Chapter 1

Introduction

1.1 Review of Ordinary Differential Equations

This course is mainly about partial differential equations (PDEs). Previously you have studied ordinary differential equations (ODEs). We will review two common types of ordinary differential equations here. If you have no difficulty with these, you have no problem with the prerequisites for this course.

1.1.1 First and Second Order Equations

Example. *Population growth:*

$$\boxed{\frac{dN}{dt} = rN}\,, \tag{1.1}$$

where $N(t)$ is the population density of a species at time t. The above equation is simply a statement that the rate of population growth, $\frac{dN}{dt}$, is proportional to the population itself, with the proportionality constant r. To solve it, we move all the N's to one side and all the t's to the other side of the equation. (This process is called "separation of variables" in the ODE literature. We will not use this term here as it may get confused with a PDE solution method with the same name which we will discuss later.) Thus:

$$\frac{dN}{N} = rdt. \tag{1.2}$$

Integrating both sides yields:

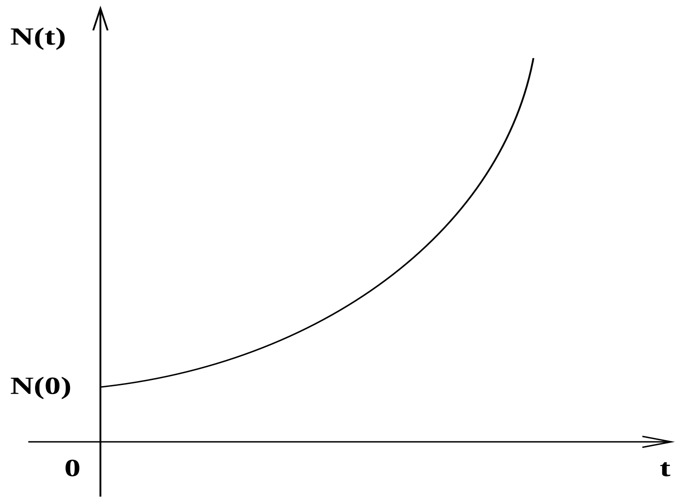

Figure 1.1: Solution to Equation 1.1

$$\ell n N = rt + \text{ constant},$$

which can be rewritten as

$$N(t) = \text{ constant } \cdot e^{rt}.$$

In order for the left-hand side to equal to the right-hand side at $t = 0$, the "constant" in the second equation must be $N(0)$. Thus,

$$\boxed{N(t) = N(0)e^{rt}}\,. \tag{1.3}$$

Population grows exponentially from an initial value $N(0)$, with an e-folding time of r^{-1}. That is, $N(t)$ increases by a factor of e with every increment of r^{-1} in t. The solution is plotted in Figure 1.1.

Equation (1.1) is perhaps an unrealistic model for most population growths. Among other things, its solution implies that the population will grow indefinitely. A better model is given by the following equation:

$$\frac{dN}{dt} = rN \cdot (1 - N/k). \tag{1.4}$$

Try solving it using the same method. The solution is plotted in Figure 1.2 for $0 < N(0) < k$.

