MATRICES, BLOCK STRUCTURES AND GAUSSIAN ELIMINATION

Numerical linear algebra lies at the heart of modern scientific computing and computational science. Today it is not uncommon to perform numerical computations with matrices having millions of components. The key to understanding how to implement such algorithms is to exploit underlying structure within the matrices. In these notes we touch on a few ideas and tools for dissecting matrix structure. Specifically we are concerned with the *block structure* matrices.

1. Rows and Columns

Let $A \in \mathbb{R}^{m\times n}$ so that A has m rows and n columns. Denote the element of A in the ith row and jth column as A_{ij}. Denote the m rows of A by $A_{1\cdot}, A_{2\cdot}, A_{3\cdot}, \ldots, A_{m\cdot}$ and the n columns of A by $A_{\cdot 1}, A_{\cdot 2}, A_{\cdot 3}, \ldots, A_{\cdot n}$. For example, if

$$A = \begin{bmatrix} 3 & 2 & -1 & 5 & 7 & 3 \\ -2 & 27 & 32 & -100 & 0 & 0 \\ -89 & 0 & 47 & 22 & -21 & 33 \end{bmatrix},$$

then $A_{2,4} = -100$,

$$A_{1\cdot} = \begin{bmatrix} 3 & 2 & -1 & 5 & 7 & 3 \end{bmatrix},\ A_{2\cdot} = \begin{bmatrix} -2 & 27 & 32 & -100 & 0 & 0 \end{bmatrix},\ A_{3\cdot} = \begin{bmatrix} -89 & 0 & 47 & 22 & -21 & 33 \end{bmatrix}$$

and

$$A_{\cdot 1} = \begin{bmatrix} 3 \\ -2 \\ -89 \end{bmatrix},\ A_{\cdot 2} = \begin{bmatrix} 2 \\ 27 \\ 0 \end{bmatrix},\ A_{\cdot 3} = \begin{bmatrix} -1 \\ 32 \\ 47 \end{bmatrix},\ A_{\cdot 4} = \begin{bmatrix} 5 \\ -100 \\ 22 \end{bmatrix},\ A_{\cdot 5} = \begin{bmatrix} 7 \\ 0 \\ -21 \end{bmatrix},\ A_{\cdot 6} = \begin{bmatrix} 3 \\ 0 \\ 33 \end{bmatrix}.$$

Exercise 1.1. *If*

$$C = \begin{bmatrix} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix},$$

what are $C_{4,4}$, $C_{\cdot 4}$ and $C_{4\cdot}$? For example, $C_{2\cdot} = \begin{bmatrix} 2 & 2 & 0 & 0 & 1 & 0 \end{bmatrix}$ *and* $C_{\cdot 2} = \begin{bmatrix} -4 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

The block structuring of a matrix into its rows and columns is of fundamental importance and is extremely useful in understanding the properties of a matrix. In particular, for $A \in \mathbb{R}^{m\times n}$ it allows us to write

$$A = \begin{bmatrix} A_{1\cdot} \\ A_{2\cdot} \\ A_{3\cdot} \\ \vdots \\ A_{m\cdot} \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} A_{\cdot 1} & A_{\cdot 2} & A_{\cdot 3} & \ldots & A_{\cdot n} \end{bmatrix}.$$

These are called the row and column block representations of A, respectively

1.1. *Matrix vector Multiplication.* Let $A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$. In terms of its coordinates (or components), we can also write $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ with each $x_j \in \mathbb{R}$. The term x_j is called the jth component of x. For example if

$$x = \begin{bmatrix} 5 \\ -100 \\ 22 \end{bmatrix},$$

then $n = 3,\ x_1 = 5,\ x_2 = -100,\ x_3 = 22$. We define the matrix-vector product Ax by

$$Ax = \begin{bmatrix} A_{1\cdot} \bullet x \\ A_{2\cdot} \bullet x \\ A_{3\cdot} \bullet x \\ \vdots \\ A_{m\cdot} \bullet x \end{bmatrix},$$

where for each $i = 1, 2, \ldots, m$, $A_{i\cdot} \bullet x$ is the dot product of the ith row of A with x and is given by

$$A_{i\cdot} \bullet x = \sum_{j=1}^{n} A_{ij} x_j \ .$$

For example, if

$$A = \begin{bmatrix} 3 & 2 & -1 & 5 & 7 & 3 \\ -2 & 27 & 32 & -100 & 0 & 0 \\ -89 & 0 & 47 & 22 & -21 & 33 \end{bmatrix} \text{ and } x = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 2 \\ 3 \end{bmatrix},$$

then

$$Ax = \begin{bmatrix} 24 \\ -29 \\ -32 \end{bmatrix}.$$

Exercise 1.2. *If*

$$C = \begin{bmatrix} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix} \textit{ and } x = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 2 \\ 3 \end{bmatrix},$$

what is Cx?

Note that if $A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$, then Ax is always well defined with $Ax \in \mathbb{R}^m$. In terms of components, the ith component of Ax is given by the dot product of the ith row of A (i.e. $A_{i\cdot}$) and x (i.e. $A_{i\cdot} \bullet x$).

The view of the matrix-vector product described above is the *row-space* perspective, where the term *row-space* will be given a more rigorous definition at a later time. But there is a very different way of viewing the matrix-vector product based on a *column-space* perspective. This view uses the notion of the linear combination of a collection of vectors.

Given k vectors $v^1, v^2, \ldots, v^k \in \mathbb{R}^n$ and k scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$, we can form the vector

$$\alpha_1 v^1 + \alpha_2 v^2 + \cdots + \alpha_k v^k \in \mathbb{R}^n \ .$$

Any vector of this kind is said to be a *linear combination* of the vectors $v^1, v^2, \ldots, v^k$ where the $\alpha_1, \alpha_2, \ldots, \alpha_k$ are called the coefficients in the linear combination. The set of all such vectors formed as linear combinations of $v^1, v^2, \ldots, v^k$ is said to be the *linear span* of $v^1, v^2, \ldots, v^k$ and is denoted

$$\operatorname{span}\left(v^1, v^2, \ldots, v^k\right) := \left\{\alpha_1 v^1 + \alpha_2 v^2 + \cdots + \alpha_k v^k \,\middle|\, \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}\right\}.$$

Returning to the matrix-vector product, one has that

$$Ax = \begin{bmatrix} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \cdots + A_{1n}x_n \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + \cdots + A_{2n}x_n \\ \vdots \qquad \vdots \qquad \vdots \\ A_{m1}x_1 + A_{m2}x_2 + A_{m3}x_3 + \cdots + A_{mn}x_n \end{bmatrix} = x_1 A_{\cdot 1} + x_2 A_{\cdot 2} + x_3 A_{\cdot 3} + \cdots + x_n A_{\cdot n},$$

which is a linear combination of the columns of A. That is, we can view the matrix-vector product Ax as taking a linear combination of the columns of A where the coefficients in the linear combination are the coordinates of the vector x.

We now have two fundamentally different ways of viewing the matrix-vector product Ax.

Row-Space view of Ax:

$$Ax = \begin{bmatrix} A_{1\cdot} \bullet x \\ A_{2\cdot} \bullet x \\ A_{3\cdot} \bullet x \\ \vdots \\ A_{m\cdot} \bullet x \end{bmatrix}$$

Column-Space view of Ax:

$$Ax = x_1 A_{\cdot 1} + x_2 A_{\cdot 2} + x_3 A_{\cdot 3} + \cdots + x_n A_{\cdot n} \ .$$

2. Matrix Multiplication

We now build on our notion of a matrix-vector product to define a notion of a matrix-matrix product which we call *matrix multiplication*. Given two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times k}$ note that each of the columns of B resides in $\mathbb{R}^n$, i.e. $B_{\cdot j} \in \mathbb{R}^n$ $i = 1, 2, \ldots, k$. Therefore, each of the matrix-vector products $AB_{\cdot j}$ is well defined for $j = 1, 2, \ldots, k$. This allows us to define a matrix-matrix product that exploits the block column structure of B by setting

$$AB := \begin{bmatrix} AB_{\cdot 1} & AB_{\cdot 2} & AB_{\cdot 3} & \cdots & AB_{\cdot k} \end{bmatrix} . \tag{1}$$

Note that the jth column of AB is $(AB)_{\cdot j} = AB_{\cdot j} \in \mathbb{R}^m$ and that $AB \in \mathbb{R}^{m\times k}$, i.e.

if $H \in \mathbb{R}^{m\times n}$ and $L \in \mathbb{R}^{n\times k}$, then $HL \in \mathbb{R}^{m\times k}$.

Also note that

if $T \in \mathbb{R}^{s\times t}$ and $M \in \mathbb{R}^{r\times \ell}$, then the matrix product TM is only defined when $t = r$.

For example, if

$$A = \begin{bmatrix} 3 & 2 & -1 & 5 & 7 & 3 \\ -2 & 27 & 32 & -100 & 0 & 0 \\ -89 & 0 & 47 & 22 & -21 & 33 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ -2 & 2 \\ 0 & 3 \\ 0 & 0 \\ 1 & 1 \\ 2 & -1 \end{bmatrix},$$

then

$$AB = \begin{bmatrix} A\begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} & A\begin{bmatrix} 0 \\ -2 \\ 3 \\ 0 \\ 1 \\ -1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -58 & 150 \\ -133 & 87 \end{bmatrix}.$$

Exercise 2.1. *if*

$$C = \begin{bmatrix} 3 & -4 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} -1 & 0 & 2 & 4 & 3 \\ 0 & -2 & -1 & 4 & 5 \\ 5 & 2 & -4 & 1 & 1 \\ 3 & 0 & 1 & 0 & 0 \end{bmatrix},$$

is CD well defined and if so what is it?

The formula (1) can be used to give further insight into the individual components of the matrix product AB. By the definition of the matrix-vector product we have for each $j = 1, 2, \ldots, k$

$$AB_{\cdot j} = \begin{bmatrix} A_{1\cdot} \bullet B_{\cdot j} \\ A_{2\cdot} \bullet B_{\cdot j} \\ A_{m\cdot} \bullet B_{\cdot j} \end{bmatrix}.$$

Consequently,

$$(AB)_{ij} = A_{i\cdot} \bullet B_{\cdot j} \quad \forall\, i = 1, 2, \ldots m,\ j = 1, 2, \ldots, k.$$

That is, the element of AB in the ith row and jth column, $(AB)_{ij}$, is the dot product of the ith row of A with the jth column of B.

2.1. **Elementary Matrices.** We define the *elementary unit coordinate matrices* in $\mathbb{R}^{m\times n}$ in much the same way as we define the elementary unit coordinate vectors. Given $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$, the elementary unit coordinate matrix $E_{ij} \in \mathbb{R}^{m\times n}$ is the matrix whose ij entry is 1 with all other entries taking the value zero. This is a slight abuse of notation since the notation E_{ij} is supposed to represent the ijth entry in the matrix E. To avoid confusion, we reserve the use of the letter E when speaking of matrices to the elementary matrices.

