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AMATH 516 FALL 2021 HW 2

2.2

2.2.1

We are trying to prove that: Q is convex, then $\mathbb{R}_+(Q)$ is convex too.

Consider 2 elements in the form of ax, by taken out from the set $\mathbb{R}_+(Q)$:

$$\begin{aligned} ax \in \mathbb{R}_+(Q), by \in \mathbb{R}_+(Q); x, y \in Q, a, b \geq 0 \\ rx + (1-r)y \in Q \quad \forall r \in (0,1) \\ \implies \forall u \geq 0 : urx + (1-r)uy \in \mathbb{R}_+(Q) \end{aligned} \tag{2.2.1.1}$$

Now let's consider a susbtitution of expressions:

$$\begin{aligned} ur &= \lambda a \quad \lambda \in (0,1), a \geq 0 \\ (1-r)u &= (1-\lambda)by \quad b \geq 0 \\ \implies u - ur &= (1-\lambda)by \\ \implies u - \lambda a &= (1-\lambda)by \\ u &= (1-\lambda)by + \lambda a \geq 0 \\ r &= \frac{\lambda a}{u} = \frac{\lambda a}{(1-\lambda)by + \lambda a} \in (0,1) \end{aligned} \tag{2.2.1.2}$$

By the choice of $\lambda \in (0,1), a, b \geq 0$, we perserve the property of $u \geq 0$ and $r \in (0,1)$.
Consider again the expression:

$$\begin{aligned} &\forall u \geq 0, r \in (0,1) : urx + (1-r)uy \in \mathbb{R}_+(Q) \\ \underset{\text{sub with 2.2.1.2}}{\implies} &\lambda ax + (1-\lambda)by \in \mathbb{R}_+(Q); \lambda \in (0,1), a, b \geq 0 \end{aligned} \tag{2.2.1.3}$$

Using the fact that $ax, by \in Q$, the last expression is a convex combinations of the 2 points, and it's presented in the set $\mathbb{R}_+$, therefore, the set $\mathbb{R}_+(Q)$ is convex.

2.2.2

We wish to prove that if Q_1, Q_2 is convex, then the set $Q_1 + Q_2$ is also a convex set. From the definition of set addition we have:

$$Q_1 + Q_2 = \{q_1 + q_2 : q_1 \in Q_1, q_2 \in Q_2\}$$

Then considering choosing $x, y \in (Q_1 + Q_2)$ and using the definition we can characterize x, y as:

$$\begin{aligned} \exists\, q_1 \in Q_1, q_2 \in Q_2 : x = q_1 + q_2 \\ \exists\, q_3 \in Q_1, q_4 \in Q_2 : y = q_3 + q_4 \end{aligned} \tag{2.2.2.1}$$

Consider the convex combinations of the 2 points x, y we have:

$$\begin{aligned} \lambda x &= \lambda(q_1 + a_2) \\ (1-\lambda)y &= (1-\lambda)(q_3 + q_4) \\ \implies \lambda x + (1-\lambda)x &= \lambda q_1 + (1-\lambda)q_3 + \lambda q_2 + (1-\lambda)q_4 \end{aligned} \tag{2.2.2.2}$$

And notice that $\lambda q_1 + (1-\lambda)q_3$ is $\in Q_1$ by convexity of Q_1, by a similar token the element $\lambda q_2 + (1-\lambda)q_4$ is also in the set Q_2 as well, therefore the convex combination of x, y is in the set $(Q_1 + Q_2)$.

2.2.3

We wish to prove that the intersections of convex set is still a convex set.

Set the intersection of convex sets be $\bigcap_{i\in I} Q_i$, where the set I is an indexing set, then we consider choosing $x, y \in \bigcap_{i\in I} Q_i$, which means that $\forall i \in I : x, y \in Q_i$. Then:

$$\begin{aligned} \lambda x + (1-\lambda)y &\in Q_i \forall i \in I \\ \lambda x + (1-\lambda)y &\in \bigcap_{i\in I} Q_i \end{aligned} \tag{2.2.3.1}$$

□

2.2.4

We wish to prove that if $Q \subset \mathbf{E}$ is convex, $L \subset \mathbf{Y}$ is convex, $A : \mathbf{E} \mapsto Y$, then $A(Q), A^{-1}(L)$ are convex, where A^{-1} is the pre-image of the linear operator.

