# 13. Stochastic and online algorithms

- stochastic gradient method
- online optimization and dual averaging method
- minimizing finite average

# Stochastic optimization problem

$$\underset{x \in X}{\text{minimize}} \quad \left\{ F(x) \stackrel{\text{def}}{=} \mathbf{E}_\xi f(x, \xi) \right\}$$

- $X \subset \mathbf{R}^n$ is a (bounded) closed convex set
- $\xi$ is a random vector whose distribution $P$ is supported on set $\Xi \subset \mathbf{R}^d$
- $f : X \times \Xi \to \mathbf{R}$, and the expectation

$$\mathbf{E}_\xi f(x, \xi) = \int_\Xi f(x, \xi) dP(\xi)$$

  is well defined and has finite value for every $x \in X$

- $F(\cdot)$ continuous and convex on $X$, and optimal value $F^\star$ attained at $x^\star$ (e.g., $F(\cdot)$ is convex if $f(\cdot, \xi)$ is convex for every $\xi \in \Xi$)

# Sample average approximation

$$\underset{x \in X}{\text{minimize}} \quad \left\{ \hat{F}_N(x) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{j=1}^{N} f(x, \xi_j) \right\}$$

- **assumption:** $\{\xi_j\}_{j=1}^N$ is a sequence of independent random outcomes
- reasonably efficient when solved by appropriate (deterministic) algorithm
- **sample complexity:** suppose $f$ has bounded variation, and let

$$V = \max\Big\{ f(x_1, \xi_1) - f(x_2, \xi_2) \ : \ x_1, x_2 \in X, \ \xi_1, \xi_2 \in \Xi \Big\}$$

then for any $\epsilon > 0$ and $\rho \in (0, 1)$, sample size $N = \lceil \frac{V^2}{2\epsilon^2} \ln \frac{2}{\rho} \rceil$ guarantees

$$\mathbf{prob}\Big( |\hat{F}_N(x) - F(x)| \leq \epsilon \Big) \geq 1 - \rho, \qquad \forall \, x \in X$$

(proved using Hoeffding inequality in probability theory)

# Stochastic approximation

choose $x^{(1)} \in X$, and iterate for $k = 1, 2, \ldots$

$$x^{(k+1)} = \pi_X\Big(x^{(k)} - t_k\, g(x^{(k)}, \xi_k)\Big)$$

- $g(x, \xi)$ is a *stochastic subgradient*, i.e., $g(x, \xi) \in \partial_x f(x, \xi)$ and

  $$F'(x) \stackrel{\text{def}}{=} \mathbf{E}_\xi g(x, \xi) \in \partial F(x)$$

  assumption: there exist a constant $G$ such that

  $$\mathbf{E}_\xi\big[\|g(x,\xi)\|_2^2\big] \leq G^2, \qquad \forall\, x \in X$$

- $\pi_X(\cdot)$ denotes projection onto $X$:

  $$\pi_X(x) = \operatorname*{argmin}_{y \in X} \|y - x\|_2^2$$

# Convergence analysis

consider squared distance to $x^\star$, and let $r_k = \mathbf{E}\big[\|x^{(k)} - x^\star\|_2^2\big]$

$$
\begin{aligned}
\big\|x^{(k+1)} - x^\star\big\|_2^2 &= \big\|\pi_X\big(x^{(k)} - t_k g(x^{(k)}, \xi_k)\big) - \pi_X(x^\star)\big\|_2^2 \\
&\leq \big\|x^{(k)} - t_k g(x^{(k)}, \xi_k) - x^\star\big\|_2^2 \\
&= \big\|x^{(k)} - x^\star\big\|_2^2 - 2t_k(x^{(k)} - x^\star)^T g(x^{(k)}, \xi_k) + t_k^2\big\|g(x^{(k)}, \xi_k)\big\|_2^2
\end{aligned}
$$

since $x^{(k)}$ is a function of $\xi_{[k-1]} = (\xi_0, \ldots, \xi_{k-1})$, it is independent of $\xi_k$

$$
\begin{aligned}
\mathbf{E}\big[(x^{(k)} - x^\star)^T g(x^{(k)}, \xi_k)\big] &= \mathbf{E}\big\{\mathbf{E}\big[(x^{(k)} - x^\star)^T g(x^{(k)}, \xi_k) \,|\, \xi_{[k-1]}\big]\big\} \\
&= \mathbf{E}\big\{(x^{(k)} - x^\star)^T \mathbf{E}\big[g(x^{(k)}, \xi_k) \,|\, \xi_{[k-1]}\big]\big\} \\
&= \mathbf{E}\big[(x^{(k)} - x^\star)^T F'(x^{(k)})\big]
\end{aligned}
$$

therefore

$$
r_{k+1} \;\leq\; r_k - 2t_k \mathbf{E}\big[(x^{(k)} - x^\star)^T F'(x^{(k)})\big] + t_k^2 G^2 \qquad (1)
$$

by convexity of $F$, it holds $F(x^\star) \geq F(x^{(k)}) + (x^\star - x^{(k)})^T F'(x^{(k)})$, hence

$$\mathbf{E}\big[(x^{(k)} - x^\star)^T F'(x^{(k)})\big] \;\geq\; \mathbf{E}\big[F(x^{(k)}) - F^\star\big]$$

combining with (1) gives

$$t_k \mathbf{E}\big[F(x^{(k)}) - F^\star\big] \;\leq\; \frac{1}{2}\big(r_k - r_{k+1} + t_k^2 G^2\big)$$

summing over $j = 1, \ldots, k$ yields

$$\sum_{j=1}^{k} t_j \mathbf{E}\big[F(x^{(j)}) - F^\star\big] \;\leq\; \frac{1}{2}\left(r_1 - r_{k+1} + G^2 \sum_{j=1}^{k} t_j^2\right) \;\leq\; \frac{1}{2}\left(r_1 + G^2 \sum_{j=1}^{k} t_j^2\right)$$

let $\nu_j^{(k)} = \dfrac{t_j}{\sum_{i=1}^k t_i}$ and $\tilde{x}^{(k)} = \sum_{j=1}^k \nu_j^{(k)} x^{(j)}$ (note $\sum_{j=1}^k \nu_j^{(k)} = 1$), then

$$\mathbf{E}\big[F(\tilde{x}^{(k)}) - F^\star\big] \;\leq\; \mathbf{E}\left[\sum_{j=1}^{k} \nu_j^{(k)} F(x^{(j)}) - F^\star\right] \;\leq\; \frac{r_1 + G^2 \sum_{j=1}^k t_j^2}{2 \sum_{j=1}^k t_j}$$

# Fixed step size

suppose the number of iterations $N$ is known in advance, then

$$\mathbf{E}\big[F(\tilde{x}^{(k)}) - F^\star\big] \;\le\; \frac{D^2 + G^2 N t^2}{2Nt}$$

where $D = \max_{x \in X} \|x - x^\star\|_2$, so that $r_1 = \mathbf{E}\|x^{(1)} - x^\star\|_2^2 \le D^2$