Exercise 2.2. *(Multiplication of square elementary matrices)Let $i, k \in \{1, 2, \ldots, m\}$ and $j, \ell \in \{1, 2, \ldots, m\}$. Show the following for elementary matrices in $\mathbb{R}^{m\times m}$ first for $m = 3$ and then in general.*

(1) $E_{ij}E_{k\ell} = \begin{cases} E_{i\ell} & \text{, if } j = k, \\ 0 & \text{, otherwise.} \end{cases}$

(2) For any $\alpha \in \mathbb{R}$, if $i \neq j$, then $(I_{m\times m} - \alpha E_{ij})(I_{m\times m} + \alpha E_{ij}) = I_{m\times m}$ so that

$$(I_{m\times m} + \alpha E_{ij})^{-1} = (I_{m\times m} - \alpha E_{ij}).$$

(3) For any $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, $(I + (\alpha^{-1} - 1)E_{ii})(I + (\alpha - 1)E_{ii}) = I$ so that

$$(I + (\alpha - 1)E_{ii})^{-1} = (I + (\alpha^{-1} - 1)E_{ii}).$$

Exercise 2.3. *(Elementary permutation matrices)Let $i, \ell \in \{1, 2, \ldots, m\}$ and consider the matrix $P_{ij} \in \mathbb{R}^{m\times m}$ obtained from the identity matrix by interchanging its i and ℓth rows. We call such a matrix an elementary permutation matrix. Again we are abusing notation, but again we reserve the letter P for permutation matrices (and, later, for projection matrices). Show the following are true first for $m = 3$ and then in general.*

(1) $P_{i\ell}P_{i\ell} = I_{m\times m}$ so that $P_{i\ell}^{-1} = P_{i\ell}$.
(2) $P_{i\ell}^T = P_{i\ell}$.
(3) $P_{i\ell} = I - E_{ii} - E_{\ell\ell} + E_{i\ell} + E_{\ell i}$.

Exercise 2.4. *(Three elementary row operations as matrix multiplication)In this exercise we show that the three elementary row operations can be performed by left multiplication by an invertible matrix. Let $A \in \mathbb{R}^{m\times n}$, $\alpha \in \mathbb{R}$ and let $i, \ell \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$. Show that the following results hold first for $m = n = 3$ and then in general.*

(1) (row interchanges) Given $A \in \mathbb{R}^{m\times n}$, the matrix $P_{ij}A$ is the same as the matrix A except with the i and jth rows interchanged.
(2) (row multiplication) Given $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, show that the matrix $(I + (\alpha - 1)E_{ii})A$ is the same as the matrix A except with the ith row replaced by α times the ith row of A.
(3) Show that matrix $E_{ij}A$ is the matrix that contains the jth row of A in its ith row with all other entries equal to zero.
(4) (replace a row by itself plus a multiple of another row) Given $\alpha \in \mathbb{R}$ and $i \neq j$, show that the matrix $(I + \alpha E_{ij})A$ is the same as the matrix A except with the ith row replaced by itself plus α times the jth row of A.

2.2. **Associativity of matrix multiplication.** Note that the definition of matrix multiplication tells us that this operation is associative. That is, if $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times k}$, and $C \in \mathbb{R}^{k\times s}$, then $AB \in \mathbb{R}^{m\times k}$ so that $(AB)C$ is well defined and $BC \in \mathbb{R}^{n\times s}$ so that $A(BC)$ is well defined, and, moreover,

$$(AB)C = \begin{bmatrix}(AB)C_{\cdot 1} & (AB)C_{\cdot 2} & \cdots & (AB)C_{\cdot s}\end{bmatrix} \tag{2}$$

where for each $\ell = 1, 2, \ldots, s$

$$\begin{aligned}(AB)C_{\cdot\ell} &= \begin{bmatrix}AB_{\cdot 1} & AB_{\cdot 2} & AB_{\cdot 3} & \cdots & AB_{\cdot k}\end{bmatrix} C_{\cdot\ell} \\ &= C_{1\ell}AB_{\cdot 1} + C_{2\ell}AB_{\cdot 2} + \cdots + C_{k\ell}AB_{\cdot k} \\ &= A\left[C_{1\ell}B_{\cdot 1} + C_{2\ell}B_{\cdot 2} + \cdots + C_{k\ell}B_{\cdot k}\right] \\ &= A(BC_{\cdot\ell})\ .\end{aligned}$$

Therefore, we may write (2) as

$$\begin{aligned}(AB)C &= \begin{bmatrix}(AB)C_{\cdot 1} & (AB)C_{\cdot 2} & \cdots & (AB)C_{\cdot s}\end{bmatrix} \\ &= \begin{bmatrix}A(BC_{\cdot 1}) & A(BC_{\cdot 2}) & \ldots & A(BC_{\cdot s})\end{bmatrix} \\ &= A\begin{bmatrix}BC_{\cdot 1} & BC_{\cdot 2} & \ldots & BC_{\cdot s}\end{bmatrix} \\ &= A(BC)\ .\end{aligned}$$

Due to this associativity property, we may dispense with the parentheses and simply write ABC for this triple matrix product. Obviously longer products are possible.

Exercise 2.5. *Consider the following matrices:*

$$A = \begin{bmatrix}2 & 3 & 1\\ 1 & 0 & -3\end{bmatrix} \quad B = \begin{bmatrix}4 & -1\\ 0 & -7\end{bmatrix} \quad C = \begin{bmatrix}-2 & 3 & 2\\ 1 & 1 & -3\\ 2 & 1 & 0\end{bmatrix}$$

$$D = \begin{bmatrix}2 & 3\\ 1 & 0\\ 8 & -5\end{bmatrix} \quad F = \begin{bmatrix}2 & 1 & 1 & 2\\ 1 & 0 & -4 & 0\\ 3 & 0 & -2 & 0\\ 5 & 1 & 1 & 1\end{bmatrix} \quad G = \begin{bmatrix}2 & 3 & 1 & -2\\ 1 & 0 & -3 & 0\end{bmatrix}\ .$$

Using these matrices, which pairs can be multiplied together and in what order? Which triples can be multiplied together and in what order (e.g. the triple product BAC is well defined)? Which quadruples can be multiplied together and in what order? Perform all of these multiplications.

3. Block Matrix Multiplication

To illustrate the general idea of block structures consider the following matrix.

$$A = \begin{bmatrix}3 & -4 & 1 & 1 & 0 & 0\\ 0 & 2 & 2 & 0 & 1 & 0\\ 1 & 0 & -1 & 0 & 0 & 1\\ 0 & 0 & 0 & 2 & 1 & 4\\ 0 & 0 & 0 & 1 & 0 & 3\end{bmatrix}\ .$$

Visual inspection tells us that this matrix has structure. But what is it, and how can it be represented? We re-write the the matrix given above *blocking* out some key structures:

$$A = \left[\begin{array}{ccc|ccc}3 & -4 & 1 & 1 & 0 & 0\\ 0 & 2 & 2 & 0 & 1 & 0\\ 1 & 0 & -1 & 0 & 0 & 1\\ \hline 0 & 0 & 0 & 2 & 1 & 4\\ 0 & 0 & 0 & 1 & 0 & 3\end{array}\right] = \left[\begin{array}{c|c}B & I_{3\times 3}\\ \hline 0_{2\times 3} & C\end{array}\right]\ ,$$

where

$$B = \begin{bmatrix}3 & -4 & 1\\ 0 & 2 & 2\\ 1 & 0 & -1\end{bmatrix},\quad C = \begin{bmatrix}2 & 1 & 4\\ 1 & 0 & 3\end{bmatrix}\ ,$$

$I_{3\times 3}$ is the 3×3 identity matrix, and $0_{2\times 3}$ is the 2×3 zero matrix. Having established this structure for the matrix A, it can now be exploited in various ways. As a simple example, we consider how it can be used in matrix multiplication.

Consider the matrix

$$M = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ -1 & -1 \\ 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{bmatrix}.$$

The matrix product AM is well defined since A is 5×6 and M is 6×2. We show how to compute this matrix product using the structure of A. To do this we must first *block decompose M conformally with the block decomposition of A.* Another way to say this is that we must give M a block structure that allows us to do block matrix multiplication with the blocks of A. The correct block structure for M is

$$M = \begin{bmatrix} X \\ Y \end{bmatrix},$$

where

$$X = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ -1 & -1 \end{bmatrix}, \quad \text{and} \quad Y = \begin{bmatrix} 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{bmatrix},$$

since then X can multiply $\begin{bmatrix} B \\ 0_{2\times 3} \end{bmatrix}$ and Y can multiply $\begin{bmatrix} I_{3\times 3} \\ C \end{bmatrix}$. This gives

$$\begin{aligned} AM &= \begin{bmatrix} B & I_{3\times 3} \\ 0_{2\times 3} & C \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} BX + Y \\ CY \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 2 & -11 \\ 2 & 12 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} -2 & 6 \\ 4 & 3 \\ -2 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 4 & -12 \\ 2 & 9 \\ 0 & 3 \\ 0 & 1 \\ -4 & -1 \end{bmatrix}. \end{aligned}$$

Block structured matrices and their matrix product is a very powerful tool in matrix analysis. Consider the matrices $M \in \mathbb{R}^{n\times m}$ and $T \in \mathbb{R}^{m\times k}$ given by

$$M = \begin{bmatrix} A_{n_1\times m_1} & B_{n_1\times m_2} \\ C_{n_2\times m_1} & D_{n_2\times m_2} \end{bmatrix}$$

and

$$T = \begin{bmatrix} E_{m_1\times k_1} & F_{m_1\times k_2} & G_{m_1\times k_3} \\ H_{m_2\times k_1} & J_{m_2\times k_2} & K_{m_2\times k_3} \end{bmatrix},$$

where $n = n_1 + n_2$, $m = m_1 + m_2$, and $k = k_1 + k_2 + k_3$. The block structures for the matrices M and T are said to be *conformal* with respect to matrix multiplication since

$$MT = \begin{bmatrix} AE + BH & AF + BJ & AG + BK \\ CE + DH & CF + DJ & CG + DK \end{bmatrix}.$$

Similarly, one can conformally block structure matrices with respect to matrix addition (how is this done?).

Exercise 3.1. *Consider the the matrix*

$$H = \begin{bmatrix} -2 & 3 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & -3 & 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -7 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 8 & -5 \end{bmatrix}.$$

Does H have a natural block structure that might be useful in performing a matrix-matrix multiply, and if so describe it by giving the blocks? Describe a conformal block decomposition of the matrix

$$M = \begin{bmatrix} 1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 1 & -2 \\ -3 & 4 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

that would be useful in performing the matrix product HM. Compute the matrix product HM using this conformal decomposition.

