HW 3 SCRATCH PAPER

§ Problem 2:

$$\begin{cases} -\partial_x^2 u + (1+x^2)u = f & 0 \le x \le 1 \\ u(0) = 0, \ u(1) = 0 \end{cases}$$

* Uniform Grid

$$h = \frac{1}{(1+m)}$$

* 2(a): Consider Finite Difference scheme:

$$\frac{2u_i - u_{i+1} - u_{i-1}}{h^2} + (1+x_i^2)u_i = f(x_i) \quad i = 1, \dots, m$$

* Find the upper and lower bound of the eigenvalues for the difference equation used for the system.

* Gerschgorin Theorem:

$$\Lambda(A) \subseteq \bigcup_{i=1}^{n} \left\{ x \in \mathbb{C} : |x - a_{i,i}| \le \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{i,j}| \right\}$$

* Our matrix: $\in \mathbb{R}^{m \times m}$, with Boundary rolling in.

$$\frac{1}{h^2}(2u_i - u_{i+1} - u_{i-1}) + (1+x_i^2)u_i = f(x_i)$$

$$\begin{cases} \frac{1}{h^2}\left((2+h^2(1+x_i^2))u_i - u_{i+1} - u_{i-1}\right) = f(x_i) & \forall i = 2, \dots, m-1 \\ \frac{1}{h^2}\left((2+h^2(1+x_i^2))u_1 - u_2\right) = f(x_1) & i = 1 \\ \frac{1}{h^2}\left((2+h^2(1+x_i^2))u_m - u_{m-1}\right) = f(x_m) & i = m \end{cases}$$

$$\begin{cases} R_i(A) = \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{i,j}| = \frac{2}{h^2} & \forall i = 2, \dots, m-1 \\ R_1(A) = \frac{1}{h^2}|-1| = \frac{1}{h^2} & i=1 \\ R_m(A) = \frac{1}{h^2}|-1| = \frac{1}{h^2} & i=m \end{cases}$$

*Consider the disks indexed by $i=2,\dots,m$

$$\left|x - \frac{2}{h^2} - (1+x_i^2)\right| \leq R_i(A) = \frac{2}{h^2}$$

$$\left|x - \left(\frac{2}{h^2} + (1+x_i^2)\right)\right| \leq \frac{2}{h^2}$$

$$-\frac{2}{h^2} \leq x - \left(\frac{2}{h^2} + (1+x_i^2)\right) \leq \frac{2}{h^2}$$ Because it's real

$$(1+x_i^2) \leq x \leq \frac{4}{h^2} + (1+x_i^2) \quad \forall i = 2, \dots, m-1$$

$$1+(ih)^2 \leq x \leq \frac{4}{h^2} + \left(1+(ih)^2\right)$$

*Consider when $i=1$ or $i=m-1$

$$\left|x - \left(\frac{2}{h^2} + (1+x_i^2)\right)\right| \leq \frac{1}{h^2}$$

$$\frac{1}{h^2} + (1+x_i^2) \leq x \leq \frac{1}{h^2} + \frac{2}{h^2} + (1+x_i^2) = \frac{3}{h^2} + (1+x_i^2)$$

* $i=m$: $$\frac{1}{h^2}\left(1+(mh)^2\right) \leq x \leq \frac{3}{h^2} + \left(1+(ih)^2\right)$$

* $i=1$

$$\frac{1}{h^2} + (1+h^2) \leq x \leq \frac{3}{h^2} + 1 + h^2$$

*Check this with Computer PLezzz...

* $\Rightarrow \Lambda(A) \in \left[1+(2h)^2, \frac{4}{h^2} + 1 + (m-1)^2 h^2 \right]$ ← A loose bound.

if we assume h to be very small, so small that:

$$1+(2h)^2 \geq \frac{1}{h^2} + (1+h^2)$$

↳ I guess this is true $\forall h \in [0,1]$ already.

*2(b) Show that the L_2 norm of the global error is of the same order as the local truncation error.