Choose x, y from the image of A: $x, y \in A(Q)$, then $\exists u \in Q : A(u) = x, \exists v \in Q : A(v) = y$, by the definition of an image of the operator A. Consider the convex combinations of x, y, we have:

$$\begin{aligned} \lambda x + (1-\lambda)y &= \lambda A(u) + (1-\lambda)A(v) \quad \forall \lambda \in (0,1) \\ &= A(\lambda u + (1-\lambda)v) \end{aligned} \tag{2.2.4.1}$$

using the fact that Q is a convex set, $\lambda u + (1-\lambda)v$ is in Q, and $\lambda x + (1-\lambda)y$ is in the range of the operator A.

Consider choices of x, y from the pre-image of A for x, y, let $U := \{u \in L : A^{-1}(u) = x\}$ and $V := \{v \in L : A^{-1}(v) = y\}$. Then consider the convex combinations of x, y:

$$\begin{aligned} \lambda x + (1-\lambda)y &= \lambda A^{-1}(U) + (1-\lambda)A^{-1}(V) \quad \forall \lambda \in (0,1) \\ &= A^{-1}(\lambda U + (1-\lambda)V) \end{aligned} \tag{2.2.4.2}$$

By the convexity of L, the set $\lambda U + (1-\lambda)V$ is a subset of Q, therefore, the pre-image of the convex combinations of x, y is still a preimage of A, therefore the set of pre-images of A is convex if L is convex.

2.6

Let $Q \subseteq \mathbf{E}$, Q convex and $k \in \mathbb{N}$, then the convex combinations of k points in the set Q is still in Q.

Inductively we assume that the convex combinations of $k-1$ points from Q is in the set Q, and we wish to show that the convex combinations of k point will be in Q, let $k \geq 2$. Define:

$$\begin{aligned} S_{k-1} &:= \left\{ \sum_{i=1}^{k-1} \lambda_i x_i : \lambda \in \Delta_{k-1}, x_i \in Q \; \forall 1 \leq i \leq k-1 \right\} \\ S_{k-1} &\subseteq Q \quad \text{Inductive Hypothesis} \end{aligned} \tag{2.6.1}$$

Then we consider the convex combinations of k points, which is any instance of the set S_k

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i x_i &= \left(\sum_{i=1}^{k-1}\lambda_i x_i\right) + \lambda_k x_k \\ &= (1-\lambda_k)\left(\sum_{i=1}^{k-1}\frac{\lambda_i x_i}{1-\lambda_k}\right) + \lambda_k x_k \\ \lambda \in \Delta_k &\implies \sum_{i=1}^{k-1}\frac{\lambda_i}{1-\lambda_k} = 1 \\ \lambda \in \Delta_k &\implies 0 \le \lambda_k \le 1\end{aligned} \tag{2.6.2}$$

We had expressed the convex combinations of k points into the convex combinations of 2 points, where one of them is from the set S_{k-1}. Using the fact that S_k is a subset of Q, both points are in Q, using the definition convex set, the convex combinations of these 2 points are in the set Q as well. Therefore, for any instance of S_k, $S_k \subseteq Q$.

The base case is S_2, if Q is convex, then $S_2 \subseteq Q$ by the definition of a convex set. Therefore, by the principle of mathematics induction, $\forall k \in \mathbb{N} : S_k \subseteq Q$.

2.8

We wish to pveo that:

$$\operatorname{conv}(Q) = T; \quad T := \left\{\sum_{i=1}^{k}\lambda_i x_i : k \in \mathbb{N}, \{x_i\}_1^k \subseteq Q, \lambda \in \Delta_k\right\}$$

The convex hull of any set Q is the convex combinations of all the points in the set Q. The convex cone is:

$$\operatorname{conv}(Q) = \bigcap\{C : C \text{ is convex and } Q \subset C\}$$

It's the intersections of all convex sets that contains Q.

$Q \in \operatorname{conv}(Q)$ by definition of $\operatorname{conv}(Q)$:

$$\begin{aligned}&\forall C, C \text{ convex } : Q \subseteq C, T \subseteq Q \subseteq C \quad \text{ by 2.8} \\ &\implies T \subseteq \operatorname{conv}(Q)\end{aligned} \tag{2.8.1}$$

If T is convex, then $\operatorname{conv}(Q)$ is a subset of T, beacuse T is one of those convex C that contains Q. T contains Q because any points in Q can be added into the convex combinations of Q by incrementing the value k. We now wish to show that T is convex so that $\operatorname{conv}(Q) \subseteq T$.

