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AMATH 585 WINTER 2022 HW 6

Problem 1

Any function with chebyshev coefficients $a_0, a_1, \cdots, a_n$, evaluated at the chbyshev node is given as:

$$p(\cos(k\pi/n)) = \sum_{j=0}^{n} a_j \cos(jk\pi/n) \tag{1}$$

Our objective here is make use hf the FFT algorithm for DFT for the objective of: Interpolation the function at chebyshev node getting the values of $a_0, a_1, \cdots, a_n$, and evaluating the function value at the chebyshev nodes using the FFT algorithm. It's implied that k, n are integers in this context.

My claim here is that, if we tiled the vector in the following format:

$$\vec{f} = [a_0, a_1, \cdots, a_{n-1}, a_n, a_{n-1}, \cdots, a_1]$$

It's symmetric exclusing the first element, and then we put this into the DFT algorithm using FFT, then we obtain the following relationsship:

$$\frac{1}{2}(F_k + a_0 + (-1)^k) = \sum_{j=0}^{n} \cos\left(\frac{\pi j k}{n}\right) a_j \quad \forall 0 \le k \le n \tag{1.1}$$

Here, we assume that the vector $\vec{F} = [F_0, F_1, \cdots F_{2n-1}]$ are the output of the DFT after we feed $\vec{f}$ into the algorithm.

Before we prove (1.1) we wish to establish some basics about the vector $\vec{f}$. Observe that the vector is symmetric if we exclude the first argument, which means that $f_j = f_{2n-j} \; \forall \; 1 \le j \le 2n-1$. Next, the vector $\vec{f}$ has a total length of $2n$. And when we index the vector $\vec{f}, \vec{F}$, we let the **index starts with zero**.

First, consider the following algebra:

$$\begin{aligned}
\exp\left(-i\frac{\pi(2n-j)k}{n}\right) &= \exp\left(-i\frac{2\pi n - j\pi k}{n}\right) \\
&= \exp\left(-i\frac{2n\pi n}{n} + \frac{ij\pi k}{n}\right) \\
&= \exp\left(i\frac{jk\pi}{n}\right) \\
\sum_{j=0}^{2n-1} \exp\left(-i\frac{i\pi jk}{n}\right) &= \sum_{j=1}^{2n} \exp\left(-i\frac{2\pi(2n-j)k}{n}\right)
\end{aligned} \tag{1.2}$$

The second equality is just a trick where I swapp the index so it starts summing in the reverse order.

Now consider the DFT on vector $\vec{f}$, which by definition would be given as:

$$\begin{aligned}
F_k &= \sum_{j=0}^{2n-1} \exp\left(-i\frac{2\pi jk}{2n}\right) f_j = \sum_{j=0}^{2n-1} \exp\left(-i\frac{\pi jk}{n}\right) f_j \\
&= \frac{1}{2}\left(\sum_{j=0}^{2n-1} \exp\left(-i\frac{\pi jk}{n}\right) f_j + \sum_{j=1}^{2n} \exp\left(-i\frac{2\pi(2n-j)k}{n}\right) \underbrace{f_{2n-j}}_{=f_j}\right) \\
&= \frac{1}{2}\left(\sum_{j=0}^{2n-1} \exp\left(-i\frac{\pi jk}{n}\right) f_j + \sum_{j=1}^{2n} \exp\left(i\frac{jk\pi}{n}\right) f_j\right) \Longleftarrow \quad \text{by: (1.2)} \\
&= \frac{1}{2}\left(2f_0 + \sum_{j=1}^{2n-1} \exp\left(-i\frac{\pi jk}{n}\right) f_j + \sum_{j=1}^{2n-1} \exp\left(i\frac{jk\pi}{n}\right) f_j\right) \\
&= f_0 + \sum_{j=1}^{2n-1} \cos\left(\frac{\pi jk}{n}\right) f_j
\end{aligned} \tag{1.3}$$

Next, please observe the fact that the term for $j = 1$ equals to $j = 2n - 1$, due to the symmetry of cos and the symmetry of vector f_j $\forall 1 \le j \le 2n-1$. And hence we obtained:

$$F_k = a_0 + \left(2\sum_{j=1}^{n-1} \cos\left(\frac{\pi jk}{n}\right) a_j\right) + (-1)^k a_n \tag{1.4}$$

Here, take note of the extra term, when $j = n$, $f_j = n$, which is right in the middle of the symmetric part of $\vec{f}$, and it only repeats once, so I take it out from the sum and it produces the term $(-1)^k a_n$. All other terms repeats 2 times and $f_0 = a_0$. Rearranging the above equation we have:

$$\begin{aligned}
\frac{1}{2}\left(F_k - a_0 - (-1)^k a_n\right) &= \sum_{j=1}^{n-1} \cos\left(\frac{\pi jk}{n}\right) a_j \\
\frac{1}{2}\left(F_k + a_0 + (-1)^k a_n\right) &= \sum_{j=0}^{n} \cos\left(\frac{\pi jk}{n}\right) a_j \\
\frac{1}{2}\left(F_k + a_0 + (-1)^k a_n\right) &= p\left(\cos\left(\frac{k\pi}{n}\right)\right)
\end{aligned} \tag{1.5}$$

From the frist line to the second line, I added $a_0, (-1)^k a_n$ to both side of the equation. At this point, we have proven that (1.1) is true, and we can make use of the algorithm fast evaluate the chebyshev series at the chebyshev nodes. Simply make the vector $\vec{f}$ as said above, and then evalute it to get F_k, and then use that above expression, for $k = 0, \cdots, n$. There will be $2n$ output vectors, but we can ignore the part where it gets symmetric.

Next, to reverse the process for looking for the chebyshee coefficients, we simply consider: "What is F_k"? And then make use of the IDFT algorithm which uses IFFT.

$$\begin{aligned}
p\left(\cos\left(\frac{k\pi}{n}\right)\right) &= \frac{1}{2}\left(F_k + a_0 + (-1)^k a_n\right) \\
2p\left(\cos\left(\frac{k\pi}{n}\right)\right) &= F_k + a_0 + (-1)^k a_n \\
2p\left(\cos\left(\frac{k\pi}{n}\right)\right) - a_0 - (-1)^k a_n &= F_k
\end{aligned} \tag{1.6}$$

Do this for $k = 0, \cdots, 2n-1$ and then invoke the IFDT using FFT, and then we get back the veoctr $\vec{f}$, and the first $n+1$ elements are the chbyshev coefficients.