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Problem (1)

Objective: Prove the identity for $\alpha \in \mathbb{R}$:

$$\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2$$

Compute it directly from left to right:

$$\begin{aligned}
&\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\| \\
&\alpha^2\|x\|^2 + (1-\alpha)^2\|y\|^2 + 2\alpha(1-\alpha)x^Ty + \alpha(1-\alpha)(\|x\|^2 + \|y\|^2 - 2x^Ty) \\
&= \|x\|^2((\alpha^2 + \alpha(1-\alpha)) + \|y\|^2((1-\alpha)^2 + \alpha(1-\alpha)) + x^Ty(2\alpha(1-\alpha) - 2\alpha(1-\alpha)) \\
&= \|x\|^2(\alpha^2 + \alpha - \alpha^2) + \|y\|^2((1-\alpha^2) + \alpha - \alpha^2) + x^Ty(0) \\
&= \alpha\|x\|^2 + (1-\alpha)\|y\|^2
\end{aligned} \tag{1}$$

Problem (2)

(a)

Objective: Show that T_λ and T has the same fixed points. x is a fixed point of the non-expansive operator T if $x \in Tx$, then we want to show that picking any $x \in Tx \implies x \in T_\lambda x$
T_λ is defined to be: $T_\lambda := (1-\lambda)I - \lambda T$.
Consider:

$$\begin{aligned}
T_\lambda(Tx) &= (1-\lambda)Tx + \lambda Tx \\
&= Tx - \lambda Tx + \lambda Tx \\
&= Tx
\end{aligned} \tag{2a1}$$

If x is a fixed for for T, then $x \in Tx$ which means $T_\lambda(Tx) \implies T_\lambda(x) = Tx$ for any x so $x = T_\lambda(x)$.

(b)

Objective: With that assumption that $\bar{z}$ is the fixed point of the operator T, show that:

$$\|T_\lambda z - \bar{z}\|^2 \le \|z - \bar{z}\|^2 - \lambda(1-\lambda)\|z - Tz\|^2$$

Because of fixed points we know that $\|T_\lambda z - \bar{z}\| = \|T_\lambda(z - \bar{z})\|$ then by definition of the T_λ:

$$\begin{aligned}
\|T_\lambda(z-\bar{z})\|^2 &= \|(1-\lambda)\underbrace{(z-\bar{z})}_{y} + \lambda\underbrace{T(z-\bar{z})}_{x}\|^2 \\
&= \|(1-\lambda)y + \lambda x\|^2 \\
\underset{\text{Using Problem 1}}{\implies} &= -\lambda(1-\lambda)\|x-y\|^2 + \lambda\|x\|^2 + (1-\lambda)\|y\|^2
\end{aligned} \tag{1b1}$$

Let's pause for a moment and consider:

$$\begin{aligned}
y - x &= z - \bar{z} - T(z - \bar{z}) \\
&= z - \bar{z} - Tz - T\bar{z} \\
&= z - \bar{z} - Tz - \bar{z} \\
&= z - T(z)
\end{aligned} \tag{2b2}$$

Let's continue on latest step from 1b1:

$$\begin{aligned}
\|T_\lambda z - \bar{z}\|^2 &= -\lambda(1-\lambda)\|z - Tz\|^2 + \lambda\|x\|^2 + (1-\lambda)\|y\|^2 \\
&= -\lambda(1-\lambda)\|z - Tz\|^2 + \underbrace{\lambda\|T(z-\bar{z})\|^2}_{\leq \lambda\|z-\bar{z}\|} + (1-\lambda)\|z-\bar{z}\|^2 \\
&\leq -\lambda(1-\lambda)\|z - Tz\|^2 + \lambda\|z-\bar{z}\|^2 + (1-\lambda)\|z-\bar{z}\|^2 \\
&= \|z-\bar{z}\|^2 - \lambda(1-\lambda)\|z - Tz\|^2
\end{aligned} \tag{1b3}$$

It has been shown.