Example. *Harmonic Oscillator Equation:*

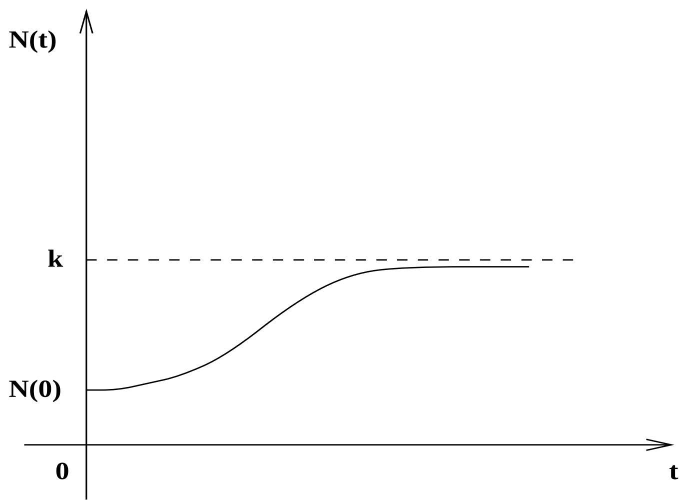

Figure 1.2: Solution to Equation 1.4

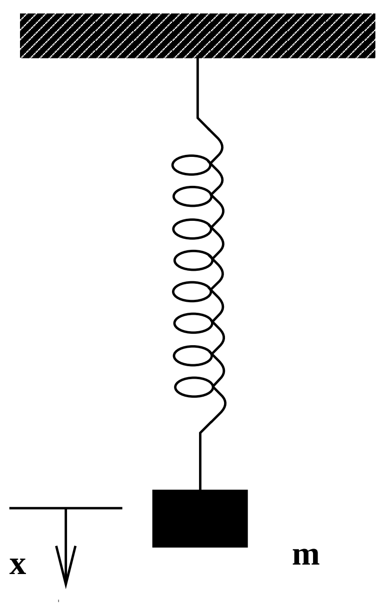

Figure 1.3: A spring of mass m suspended under gravity and in equilibrium. x is the displacement from the equilibrium position.

$$\boxed{m\frac{d^2x}{dt^2} + kx = 0}\,. \tag{1.5}$$

Here x is the vertical displacement from the equilibrium position of the spring, which has a mass m and a spring constant k. Equation (1.5) is a statement of Newton's Law of Motion: $m\frac{d^2x}{dt^2}$ is mass times acceleration. This is required to be equal to the spring's restoring force $-kx$. This force is assumed to be proportional to the displacement from equilibrium.

It is harder to solve a second order ODE, although this particular equation is so common that we were taught to try exponential solutions wherever we see linear equations with constant coefficients.

Solution by trial method: Try $e^{\alpha t}$.

Plugging the guess in Equation (1.5) for x then suggests that α must satisfy

$$m\alpha^2 + k = 0,$$

which means

$$\alpha = i\sqrt{k/m}, \quad \text{or } \alpha = -i\sqrt{k/m},$$

where $i \equiv \sqrt{-1}$ is the imaginary number. There are two different values of α which will make our trial solution satisfy equation (1.5). So the general solution should be a linear combination of the two:

$$\boxed{x(t) = c_1 e^{i\sqrt{k/m}t} + c_2 e^{-i\sqrt{k/m}t}}\,, \tag{1.6}$$

where c_1 and c_2 are two arbitrary (complex) constants.

If you do not like using complex notations (numbers that involve i), you can rewrite (1.6) in real notation, making use of the Euler's Identity, which we will derive a little later:

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{1.7}$$

Thus the rest solution (1.6) can be rewritten as

$$\boxed{x(t) = A\sin(\sqrt{k/m}t) + B\cos(\sqrt{k/m}t)}\,, \tag{1.8}$$

where A and B are some arbitrary real constants (since c_1 and c_2 were undetermined).

We can verify that (1.8) is indeed the solution to the harmonic oscillator equation (1.5) by noting, from calculus:

$$\frac{d}{dt}\sin\omega t = \omega\cos\omega t, \quad \frac{d}{dt}\cos\omega t = -\omega\sin\omega t$$

so

$$\frac{d^2}{dt^2}\sin\omega t = \frac{d}{dt}(\omega\cos\omega t) = -\omega^2\sin\omega t$$

and

$$\frac{d^2}{dt^2}\cos\omega t = \frac{d}{dt}(-\omega\sin\omega t) = -\omega^2\cos\omega t.$$

Therefore, the sum (1.8) satisfies

$$\frac{d^2}{dt^2}x = -\omega^2 x, \tag{1.9}$$

which is the same as (1.5), provided that $\omega^2 = k/m$.

Euler's Identity:

Euler's Identity, as used in (1.7), deserves some comment. We will also need this identity later when we deal with Fourier series and transforms.

In calculus, we learned how to differentiate an exponential

$$\frac{d}{d\theta}e^{a\theta} = ae^{a\theta}.$$

Although you have always assumed a to be a real number, it does not make any difference if a is complex. So letting $a = i$, we find

$$\frac{d}{d\theta}e^{i\theta} = ie^{i\theta}$$

$$\frac{d^2}{d\theta^2}e^{i\theta} = \frac{d}{d\theta}(ie^{i\theta}) = i^2e^{i\theta} = -e^{i\theta}.$$