$$\begin{aligned}&u, v \in T \\ &\implies u \in \sum_{i=1}^{k_1}\beta_i x_i,\ v \in \sum_{i=1}^{k_2}\alpha_i y_i \\ \implies\ &\lambda u + (1-\lambda)v = \lambda\sum_{i=1}^{k_1}\beta_i x_i + (1-\lambda)\sum_{i=1}^{k_2}\alpha_i y_i \quad \forall\lambda \in (0,1)\end{aligned} \tag{2.8.2}$$

Notice that:

$$\begin{aligned}
&\lambda\sum_{i=1}^{k_1}\beta_i+(1-\lambda)\sum_{i=1}^{k_2}\alpha_i=\lambda(1)+(1-\lambda)(1)=1 \qquad (2.8.3)\\
\implies &\forall\lambda\in(0,1):\begin{bmatrix}\lambda\beta\\(1-\lambda)\alpha\end{bmatrix}\in\Delta_{k1+k2}\\
\implies &\lambda u+(1-\lambda)v\in T\\
\implies &T\text{ is convex}
\end{aligned}$$

Therefore $\text{conv}(Q)\subseteq T$, therefore $T=\text{conv}(Q)$.

2.16

We wish to prove that the dist_Q is a 1-Liptshitz continuous funciton:

$$|\text{dist}_Q(x)-\text{dist}_Q(y)|\leq\|x-y\|_2$$

Choose any $x,y\in\mathbf{E}$, $z\in Q$. Then we have:

$$\begin{aligned}
&\text{dist}_Q(x)\leq\|x-z\|_2\leq\|x-y\|_2+\|y-z\|_2 \qquad (2.16.1)\\
&\text{dist}_Q(x)\leq\|x-y\|_2+\|y-z\|_2
\end{aligned}$$

This is triangular inequality. Consider for y:

$$\begin{aligned}
&\text{dist}_Q(y)\leq\|y-z\|_2\leq\|y-x\|_2+\|x-z\|_2 \qquad (2.16.2)\\
&\text{dist}_Q(y)\leq\|y-x\|_2+\|x-z\|_2
\end{aligned}$$

Take the difference between the 2 expression, their absolute value is bounded:

$$\begin{aligned}
&|\text{dist}_Q(x)-\text{dist}_Q(y)|\leq\|y-z\|_2-\|x-z\|_2 \qquad (2.16.3)\\
&|\text{dist}_Q(x)-\text{dist}_Q(y)|\leq\|x-y\|_2
\end{aligned}$$

because $\|y-z\|\leq\|x-z\|+\|x-y\|$. And the last inequality is imposed using triangular inequality.

2.18

We wish to prove that the projection function $\text{proj}_Q(x)$ is a 1-Lipschitz function when the set Q is closed and convex.

We wish to prove this claim:

$$\|\underset{Q}{\text{proj}}(x_1)-\underset{Q}{\text{proj}}(x_2)\|_2^2\leq\left\langle\underset{Q}{\text{proj}}(x_1)-\underset{Q}{\text{proj}}(x_2),x_1-x_2\right\rangle$$

If this is true, then we can use the Cuachy Schwartz inequality, which gives us the 1-Lipschitz continuity:

$$\begin{aligned}
&\left\langle\underset{Q}{\text{proj}}(x_1)-\underset{Q}{\text{proj}}(x_2),x_1-x_2\right\rangle\leq\|\underset{Q}{\text{proj}}(x_1)-\underset{Q}{\text{proj}}(x_2)\|\|x_1-x_2\| \qquad (2.18.1)\\
\implies &\|\underset{Q}{\text{proj}}(x_1))-\underset{Q}{\text{proj}}(x_2)\|_2\leq\|x_1-x_2\|_2
\end{aligned}$$

Let's proe the claim using the obstuse angle theorem for the porjections of any convex set Q, we have:

$$\begin{aligned}
&\forall x_1,x_2\in Q: \qquad (2.18.2)\\
&\left\langle\underset{Q}{\text{proj}}(x_2)-\underset{Q}{\text{proj}}(x_1),x_1-\underset{Q}{\text{proj}}(x_1)\right\rangle\leq 0\\
&\left\langle\underset{Q}{\text{proj}}(x_1)-\underset{Q}{\text{proj}}(x_2),x_2-\underset{Q}{\text{proj}}(x_1)\right\rangle\leq 0
\end{aligned}$$

