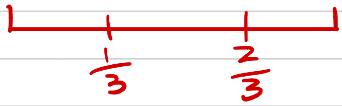
HW2

§ Problem 2

#2(a)

$*$ $u''(x) = f(x)$, $u(0) = u(1) = 1$, $h = \frac{1}{3}$

$$A = 9\begin{bmatrix} 1/9 & & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & & 1/9 \end{bmatrix}$$



$$\vec{u} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad \hat{f} = \begin{bmatrix} 1 \\ f(x_1) \\ f(x_2) \\ 1 \end{bmatrix}$$

#2(b)

Let B be the inverse of A, then columns of B are $hG(x_i, x_j)$, the Greens function to $h\delta(x-x_j)$.

$$B_{i,j} = hG(x_i, x_j) = \begin{cases} h(x_j-1)x_i & i=1,2,\cdots j \quad (i\le j) \\ h(x_i-1)x_j & i=j,j+1\cdots m \quad (i\ge j) \\ 0 & i=m+1, 0 \end{cases}$$

for the interior points:

$$B_{i,0} = G_0(x_i) = 1-x_i \qquad B_{i,m+1} = G_1(x_i) = x_i$$

for the first

* A^{-1} can be written out as:

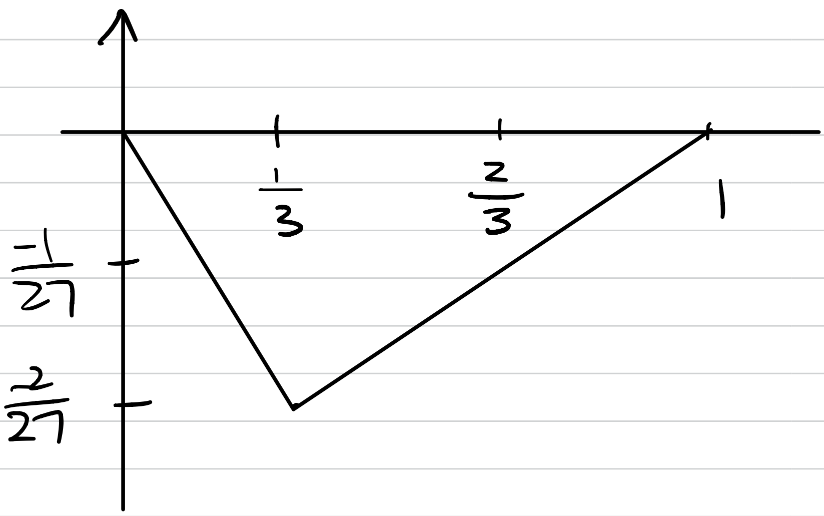
$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2/3 & 1/3\left(\frac{-2}{3}\right)\left(\frac{1}{3}\right) & 1/3\left(\frac{-1}{3}\right)\frac{1}{3} & 1/3 \\ 1/3 & 1/3\left(\frac{-1}{3}\right)\left(\frac{1}{3}\right) & 1/3\left(\frac{-1}{3}\right)\frac{2}{3} & 2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2/3 & \frac{-2}{27} & \frac{-1}{27} & 1/3 \\ 1/3 & \frac{-1}{27} & \frac{-2}{27} & 2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

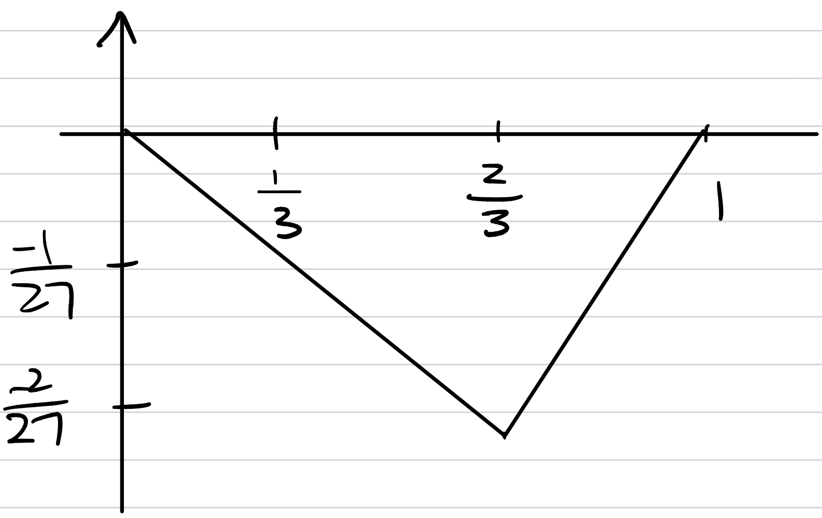
#2(c)

* Let $f(x) = x$, determine discrete approx to this BVP problem.
* Sketch the solution
* Sketch the 4 Green's function.

