Lorum ipsum

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Abstract

Lorem ipsum dolor sit amet, dicta iudicabit consequat ex vix, veniam legimus appetere has id, an pri graece epicuri detraxit. Ea aliquam expetendis posidonium eos, nam invenire corrumpit imperdiet ei. Et constituto dissentias usu, mel solum erant et. Mel dolorem menandri in.

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1 Notations Demonstrations

Read the source to understand these user defined latex symbols. In the source, `wang_scientific.tex` contains these macros:

(i) `\To`: $\rightrightarrows$

(ii) `\GX`: Γ

(iii) `\mal`: $\mathfrak{m}$

(iv) `\mumu`: $\mu\mu$

(v) `\paver`: $\mathcal{P}$

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(vi) \zzz: $X \times X^*$

(vii) \RRR: $\mathbb{R} \times \mathbb{R}$

(viii) \todo: $\hookrightarrow$ TO DO:

(ix) \lev{#1}: $\operatorname{lev}_{\leq \#1}$

(x) \moyo{1}{2}: ${}_2 1$

(xi) \emp: $\varnothing$

(xii) \infconv: $\square$

(xiii) \pair: $\langle x, y \rangle$

(xiv) \scal{#1}{#2}: $\langle \#1, \#2 \rangle$

(xv) \Scal{#1}{#2}: $\langle \#1, \#2 \rangle$

(xvi) \pscal: $\langle \cdot, \cdot \rangle$

(xvii) \Tt: $\mathfrak{T}$

(xviii) \YY: $\mathcal{Y}$

(xix) \XX: $\mathcal{X}$

(xx) \HH: $\mathcal{H}$

(xxi) \XP: $\mathcal{X}^*$

(xxii) \st: $\mid$

(xxiii) \zeroun: $]0, 1[$

(i) \yosida

(ii) \exi

(iii) \GG

(iv) \RR

(v) \SSS

(vi) \CC

(vii) `\Real`

(viii) `\ii`

(ix) `\RP`

(x) `\RPX`

(xi) `\RPP`

(xii) `\RX`

(xiii) `\RXX`

(xiv) `\KK`

(xv) `\NN`

(xvi) `\nnn`

(xvii) `\thalb`

(xviii) `\zo`

(xix) `\lzo`

2 Schur's Cone and Its Polar

Example 2.1 Define

$$Q = \left\{ x \in \mathbb{R}^n \left| \sum_{i=1}^{k} x_i \geq 0,\ k = 1, \cdots, n-1,\ x_1 + \cdots + x_n = 0 \right. \right\}. \tag{1}$$

Then

$$Q^\circ = \{ y \in \mathbb{R}^n | \langle y, x \rangle \geq 0\ \forall x \in Q \} \tag{2}$$

$$= \{ y \in \mathbb{R}^n | y_1 \geq y_2 \geq \cdots \geq y_n \}. \tag{3}$$

Proof. We have $Q = H(\mathbb{R}_+^{n-1})$, where

$$H = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & & \\ & & \ddots & 1 \\ & & & -1 \end{bmatrix}, \tag{4}$$

H is a bi-diagonal matrix with zeros on the diagonal and -1 on its sub-diagonals. Indeed, for $x = (x_1, \cdots, x_{n-1}) \in \mathbb{R}_+^{n-1}$, we have

$$\begin{align} y &= Hx \tag{5}\\ &= \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{bmatrix} x_2 + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -1 \end{bmatrix} x_{n-1} \tag{6}\\ &= \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ -x_{n-2} + x_{n-1} \\ -x_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}, \tag{7} \end{align}$$

where $y_1 \geq 0, y_2 + y_1 \geq 0, \cdots, y_{n-1} + \cdots + y_1 \geq 0$, and $y_1 + \cdots + y_n = 0$. So $y \in Q$. Hence $H(\mathbb{R}_+^{n-1}) \subseteq Q$. Similarly, one can write for $y \in Q$, $\exists x \in \mathbb{R}_+^{n-1}$ with $y = Hx$. In fact, $x = y_1, x_2 = y_1 + y_2, \cdots, x_{n-1} = y_1 + \cdots + y_{n-1}, -x_{n-1} = y_n$ because $y_1 + \cdots + y_n = 0$. In particular, $x \in \mathbb{R}_+^{n-1}$.

Note that $y \in Q^\circ \Leftrightarrow y^\intercal a_i \geq 0$ for $i = 1, \cdots, n-1$. Where

$$a_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0, \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0, \end{bmatrix}, \cdots, a_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{bmatrix}. \tag{8}$$

Now $y^\intercal a_i \geq 0 \Leftrightarrow y_i - y_{i+1} \geq 0$, for $i = 1, \cdots, n-1$. Hence $Q^\circ = \{y \in \mathbb{R}^n | y_1 \geq y_2 \geq \cdots \geq y_n\}$.

■

3 Lorentz Cone and its Polar

Example 3.1

$$L = \left\{ (\xi, t) \in \mathbb{R}^n \;\middle|\; \|\xi\| \leq t, \xi \in \mathbb{R}^{n-1} \right\}. \tag{9}$$

Then the polar cone of L is $L^\circ = \{(x, s) \mid \langle (x, s), (\xi, t) \rangle \geq 0\}\} = L$.

Proof.

Let $(x, s) \in L^\circ$. Then $\langle (x,s),(\xi,t)\rangle \geq 0$, $\forall (\xi,t) \in L$, i. e. $\langle x,\xi\rangle + st \geq 0$.

For $t > 0$, we have $\langle x, \xi/t\rangle \geq -s$, i.e. $\langle x, -\xi/t\rangle \leq s$. Using $\|\xi\| \leq t$ and by taking supremum over $\|\xi/t\| \leq 1$, we obtain $\|x\| \leq s$.

Hence $L^\circ = L$. ■

Corollary 3.2 *Let $A \in \mathbb{R}^{n\times n}$ be positive definite. Define*

$$L_A = \left\{ (\xi, t) \in \mathbb{R}^n \,\middle|\, \sqrt{\langle \xi, A\xi\rangle} \leq t \right\}.$$

Then $(L_A)^\circ = \{(\xi,t) \mid \sqrt{\langle \xi, A^{-1}\xi\rangle} \leq t\}$.

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