1. Introduction

- performance of numerical methods
- complexity bounds
- structural convex optimization
- course goals and topics

Some course info

Welcome to EE 546!

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please see webpage for details:
`http://www.ee.washington.edu/class/546/2016spr/`

a few notes:

- pre-requisites: ee 578 or math 516 (if you have not taken these, consent of instructor is strictly needed)
- requirements: homeworks (3), course project (proposal, poster, mid-quarter+final reports)
- Maryam's office hours: Wednesdays 10:30-11:45am; Reza's TBA

General formulation

(mathematical) optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_j(x) \leq 0, \quad j = 1, \ldots, m \\ & x \in S \end{array}$$

- $x = (x_1, \ldots, x_n)$: optimization variables
- $f_0 : \mathbf{R}^n \to \mathbf{R}$: objective function
- $f_j : \mathbf{R}^n \to \mathbf{R}$, $j = 1, \ldots, m$: constraint functions
- S: "structural" constraints (like nonnegativity or boundedness)

optimal solution $x^\star$ satisfies $f_0(x^\star) \leq f_0(x)$ for all *feasible* x

Performance of a numerical method

$$\text{numerical method } \mathcal{M} \quad \Longleftrightarrow \quad \text{problem } \mathcal{P}$$

performance of $\mathcal{M}$ on $\mathcal{P}$: total amount of *computational efforts* required by method $\mathcal{M}$ to *solve* the problem $\mathcal{P}$

- **to solve the problem** could mean
  - find the *exact* solution (impossible for most problems in finite time)
  - find an *approximate* solution with a small accuracy $\epsilon > 0$
- performance of $\mathcal{M}$ with respect to a *single* problem is meaningless
- need to define a model $(\mathcal{F}, \mathcal{O})$ consisting of
  - a *class* of problems $\mathcal{F}$, which have some common properties
  - an *oracle* $\mathcal{O}$, which provides $\mathcal{M}$ some information about $\mathcal{P}$ in $\mathcal{F}$

performance of $\mathcal{M}$ on $(\mathcal{F}, \mathcal{O})$: its performance on the *worst* problem from $\mathcal{F}$ (which may depend on $\mathcal{M}$)

General iterative scheme

input: a starting point $x^{(0)}$ and an accuracy $\epsilon > 0$

initialization: set $k = 0$, $I_{-1} = \emptyset$

- k is iteration count
- I_k is accumulated information set

main loop:

1. call oracle $\mathcal{O}$ at $x^{(k)}$
2. update information set $I_k = I_{k-1} \cup \{x^{(k)}, \mathcal{O}(x^{(k)})\}$
3. apply rules of method $\mathcal{M}$ to I_k and form new point $x^{(k+1)}$
4. check stopping criterion:
   - if yes then form an output $\bar{x}$
   - otherwise set $k = k + 1$ and go to 1

Measuring computational effort

- **analytical complexity:** number of calls of oracle required to solve problem $\mathcal{P}$ upto accuracy ϵ (also called *informational complexity*)
- **arithmetical complexity:** total number of arithmetic operations (including work of oracle and method itself) required to solve problem $\mathcal{P}$ upto accuracy ϵ

relationships

- arithmetical complexity is more useful in practice; usually easily obtained from analytical complexity and complexity of oracle

we will mainly work with *upper/lower bounds* on analytical complexity

Black box oracle

local black box

- only information available for numerical method is answer of oracle
- oracle is *local:* small variation of problem far enough from query point x does not change answer at x

examples of oracle $\mathcal{O}(x)$

- *zero-order oracle:* returns function value $f(x)$
- *first-order oracle:* returns $f(x)$ and gradient $\nabla f(x)$
- *second-order oracle:* returns $f(x)$, $\nabla f(x)$ and Hessian $\nabla^2 f(x)$

