Sample Solutions for Assignment 2.

Reading: Through Sec. 2.15. (Page numbers and equation numbers in this assignment refer to the text.)

1. A rod of length 1 meter has a heat source applied to it and it eventually reaches a steady-state where the temperature is not changing. The conductivity of the rod is a function of position x and is given by $c(x) = 1 + x^2$. The left end of the rod is held at a constant temperature of 1 degree. The right end of the rod is insulated so that no heat flows in or out from that end of the rod. This problem is described by the boundary value problem:

$$\frac{d}{dx}\left((1+x^2)\frac{du}{dx}\right) = f(x), \quad 0 \le x \le 1,$$

$$u(0) = 1, \quad u'(1) = 0.$$

(a) Write down a set of difference equations for this problem. Be sure to show how you do the differencing at the endpoints. [Note: It is better **not** to rewrite $\frac{d}{dx}((1+x^2)\frac{du}{dx})$ as $(1+x^2)u''(x) + 2xu'(x)$; leave the equation in the form above.]

Let $x_{j\pm 1/2} = x_j \pm h/2$. At the interior nodes, $j = 1, \ldots, m$, we can write

$$\frac{-((1+x_{j+1/2}^2) + (1+x_{j-1/2}^2))u_j + (1+x_{j+1/2}^2)u_{j+1} + (1+x_{j-1/2}^2)u_{j-1}}{h^2} = f(x_j).$$

We know the value of u_0, and we can add an $(m+1)$st equation for u_{m+1} by writing a second order accurate approximation to $u'(1)$ and setting that equal to 0:

$$\frac{3u_{m+1} - 4u_m + u_{m-1}}{2h} = 0.$$

In matrix form, these equations take the form $A\mathbf{u} = \mathbf{f}$, where

$$A = \frac{1}{h^2}\begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & \ddots & & \\ & \ddots & \ddots & b_{m-1} & \\ & & b_{m-1} & a_m & b_m \\ & & \frac{1}{2}h & -2h & \frac{3}{2}h \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_m \\ u_{m+1} \end{pmatrix},$$

$$\mathbf{f} = \begin{pmatrix} f(x_1) - (1+(h/2)^2)/h^2 \\ f(x_2) \\ \cdots \\ f(x_m) \\ 0 \end{pmatrix},$$

and

$$a_j = -(2 + x_{j+1/2}^2 + x_{j-1/2}^2), \quad b_j = (1 + x_{j+1/2}^2),$$

(b) Write a MATLAB code to solve the difference equations. You can test your code on a problem where you know the solution by choosing a function $u(x)$ that satisfies the boundary conditions and determining what $f(x)$ must be in order for $u(x)$ to solve the problem. Try $u(x) = (1-x)^2$. Then $f(x) = 2(3x^2 - 2x + 1)$.

See `prob1hw2.m`.

(c) Try several different values for the mesh size h. Based on your results, what would you say is the order of accuracy of your method?

The method is second order accurate in the L_2-norm as illustrated below: When h is reduced by a factor of 2, the error goes down by about a factor of 4.

h	L_2-norm of error	error(2h)/error(h)
$\frac{1}{10}$	1.7565e-03	
$\frac{1}{20}$	4.2625e-04	4.1207e+00
$\frac{1}{40}$	1.0522e-04	4.0512e+00
$\frac{1}{80}$	2.6150e-05	4.0236e+00
$\frac{1}{160}$	6.5190e-06	4.0113e+00

2. (Inverse matrix and Green's functions)

(a) Write out the 4×4 matrix A from (2.43) for the boundary value problem $u''(x) = f(x)$ with $u(0) = u(1) = 1$ for $h = 1/3$.

$$A = 9 \begin{pmatrix} 1/9 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1/9 \end{pmatrix}$$

(b) Write out the 4×4 inverse matrix A^{-1} explicitly for this problem.

Since $B_{i0} = 1 - x_i$, the first column of A^{-1} is

$$A^{-1}(:,1) = \begin{pmatrix} 1 \\ 2/3 \\ 1/3 \\ 0 \end{pmatrix}.$$

Since

$$B_{ij} = hG(x_i; x_j) = \begin{cases} h(x_j - 1)x_i, & i = 1, \ldots, j \\ h(x_i - 1)x_j, & i = j, j+1, \ldots, m \end{cases}, \quad j = 1, \ldots, m,$$

columns 2 and 3 of A^{-1} are:

$$A^{-1}(:, 2:3) = \frac{1}{3} \begin{pmatrix} 0 & 0 \\ -2/9 & -1/9 \\ -1/9 & -2/9 \\ 0 & 0 \end{pmatrix}.$$