*Let's figure out the LTE first how about that?

$$\frac{2u_i - u_{i+1} - u_{i-1}}{h^2} + (1+x_i^2)u_i - f(x_i) = O(h^2) = \tau_i$$

* $\check{u}$ :: Computed solution from $A\check{u} = f(x_i)$
$\hat{u}$ :: exact solution where $\hat{u}_i = u(x_i)$ and $u(x)$ solves the system.

$$A\hat{u} - f = \vec{\tau}$$
$$A\check{u} = f$$
$$\Rightarrow A\hat{u} - A\check{u} = \vec{\tau}$$
$$A(\hat{u} - \check{u}) = \vec{\tau} \qquad (\vec{\eta})_i = u_i - u(x_i)$$
$$A\vec{\eta} = \vec{\tau}$$

$$(A\vec{e})_i = \frac{2\eta_i - \eta_{i+1} - \eta_{i-1}}{h^2} + (1+x_i^2)\eta_i = O(h^2)$$

*A's most extreme eigenvalue: $\frac{4}{h^2} + (1+(ih)^2)$

$$\vec{\eta} = A^{-1}\vec{\tau}$$

Note: $\|x\|_{\ell_2} = \sqrt{h}\|x\|_2$

$$\|\vec{\eta}\|_2 \le \|A^{-1}\|_2 \|\vec{\tau}\|_2$$

Spectral Norm

$$A\vec{\eta} = \vec{\tau}$$

$$\|A\|_2^2 \|\vec{\eta}\|_2^2 \le \|A\vec{\eta}\|_2^2 = \|\vec{\tau}\|_2$$

$$\Longrightarrow \|A\|_2 \|\vec{\eta}\|_2 \le \|\vec{\tau}\|_2$$

$$\|A\|_2^2 = \frac{\lambda_{max}(A^TA)}{\lambda_{min}(A^TA)} = \frac{\lambda_{max}(A^2)}{\lambda_{min}(A^2)} = \left(\frac{\lambda_{max}(A)}{\lambda_{min}(A)}\right)^2$$

$$\Rightarrow \|A\|_2 = \frac{\lambda_{max}(A)}{\lambda_{min}(A)}$$

$$= \frac{\frac{4}{h^2} + 1 + (m-1)^2 h^2}{1+(2h)^2} = \frac{4 + h^2 + (m-1)^2 h^4}{h^2 + 4h^4}$$

* Consider $\|\vec{\tau}\|_2$;

$$\|\vec{\tau}\|_2^2 = \sum_{j=1}^{m} \left(O(h^2)\right)^2 = \sum_{j=1}^{m} O(h^4)$$

$$= O(h^3)$$

Because: $m+1 = 1/h$

$m = 1/h - 1$

$$\Longrightarrow \|\vec{\tau}\|_2 = O(h^{3/2})$$

$$\|\vec{\eta}\|_2 \leq \frac{\|\vec{\tau}\|_2}{\|A\|_2}$$

$$\|\vec{\eta}\|_{L_2} = \sqrt{h}\,\|\vec{\eta}\|_2 \leq \frac{\sqrt{h}\,\|\vec{\tau}\|_2}{\|A\|_2}$$

$$\sqrt{h}\,\|\vec{\tau}\|_2 = \sqrt{h}\,O(h^{3/2}) = O(h^2)$$

$$\Rightarrow \frac{\sqrt{h}\,\|\vec{\tau}\|_2}{\|A\|_2} = O(h^2)\,\frac{h^2+4h^4}{4+h^2+(m-1)^2h^4}$$

$$= O(h^2)\,\frac{1+4h^2}{4/h^2+1+(m-1)^2h^2}$$

$$= O(h^2)\,O(1) = O(h^2)$$

*Done. $\|\vec{\eta}\|_{L_2} = O(h^2)$ same as the LTE.

§Problem 3

This one is a complicated one...

*We need the actual Error from the scheme.

$$\partial_x \left(c(x)\, \partial_x u(x) \right) = f(x)$$

$$D_\pm \left[c(x_{i+1/2})\, D_\pm [u(x_i)] \right] = f(x_i)$$

$D_\pm$ :: Second Order, central diff, first derivative.