Complexity bound for global optimization

problem class $\mathcal{F}$ (formulation and assumptions)

$$\text{minimize}_{x \in B_n} \quad f(x)$$

- $B_n = \{x \in \mathbf{R}^n \mid 0 \leq x_i \leq 1,\ i = 1, \ldots, n\}$
- $f(x)$ Lipschitz continuous on B_n: there exist $L > 0$ such that

$$|f(x) - f(y)| \leq L\|x - y\|_2, \quad \forall\, x, y \in B_n$$

zero-order oracle: $\mathcal{O}(x) = f(x)$

goal: find $\bar{x} \in B_n$ such that $f(\bar{x}) - f^\star \leq \epsilon$

Uniform grid method

method $\mathcal{G}(\epsilon)$

1. let $p = \lfloor \frac{L\sqrt{n}}{2\epsilon} \rfloor + 1$ and form $(p+1)^n$ points

$$x^{(k_1,\ldots,k_n)} = \left(\frac{k_1}{p}, \cdots, \frac{k_n}{p}\right), \quad k_1 = 0, \ldots, p, \quad \ldots, \quad k_n = 0, \ldots, p$$

2. among all points $x^{(k_1,\ldots,k_n)}$, find $\bar{x}$ that has minimal objective value
3. return the pair $(\bar{x}, f(\bar{x}))$ as a result

(can be treated as an iterative process with $(p+1)^n$ iterations)

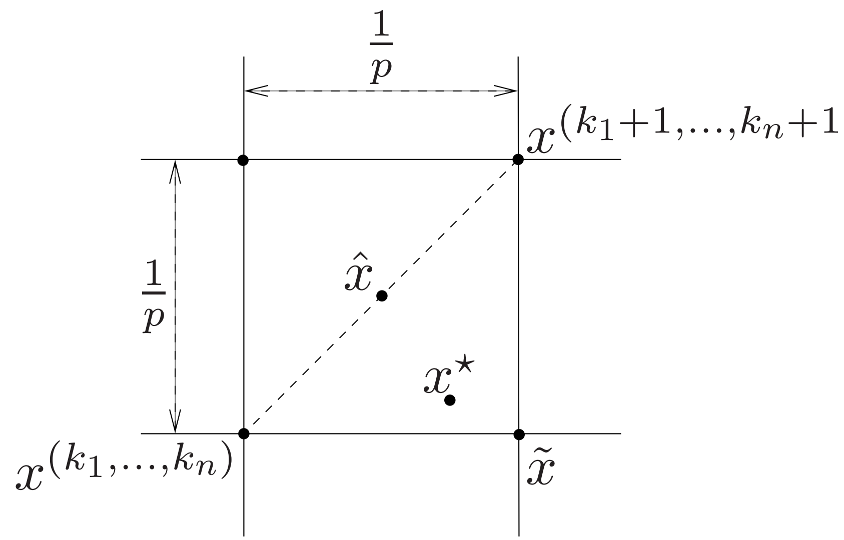
theorem: analytical complexity of $\mathcal{G}$ on model $(\mathcal{F}, \mathcal{O})$ is $\left(\dfrac{L\sqrt{n}}{2\epsilon} + 2\right)^n$

proof: let $x^\star$ be a global solution, then there exist $(k_1, \ldots, k_n)$ such that

$$x^{(k_1,\ldots,k_n)} \leq x^\star \leq x^{(k_1+1,\ldots,k_n+1)} \qquad \text{(element-wise inequality)}$$

let $\hat{x} = \frac{1}{2}(x^{(k_1,\ldots,k_n)} + x^{(k_1+1,\ldots,k_n+1)})$ and

$$\tilde{x} = \begin{cases} x_i^{(k_1+1,\ldots,k_n+1)}, & \text{if } x_i^\star \geq \hat{x}_i \\ x_i^{(k_1,\ldots,k_n)}, & \text{otherwise} \end{cases}$$



then $|\tilde{x}_i - x_i^\star| \leq \frac{1}{2p}$ for all i, therefore

$$\|\tilde{x} - x^\star\|_2^2 = \sum_{i=1}^{n} (\tilde{x}_i - x_i^\star)^2 \leq \frac{n}{4p^2}$$

since $\tilde{x}$ belongs to the grid, and $p = \lfloor \frac{L\sqrt{n}}{2\epsilon} \rfloor + 1 \geq \frac{L\sqrt{n}}{2\epsilon}$, we conclude

$$f(\bar{x}) - f^\star \;\leq\; f(\tilde{x}) - f^\star \;\leq\; L\|\tilde{x} - x^\star\|_2 \;\leq\; \frac{L\sqrt{n}}{2p} \leq \epsilon$$