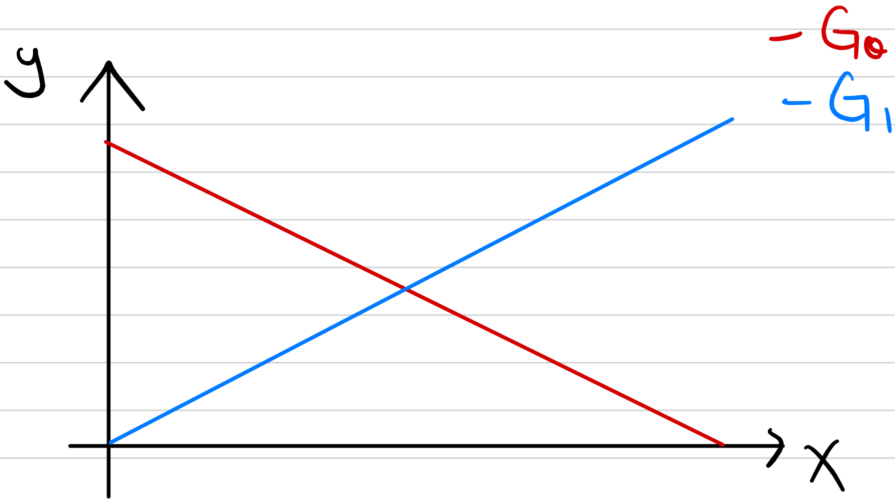
$G(x, \frac{1}{3})$:



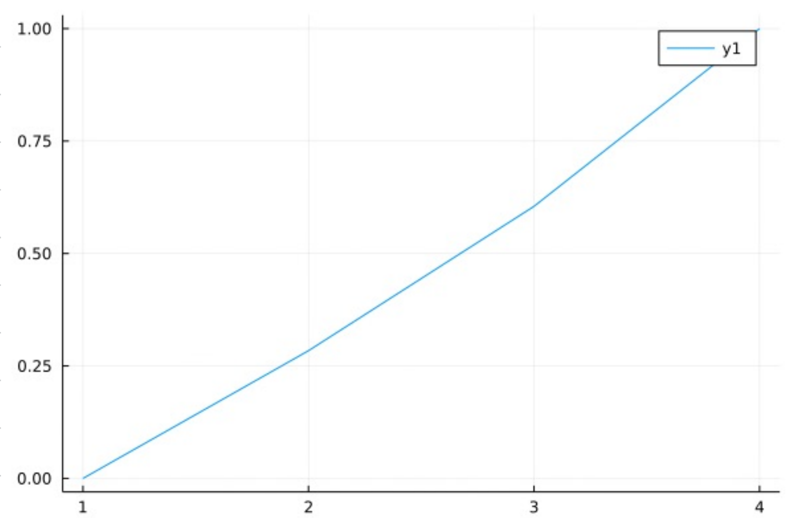
$G(x, \frac{2}{3})$



$G_0(x)$ and $G_1(x)$



u :: Approximated:



§ Problem 3

* 3(a)

Consider:

$$h\sum_{j=0}^{m}\frac{g(x_j)+g(x_{j+1})}{2} \approx \int_a^b g(x)\,dx$$

g is smooth, then consider taylor series:

$$g(x+h)=g(x)+hg'(x)+\frac{h^2}{2}g''(x)+O(h^3)$$

Consider each term in the sum using taylor expansion:

$$h\frac{g(x_j)+g(x_j+h)}{2}$$

$$=\frac{h}{2}\left(2g(x_j)+hg'(x_j)+\frac{h^2}{2}g''(x_j)+O(h^3)\right)$$

$$=hg(x_j)+\frac{h}{2}\left(hg'(x_j)+\frac{h^2}{2}g''(x_j)+O(h^3)\right)$$

$$=hg(x_j)+\frac{1}{2}\left(h^2g'(x_j)+\frac{h^3}{2}g''(x_j)+O(h^4)\right)$$

which serves an approximation to $\int_{x_j}^{x_j+h} g(x)\,dx$.

