Chapter 10

Some Special Functions

10.1 Introduction

In the next few chapters we will study partial differential equations in multi-dimensions. Instead of sines and cosines we encountered in one dimensions (and also in multi-dimensions in Cartesian geometry), we will often run into special functions, such as the Bessel function (in cylindrical coordinates), Legendre function and spherical harmonics (in spherical geometry). These special functions will be introduced here. We will make extensive use of the method of Frobenius.

10.2 Legendre differential equation

Legendre equation arises in polar coordinates, where $x = \cos\theta$:

$$\frac{d}{dx}[(1-x^2)\frac{d}{dx}y] + \mu y = 0 \tag{10.1}$$

This equation has singularities at $x = \pm 1$. We seek a series solution about $x = 0$, which is an ordinary point (not singular), and so the solution can be written in the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{10.2}$$

Differentiating:

$$\begin{aligned} y'(x) &= \sum n a_n x^{n-1} \\ y''(x) &= \sum n(n-1) a_n x^{n-1}. \end{aligned}$$

Substituting these series into (1) yields

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n x^n - 2\sum_{n=0}^{\infty} n a_n x^n + \mu \sum_{n=0}^{\infty} a_n x^n = 0$$

Making the substitution $m = n - 2$ in the first sum and $m = n$ in the other terms:

$$\sum_{m=-2}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=0}^{\infty} m(m-1)a_m x^m - 2\sum_{m=0}^{\infty} m a_m x^m + \mu \sum_{m=0}^{\infty} a_m x^m = 0$$

which is

$$\sum_{m=0}^{\infty} \{(m+2)(m+1)a_{m+2} - (m(m+1) - \mu)a_m\}x^m = 0. \tag{10.3}$$

Since (10.3) is to hold for all x in certain domain, the terms in $\{\quad\}$ must add up to zero. Thus

$$a_{m+2} = \frac{m(m+1) - \mu}{(m+2)(m+1)} a_m, \quad m = 0, 1, 2, 3 \ldots \tag{10.4}$$

(10.4) is a second-order recurrent relationship. The coefficients a_m for m even will all be proportional to a_0:

$$a_2 = \frac{-\mu}{2} a_0, \quad a_4 = \frac{2 \cdot 3 - \mu}{4.3} a_2 = \frac{-(2 \cdot 3 - \mu)\mu}{4!} a_0$$

$$\ldots .$$

$$a_{2j} = \frac{-\mu(2 \cdot 3 - \mu)(4 \cdot 5 - \mu) \ldots ((2j-2)(2j-1) - \mu)}{(2j)!} a_0$$

And the coefficients a_m for m odd will all be proportional to a_1:

$$a_3 = \frac{1 \cdot 2 - \mu}{3 \cdot 2} a_1, \quad a_5 = \frac{(3 \cdot 4 - \mu)}{5 \cdot 4} a_3 = \frac{(3 \cdot 4 - \mu)(1 \cdot 2 - \mu)}{5!} a_1$$

$$\ldots .$$

$$a_{2j+1} = \frac{(1 \cdot 2 - \mu)(3 \cdot 4 - \mu) \ldots ((2j-1)(2j) - \mu)}{(2j+1)!} a_1$$

The general solution is

$$y(x) = a_0 \sum_{j=0}^{\infty} (a_{2j}/a_0) x^{2j} + a_1 \sum_{j=0}^{\infty} (a_{2j+1}/a_1) x^{2j+1}$$

Both solutions converge at $x = 0$ and in its neighborhood as long as $|x| < 1$. This can be shown using the ratio test. For convergence, we must have

$$\lim_{m\to\infty}\left|\frac{a_{m+2}x^{m+2}}{a_m x^m}\right| < 1,$$

but from (10.4)

$$\lim_{m\to\infty}|a_{m+2}/a_m| = 1.$$

Therefore both series converge for $|x| < 1$. That the series diverges at $x = \pm 1$ is not surprising in view of the presence of a singularity there.

Notice however, that the series converge for any value of x if they terminate. This happens for one of the series when μ is equal to the product of two consecutive integers:

$$\mu = n(n+1), \quad n = \text{ integer.}$$

This fact can be used to find the following *eigenvalue* problem: Find the eigenvalue μ and the eigenfunction $y(x)$ such that

$$\frac{d}{dx}[(1-x^2)\frac{d}{dx}y] + \mu y = 0, \qquad -1 \le x \le 1$$

subject to the boundary condition:

$$y(x) \text{ bounded at } x = -1 \text{ and } x = 1. \tag{10.5}$$

[Later on we will show that $x = \pm 1$ corresponds to the north and south poles; and we don't want our solution to blow up there (or anywhere else).]