Lower complexity bound

questions:

- how good is this bound? (maybe our proof is too rough)
- how good is this method? (there may exist much better algorithms)

lower complexity bound

- based on *black box* concept
- valid for all reasonable iterative schemes working with the model $(\mathcal{F}, \mathcal{O})$
- often use the idea of a *resisting oracle*
  - tries to create a *worst* problem for a particular method
  - starts from an "empty" function and tries to answer each call in worst possible way
  - however, must be compatible with previous answers and $\mathcal{F}$ (after termination, it is possible to *reconstruct* the problem)

Lower bound for global optimization

problem class $\mathcal{F}$ (formulation and assumptions)

$$\text{minimize}_{x \in B_n} \quad f(x)$$

- $B_n = \{x \in \mathbf{R}^n \mid 0 \le x_i \le 1,\ i = 1, \ldots, n\}$
- $f(x)$ Lipschitz continuous on B_n: there exist $L > 0$ such that

$$|f(x) - f(y)| \le L\|x - y\|_2, \quad \forall\, x, y \in B_n$$

zero-order oracle: $\mathcal{O}(x) = f(x)$

theorem: analytical complexity of this model $(\mathcal{F}, \mathcal{O})$ is at least $\left(\left\lfloor \dfrac{L}{2\epsilon} \right\rfloor\right)^n$

Proof of lower bound

define resisting oracle

$$\mathcal{O}(x) \text{ returns } f(x) = 0 \text{ at any test point } x$$

therefore *any* method can only return $\bar{x}$ with $f(\bar{x}) = 0$

construct worst function

- let $p = \lfloor \frac{L}{2\epsilon} \rfloor \geq 1$, then for any method that takes less than p^n calls, there exist $x^\star \in B_n$ such that there is no test point in the box

$$B = \left\{ x \;\middle|\; \|x - x^\star\|_\infty \leq \frac{1}{2p} \right\}$$

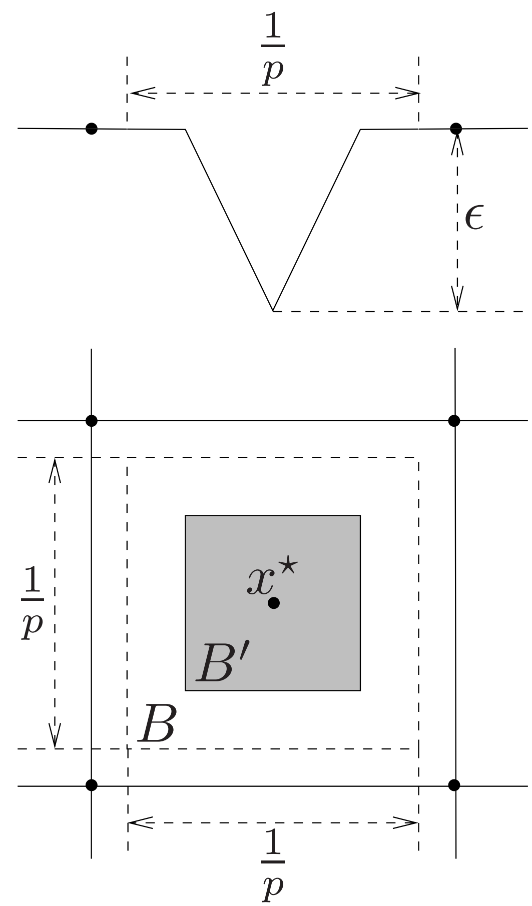
- consider the function

$$\bar{f}(x) = \min\{0,\ L\|x - x^\star\|_\infty - \epsilon\}$$

optimal value: $\min\limits_{x \in B_n} \bar{f}(x) = \bar{f}(x^\star) = -\epsilon$