$$* \quad D_\pm [u] \Big|_{x=x_i} := \frac{u(x_{i+1}) - u(x_{i-1})}{2h}$$

*Error Terms:

$$u(x_{i+1}) = u(x_i + h)$$

$$= u(x_i) + h u'(x_i) + \frac{h^2}{2} u''(x_i) + \sum_{j=3}^{\infty} \frac{h^j}{(j)!} u^{(j)}(x_i)$$

$$u(x_{i-1})$$

$$= u(x_i) - h u'(x_i) + \frac{h^2}{2} u''(x_i) + \sum_{j=3}^{\infty} \frac{(-1)^j h^j}{(j)!} u^{(j)}(x_i)$$

$$u(x_i + h) - u(x_i - h)$$

$$= 2h u'(x_i) + \sum_{j=3}^{\infty} \left(\frac{(1 - (-1)^j) h^j}{(j)!} u^{(j)}(x_i) \right)$$

j^{th} even terms cancelled out

$$D_\pm[u]\Big|_{x=x_i} = u'(x_i) + (2h)^{-1} \sum_{j=3}^{\infty} \left(\frac{(1-(-1)^j)h^j}{(j)!} u^{(j)}(x_i) \right)$$

$$= u'(x_i) + (2h)^{-1} \sum_{j=1}^{\infty} \left(\frac{2h^{2j+1}}{(2j+1)!} u^{(2j+1)}(x_i) \right)$$

$$= u'(x_i) + \sum_{j=1}^{\infty} \frac{h^{2j}}{(2j+1)!} u^{(2j+1)}(x_i)$$

$$= u'(x_i) + \frac{h^2}{3!} u^{(3)}(x_i) + \sum_{j=2}^{\infty} \frac{h^{2j}}{(2j+1)!} u^{(2j+1)}(x_i)$$

* OK what's the error on that particular problem:

$$\forall$$

$$D_\pm\left[\underbrace{c(x_{i+1/2})}\, D_\pm[u(x_i)]\Big|_{x=x_i} \right]\Bigg|_{x=x_i} = f(x_i)$$

$$= D_\pm\left[\underbrace{C_{i+1/2}\left(u'(x_i) + \frac{h^2}{3!} + \mathcal{O}(h^4) \right)} \right]\Bigg|_{x=x_i} = f(x_i)$$

* treat it as a function of x

$$\Rightarrow g(x) := C\left(x+\frac{h}{2}\right)\left(u'(x) + \frac{h^2}{3!} + \mathcal{O}(h^4) \right)$$

$$= D_\pm[g(x)]\Big|_{x=x_i}$$

$$= g'(x) + \frac{h^2}{3!} g^{(3)}(x_i) + \mathcal{O}(h^4 | D_\pm, g)$$

$$D_{\pm}\left[c(x_{i+1/2})D_{\pm}[u(x_i)]\Big|_{\substack{x=x_i\\h}}\right]\Bigg|_{\substack{x=x_i\\h}} = f(x_i)$$

$$g'(x) + \frac{h^2}{3!}g^{(3)}(x_i) + O(h^4 | D_{\pm}, g) = f(x_i)$$

* Consider the statement:

$$\underbrace{D_{\pm}\left[c(x_{i+1/2})D_{\pm}[u(x_i)]\Big|_{\substack{x=x_i\\h}}\right]\Bigg|_{\substack{x=x_i\\h}}}_{\varphi_1(h) \,::\, \text{Richardson Extrapolant.}} = \ldots$$

* Consider:

$$\Rightarrow \varphi_1(h) = g'(x) + \frac{h^2}{3!}g^{(3)}(x_i) + O(h^4 | D_{\pm}, g)$$

$$\varphi_1(h/2) = g'(x) + \frac{h^2}{4(3!)}g^{(3)}(x_i) + O\left(\frac{h^2}{4} \Big| D_{\pm}, g\right)$$

$$\varphi_1(h) - 4\varphi_1\left(\frac{h}{2}\right) = -3g'(x) + \ldots$$

$$O(h^4 | D_{\pm}, g) - 4O\left(\frac{h^2}{4} \Big| D_{\pm}, g\right)$$

$$\frac{4\varphi_1\left(\frac{h}{2}\right) - \varphi_1(h)}{3} = g'(x) + \ldots$$

$$O(h^4 | D_{\pm}, g) - 4O\left(\frac{h^2}{4} \Big| D_{\pm}, g\right)$$

*It's the same...

The linear combo of interpolant is:

$$\frac{4\varphi_1(\frac{h}{2})-\varphi_1(h)}{3}$$, then the error is eliminated...

*Wait, but what I got is a new linear System....

$$\varphi_1(h) = g'(x) + \frac{h^2}{3!} g^{(3)}(x_i) + O(h^4 | D_\pm, g)$$

$$g'(x) =$$

§ Problem 3 Second Trial

A_h :: The finite Diff matrix paramaterized by h

$$\hat{u}_i(h) = (A_h^{-1}\vec{f})_i$$ ⇐ A particular Approximated solution at x_i

* The solution from finite diff at x_i can be view as a function wrt to h.