Since $B_{i,4} = x_i$, the last column of A^{-1} is

$$A^{-1}(:,4) = \begin{pmatrix} 0 \\ 1/3 \\ 2/3 \\ 1 \end{pmatrix}.$$

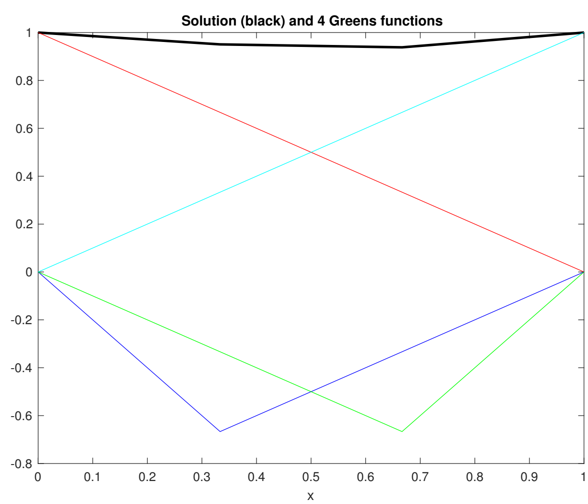
Putting all of this together,

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2/3 & -2/27 & -1/27 & 1/3 \\ 1/3 & -1/27 & -2/27 & 2/3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(c) If $f(x) = x$, determine the discrete approximation to the solution of the boundary value problem on this grid and sketch this solution and the four Green's functions from which the solution is obtained.

The approximate solution is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2/3 & -2/27 & -1/27 & 1/3 \\ 1/3 & -1/27 & -2/27 & 2/3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1/3 \\ 2/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 77/81 \\ 76/81 \\ 1 \end{pmatrix}.$$



3. (Another way of analyzing the error using Green's functions) The *composite trapezoid rule* for integration approximates the integral from a to b of a function g by dividing the interval into segments of length h and approximating the integral over each segment by the integral of the linear function that matches g at the endpoints of the segment. (For $g > 0$, this is the area of the trapezoid with height $g(x_j)$ at the left endpoint x_j and height $g(x_{j+1})$ at the right endpoint x_{j+1}.) Letting $h = (b-a)/(m+1)$ and $x_j = a + jh$, $j = 0, 1, \ldots, m, m+1$:

$$\int_a^b g(x)\,dx \approx h \sum_{j=0}^{m} \frac{g(x_j) + g(x_{j+1})}{2} = h \left[\frac{g(x_0)}{2} + \sum_{j=1}^{m} g(x_j) + \frac{g(x_{m+1})}{2} \right].$$

(a) Assuming that g is sufficiently smooth, show that the error in the composite trapezoid rule approximation to the integral is $O(h^2)$. [Hint: Show that the error on each subinterval is $O(h^3)$.]

Consider the integral from x_j to x_{j+1}. For $x \in [x_j, x_{j+1}]$, expand $g(x)$ in Taylor series about x_j and x_{j+1}:

$$\begin{aligned} g(x) &= g(x_j) + (x - x_j)g'(x_j) + O(h^2) \\ g(x) &= g(x_{j+1}) + (x - x_{j+1})g'(x_{j+1}) + O(h^2) \end{aligned}$$

Add these two equations, divide by 2, and integrate over the subinterval to find

$$\int_{x_j}^{x_{j+1}} g(x)\,dx = \frac{h}{2}[g(x_j) + g(x_{j+1})] + \frac{h^2}{4}[g'(x_j) - g'(x_{j+1})] + O(h^3).$$

Since $g'(x_{j+1}) = g'(x_j) + hg''(x_j) + O(h^2)$, we have

$$\int_{x_j}^{x_{j+1}} g(x)\,dx = \frac{h}{2}[g(x_j) + g(x_{j+1})] + O(h^3).$$

Summing over the $m+1 = (b-a)/h$ subintervals, we obtain an error that is $\frac{b-a}{h}O(h^3) = O(h^2)$.

(b) Recall that the true solution of the boundary value problem $u''(x) = f(x)$, $u(0) = u(1) = 0$ can be written as

$$u(x) = \int_0^1 f(\bar{x})G(x;\bar{x})\,d\bar{x}, \tag{1}$$

where $G(x;\bar{x})$ is the Green's function corresponding to $\bar{x}$. The finite difference approximation u_i to $u(x_i)$, using the centered finite difference scheme in (2.43), is

$$u_i = h\sum_{j=1}^{m} f(x_j)G(x_i;x_j), \quad i = 1, \ldots, m. \tag{2}$$

Show that formula (2) is the trapezoid rule approximation to the integral in (1) when $x = x_i$, and conclude from this that the error in the finite difference approximation is $O(h^2)$ at each node x_i. [Recall: The Green's function $G(x;x_j)$ has a *discontinuous* derivative at $x = x_j$. Why does this not degrade the accuracy of the composite trapezoid rule?]