- minimizing upper bound over $t > 0$ gives $t = \dfrac{D}{G\sqrt{N}}$ and

$$\mathbf{E}\big[F(\tilde{x}^{(k)}) - F^\star\big] \le \frac{DG}{\sqrt{N}}$$

- if $t = \dfrac{\theta D}{G\sqrt{N}}$ for some constant $\theta > 0$, then

$$\mathbf{E}\big[F(\tilde{x}^{(k)}) - F^\star\big] \le \max\{\theta, \theta^{-1}\}\frac{DG}{\sqrt{N}}$$

  therefore, $O(1/\sqrt{N})$ convergence *robust* against step size choices

# Diminishing step size

following the halving trick in deterministic subgradient method, redefine

$$\tilde{x}^{(k)} = \frac{\displaystyle\sum_{k/2 \le j \le k} t_j x^{(j)}}{\displaystyle\sum_{k/2 \le j \le k} t_j}$$

if the step sizes are chosen as

$$t_k = \frac{\theta D}{G\sqrt{k}}$$

then the following holds with a constant $C > 1$

$$\mathbf{E}\big[F(\tilde{x}^{(k)}) - F^\star\big] \le C \max\{\theta, \theta^{-1}\} \frac{DG}{\sqrt{k}}$$

$O(1/\sqrt{k})$ convergence rate is optimal for general convex functions

## Analysis for strongly convex functions

assume $F = \mathbf{E}_\xi f(x, \xi)$ is differentiable and strongly convex

$$F(y) \geq F(x) + \nabla F(x)^T(y - x) + \frac{\mu}{2}\|y - x\|_2^2 \qquad \forall\, x, y \in X$$

or equivalently

$$(x - y)^T(\nabla F(x) - \nabla F(y)) \geq \mu\|x - y\|_2^2, \qquad \forall\, x, y \in X$$

by optimality of $x^\star$,

$$(x - x^\star)^T \nabla F(x^\star) \geq 0, \qquad \forall\, x \in X$$

therefore

$$(x - x^\star)^T \nabla F(x) \geq \mu\|x - x^\star\|_2^2, \qquad \forall\, x \in X \tag{2}$$

combining (1) and (2) gives

$$r_{k+1} \ \leq \ (1 - 2\mu t_k) r_k + t_k^2 G^2$$

let's take step size $t_k = \theta/k$ for some constant $\theta > 1/(2\mu)$, then

$$r_{k+1} \ \leq \ (1 - 2\mu\theta/k) r_k + \theta^2 G^2/k^2$$

- it follows by induction that (Nemirovski et al. 2009)

$$\mathbf{E}\big[\|x^{(k)} - x^\star\|_2^2\big] = r_k \leq \frac{Q(\theta)}{k}$$

  where $Q(\theta) = \max\{\theta^2 G^2 (2\mu\theta - 1)^{-1}, \|x^{(1)} - x^\star\|_2^2\}$

- if in addition $\nabla F$ is Lipschitz continuous with constant $L > 0$, then

$$\mathbf{E}\big[F(x^{(k)}) - F^\star\big] \ \leq \ \frac{L}{2}\mathbf{E}\big[\|x^{(k)} - x^\star\|_2^2\big] \ \leq \ \frac{LQ(\theta)}{2k}$$

## Sensitivity to priori knowledge of $\mu$

example: let $F(x) = x^2/10$, $X = [-1, 1]$, $\mu = 0.2$, and there is no noise

- if $\theta = 1$ (which violates the condition $\theta > 1/(2\mu)$), then

$$x^{(k+1)} = x^{(k)} - \frac{1}{k} F'(x^{(k)}) = \left(1 - \frac{1}{5k}\right) x^{(k)}$$

  starting with $x^{(1)} = 1$ leads to

$$x^{(k)} > 0.8 k^{-1/5}$$

  error is larger than $0.015$ even after $10^9$ iterations!

- if $\theta = 1/\mu = 5$, then $x^\star = 0$ is obtained in one iteration

- step size $t_k = \theta/k$ too small if $F$ is not strongly convex

# Outline

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# Online convex optimization

- explained as online game: for $k = 1, 2, 3, \ldots$,
  - player chooses $x^{(k)} \in X$ based on previous information
  - adversary reveals cost function $f_k$, and player encurs loss $f_k(x^{(k)})$

  assumptions: $f_k$ convex; $X$ bounded, closed and convex

- player wants to minimize *regret*:

$$R_N \triangleq \sum_{k=1}^{N} \big(f_k(x^{(k)})\big) - \min_{x \in X} \left\{ \sum_{k=1}^{N} f_k(x) \right\}$$

- online subgradient method

$$x^{(k+1)} = \pi_X\big(x^{(k)} - t_k g^{(k)}\big), \qquad g^{(k)} \in \partial f_k(x^{(k)})$$

with appropriate step size, can show $R_N \leq O(\sqrt{N})$

# Connection to stochastic approximation

- a more general framework without stochastic assumptions
- suppose $f_k(x) \stackrel{\text{def}}{=} f(x, \xi_k)$, and let $\bar{x}^{(N)} = \frac{1}{N}\sum_{k=1}^N x^{(k)}$, then

$$F(\bar{x}^{(N)}) - F^\star \;\leq\; \frac{1}{N}\mathbf{E}[R_N]$$

proof:

$$\begin{aligned} F(\bar{x}^{(N)}) - F^\star &\leq \frac{1}{N}\sum_{k=1}^N \Big(F(x^{(k)}) - F^\star\Big) \\ &= \frac{1}{N}\sum_{k=1}^N \Big(\mathbf{E}\big[f(x^{(k)}, \xi_k)\big] - \min_x \mathbf{E}\big[f(x, \xi_k)\big]\Big) \\ &= \frac{1}{N}\mathbf{E}\Bigg[\sum_{k=1}^N \Big(f(x^{(k)}, \xi_k) - f(x^\star, \xi_k)\Big)\Bigg] \end{aligned}$$

# Dual averaging method (Nesterov)

initialize: choose $x^{(1)} \in \mathbf{R}^n$ and set $s^{(0)} = 0$

iterate for $k = 0, 1, 2, \ldots$

1. compute $g^{(k)} \in \partial f_k(x^{(k)})$ and set
$$s^{(k)} = s^{(k-1)} + g^{(k)}$$

2. update:
$$\begin{aligned} x^{(k+1)} &= \operatorname*{argmin}_{x \in X} \left\{ \langle s^{(k)}, x \rangle + \frac{\beta_k}{2} \|x - x^{(0)}\|_2^2 \right\} \\ &= \pi_X \left( x^{(0)} - \frac{1}{\beta_k} s^{(k)} \right) \end{aligned}$$

- choice of $\{\beta_k\}$: e.g., $\beta_k = \gamma\sqrt{k}$ with $\gamma > 0$
- can also work with composite objectives: $\mathrm{minimize}_x \;\; f(x) + \Psi(x)$