Problem (3)

(a)

Objective: If T is a firmly non-expansive operator, then we want to show that:

$$\|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 \leq \|x-y\|^2 \iff \langle x-y, Tx-Ty\rangle \geq \|Tx-Ty\|^2$$

Let's focus on one of the terms that is one the LHS:

$$\begin{aligned}
&= \|(I-T)x - (I-T)y\|^2 \\
&= \|x - Tx - y + Ty\|^2 \\
&= \|x - Tx - y + Ty\|^2 \\
&= \|x - y + Ty - Tx\|^2
\end{aligned} \tag{3a1}$$

Now, notice that I can move this term to the RHS to the definition of the non-expansive operator, and then we will have:

$$\begin{aligned}
&\|x-y\|^2 - \|(x-y) - (Ty-Tx)\|^2 \\
&= \|x-y\|^2 - \left[\|x-y\|^2 + \|Ty-Tx\|^2 + 2(x-y)^T(Ty-Tx)\right] \\
&= -\|Ty-Tx\|^2 - 2(x-y)^T(Ty-Tx)
\end{aligned} \tag{3a2}$$

So then, we have the expression:

$$\begin{aligned}
&-\|Ty-Tx\|^2 - 2(x-y)^T(Ty-Tx) \geq \|Tx-Ty\|^2 \\
&\iff -2(x-y)^T(Ty-Tx) \geq 2\|Tx-Ty\|^2 \\
&\iff (x-y)^T(Tx-Ty) \geq \|Tx-Ty\|^2 \\
&\iff \langle x-y, Tx-Ty\rangle \geq \|Tx-Ty\|^2
\end{aligned} \tag{3a3}$$

(b)

Objective: Show that:

$$\langle Tx-Ty, (I-T)x - (I-T)y\rangle \geq 0$$

For this part we can choose to continue what we derived before and them show the equivalency of this statement and the previous statement.

$$\begin{aligned}
\langle x-y, Tx-Ty\rangle &\geq \|Tx-Ty\|^2 \\
\langle x-y, Tx-Ty\rangle &\geq \langle Tx-Ty, Tx-Ty\rangle \\
\langle x-y, Tx-Ty\rangle - langle Tx-Ty, Tx-Ty\rangle &\geq 0 \\
\langle Tx-Ty, x-y-Tx+Ty\rangle &\geq 0 \\
\langle Tx-Ty, (I-T)x + (I-T)y\rangle &\geq 0
\end{aligned} \tag{3b1}$$

(c)

Objective: Suppose that $S = 2T - I$, then we let:

$$\mu = \|Tx - Ty\|^2 + \underbrace{\|(I-T)x - (I-T)y\|^2}_{(1)} - \|x-y\|^2$$

$$v = \|Sx - Sy\|^2 - \|x-y\|^2$$

Then sow that $2\mu = v$
Consider the expression (1), we should have:

$$\begin{aligned}
&\|(I-T)x - (I-T)y\|^2 \\
&= \|x - Tx - y + Ty\| \\
&= \|(Ty - Tx) + (x-y)\|^2 \\
&= \|Ty - Tx\|^2 + \|x-y\|^2 + 2(x-y)^T(Ty - Tx)
\end{aligned} \tag{3c1}$$

And notice that substituting in we have:

$$\begin{aligned}
\mu &= 2\|Ty - Tx\|^2 + 2(x-y)^T(Ty-Tx) \\
2\mu &= 4\|Ty - Tx\|^2 + 4(x-y)^T(Ty-Tx) \\
&= \|2Ty - 2Tx\|^2 + 2(x-y)^T(2Ty - 2Tx) + \|x - y^2\| - \|x-y\|^2 \\
&= \|2Ty - 2Tx + x - y\|^2 - \|x-y\|^2 \\
&= \|2Tx - 2Ty + y - x\|^2 - \|x-y\|^2 \\
&= \|Sx - Sy\|^2 - \|x-y\|^2 \\
&= v
\end{aligned} \tag{3c2}$$

By the non-expansive property for the operator S we have:

$$\begin{aligned}
v = \|Sx - Sy\|^2 - \|x-y\|^2 &\leq 0 \\
2\mu &\leq 0 \\
\mu &\leq 0 \\
\|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 &\leq \|x-y\|^2
\end{aligned} \tag{3c3}$$

Therefore, we can conclude that, if the operator S is non expansive, meaning that $2T - I$ is non-expansive, the operator T is firmly non-expansive.