check compatibility

$$\bar{f}(x) = \min\{0,\ L\|x - x^\star\|_\infty - \epsilon\}$$



- $\bar{f}(x)$ is Lipschitz continuous with parameter L

$$|\bar{f}(x) - \bar{f}(y)| \leq L\|x - y\|_\infty \leq L\|x - y\|_2$$

- function $\bar{f}(x)$ is non-zero only inside the box

$$B' = \{x \mid \|x - x^\star\|_\infty \leq \epsilon/L\}$$

- since $p = \lfloor \frac{L}{2\epsilon} \rfloor \leq \frac{L}{2\epsilon}$, we conclude that

$$B' \subseteq B$$

therefore $\bar{f}(x)$ equals zero at all test points

conclusion: accuracy no less than ϵ if number of oracle calls less than p^n

Complexity of global optimization

	uniform grid	lower bound
complexity	$\left(\dfrac{L\sqrt{n}}{2\epsilon}\right)^n$	$\left(\dfrac{L}{2\epsilon}\right)^n$

- dependence on ϵ is *optimal*
- dependence on n is *not optimal*

the conclusion depends on the problem class $\mathcal{F}$: if we assume

$$|f(x) - f(y)| \leq L\|x - y\|_\infty, \quad \forall\, x, y \in B_n$$

then uniform grid method has complexity $\left(\dfrac{L}{2\epsilon}\right)^n$, and it is *optimal*

question: will higher-order oracles help improve complexity results?

Common classes and features

- **global optimization**
  - *goal:* find a global minimum
  - *problem class:* continuous functions
  - *oracle:* 0-1-2 order black box
  - *features:* no guarantee
- **nonlinear optimization**
  - *goal:* find a local minimum (not always acceptable)
  - *problem class:* differentiable functions
  - *oracle:* 1-2 order black box
  - *features:* variety of approaches, widespread software
- **convex optimization**
  - *goal:* find a global minimum
  - *problem class:* convex sets, convex functions (sometimes restrictive)
  - *oracle:* 1-2 order black box, and beyond
  - *features:* efficient practical methods, complete complexity theory

Complexities for convex optimization

minimize$_{x \in Q}$ $\quad f(x), \qquad$ where $Q \subseteq \mathbf{R}^n$ is bounded, closed and convex

problem class	lower bound	optimal methods?
nonsmooth	$O\left(1/\epsilon^2\right)$	yes
smooth	$O\left(1/\sqrt{\epsilon}\right)$	yes
smooth and strongly convex	$O\left(\log(1/\epsilon)\right)$	yes

- based on **local black-box** first-order oracle
- independent of dimension (good for high-dimensional problems)

big O notation: $a(\epsilon) = O(b(\epsilon))$ means there exists $M > 0$ such that $a(\epsilon) \leq Mb(\epsilon)$ for all ϵ sufficiently small

A conceptual contradiction

convexity is a global structure

- usually checked by inspection: e.g., composition of basic convex functions
- numerical verification of convexity is extremely difficult

but numerical methods use local black-box

beyond block box: structural convex optimization

- exploiting structure to improve performance of numerical methods
- recent developments:
  - interior-point methods (2nd-order oracle)
  - smoothing
  - minimization of composite objective

Minimization of composite objective

problem class:

$$\text{minimize}_{x \in \mathbf{R}^n} \quad \left\{ \phi(x) \triangleq f(x) + \Psi(x) \right\}$$

- f is convex and smooth (having Lipschitz-continuous gradient)
- Ψ is convex, but may be nondifferentiable
- using black-box first-order oracle, complexity is $O(1/\epsilon^2)$

structural convex optimization

- assume Ψ is *simple*, e.g., can solve explicitly the auxiliary problem

$$\text{minimize}_{x \in \mathbf{dom}\,\Psi} \left\{ f(y) + \langle \nabla f(y), x - y \rangle + \frac{L}{2}\|x - y\|^2 + \Psi(x) \right\}$$

- accelerated gradient methods achieve reduced complexity $O(1/\sqrt{\epsilon})$

Example: sparse least-squares

$$\text{minimize}_{x \in \mathbf{R}^n} \quad \frac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1 \qquad (\text{where } A \in \mathbf{R}^{m\times n})$$