# A soft support function

for any $\beta \geq 0$ and any $x^{(0)} \in X$, define

$$V_\beta(s) = \max_{x \in X} \left\{ \langle s, x - x^{(0)} \rangle - \frac{\beta}{2} \|x - x^{(0)}\|_2^2 \right\}$$

- $V_\beta(s) \geq 0$ for any $\beta \geq 0$; if $\beta_2 \geq \beta_1 > 0$, then $V_{\beta_2}(s) \leq V_{\beta_1}(s)$
- $V_\beta(\cdot)$ is convex and differentiable
- $\nabla V_\beta$ is Lipschitz continuous with constant $1/\beta$

$$\|\nabla V_\beta(s_1) - \nabla V_\beta(s_2)\|_2 \;\leq\; \frac{1}{\beta}\|s_1 - s_2\|_2, \qquad \forall\, s_1, s_2 \in \mathbf{R}^n$$

therefore

$$V_\beta(s + \delta) \;\leq\; V_\beta(s) + \langle \delta, \nabla V_\beta(s) \rangle + \frac{1}{2\beta}\|\delta\|_2^2$$

**lemma:** let $D = \max_{x\in X} \|x - x^{(0)}\|_2$, then

$$\max_{x\in X}\langle s, x - x^{(0)}\rangle \ \le\ \frac{\beta D^2}{2} + V_\beta(s)$$

proof:

$$\begin{aligned}
\max_{x\in X}\langle s, x - x^{(0)}\rangle &= \max_{x\in X}\left\{\langle s, x - x^{(0)}\rangle \ :\ \frac{1}{2}\|x - x^{(0)}\|_2^2 \le \frac{1}{2}D^2\right\} \\
&= \max_{x\in X}\ \min_{\beta\ge 0}\left\{\langle s, x - x^{(0)}\rangle + \frac{\beta}{2}(D^2 - \|x - x^{(0)}\|_2^2)\right\} \\
&\le \min_{\beta\ge 0}\ \max_{x\in X}\left\{\langle s, x - x^{(0)}\rangle + \frac{\beta}{2}(D^2 - \|x - x^{(0)}\|_2^2)\right\} \\
&\le \max_{x\in X}\left\{\langle s, x - x^{(0)}\rangle + \frac{\beta}{2}(D^2 - \|x - x^{(0)}\|_2^2)\right\} \\
&\le \frac{\beta D^2}{2} + V_\beta(s)
\end{aligned}$$

# Convergence analysis

$$\begin{aligned}
V_{\beta_k}(-s^{(k)}) &\leq V_{\beta_{k-1}}(-s^{(k)}) \\
&\leq V_{\beta_{k-1}}(-s^{(k-1)}) + \langle -g^{(k)}, \nabla V_{\beta_{k-1}}(-s^{(k-1)})\rangle + \frac{1}{2\beta_{k-1}}\|g^{(k)}\|_2^2 \\
&= V_{\beta_{k-1}}(-s^{(k-1)}) - \langle g^{(k)}, x^{(k)} - x^{(0)}\rangle + \frac{1}{2\beta_{k-1}}\|g^{(k)}\|_2^2
\end{aligned}$$

therefore

$$\langle g^{(k)}, x^{(k)} - x^{(0)}\rangle \leq V_{\beta_{k-1}}(-s^{(k-1)}) - V_{\beta_k}(-s^{(k)}) + \frac{1}{2\beta_{k-1}}\|g^{(k)}\|_2^2$$

summing over $k = 2, \ldots, N$ and choose $x^{(0)} = x^{(1)}$ results in

$$\sum_{k=1}^{N}\langle g^{(k)}, x^{(k)} - x^{(0)}\rangle \leq V_{\beta_1}(-s^{(1)}) - V_{\beta_N}(-s^{(N)}) + \sum_{k=2}^{N}\frac{1}{2\beta_{k-1}}\|g^{(k)}\|_2^2$$

$$
\begin{aligned}
\delta_N & \stackrel{\text{def}}{=} \max_{x \in X} \sum_{k=1}^{N} \langle g^{(k)}, x^{(k)} - x \rangle \\
& = \sum_{k=1}^{N} \langle g^{(k)}, x^{(k)} - x^{(0)} \rangle + \max_{x \in X} \sum_{k=1}^{N} \langle g^{(k)}, x^{(0)} - x \rangle \\
& = \sum_{k=1}^{N} \langle g^{(k)}, x^{(k)} - x^{(0)} \rangle + \max_{x \in X} \langle -s^{(N)}, x - x^{(0)} \rangle \\
& \leq V_{\beta_1}(-s^{(1)}) - V_{\beta_N}(-s^{(N)}) + \sum_{k=2}^{N} \frac{1}{2\beta_{k-1}} \|g^{(k)}\|_2^2 + \frac{\beta_N D^2}{2} + V_{\beta_N}(-s^{(N)}) \\
& \leq \frac{1}{2\beta_1} \|g^{(1)}\|_2^2 + \sum_{k=2}^{N} \frac{1}{2\beta_{k-1}} \|g^{(k)}\|_2^2 + \frac{\beta_N D^2}{2} \\
& \leq \frac{\beta_N D^2}{2} + \sum_{k=0}^{N-1} \frac{G^2}{2\beta_k} \qquad \text{(for convenience, define } \beta_0 = \beta_1\text{)}
\end{aligned}
$$

by convexity,

$$
\begin{aligned}
\delta_N &\stackrel{\text{def}}{=} \max_{x\in X} \sum_{k=1}^{N} \langle g^{(k)}, x^{(k)} - x \rangle \\
&\geq \max_{x\in X} \sum_{k=1}^{N} \big( f_k(x^{(k)}) - f_k(x) \big) \\
&= \sum_{k=1}^{N} f_k(x^{(k)}) - \min_{x\in X} \sum_{k=1}^{N} f_k(x)
\end{aligned}
$$

therefore, $R_N \leq \delta_N$, so

$$
R_N \stackrel{\text{def}}{=} \sum_{k=1}^{N} f_k(x^{(k)}) - \min_{x\in X} \sum_{k=1}^{N} f_k(x) \leq \frac{\beta_N D^2}{2} + \sum_{k=0}^{N-1} \frac{G^2}{2\beta_k}
$$

choose parameters

$$\beta_k = \gamma\sqrt{k}, \quad k \geq 1$$

and let $\beta_0 = \beta_1$, then

$$\sum_{k=0}^{N-1} \frac{G^2}{2\beta_k} \;=\; \frac{G^2}{2\gamma}\left(1 + \sum_{k=1}^{N-1} \frac{1}{\sqrt{k}}\right) \;\leq\; \frac{G^2}{2\gamma}\left(2 + \int_1^N \frac{1}{\sqrt{t}}dt\right) = \frac{G^2\sqrt{N}}{\gamma}$$

finally,

$$R_N \;\leq\; \left(\gamma\frac{D^2}{2} + \frac{G^2}{\gamma}\right)\sqrt{N}$$

upper bound is minimized by choosing

$$\gamma^\star = \sqrt{2}\,\frac{G}{D}$$

which yields

$$R_N \leq \sqrt{2}\,GD\sqrt{N}$$

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# Minimizing finite average of convex functions

problem

$$\text{minimize} \quad F(x) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^{n} f_i(x)$$

stochastic gradient method: pick $i_k \in \{1, \ldots, n\}$ randomly and update

$$x_{k+1} = x_k - \eta_k \nabla f_{i_k}(x_k)$$

**two perspectives:**

- *stochastic optimization:* viewed as trying to minimize $\mathbf{E}_\xi f(x, \xi)$
- *deterministic optimization:* a randomized incremental gradient method for a structured convex problem