Compute this exact quantity:

$$\int_{x_j}^{x_j+h} g(x)\,dx = \int_0^h g(x_j+\xi)\,d\xi$$

usub: $x=x_j+\xi$

$$=\int_0^h g(x_j)+\xi g'(x_j)+\frac{\xi^2}{2}g''(x_j)+O(\xi^3)\,d\xi$$

$$= hg(x_j) + \frac{h^2}{2} g'(x_j) + \frac{h^3}{3!} g''(x_j) + O(h^4)$$

We used tayler series and its uniform convergence. Whats the local error betwen integral and discrete sum?

$$\int_{x_j}^{x_j+h} g(x)\,dx - \frac{h}{2}\left(g(x_j) + g(x_j+h)\right)$$

$$= hg(x_j) + \frac{h^2}{2} g'(x_j) + \frac{h^3}{3!} g''(x_j) + O(h^4) - \ldots$$

$$hg(x_j) - \frac{1}{2}\left(h^2 g'(x_j) + \frac{h^3}{2} g''(x_j) + O(h^4)\right)$$

$$= \frac{h^3}{3!} g''(x_j) - \frac{h^3}{2} g''(x_j) + O(h^4)$$

$$\in O(h^3)$$

Then the error Golabally loses one degree of accuracy because:

$$\int_a^b g(x)\,dx - \sum_{j=0}^{m} h \frac{g(x_j) + g(x_{j+1})}{2}$$

$$= \sum_{j=0}^{m} \int_{x_j}^{x_j+h} g(x)\,dx - \frac{h}{2}\left(g(x_j) + g(x_j+h)\right)$$

$$= \sum_{j=0}^{m} \frac{h^3}{3!} g''(x_j) - \frac{h^3}{2} g''(x_j) + O(h^4)$$

$$= \sum_{j=0}^{m} g''(x_j) h^3 \left(\frac{-1}{3}\right) + O(h^4)$$

* Suppose $\sup_{x \in [0,1]} |g''(x)| \le M$,

$$\Rightarrow \left| \sum_{j=0}^{m} g''(x_j) h^3 \left(\frac{-1}{3}\right) + O(h^4) \right| \le h^3 \left(\frac{-1}{3}\right) m M + O(h^3)$$

observe: $m = \frac{b-a}{h}$

$$\Rightarrow h^3 \left(\frac{-1}{3}\right) m M + O(h^3) \in O(h^2)$$

If the second derivative is bounded then the global error is $O(h^2)$.

*3(b)

The BVP is: $\begin{cases} u''(x) = f(x) \\ u(0) = u(1) = 0 \end{cases}$

Solution can be written as:

$$u(x) = \int_0^1 f(\bar{x}) G(x;\bar{x}) \, d\bar{x} \qquad (1)$$

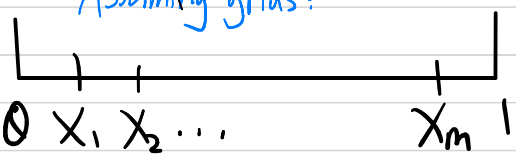
Using central diff, and solving it gives approximation:

$$u_i = h \sum_{j=1}^{m} f(x_j) G(x_i | x_j), \quad i = 1 \ldots m \qquad (2)$$

*Show (2) is the trapz approximation to (1)

$$u(x_i) = \int_0^1 f(\bar{x}) G(x|\bar{x}) \, d\bar{x}$$

Assuming grids:



$$\approx \sum_{j=0}^{m} \frac{h}{2} \Big(f(x_j) G(x_i | x_j) + f(x_j + h) G(x_i | x_j + h) \Big)$$

$$= \sum_{j=0}^{m} \frac{h}{2} \big(f(x_j) G(x_i | x_j) \big) + \sum_{j=1}^{m+1} \frac{h}{2} \big(f(x_j) G(x_i | x_j) \big)$$

$$= \frac{h}{2} \big(f(x_0) G(x_i | x_0) + f(x_{m+1}) G(x_i | x_{m+1}) \big) + \ldots$$

$$\sum_{j=1}^{m} h \, f(x_j) G(x_i | x_j)$$

$$= 0 + \sum_{j=1}^{m} h \big(f(x_j) G(x_i | x_j) \big) = u_i$$

* Therefore (2) is trapz for $u(x_i)$

Since trapz has $O(h^2)$; then $u_i - u(x_i) \in O(h^2)$ as well.