Exercise 3.2. *Let $T \in \mathbb{R}^{m\times n}$ with $T \neq 0$ and let I be the $m \times m$ identity matrix. Consider the block structured matrix $A = [\, I \quad T\,]$.*

(i) If $A \in \mathbb{R}^{k\times s}$, what are k and s?
(ii) Construct a non-zero $s \times n$ matrix B such that $AB = 0$.

The examples given above illustrate how block matrix multiplication works and why it might be useful. One of the most powerful uses of block structures is in understanding and implementing standard *matrix factorizations* or reductions.

4. Gauss-Jordan Elimination Matrices and Reduction to Reduced Echelon Form

In this section, we show that Gaussian-Jordan elimination can be represented as a consequence of left multiplication by a specially designed matrix called a *Gaussian-Jordan elimination matrix.*

Consider the vector $v \in \mathbb{R}^m$ block decomposed as

$$v = \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix}$$

where $a \in \mathbb{R}^s$, $\alpha \in \mathbb{R}$, and $b \in \mathbb{R}^t$ with $m = s + 1 + t$. In this vector we refer to the α entry as the *pivot* and assume that $\alpha \neq 0$. We wish to determine a matrix G such that

$$Gv = e_{s+1}$$

where for $j = 1, \ldots, n$, e_j is the unit coordinate vector having a one in the jth position and zeros elsewhere. We claim that the matrix

$$G = \begin{bmatrix} I_{s\times s} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{t\times t} \end{bmatrix}$$

does the trick. Indeed,

$$Gv = \begin{bmatrix} I_{s\times s} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{t\times t} \end{bmatrix} \begin{pmatrix} a \\ \alpha \\ b \end{pmatrix} = \begin{bmatrix} a - a \\ \alpha^{-1}\alpha \\ -b + b \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e_{s+1}. \tag{3}$$

The matrix G is called a *Gaussian-Jordan Elimination Matrix*, or GJEM for short. Note that G is invertible since

$$G^{-1} = \begin{bmatrix} I & a & 0 \\ 0 & \alpha & 0 \\ 0 & b & I \end{bmatrix},$$

Moreover, for any vector of the form $w = \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}$ where $x \in \mathbb{R}^s$ $y \in \mathbb{R}^t$, we have

$$Gw = w.$$

The GJEM matrices perform precisely the operations required in order to execute Gauss-Jordan elimination. That is, each elimination step can be realized as left multiplication of the augmented matrix by the appropriate GJEM.

For example, consider the linear system

$$\begin{array}{ccccccc} 2x_1 & + & x_2 & + & 3x_3 & = & 5 \\ 2x_1 & + & 2x_2 & + & 4x_3 & = & 8 \\ 4x_1 & + & 2x_2 & + & 7x_3 & = & 11 \\ 5x_1 & + & 3x_2 & + & 4x_3 & = & 10 \end{array}$$

and its associated augmented matrix

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 2 & 2 & 4 & 8 \\ 4 & 2 & 7 & 11 \\ 5 & 3 & 4 & 10 \end{bmatrix}.$$

The first step of Gauss-Jordan elimination is to transform the first column of this augmented matrix into the first unit coordinate vector. The procedure described in (3) can be employed for this purpose. In this case the pivot is the $(1,1)$ entry of the augmented matrix and so

$$s = 0, \ a \text{ is void}, \ \alpha = 2, \ t = 3, \text{ and } b = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix},$$

which gives

$$G_1 = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -5/2 & 0 & 0 & 1 \end{bmatrix}.$$

Multiplying these two matrices gives

$$G_1 A = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -5/2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 & 5 \\ 2 & 2 & 4 & 8 \\ 4 & 2 & 7 & 11 \\ 5 & 3 & 4 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 3/2 & 5/2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1/2 & -7/2 & -5/2 \end{bmatrix}.$$

We now repeat this process to transform the second column of this matrix into the second unit coordinate vector. In this case the $(2,2)$ position becomes the pivot so that

$$s = 1, \ a = 1/2, \ \alpha = 1, \ t = 2, \text{ and } b = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

yielding

$$G_2 = \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1/2 & 0 & 1 \end{bmatrix}.$$

Again, multiplying these two matrices gives

$$G_2 G_1 A = \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 3/2 & 5/2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1/2 & -7/2 & -5/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -4 & -4 \end{bmatrix}.$$

Repeating the process on the third column transforms it into the third unit coordinate vector. In this case the pivot is the $(3,3)$ entry so that

$$s = 2,\ a = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \alpha = 1,\ t = 1,\ \text{and } b = -4$$

yielding

$$G_3 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix}.$$

Multiplying these matrices gives

$$G_3G_2G_1A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -4 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is in reduced echelon form. Therefore the system is consistent and the unique solution is

$$x = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

Observe that

$$G_3G_2G_1 = \begin{bmatrix} 3 & -1/2 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ -2 & 0 & 1 & 0 \\ -10 & -1/2 & 4 & 1 \end{bmatrix}$$

and that

$$\begin{aligned} (G_3G_2G_1)^{-1} &= G_1^{-1}G_2^{-1}G_3^{-1} \\ &= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & 3 & 0 \\ 2 & 2 & 4 & 0 \\ 4 & 2 & 7 & 0 \\ 5 & 3 & 4 & 1 \end{bmatrix}. \end{aligned}$$

In particular, reduced Gauss-Jordan form can always be achieved by multiplying the augmented matrix on the left by an invertible matrix which can be written as a product of Gauss-Jordan elimination matrices.

Exercise 4.1. *What are the Gauss-Jordan elimination matrices that transform the vector* $\begin{bmatrix} 2 \\ 3 \\ -2 \\ 5 \end{bmatrix}$ *in to* e_j *for* $j = 1, 2, 3, 4$*, and what are the inverses of these matrices?*

5. Some Special Square Matrices

We say that a matrix A is square if there is a positive integer n such that $A \in \mathbb{R}^{n\times n}$. For example, the Gauss-Jordan elimination matrices are a special kind of square matrix. Below we give a list of some square matrices with special properties that are very useful to our future work.

Diagonal Matrices: The diagonal of a matrix $A = [A_{ij}]$ is the vector $(A_{11}, A_{22}, \ldots, A_{nn})^T \in \mathbb{R}^n$. A matrix in $\mathbb{R}^{n\times n}$ is said to be diagonal if the only non-zero entries of the matrix are the diagonal entries. Given a vector $v \in \mathbb{R}^n$, we write $\text{diag}(v)$ the denote the diagonal matrix whose diagonal is the vector v.

The Identity Matrix: The identity matrix is the diagonal matrix whose diagonal entries are all ones. We denote the identity matrix in $\mathbb{R}^k$ by I_k. If the dimension of the identity is clear, we simply write I. Note that for any matrix $A \in \mathbb{R}^{m\times n}$ we have $I_mA = A = AI_n$.

Inverse Matrices: The inverse of a matrix $X \in \mathbb{R}^{n\times n}$ is any matrix $Y \in \mathbb{R}^{n\times n}$ such that $XY = I$ in which case we write $X^{-1} := Y$. It is easily shown that if Y is an inverse of X, then Y is unique and $YX = I$.

Permutation Matrices: A matrix $P \in \mathbb{R}^{n\times n}$ is said to be a permutation matrix if P is obtained from the identity matrix by either permuting the columns of the identity matrix or permuting its rows. It is easily seen that $P^{-1} = P^T$.

Unitary Matrices: A matrix $U \in \mathbb{R}^{n\times n}$ is said to be a unitary matrix if $U^TU = I$, that is $U^T = U^{-1}$. Note that every permutation matrix is unitary. But the converse is note true since for any vector u with $\|u\|_2 = 1$ the matrix $I - 2uu^T$ is unitary.

Symmetric Matrices: A matrix $M \in \mathbb{R}^{n\times n}$ is said to be symmetric if $M^T = M$.

Skew Symmetric Matrices: A matrix $M \in \mathbb{R}^{n\times n}$ is said to be skew symmetric if $M^T = -M$.

6. The LU Factorization

In this section we revisit the reduction to echelon form, but we incorporate permutation matrices into the pivoting process. Recall that a matrix $P \in \mathbb{R}^{m\times m}$ is a *permutation matrix* if it can be obtained from the identity matrix by permuting either its rows or columns. It is straightforward to show that $P^TP = I$ so that the inverse of a permutation matrix is its transpose. Multiplication of a matrix on the left permutes the rows of the matrix while multiplication on the right permutes the columns. We now apply permutation matrices in the Gaussian elimination process in order to avoid zero pivots.

Let $A \in \mathbb{R}^{m\times n}$ and assume that $A \neq 0$. Set $\widetilde{A}_0 := A$. If the $(1,1)$ entry of $\widetilde{A}_0$ is zero, then apply permutation matrices P_{l0} and P_{r0} to the left and and right of $\widetilde{A}_0$, respectively, to bring *any* non-zero element of $\widetilde{A}_0$ into the $(1,1)$ position (e.g., the one with largest magnitude) and set $A_0 := P_{l0}\widetilde{A}_0P_{r0}$. Write A_0 in block form as

$$A_0 = \begin{bmatrix} \alpha_1 & v_1^T \\ u_1 & \widehat{A}_1 \end{bmatrix} \in \mathbb{R}^{m\times n},$$

with $0 \neq \alpha_1 \in \mathbb{R}$, $u_1 \in \mathbb{R}^{n-1}$, $v_1 \in \mathbb{R}^{m-1}$, and $\widetilde{A}_1 \in \mathbb{R}^{(m-1)\times(n-1)}$. Then using α_1 to zero out u_1 amounts to left multiplication of the matrix A_0 by the Gaussian elimination matrix

$$\begin{bmatrix} 1 & 0 \\ -\frac{u_1}{\alpha_1} & I \end{bmatrix}$$

to get

$$\begin{bmatrix} 1 & 0 \\ -\frac{u_1}{\alpha_1} & I \end{bmatrix}\begin{bmatrix} \alpha_1 & v_1^T \\ u_1 & \widehat{A}_1 \end{bmatrix} = \begin{bmatrix} \alpha_1 & v_1^T \\ 0 & \widetilde{A}_1 \end{bmatrix} \in \mathbb{R}^{m\times n}\ , \tag{4}$$

where

$$\widetilde{A}_1 = \widehat{A}_1 - u_1v_1^T/\alpha_1\ .$$

Define

$$\widetilde{L}_1 = \begin{bmatrix} 1 & 0 \\ \frac{u_1}{\alpha_1} & I \end{bmatrix} \in \mathbb{R}^{m\times m} \quad \text{and} \quad \widetilde{U}_1 = \begin{bmatrix} \alpha_1 & v_1^T \\ 0 & \widetilde{A}_1 \end{bmatrix} \in \mathbb{R}^{m\times n}\ .$$

and observe that

$$\widetilde{L}_1^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{u_1}{\alpha_1} & I \end{bmatrix}$$

Hence (4) becomes

$$\widetilde{L}_1^{-1}P_{l0}\widetilde{A}_0P_{r0} = \widetilde{U}_1, \text{ or equivalently,} \quad A = P_{l0}\widetilde{L}_1\widetilde{U}_1P_{r0}^T\ . \tag{5}$$

Note that $\widetilde{L}_1$ is *unit* lower triangular (ones on the mail diagonal) and $\widetilde{U}_1$ is block upper-triangular with one nonsingular 1×1 block and one $(m-1)\times(n-1)$ block on the block diagonal.

