# Reading Notes

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**Abstract**

Reports on papers read. This is a LaTEX file for my own notes taking. It may accelerate the process of writing my thesis for my PhD degree.

This is just pseudo text. This is just pseudo text. This is just pseudo text. This is just pseudo text [1].

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# Chapter 1

# The Basics of Optimization Theories

Notations in this chapter are not shared and they are for this chapter only.

# Chapter 2

# Linear Convergence of First Order Method

In this chapter, we are specifically interested in characterizing linear convergence of well known first order optimization algorithms.

## 2.1 Necoara's et al's Paper

### 2.1.1 The Settings

{ass:necoara-2019-settings} The assumption follows give the same setting as Necoara et al. [1].

**Assumption 2.1.1** Consider optimization problem:

$$-\infty < f^+ = \min_{x \in X} f(x). \tag{2.1.1}$$

{problem:necoara-2019} $X \subseteq \mathbb{R}^n$ is a closed convex set. Assume projection onto $X$, denoted by $\Pi_X$ is easy. Denote $X^+ = \operatorname*{argmin}_{x \in X} f(x) \neq \emptyset$, assume it's a closed set. Assume $f$ has $L_f$ Lipschitz continuous gradient, i.e: for all $x, y \in X$:

$$\|\nabla f(x) - \nabla f(y)\| \leq L_f \|x - y\|.$$

**Definition 2.1.2 (Bregman Divergence)** *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a differentiable function. Define Bregman Divergence:*

$$D_f : \mathbb{R}^n \times \operatorname{dom} \nabla f \to \overline{\mathbb{R}} := (x, y) \mapsto f(x) - f(y) - \langle \nabla f(y), x - y \rangle.$$

Some immediate consequences of Assumption 2.1.1 now follows. The variational inequality characterizing optimal solution has:

$$x^+ \in X^+ \implies (\forall x \in X)\ \langle \nabla f(x^+), x - x^+\rangle \geq 0.$$

The converse is true if $f$ is convex. The gradient mapping in this case is:

{def:necoara-scnvx}

$$\mathcal{G}_{L_f} x = L_f(x - \Pi_X x).$$

**Definition 2.1.3 (Strong convexity)** *Suppose $f$ satisfies Assumption 2.1.1. Then $f \in \mathbb{S}(L_f, \kappa_f, X)$ is strongly convex iff*

$$(\forall x, y \in X)\ \kappa_f \|x - y\|^2 \leq D_f(x, y) \leq L_f \|x - y\|^2.$$

{def:necoara-weaker-scnvx} Then it's not hard to imagine the following natural relaxation of the above conditions.

**Definition 2.1.4 (Relaxations of Strong convexity)** *Suppose $f$ satisfies Assumption 2.1.1. Let $L_f \geq \kappa_f \geq 0$ such that for all $x \in X$, $\bar{x} = \Pi_{X^+} x$. We define the following:*

{def:neocara-qscnvx}

(i) *Quasi-strong convexity (Q-SCNVX): $0 \leq D_f(\bar{x}, x) - \frac{\kappa_f}{2}\|x - \bar{x}\|^2$. Denoeted by $\mathbb{S}'(L_f, \kappa_f, X)$.*

{def:necoara-qup}

(ii) *Quadratic under approximation (QUA): $0 \leq D_f(x, \bar{x}) - \frac{\kappa_f}{2}\|x - \bar{x}\|^2$. Denoeted by $\mathbb{U}(L_f, \kappa_f, X)$.*

{def:necoara-qgg}

(iii) *Quadratic Gradient Growth (QGG): $0 \leq D_f(x, \bar{x}) + D_f(\bar{x}, x) - \kappa_f/2\|x - \bar{x}\|^2$. Denoted by $\mathbb{G}(L_f, \kappa_f, X)$.*

{def:necoara-qfg}

(iv) *Quadratic Function Growth (QFG): $0 \leq f(x) - f^* - \kappa_f/2\|x - \bar{x}\|^2$. Denoted by $\mathbb{F}(L_f, \kappa_f, X)$.*

{def:necoara-eb}

(v) *Error Bound (EB): $\|\mathcal{G}_{L_f} x\| \geq \kappa_f \|x - \bar{x}\|$. Denoted by $\mathbb{E}(L_f, \kappa_f, X)$.*

**Remark 2.1.5** The error bound condition in Necoara et al. is sometimes referred to as the "Proximal Error Bound".

### 2.1.2 Major Results in the paper

In Necoara's et al, major results assume convexity of $f$.