Let me add a negative sign so we can combine them:

$$
\begin{aligned}
&\forall x_1, x_2 \in Q: \\
&\left\langle \underset{Q}{\mathrm{proj}}(x_2) - \underset{Q}{\mathrm{proj}}(x_1), x_1 - \underset{Q}{\mathrm{proj}}(x_1) \right\rangle \le 0 \\
&\left\langle \underset{Q}{\mathrm{proj}}(x_2) - \underset{Q}{\mathrm{proj}}(x_1), \underset{Q}{\mathrm{proj}}(x_1) - x_2 \right\rangle \le 0 \\
\underset{\text{add them}}{\Longrightarrow} &\left\langle \underset{Q}{\mathrm{proj}}(x_2) - \underset{Q}{\mathrm{proj}}(x_1), x_1 - x_2 + \underset{Q}{\mathrm{proj}}(x_2) - \underset{Q}{\mathrm{proj}}(x_1) \right\rangle \le 0 \\
&\|\underset{Q}{\mathrm{proj}}(x_2) - \underset{Q}{\mathrm{proj}}(x_1)\|_2^2 - \left\langle \underset{Q}{\mathrm{proj}}(x_2) - \underset{Q}{\mathrm{proj}}(x_1), x_2 - x_1 \right\rangle \le 0
\end{aligned}
\tag{2.18.3}
$$

The claim is proven, if follows from 2.18.1 that the projection function is 1-Lipschitz continuous.

2.23

Let $K \subseteq \mathbf{E}$ then K is a convex cone iff the point $\lambda x + uy$ lies in K for any 2 points $x, y \in K$, $u, \lambda \ge 0$.

We wish to prove it in 2 directions. Consier the cone C. First, we wish to show that if the set is a convex cone, then $\lambda x + uy \in C$ forall $\lambda, u \ge 0$.
If the set C is a cone, then:

$$
\begin{aligned}
\lambda x \in C \quad \forall x \in C, \lambda \ge 0 \\
uy \in C \quad \forall y \in C, uy \ge 0
\end{aligned}
\tag{2.23.1}
$$

Using the fact that C is also convex, then:

$$
\begin{aligned}
r\lambda x + (1-r)uy \in C \quad \forall r \in (0,1) \\
t := r\lambda \ge 0 \quad k := (1-r)u \ge 0 \\
\forall t, k \ge 0 : tx + ky \in C
\end{aligned}
\tag{2.23.2}
$$

Done.

Now we wish to prove that if $\lambda x + uy \in C$ for all points $x, y \in C$ and $\lambda, u \ge 0$, then the set C is a convex cone.
Suppoe that $x, y \in C; \lambda, u \ge 0$, $\lambda x + uy \in C$, choose $\lambda = 0, u \ge 0$, then $uy \in C\ \forall u \ge 0$. And this indicates that the set C is a cone.
Choose $\lambda \in (0,1), u = (1-\lambda)$, then both $\lambda, u \ge 0$ still, but $\lambda x + (1-\lambda)y \in C$ which implies that the set C is convex.

2.27

Double Polar Theorem. Given a set K that is non-empty and a cone, then:

$$
(K^\circ)^\circ = (\mathrm{cl} \circ \mathrm{conv})(K)
$$

The convex closure of the cone is the same as the polar polar cone.
The set of halfplanes supporting the cone K is:

$$
\mathcal{F}_K = \{a \in \mathbf{E} : \langle x, a \rangle \le 0\ \forall x \in K\} \subset K^\circ
\tag{2.27.1}
$$

We can construct the closure of the convex hull by intersecting all the supporing halfplanes for the set K:

$$\begin{aligned} \mathrm{cl} \circ \mathrm{conv}(K) &= \bigcap_{a \in \mathcal{F}_k} \{x \in \mathbf{E} : \langle a, x \rangle \leq 0\} \\ &= \bigcap_{a \in K^\circ} \{x \in \mathbf{E} : \langle a, x \rangle \leq 0\} \\ &= \{x \in \mathbf{E} : \langle a, x \rangle \leq 0, a \in K^\circ\} \\ &= (K^\circ)^\circ \end{aligned} \tag{2.27.2}$$