We construct the eigenfunction

$$y(x) = P_n(x)$$

corresponding to the eigenvalue

$$\mu = +n(n+1)$$

in the following way.

For $n = 0$, $\mu = 0$, the even series terminates after one term. We set $a_0 = 1$ and $a_1 = 0$ to get

$$P_0(x) = 1$$

For $n = 1$, $\mu = 2$, the odd series terminates after one term. We set $a_0 = 0$ and $a_1 = 1$ to get

$$P_1(x) = x$$

For $n = 2$, $\mu = 6$, we set $a_1 = 0$ and $a_0 = -\frac{1}{2}$:

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

For $n = 3$, $\mu = 12$, we set $a_0 = 0$, $a_1 = -\frac{3}{2}$:

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x.$$

For general integer n:

$$P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}, \quad [n/2] \text{ greatest integer } \leq n/2$$

These solutions are known as the Legendre polynomials. The solution to (10.5) can simply be written as

$$y(x) = P_n(x), \quad \mu = n(n+1), \quad n = 0, 1, 2, 3, \ldots \tag{10.6}$$

The overall constant is chosen so that the eigenfunction is equal to 1 at $x = 1$, the north pole. The solution can be multiplied by any arbitrary constant.

There is a useful formula, called Rodrigue's Formula, which allows us to generate all P_n's simply:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n. \tag{10.7}$$

One can show by differentiation that (10.7) satisfies

$$\frac{d}{dx}[(1 - x^2)\frac{d}{dx}P_n] + n(n+1)P_n = 0,$$

and that it is a polynomial of degree n. Therefore the function defined in (10.7) must be the same (within an overall constant) of the required eigenfunction defined in (10.6). It can be shown that both function has the normalization such that $P_n(1) = 1$. Thus (10.6) and (10.7) must be the same.

The Legendre polynomials are orthogonal to each other, i.e.:

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases} \tag{10.8}$$

Proof: [see later for an easier proof using Sturm-Liouville theory.]

To show the first part for $m \neq n$, we assume, without loss of generality, $m < n$. Let $f(x)$ be any function with at least n continuous derivatives in $-1 \leq x \leq 1$. Consider the integral

$$I \equiv \int_{-1}^{1} f(x) P_n(x) dx$$

Using Rodrigue's Formula:

$$I = \frac{1}{2^n n!} \int_{-1}^{1} f(x) \frac{d^n}{dx^n} (x^2 - 1)^n dx,$$

and an integration by parts, we find

$$I = \frac{1}{2^n n!} [f(x) \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n]_{-1}^{1} - \frac{1}{2^n n!} \int_{-1}^{1} f'(x) \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n dx$$

The boundary terms vanish, because after $(n-1)$ derivatives, there is still a $(x^2 - 1)$ term left over undifferentiated. That term vanishes at $x = \pm 1$.

This integration by parts can be continued to give in the end:

$$I = \frac{(-1)^n}{2^n n!} \int_{-1}^{1} f^{(n)}(x) (x^2 - 1)^n dx.$$

If $f(x) = P_m(x)$, $m < n$, which is a polynomial of degree less than n, then its nth derivative, $f^{(n)}(x)$, must vanish. Thus $I = 0$ for $m < n$, (and also for $m > n$, by interchanging m and n in the above proof).

For $m = n$, $f(x) = P_n(x)$, we have, since

$$P_n^{(n)}(x) = (2n)!/2^n n!,$$

$$I = \frac{(2n)!}{2^{2n} (n!)^2} \int_{-1}^{1} (1 - x^2)^n dx$$

This integral can be evaluated with a change of variable $x = \sin\theta$ and repeated integration of $\cos^{2n+1}\theta$:

$$\int_{-1}^{1} (1 - x^2)^n dx = 2 \int_{0}^{1} (1 - x^2)^n dx = \int_{0}^{\pi/2} \cos^{2n+1}\theta d\theta = \frac{2^{2n+1} (n!)^2}{(2n)!(2n+1)}$$

Thus $I = \frac{2}{2n+1}$ when $m = n$.