{thm:qscnvx-means-qua} **Theorem 2.1.6** *(Q-SCNVX implies QUA) Let $f$ satisfies Assumption 2.1.1 and assume $f$ is convex:*

$$\mathbb{S}'(L_f, \kappa_f, X) \subseteq \mathbb{U}(L_f, \kappa_f, X).$$

*Proof.* We proof by induction. Convexity of $f$ makes $X^+$ convex and $\Pi_{X^+}X$ unique for all $x \in X$. Make inductive hypothesis that there exists $\kappa^{(k)} \geq 0$ such that

$$(\forall x \in X) \quad f(x) \geq f^+ + \langle \nabla f(\Pi_{X^+}x), x - \Pi_{X^+}x\rangle + \kappa^{(k)}/2\|x - \Pi_{X^+}x\|^2.$$

The base case is true by convexity of $f$ with $\kappa_f^{(0)} = 0$. Choose any $x \in X$ define $\bar{x} = \Pi_{X^+}x$. Consider $x_\tau = \bar{x} + \tau(x - \bar{x})$ for $\tau \in [0, 1]$. Calculus rule has

$$\begin{aligned}
f(x) &= f(\bar{x}) + \int_0^1 \langle \nabla f(x_\tau), x - \bar{x}\rangle d\tau \\
&= f(\bar{x}) + \int_0^1 \tau^{-1}\langle \nabla f(x_\tau), \tau(x - \bar{x})\rangle d\tau \\
&= f(\bar{x}) + \int_0^1 \tau^{-1}\langle \nabla f(x_\tau), x_\tau - \bar{x}\rangle d\tau.
\end{aligned}$$

$f$ is Q-SCNVX so

$$\begin{aligned}
f^+ - f(x_\tau) &\geq \langle \nabla f(x_\tau), \Pi_{X^+}x_\tau - x_\tau\rangle + \kappa_f/2\|x_\tau - \Pi_{X^+}x_\tau\|^2 \\
&= \langle \nabla f(x_\tau), \bar{x} - x_\tau\rangle + \kappa_f/2\|x_\tau - \bar{x}\|^2 \\
\iff \langle \nabla f(x_\tau), x_\tau - \bar{x}\rangle &\geq f(x_\tau) - f^+ + \kappa_f/2\|x_\tau - \bar{x}\|^2.
\end{aligned}$$

We used $\Pi_{X^+}x_\tau = \bar{x}$ by convexity of $f$. Therefore:

$$\begin{aligned}
f(x) &\geq f(\bar{x}) + \int_0^1 \tau^{-1}\left(f(x_\tau) - f^+ + \frac{\kappa_f}{2}\|x_\tau - \bar{x}\|^2\right) d\tau \\
&= f(\bar{x}) + \int_0^1 \tau^{-1}\left(f(x_\tau) - f^+\right) + \frac{\tau\kappa_f}{2}\|x - \bar{x}\|^2 d\tau \\
&\geq f(\bar{x}) + \int_0^1 \tau^{-1}\left(\langle \nabla f(\Pi_{X^+}x_\tau), x_\tau - \Pi_{X^+}x_\tau\rangle + \frac{\kappa_f^{(k)}}{2}\|x_\tau - \Pi_{X^+}x_\tau\|^2\right) + \frac{\tau\kappa_f}{2}\|x - \Pi_{X^+}x_\tau\|^2 d\tau \\
&= f(\bar{x}) + \int_0^1 \tau^{-1}\left(\langle \nabla f(\bar{x}), x_\tau - \bar{x}\rangle + \frac{\kappa_f^{(k)}}{2}\|x_\tau - \bar{x}\|^2\right) + \frac{\tau\kappa_f}{2}\|x - \bar{x}\|^2 d\tau \\
&= f(\bar{x}) + \int_0^1 \langle \nabla f(\bar{x}), x - \bar{x}\rangle + \frac{\tau\kappa_f^{(k)}}{2}\|x - \bar{x}\|^2 + \frac{\tau\kappa_f}{2}\|x - \bar{x}\|^2 d\tau \\
&= f(\bar{x}) + \langle \nabla f(\bar{x}), x - \bar{x}\rangle + \frac{\kappa_f^{(k)} + \kappa_f}{4}\|x - \bar{x}\|^2.
\end{aligned}$$