Next consider the matrix $\widetilde{A}_1$ in $\widetilde{U}_1$. If the $(1,1)$ entry of $\widetilde{A}_1$ is zero, then apply permutation matrices $\widetilde{P}_{l1} \in \mathbb{R}^{(m-1)\times(m-1)}$ and $\widetilde{P}_{r1} \in \mathbb{R}^{(n-1)\times(n-1)}$ to the left and and right of $\widetilde{A}_1 \in \mathbb{R}^{(m-1)\times(n-1)}$, respectively, to bring *any* non-zero element of $\widetilde{A}_0$ into the $(1,1)$ position (e.g., the one with largest magnitude) and set $A_1 := \widetilde{P}_{l1}\widetilde{A}_1P_{r1}$. If the element of $\widetilde{A}_1$ is zero, then stop. Define

$$P_{l1} := \begin{bmatrix} 1 & 0 \\ 0 & \widetilde{P}_{l1} \end{bmatrix} \quad \text{and} \quad P_{r1} := \begin{bmatrix} 1 & 0 \\ 0 & \widetilde{P}_{r1} \end{bmatrix}$$

so that P_{l1} and P_{r1} are also permutation matrices and

$$P_{l1}\widetilde{U}_1 P_{r1} = \begin{bmatrix}1 & 0\\ 0 & \widetilde{P}_{l1}\end{bmatrix}\begin{bmatrix}\alpha_1 & v_1^T\\ 0 & \widetilde{A}_1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \widetilde{P}_{r1}\end{bmatrix} = \begin{bmatrix}\alpha_1 & v_1^T P_{r1}\\ 0 & \widetilde{P}_{l1}\widetilde{A}_1 P_{r1}\end{bmatrix} = \begin{bmatrix}\alpha_1 & \tilde{v}_1^T\\ 0 & A_1\end{bmatrix}, \tag{6}$$

where $\tilde{v}_1 := P_{r1}^T v_1$. Define

$$U_1 := \begin{bmatrix}\alpha_1 & \tilde{v}_1^T\\ 0 & A_1\end{bmatrix}, \quad \text{where} \quad A_1 = \begin{bmatrix}\alpha_2 & v_2^T\\ u_2 & \widehat{A}_2\end{bmatrix} \in \mathbb{R}^{(m-1)\times(n-1)},$$

with $0 \neq \alpha_2 \in \mathbb{R}$, $u_2 \in \mathbb{R}^{n-2}$, $v_1 \in \mathbb{R}^{m-2}$, and $\widetilde{A}_2 \in \mathbb{R}^{(m-2)\times(n-2)}$. In addition, define

$$L_1 := \begin{bmatrix}1 & 0\\ \widetilde{P}_{l1}\frac{u_1}{\alpha_1} & I\end{bmatrix},$$

so that

$$\begin{aligned}P_{l1}^T L_1 &= \begin{bmatrix}1 & 0\\ 0 & \widetilde{P}_{l1}^T\end{bmatrix}\begin{bmatrix}1 & 0\\ \widetilde{P}_{l1}\frac{u_1}{\alpha 1} & I\end{bmatrix}\\ &= \begin{bmatrix}1 & 0\\ \frac{u_1}{\alpha 1} & \widetilde{P}_{l1}^T\end{bmatrix}\\ &= \begin{bmatrix}1 & 0\\ \frac{u_1}{\alpha 1} & I\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \widetilde{P}_{l1}^T\end{bmatrix}\\ &= \widetilde{L}_1 P_{l1}^T,\end{aligned}$$

and consequently

$$L_1^{-1}P_{l1} = P_{l1}\widetilde{L}_1^{-1}\ .$$

Plugging this into (5) and using (6), we obtain

$$L_1^{-1}P_{l1}P_{l0}\widetilde{A}_0 P_{r0}P_{r1} = P_{l1}\widetilde{L}_1^{-1}P_{l0}\widetilde{A}_0 P_{r0}P_{r1} = P_{l1}\widetilde{U}_1 P_{r1} = U_1,$$

or equivalently,

$$P_{l1}P_{l0}AP_{r0}P_{r1} = L_1 U_1.$$

We can now repeat this process on the matrix A_1 since the $(1,1)$ entry of this matrix is non-zero. The process can run for no more than the number of rows of A which is m. However, it may terminate after $k < m$ steps if the matrix $\widehat{A}_k$ is the zero matrix. In either event, we obtain the following result.

Theorem 6.1. *[The LU Factorization] Let $A \in \mathbb{R}^{m\times n}$. If $k = \operatorname{rank}(A)$, then there exist permutation matrices $P_l \in \mathbb{R}^{m\times m}$ and $P_r \in \mathbb{R}^{n\times n}$ such that*

$$P_l A P_r = LU,$$

where $L \in \mathbb{R}^{m\times m}$ is a lower triangular matrix having ones on its diagonal and

$$U = \begin{bmatrix}U_1 & U_2\\ 0 & 0\end{bmatrix}$$

with $U_1 \in \mathbb{R}^{k\times k}$ a nonsingular upper triangular matrix.

Note that a column permutation is only required if the first column of $\hat{A}_k$ is zero for some k before termination. In particular, this implies that the $\operatorname{rank}(A) < m$. Therefore, if $\operatorname{rank}(A) = m$, column permutations are not required, and $P_r = I$. If one implements the LU factorization so that a column permutation is *only* employed in the case when the first column of $\hat{A}_k$ is zero for some k, then we say the LU factorization is obtained through partial pivoting.

Example 6.1. *We now use the procedure outlined above to compute the LU factorization of the matrix*

$$A = \begin{bmatrix}1 & 1 & 2\\ 2 & 4 & 2\\ -1 & 1 & 3\end{bmatrix}.$$

$$\begin{aligned} L_1^{-1}A &= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -3 \\ 0 & 2 & 5 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} L_2^{-1}L_1^{-1}A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -3 \\ 0 & 2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 8 \end{bmatrix} \end{aligned}$$

We now have

$$U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 8 \end{bmatrix},$$

and

$$L = L_1L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

7. Solving Equations with the LU Factorization

Consider the equation $Ax = b$. In this section we show how to solve this equation using the LU factorization. Recall from Theorem 6.1 that the algorithm of the previous section produces a factorization of A of the form $P_l \in \mathbb{R}^{m\times m}$ and $P_r \in \mathbb{R}^{n\times n}$ such that

$$A = P_l^T L U P_r^T,$$

where $P_l \in \mathbb{R}^{m\times m}$ and $P_r \in \mathbb{R}^{n\times n}$ are permutation matrices, $L \in \mathbb{R}^{m\times m}$ is a lower triangular matrix having ones on its diagonal, and

$$U = \begin{bmatrix} U_1 & U_2 \\ 0 & 0 \end{bmatrix}$$

with $U_1 \in \mathbb{R}^{k\times k}$ a nonsingular upper triangular matrix. Hence we may write the equation $Ax = b$ as

$$P_l^T L U P_r^T x = b.$$

Multiplying through by P_l and replacing $UP_r^T x$ by w gives the equation

$$Lw = \hat{b}, \quad \text{where} \quad \hat{b} := P_l b\ .$$

This equation is easily solved by forward substitution since L is a nonsingular lower triangular matrix. Denote the solution by $\overline{w}$. To obtain a solution x we must still solve $UP_r^T x = \overline{w}$. Set $y = P_r x$. The this equation becomes

$$\overline{w} = Uy = \begin{bmatrix} U_1 & U_2 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where we have decomposed y to conform to the decomposition of U. Doing the same for $\overline{w}$ gives

$$\begin{pmatrix} \overline{w}_1 \\ \overline{w}_2 \end{pmatrix} = \begin{bmatrix} U_1 & U_2 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

or equivalently,

$$\begin{aligned} \overline{w}_1 &= U_1 y_1 + U_2 y_2 \\ \overline{w}_2 &= 0. \end{aligned}$$

Hence, if $\overline{w}_2 \neq 0$, the system is inconsistent, i.e., no solution exists. On the other hand, if $\overline{w}_2 = 0$, we can take $y_2 = 0$ and solve the equation

$$\overline{w}_1 = U_1 y_1 \tag{7}$$

for $\overline{y}_1$, then

$$\overline{x} = P_r^T \begin{pmatrix} \overline{y}_1 \\ 0 \end{pmatrix}$$

is a solution to $Ax = b$. The equation (7) is also easy to solve since U_1 is an upper triangular nonsingular matrix so that (7) can be solved by back substitution.

8. The Four Fundamental Subspaces and Echelon Form

Recall that a subset W of $\mathbb{R}^n$ is a subspace if and only if it satisfies the following three conditions:

(1) The origin is an element of W.
(2) The set W is closed with respect to addition, i.e. if $u \in W$ and $v \in W$, then $u + v \in W$.
(3) The set W is closed with respect to scalar multiplication, i.e. if $\alpha \in \mathbb{R}$ and $u \in W$, then $\alpha u \in W$.

Exercise 8.1. *Given $v^1, v^2, \ldots, v^k \in \mathbb{R}^n$, show that the linear span of these vectors,*

$$\mathrm{span}\left(v^1, v^2, \ldots, v^k\right) := \left\{\alpha_1 v^1 + \alpha_2 v^2 + \cdots + \alpha_k v^k \,\middle|\, \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}\right\}$$

is a subspace.

Exercise 8.2. *Show that for any set S in $\mathbb{R}^n$, the set*

$$S^\perp = \{v \,:\, w^T v = 0 \text{ for all } w \in S\}$$

is a subspace. If S is itself a subspace, then $S^\perp$ is called the subspace orthogonal (or perpendicular) to the subspace S.

Exercise 8.3. *If S is any suset of $\mathbb{R}^n$ (not necessarily a subspace), show that $(S^\perp)^\perp = \mathrm{span}\,(S)$.*

Exercise 8.4. *If $S \subset \mathbb{R}^n$ is a subspace, show that $S = (S^\perp)^\perp$.*

A set of vectors $v^1, v^2, \ldots, v^k \in \mathbb{R}^n$ are said to be *linearly independent* if $0 = a_1 v^1 + \cdots + a_k v^k$ if and only if $0 = a_1 = a_2 = \cdots = a_k$. A *basis* for a subspace in any maximal linearly independent subspace. An elementary fact from linear algebra is that the subspace equals the linear span of any basis for the subspace and that every basis of a subspace has the same number of vectors in it. We call this number the *dimension* for the subspace. If S is a subspace, we denote the dimension of S by $\dim S$.