2.33

In this section we wish to prove that, $\forall Q \in \mathbf{E}$, where Q is a convex set; and a point $\bar{x} \in Q$:

$$T_Q(\bar{x}) = \mathrm{cl}\mathbb{R}_+(Q - \bar{x})$$

The tagent cone generated at the point $\bar{x}$ is the closure of the cone generated by offsetting the set Q by $\bar{x}$. Firstly if $\bar{x} \in \mathrm{int}(Q)$, then the tangent cone is $\mathbf{E}$, and $Q - \bar{x}$ interset with an ϵ around the origin, making $\mathrm{cl}\mathbb{R}_+(Q - \bar{x}) = \mathbf{E}$ as well.
Now choose a point $\bar{x} \in \mathrm{cl}(Q) \setminus \mathrm{int}(Q)$, from the boundary of the set Q, then we want to show that $T_Q(\bar{x}) \supseteq \mathrm{cl}\mathbb{R}_+(Q - \bar{x})$.

$$\begin{aligned} &\bar{x} \in Q && \text{By def} \\ &\forall x \in Q && \text{Our choice} \\ &\frac{1}{n}x\left(1 - \frac{1}{n}\right)\bar{x} \in Q && \text{By Q Convex} \\ &\forall u_i : \lim_{i\to\infty} u_i = 0 \wedge u_i \in [0, 1] \\ &r_n := u_n x + (1 - u_n)\bar{x} && \text{We defined it} \\ &\lim_{n\to\infty} r_n = \bar{x}, r_n \in Q \\ &\tau_n = \frac{u_n}{\lambda}, \lambda \geq 0 \implies \tau_n \searrow 0 && \text{Our choice} \\ &\lim_{n\to\infty} \tau_n^{-1}(r_n - \bar{x}) \in T_Q(\bar{x}) && \text{Def of } T_Q(\bar{x}) \\ &\lim_{n\to\infty} \tau_n^{-1}(r_n - \bar{x}) \\ = &\lim_{n\to\infty} \frac{\lambda}{u_n}(u_n x + (1 - u_n)\bar{x} - \bar{x}) \\ = &\lim_{n\to\infty} (\lambda x - \lambda\bar{x}) = \lambda(x - \bar{x}) \in \mathbb{R}_+(Q - \bar{x}) && \text{by Def} \end{aligned} \tag{2.33.1}$$

Note that, the convexity of Q is important here. x_n is a sequence of points connecting the between $x, \bar{x}$, which are both in the set Q by defintion. Therefore any points that linearly interpolating between then will still be in the set Q by the convexity of the set Q.

Next we wish to prove that $\mathrm{cl}\mathbb{R}_+(Q - \bar{x}) \supseteq T_Q(\bar{x})$. Fix any $\bar{x} \in \mathrm{cl}(Q) \setminus \mathrm{int}(Q)$
Start with the tangent cone at the point $\bar{x}$:

$$\begin{aligned} v \in T_Q(\bar{x}) &\iff v = \lim_{i\to\infty} \tau_i(x_i - \bar{x}), \tau_1 \searrow 0, x_i \in Q\ \forall i \\ &\quad \lambda_i := \tau_i^{-1} \geq 0 \implies \tau_i^{-1}(x - \bar{x}) = \lambda_i(x_i - \bar{x}) \\ x_i \in Q &\implies x_i - \bar{x} \in Q - \bar{x} \quad \forall i \\ &\implies \lambda_i(x_i - \bar{x}) \in \mathrm{cl}\mathbb{R}_+(Q - \bar{x}) \quad \forall i \\ &\implies v \in \mathrm{cl}\mathbb{R}_+(Q - \bar{x}) \end{aligned} \tag{2.33.2}$$

2.36

We wish to prove that, for $Q \subseteq \mathbf{E}$ with Q being convex and a point $\bar{x} \in Q$ then the equality:

$$N_Q(\bar{x}) = \{v \in \mathbf{E} : \langle v, x - \bar{x}\rangle \leq 0 \; \forall x \in Q\}$$

Let's start by choosing $v \in$ RHS then:

$$\begin{aligned}
\langle v, x - \bar{x}\rangle \leq 0 \; \forall x \in Q \\
\text{let: } x_n \to \bar{x}, x_n \in Q \; \forall n \\
\langle v, x_n - \bar{x}\rangle \leq 0 \; \forall x_n \in Q \\
\langle v, x_n - \bar{x}\rangle \leq 0 \leq o(\|x - \bar{x}\|_2) \; \forall x_n \in Q \\
\implies v \in N_Q(\bar{x})
\end{aligned} \tag{2.36.1}$$