- important applications in signal processing, statistics, machine learning
- focus on problem class: $m < n$ and $x^\star$ sparse (compressed sensing)

complexities of structural convex optimization

numerical method	analytical complexity	oracle complexity
subgradient method	$O(1/\epsilon^2)$	$O(mn)$
proximal gradient method	$O(1/\epsilon)$	$O(mn)$
accelerated gradient method	$O(1/\sqrt{\epsilon})$	$O(mn)$
interior-point method	$O(\log(1/\epsilon))$	$O(m^2n)$
prox gradient homotopy (under RIP)	$O(\log(1/\epsilon))$	$O(mn)$

Applications of smoothing

- piecewise-linear approximation

$$\text{minimize}_{x \in \mathbf{R}^n} \quad \max_{i=1,\ldots,m} (a_i^T x + b_i)$$

- one-norm approximation

$$\text{minimize}_x \quad \|x\|_1 \quad \text{subject to} \quad ||Ax - b\|_2 \leq \delta$$

- group regularization:

$$\text{minimize}_{x \in \mathbf{R}^n} \quad \frac{1}{2}\|Ax - b\|_2^2 + \lambda \|x\|_1 + \rho \sum_{g \in \mathcal{G}} w_g \|x_g\|_2$$

Example: low-rank matrix recovery

find a low-rank matrix given noisy linear constraints

$$\text{minimize}_{X \in \mathbf{R}^{m \times n}} \quad \frac{1}{2}\|\mathcal{A}(X) - b\|_2^2 + \lambda \|X\|_*$$

where $\mathcal{A} : \mathbf{R}^{m \times n} \to \mathbf{R}^p$ is a linear map, $b \in \mathbf{R}^p$. $\|X\|_* = \sum_i \sigma_i(X)$ is the nuclear norm or trace norm, sum of singular values

- **special case:** when $X = \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{bmatrix}$,

  $\mathbf{rank}\, X = \#$ of nonzero x_i
  reduces to sparse least squares, $\|X\|_*$ reduces to ℓ_1 norm

- many applications in machine learning, controls, signal processing, e.g., matrix completion problem (recommender systems, e.g. Netflix)

- more later. . .

Course goals and course work

- optimization algorithms along with their complexity analysis
- experience with implementations and applications
- methodologies of structural convex optimization
- exposure to research frontiers in convex optimization and applications

course work

- lectures focus on algorithms and complexity analysis
- 3 homeworks (lag implementation & theory)
- substantial project

Syllabus

tentative:

- **smooth optimization:** gradient method, quasi-Newton methods, Nesterov's optimal methods
- **nonsmooth optimization:** subgradient calculus, subgradient methods
- **accelerated gradient methods:** proximal mapping, accelerated proximal gradient methods, smoothing
- **decomposition and coordinate descent:** dual decomposition, alternating direction multiplier method, randomized coordinate descent
- **stochastic and online optimization:** convergence and regret analysis, applications in large-scale machine learning
- **interior-point methods:** self-concordant barriers, path-following methods, efficient implementations

On the role of complexity analysis

complexity analysis plays an important role in convex optimization

- many ideas appeared early, but did not result in significant impact due to lack of convincing complexity analysis

modern algorithms	early prototypes
accelerated gradient methods	heavy ball method
polynomial-time IPMs	classical barrier methods
smoothing	smoothing

- quote from Yurii Nesterov (in 2004 book)

  ... more and more common that the new methods were provided with a complexity analysis, which is considered a better justification of their efficiency than computational experiments ...

References

- Yu. Nesterov, *Introductory Lectures on Convex Optimization. A Basic Course* (2004), Section 1.1.

  (The global optimization example with Lipschtiz continuous function in the Euclidean norm is from Nesterov's lecture notes for INMA2460: Nonlinear Optimization, Catholic University of Louvain)

- Yu. Nesterov, *How to advance in Structural Convex Optimization* (November 2008), OPTIMA 78, Mathematical Programming Society Newsletter, pages 2-5.