* $G(x|\bar{x})$ has discontinous derivative at $x = x_j$, why does this not degrade the accuracy of the composite trapzoidal rule?

Short answer: Because Green's function is discontinuous on one of the grid points x_j.

Long answer:

There is $O(h^2)$ error when approximating the integral using trapzoid rule locally because:

$$\int_{x_j}^{x_{j+1}} f(\bar{x}) G(x_i|\bar{x}) d\bar{x} = \frac{h}{2}\Big(f(x_j)G(x_i|x_j) + f(x_j)G(x_i|x_j)\Big) + O(h^2)$$

This is true because $G(x_i|x_j)$ is a linear function in the interval (x_j, x_{j+1}), and if $f(\bar{x})$ has bounded second derivative, then using the analysis from 3(a), the error is $O(h^3)$ locally, and $O(h^2)$ globally.

§ Problem 4: Green's function With Neuman Boundary

4(a)

$u'' = f(x)$, Neuman Boundary at $x=0$, Dirichilet B.C at $x=1$

$$\begin{cases} u''(x) = \delta(x-\bar{x}) \\ u'(0) = 0,\ u(1) = 0 \end{cases}$$

G_0 solves:

$$\begin{cases} u''(x) = 0 \\ u'(0) = 1,\ u(1) = 0 \end{cases}$$

$$\int\int u''(x)\,dx\,dx = \int\int 0\,dx\,dx$$

$$= \int c\,dx = Cx + D$$

$$\partial_x (Cx+D)\Big|_{x=0} = 1$$

$$C = 1$$

$$(Cx+D)\Big|_{x=1} = 0$$

$$C + D = 0$$

$$1 + D = 0$$

$$D = -1$$

$$\Rightarrow G_0(x) = x - 1$$

$G_1(x)$ Solves :

$$\begin{cases} u''(x) = 0 \\ u'(0) = 0,\ u(1) = 1 \end{cases}$$

Similar to previous case, $G_1(x)$ takes the form: $Cx + D$

$$\partial_x (Cx + D)\Big|_{x=0} = 0$$

$$C = 0$$

$$Cx + D\Big|_{x=1} = 1$$

$$D = 1$$

$$D = 1$$

$$\Rightarrow G_1(x) = 1$$

$G(x|\bar{x})$?

$$\Longrightarrow \quad \begin{cases} u''(x) = \delta(x - \bar{x}) \\ u'(0) = 0,\ u(1) = 0 \end{cases}$$

$$\partial_x G(x|\bar{x})\Big|_{x=0} = 0$$

$$G(1|\bar{x}) = 0$$

$$\forall \varepsilon > 0 : \int_{\bar{x}-\varepsilon}^{\bar{x}+\varepsilon} G''(x|\bar{x})dx = 1$$

$$= G'(\bar{x}+\varepsilon|\bar{x}) - G'(\bar{x}-\varepsilon|\bar{x}) = 1$$

* $\int_0^{\bar{x}-\varepsilon}\int_0^{\bar{x}-\varepsilon} G(x|\bar{x})\,dx\,dx \quad \forall x \in [0, \bar{x}-\varepsilon]$

$= C_1 x + C_2$

$\int_{\bar{x}+\varepsilon}^{1}\int_{\bar{x}+\varepsilon}^{1} G(x|\bar{x})\,dx\,dx = C_3 x + C_4$

as $\varepsilon \to 0$

$$\Rightarrow G(x|\bar{x}) = \begin{cases} C_1 x + C_2 & x \in [0, \bar{x}] \\ C_3 x + C_4 & x \in [\bar{x}, 1] \end{cases}$$

Solving for the Coefficients!

