Name:

AMATH 515 Homework Set 3

Due: Wednesday, March 3rd by midnight.

Let f be a closed proper convex function. The convex conjugate of f, called f^*, is defined by

$$f^*(z) = \sup_x \left\{ z^T x - f(x) \right\}.$$

(1) Compute the conjugates of the following functions.
  (a) $f(x) = \delta_{\mathbb{B}_\infty}(x)$.
  (b) $f(x) = \delta_{\mathbb{B}_2}(x)$.
  (c) $f(x) = \exp(x)$.
  (d) $f(x) = \log(1 + \exp(x))$
  (e) $f(x) = x \log(x)$

(2) Let g be any convex function; f is formed using g. Compute f^* in terms of g^*.
  (a) $f(x) = \lambda g(x)$.
  (b) $f(x) = g(x - a) + \langle x, b \rangle$.
  (c) $f(x) = \inf_z \{g(x, z)\}$.
  (d) $f(x) = \inf_z \left\{ \frac{1}{2}\|x - z\|^2 + g(z) \right\}$

(3) Moreau Identities.
  (a) Derive the Moreau Identity:

$$\text{prox}_f(z) + \text{prox}_{f^*}(z) = z.$$

  (b) Use the Moreau identity and 1a, 1b to check your formulas for

$$\text{prox}_{\|\cdot\|_1}, \quad \text{prox}_{\|\cdot\|_2}$$

  from last week's homework.

(4) Duals of regularized GLM. Consider the Generalized Linear Model family:

$$\min_x \sum_{i=1}^{n} g(\langle a_i, x\rangle) - b^T A x + R(x),$$

Where g is convex and R is any regularizer.

(a) Write down the dual obtained by dualizing g.

(b) Specify your formula to Ridge-regularized logistic regression:

$$\min_x \sum_{i=1}^{n} \log(1+\exp(\langle a_i, x\rangle)) - b^T A x + \frac{\lambda}{2}\|x\|^2.$$

(c) Specify your formula to 1-norm regularized Poisson regression:

$$\min_x \sum_{i=1}^{n} \exp(\langle a_i, x\rangle) - b^T A x + \lambda\|x\|_1.$$

Coding Assignment

Please download `515Hw3_Coding.ipynb` and `proxes.py` to complete problem (5).

(5) In this problem you will write a routine to project onto the capped simplex.
The Capped Simplex Δ_k is defined as follows:

$$\Delta_k := \left\{x : 1^T x = k, \quad 0 \le x_i \le 1 \quad \forall i.\right\}$$

This is the intersection of the k-simplex with the unit box.
The projection problem is given by

$$\mathrm{proj}_{\Delta_k}(z) = \arg\min_{x\in\Delta_k} \frac{1}{2}\|x-z\|^2.$$

(a) Derive the (1-dimensional) dual problem by focusing on the $1^T x = k$ constraint.

(b) Implement a routine to solve this dual. It's a scalar root finding problem, so you can use the root-finding algorithm provided in the code.

(c) Using the dual solution, write down a closed form formula for the projection. Use this formula, along with your dual solver, to implement the projection. You can use the unit test provided to check if your code id working correctly.