# Note the problem structure

stochastic optimization perspective:

- complexity theory: $O(\frac{1}{\epsilon^2})$, or $O(\frac{1}{\epsilon})$ with strong convexity

deterministic optimization perspective:

- sanity check: should at least beat full gradient methods:
  complexity $O(n\frac{L}{\mu}\log\frac{1}{\epsilon})$ or $O(n\sqrt{\frac{L}{\mu}}\log\frac{1}{\epsilon})$
- recent progresse: SAG and SVRG by exploiting finite average structure

# Stochastic average gradient (SAG)

- SAG method (Le Roux, Schmidt, Bach 2012)

$$x_{k+1} = x_k - \frac{\alpha_k}{n} \sum_{i=1}^{n} g_k^{(i)}$$

  where

$$g_k^{(i)} = \begin{cases} \nabla f_i(x_k) & \text{if } i = i_k \\ g_{k-1}^{(i)} & \text{otherwise} \end{cases}$$

- a randomized variant of incremental aggregated gradient (IAG) of Blatt, Hero, & Gauchman (2007)
- complexity (# component gradient evaluations): $O(\max\{n, \frac{L}{\mu}\} \log \frac{1}{\epsilon})$
  cf. full gradient method: $O(n\frac{L}{\mu} \log \frac{1}{\epsilon})$, and stochastic gradient: $O(\frac{1}{\epsilon})$
- need to store most recent gradient of each component, but can be avoided for some structured problems

# Stochastic variance reduced gradient (SVRG)

- SVRG (Johnson & Zhang 2013, Mahdavi, Zhang & Jin 2013)

$$x_{k+1} = x_k - \eta(\nabla f_{i_k}(x_k) - \nabla f_{i_k}(\tilde{x}) + \nabla F(\tilde{x}))$$

  and update $\tilde{x}$ periodically (every few passes)

- still a stochastic gradient method

$$\begin{aligned} & \mathbf{E}[\nabla f_{i_k}(x_k) - \nabla f_{i_k}(\tilde{x}) + \nabla F(\tilde{x})] \\ = \ & \nabla F(x_k) - \nabla F(\tilde{x}) + \nabla F(\tilde{x}) \\ = \ & \nabla F(x_k) \end{aligned}$$

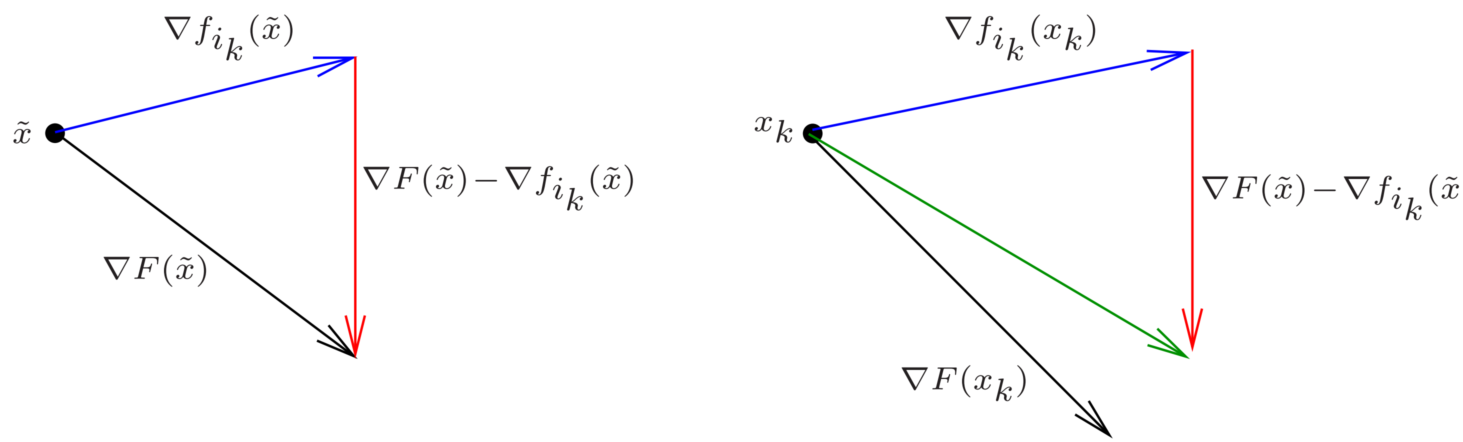
  - expected update direction is the same as $\mathbf{E} f_{i_k}(x_k)$
  - variance can be diminishing if $\tilde{x}$ updated periodically

- complexity: $O\left((n + \frac{L}{\mu}) \log \frac{1}{\epsilon}\right)$, cf. SAG: $O\left(\max\{n, \frac{L}{\mu}\} \log \frac{1}{\epsilon}\right)$

# Stochastic variance reduced gradient (SVRG)

- computational cost per iteration:
  - unlike SAG, no need to store gradients for each component
  - need to compute two gradients at each iteration, and also full gradient periodically
  - for many structured problems, two gradients at each iteration can be reduced to only one

- intuition of variance reduction



# Problem statement and assumptions

$$\underset{x \in \mathbf{R}^d}{\text{minimize}} \quad F(x) = \frac{1}{n}\sum_{i=1}^{n} f_i(x)$$

assumptions:

- each $f_i(x)$, for $i = 1, \ldots, n$, is convex
- each $f_i(x)$ is smooth with Lipschitz constant $L$

$$\|\nabla f_i(x) - \nabla f_i(y)\| \leq L\|x - y\|$$

  (which implies that $\nabla F(x)$ also has Lipschitz constant $L$)

- $F(x)$ strongly convex: for all $x, y \in \mathbf{R}^d$,

$$F(y) \geq F(x) + \nabla F(x)^T(y - x) + \frac{\mu}{2}\|y - x\|^2$$

# SVRG method

**input:** $\tilde{x}_0$, $\eta$, $m$
**iterate:** for $s = 1, 2, \ldots$

$\tilde{x} = \tilde{x}_{s-1}$

$\tilde{v} = \nabla F(\tilde{x})$

$x_0 = \tilde{x}$

**iterate:** for $k = 1, 2, \ldots, m$

pick $i_k \in \{1, \ldots, n\}$ uniformly at random
$x_k = x_{k-1} - \eta\big(\nabla f_{i_k}(x_{k-1}) - \nabla f_{i_k}(\tilde{x}) + \tilde{v}\big)$