* The error bewteen $\hat{u}_i$ and $u(x_i)$ is $\mathcal{O}(h^2)$, which would mean that:

$$\hat{u}_i - u(x_i) = \mathcal{O}(h^2)$$

$$\Rightarrow \lim_{h\to 0} \frac{\hat{u}_i - u(x_i)}{h^2} = C$$

Constant that doesn't relate to h.

* Assume that $\hat{u}_i(h)$ is analytic and has taylor series of the form:

$$\hat{u}_i(h) = \hat{u}_i(0) + h\hat{u}_i'(0) + \frac{h^2}{2}\hat{u}_i''(0) + \underbrace{\sum_{j=3}^{\infty} \frac{h^j}{(j)!}\hat{u}_i^{(j)}(0)}_{\mathcal{O}(h^3)}$$

$$\lim_{h\to 0} \frac{\hat{u}_i(h) - u(x_i)}{h^2} = C$$

$$\Rightarrow \hat{u}_i'(0) = 0 \quad \& \quad \frac{h^2}{2}\hat{u}_i''(0) = 0$$

$$\lim_{h\to 0} \frac{\hat{u}_i(h) - u(x_i)}{h^2} = \text{Constant} \neq 0$$

* Consider the taylor series of $\hat{u}_i(h)$

$$\Rightarrow \hat{u}_i(h) - u(x_i) = \frac{h^2}{2}\hat{u}_i''(0) + \underbrace{\sum_{n=3}^{\infty} \frac{h^n}{n!}\hat{u}_i^{(n)}(0)}_{O(h^3)}$$

$$u(x_i) = \hat{u}_i(h) - \frac{h^2}{2}\hat{u}_i''(0) + O(h^3)$$

$$u(x_i) = \hat{u}_i\left(\frac{h}{2}\right) - \frac{h^2}{8}\hat{u}_i''(0) + O(h^3)$$

costant

$$\lim_{h\to 0} \frac{u_i - u(x_i)}{h^2} = C \neq 0$$

* Assume a tayor series representation of u_i, so u_i is analytic as a function of h.

$$\begin{cases} \hat{u}_i(h) = \left[\hat{u}_i(0) - u(x_i)\right] + \dfrac{h^3}{3!}\hat{u}_i^{(3)}(0) + O(h^4) \\ \hat{u}_i\left(\dfrac{h}{2}\right) = \left[\hat{u}_i(0) - u(x_i)\right] + \dfrac{h^3}{8(3!)}\hat{u}_i^{(3)}(0) + O\left(\dfrac{h^4}{16}\right) \end{cases}$$

$$8\hat{u}_i\left(\frac{h}{2}\right) - \hat{u}_i(h)$$

$$= 7\left[\hat{u}_i(0) - u(x_i)\right] + 8\,O\left(\frac{h^4}{16}\right) - O(h^4)$$

$$\frac{8\hat{u}_i\left(\frac{h}{2}\right) - \hat{u}_i(h)}{7} = \left[\hat{u}_i(0) - u(x_i)\right] + O_2(h^4)$$

$$u(x_i) = \hat{u}_i(0) - \frac{8\hat{u}_i\left(\frac{h}{2}\right) - \hat{u}_i(h)}{7} + O_2(h^4)$$

$$= \frac{7\hat{u}_i(0) - 8\hat{u}_i\left(\frac{h}{2}\right) + \hat{u}_i(h)}{7} + O(h^4)$$

* We can do it with $-h$ and consider:

$$\begin{cases} \hat{u}_i(-h) = \left[\hat{u}_i(0) - u(x_i)\right] - \dfrac{h^3}{3!}\hat{u}_i^{(3)}(0) + O(h^4) \\ \hat{u}_i\left(\dfrac{-h}{2}\right) = \left[\hat{u}_i(0) - u(x_i)\right] - \dfrac{h^3}{8(3!)}\hat{u}_i^{(3)}(0) + O\left(\dfrac{h^4}{16}\right) \end{cases}$$

$$\hat{u}_i(-h) - 8\hat{u}_i\left(\frac{-h}{2}\right) = -7\left[\hat{u}_i(0) - u(x_i)\right] + O(h^4) - 8O\left(\frac{h^4}{16}\right)$$

$$\frac{8\hat{u}_i\left(\frac{-h}{2}\right) - \hat{u}_i(-h)}{7} = \left[\hat{u}_i(0) - u(x_i)\right] + O_2(h^4)$$

$$u(x_i) = \hat{u}_i(0) - \frac{8\hat{u}_i\left(\frac{-h}{2}\right) - \hat{u}_i(-h)}{7} + O_4(h^4)$$