10.3 Associated Legendre differential equation

The associated Legendre differential equation is

$$\frac{d}{dx}[(1-x^2)\frac{d}{dx}y] + [\mu - \frac{m^2}{1-x^2}]y = 0, \quad -1 \leq x \leq 1. \tag{10.9}$$

The case of $m = 0$ reduces to the Legendre differential equation. For physical reasons (to be discussed in the next chapter) we are interested only in the case where m is an integer

$$m = 0, \quad \pm 1, \quad \pm 2, \quad \pm 3, \ldots$$

Since the equation has a singular point at $x = \pm 1$, the general solution is probably not bounded at the boundaries $x = \pm 1$ of the domain in (10.9). The eigenvalue problem is to find μ such that the eigenfunctions $y(x)$ is bounded at $x = \pm 1$. The procedure for solving this eigenvalues problem is quite similar to that for the Legendre differential equation. The eigenvalue is found to be

$$\mu = n(n+1), \quad n = 0, 1, 2, 3, \ldots \tag{10.10}$$

and the eigenfunction is the associated Legendre functions:

$$y(x) = P_n^m(x). \tag{10.11}$$

They are related to the Legendre polynomials by, for $m \geq 0$:

$$P_n^m(x) = (x^2-1)^{m/2}\frac{d^m}{dx^m}P_n(x), \quad m \leq n. \tag{10.12}$$

$$P_n^m(x) \equiv 0 \quad \text{for } m > n$$

because $P_n(x)$ is a polynomial of degree n, and its mth derivative vanishes if $m > n$. (10.12) is also called Rodrigue's Formula. We will prove it in a moment.

The definition of $P_n^m(x)$ is extended to negative m's by this formula:

$$P_n^m(x) = (-1)^m \frac{(n+m)!}{(n-m)!} P_n^{-m}(x), \tag{10.13}$$

so that $P_n^m(x)$ is simply a scalar multiple of $P_n^{-m}(x)$.

For each n ($n = 0, 1, 2, 3, \ldots$) we have $2n+1$ associated Legendre functions $P_n^m(x)$, where m runs from $-n$ to n.

The first few of $P_n^m(x)$ are, with $x = \cos\theta$

$$P_1^1 = \sin\theta, \quad P_2^1 = \frac{3}{2}\sin 2\theta, \quad P_3^1 = \frac{3}{8}(\sin\theta + 5\sin 3\theta)$$
$$P_1^2 = 0, \quad P_2^2 = \frac{3}{2}(1 - \cos 2\theta), \quad P_3^2 = \frac{15}{4}(\cos\theta - \cos 3\theta)$$

The associated Legendre functions are also orthogonal to each other:

$$\int_{-1}^{1} P_k^m(x) P_n^m(x) dx = \begin{cases} 0 & \text{if } k \neq n \\ \frac{2}{(2n+1)} \frac{(n+m)!}{(n-m)!} & \text{if } k = n. \end{cases} \tag{10.14}$$

10.4 Proof of Rodrigue's Formula

First we show that

$$P_n^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m P_n(x)}{dx^m}$$

satisfies the associated Legendre differential equation

$$(1 - x^2)y'' - 2xy' + [n(n+1) - \frac{m^2}{1 - x^2}]y = 0, \quad -1 < x < 1.$$

It suffices to consider $m > 0$. We will start with the equation satisfied by $P_n^0 = P_n(x)$:

$$(1 - x^2)P_n^{(2)} - 2xP_n^{(1)} + [n(n+1)]P_n = 0$$

and differentiate it m times to get (trust me!).

$$(1 - x^2)P_n^{(m+2)} - 2(m+1)xP_n^{(m+1)} + (n - m)(n + m + 1)P_n^{(m)} = 0,$$

where $P_n^{(m)} \equiv \frac{d^m}{dx^m} P_n(x)$. [If you don't believe me, you can prove this by induction. First show that it is true for $m = 1$, by simply differentiating the equation for P_n once to get

$$(1 - x^2)P_n^{(3)} - 2 \cdot 2 \cdot xP_n^{(2)} + (n - 1)(n + 1 + 1)P_n^{(1)} = 0$$

Now assume that it is true for m, and show that it is also true for $m + 1$. This is done by differentiating the equation for $P_n^{(m)}$ once, to yield

$$(1 - x^2)P_n^{(m+3)} - 2(m + 2)xP_n^{(m+2)} + (n - m - 1)(n + m + 2)P_n^{(m+1)} = 0$$

This is the same equation as that for $P_n^{(m+1)}$. So the result is proved.]