We have thus shown that the function

$$y(\theta) = e^{i\theta} \tag{1.10}$$

satisfies the second-order ODE:

$$\frac{d^2}{d\theta^2}y + y = 0. \tag{1.11}$$

$e^{i\theta}$ also happens to satisfy the *initial conditions*:

$$y(0) = 1, \quad \frac{d}{d\theta}y(0) = i. \tag{1.12}$$

On the other hand, we have just verified in (1.9) that

$$y = A\sin\theta + B\cos\theta \tag{1.13}$$

also satisfies (1.11), which is the same as (1.9) if we replace t by θ and ω by 1. If we furthermore require the sum (1.13) to also satisfy the initial condition (1.12), we will find that $B = 1$ and $A = i$. Since,

$$y(\theta) = \cos\theta + i\sin\theta \tag{1.14}$$

satisfies the same ODE (1.11) and the same initial conditions (1.12) as (1.10), (1.14) and (1.10) must be the same by the Uniqueness Theorem for ODEs. What we have outlined is one way for proving the Euler Identity (1.7):

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}\ .$$

Solution from First Principles:

If you prefer to find the solution from first principles, i.e. not by guessing that it should be in the form of an exponential, it can be done by "reduction of order", although we normally do not bother to do it this way:

We recognize that Equation (1.5) is a type of ODE with its "independent variable missing". The method of reduction of order suggests that we let

$$p \equiv \frac{dx}{dt},$$

and write

$$\frac{d^2x}{dt^2} = \frac{dp}{dt} = \frac{dx}{dt}\frac{dp}{dx} = p\frac{dp}{dx}.$$

Treating p now as a function of x, Equation (1.5) becomes

$$p\frac{dp}{dx} + \omega^2 x = 0, \tag{1.15}$$

where we have used ω^2 for k/m.

Equation (1.15), a first order ODE, can be solved by the same method we used in Example 1:

Integrating $pdp + \omega^2 xdx = 0$ yields,

$$p^2 + \omega^2 x^2 = \omega^2 a^2.$$

with $\omega^2 a^2$ being an arbitrary constant of integration. From $p = \pm\omega\sqrt{a^2 - x^2}$, we have, since $p = \frac{d}{dt}x$,

$$\frac{dx}{dt} = \pm\alpha\sqrt{a^2 - x^2}.$$

This is again a first order ODE, which we solve as before.

Integrating both sides of

$$\frac{dx}{\sqrt{a^2 - x^2}} = \pm\omega dt$$

and using the integral formula:

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + b, \quad b \text{ being a constant,}$$

we find

$$\sin^{-1}\frac{x}{a} = \pm\omega t + b.$$

Inverting, we find:

$$\frac{x}{a} = \sin(\pm\omega t + b),$$

which can (finally!) be rewritten as

$$x(t) = A\sin\omega t + B\cos\omega t \ . \tag{1.16}$$

1.2 Nonhomogeneous Ordinary Differential Equations

1.2.1 First-Order Equations:

A nonhomogeneous version of the example in (1.1) is

$$\boxed{\frac{dN}{dt} - rN = f(t),} \tag{1.17}$$

where $f(t)$ is a (known) specified function of t, independent of the "unknown" N. It is called the "inhomogeneous term" or the "forcing term". In the population growth example we discussed earlier, $f(t)$ can represent the rate of population growth of the species due to migration.

We are here concerned with the method of solution of (1.17) for any given $f(t)$. We proceed to multiply both sides of (1.17) by a yet-to-be-determined function $\mu(t)$, called the *integrating factor*:

$$\mu\frac{dN}{dt} - r\mu N = \mu f. \tag{1.18}$$

We choose $\mu(t)$ such that the product on the left-hand side of (1.18) is a perfect derivative, i.e.

$$\mu\frac{dN}{dt} - r\mu N = \frac{d}{dt}(\mu N). \tag{1.19}$$

If this can be done, then (1.10) would become:

$$\frac{d}{dt}(\mu N) = \mu f,$$

which can be integrated from $t = 0$ to t to yield:

$$\mu(t)N(t) - \mu(0)N(0) = \int_0^t \mu(t)f(t)dt. \tag{1.20}$$

The notation on the right-hand side of (1.20) is rather confusing. A better way is to use a different symbol, say, τ, in place of t as the dummy variable of integration. Then, (1.20) can be rewritten as

$$N(t) = N(0)\mu(0)\mu^{-1}(t) + \mu^{-1}(t)\int_0^t \mu(\tau)f(\tau)d\tau. \tag{1.21}$$