~~*Can we do this richardson Extrapolation for a finer grid points, say at~~ $\cancel{x_i + h/2}$ ~~?~~

$$\hat{u}_i(h) = \left[\hat{u}_i(0) - u(x_i)\right] + \frac{h^3}{3!}\hat{u}_i^{(3)}(0) + \mathcal{O}(h^4)$$

$$=$$

$u_i - u(x_i)$ has error $O(h^2)$

$$\Rightarrow \quad u_i - u(x_i) = C_2 h^2 + C_3 h^3 + O(h^4)$$

$$u(x_i) = u_i + C_2 h^2 + C_3 h^3 + O(h^4)$$

C_2, C_3 are just generic constants.

$$u(x_i) = u_i + C_2 h^2 + C_3 h^3 + O(h^4)$$

let $h = \frac{h}{2}$, So we decreased the step size for the system, assuming the error func stays the same then:

$$\Rightarrow u(x_i) = u_i + C_2\left(\frac{h}{2}\right)^2 + C_3\left(\frac{h}{2}\right)^3 + O(h^4)$$

$$\begin{cases} 4u(x_i) = 4u_i + C_2 h^2 + 4C_3\left(\frac{h}{2}\right)^3 + O(h^4) \\ u(x_i) = u_i + C_2 h^2 + C_3 h^3 + O(h^4) \end{cases}$$

$$4u(x_i) - u(x_i) = 3u_i + O(h^3)$$

$$\frac{4u(x_i) - u(x_i)}{3} = u_i + O(h^3)$$

* $\hat{u}(h)$ :: Solution of finite diff with grid point width h near a particular point of interest : x, fixing its position.

* When $h = \frac{1}{2^k}$, we can find $\hat{u}(h)$ exactly at x_i for each k by solving the F.D. approx system.

* Assume a power series representation of the error:

$$\hat{u}(h) = u(x) + C_2 h^2 + C_3 h^3 + \mathcal{O}(h^4)$$

$$\hat{u}\left(\frac{h}{2}\right) = u(x) + C_2\left(\frac{h}{2}\right)^2 + C_3\left(\frac{h}{2}\right)^3 + \mathcal{O}(h^4)$$

$$4\hat{u}\left(\frac{h}{2}\right) = 4u(x) + C_2 h^2 + 4C_3\left(\frac{h}{2}\right)^3 + \mathcal{O}(h^4)$$

$$\Rightarrow \frac{4\hat{u}(h/2) - \hat{u}(h)}{3} = u(x) + \frac{4C_3\left(\frac{h}{2}\right)^3 - C_3 h^3}{3} + \mathcal{O}(h^4)$$

$$= u(x) + \frac{\frac{1}{2}C_3 h^3 - C_3 h^3}{3} + \mathcal{O}(h^4)$$

$$\varphi_1(h) = u(x) + C_3 h^3\left(\frac{-1}{6}\right) + \mathcal{O}(h^4)$$

*A third order accuracy is obtained using:

$\hat{u}(h)$, $\hat{u}\left(\pm\frac{h}{2}\right)$ solved at x.

And we only made use of the coarse grid point.

* Can't do that with average interpolated point, the series at different x is just different.