**end**

set $\tilde{x}_s = \frac{1}{m}\sum_{k=1}^{m} x_{k-1}$

**end**

# Convergence analysis of SVRG

- **theorem:** suppose $0 < \eta \leq 1/2L$ and $m$ sufficiently large so that

$$\rho = \frac{1}{\mu\eta(1-2L\eta)m} + \frac{2L\eta}{1-2L\eta} < 1$$

then we have geometric convergence in expectation:

$$\mathbf{E}F(\tilde{x}_s) - F(x_\star) \leq \rho^s[F(\tilde{x}_0) - F(x_\star)]$$

- *more concretely,* if $\eta = \theta/L$, then

$$\rho = \frac{L/\mu}{\theta(1-2\theta)m} + \frac{2\theta}{1-2\theta}$$

choosing $\theta = 0.1$ and $m = 50(L/\mu)$ results in $\rho = 1/2$

- overall complexity: $O\left(\left(\frac{L}{\mu} + n\right)\log\left(\frac{1}{\epsilon}\right)\right)$

# Proof

- let $g_k = \nabla f_{i_k}(x_{x-1}) - \nabla f_{i_k}(\tilde{x}) + \nabla F(\tilde{x})$, then

$$x_k = x_{k-1} - \eta g_k, \qquad \text{and} \quad \mathbf{E}_{i_k}[g_k] = \nabla F(x_{k-1})$$

- similar as in classical analysis of stochastic gradient methods

$$\begin{aligned}
\mathbf{E}\|x_k - x_\star\|^2 &= \mathbf{E}\|x_{k-1} - \eta g_k - x_\star\|^2 \\
&= \|x_{k-1} - x_\star\|^2 - 2\eta(x_{k-1} - x_\star)^T \mathbf{E}[g_k] + \eta^2 \mathbf{E}[\|g_k\|^2] \\
&= \|x_{k-1} - x_\star\|^2 - 2\eta(x_{k-1} - x_\star)^T \nabla F(x_{k-1}) + \eta^2 \mathbf{E}[\|g_k\|^2] \\
&\le \|x_{k-1} - x_\star\|^2 - 2\eta(F(x_{k-1}) - F(x_\star)) + \eta^2 \mathbf{E}[\|g_k\|^2]
\end{aligned}$$

then need to bound $\mathbf{E}[\|g_k\|^2]$ carefully using the finite average structure

- by smoothness of $f_i(x)$,

$$\left\|\nabla f_i(x) - \nabla f_i(x_\star)\right\|^2 \le 2L\left[f_i(x) - f_i(x_\star) - \nabla f_i(x_\star)^T(x - x_\star)\right]$$

- summing above inequalities over $i = 1, \ldots, n$ and using $\nabla F(x_\star) = 0$,

$$\frac{1}{n}\sum_{i=1}^{n}\left\|\nabla f_i(x) - \nabla f_i(x_\star)\right\|^2 \le 2L\left[F(x) - F(x_\star)\right]$$

$$\begin{aligned}
\mathbf{E}\|g_k\|^2 &= \mathbf{E}\left\|\nabla f_{i_k}(x_{k-1}) - \nabla f_{i_k}(x_\star) + \nabla f_{i_k}(x_\star) - \nabla f_{i_k}(\tilde{x}) + \nabla F(\tilde{x})\right\|^2 \\
&\le 2\mathbf{E}\left\|\nabla f_{i_k}(x_{k-1}) - \nabla f_{i_k}(x_\star)\right\|^2 + 2\mathbf{E}\left\|\nabla f_{i_k}(\tilde{x}) - \nabla f_{i_k}(x_\star) - \nabla F(\tilde{x})\right\|^2 \\
&= 2\mathbf{E}\left\|\nabla f_{i_k}(x_{k-1}) - \nabla f_{i_k}(x_\star)\right\|^2 \\
&\quad + 2\mathbf{E}\left\|\nabla f_{i_k}(\tilde{x}) - \nabla f_{i_k}(x_\star) - \mathbf{E}[\nabla f_{i_k}(\tilde{x}) - \nabla f_{i_k}(x_\star)]\right\|^2 \\
&\le 2\mathbf{E}\left\|\nabla f_{i_k}(x_{k-1}) - \nabla f_{i_k}(x_\star)\right\|^2 + 2\mathbf{E}\left\|\nabla f_{i_k}(\tilde{x}) - \nabla f_{i_k}(x_\star)\right\|^2 \\
&\le 4L\left[F(x_{k-1}) - F(x_\star) + F(\tilde{x}) - F(x_\star)\right]
\end{aligned}$$

continue derivation on page 13–29

$$\mathbf{E}\|x_k - x_\star\|^2 \le \|x_{k-1} - x_\star\|^2 - 2\eta(1-2L\eta)[F(x_{k-1}) - F(x_\star)] + 4L\eta^2\left[F(\tilde{x}) - F(x_\star)\right]$$

summing over $k = 1, \dots, m$, and take expectation w.r.t. whole history

$$\begin{aligned}
& \mathbf{E}\|x_m - x_\star\|^2 + 2\eta(1-2L\eta)\sum_{k=0}^{m-1}\mathbf{E}[F(x_k) - F(x_\star)] \\
\le\ & \mathbf{E}\|x_0 - x_\star\|^2 + 4Lm\eta^2\mathbf{E}[F(x_0) - F(x_\star)] \\
\le\ & \frac{2}{\mu}\mathbf{E}[F(x_0) - F(x_\star)] + 4Lm\eta^2\mathbf{E}[F(x_0) - F(x_\star)]
\end{aligned}$$

therefore, for each stage $s$

$$\begin{aligned}
\mathbf{E}[F(\tilde{x}_s) - F(x_\star)] &\le \frac{1}{m}\sum_{k=0}^{m-1}\mathbf{E}[F(x_k) - F(x_\star)] \\
&\le \frac{1}{2\eta(1-2L\eta)m}\left(\frac{2}{\mu} + 4Lm\eta^2\right)\mathbf{E}[F(x_0) - F(x_\star)]
\end{aligned}$$

## Numerical experiments

- binary classification: $(a_1, b_1), \ldots, (a_n, b_n)$ with $a_i \in \mathbf{R}^d$, $b_i \in \{+1, -1\}$
- regularized logistic regression