For $x = x_i$, equation (1) becomes

$$u(x_i) = \int_0^1 f(\bar{x})G(x_i;\bar{x})\,d\bar{x},$$

and the composite trapezoid rule approximation for this integral is

$$h\left[\frac{f(x_0)G(x_i;x_0)}{2} + \sum_{j=1}^{m} f(x_j)G(x_i;x_j) + \frac{f(x_{m+1})G(x_i;x_{m+1})}{2}\right].$$

Since $G(x_i; x_0) = G(x_i; x_{m+1}) = 0$, this is just the formula in (2). Thus, at each node i, the difference between u_i and the true solution $u(x_i)$ is the error in the trapezoid rule approximation to the integral in (1), which is $O(h^2)$. Note that while the Green's function $G(x; x_j)$ has a discontinuous derivative at x_j, this does not degrade the accuracy of the composite trapezoid rule. As shown in (a), its accuracy depends on the integrand being smooth *within* each subinterval, since that is where Taylor series are used, but the integrand or its derivatives can be discontinuous at the nodes.

4. (Green's function with Neumann boundary conditions)

   (a) Determine the Green's functions for the two-point boundary value problem $u''(x) = f(x)$ on $0 < x < 1$ with a Neumann boundary condition at $x = 0$ and a Dirichlet condition at $x = 1$, i.e, find the function $G(x, \bar{x})$ solving

$$u''(x) = \delta(x - \bar{x}), \quad u'(0) = 0, \quad u(1) = 0$$

   and the functions $G_0(x)$ solving

$$u''(x) = 0, \quad u'(0) = 1, \quad u(1) = 0$$

   and $G_1(x)$ solving

$$u''(x) = 0, \quad u'(0) = 0, \quad u(1) = 1.$$

   In order to satisfy $G''(x; \bar{x}) = \delta(x - \bar{x})$, the Green's function $G(x; \bar{x})$ must have the form

$$G(x; \bar{x}) = \begin{cases} a(x - \bar{x}) + b & 0 < x \le \bar{x} \\ c(x - \bar{x}) + b & \bar{x} \le x < 1 \end{cases},$$

   where the jump $c - a$ in G' at $x = \bar{x}$ is 1; i.e., $c = a + 1$. In order to satisfy $G'(0; \bar{x}) = 0$, we must have $a = 0$; hence $c = 1$. In order to satisfy $G(1; \bar{x}) = 0$, we must have $b = \bar{x} - 1$. Thus,

$$G(x; \bar{x}) = \begin{cases} \bar{x} - 1, & 0 < x \le \bar{x} \\ x - 1 & \bar{x} \le x < 1 \end{cases}.$$

   In order to satisfy $G_0'' = 0$, G_0 must have the form $a + bx$. In order that $G_0'(0) = 1$, we must have $b = 1$, and in order that $G_0(1) = 0$, we must have $a = -b$. Thus, $G_0(x) = x - 1$. Similarly, G_1 has the form $a + bx$, and in order that $G_1'(0) = 0$, we must have $b = 0$, and in order that $G_1(1) = 1$, we must have $a = 1 - b$. Thus, $G_1(x) = 1$.

   (b) Using this as guidance, find the general formulas for the elements of the inverse of the matrix in equation (2.54). Write out the 4×4 matrices A and A^{-1} for the case $h = 1/3$.

Substituting $h = 1/3$ into formula (2.54),

$$A = 9 \begin{pmatrix} -1/3 & 1/3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1/9 \end{pmatrix}.$$

Guessing that the columns of A^{-1} will be the values of the Green's functions at the nodes (multiplied by h for the interior nodes), we find that

$$\mathbf{B_0} = \begin{pmatrix} -1 \\ -2/3 \\ -1/3 \\ 0 \end{pmatrix}, \quad \mathbf{B_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$h\mathbf{B_1} = \begin{pmatrix} -2/9 \\ -2/9 \\ -1/9 \\ 0 \end{pmatrix}, \quad h\mathbf{B_2} = \begin{pmatrix} -1/9 \\ -1/9 \\ -1/9 \\ 0 \end{pmatrix}.$$