The remaining task is to find the integrating factor $\mu(t)$. In order for (1.19) to hold, we must have the right-hand side

$$\mu\frac{dN}{dt} + N\frac{d\mu}{dt}$$

equal the left-hand side, implying

$$\frac{d\mu}{dt} = -r\mu. \tag{1.22}$$

The solution to (1.22) is simply

$$\mu(t) = \mu(0)e^{-rt}. \tag{1.23}$$

Substituting (1.23) back into (1.21) then yields

$$\boxed{N(t) = N(0)e^{rt} + e^{rt}\int_0^t e^{-r\tau} f(\tau)d\tau}. \tag{1.24}$$

This then completes the solution of (1.17). If r is not a constant, but is a function t, the procedure remains the same up to, and including (1.22). The solution to (1.22) should now be

$$\mu(t) = \mu(0)e^{-\int_0^t r(t')dt'}. \tag{1.25}$$

The final solution is obtained by substituting (1.25) into (1.21).

Notice that the solution to the (linear) nonhomogeneous equation consists of two parts: a part satisfying the general homogeneous equation and a part that is a particular solution of the nonhomogeneous equation (the first and second terms on the right-hand side of (1.24) respectively). For some simple forcing functions $f(t)$, there is no need to use the general procedure of integrating factors if we can somehow guess a particular solution. For example, suppose we want to solve

$$\boxed{\frac{dN}{dt} - rN = 1}. \tag{1.26}$$

We write

$$N(t) = N_h(t) + N_p(t),$$

where $N_h(t)$ satisfies the homogeneous equation

$$\frac{dN_h}{dt} - rN_h = 0$$

and so is

$$N_h(t) = ke^{rt},$$

for some constant k. $N_p(t)$ is any solution of the Eq. (1.26). By the "method of judicious guessing", we try selecting a constant for $N_p(t)$. Upon substituting $N_p(t) = a$ into (1.26), we find the only possibility: $a = -\frac{1}{r}$. Thus the full solution is

$$N(t) = ke^{rt} - \frac{1}{r} = N(0)e^{rt} + \frac{1}{r}(e^{rt} - 1).$$

1.2.2 Second-Order Equations:

For our purpose here we will be using only the "method of judicious guessing" for linear second-order equations. The more general method of "variation of parameters" is too cumbersome for our limited purposes.

Example: Solve

$$\frac{d^2}{dt^2}x + \omega^2 x = 1. \tag{1.27}$$

We write the solution as a sum of a homogeneous solution x_h and a particular solution x_p, i.e.

$$x(t) = x_h(t) + x_p(t).$$

We already know that the homogeneous solution (to (1.5)) is of the form

$$x_h(t) = A\sin\omega t + B\cos\omega t. \tag{1.28}$$

We guess that a particular solution to (1.27) is a constant

$$x_p(t) = a. \tag{1.29}$$

Subsituting (1.29) into (1.27) then shows that constant is $1/\omega^2$. Thus the general solution to (1.27) is

$$\boxed{x(t) = A\sin\omega t + B\cos\omega t + 1/\omega^2}\,. \tag{1.30}$$

The arbitrary constants A and B are to be determined by initial conditions.

Example: Solve

$$\boxed{\frac{d^2}{dt^2}x + \omega^2 x = \sin\omega_0 t}\,. \tag{1.31}$$

Again we write the solution as a sum of the homogeneous solution and a particular solution. For the particular solution, we try:

$$x_p(t) = a\sin\omega_0 t. \tag{1.32}$$

Upon substitution of (1.32) into (1.31), we find

$$a = (\omega^2 - \omega_0^2)^{-1},$$

and so the general solution is

$$\boxed{x(t) = A \sin \omega t + B \cos \omega t + \frac{\sin \omega_0 t}{(\omega^2 - \omega_0^2)}.} \tag{1.33}$$

The solution in (1.33) is valid as is for $\omega_0 \neq \omega$. Some special treatment is helpful when the forcing frequency ω_0 approaches the natural frequency ω. Let us write

$$\omega_0 = \omega + \epsilon$$

and let $\epsilon \to 0$. The particular solution can be written as

$$\begin{aligned} x_p(t) &= \frac{\sin \omega_0 t}{(\omega^2 - \omega_0^2)} = \frac{\sin(\omega t + \epsilon t)}{\omega^2 - (\omega + \epsilon)^2} \\ &= \frac{\sin \omega t \cos \epsilon t + \cos \omega t \sin \epsilon t}{-2\omega\epsilon - \epsilon^2} \to -\frac{\cos \omega t}{2\omega} \cdot t - \frac{\sin \omega t}{2\omega\epsilon} \text{ as } \epsilon \to 0 \end{aligned}$$

Thus for the case of *resonance*, $\omega_0 = \omega$, the solution (1.33) becomes

$$x(t) = A' \sin \omega t + B \cos \omega t - \frac{1}{2\omega} t \cos \omega t,$$

where we have written $A = A' + \frac{1}{2\omega\epsilon}$, with A' being a (finite) arbitrary constant. The solution grows secularly in t.

