vertex cover F with $|F| = |M|$.

Define the sets and directed edges in the following way:

$$M :: \text{The maximum Matching!} \tag{1.0.1}$$

$$L := \left\{ v \in U : v \notin \bigcup_{e\in M} e \right\} \tag{1.0.2}$$

$$S := \text{cc}(L) \cap V \tag{1.0.3}$$

$$e \in M, e = (v_1, v_2) \implies v_1 \in V, v_2 \in U \tag{1.0.4}$$

$$e \notin M, e = (v_1, v_2) \implies v_1 \in U, v_2 \in V \tag{1.0.5}$$

(1.0.2) : L is the set of vertices in U that are not covered by the matching.

(1.0.3) : S is the set of reachable vertices from all vertices in L.

(1.0.4) : An edge in matching goes from V to U.

(1.0.5) : an edge not in matching goes from U to V.

Lemma 1.0.1 (Lemma 2)**.** It's impossible to have a path going from L to S to $U \setminus L$ to $V \setminus S$.

Proof. This is true because S by definition is set of all vertices reachable from L in V. And if we reached some vertices in $V \setminus S$, then it's not in S, which violate the definition of S. □

Lemma 1.0.2 (Lemma 3)**.** All vertices in S are covered by M.

Proof. If not, there exists a path going from $u \in L$ to $v \in S$ such that v not covered by M, since u not covered by M by definition of L; an augmented path is found, therefore M is not maximum. $\square$

Lemma 1.0.3 (Lemma 4)**.** No edges, in any directions exists between the set $V \setminus S$, L.

Proof. For contradiction, suppose there is such an edge and denote that edge as e^+. Then the contradiction is:

$$e^+ \notin M \wedge e^+ \in M \tag{1.0.6}$$

Because $V \setminus S$ is the set of vertices in V that can't be reached by L, therefore there are no direct edges going from $L \subseteq U$ to $(V \setminus S) \subseteq V$, therefore, $e^+ \notin M$; which also means e^+ will go from $(V \setminus S) \subseteq V$ to $L \subseteq U$, therefore $e^+ \in M$. Which is impossible because by definition L is not covered by M. $\square$

Proposition 1.1. Let $\overline{F} := U \setminus L$. The claim is the I can keep the $|\overline{F}|$ fixed and exchange vertices to make this into a vertex cover.

If $L = \emptyset$, then $\overline{F}$ is a vertex cover because $\overline{F} = U$. Using the fact that G is bipartite, $\overline{F}$ covers all edges. And that means M covers all U because $L = \emptyset$; implying $|\overline{F}| = |M|$

If $L \neq \emptyset$, then for all $e \in E, e = \{u, v\}$ (direction doesn't matter). Then there are 3 cases:

(1.) e goes from $u \in L$ to $v \in S$, let $e = (u, v)$. $e \notin M$ because $u \in L$ by def of L, u not covered by M. However, v is covered by M because $v \in S$ and we use lemma 1.0.2. Therefore $\exists! u' \in U \setminus L : \{u', v\} \in M$.

I can then construct $\overline{F} := (F \setminus \{u'\}) \cup \{v\}$ to be a minimum vertex cover, without losing edges. u' can be removed from $\overline{F}$ by lemma 1.0.1. To convince you further, assuing it's not the case, suppose that removing u' expose an edge $e' = (u', v')$ that I am unbale to cover. Observe that v' must be in $V \setminus S$ because S are already all covered by M. Then the path is possible:

$$u \rightarrow v \rightarrow u' \rightarrow v' \tag{1.0.7}$$
$$u \in L \tag{1.0.8}$$
$$v \in S \tag{1.0.9}$$
$$u' \in U \setminus L \tag{1.0.10}$$
$$v' \in V \setminus S \tag{1.0.11}$$

Which contradicts lemma 1.0.1. Therefore $(F \setminus \{u'\}) \cup \{v\}$ now covers the additional edge: e without exposing any other edges.