Now consider choosing $v \in N_Q(\bar{x})$, consider:

$$\begin{aligned}
& N_Q(\bar{x}) = (T_Q(\bar{x}))^\circ && \text{By Lemma 2.35} \\
& N_Q(\bar{x}) = \{v \in \mathbf{E} : \langle v, x\rangle \leq 0 \; \forall x \in T_Q(\bar{x})\} && \text{Def of Polar Cone} \\
& x \in T_Q(\bar{x}) = \mathrm{cl}\mathbb{R}_+(Q - \bar{x}) && \text{Q convex, use 2.33} \\
& x = \lambda(y - \bar{x}), \lambda \geq 0, y \in Q && \text{Def of } \mathbb{R}_+(Q - \bar{x}) \text{ form match} \\
\implies & N_Q(\bar{x}) = \{v \in \mathbf{E} : \lambda \langle v, y - \bar{x}\rangle \leq 0 \; \forall y \in Q\} && \text{Arrived at LHS}
\end{aligned} \tag{2.36.2}$$

The proof is done.

2.37

We wish to prove that the following statements are all equivalent:

(a) $v \in N_Q(\bar{x})$

(b) $\bar{x} \in \arg\max_{x \in Q} \langle v, x\rangle$

(c) $\mathrm{proj}_Q(\bar{x} + \lambda v) = \bar{x} \; \forall \lambda \geq 0$

(d) $\mathrm{proj}_Q(\bar{x} + \lambda v) = \bar{x}$ for some $\bar{\lambda} \geq 0$

We wish to prove $(a) \implies (b) \implies (c) \implies (d) \implies (a)$ to show the equivalence of the statement.

verify (a) $\implies$ (b)

$$\begin{aligned}
& v \in N_Q(\bar{x}) \\
& \forall x \in Q : \langle x - \bar{x}, v\rangle \leq 0 && \text{by 2.36} \\
& \forall x \in Q : \langle x, v\rangle \leq \langle \bar{x}, v\rangle \\
& \bar{x} \in \arg\max_{x \in Q} \langle v, x\rangle
\end{aligned} \tag{2.37.1}$$

Verify (b) $\implies$ (c) We wish to say that:

$$\bar{x} \in \arg\max_{x \in Q} \langle v, x\rangle \implies \mathrm{proj}_Q(\bar{x} + \lambda v) = \bar{x} \; \forall \lambda \geq 0$$

From the definition that $\bar{x}$ is the maximizer for the dot product on v we have:

$$\begin{aligned}
\bar{x} \in \arg\max_{x \in Q} \langle v, x\rangle \implies \forall x \in Q : \langle x, v\rangle \leq \langle \bar{x}, v\rangle \\
\forall x \in Q \, \langle x - \bar{x}, v\rangle \leq 0 \\
\forall x \in Q : \langle x - \bar{x}, \lambda v\rangle \leq 0
\end{aligned} \tag{2.37.2}$$

Take not that the last expression is the Obtuse Angle characterization of projections on to the convex set Q, consider $\lambda v = u - \bar{x}$, then we have:

$$\begin{aligned} & \forall x \in Q : \langle x - \bar{x}, u - \bar{x} \rangle \le 0 \; \forall \lambda \ge 0 \\ \implies & \underset{Q}{\text{proj}}(u) = \bar{x} \\ & = \underset{Q}{\text{proj}}(\bar{x} + \lambda v) \; \forall \ge 0 \end{aligned} \tag{2.37.3}$$

Verify $(c) \implies (d)$:
This is trival, simply by choose some λ to be $\bar{\lambda}$ and the statement follows.
Verify $(d) \implies (a)$
We wish to show that

$$\exists \bar{\lambda} > 0 : \underset{Q}{\text{proj}}(\bar{x} + \bar{\lambda} v) = \bar{x} \implies v \in N_Q(\bar{x})$$

Consider:

$$\begin{aligned} & \forall x \in Q : \left\langle x - \bar{x}, \bar{\lambda} v \right\rangle \le 0 && \text{Proj Obtuse Angle} \\ & \forall x \in Q : \langle x - \bar{x}, v \rangle \le 0 && \bar{\lambda} \ge 0 \\ & \forall x \in Q : \langle x - \bar{x}, v \rangle \le o(\|x - \bar{x}\|_2) \\ & x_n \to \bar{x}, x_n \in Q \; \forall n \\ & \lim_{n \to \infty} \langle x_n - \bar{x}, v \rangle \le o(\|x_n - \bar{x}\|_2) \\ \implies & v \in N_Q(\bar{x}) \end{aligned} \tag{2.37.4}$$