Therefore

$$w(x) \equiv P_n^{(m)}(x)$$

satisfies

$$(1-x^2)w'' - 2(m+1)xw' + (n-m)(n+m+1)w = 0$$

Now let

$$y = (-1)^m(1-x^2)^{m/2}w$$

(the $(-1)^m$ factor is immaterial).

Substituting

$$w = (-1)^m(1-x^2)^{-m/2}y(x)$$

into the w equation above shows that $y(x)$ satisfies the associated Legendre equation:

$$(1-x^2)y'' - 2xy' + [n(n+1) - \frac{m^2}{1-x^2}]y = 0.$$

Therefore the solution is

$$y = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)$$

For $m > 0$, $y(x) = 0$ at $x = \pm 1$, and so are bounded there. Also, since $\frac{d^m}{dx^m}P_n(x)$ is the mth derivative of an nth order polynomial, it is also a polynomial (and therefore are bounded in $-1 \le x \le 1$).

For $m < 0$, $P_n^m(x)$ is not defined through the Rodrigues' Formula because $\frac{d^m}{dx^m}$ is not defined. $P_n^m(x)$ is then defined as a scalar multiple of $P_n^{-m}(x)$, which also satisfies the associated Legendre equation for negative m.

For m even, $P_n^m(x)$ is a polynomial. For m odd P_n^m is not a polynomial because of the factor $(1-x^2)^{m/2}$. $P_n^m(x)$ is even if $n+m$ is even and odd if $n+m$ is odd.

10.5 Bessel's differential equation

Bessel equation is one of the important special functions. It arises most commonly from separation of variables in cylindrical coordinates, and also in spherical coordinates, in the radial coordinate r:

$$r^2\frac{d^2R}{dr^2} + r\frac{d}{dr}R + (\lambda^2r^2 - p^2)R = 0 \tag{10.15}$$

It can be put into a standard form by letting

$$R(r) = y(x), \quad x \equiv \lambda r.$$

Then,

$$x^2 \frac{d^2}{dx^2} y + x \frac{d}{dx} y + (x^2 - p^2) y = 0 \tag{10.16}$$

This is called Bessel equation of oder p. In cylindrical coordinates, p is usually an integer m, $m = 0, \pm 1, \pm 2, \ldots$. In spherical coordinates, p is usually a half integer, $p = n + \frac{1}{2}$, $n = 0, 1, 2, \ldots$.

The origin $x = 0$ is a regular singular point of the Bessel equation. A test with

$$y(x) \sim x^s$$

yields the indicial equation

$$s(s-1) + s - p^2 = 0$$

Thus the two indices are

$$s_1 = p \quad \text{and} \quad s_2 = -p.$$

In the application we will be dealing with, s_1 and s_2 differ by an integer. Frobenius method tells us that we will have one series solution, and another solution which blows up at $x = 0$.

The two solutions are denoted by $J_p(x)$ and $Y_p(x)$ and are called Bessel function of the first kind and the second kind, respectively. Near $x = 0$, they behave like, for $p = m$, an integer

$$J_m(x) \sim \frac{1}{2^m m!} x^m$$

$$Y_m(x) \sim \begin{cases} -2^m \frac{(m-1)!}{\pi} x^{-m}, & m > 0 \\ \frac{2}{\pi} \ell n x, & m = 0 \end{cases}$$

The general solution to (10.15) is, for $p = m$

$$R(r) = c_1 J_m(\lambda r) + c_2 Y_m(\lambda r).$$

Boundedness at $r = 0$ implies $c_2 = 0$. Thus (with $c_1 = 1$)

$$R(r) = J_m(\lambda r)$$

The boundary condition at $r = a$ is:

$$0 = R(a) = J_m(\lambda a)$$

Let z_{mn} be the nth zero of $J_m(x)$. The eigenvalue λ is then determined as

$$\lambda = \lambda_{mn} = \frac{z_{mn}}{a}$$

The values of the zeros of $J_m(x)$ (i.e. $J_m(z_{mn}) = 0$) are tabulated.