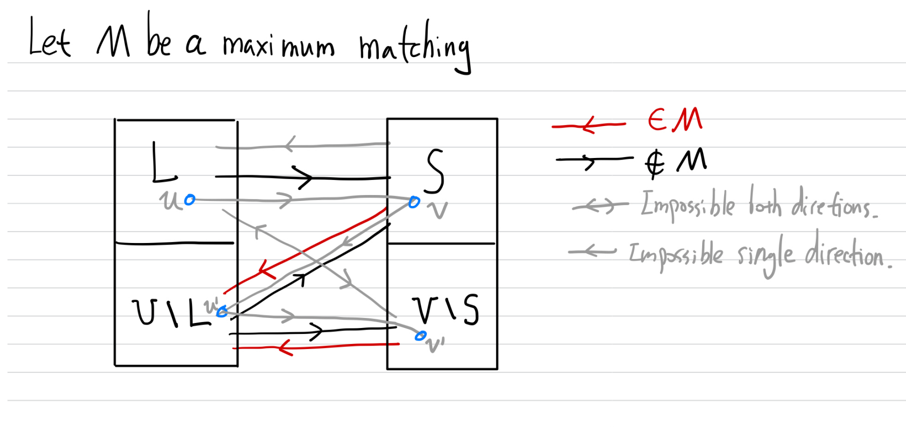
(2.) $e = \{u, v\}$, direction doesn't matter, it goes between S and $U \setminus L$. Then $\overline{F}$ covers the edge: e because $v \in U \setminus L$, and $\overline{F} = U \setminus L$ at the start, and in case (1.), we move vertices to S, therefore, such an edge is always gonna be covered by $\overline{F}$.

(3.) e goes between $U \setminus L$ and $V \setminus S$. This is covered by $\overline{F}$ because $U \setminus L$ originally covers all edges incident to $U \setminus L$, and in case (1.) above, when we remove u', we never expose any edges going between $U \setminus L$ and $V \setminus S$.

(4.) e goes between the set L and $V \setminus S$. This is impossible by 1.0.3.

For all cases, I can re-arrange $\overline{F}$ such that its cardinality remains unchanged and all the edges are covered. I started with $|\overline{F}| = |M|$, therefore, we have a vertex cover $|\overline{F}| = |M|$ in the end.

HEEEEEEY! Here is picture to get my point across fig: 1: $\square$



2 Problem 3.25

Theorem 1. Let $G = (V, E)$ be a bipartite graph. Then the perfect matching polytope P is equal to the set of verctors $x \in \mathbb{R}^{|E|}$ satisfying:

$$Q := \begin{cases} x_e \geq 0 & \forall e \in E \\ \sum_{e \ni v} x_e = 1 & \forall v \in V \end{cases} \tag{2.0.1}$$

Here let $G = (V \dot\cup U, E)$ be a bipartite graph and we denote P as the polytope that is the convex hull of all possible perfect matching solution vectors.

Assuming that $|V| = |U| = n$ so that perfect matching is at least possible, notice that the perfect matching vector is a solution to the perfect matching problem we denote it as $\chi_M \in \{0, 1\}^{|E|}$:

$$\forall e \in E : (\chi_M)_e \begin{cases} 1 & e \in M \\ 0 & \text{else} \end{cases} \tag{2.0.2}$$

Observe that if $P = \emptyset$ then $Q = \emptyset$. This is obvious because if there is no perfect matching, then it's impossible to sum up all the edges incident to a vertex and sum up to 1.

Proof. We first prove the fact that $P \subseteq Q$ $\square$

3 Problem 8.4

Let $G = (V, E)$ be a graph. Describe the problem of finding a clique (= complete subgraph) of maximum cardinality as an integer linear programming problem.

We consider decision variables of both vertices and edges. Let $x \in [0, 1]^{|V|}$, then:

$$P := \forall u, v \notin E : x_u + x_v \leq 1 \tag{3.0.1}$$

$$P_I := P \cap \mathbb{Z}^{|E|+|V|} \leftarrow \text{This is what we want} \tag{3.0.2}$$

For every vertices chosen, there must exist an edge $e \in E$ between them, then it will be a clique on the graph. To assert it we prevent the case where u, v are chosen and there is no edges betwee them (which is (2.0.1)). The second line (2.0.2) asserts the conditions that we want the integral solutions.