Consider the Boundary Conditions first:

* $G'(0) = 0 \Rightarrow \partial_x [C_1 x + C_2]\Big|_{x=0} = 0$

$C_1 = 0$

* $G(1) = 0 \Rightarrow C_3 + C_4 = 0 \Rightarrow$

$$\Rightarrow G(x|\bar{x}) = \begin{cases} C_2 & x \in [0, \bar{x}] \\ C_3 x - C_3 & x \in [\bar{x}, 1] \end{cases}$$

$G'(\bar{x}+\varepsilon|\bar{x}) - G'(\bar{x}-\varepsilon|\bar{x}) = 1$

$= C_3 - 0 = 1 \Rightarrow C_3 = 1$

$$\Rightarrow G(x|\bar{x}) = \begin{cases} C_2 & x \in [0, \bar{x}] \\ x-1 & x \in [\bar{x}, 1] \end{cases}$$

$G(x|\bar{x})$ continuous!

$$\Rightarrow C_2 = \bar{x} - 1$$

then $G(x|\bar{x}) = \begin{cases} \bar{x}-1 & x \in [0, \bar{x}] \\ x-1 & x \in [\bar{x}, 1] \end{cases}$

Combining the Green's function to get the solutions.

$G_0(x)$ is like $\begin{cases} G_0''(x) = 0 \\ G_0'(0) = 1, \quad G_0(1) = 0 \end{cases}$

$G_1(x)$ is like: $\begin{cases} G_1''(x) = 0 \\ u'(0) = 0, \quad u(1) = 1 \end{cases}$

$G(x|\bar{x})$ is like: $\begin{cases} G''(x|\bar{x}) = \delta(x-\bar{x}) \\ u'(0) = 0, \quad u(1) = 0 \end{cases}$

In general Nauman at $x=0$ and Dirichilet at $x=1$ is the system:

$$\begin{cases} u''(x) = f(x) \\ u'(0) = \alpha, \quad u(1) = \beta \end{cases}$$

Then the solution expressed using Green's function will be:

$$u(x) = \alpha G_0(x) + \int_0^1 f(\bar{x}) G(x|\bar{x}) d\bar{x} + \beta G_1(x)$$

Discretizing it at grid points: $x_0 x_1 \cdots x_m x_{m+1}$, width: h

$$u_i = \alpha G_0(x_i) + \sum_{j=1}^{m} h f(x_j) G(x_i|x_j) + \beta G_1(x_i)$$

$$\vec{u} = \underbrace{\alpha G_0(\vec{x})}_{\text{first column of } A^{-1}} + \underbrace{\sum_{j=1}^{m} h f(\vec{x}) G(\vec{x}|x_j)}_{\text{Columns in the middle of } A^{-1}} + \underbrace{\beta G_1(\vec{x})}_{\text{last column of } A^{-1}.}$$

* 4(b)

\# The inverse for (2.54) should follow the same rule as the inverse for (2.43) in the text which we derived in the lecture.

(2.54) the $\mathcal{O}(h)$ one

$$A = \frac{1}{h^2}\begin{bmatrix} -h & h & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \\ & & & & 0 & h^2 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \\ u_m \\ u_{m+1} \end{bmatrix} = \begin{bmatrix} \sigma \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) \\ \beta \end{bmatrix}$$

\# Similar to (2.43), index starts with zero:

$$(A^{-1})_{:,1} = G_0(\vec{x}) \qquad (A^{-1})_{:,m+1} = G_1(\vec{x})$$

$$(A^{-1})_{:,j} = h\,G(\vec{x} \mid x_j)$$

\# When $h = \frac{1}{3}$, $x_0, x_1, x_2, x_3 = 0, \frac{1}{3}, \frac{2}{3}, 1$

$$A = 9\begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & 0 & \frac{1}{9} \end{bmatrix}$$

Let $h=\frac{1}{3}$, then: $x_0, x_1, x_2, x_3 = 0, \frac{1}{3}, \frac{2}{3}, 1$

$$(A^{-1})_{:,0} = \begin{bmatrix} 0 \\ 1/3 \\ 2/3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2/3 \\ -1/3 \\ 0 \end{bmatrix}$$

$$(A^{-1})_{:,3} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Recall:

$$G(x|\bar{x}) = \begin{cases} \bar{x}-1 & x \in [0, \bar{x}] \\ x-1 & x \in [\bar{x}, 1] \end{cases}$$

$$(A^{-1})_{:,1} = \frac{1}{3} G(\vec{x} \mid \tfrac{1}{3}) \qquad (A^{-1})_{:,2} = \frac{1}{3} G(\vec{x} \mid \tfrac{2}{3})$$

$$= \frac{1}{3} \begin{bmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix} \qquad = \frac{1}{3} \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 & \frac{-2}{9} & \frac{-1}{9} & 1 \\ -2/3 & \frac{-2}{9} & \frac{-1}{9} & 1 \\ -1/3 & \frac{-1}{9} & \frac{-1}{9} & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