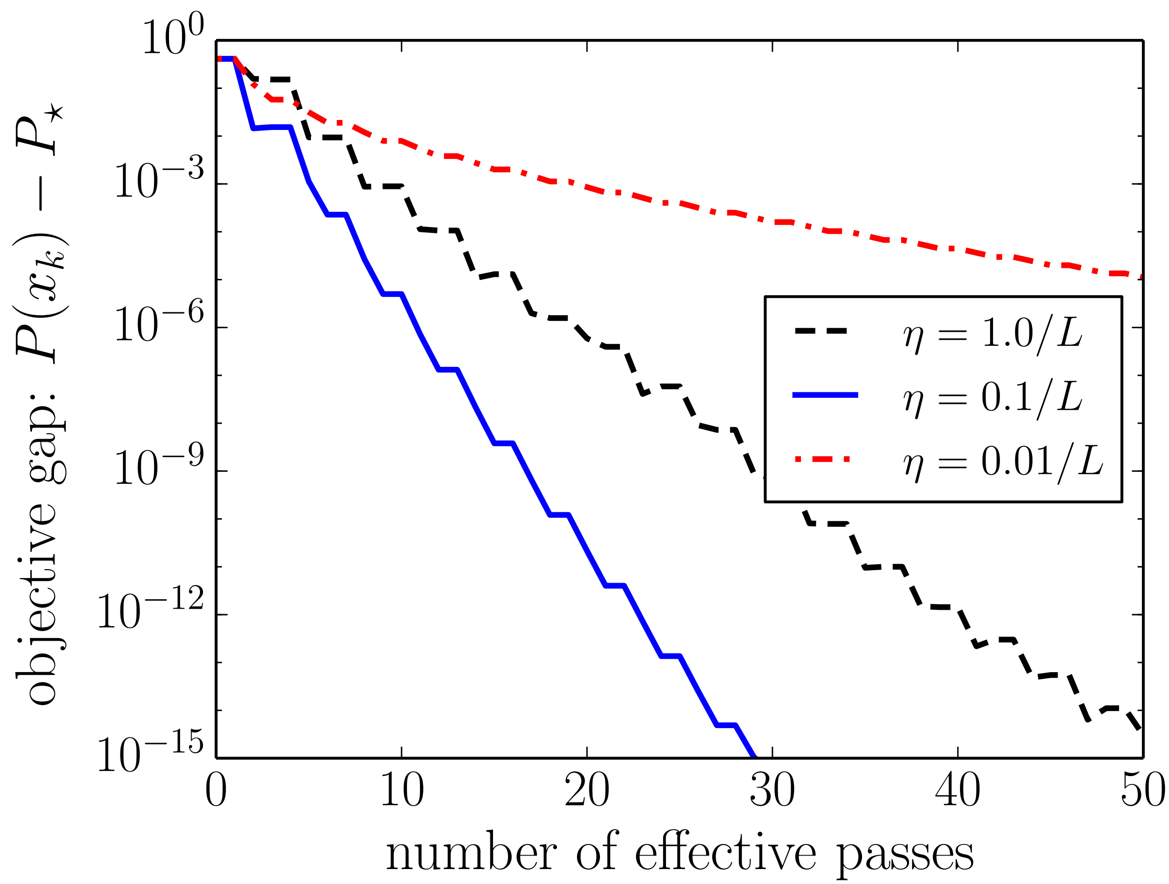
$$\underset{x \in \mathbf{R}^d}{\text{minimize}} \quad \frac{1}{n} \sum_{i=1}^{n} \log\big(1 + \exp(-b_i a_i^T x)\big) + \frac{\lambda_2}{2} \|x\|_2^2 + \lambda_1 \|x\|_1$$

  nonsmooth term $\|x\|_1$ handled by proximal gradient methods

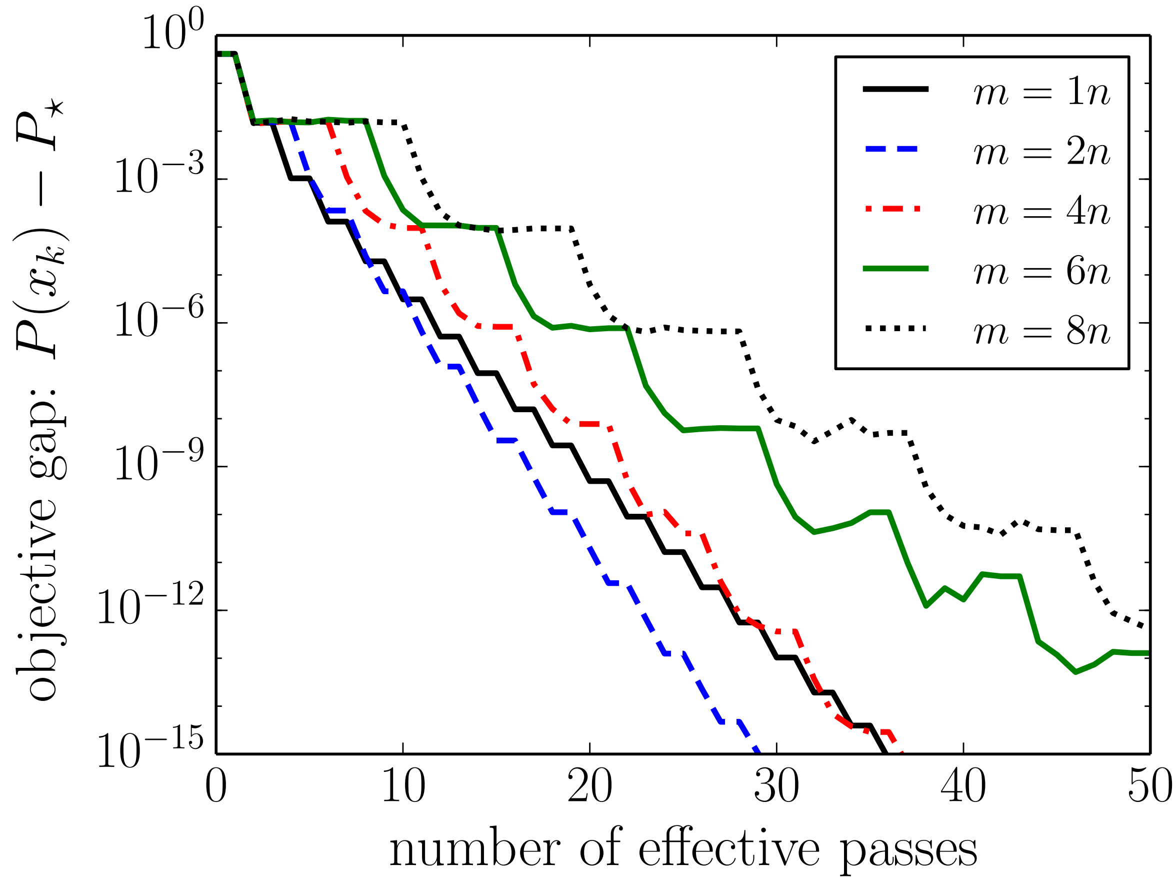
- data sets and characteristics:

| data sets | $n$ | $d$ | $\lambda_2$ | $\lambda_1$ |
|---|---|---|---|---|
| `rcv1` | 20,242 | 47,236 | $10^{-4}$ | $10^{-5}$ |
| `covertype` | 581,012 | 54 | $10^{-5}$ | $10^{-4}$ |
| `sido0` | 12,678 | 4,932 | $10^{-4}$ | $10^{-4}$ |