1.3 Summary of ODE solutions

In this course we will be mostly dealing with linear differential equations with constant coefficients. For these, simply try an exponential solution. This is the easiest way. You are not expected to have to repeat each time the derivation given in the previous sections on why the exponentials are the right solutions to try. Just do the following:

(a) $\boxed{\frac{d}{dt} N = rN}$

Try $N(t) = ae^{\alpha t}$ and find $\alpha = r$ so the solution is

$$\boxed{N(t) = ae^{rt}}.$$

(b) $\boxed{\frac{d^2}{dt^2} x + \omega^2 x = 0}$

Try $x(t) = ae^{\alpha t}$ and find $\alpha = \pm i\omega$ so the complex solution is

$$\boxed{x(t) = a_1 e^{i\omega t} + a_2 e^{-i\omega t}},$$

and the real solution

$$x = A\cos\omega t + B\sin\omega t$$

1.4 Partial Derivatives

The ordinary differential equations we discussed in the last section describe functions of only one independent variable. For example, the unknown N in (1.1) is a function of t only, and Eq. (1.1) describes the rate of change of $N(t)$ with respect to t. It is not hard to imagine a physical situation where the population N depends not only on time t, but also on space x (more realistically on all three spatial dimensions, x, y, z). For a function of more than one independent variables, for example,

$$N = N(x, t),$$

we need to distinguish the derivative with respect to t from the derivative with respect to x. For this purpose, we define the *partial derivatives* in the following way.

The partial derivative of $N(x, t)$ with respect to t, denoted by $\frac{\partial}{\partial t}N(x, t)$, or $N_t(x, t)$ for short, is defined as the derivative of N with respect to t *holding all other independent variables—in this case x—constant*:

$$\boxed{\frac{\partial}{\partial t}N(x,t) = \lim_{\substack{\Delta t \to 0 \\ x \text{ held constant}}} \frac{N(x, t+\Delta t) - N(x,t)}{\Delta t}}. \tag{1.34}$$

Similarly, the partial derivative of $N(x, t)$ with respect to x, denoted by $\frac{\partial}{\partial x}N(x, t)$, or $N_x(x, t)$ for short, is defined as:

$$\boxed{\frac{\partial}{\partial x}N(x,t) = \lim_{\substack{\Delta x \to 0 \\ t \text{ held constant}}} \frac{N(x+\Delta x, t) - N(x,t)}{\Delta x}.} \tag{1.35}$$

Compare the partial derivatives with the ordinary derivatives:

$$\frac{d}{dt}N(t) = \lim_{\Delta t \to 0} \frac{N(t+\Delta t) - N(t)}{\Delta t},$$

and you will see that the partial derivative is the same as the ordinary derivative if you can just pretend that the other independent variables were constants.

Example: The (first) partial derivatives of

$$N(x,t) = x^3 + t^2 \tag{1.36}$$

are

$$N_t = 2t, \qquad \text{and} \qquad N_x = 3x^2. \tag{1.37}$$

When N is a function of x and t, its integral with respect to t is done by pretending that x is a constant, as in the following example.

Example: For $N(x,t)$ given by (1.36)

$$\int^t N(x,t)dt = x^3 t + \frac{1}{3}t^3 + f(x), \tag{1.38}$$

where $f(x)$ plays the role of a "constant of integration" with respect to t and is actually an arbitrary function of x. To verify this, we can take the (partial) derivative of (1.38) with respect to t and recover $N(x,t)$ in (1.36).

Example: The solution $N(x,t)$ of the PDE:

$$\frac{\partial}{\partial t}N = rN \tag{1.39}$$

for $t > 0$ is

$$N(x,t) = a(x)e^{rt}. \tag{1.40}$$

Here $a(x)$ plays the role of the "constant" in the solution to the ODE (1.1). Setting $t = 0$, we find

$$a(x) = N(x,0),$$

which is to be given by the initial distribution of the population.

