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AMATH 585 2022 WINTER HW7

Problem 1

The conjugate gradient algorithm is given as:

$$
\begin{aligned}
& p_0 = b - Ax_0 \\
& \text{For } i = 0, 1, \cdots \\
& \quad a_i = \frac{\|r_i\|^2}{\|p_i\|_A^2} \\
& \quad x_{i+1} = x_i + a_i p_i \\
& \quad r_{i+1} = r_i - a_i A p_i \\
& \quad b_i = \frac{\|r_{i+1}\|_2^2}{\|r_i\|_2^2} \\
& \quad p_{i+1} = r_{i+1} + b_i p_i
\end{aligned}
$$

The algorithm has been rephrased. And we make the assumption that the matrix A is symmetric poisitive definite. In addition, observe that a_i, b_i are non-negative real numbers. This means that we can move then around even if the vector in the inner products can be complex. We wish to prove 3 hypothesis about the algorithm inductively:

$$
\begin{aligned}
\mathcal{H}_1(k) &\equiv \forall\, 0 \le j \le k-1 : \langle r_k, p_j \rangle = 0 \\
\mathcal{H}_2(k) &\equiv \forall\, 0 \le j \le k-1 : \langle p_k, Ap_j \rangle = 0 \\
\mathcal{H}_3(k) &\equiv \forall\, 0 \le j \le k-1 : \langle r_k, r_j \rangle = 0
\end{aligned} \tag{1.1}
$$

First we verify the basecase by considering: $\mathcal{H}_1(1), \mathcal{H}_2(1), \mathcal{H}_3(1)$.

$$
\begin{aligned}
\langle r_1, r_0 \rangle &= \langle r_0 - a_0 A p_0, r_0 \rangle \\
&= \langle r_0, p_0 \rangle - a_0 \langle r_0, A p_0 \rangle \\
&= \langle r_0, r_0 \rangle - a_0 \langle r_0, A r_0 \rangle \\
&= 0 \\
&\implies \mathcal{H}_3(1) \text{ is true} \\
\langle p_1, A p_0 \rangle &= \langle r_1, A p_0 \rangle + \frac{\langle r_1, r_1 \rangle}{\langle r_0, r_0 \rangle} \langle p_0, A p_0 \rangle \\
&= \langle r_1, a_0^{-1}(r_0 - r_1) \rangle + a_0^{-1} \langle r_1, r_1 \rangle \\
\mathcal{H}_3(1) \implies &= -a_0^{-1} \langle r_1, r_1 \rangle + a_0^{-1} \langle r_1, r_1 \rangle \\
&\implies \mathcal{H}_2(1) \text{ is true} \\
\langle r_1, p_0 \rangle &= \langle r_1, r_0 \rangle \\
\mathcal{H}_3(1) \implies &= 0 \\
&\implies \mathcal{H}_1(1) \text{ is true}
\end{aligned} \tag{1.2}
$$

Basecase is asserted by the definition of the starting conditions and the coefficient a_0. next we assume that $\mathcal{H}_1(k), \mathcal{H}(k), \mathcal{H}(k)$ are all true, and then we wish to prove inductively that they remainds to be true. First, we establish some equalities to simplify the proof, and then we prove it.

$$
\begin{aligned}
\langle p_k, A p_k \rangle &= \langle r_k + b_{k-1} p_{k-1}, A p_k \rangle \\
&= \langle r_k, A p_k \rangle \quad \text{by: } \mathcal{H}_2(k) \\
\langle r_k, p_k \rangle &= \langle r_k, r_k + b_{k-1} p_{k-1} \rangle \\
&= \langle r_k, r_k \rangle \quad \text{by: } \mathcal{H}_1(k)
\end{aligned} \tag{1.3}
$$

The first is implied by $\mathcal{H}_2(k)$ and the second one is asserted by $\mathcal{H}_1(k)$. Next, we prove that $\mathcal{H}_3(k+1)$ is true.

$$\begin{aligned}
\langle r_{k+1}, r_k \rangle &= \langle r_k, r_k \rangle - a_k \langle r_k, Ap_k \rangle \\
&= \langle r_k, r_k \rangle - a_k \langle p_k, Ap_k \rangle \quad \text{by (1.3)} \\
&= \langle r_k, r_k \rangle - \langle r_k, r_k \rangle \\
&= 0 \\
\forall\, 0 \leq j \leq k-1: \quad \langle r_{k+1}, r_j \rangle &= \langle r_k - a_k A p_k, r_j \rangle \\
&= \langle r_k, r_j \rangle - a_k \langle Ap_k, r_j \rangle \\
&= -a_k \langle Ap_k, r_j \rangle \\
&= -a_k \langle Ap_k, p_j - b_{j-1} p_{j-1} \rangle \\
&= 0 \quad \text{by } \mathcal{H}_2(k) \\
&\implies \mathcal{H}_3(k+1) \text{ is true.}
\end{aligned} \tag{1.4}$$

Next, we consider:

$$\begin{aligned}
\langle r_{k+1}, p_k \rangle &= \langle r_k, p_k \rangle - a_k \langle Ap_k, p_k \rangle \\
&= \langle r_k, r_k \rangle - a_k \langle Ap_k, p_k \rangle \quad \text{By: (1.3)} \\
&= 0 \\
\forall\, 0 \leq j \leq k-1: \quad \langle r_{k+1}, p_j \rangle &= \langle r_k - a_k A p_k, p_j \rangle \\
&= \langle r_k, p_j \rangle - a_k \langle Ap_k, p_j \rangle \\
&= 0 \quad \text{by: } \mathcal{H}_1(k) \wedge \mathcal{H}_2(k) \\
&\implies \mathcal{H}_1(k+1) \text{ is true}
\end{aligned} \tag{1.5}$$

One last hyphothesis to prove. Consider:

$$\begin{aligned}
\langle p_{k+1}, Ap_k \rangle &= \langle r_{k+1}, Ap_k \rangle + b_k \langle p_k, Ap_k \rangle \\
&= \langle r_{k+1}, Ap_k \rangle + \frac{\langle r_{k+1}, r_{k+1} \rangle}{\langle r_k, r_k \rangle} \langle p_k, Ap_k \rangle \\
&= \langle r_{k+1}, Ap_k \rangle + a_k^{-1} \langle r_{k+1}, r_{k+1} \rangle \\
&= \langle r_{k+1}, a_k^{-1}(r_k - r_{k+1}) \rangle + a_k^{-1} \langle r_{k+1}, r_{k+1} \rangle \\
&= \langle r_{k+1}, -a_k^{-1} r_{k+1} \rangle + a_k^{-1} \langle r_{k+1}, r_{k+1} \rangle \quad \text{by: } \mathcal{H}_3(k+1) \\
&= 0 \\
\forall\, 0 \leq j \leq k-1: \quad \langle p_{k+1}, Ap_j \rangle &= \langle r_{k+1} + b_k p_k, Ap_j \rangle \\
&= \langle r_{k+1}, a_j^{-1}(r_j - r_{j+1}) \rangle \\
&= 0 \quad \text{by: } \mathcal{H}_3(k+1) \\
&\implies \mathcal{H}_2(k+1) \text{ is true.}
\end{aligned} \tag{1.6}$$

All hypotheses fall through, and the base case is true by the algorithm. The proof has been completed.