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B.1: Intro to Sample Complexity

B.1.a

Let's exam the statement:

$$\mathbb{P}\left[\hat{R}_n(f) = 0\right] = \mathbb{P}\left[\frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\{f(x_i) \neq y_i\} = 0\right] \tag{B.1.a}$$
$$= \prod_{i=1}^{n} 1 - \mathbb{P}\left[f(x_i) \neq y_i\right]$$

Let's use the statement in the hypothesis. The statement was $R(f) > \epsilon$, which describes the event that $\mathbb{E}\left[\mathbf{1}\{f(x) \neq Y\}\right]$. so then $1 - R(f)$ describes exepected value of the event that: $1 - \mathbb{P}\left(f(x_i) \neq y\right)$. And notice that $1 - R(f) < 1 - \epsilon < \exp(epsilon)$, so then we can simplify the above expression into:

$$\prod_{i=1}^{n} 1 - \mathbb{P}\left[f(x_i) \neq y_i\right] = (1 - R(f)) \leq (\exp(\epsilon))^n = \exp(n\epsilon) \tag{B.1.a.1}$$
$$\implies \mathbb{P}\left[\hat{R}_n(f) = 0\right] \leq \exp(n\epsilon)$$

B.1.b

The results from the previous involves the hypothesis that $R(f) \geq \epsilon$, the theoretical risk of the model is larger than ϵ, therefore, under a larger scope the more appropriate inequality to make should be:

$$\mathbb{P}\left[\hat{R}_n(f) = 0 \wedge R(f) \geq \epsilon\right] \leq \exp(-n\epsilon) \tag{B.1.b.1}$$

For the proof, let's start with the following statement:

$$\mathbb{P}\left[f(f) > \epsilon \wedge \hat{R}_n(f) = 0\right] = \mathbb{P}\left[\hat{R}_n(f) = 0 \middle| R(f) > \epsilon\right] \mathbb{P}\left[R(f) > \epsilon\right] \tag{B.1.b.2}$$
$$\implies \mathbb{P}\left[f(f) > \epsilon \wedge \hat{R}_n(f) = 0\right] \leq \mathbb{P}\left[\hat{R}_n(f) = 0 \middle| R(f) > \epsilon\right] \leq \exp(-n\epsilon)$$

Now, the statement $\exists f \in \mathcal{F} : R(f) > \epsilon \wedge \hat{R}_n(f) = 0$ implies that occurence of at least one events, occurence of many events is the includes the case of at least one event, therefore we can say that:

$$\mathbb{P}\left[\exists f \in \mathcal{F} : R(f) > \epsilon \wedge \hat{R}_n(f) = 0\right] \leq \mathbb{P}\left[\bigcup_{f \in \mathcal{F}} \left\{R(f) > \epsilon \wedge \hat{R}_n(f) = 0\right\}\right] \tag{B.1.b.3}$$
$$\underset{\text{Union Bound}}{\leq} \sum_{f \in \mathcal{F}} \mathbb{P}\left[R(f) > \epsilon \wedge \hat{R}_n(f) = 0\right]$$
$$\leq |\mathcal{F}| \exp(-n\epsilon)$$