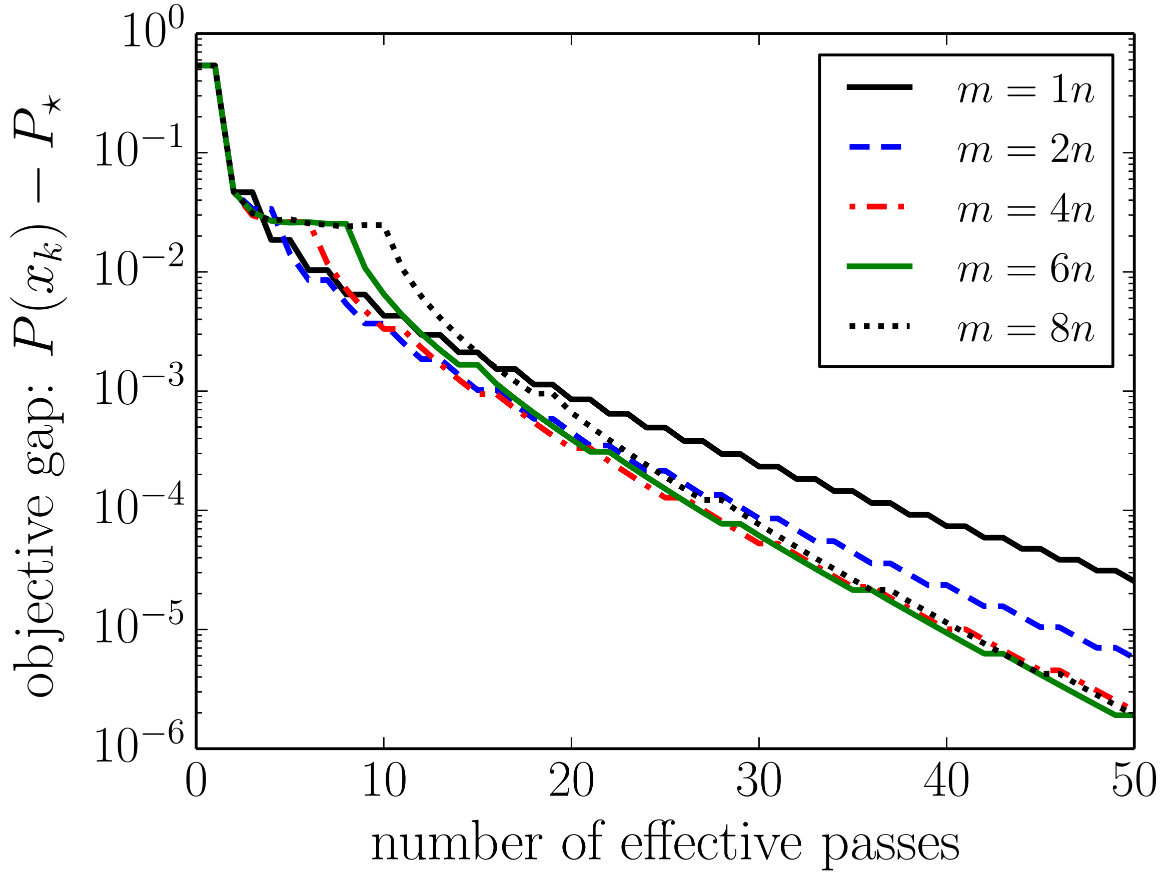
(thanks to Lin Xiao for the experiments)



