A2.b.(a)

$$
\begin{aligned}
&\partial\left[\sum_{i=1}^{n}|x_i|\right] \\
&= \sum_{i=1}^{n}\partial[|x_i|] \\
&= \sum_{i=1}^{n} g_i\mathbf{e}_i
\end{aligned}
\tag{a2.b.a.1}
$$

g_i is essentially:

$$
g_i \in \partial[|x_i|] = \begin{cases} \{1\} & x_i \geq 1 \\ [-1, 1] & x_i = 0 \\ \{-1\} & x_i \leq 0 \end{cases}
\tag{a2.b.a.2}
$$

And using the hint from the next part, the sub gradient of $\|x\|_1$ is the convex combinations of all $g_i\mathbf{e}_i$:

$$
\sum_{i=1}^{n}\lambda_i g_i \mathbf{e}_i \in \partial[\|x\|_1] \quad \sum_{i=1}^{n}\lambda_i \leq 1 \wedge \lambda_i \geq 0
\tag{a2.b.a.3}
$$

And the span of all sub gradient for each $|x_i|$ will make up the set of sub-gradient for the original function, and hence, let v_j be the j th element of the sub gradient of $\|x\|_1$, the closed form will be:

$$
v_j \in \begin{cases} \{1\} & x_j > 0 \\ [-1, 1] & x_j = 0 \\ \{-1\} & x_j \leq 0 \end{cases}
\tag{A2.b.1.3}
$$

A2.b.(b)

Let λ_i be the set of coefficients for a convex combinations, meaning that $\sum_{i=1}^{n}\lambda_i = 1$ and $\lambda_i \geq 0$, implying that $\lambda_i \in (0, 1)$. Using this fact and the definition of $f(x) := \max\{f_i(x)\}_i^m$, consider the following:

$$
\begin{aligned}
f(y) &\geq f_i(y) \quad \forall\, i \\
\lambda_i f(y) &\geq \lambda_i f_i(y) \quad \forall\, i \\
\sum_{i=1}^{m}\lambda_i f(y) &\geq \sum_{i=1}^{m}\lambda_i f_i(y) \\
\underset{(1)}{\Longrightarrow}\quad f(y) &\geq \sum_{i=1}^{m}\lambda_i f_i(y) \\
f(y) &\geq \left(\underbrace{\sum_{i=1}^{m}\lambda_i f_i(x)}_{\leq f(x)}\right) + \lambda_i \nabla[f_i](x)^T(y - x) \\
\underset{(2)}{\Longrightarrow}\quad f(y) &\geq f(x) + \lambda_i \nabla[f_i](x)^T(y - x) \quad \forall\, i
\end{aligned}
\tag{A2.b.b.1}
$$

(1) : True because the convex combinations coefficients $\sum_{i=1}^{m} \lambda_i = 1$ and $f(y)$ is independent of the summation.

(2) : True because the $\sum_{i=1}^{m} \lambda_i f_i(x) \le f(x)$ is already proven in (1).

Now, we are free to choose λ_i to find the bound of the all the convex combinations of the sub gradient on f_i at x. Therefore, the sub-gradient is the set defined as the following:

$$(\partial[f](x))_j = (\inf \{(\nabla[f_i](x))_j : f_i(x) = f(x)\}, \sup \{(\nabla[f_i](x))_j : f_i(x) = f(x)\}) \tag{A2.b.b.2}$$

Note: The notation of $(\bullet)_j$ is denoting the j th element of a vector, in this case, we are saying that the j th element of the sub gradient vector for f is bounded by the sup and inf of the j th element of the gradient of the smooth function f_i.

A2.c

In this case $f_i(x) = |x_i - (1 + \eta/i)|$ hence we can say v_i is a subgradient of f_i if:

$$\begin{aligned} v_i \in \partial[|x_i - (1+\eta/i)|] = &\begin{cases} \{1\} & x > 1 + \frac{\eta}{i} \\ [-1, 1] & x_i = 1 + \frac{\eta}{i} \\ \{-1\} & x_i < 1 + \frac{\eta}{i} \end{cases} \\ \implies \forall x \in \mathrm{dom}(f), i \in [n] : \quad &-1 \le v_i \le 1 \\ \implies &\|v_i \mathbf{e_i}\|_\infty \le 1 \end{aligned} \tag{A2.c.1}$$

Therefore, we know that the convex combinations will be bounded too and it's like:

$$\forall \lambda_i \ge 0 \wedge \sum_{i=1}^{n} \lambda_i \le 1 : \quad \left\| \underbrace{\sum_{i=1}^{n} \lambda_i v_i \mathbf{e}_i}_{\in \partial[f]} \right\|_\infty \in [0, 1] \tag{A2.c.2}$$

Therefore, the infinity norm of the sub gradient of the function f is in the set interval $[0, 1]$[1].

[1]The infinity norm has only positive part, so it's less than one in the end