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B.1

Objective: Given the definition for the L2, L1 and the Infinity norm of real vector, show that $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$.
First we are going to show that $\|x\|_2^2 \le \|x\|_1^2$, starting from the definition of the norms we have:

$$\begin{aligned}
\|x\|_1^2 &= \left(\sum_{i=1}^{n}|x_i|\right)^2 \\
&= \sum_{i=1}^{n}\left(|x_i|\sum_{j=1}^{n}|x_j|\right) \\
&= \sum_{i=1}^{n}\left(|x_i|^2 + |x_i|\sum_{j=1,j\neq i}^{n}|x_j|\right) \\
&= \sum_{i=1}^{n}|x_i|^2 + \sum_{i=1}^{n}|x_i|\left(\sum_{j=1,j\neq i}^{n}|x_j|\right) \\
&= \|x\|_2^2 + \underbrace{\sum_{i=2}^{n}\sum_{j=1}^{i-1}2|x_i||x_j|}_{\ge 0} \\
\implies& \|x\|_2^2 \le \|x\|_1^2
\end{aligned} \tag{B.1.1}$$

And now we are going to shoe that $\|x\|_\infty^2 \le \|x\|_2^2$. By the definition of the infinity norm, we know that therde exists $1 \le m \le n$ such that $x_m = \|x\|_\infty = \max_{1\le i\le n}(x_i)$. Then it can be said that:

$$\begin{aligned}
x_m^2 \le& x_m^2 + \underbrace{\sum_{i=1,i\neq m}^{n} x_i^2}_{\ge 0} \\
x_m^2 =& \|x\|_\infty \le \sum_{i=1}^{n} x_i^2 = \|x\|_2^2
\end{aligned} \tag{B.1.2}$$

And then combing together, we can take the square root because the function $\sqrt{\bullet}$ is monotone increase, hence it preserves the inequality, which will give us $\|x\|_\infty \le \|x\|_2^2 \le \|x\|_1$.

B.2

B.2.a

Objective: The function $\|x\|$ is a convex function.

$$\begin{aligned}
\|\lambda x + (1-\lambda)y\| &\le \|\lambda x\| + \|(1-\lambda)y\| \\
&= \lambda\|x\| + (1-\lambda)\|y\|
\end{aligned} \tag{B.2.a.1}$$

Note, I just directly apply the Triangular inequality of the norm to get the inequality, and then becaues $\lambda \in [0,1]$, so there is no absolute value, and notice that the resulting expression is the definition of Convexity the given function.

B.2.b

Objective: Show that the set $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is a convex set. Let the set be denoted as S Let's take any 2 points in the set like $x \in S$, $y \in s$, then $\|x\| \leq 1$ and $\|y\| \leq 1$ for any line defined by the 2 points:

$$\begin{aligned} \|\lambda x + (1-\lambda)y\| &\leq \lambda \underbrace{\|x\|}_{\leq \lambda} + \underbrace{(1-\lambda)\|y\|}_{\leq 1-\lambda} \\ \implies \|\lambda x + (1-\lambda)y\| &\leq 1 \\ \implies \lambda x + (1-\lambda)y &\in S \end{aligned} \tag{B.2.b.1}$$

The first by the inequality of norm, and the second is by the definition of the fact that $x, y \in S$, and the third is by the definition of the set.