10.5.1 Frobenius solution to the Bessel equation

$$x^2 y''(x) + xy'(x) + (x^2 - p^2)y(x) = 0$$

Let's consider the series solution corresponding to the index $s_1 = p$:

$$\begin{aligned} y(x) &= x^p \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+p} \\ y'(x) &= \sum (n+p) a_n x^{n+p-1} \\ y''(x) &= \sum (n+p-1)(n+p) a_n x^{n+p-2}. \end{aligned}$$

Substituting these into the Bessel equation implies

$$\sum_{n=0}^{\infty} [(n+p-1)(n+p) + (n+p) - p^2] a_n x^{n+p} + \sum_{n=0}^{\infty} a_n x^{n+p+2} = 0$$

We change the dummy index in the second sum to $n + 2 = m$

$$\sum_{n=0}^{\infty} a_n x^{n+p+2} = \sum_{m=2}^{\infty} a_{m-2} x^{m+p} = \sum_{n=2}^{\infty} a_{n-2} x^{n+p}.$$

So the equation becomes

$$\sum_{n=2}^{\infty} \{[(n+p)^2 - p^2] a_n + a_{n-2}\} x^{n+p} + 0 \cdot a_0 + (2p+1) a_1 x^{1+p} = 0$$

Therefore we have, by equating the coefficient of each power of x to zero:

$$a_0 = \text{arbitrary}, \quad a_1 = 0$$

and

$$a_n = -\frac{a_{n-2}}{n(2p+n)}, \quad n = 2, 3, 4, \ldots.$$

All $a_n = 0$ for n odd.

$$\begin{aligned} a_2 &= -\frac{a_0}{2(2p+2)} \\ a_4 &= -\frac{a_2}{4(2p+4)} = \frac{a_0}{2 \cdot 4(2p+2)(2p+4)} \\ a_6 &= -\frac{a_4}{6(2p+6)} = -\frac{a_0}{2 \cdot 4 \cdot 6(2p+2)(2p+4)(2p+6)}, \end{aligned}$$

$$\begin{aligned}
y(x) &= a_0 x^p [1 - \frac{x^2}{2^2(p+1)} + \frac{x^4}{2^4 2!(p+1)(p+2)} \\
&\quad - \frac{x^6}{2^6 3!(p+1)(p+2)(p+3)} + \ldots] \\
&= a_0 x^p \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} n!(p+1)\ldots(p+n)}
\end{aligned}$$

The Bessel function of the first kind of order p, denoted by $J_p(x)$, is defined with $a_0 = \frac{1}{2^p \Gamma(p+1)}$

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{n!\Gamma(p+1+n)},$$

where the gamma function is defined by: $\Gamma(z+1) = z\Gamma(z)$, and so $\Gamma(1+p) \cdot [(1+p)(2+p)\ldots(n+p)] = \Gamma(2+p) \cdot [(2+p)\ldots(n+p)] = \Gamma(3+p) \cdot [(3+p)\ldots(n+p)] = \ldots = \Gamma(n+p+1)$. For $p = m$, a positive integer, $\Gamma(m+1+n) = (n+m)!$. For $p = -m$, a negative integer, $1/\Gamma(n+1-m) = 0$, for $n > m$.

10.5.2 Some identities

The Bessel function $J_p(x)$ is defined for any real number p by

$$\boxed{J_p(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n+p}}{n!\Gamma(p+1+n)}}.$$

Some useful identities follow from this definition:

$$\begin{aligned}
\frac{d}{dx}[x^p J_p(x)] &= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2p}}{2^{2n+p} n!\Gamma(p+1+n)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2p-1}}{2^{2n+p-1} n!\Gamma(p+n)} \\
&= x^p \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p-1}}{n!\Gamma(p+n)} \\
&= x^p J_{p-1}(x).
\end{aligned}$$

Therefore we have the identity:

$$\boxed{\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x)}$$

Similarly, we can show that

$$\boxed{\frac{d}{dx}[x^{-p}J_p(x)] = -x^{-p}J_{p+1}(x)}$$

Combining the last two formula, we have

$$2\frac{d}{dx}J_p(x) = J_{p-1}(x) - J_{p+1}(x).$$

10.5.3 Linear independence

If p is not an integer, a second independent solution to the Bessel's equation is given by

$$J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n-p}}{n!\Gamma(-p+1+n)}.$$

