Name: Hongda Li
AMATH 585 HW4 2022 WINTER

Problem 1

Consider the boundary value problem:

$$\begin{cases} -\frac{d}{dx}\left((1+x^2)\frac{du}{dx}\right) = f(x) & \forall x \in [0,1] \\ u(0) = u(1) = 0 \end{cases}$$

(a)

Objective: Derive the Matrix equation that we will need to solve the problem, using Galerkin Finite Element Method, With continuous piecewise linear basis functions.

Introduce $\mathcal{L}[\cdot] := \partial_x[r(x)\partial[\cdot]]$ as a differential operator, to derive, consider the weak form:

$$\begin{aligned} \langle \mathcal{L}[u], \varphi\rangle &\equiv \int_0^1 (\partial_x[r(x)\partial_x[u(x)]])\varphi(x)dx = \int_0^1 f(x)\varphi(x)dx \equiv \langle f, \varphi\rangle \\ &= \int_0^1 \varphi(x)d(r(x)\partial_x[u(x)])dx \\ &= \varphi(x)(r(x)\partial_x[u(x)])|_0^1 - \int_0^1 r(x)\partial_x[u(x)]\varphi'(x)dx \\ &= \varphi(1)r(1)\partial_x[u](1) - \varphi(0)r(0)\partial_x[u](0) - \int_0^1 r(x)\partial_x[u(x)]\varphi'(x)dx \end{aligned} \tag{1.a.1}$$

Where, $\varphi(x)$ is some basis function in $\mathcal{S}$, in our cause, $\mathcal{S}$ is finite and has $n-1$ elements (Because it's constructed on an discretized grid points), and let $\varphi_i(x)$ be the index basis function, then we have:

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ 0 & \text{else} \end{cases} \tag{1.a.2}$$

The objective of the finite elements method is to represents the solution, $u(x)$ as a linear combinations of the basis function $\varphi_i(x)$. Suppose that $\hat{u} = \sum_{j=1}^{n-1} c_j\varphi_j(x)$, then reconsidering (1.a.1) the weak form and convert it to a matrix vector equation:

$$\tag{1.a.3}$$