Exercise 8.5. *If $s \subset \mathbb{R}^n$ is a subspace, then any basis of S can contain only finitely many vectors.*

Exercise 8.6. *Show that every subspace can be represented as the linear span of a basis for that subspace.*

Exercise 8.7. *Show that every basis for a subspace contains the same number of vectors.*

Exercise 8.8. *If $S \subset \mathbb{R}^n$ is a subspace, show that*

$$\mathbb{R}^n = S + S^\perp \tag{8}$$

and that

$$n = \dim S + \dim S^\perp. \tag{9}$$

Let $A \in \mathbb{R}^{m \times n}$. We associate with A its four fundamental subspaces:

$$\begin{array}{ll} \mathrm{Ran}(A) := \{Ax \,|\, x \in \mathbb{R}^n\} & \mathrm{Null}(A) := \{x \,|\, Ax = 0\} \\ \mathrm{Ran}(A^T) := \left\{A^T y \,\middle|\, y \in \mathbb{R}^m\right\} & \mathrm{Null}(A^T) := \left\{y \,\middle|\, A^T y = 0\right\}. \end{array}$$

where

$$\begin{array}{ll} \mathrm{rank}(A) := \dim \mathrm{Ran}(A) & \mathrm{nullity}(A) := \dim \mathrm{Null}(A) \\ \mathrm{rank}(A^T) := \dim \mathrm{Ran}(A^T) & \mathrm{nullity}(A^T) := \dim \mathrm{Null}(A^T) \end{array} \tag{10}$$

Exercise 8.9. *Show that the four fundamental subspaces associated with a matrix are indeed subspaces.*

Observe that

$$\begin{aligned}
\mathrm{Null}(A) &:= \{x \,|\, Ax = 0\} \\
&= \{x \,|\, A_{i\cdot} \bullet x = 0,\ i = 1, 2, \ldots, m\} \\
&= \{A_{1\cdot}, A_{2\cdot}, \ldots, A_{m\cdot}\}^{\perp} \\
&= \mathrm{span}\,(A_{1\cdot}, A_{2\cdot}, \ldots, A_{m\cdot})^{\perp} \\
&= \mathrm{Ran}(A^T)^{\perp}\ .
\end{aligned}$$

Since for any subspace $S \subset \mathbb{R}^n$, we have $(S^{\perp})^{\perp} = S$, we obtain

$$\mathrm{Null}(A)^{\perp} = \mathrm{Ran}(A^T) \text{ and } \mathrm{Null}(A^T) = \mathrm{Ran}(A)^{\perp}. \tag{11}$$

The equivalences in (11) are called the *Fundamental Theorem of the Alternative*.

One of the big consequences of echelon form is that

$$n = \mathrm{rank}(A) + \mathrm{nullity}(A). \tag{12}$$

By combining (12), (9) and (11), we obtain the equivalence

$$\mathrm{rank}(A^T) = \dim \mathrm{Ran}(A^T) = \dim \mathrm{Null}(A)^{\perp} = n - \mathrm{nullity}(A) = \mathrm{rank}(A).$$

That is, the row rank of a matrix equals the column rank of a matrix, i.e., the dimensions of the row and column spaces of a matrix are the same!

Math 407 — Linear Optimization

1 Introduction

1.1 What is optimization?

Broadly speaking, a mathematical optimization problem is one in which a given real value function is either maximized or minimized relative to a given set of alternatives. The function to be minimized or maximized is called the *objective function* and the set of alternatives is called the feasible region (or constraint region). In this course, the feasible region is always taken to be a subset of $\mathbb{R}^n$ (real n-dimensional space) and the objective function is a function from $\mathbb{R}^n$ to $\mathbb{R}$.

The process of formulating a problem from engineering, business, science, medicine, or elsewhere as a mathematical problem is called mathematical modeling. A *modeling class* is a flexible mathematical structure that allows one to model a wide range of physical phenomenon. The scope of potential mathematical models is enormous. Whole fields of mathematics have emerged over the years to uncover the underlying mathematics associated with powerful model classes. A model class is said to powerful if it yields fundamental and important insight into the underlying physical phenomenon.

Although there are many optimization model classes, historically, two stand out as the most powerful. The first, *linear least squares* (LLS), is usually credited to Legendre and Gauss for their careful study of the method around 1800. But the method was in common practice for at least 50 years prior to their work. The second, *linear programming* (LP), is of a more recent vintage and arose almost simultaneously in the Soviet Union, Europe, and the US in response to the need to efficiently allocate resources during the second world war. The Soviet economist Leonid Kantorovich was the first to propose a model broad enough to capture all of LP in 1939. He also proposed a method of solution for his models. At about the same time the Dutch-American economist Tjalling C. Koopmans formulated the classical problem of linear economic models as a linear program. In 1941, Frank Lauren Hitchcock formulated transportation problems as linear programs and developed a solution method close in spirit to the *simplex algorithm*. The 1975 Nobel Prize in Economics was awarded to Kantorovich and Koopmans for their seminal work on linear programming. Hitchcock did not share in this prize since he died in 1957 and the Nobel Prize is not awarded posthumously.

In 1946-47, George B. Dantzig independently developed a general linear programming formulation for planning problems in the US Air Force, and in 1947 Danzig invented the simplex algorithm to solve general LPs. The simplex algorithm is the first computationally efficient method for solving this class of problems. Today it remains one of the workhorses in the solution of general LPs. It is the focus of our numerical study of linear programming.

In 1947-48, Danzig discussed linear programming and the simplex algorithm with John von Neumann who immediately conjectured the theory of *linear programing duality*. His conjecture was based on his invention of game theory with Oskar Morgenstern at Princeton. In addition, von Neumann demonstrated the equivalence of two person zero-sum games with linear programs. Danzig provided the formal proof of LP duality in 1948.

An LP is an optimization problem over $\mathbb{R}^n$ wherein the objective function is a linear function, that is, the objective has the form

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

for some $c_i \in \mathbb{R}$ $i = 1, \ldots, n$, and the feasible region is the set of solutions to a finite number of linear inequality and equality constraints, of the form

$$a_{i1}x_i + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i \qquad i = 1, \ldots, s$$

and

$$a_{i1}x_i + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \qquad i = s+1, \ldots, m.$$

Linear programming is an extremely powerful tool for addressing a wide range of applied optimization problems. A short list of application areas is resource allocation, production scheduling, warehousing, layout, transportation scheduling, facility location, flight crew scheduling, portfolio optimization, parameter estimation, machine learning, compressed sensing, robust data analysis, ...

1.2 An Example

We begin our study of LP with a simple two-dimensional example, and then use this example to illustrate a few properties share by LPs in general. To modeling this example, we use four basic steps:

1. Identify and label the *decision variables*.
2. Determine the objective and use the decision variables to write an expression for the *objective function* as a linear function of the decision variables.
3. Determine the *explicit constraints* and write a functional expression for each of them as either a linear equation or a linear inequality in the decision variables.
4. Determine the *implicit constraints*, and write each as either a linear equation or a linear inequality in the decision variables.

   PLASTIC CUP FACTORY

   A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is \$25 while the profit on a case of champagne glasses is \$20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

We model the problem of maximizing the profit for this company as an LP. The first step is to identify and label the *decision variables*. These are the variables that represent the quantifiable decisions that must be made in order to determine the daily production schedule. That is, we need to specify those quantities whose values completely determine a production schedule and its associated profit. In order to determine these quantities, one should try to personalize the decision making process by asking yourself such questions as "What must I know in order to implement a production schedule?" The decision variables are best arrived at by putting oneself in the shoes of the decision maker and then ask the question "What are the very practical, concrete pieces of information I need to know in order to make this thing work?" In the modeling process, it is equally important to purge all thoughts of optimization from your thoughts as this confuses and complicates the modeling process and almost always leads to an incorrect model.

In the case of the plastic cup factory, everything is determined once it is known how many cases of beer mugs and champagne glasses are to be produced each day.

Decision Variables:

$B = \#$ of cases of beer mugs to be produced daily.

$C = \#$ of cases of champagne glasses to be produced daily.

You will soon discover that the most difficult part of any modeling problem is identifying the decision variables. Once these variables are correctly identified then the remainder of the modeling process usually goes smoothly. But be aware that there is rarely a *unique* choice of decision variables. For example, in this problem one could just as well let B and C be the *hours* devoted to beer mug and champagne glass production each day, respectively. The choice of decision variables becomes more complicated as the complexity of the problems grows. In general, one should try to hew as closely to the problem description as possible when specifying these variables.

After identifying and labeling the decision variables, one then specifies the problem objective. That is, write an expression for the objective function as a linear function of the decision variables.

Objective Function:

Maximize profit where profit $= 25B + 20C$

The next step in the modeling process is to express the feasible region as the solution set of a finite collection of linear inequality and equality constraints. We separate this process into two steps:

1. determine the explicit constraints, and
2. determine the implicit constraints.

The explicit constraints are those that are explicitly given in the problem statement. In the problem under consideration, there are explicit constraints on the amount of resin and the number of work hours that are available on a daily basis.

Explicit Constraints:

resin constraint: $20B + 12C \leq 1800$

work hours constraint: $\frac{1}{15}B + \frac{1}{15}C \leq 8$.

This problem has other constraints called implicit constraints. These are constraints that are not explicitly given in the problem statement but are present nonetheless. Typically these constraints are associated with "natural" or "common sense" restrictions on the decision variables. In the cup factory problem it is clear that one cannot have negative cases of beer mugs and champagne glasses. That is, both B and C must be non-negative quantities.