This is the new inductive hypothesis, and it has $\kappa_f^{(k+1)} = (\kappa_f^{(k)} + \kappa_f)/2$. The induction admits recurrence:

$$\kappa_f^{(n)} = (1/2^n)(\kappa_f^{(0)} + (2^n - 1)\kappa_f).$$

Inductive hypothesis is true for $\kappa_f^{(0)} = 0$ and $f$ being convex is sufficient. It has $\lim_{n\to\infty} \kappa_f^{(n)} = \kappa_f$. ∎

**Remark 2.1.7** This is Theorem 1 in the paper. Convexity assumption of $f$ makes $X^+$ convex, so the projection is unique, and it has $\Pi_{X^+}x_\tau = \bar{x}$ for all $\tau \in [0,1]$. In addition, the inductive hypothesis has $\kappa_f^{(n)} \geq 0$, which is not sufficient for convexity, but necessary. The projectin property remains true for nonconvex $X^+$, however the base case require rethinking.

**Theorem 2.1.8 (Q-SCNVX implies QGG)** *Under Assumption 2.1.1 and convexity of $f$, it has*

$$\mathbb{S}'(L_f, \kappa_f, X) \subseteq \mathbb{G}(L_f, \kappa_f, X).$$

*Proof.* If $f \in \mathbb{S}'(L_f, \kappa_f, X)$ then Theorem 2.1.6 has $f \in \mathbb{U}(L_f, \kappa_f, X)$. Then, add (ii), (i) in Definition 2.1.4 yield the results. ■

**Remark 2.1.9** This is Theorem 2 in the Necoara et al. [1], right after it claims $\mathbb{U}(L_f, \kappa_f, X) \subseteq \mathbb{G}(L_f, \kappa_f/2, X)$ under convexity.

{thm:qfg-suff}

**Theorem 2.1.10 (sufficiency of QFG)** *Let $f$ satisfies Assumption 2.1.1. For all $0 < \beta < 1$, $x \in X$, let $x^+ = \Pi_X(x - L_f^{-1}\nabla f(x))$. If*

$$\|x^+ - \Pi_{X^+}x^+\| \leq \beta\|x - \Pi_{X^+}x\|,$$

*then $f$ satisfies the QFG condition with $\kappa_f = L_f(1-\beta)^2$.*

*Proof.* The proof is direct.

$$\begin{aligned}
\|x - \Pi_{X^+}x\| &\leq \|x - \Pi_{X^+}x^+\| && (2.1.2)\\
&\leq \|x - x^+\| + \|x^+ - \Pi_{X^+}x^+\| && (2.1.3)\\
&\leq \|x - x^+\| + \beta\|x - \Pi_{X^+}x\| && (2.1.4)\\
\iff 0 &\leq \|x - x^+\| - (1-\beta)\|x - \Pi_{X^+}x\|. && (2.1.5)
\end{aligned}$$

$x^+$ has descent lemma hence we have

$$f^+ - f(X) \leq f(x^+) - f(x) \leq -\frac{L_f}{2}\|x^+ - x\|^2 \leq -\frac{L_f}{2}(1-\beta)^2\|x - \Pi_{X^+}\|^2.$$

Hence it gives the quadratic growth condition. ■

**Remark 2.1.11** It's unclear where convexity is used. However, it' still assumed in Necoara et al paper.

The following theorems are about the relation between EB and QFG.

# Bibliography

[1] I. Necoara, Y. Nesterov, and F. Glineur, *Linear convergence of first order methods for non-strongly convex optimization*, Mathematical Programming, 175 (2019), pp. 69–107.