1.5 Exercise I

Review of ordinary differential equations:

1. Find the most general solution to:

   (a) $\frac{d}{dt}N = rN + b$; r, b are constants,

   (b) $\frac{d^2}{dt^2}x + \beta\frac{d}{dt}x + \omega^2 x = 0$; β, ω^2 are constants,

   (c) $\frac{d^2}{dt^2}x + \omega^2 x = \cos\omega_0 t$, $\omega \neq \omega_0$,

   (d) $\frac{d^2}{dt^2}x + \omega_0^2 x = \cos\omega_0 t$.

2. Find the solution satisfying the specified initial conditions:

   (a) $\frac{d}{dt}N = rN + b$
   $N(0) = 0$.

   (b) $\frac{d^2}{dt^2}x + \beta\frac{d}{dt}x + \omega^2 x = 0$
   $x(0) = 1$, $\frac{d}{dt}x(0) = 0$.

   (c) $\frac{d^2}{dt^2}x + \omega^2 x = \cos\omega_0 t$, $\omega \neq \omega_0$
   $x(0) = 0$, $\frac{d}{dt}x(0) = 0$.

   (d) $\frac{d^2}{dt^2}x + \omega_0^2 x = \cos\omega_0 t$
   $x(0) = 0$, $\frac{d}{dt}x(0) = 0$.

1.6 Solutions to Exercise I

1. (a) $N(t) = N_h(t) + N_p(t)$, where $N_h(t)$ is the solution to the homogeneous equation and $N_p(t)$ is a particular solution to the non-homogeneous equation. For $N_p(t)$, we try a constant, i.e. $N_p(t) = c$. Substituting it into $\frac{d}{dt}N = rN + b$ yields $0 = rc + b$, implying $c = -b/r$.

The solution to the homogeneous equation $\frac{d}{dt}N = rN$ is: $N_h(t) = ae^{rt}$, where a is an arbitrary constant.

Combining:

$$N(t) = ae^{rt} - b/r.$$

(b) Try

$$x(t) = ae^{\alpha t}.$$

Substituting into the ODE yields:

$$\alpha^2 + \alpha\beta + \omega^2 = 0.$$

So $\alpha = \alpha_1$ or $\alpha = \alpha_2$, where

$$\alpha_1 \equiv -\frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} - \omega^2} \quad \text{and} \quad \alpha_2 \equiv -\frac{\beta}{2} - \sqrt{\frac{\beta^2}{4} - \omega^2}$$

The general solution is

$$x(t) = a_1 e^{\alpha_1 t} + a_2 e^{\alpha_2 t}$$

(c) $x(t) = x_h(t) + x_p(t)$.

For $x_p(t)$, try $x_p(t) = a\cos\omega_0 t$. Substituting into the ODE yields

$$-\omega_0^2 a + \omega^2 a = 1.$$

So $a = 1/(\omega^2 - \omega_0^2)$. For $x_h(t)$, we know the general solution to the homogeneous ODE is

$$x_h(t) = A\sin\omega t + B\cos\omega t; A, B \text{ are arbitrary constants.}$$

The full solution is

$$x(t) = A\sin\omega t + B\cos\omega t + \cos\omega_0 t/(\omega^2 - \omega_0^2).$$

(d) This is the resonance case. Still try $x(t) = x_h(t) + x_p(t)$.

For $x_p(t)$, try $x_p(t) = at\sin\omega_0 t$. Substituting into the nonhomogeneous ODE yields

$$2a\omega_0\cos\omega_0 t - a\omega_0^2 t\sin\omega_0 t + \omega_0^2 at\sin\omega_0 t = \cos\omega_0 t.$$

