Assignment 7.

Due Wednesday, Mar. 9.

Reading: Ch. 4 in text. Supplementary material can be found in *Iterative Methods for Solving Linear Systems*, by A. Greenbaum, SIAM, 1997, chs. 2 and 3.

1. The conjugate gradient method for solving a symmetric positive definite linear system $Ax = b$ can be written as below:

   Given x_0, compute $r_0 = b - Ax_0$, and set $p_0 = r_0$.
   For $k = 1, 2, \ldots,$
     Compute Ap_{k-1}.
     Set $x_k = x_{k-1} + a_{k-1}p_{k-1}$, where $a_{k-1} = \frac{\langle r_{k-1}, r_{k-1}\rangle}{\langle p_{k-1}, Ap_{k-1}\rangle}$.
     Compute $r_k = r_{k-1} - a_{k-1}Ap_{k-1}$.
     Set $p_k = r_k + b_{k-1}p_{k-1}$, where $b_{k-1} = \frac{\langle r_k, r_k\rangle}{\langle r_{k-1}, r_{k-1}\rangle}$.
   Endfor

   Show that the residual vectors $r_0, \ldots, r_k$ are orthogonal to each other ($\langle r_i, r_j\rangle = 0$ if $i \neq j$) and that the direction vectors $p_0, \ldots, p_k$ are A-orthogonal ($\langle p_i, Ap_j\rangle = 0$ if $i \neq j$). [Hint: First show that $\langle r_1, r_0\rangle = \langle p_1, Ap_0\rangle = 0$ and then use induction on k.]

2. Repeat the experiments on p. 103 of the text, leading to Figures 4.8 and 4.9, but use the Gauss-Seidel method and (unpreconditioned) CG instead of Jacobi iteration. That is, set up difference equations for the problem

   $$u''(x) = f(x), \quad u(0) = 1,\ u(1) = 3,$$

   where

   $$f(x) = -20 + a\phi''(x)\cos(\phi(x)) - a(\phi'(x))^2\sin(\phi(x)),$$

   where $a = 0.5$ and $\phi(x) = 20\pi x^3$. The true solution is

   $$u(x) = 1 + 12x - 10x^2 + a\sin(\phi(x)).$$

   Starting with initial guess $u^{(0)}$ with components $1 + 2x_i$, $i = 1, \ldots, 255$, run, say, 20 Gauss-Seidel iterations and then 20 CG iterations, plotting the true solution to the linear system and the approximate solution, say, at steps 0, 5, 10, and 20, and also plotting the error (the difference between true and approximate solution). Print the size of the error (the L_2-norm or the ∞-norm) at these steps too. Based on your results, would you say that Gauss-Seidel and CG would be effective smoothers for a multigrid method?

3. Implement a 2-grid method for solving the 1D model problem with homogeneous Dirichlet boundary conditions:

$$u_{xx} = f(x), \quad u(0) = u(1) = 0.$$

Use linear interpolation to go from the coarse grid with spacing $2h$ to the fine grid with spacing h. Take the projection matrix I_h^{2h}, going from the fine grid to the coarse grid, to be 0.5 times the transpose of the interpolation matrix: $I_h^{2h} = \frac{1}{2}(I_{2h}^h)^T$. Use a multigrid V-cycle with 1 smoothing step on each visit to each grid level. Try weighted Jacobi and Gauss-Seidel as the smoothing step. Try several different values of the mesh spacing h and show that you achieve convergence to a fixed tolerance in a number of cycles that is independent of the mesh size.

4. (a) Consider an iteration of the form

$$x_k = x_{k-1} + M^{-1}(b - Ax_{k-1}),$$

for solving a nonsingular linear system $Ax = b$. Note that the error $e_k := A^{-1}b - x_k$ satisfies

$$e_k = (I - M^{-1}A)e_{k-1} = \ldots = (I - M^{-1}A)^k e_0.$$

Assume that $\|e_0\|_2 = 1$ and that $\|I - M^{-1}A\|_2 = \frac{1}{2}$. Estimate the number of iterations required to reduce the 2-norm of the error below 2^{-20}. Show how you obtain your estimate. Now suppose you know only that the spectral radius $\rho(I - M^{-1}A) = \frac{1}{2}$. Can you give an estimate of the number of iterations required to reduce the 2-norm of the error below 2^{-20}? Explain why or why not.

(b) Consider the GMRES algorithm applied to an n by n matrix A with the sparsity pattern pictured below:

$$\begin{bmatrix} * & * & \cdots & * & * \\ * & * & \cdots & * & 0 \\ 0 & * & \cdots & * & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & * & 0 \end{bmatrix},$$

where the $*$'s represent nonzero entries. Show that if the initial residual is the nth unit vector $(0, \ldots, 0, 1)^T$, then the algorithm makes no progress until step n. Show that a matrix with this sparsity pattern can have *any* eigenvalues. Conclude that eigenvalue information alone cannot be enough to ensure fast convergence of the GMRES algorithm.