Implicit Constraints:

$$0 \leq B, \qquad 0 \leq C.$$

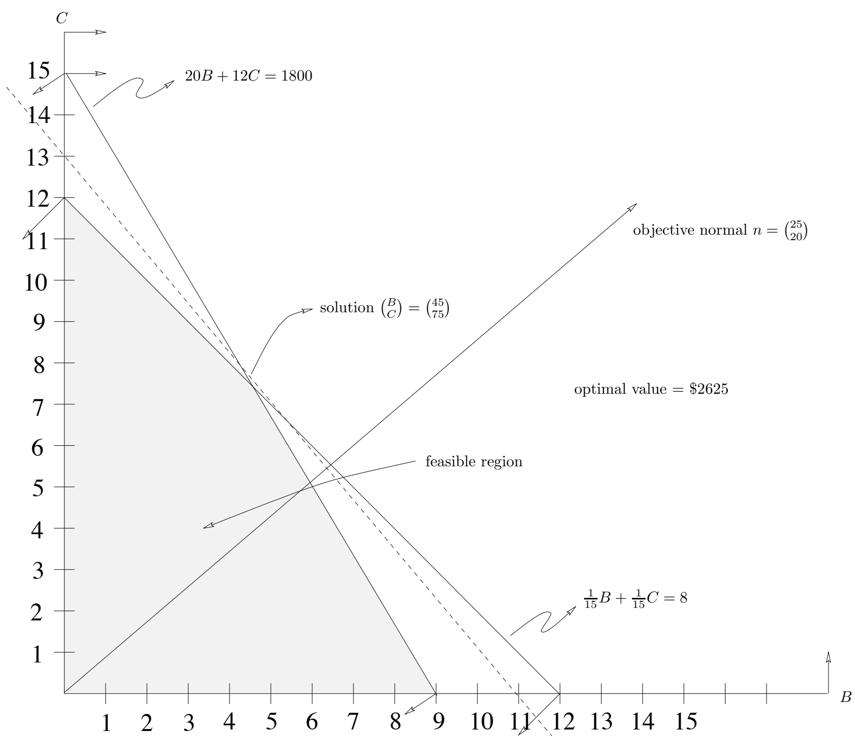
The entire model for the cup factory problem can now be succinctly stated as

$$\begin{array}{rlrcl} \mathcal{P} : & \max 25B + 20C & & & \\ & \text{subject to} & 20B + 12C & \leq & 1800 \\ & & \frac{1}{15}B + \frac{1}{15}C & \leq & 8 \\ & & 0 & \leq & B, C \end{array}$$

In practice, one usually formulates a model with the assistance of a *domain expert*, that is, someone who is an expert in the phenomenon being modeled. It often happens that models fail because what is "natural", "common sense", or "obvious" to the domain expert is not at all "natural", "common sense", or "obvious" to you. It can be difficult to overcome these barriers of language, culture, and expertise. A good rule is to ask the expert to think of you as a very intelligent fourth grader that rapidly absorbs information, but to who everything must be explained from basic definitions and principles.

Since the Plastic Cup Factory problem is an introductory example, it is particularly easy to model. As the course progresses you will be asked to model problems of increasing difficulty and complexity. In this regard, we again emphasize that the first step in the modeling process, identification of the decision variables, is always the most difficult. In addition, the 4 step modeling process outlined above is not intended to be a process that one steps through in a linear fashion. As the model unfolds it is often necessary to revisit earlier steps, for example by changing or adding in more decision variables (a very common requirement). Moving between these steps several times is often required before the model is complete. In this process, the greatest stumbling block experienced by students is the overwhelming desire to try to solve the problem as it is being modeled. Indeed, every student or this subject has made this error. A related common error is to try to reduce the total number of decision variables required. This often complicates the modeling process, blocks the ability to fully characterize all of the variability present, makes it difficult to interpret the solution and understand its robustness, and makes it difficult to modify the model as it evolves. In addition, one should not be afraid of adding more variables to simplify the development of the model. The addition of variables can often clarify the model and improve its flexibility, while modern LP software easily solves problems with tens of thousands of variables, and in some cases tens of millions of variables. It is far more important to get a correct, easily interpretable, and flexible model then to provide a compact minimalist model.

We now turn to solving the Plastic Cup Factory problem. Since this problem is two dimensional, it is possible to provide a graphical representation and solution. The first step is to graph the feasible region.



Graph the line associated with each of the linear inequality constraints. Then determine on which side of each of these lines the feasible region must lie (don't forget the implicit constraints!). To determine the correct side, locate a point not on the line that determines the constraint (for example, the origin is often not on the line, and it is particularly easy to use). Plug this point in and see if it satisfies the constraint. If it does, then it is on the correct side of the line. If it does not, then the other side of the line is correct. Once the correct side is determined put little arrows on the line to indicate the correct side. Then shade in the resulting feasible region which is the set of points satisfying all of the linear inequalities.

The next step is to draw in the vector representing the gradient of the objective function. This vector may be placed anywhere on your graph, but, in simple examples, it is often convenient to draw it emanating from the origin. Since the objective function has the form

$$f(x_1, x_2) = c_1x_1 + c_2x_2,$$

the gradient of f is the same at every point in $\mathbb{R}^2$;

$$\nabla f(x_1, x_2) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Recall from calculus that the gradient always points in the direction of increasing function values. Moreover, since the gradient is constant on the whole space, the level sets of f associated with different function values (i.e. for k =constant, the associated level set is $\{x \mid f(x) = k\}$) are given by the lines perpendicular to the gradient. For example, in the Plastic Cup Factory problem, we have

$$f(B, C) = 20B + 25C \text{ and } \nabla f(B, C) = \begin{pmatrix} 25 \\ 20 \end{pmatrix} \text{ with}$$
$$\{(B, C) \mid f(B, C) = 2\} = \{(B, C) \mid 25B + 20C = 2\}.$$

Consequently, to obtain the location of the point at which the objective is maximized we simply set a ruler perpendicular to the gradient and then move the ruler in the direction of the gradient until we reach the last point (or points) at which the line determined by the ruler intersects the feasible region. In the case of the cup factory problem this gives the solution to the LP as $\binom{B}{C} = \binom{45}{75}$

We now recap the steps described in the solution procedure given above:

Step 1: Graph each of the linear constraints indicating on which side of the constraint the feasible region must lie with an arrow. Don't forget the implicit constraints!

Step 2: Shade in the feasible region.

Step 3: Draw the gradient vector of the objective function.

Step 4: Place a straight-edge perpendicular to the gradient vector and move the straight-edge in the direction of the gradient vector for maximization (or in the opposite direction of the gradient vector for minimization) to the last point for which the straight-edge intersects the feasible region. The set of points of intersection between the straight-edge and the feasible region is the set of optimal solutions to the LP.

Step 5: Compute the exact optimal corner point (or vertex) solutions to the LP as the points of intersection of the lines on the boundary of the feasible region indicated in Step 4. Then compute the resulting optimal value associated with these points.

The solution procedure described above for two dimensional problems reveals a great deal about the geometric structure of LPs that remains true in n dimensions. We will explore this geometric structure more fully as the course evolves. For the moment, note that the solution to the Plastic Cup Factory problem lies at a *corner point* of the feasible region. Indeed, it is easy to convince oneself that every 2 dimensional LP has an optimal solution that is such a *corner point*. The notion of a corner point can be generalized to n dimensional space where it is referred to as a *vertex*. These vertices play a big role in understanding the geometry of linear programming.

Before leaving this section, we make a final comment on the modeling process described above. We again emphasize that there is not one and only one way to model the Cup Factory problem, or any problem for that matter. In particular, there are many ways to choose the decision variables for this problem. Clearly, it is sufficient for the shop manager to know how many hours each day should be devoted to the manufacture of beer mugs and how many hours to champagne glasses. From this information everything else can be determined. For example, the number of cases of beer mugs produced is 15 times the number of hours devoted to the production of beer mugs. However, in the end, all choice of decision variables yield the same optimal process.

1.3 Sensitivity Analysis

One of the most important things to keep in mind about "real world" LPs is that the input data associated with the problem specification is often uncertain and variable. That is, it is subject to measurement error, it is often the product of educated guesses (another name for fudging), and it can change over time. For example, in the case of the cup factory the profit levels for both beer mugs and champagne glasses are subject to seasonal variations. Prior to the New Year, the higher demand for champagne glasses forces up the sale price and consequently their profitability. As St. Patrick's Day approaches the demand for champagne glasses drops, but the demand for beer mugs soars. In June, demand for champagne glasses again rises due to the increase in marriage celebrations. Then, just before the Fourth of July, the demand for beer mugs returns. These seasonal fluctuations may effect the optimal solution and the optimal value. Similarly, the availability of the resources required to produce the beer mugs and champagne glasses as well as their purchase prices vary with time due to changes and innovations in the market place. In this context, it is natural, and often essential, to ask how the optimal value and optimal solutions change as the input data for the problem changes. The mathematical study of these changes is called *sensitivity analysis*. This is a vital area of study in linear programming. Although we delay a detailed study of this topic to later in the the course, it is useful to introduce some of these ideas now to motivate several important topics. The most important of these being *duality theory*. We begin with the optimal value function and marginal values.

1.3.1 The Optimal Value Function and Marginal Values

Consider the effect of fluctuations in the availability of resources on both the optimal solution and the optimal value. In the case of the cup factory there are two basic resources consumed by the production process: plastic resin and labor hours. In order to analyze the behavior of the problem as the availability of these resources change, recall our observation that if an optimal solution exists, then at least one vertex, or corner point, optimal solution exists. We make this intuition rigorous in a future section. Next note that as the availability of a resource is changed the constraint line associated with that resource moves in a parallel fashion along a line perpendicular, or normal, to the constraint. Thus, at least for a small range of perturbations to the resources, the vertex associated with the current optimal solution moves but remains optimal. (We caution that this is only a generic property of an optimal vertex and there are examples for which it fails; for example, in some models the feasible region can be made empty under arbitrarily small perturbations of the resources.) These observations lead us to conjecture that the solution to the LPs

$$\begin{array}{rcl} v(\epsilon_1,\epsilon_2) & = & \max 25B+20C \\ & & \text{subject to } \begin{array}{rcl} 20B+12C & \leq & 1800+\epsilon_1 \\ \frac{1}{15}B+\frac{1}{15}C & \leq & 8+\epsilon_2 \\ 0 & \leq & B,C \end{array} \end{array}$$

lies at the intersection of the two lines $20B+12C = 1800+\epsilon_1$ and $\frac{1}{15}B+\frac{1}{15}C = 8+\epsilon_2$ for small values of ϵ_1 and ϵ_2; namely

$$\begin{array}{rcl} B & = & 45 - \frac{45}{2}\epsilon_2 + \frac{1}{8}\epsilon_1 \\ C & = & 75 + \frac{75}{2}\epsilon_2 - \frac{1}{8}\epsilon_1, \text{ and} \\ v(\epsilon_1,\epsilon_2) & = & 2625 + \frac{375}{2}\epsilon_2 + \frac{5}{8}\epsilon_1. \end{array}$$

It can be verified by direct computation that this indeed yields the optimal solution for "small" values of ϵ_1 and ϵ_2.

Next observe that the value $v(\epsilon_1,\epsilon_2)$ can now be viewed as a function of ϵ_1 and ϵ_2 and that this function is differentiable at $\binom{\epsilon_1}{\epsilon_2} = \binom{0}{0}$ with

$$\nabla v(\epsilon_1,\epsilon_2) = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}.$$

The number $\frac{5}{8}$ is called the marginal value of the resin resource at the optimal solution $\binom{B}{C} = \binom{45}{75}$, and the number $\frac{375}{2}$ is called the marginal value of the labor time resource at the optimal solution $\binom{B}{C} = \binom{45}{75}$. We have the following interpretation for these marginal values: each additional pound of resin beyond the base amount of 1800 lbs. contributes $\$\frac{5}{8}$ to the profit and each additional hour of labor beyond the base amount of 8 hours contributes $\$\frac{375}{2}$ to the profit.