§Problem 5:

$$A = \frac{1}{h^2}\begin{bmatrix} -h & h & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \\ & & & & h & -h \end{bmatrix}$$

$$A = \frac{1}{h^2}\begin{bmatrix} -h & h(e_1^{(m)})^T & 0 \\ \vec{e}_1^{(m)} & \hat{A} & \vec{e}_m^{(m)} \\ 0 & h(\vec{e}_m^{(m)})^T & -h \end{bmatrix} \qquad \tilde{A} = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

$\tilde{A}$ is invertible $\qquad \tilde{A} = A^T \quad A^{-1}$ D.N.E

$$A^T = \frac{1}{h^2}\begin{bmatrix} -h & (\vec{e}_1^{(m)})^T & 0 \\ h(e_1^{(m)}) & \tilde{A} & h\vec{e}_m^{(m)} \\ 0 & (\vec{e}_m^{(m)})^T & -h \end{bmatrix}$$

$$= \frac{1}{h^2}\begin{bmatrix} -h & (\vec{e}_1^{(m)})^T & 0 \\ h(e_1^{(m)}) & \tilde{A} & h\vec{e}_m^{(m)} \\ 0 & (\vec{e}_m^{(m)})^T & -h \end{bmatrix}\begin{bmatrix} x_1 \\ \vec{y} \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \vec{0} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -hx_1 + y_1 \\ hx_1 e_1^{(m)} + \tilde{A}\vec{y} + hx_2 \vec{e}_m^{(m)} \\ y_m - x_2 h \end{bmatrix} = \begin{bmatrix} 0 \\ \vec{0} \\ 0 \end{bmatrix}$$

$$(y_1) = hx_1 \quad x_1 = (y_1)/h$$

$$(y_m) = hX_2 \quad x_2 = (y_m)/h$$

$$hx_1 e_1^{(m)} + \tilde{A}\vec{y} + hx_2 \vec{e}_m^{\,(m)}$$

$$= y_1 e_1^{(m)} + \tilde{A}\vec{y} + y_m \vec{e}_m^{\,(m)} = \vec{0}$$

$$\tilde{A}y = -(\vec{y}_1)e_1^{(m)} - (y_m)\vec{e}_m^{\,(m)}$$

$$(\tilde{A} + e_1 e_1^T + e_m e_m^T)y = \vec{0}$$

$$\tilde{A} = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

$$\Rightarrow \tilde{A} + e_1 e_1^T + e_m^T e_m = \begin{bmatrix} -1 & 1 & & & \\ & 1 & -2 & 1 & \\ & & & \ddots & 1 \\ & & & 1 & -1 \end{bmatrix}$$

The null space of above system is:

$$y = c\vec{\mathbb{1}}$$

let $y = c\vec{\mathbb{1}}$, then

$$x_1 = c/h = x_2 = c/h$$

\# $N(A^T) = \mathrm{span}\left\{ \begin{bmatrix} h \\ \vdots \\ h \end{bmatrix} \right\}$

§ Problem 6

6(a)

$$A = \frac{1}{h^2}\begin{bmatrix} -h & h & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

$$DAD^{-1} = D\frac{1}{h^2}\begin{bmatrix} -\frac{h}{d_1} & \frac{h}{d_2} & & & \\ \frac{1}{d_1} & \frac{-2}{d_2} & \frac{1}{d_3} & \ddots & \\ & \frac{1}{d_2} & \frac{-2}{d_3} & \ddots & \frac{1}{d_m} \\ & & \ddots & & \\ & & & \frac{1}{d_{m-1}} & \frac{-2}{d_m} \end{bmatrix}$$