Thus

$$2a\omega_0 = 1.$$

So $x_p(t) = t \sin \omega_0 t / 2\omega_0$. The general solution to the homogeneous ODE is:

$$x_h(t) = A \sin \omega_0 t + B \cos \omega_0 t.$$

The full solution is

$$x = A \sin \omega_0 t + B \cos \omega_0 t + t \sin \omega_0 t / 2\omega_0.$$

2. (a) $N(t) = ae^{rt} - b/r$

$$N(0) = a - b/r = 0 \text{ implies } a = b/r.$$

So,

$$N(t) = \frac{b}{r}(e^{rt} - 1)$$

(b) $x(t) = a_1 e^{\alpha_1 t} + a_2 e^{\alpha_2 t}$

$$x(0) = a_1 + a_2 = 1$$

$$\frac{d}{dt}x(0) = \alpha_1 a_1 + \alpha_2 a_2 = 0$$

Thus $a_1 = -\alpha_2/(\alpha_1 - \alpha_2)$, $a_2 = \alpha_1/(\alpha_1 - \alpha_2)$.
Finally

$$x(t) = \frac{1}{(\alpha_1 - \alpha_2)}[-\alpha_2 e^{\alpha_1 t} + \alpha_1 e^{\alpha_2 t}]$$

(c) $x(t) = A \sin \omega t + B \cos \omega t + \cos \omega_0 t / (\omega^2 - \omega_0^2)$

$$x(0) = B + 1/(\omega^2 - \omega_0^2)$$

$$\frac{d}{dt}x(0) = A\omega = 0$$

Thus $A = 0$, $B = -1/(\omega^2 - \omega_0^2)$ and

$$x(t) = \frac{1}{(\omega^2 - \omega_0^2)}[\cos \omega_0 t - \cos \omega t]$$

(d) $x(t) = A \sin \omega_0 t + B \cos \omega_0 t + t \sin \omega_0 t / \omega_0$

$$x(0) = B = 0$$

$$\frac{d}{dt}x(0) = A\omega_0 = 0.$$

Thus $A = 0$, $B = 0$, and

$$x(t) = t \sin \omega_0 t / \omega_0.$$

1.7 Exercise II

Partial derivatives:

1. Evaluate $\frac{\partial}{\partial x}u(x,y)$ for

   (a) $u(x,y) = e^{xy}$

   (b) $u(x,y) = (x+y)^2$

   (c) $u(x,y) = x^2 + y^2$

2. Evaluate $\int^y u(x,\eta)d\eta$ (as an indefinite integral) for $u(x,y)$ given by (a), (b) and (c) from Problem 1 above.

3. (a) Find the general solution $u(t)$ to the ODE

$$m\frac{d^2}{dt^2}u + ku = 0,$$

   where m and k are constants.

   (b) Find the general solution $u(x,y)$ to the PDE

$$\frac{\partial^2}{\partial x^2}u + (1+y^2)u = 0$$

1.8 Solutions to Exercise II

1. Evaluate $\frac{\partial}{\partial x}u(x,y)$ for

   $u(x,y) = e^{xy}, \qquad \frac{\partial}{\partial x}u = ye^{xy}$

   $u(x,y) = (x+y)^2, \quad \frac{\partial}{\partial x}u = 2(x+y)$

   $u(x,y) = x^2 + y^2, \quad \frac{\partial}{\partial x}u = 2x.$

2. Evaluate $\int^y u(x,\eta)d\eta$ from problem 1

   $\int^y e^{x\eta}d\eta = \frac{1}{x}e^{xy} + f(x)$

   $\int^y (x+\eta)^2 d\eta = \frac{(x+y)^3}{3} + g(x)$

   $\int^y (x^2+\eta^2)d\eta = x^2y + \frac{y^3}{3} + h(x),$

   where f,g, and h are arbitrary functions of x.

3a. Find the general solution u(t) to $m\frac{d^2}{dt^2}u + ku = 0$, where m and k are constants.

   Try $u(t) = A\cos\alpha t + B\sin\alpha t$.

   Substitute into $m\frac{d^2}{dt^2}u + ku = 0$ to find $\alpha = \sqrt{k/m}$.

   So the general solution is $u(t) = A\cos\sqrt{k/m}t + B\sin\sqrt{k/m}t$ with A and B arbitrary constants.

3b. Find the general solution $u(x,y)$ to $\frac{\partial^2}{\partial x^2}u + (1+y^2)u = 0$.

   Treat y as a constant with respect to x-partial derivative.

   Try $u(x,y) = A(y)\cos(\alpha(y)x) + B(y)\sin(\alpha(y)x)$.

   Find $\alpha(y) = \sqrt{(1+y^2)}$.

   Thus we have $u(x,y) = A(y)\cos(\sqrt{(1+y^2)}x) + B(y)\sin(\sqrt{(1+y^2)}x)$, with A and B arbitrary functions of y.