Using this information one can answer certain questions concerning how one might change current operating protocol. For example, if we can buy additional resin from another supplier, how much more per pound are we willing to pay than we are currently paying? (Answer: $\$\frac{5}{8}$ per pound is the most we are willing to pay beyond what we now pay, why?) Or, if we are willing to add overtime hours, what is the greatest overtime salary we are willing to pay? Of course, the marginal values are only good for a certain range of fluctuation in the resources, but within that range they provide valuable information.

1.4 Duality Theory

We now briefly discuss the economic theory behind the marginal values and how the "hidden hand of the market place" gives rise to them. This leads in a natural way to a mathematical theory of duality for linear programming.

Think of the cup factory production process as a black box through which the resources flow. Raw resources go in one end and exit the other. When they come out the resources have a different form, but whatever comes out is still comprised of the entering resources. However, something has happened to the value of the resources by passing through the black box. The resources have been purchased for one price as they enter the box and are sold in their new form when they leave. The difference between the entering and exiting prices is called the profit. If the profit is positive, then the resources have increased in value as they pass through the production process. That is, you are selling the resources for more than you paid for them.

Now consider how the market introduces pressures on profitability and the value of the resources available to the market place. Take the perspective of the cup factory *vs* the market place. The market place does not want the cup factory to go out of business. On the other hand, it wants to reset the price of the resources so as to take away as much of the cup factory profit as possible. That is, the market wants to keep all the profit for itself and only let the cup factory just break even. That is, market's goal is to reset the prices for plastic resin and labor so that the cup factory sees no profit and just breaks even. Since the cup factory is now seeing a profit, the market must figure out by how much the sale price of resin and labor must be raised to reduce this profit to zero. This is done by minimizing the value of the available resources over all price increments that guarantee that the cup factory either loses money or sees no profit from both of its products. If we denote the per unit price increment for resin by R and that for labor by L, then the profit for beer mugs is eliminated as long as

$$\left.\begin{array}{c}\text{increase in cost of resources}\\ \text{per case of beer mugs}\end{array}\right\} = 20R + \frac{1}{15}L \geq 25 = \text{profit per case of beer mugs}$$

since the left hand side is the increased value of the resources consumed in the production of one case of beer mugs and the right hand side is the current profit on a case of beer mugs.

Similarly, for champagne glasses, the market wants to choose R and L so that

$$\left.\begin{array}{c}\text{increase in cost of resources}\\ \text{per case of champagne glasses}\end{array}\right\} = 12R+\frac{1}{15}L \geq 20 = \text{profit per case of champagne glasses.}$$

Now in order to maintain equilibrium in the market place, that is, not drive the cup factory out of business (since then the market realizes no profit at all), the market chooses R and L so as to minimize the increased value of the available resources. That is, the market chooses R and L to solve the problem

$$\begin{array}{rlrcl}\mathcal{D} & : & \text{minimize}\,1800R + 8L & & \\ & & \text{subject to}\ \ 20R + \frac{1}{15}L & \geq & 25 \\ & & 12R + \frac{1}{15}L & \geq & 20 \\ & & 0 & \leq & R, L\end{array}$$

This is just another LP. It is called the "dual" to the LP $\mathcal{P}$ in which the cup factory tries to maximize profit. Observe that if $\binom{B}{C}$ is feasible for $\mathcal{P}$ and $\binom{R}{L}$ is feasible for $\mathcal{D}$, then

$$\begin{array}{rcl}25B + 20C & \leq & [20R + \frac{1}{15}L]B + [12R + \frac{1}{15}L]C \\ & = & R[20B + 12C] + L[\frac{1}{15}B + \frac{1}{15}C] \\ & \leq & 1800R + 8L.\end{array}$$

Thus, the value of the objective in $\mathcal{P}$ at a feasible point in $\mathcal{P}$ is bounded above by the objective in $\mathcal{D}$ at any feasible point for $\mathcal{D}$. In particular, the optimal value in $\mathcal{P}$ is bounded above by the optimal value in $\mathcal{D}$. The "strong duality theorem" states that if either of these problems has a finite optimal value, then so does the other and these values coincide. In addition, we claim that the solution to $\mathcal{D}$ is given by the marginal values for $\mathcal{P}$. That is, $\binom{R}{L} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$ is the optimal solution for $\mathcal{D}$. In order to show this we need only show that $\binom{R}{L} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$ is feasible for $\mathcal{D}$ and that the value of the objective in $\mathcal{D}$ at $\binom{R}{L} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$ coincides with the value of the objective in $\mathcal{P}$ at $\binom{B}{C} = \binom{45}{75}$. First we check feasibility:

$$0 \leq \frac{5}{8}, \qquad 0 \leq \frac{375}{2}$$

$$20 \cdot \frac{5}{8} + \frac{1}{15} \cdot \frac{375}{2} \geq 25$$

$$12 \cdot \frac{5}{8} + \frac{1}{15} \cdot \frac{375}{2} \geq 20.$$

Next we check optimality

$$25 \cdot 45 + 20 \cdot 75 = 2625 = 1800 \cdot \frac{5}{8} + 8 \cdot \frac{375}{2}.$$

This is a most remarkable relationship! We have shown that the marginal values have three distinct and seemingly disparate interpretations:

1. The marginal values are the partial derivatives of the value function for the LP with respect to resource availability,

2. The marginal values give the per unit increase in value of each of the resources that occurs as a result of the production process, and

3. The marginal values are the solutions to a dual LP, $\mathcal{D}$.

1.5 LPs in Standard Form and Their Duals

Recall that a linear program is an optimization problem in a finite number of variables wherein one either maximizes or minimizes a linear function subject to a finite number of linear inequality and/or equality constraints. This general definition leads to an enormous variety of possible formulations. In this section we propose one fixed formulation for the purposes of developing an algorithmic solution procedure. We will show that every LP can be recast in this one fixed form.

We say that an LP is in *standard form* if it takes the form

$$\begin{array}{rll} \mathcal{P}: & \text{maximize} & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ & \text{subject to} & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \ \leq \ b_i \quad \text{for } i = 1, 2, \ldots, m \\ & & 0 \leq x_j \quad \text{for } j = 1, 2, \ldots, n \ . \end{array}$$

Using matrix notation, we can rewrite this LP as

$$\begin{array}{rll} \mathcal{P}: & \text{maximize} & c^T x \\ & \text{subject to} & Ax \leq b \\ & & \quad 0 \leq x \ , \end{array}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$ and the inequalities $Ax \leq b$ and $0 \leq x$ are to be interpreted componentwise.

Following the results of the previous section on LP duality, we claim that the dual LP to $\mathcal{P}$ is the LP

$$\begin{array}{rll} \mathcal{D}: & \text{minimize} & b_1y_1 + b_2y_2 + \cdots + b_my_m \\ & \text{subject to} & a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \ \geq \ c_j \text{ for } j = 1, 2, \ldots, n \\ & & 0 \leq y_i \text{ for } i = 1, 2, \ldots, m \ , \end{array}$$

or, equivalently, using matrix notation we have

$$\begin{array}{rll} \mathcal{D}: & \text{minimize} & b^T y \\ & \text{subject to} & A^T y \geq c \\ & & 0 \leq y \ . \end{array}$$

We say that the problem $\mathcal{P}$ is *infeasible* if the feasible region $\{x \mid Ax \leq b,\ 0 \leq x\}$ is empty. We say that $\mathcal{P}$ is *unbounded* if for every $\alpha \in \mathbb{R}$ there is an $x \in \{x \mid Ax \leq b,\ 0 \leq x\}$

such that $c^T x \geq \alpha$. In particular, unboundedness implies feasibility. If $\mathcal{P}$ is infeasible, we say that its optimal value is $-\infty$, while if it is unbounded, we say that its optimal value is $+\infty$. Similarly, we say that $\mathcal{D}$ is *infeasible* if its feasible region $\left\{y \;\middle|\; A^T y \geq c,\ 0 \leq y\right\}$ is empty, and we say that $\mathcal{D}$ is *unbounded* if for every $\beta \in \mathbb{R}$ there is an $y \in \left\{y \;\middle|\; A^T y \geq c,\ 0 \leq y\right\}$ such that $b^T y \leq \beta$. The optimal value in $\mathcal{D}$ is $+\infty$ if it is infeasible and $-\infty$ if it is unbounded.

Just as for the cup factory problem, the primal-dual pair of LPs $\mathcal{P}$ and $\mathcal{D}$ are related via the *Weak Duality Theorem* for linear programming.

Theorem 1.1 (Weak Duality Theorem) *If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then*

$$c^T x \leq y^T A x \leq b^T y.$$

Thus, if $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is necessarily infeasible, and if $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is necessarily infeasible. Moreover, if $c^T \bar{x} = b^T \bar{y}$ with $\bar{x}$ feasible for $\mathcal{P}$ and $\bar{y}$ feasible for $\mathcal{D}$, then $\bar{x}$ must solve $\mathcal{P}$ and $\bar{y}$ must solve $\mathcal{D}$.

Proof: Let $x \in \mathbb{R}^n$ be feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ be feasible for $\mathcal{D}$. Then

$$\begin{array}{rcll}
c^T x & = & \sum_{j=1}^{n} c_j x_j & \\
& \leq & \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i) x_j & [\text{since } 0 \leq x_j \text{ and } c_j \leq \sum_{i=1}^{m} a_{ij} y_i, \text{ so } c_j x_j \leq (\sum_{i=1}^{m} a_{ij} y_i) x_j] \\
& = & y^T A x & \\
& = & \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) y_i & \\
& \leq & \sum_{i=1}^{m} b_i y_i & [\text{since } 0 \leq y_i \text{ and } \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \text{ so } (\sum_{j=1}^{n} a_{ij} x_j) y_i \leq b_i y_i] \\
& = & b^T y &
\end{array}$$

To see that $c^T \bar{x} = b^T \bar{y}$ plus $\mathcal{P}$–$\mathcal{D}$ feasibility implies optimality, simply observe that for every other $\mathcal{P}$–$\mathcal{D}$ feasible pair (x, y) we have

$$c^T x \leq b^T \bar{y} = c^T \bar{x} \leq b^T y\ .$$

■

We caution that the infeasibility of either $\mathcal{P}$ or $\mathcal{D}$ does not imply the unboundedness of the other. Indeed, it is possible for both $\mathcal{P}$ and $\mathcal{D}$ to be infeasible as is illustrated by the following example.