We can now check that the formula for A^{-1} is

$$A^{-1} = \begin{pmatrix} -1 & -2/9 & -1/9 & 1 \\ -2/3 & -2/9 & -1/9 & 1 \\ -1/3 & -1/9 & -1/9 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

5. (Solvability condition for Neumann problem) Determine the null space of the matrix A^T, where A is given in equation (2.58), and verify that the condition (2.62) must hold for the linear system to have solutions.

$$A^T = \frac{1}{h^2} \begin{pmatrix} -h & 1 & & & & & \\ h & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & h \\ & & & & & 1 & -h \end{pmatrix}.$$

Assume that the first entry v_1 of some vector $\mathbf{v}$ in the null space of A^T is nonzero; wlog we can set it to 1. Using the first equation to determine v_2, then the second equation to determine v_3, etc., we find that

$$\mathbf{v} = \begin{pmatrix} 1 \\ h \\ h \\ \vdots \\ h \\ h \\ 1 \end{pmatrix}.$$

On the other hand, if the first entry w_1 of some vector $\mathbf{w}$ in the null space of A^T were 0, then it would follow from the first equation that $w_2 = 0$, from the second equation that $w_3 = 0$, etc., so that $\mathbf{w}$ would have to be the zero vector. Therefore the null space of A^T consists of all scalar multiples of the vector $\mathbf{v}$ given above.

The linear system $A\mathbf{u} = \mathbf{f}$ has solutions if and only if the right-hand side vector $\mathbf{f}$ is orthogonal to the null space of A^T. For the right-hand side vector in (2.58), this is exactly the condition (2.62).

6. (Symmetric tridiagonal matrices)

(a) Consider the **Second approach** described on p. 31 for dealing with a Neumann boundary condition. If we use this technique to approximate the solution to the boundary value problem $u''(x) = f(x)$, $0 \le x \le 1$, $u'(0) = \sigma$, $u(1) = \beta$, then the resulting linear system $A\mathbf{u} = \mathbf{f}$ has the following form:

$$\frac{1}{h^2}\begin{pmatrix} -h & h & & & \\ 1 & -2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \\ u_m \end{pmatrix} = \begin{pmatrix} \sigma + (h/2)f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) - \beta/h^2 \end{pmatrix}.$$

Show that the above matrix is similar to a symmetric tridiagonal matrix via a *diagonal* similarity transformation; that is, there is a diagonal matrix D such that DAD^{-1} is symmetric.

Let $D = \mathrm{diag}(1/\sqrt{h}, 1, \ldots, 1)$. Then

$$DAD^{-1} = \frac{1}{h^2}\begin{pmatrix} -h & \sqrt{h} & & & \\ \sqrt{h} & -2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{pmatrix}$$

(b) Consider the **Third approach** described on pp. 31-32 for dealing with a Neumann boundary condition. **Note:** There is a typo in the matrix (2.57) on p. 32. There should be a row above what is written there that has entries $\frac{3}{2}h$, $-2h$, and $\frac{1}{2}h$ in columns 1 through 3 and 0's elsewhere. Show that if we use that first equation (given at the bottom of p. 31) to eliminate u_0 and we also eliminate u_{m+1} from the equations by setting it equal to β and modifying the right-hand side vector accordingly, then we obtain an m by m linear system $A\mathbf{u} = \mathbf{f}$, where A is similar to a symmetric tridiagonal matrix via a diagonal similarity transformation.

Solving for u_0, we find

$$u_0 = \frac{2h}{3}\sigma + \frac{4}{3}u_1 - \frac{1}{3}u_2.$$

Substituting this into the equation at x_1, we have

$$\frac{1}{h^2}[u_0 + u_2 - 2u_1] = \frac{1}{h^2}\left[\frac{2h}{3}\sigma - \frac{2}{3}u_1 + \frac{2}{3}u_2\right] = f(x_1),$$

or,

$$\frac{1}{h^2}\left[-\frac{2}{3}u_1 + \frac{2}{3}u_2\right] = f(x_1) - \frac{2}{3h}\sigma.$$

Thus, after eliminating the equation for u_{m+1} as well, the matrix takes the form

$$A = \frac{1}{h^2}\begin{pmatrix} -2/3 & 2/3 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}.$$

Let $D = \mathrm{diag}(1/\sqrt{2/3}, 1, \ldots, 1)$. Then

$$DAD^{-1} = \frac{1}{h^2}\begin{pmatrix} -2/3 & \sqrt{2/3} & & & \\ \sqrt{2/3} & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}.$$