Name: Hongda Li Class: CSE 546 HW2B

B.1

Objective: Given the definition for the L2, L1 and the Infinity norm of real vector, show that $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$.
First we are going to show that $\|x\|_2^2 \le \|x\|_1^2$, starting from the definition of the norms we have:

$$
\begin{aligned}
\|x\|_1^2 &= \left(\sum_{i=1}^{n}|x_i|\right)^2 \qquad \text{(B.1.1)} \\
&= \sum_{i=1}^{n}\left(|x_i|\sum_{j=1}^{n}|x_j|\right) \\
&= \sum_{i=1}^{n}\left(|x_i|^2 + |x_i|\sum_{j=1,j\neq i}^{n}|x_j|\right) \\
&= \sum_{i=1}^{n}|x_i|^2 + \sum_{i=1}^{n}|x_i|\left(\sum_{j=1,j\neq i}^{n}|x_j|\right) \\
&= \|x\|_2^2 + \underbrace{\sum_{i=2}^{n}\sum_{j=1}^{i-1}2|x_i||x_j|}_{\ge 0} \\
&\implies \|x\|_2^2 \le \|x\|_1^2
\end{aligned}
$$

And now we are going to shoe that $\|x\|_\infty^2 \le \|x\|_2^2$. By the definition of the infinity norm, we know that therde exists $1 \le m \le n$ such that $x_m = \|x\|_\infty = \max_{1\le i\le n}(x_i)$. Then it can be said that:

$$
\begin{aligned}
x_m^2 \le& x_m^2 + \underbrace{\sum_{i=1,i\neq m}^{n}x_i^2}_{\ge 0} \qquad \text{(B.1.2)} \\
x_m^2 =& \|x\|_\infty \le \sum_{i=1}^{n}x_i^2 = \|x\|_2^2
\end{aligned}
$$

And then combing together, we can take the square root because the function $\sqrt{\bullet}$ is monotone increase, hence it preserves the inequality, which will give us $\|x\|_\infty \le \|x\|_2^2 \le \|x\|_1$.