$$= \frac{1}{h^2}\begin{bmatrix} -\frac{hd_1}{d_1} & \frac{hd_1}{d_2} & & & \\ \frac{d_2}{d_1} & \frac{-2d_2}{d_2} & \frac{d_2}{d_3} & \ddots & \\ & \frac{d_3}{d_2} & \frac{-2d_3}{d_3} & \ddots & \frac{d_{m-1}}{d_m} \\ & & \ddots & & \\ & & & \frac{d_m}{d_{m-1}} & \frac{-2d_m}{d_m} \end{bmatrix}$$

$$
= \frac{1}{h^2}\begin{bmatrix}
-h & \frac{hd_1}{d_2} & & & \\
\frac{d_2}{d_1} & -2 & \frac{d_2}{d_3} & & \\
 & \frac{d_3}{d_2} & -2 & \ddots & \\
 & & \ddots & \ddots & \frac{d_{m-1}}{d_m} \\
 & & & \frac{d_m}{d_{m-1}} & -2
\end{bmatrix}
$$

$$\frac{hd_1}{d_2} = \frac{d_2}{d_1} \qquad hd_1^2 = d_2^2 \qquad \text{Let } d_1 = 1$$

$$\frac{d_2}{d_3} = \frac{d_3}{d_2} \qquad d_2^2 = d_3^2 \qquad \Longrightarrow \quad h = d_2^2$$

$$h^2 = d_3^2$$

$$\vdots \qquad \vdots \qquad h^4 = d_4^2$$

$$h^8 = d_4^2$$

$$\vdots$$

$$\frac{d_{j-1}}{d_j} = \frac{d_j}{d_{j-1}} \qquad d_{j-1}^2 = d_j^2 \qquad h^{2^{j-1}} = d_j^2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{d_{m-1}}{d_m} = \frac{d_m}{d_{m-1}} \qquad d_{m-1}^2 = d_m^2 \qquad h^{2^{m-1}} = d_m^2$$

$$d_j = h^{\frac{2^{j-1}}{2}} \qquad 2 \le j \le m-1$$

$$d_1 = 1$$

§6(b)

Consider the first 2 rows.

$$\begin{cases} \frac{1}{h}\left(\frac{3}{2}u_0 - 2u_1 + \frac{1}{2}u_2\right) = \sigma \\ \frac{1}{h^2}(u_0 - 2u_1 + u_2) = f(x_1) \end{cases}$$

$$\begin{cases} 3u_0 - 4u_1 + u_2 = 2h\sigma \\ u_0 - 2u_1 + u_2 = h^2 f(x_1) \end{cases}$$

$$3u_0 - 4u_1 + u_2 = 2h\sigma$$

$$u_0 = \frac{1}{3}(2h\sigma + 4u_1 - u_2)$$

$$u_0 - 2u_1 + u_2 = h^2 f(x_1)$$

$$\frac{1}{3}(2h\sigma + 4u_1 - u_2) - 2u_1 + u_2 = h^2 f(x_1)$$

$$2h\sigma + 4u_1 - u_2 - 6u_1 + 3u_2 = 3h^2 f(x_1)$$

$$2h\sigma - 2u_1 + 2u_2 = 3h^2 f(x_1)$$

$$-2u_1 + 2u_2 = 3h^2 f(x_1) - 2h\sigma$$

$$u_2 - u_1 = \frac{3}{2}h^2 f(x_1) - h\sigma$$

$$\Rightarrow \frac{1}{h^2}(u_2 - u_1) = \frac{3}{2}f(x_1) - \frac{\sigma}{h}$$

$$\frac{1}{h^2}(hu_2 - hu_1) = \frac{3h}{2}f(x_1) - \sigma$$

last 2 Rows of the matrix

$$\begin{cases} u_{m-1} - 2u_m + u_{m+1} = h^2 f(x_m) \\ u_{m+1} = \beta \end{cases}$$

$$\hookrightarrow u_{m+1} = \beta$$

$$u_{m-1} - 2u_m = h^2 f(x_m) - \beta$$

$$\frac{1}{h^2}\left(u_{m-1} - 2u_m\right) = f(x_m) - \beta/h^2$$

The system now is:

$$A = \frac{1}{h^2}\begin{bmatrix} -h & h & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} \frac{3h}{2} f(x_1) - \sigma \\ f(x_2) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) - \beta/h^2 \end{bmatrix}$$

Yes it's diagonally similar to a symmetric matrix from the last part (a).