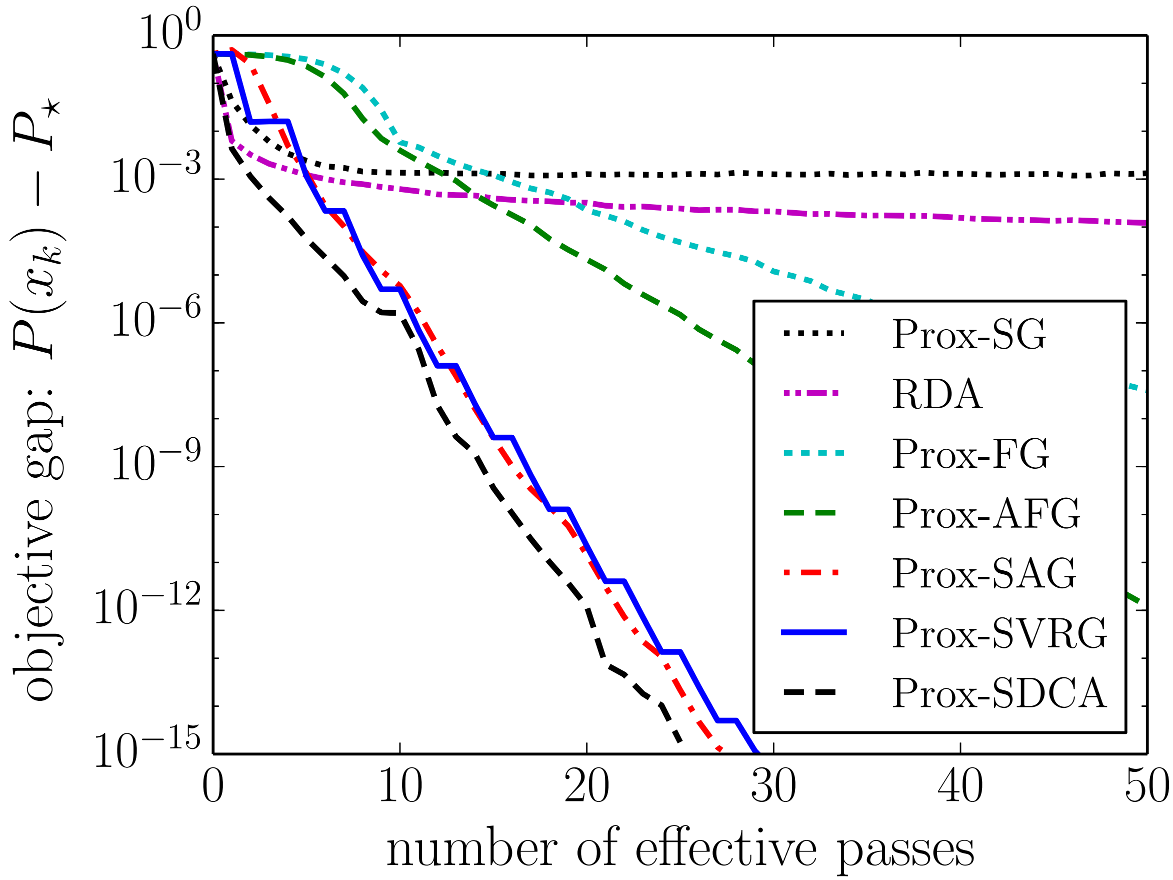

SVRG on `rcv1` dataset: varying step size $\eta$ with $m = 2n$



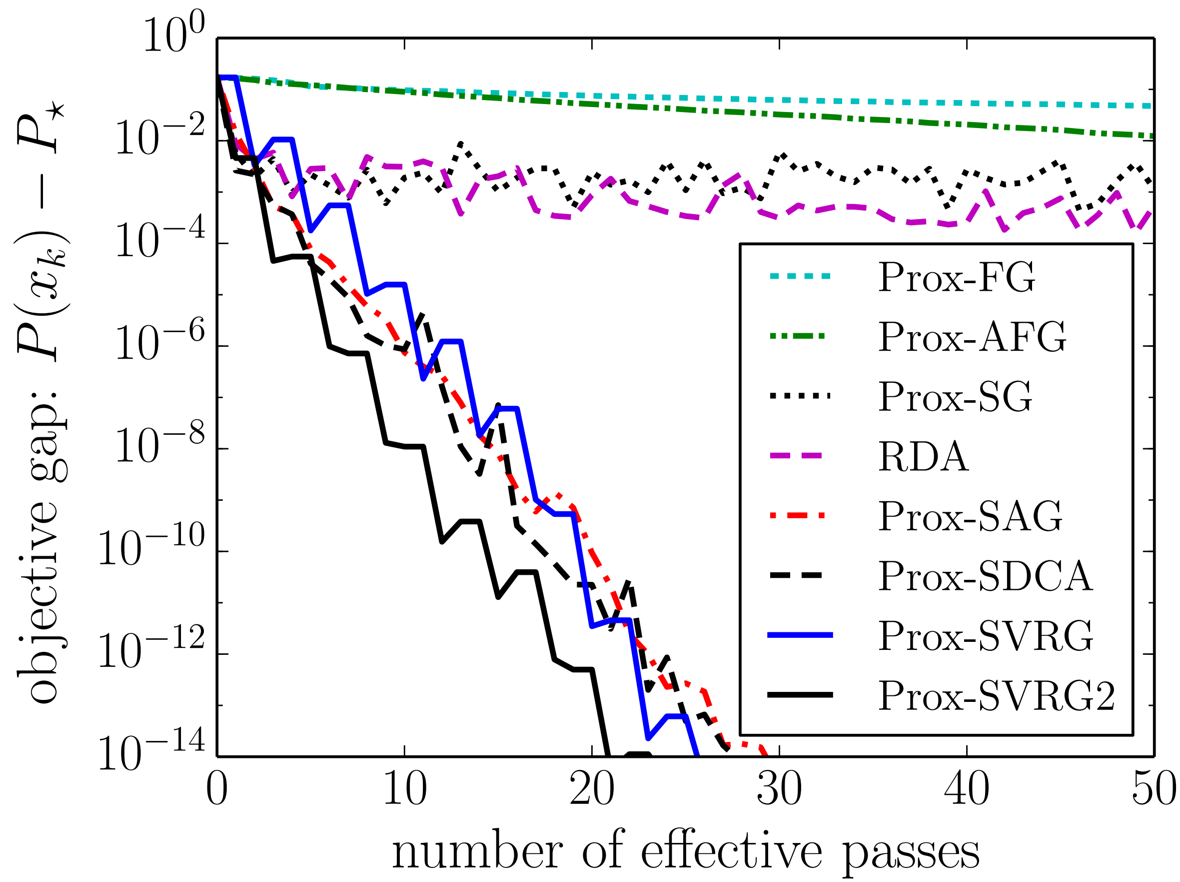

SVRG on `rcv1` dataset with $\lambda_2 = 10^{-4}$ and stepsize $\eta = 0.1/L$:
varying the period $m$ between full gradient evaluations



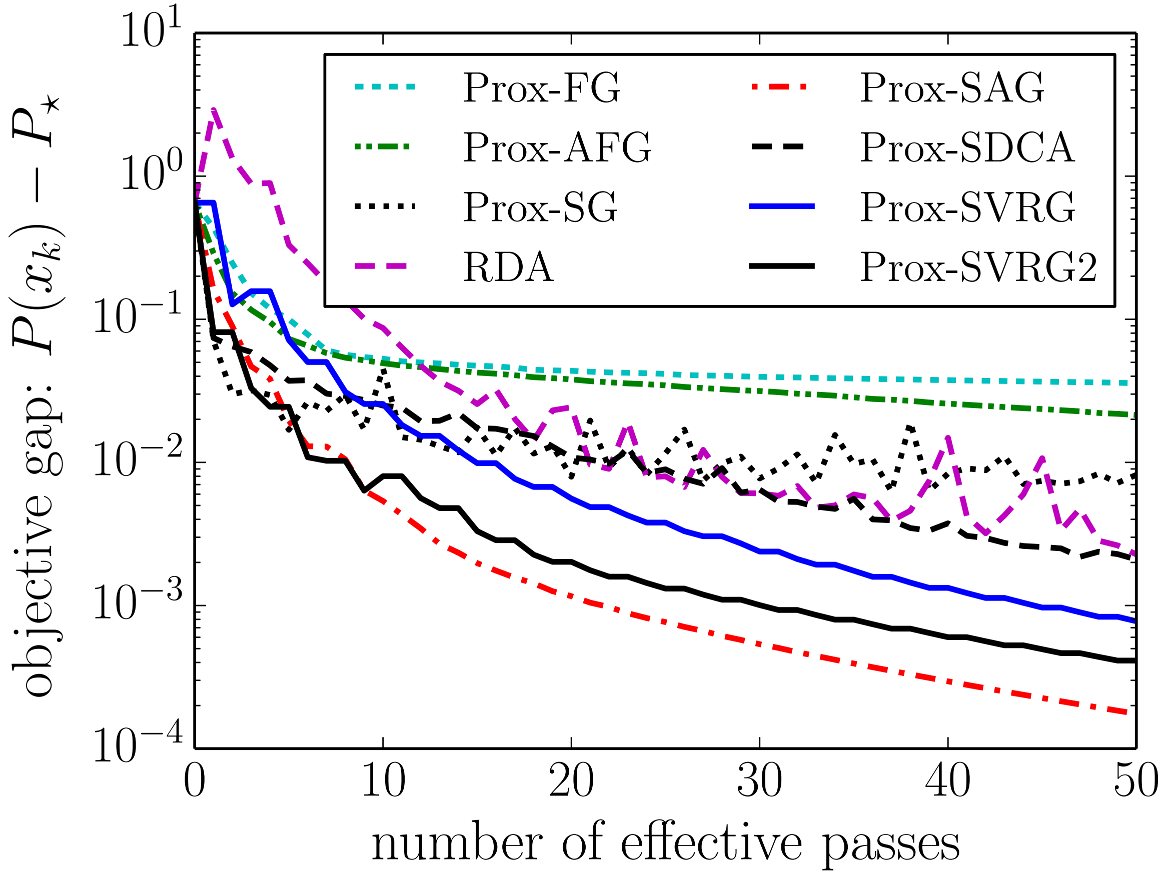

SVRG on `rcv1` dataset with $\lambda_2 = 10^{-5}$ and stepsize $\eta = 0.1/L$:
varying the period $m$ between full gradient evaluations




comparison with related algorithms on `rcv1` datasets




comparison with related algorithms on `covertype` datasets




comparison with related algorithms on `sido0` datasets

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