* Say we are interested in $x = \frac{1}{2}$, then:

$$h = \frac{1}{2} \quad \hat{u}\left(\frac{1}{2}\right) = \left(A_{\frac{1}{2}}^{-1} \vec{f}\right)_1$$

$$h = \frac{1}{4} \quad \hat{u}\left(\frac{1}{4}\right) = \left(A_{\frac{1}{4}}^{-1} \vec{f}\right)_3$$

$$h = \frac{1}{8} \quad \hat{u}\left(\frac{1}{2^3}\right) = \left(A_{\frac{1}{8}}^{-1} \vec{f}\right)_7$$

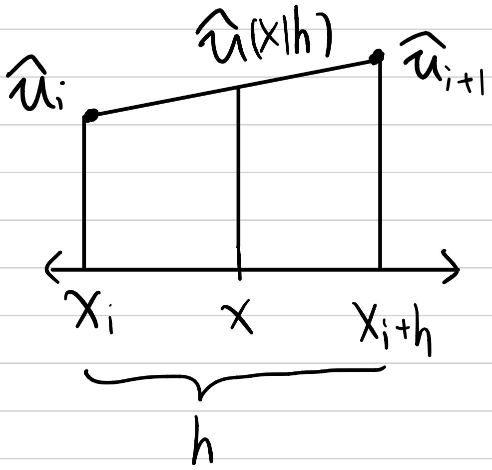
$$\vdots$$

$$h = \frac{1}{2^k} \quad \hat{u}\left(\frac{1}{2^k}\right) = \left(A_{\frac{1}{2^k}}^{-1} \vec{f}\right)_{2^k - 1}$$

* has a fixed power series representation.

$$\hat{u}_i(h)$$
$$\hat{u}_{2i}\left(\frac{h}{2}\right)$$

* Let's formalize $\hat{u}(x|h)$ a bit.



$$\hat{u}(x|h) = \frac{\hat{u}_{i+1} - \hat{u}_i}{h}(x - x_i) + \hat{u}_i$$

s.t: $x \in [ih, (i+1)h]$

i is a function of h, x as well.

* Here is what we assumed for the error. for a fix value of x, the error has a power series representation wrt to h.

$$\hat{u}(x|h) = u(x) + C_1 h^2 + O(h^3)$$

* C_1 is paramaterized by x.

$$\hat{u}(x|\tfrac{h}{2}) = u(x) + C_1 \frac{h^2}{4} + O(h^3)$$

$$4\hat{u}(x|h/2) = 4u(x) + C_1 h^2 + O(h^3)$$

$$4\hat{u}(x|h/2) - \hat{u}(x|h) = 3u(x) + O(h^3)$$

$$\frac{4\hat{u}(x|h/2) - \hat{u}(x|h)}{3} = u(x)$$

* That's the interpolant. it only uses coarse point.

§ Errors Between Intervals...

$$\hat{u}(x|h) = \frac{\hat{u}_{i+1} - \hat{u}_i}{h}(x - x_i) + \hat{u}_i$$

$$s.t: x \in [ih, (i+1)h]$$

$$\hat{u}_{i+1} = u(x_{i+1}) + O(h^2)$$

$$\hat{u}_i = u(x_i) + O(h^2)$$

* You know what, consider $(x - x_i) = ch$ where $c \in (0, 1)$

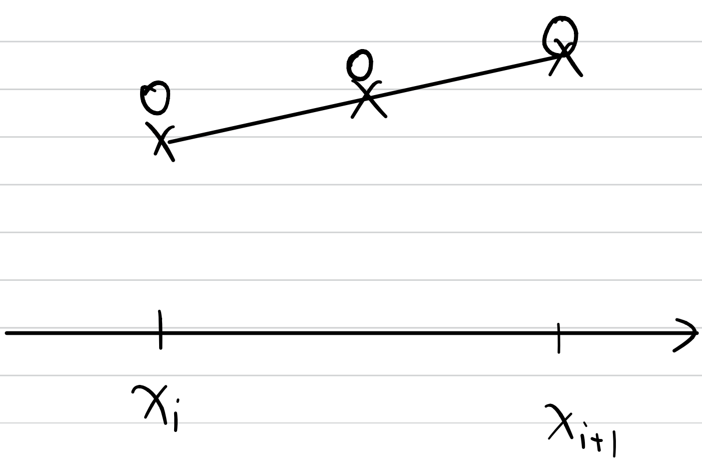
$$\hat{u}(x|h) = \frac{\hat{u}_{i+1} - \hat{u}_i}{h}(x - x_i) + \hat{u}_i$$

$$= \frac{\hat{u}_{i+1} - \hat{u}_i}{h} ch + \hat{u}_i$$

$$= c(\hat{u}_{i+1} - \hat{u}_i) + \hat{u}_i$$

* Error stays at the same order.

* fine grid and interpolated value.



o :: fine grid

x :: coarse grid interpolated.

* Can't interpolate because

$u($