Lorum ipsum

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Abstract

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1 Schur's Cone and Its Polar

Example 1.1 Define

$$Q = \left\{ x \in \mathbb{R}^n \middle| \sum_{i=1}^{k} x_i \geq 0,\ k = 1, \cdots, n-1,\ x_1 + \cdots + x_n = 0 \right\}. \tag{1.1}$$

Then

$$Q^\circ = \{ y \in \mathbb{R}^n | \langle y, x \rangle \geq 0\ \forall x \in Q \} \tag{1.2}$$
$$= \{ y \in \mathbb{R}^n | y_1 \geq y_2 \geq \cdots \geq y_n \}. \tag{1.3}$$

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Proof. We have $Q = H(\mathbb{R}_+^{n-1})$, where

$$H = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & & \\ & & \ddots & 1 \\ & & & -1 \end{bmatrix}, \tag{1.4}$$

H is a bi-diagonal matrix with zeros on the diagonal and -1 on its sub-diagonals. Indeed, for $x = (x_1, \cdots, x_{n-1}) \in \mathbb{R}_+^{n-1}$, we have

$$\begin{aligned} y &= Hx \qquad (1.5) \\ &= \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{bmatrix} x_2 + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -1 \end{bmatrix} x_{n-1} \qquad (1.6) \\ &= \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ -x_{n-2} + x_{n-1} \\ -x_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}, \qquad (1.7) \end{aligned}$$

where $y_1 \geq 0, y_2 + y_1 \geq 0, \cdots, y_{n-1} + \cdots + y_1 \geq 0$, and $y_1 + \cdots + y_n = 0$. So $y \in Q$. Hence $H(\mathbb{R}_+^{n-1}) \subseteq Q$. Similarly, one can write for $y \in Q$, $\exists x \in \mathbb{R}_+^{n-1}$ with $y = Hx$. In fact, $x = y_1, x_2 = y_1 + y_2, \cdots, x_{n-1} = y_1 + \cdots + y_{n-1}, -x_{n-1} = y_n$ because $y_1 + \cdots + y_n = 0$. In particular, $x \in \mathbb{R}_+^{n-1}$.

Note that $y \in Q^\circ \iff y^\intercal a_i \geq 0$ for $i = 1, \cdots, n-1$. Where

$$a_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0, \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0, \end{bmatrix}, \cdots, a_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{bmatrix}. \tag{1.8}$$

Now $y^\intercal a_i \geq 0 \iff y_i - y_{i+1} \geq 0$, for $i = 1, \cdots, n-1$. Hence $Q^\circ = \{y \in \mathbb{R}^n | y_1 \geq y_2 \geq \cdots \geq y_n\}$.

■

2 Lorentz Cone and its Polar

Example 2.1

$$L = \left\{(\xi, t) \in \mathbb{R}^n \,\middle|\, \|\xi\| \le t, \xi \in \mathbb{R}^{n-1}\right\}. \tag{2.1}$$

Then the polar cone of L is $L^\circ = \{(x,s) \mid \langle (x,s), (\xi,t)\rangle \ge 0\}\} = L$.

Proof.

Let $(x,s) \in L^\circ$. Then $\langle (x,s), (\xi,t)\rangle \ge 0$, $\forall (\xi,t) \in L$, i. e. $\langle x, \xi\rangle + st \ge 0$.

For $t > 0$, we have $\langle x, \xi/t\rangle \ge -s$, i.e. $\langle x, -\xi/t\rangle \le s$. Using $\|\xi\| \le t$ and by taking supremum over $\|\xi/t\| \le 1$, we obtain $\|x\| \le s$.

Hence $L^\circ = L$. ∎

Corollary 2.2 *Let $A \in \mathbb{R}^{n\times n}$ be positive definite. Define*

$$L_A = \left\{(\xi, t) \in \mathbb{R}^n \,\middle|\, \sqrt{\langle \xi, A\xi\rangle} \le t\right\}.$$

Then $(L_A)^\circ = \{(\xi,t) \mid \sqrt{\langle \xi, A^{-1}\xi\rangle} \le t\}$.

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