When $p = m$ an integer, howeever, $J_m(x)$ and $J_{-m}(x)$ cease to be linearly independent because it can be shown that

$$J_{-m}(x) = (-1)^m J_m(x), \quad m \text{ integer} \geq 0.$$

To show this, we start with

$$J_{-m}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n-m}}{n!\Gamma(n-m+1)}$$

and note that for the factorial $1/\Gamma(n-m+1) = 0$ if $n < m$. $\Gamma(n-m+1) = (n-m)!$ for $n > m$. Thus

$$J_{-m}(x) = \sum_{n=m}^{\infty} \frac{(-1)^n (x/2)^{2n-m}}{n!(n-m)!}$$

Let $k = n - m$,

$$\begin{aligned} J_{-m}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^{k+m} (x/2)^{2k+m}}{(k+m)!k!} = (-1)^m \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+m}}{n!(n+m)!} \\ &= (-1)^m J_m(x). \end{aligned}$$

Since $J_m(x)$ and $J_{-m}(x)$ are not linearly independent, the general solution of the Bessel's equation should be

$$y(x) = AJ_p(x) + BJ_{-p}(x) \text{ if } p \text{ is not an integer}$$

but

$$y(x) = AJ_p(x) + B(Y_m(x) \text{ if } p = m \text{ an integer}$$

10.5.4 Generating function

We multiply $J_m(z)$ by t^m for another variable t, and form the following infinite series:

$$\sum_{m=-\infty}^{\infty} J_m(z)t^m \equiv \varphi(z,t)$$

We called this series $\varphi(z,t)$ above. We note that $\varphi(0,t) = 1$. Assuming that this series converges, we can differentiate it

$$\begin{aligned}\frac{\partial}{\partial z}\varphi(z,t) &= \sum_{m=-\infty}^{\infty} J'_m(z)t^m \\ &= \sum_{m=-\infty}^{\infty} \frac{1}{2}(J_{m-1}(z) - J_{m+1}(z_1)t^m,\end{aligned}$$

using the formula derived in 10.5.2 for the derivative.

We can rewrite the series by letting $n = m-1$ in the first and $n = m+1$ in the second, to yield

$$\frac{\partial}{\partial z}\varphi(z,t) = \sum_{n=-\infty}^{\infty} \frac{1}{2}(t - \frac{1}{t})J_n(z)t^n.$$

So the function $\varphi(z,t)$ satisfies the ordinary differential equation

$$\frac{\partial}{\partial z}\varphi(z,t) = \frac{1}{2}(t - \frac{1}{t})\varphi(z,t).$$

Treating t as a "constant" as far as the z-derivative is concerned, we find the solution to this first order ordinary differential equation as an exponential:

$$\varphi(z,t) = f(t)e^{\frac{1}{2}(t-1/t)z},$$

where $f(t)$ is an arbitrary "constant". Setting $z = 0$ we see that $f(t) = 1$.

Finally we obtained the so-called generating function for Bessel functions:

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(z)t^n$$

Some useful forms can also be derived. We let $t = e^{i\theta}$, $-\pi \le \theta \le \pi$:

$$e^{z(e^{i\theta}-e^{-i\theta})/2} = \sum_{n=-\infty}^{\infty} J_n(z)e^{in\theta}$$

Therefore

$$e^{iz\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(z)e^{in\theta}, \quad -\pi \le \theta \le \theta$$

which is in the form of a Fourier series. So $J_n(z)$ must be the Fourier series coefficient c_n of the function $e^{iz\sin\theta}$

$$\begin{aligned} J_n(z) = c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\sin\theta}e^{-in\theta}d\theta \\ &= \frac{1}{2\pi}\left[\int_0^{\pi} + \int_{-\pi}^{0}\right] \\ &= \frac{1}{2\pi}\int_0^{\pi}[e^{-i(n\theta - z\sin\theta)} + e^{i(n\theta - z\sin\theta)}]d\theta \end{aligned}$$

after making the switch of θ to $-\theta$ in the second integral. Finally,

$$\boxed{J_n(z) = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - z\sin\theta)d\theta,}$$

which is called Bessel's integral.

An immediate consequence of this formula is that

$$|J_n(x)| \le 1 \text{ for all real } x$$

10.5.5 Qualitative properties of Bessel functions

We have enough information now to derive some asymptotic results for the Bessel functions. For large x, $J_m(x)$ and $Ym(x)$ are like cosines and sines, except with a decaying amplitude (we will show in a moment):

$$\begin{aligned} J_m(x) &\sim \sqrt{\frac{2}{\pi x}}\cos(x - \frac{\pi}{4} - m\pi/2) \\ Y_m(x) &\sim \sqrt{\frac{2}{\pi x}}\sin(x - \frac{\pi}{4} - m\pi/2), \quad \text{for } |x| \gg 1 \end{aligned}$$

From this expression, we know that the kth zero of $J_m(x)$ is, approximately

$$z_{mk} \cong m\pi/2 - \pi/4 + k\pi.$$

For small x, we know already (from the Frobenius solution)

$$J_0(x) \cong 1, \quad Y_0(x) \cong \frac{2}{\pi}\ell n x$$
$$J_1(x) \cong \frac{1}{2}x, \quad Y_1(x) \cong -\frac{2}{\pi}x^{-1}$$
$$J_2(x) \cong \frac{1}{8}x^2, \quad Y_2(x) \cong -\frac{\pi}{4}x^{-2}, \qquad \text{for } |x| \ll 1$$

These are sufficient for us to obtain a sketch of these functions.

Method of stationary phase

To obtain the asymptotic behavior of the Bessel functions for large values of x, we start with the integral representation of $J_m(x)$:

$$J_m(x) = \frac{1}{\pi}\int_0^\pi \cos\phi(\theta)d\theta,$$

where $\phi(\theta) \equiv x\sin\theta - m\theta$ is the phase of the oscillation in the integrand. When x is large, the oscillation is rapid, with positive and negative values of $\cos\phi$ cancels themselves out when integrated. The only place where there is no cancellation is where $\phi(\theta)$ does not vary with θ. This occurs at *stationary phase* points, θ_0, given by

$$\phi'(\theta_0) = x\cos\theta_0 - m = 0.$$

This is

$$\cos\theta_0 = m/x \cong 0, \quad \text{for large } x.$$

Therefore

$$\theta_0 \cong \pi/2, \quad \phi(\theta_0) = x - m\pi/2$$

since we are only interested in values of θ_0 inside $0 < \theta < \pi$ of the integral. In the neighborhood of this point, $\phi(\theta)$ can be approximated in a Taylor series by:

$$\phi(\theta) \cong \phi(\theta_0) + \frac{1}{2}\phi''(\theta_0)(\theta - \theta_0)^2,$$

where

$$\phi''(\theta_0) = -x\sin\theta_0 \cong -x$$

The integral is approximated by

$$J_m(x) \cong \frac{1}{\pi}\int_0^\pi \cos(\phi(\theta_0) + \frac{1}{2}\phi''(\theta_0)(\theta-\theta_0)^2)d\theta$$
$$\cong \frac{1}{\pi}\int_{-\infty}^\infty \cos(\phi(\theta_0) + \frac{1}{2}\phi''(\theta_0)(\theta-\theta_0)^2)d\theta$$

Since the integrand does not contribute much to the integral for θ away from $\theta_0 \cong \pi/2$, we are making only a small error by extending the integral to $\pm\infty$. This latter integral can be performed exactly, since from table of integrals:

$$\int_0^\infty \sin ax^2 dx = \int_0^\infty \cos ax^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2|a|}}, \quad a > 0.$$

$$\begin{aligned}
J_m(x) &\cong \frac{1}{\pi}\int_{-\infty}^{\infty} \cos(\phi(\theta_0) + \frac{1}{2}\phi''(\theta_0)\theta^2) d\theta \\
&= \frac{2}{\pi}\int_0^\infty \cos(\phi(\theta_0) + \frac{1}{2}\phi''(\theta_0)\theta^2) d\theta \\
&= \frac{2}{\pi}\int_0^\infty \cos(\phi(\theta_0)) \cos(\frac{1}{2}|\phi''(\theta_0)|\theta^2) d\theta \\
&\quad + \frac{2}{\pi}\int_0^\infty \sin(\phi(\theta_0)) \sin(\frac{1}{2}|\phi''(\theta_0)|\theta^2) d\theta \\
&= \sqrt{\frac{1}{|\phi''(\theta_0)|\pi}}[\cos(\phi(\theta_0)) + \sin(\phi(\theta_0))] \\
&= \sqrt{\frac{2}{\pi|\phi''(\theta_0)|}} \cos[\phi(\theta_0) - \frac{\pi}{4}] \\
&= \sqrt{\frac{2}{\pi x}} \cos(x - m\pi/2 - \pi/4).
\end{aligned}$$