Example:

$$\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
& x_1 & - & x_2 & \leq & 1 \\
& -x_1 & + & x_2 & \leq & -2 \\
& 0 & \leq & x_1, & x_2 &
\end{array}$$

1.5.1 Transformation to Standard Form

Every LP can be transformed to an LP in standard form. This process usually requires a transformation of variables and occasionally the addition of new variables. We provide a step-by-step procedure for transforming any LP to one in standard form.

minimization $\to$ maximization

To transform a minimization problem to a maximization problem just multiply the objective function by -1.

linear inequalities

If an LP has an equality constraint of the form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i,$$

it can be transformed to one in standard form by multiplying the inequality through by -1 to get

$$-a_{i1}x_1 - a_{i2}x_2 - \cdots - a_{in}x_n \leq -b_i.$$

linear equation

The linear equation

$$a_{i1}x_i + \cdots + a_{in}x_n = b_i$$

can be written as two linear inequalities

$$a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$$

and

$$a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i.$$

The second of these inequalities can be transformed to standard form by multiplying through by -1.

variables with lower bounds

If a variable x_i has lower bound l_i which is not zero ($l_i \leq x_i$), one obtains a non-negative variable w_i with the substitution

$$x_i = w_i + l_i.$$

In this case, the bound $l_i \leq x_i$ is equivalent to the bound $0 \leq w_i$.

variables with upper bounds

If a variable x_i has an upper bound u_i ($x_i \leq u_i$) one obtains a non-negative variable w_i with the substitution

$$x_i = u_i - w_i.$$

In this case, the bound $x_i \leq u_i$ is equivalent to the bound $0 \leq w_i$.

variables with interval bounds

An interval bound of the form $l_i \leq x_i \leq u_i$ can be transformed into one non-negativity constraint and one linear inequality constraint in standard form by making the substitution

$$x_i = w_i + l_i.$$

In this case, the bounds $l_i \leq x_i \leq u_i$ are equivalent to the constraints

$$0 \leq w_i \qquad \text{and} \qquad w_i \leq u_i - l_i.$$

free variables

Sometimes a variable is given without any bounds. Such variables are called free variables. To obtain standard form every free variable must be replaced by the difference of two non-negative variables. That is, if x_i is free, then we get

$$x_i = u_i - v_i$$

with $0 \leq u_i$ and $0 \leq v_i$.

To illustrate the ideas given above, we put the following LP into standard form.

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & 3x_1 & - & x_2 & & & & & & \\
\text{subject to} & -x_1 & + & 6x_2 & - & x_3 & + & x_4 & \geq & -3 \\
 & & & 7x_2 & & & + & x_4 & = & 5 \\
 & & & & & x_3 & + & x_4 & \leq & 2
\end{array}$$

$$-1 \leq x_2, x_3 \leq 5, -2 \leq x_4 \leq 2.$$

The hardest part of the translation to standard form, or at least the part most susceptible to error, is the replacement of existing variables with non-negative variables. For this reason, it is advantageous to make the translation in two steps. In the first step make all of the changes that do not involve variable substitution, and then, in the second step, start again and make all of the variable substitutions. Following this procedure, let us start with all of the transformations that do not require variable substitution. First, turn the minimization problem into a maximization problem by rewriting the objective as

$$\text{maximize } -3x_1 + x_2.$$

Next replace the first inequality constraint by the constraint

$$x_1 - 6x_2 + x_3 - x_4 \leq 3.$$

The equality constraint is replaced by the two inequality constraints

$$\begin{array}{rcrcr} 7x_2 & + & x_4 & \leq & 5 \\ -7x_2 & - & x_4 & \leq & -5. \end{array}$$

Finally, split the double bound $-2 \leq x_4 \leq 2$ into two pieces $-2 \leq x_4$ and $x_4 \leq 2$ and group the bound $x_4 \leq 2$ with the linear inequalities. All of these changes give the LP

$$\begin{array}{lrcrcrcrcr} \text{maximize} & -3x_1 & + & x_2 & & & & & & \\ \text{subject to} & x_1 & - & 6x_2 & + & x_3 & - & x_4 & \leq & 3 \\ & & & 7x_2 & & & + & x_4 & \leq & 5 \\ & & - & 7x_2 & & & - & x_4 & \leq & -5 \\ & & & & & x_3 & + & x_4 & \leq & 2 \\ & & & & & & & x_4 & \leq & 2 \end{array}$$

$$-1 \leq x_2, x_3 \leq 5, -2 \leq x_4.$$

We now move on to variable replacement. Observe that the variable x_1 is free, so we replace it by

$$x_1 = z_1^+ - z_1^- \text{ with } 0 \leq z_1^+, 0 \leq z_1^-.$$

The variable x_2 has a non-zero lower bound so we replace it by

$$z_2 = x_2 + 1 \quad \text{or} \quad x_2 = z_2 - 1 \quad \text{with} \quad 0 \leq z_2.$$

The variable x_3 is bounded above, so we replace it by

$$z_3 = 5 - x_3 \quad \text{or} \quad x_3 = 5 - z_3 \quad \text{with} \quad 0 \leq z_3.$$

The variable x_5 is bounded below, so we replace it by

$$z_4 = x_4 + 2 \quad \text{or} \quad x_4 = z_4 - 2 \quad \text{with} \quad 0 \leq z_4.$$

After making these substitutions, we get the following LP in standard form:

$$\begin{array}{lrcrcrcrcrcr} \text{maximize} & -3z_1^+ & + & 3z_1^- & + & z_2 & & & & & & \\ \text{subject to} & z_1^+ & - & z_1^- & - & 6z_2 & - & z_3 & - & z_4 & \leq & -10 \\ & & & & & 7z_2 & & & + & z_4 & \leq & 14 \\ & & & & - & 7z_2 & & & - & z_4 & \leq & -14 \\ & & & & & & - & z_3 & + & z_4 & \leq & -1 \\ & & & & & & & & & z_4 & \leq & 4 \end{array}$$

$$0 \leq z_1^+, z_1^-, z_2, z_3, z_4.$$

2 Solving LPs: The Simplex Algorithm of George Dantzig

2.1 Simplex Pivoting: Dictionary Format

We illustrate a general solution procedure, called the *simplex algorithm*, by implementing it on a very simple example. Consider the LP

$$\begin{array}{rrcl} \max & 5x_1 + 4x_2 + 3x_3 & & \\ \text{s.t.} & 2x_1 + 3x_2 + x_3 & \leq & 5 \\ & 4x_1 + x_2 + 2x_3 & \leq & 11 \\ & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\ & 0 \leq x_1, x_2, x_3 & & \end{array} \tag{2.1}$$

In devising our approach we use a standard mathematical approach; reduce the problem to one that we already know how to solve. Since the structure of this problem is essentially linear, we try to reduce it to a problem of solving a system of linear equations, or perhaps a series of such systems. By encoding the problem as a system of linear equations we bring into play our knowledge and experience with such systems in the new context of linear programming.

In order to encode the LP (2.1) as a system of linear equations we first transform the linear inequalities into linear equations. This is done by introducing a new non-negative variable, called a *slack variable*, for each inequality:

$$\begin{array}{rcrcll} x_4 & = & 5 & - & [2x_1 + 3x_2 + x_3] & \geq 0, \\ x_5 & = & 11 & - & [4x_1 + x_2 + 2x_3] & \geq 0, \\ x_6 & = & 8 & - & [3x_1 + 4x_2 + 2x_3] & \geq 0. \end{array}$$

To handle the objective, we introduce a new variable z:

$$z = 5x_1 + 4x_2 + 3x_3.$$

Then all of the information associated with the LP (2.1) can be coded as follows:

$$\begin{array}{rcccccccl} 2x_1 + 3x_2 + x_3 & + & x_4 & & & & & = & 5 \\ 4x_1 + x_2 + 2x_3 & & & + & x_5 & & & = & 11 \\ 3x_2 + 4x_2 + 2x_3 & & & & & + & x_6 & = & 8 \\ -z + 5x_1 + 4x_2 + 3x_3 & & & & & & & = & 0 \\ \multicolumn{9}{c}{0 \leq x_1, x_2, x_3, x_4, x_5, x_6.} \end{array} \tag{2.2}$$

The new variables x_4, x_5, and x_6 are called slack variables since they take up the *slack* in the linear inequalities. This system can also be written using block structured matrix notation:

$$\begin{bmatrix} 0 & A & I \\ -1 & c^T & 0 \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix},$$

where

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix}, \; I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix}, \text{ and } c = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}.$$

The augmented matrix associated with the system (2.2) is

$$\left[\begin{array}{ccc|c} 0 & A & I & b \\ -1 & c & 0 & 0 \end{array}\right] \tag{2.3}$$

and is referred to as the *initial simplex tableau* for the LP (2.1).

Now return to the system

$$\begin{array}{rcl} x_4 & = & 5 - 2x_1 - 3x_2 - x_3 \\ x_5 & = & 11 - 4x_1 - x_2 - 2x_3 \\ x_6 & = & 8 - 3x_1 - 4x_2 - 2x_3 \\ z & = & 5x_1 + 4x_2 + 3x_3. \end{array} \tag{2.4}$$

This system defines the variables x_4, x_5, x_6 and z as linear combinations of the variables x_1, x_2, and x_3. We call this system a *dictionary* for the LP (2.1). More specifically, it is the *initial* dictionary for the the LP (2.1). This initial dictionary defines the objective value z and the slack variables as a linear combination of the initial decision variables. The variables that are "defined" in this way are called the *basic variables*, while the remaining variables are called *nonbasic*. The set of all basic variables is called the *basis*. A particular solution to this system is easily obtained by setting the non-basic variables equal to zero. In this case, we have

$$\begin{array}{rcl} x_1 & = & 0 \\ x_2 & = & 0 \\ x_3 & = & 0 \end{array} \quad \text{giving} \quad \begin{array}{rcl} x_4 & = & 5 \\ x_5 & = & 11 \\ x_6 & = & 8 \\ z & = & 0. \end{array}$$

Note that the solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 11 \\ 8 \end{pmatrix} \tag{2.5}$$

is feasible for the extended system (2.2) since all components are non-negative. We call this solution the *basic feasible solution* (BFS) associated with the dictionary (2.4). Moreover, we call the dictionary (2.4) a *feasible dictionary* for the LP (2.1), and we say that this LP has *feasible origin.*

In general, a dictionary for the LP (2.1) is any system of 4 linear equations that defines three of the variables $x_1, \ldots, x_6$ and z in terms of the remaining 3 variables and has the same solution set as the initial dictionary. The variables other than z that are being defined in the dictionary are called the basis for the dictionary, and the remaining variables are said to be non-basic in the dictionary. Every dictionary identifies a particular solution to the linear system obtained by setting the non-basic variables equal to zero. Such a solution is said to be a *basic feasible solution* (BFS) for the LP (2.1) if it componentwise non-negative, that is, all of the numbers in the vector are non-negative so that the point lies in the feasible region for the LP.

The grand strategy of the simplex algorithm is to move from one feasible dictionary representation of the system (2.2) to another (and hence from one BFS to another) while simultaneously increasing the value of the objective variable z at the associated BFS. In the current setting, beginning with the dictionary (2.4), what strategy might one employ in order to determine a new dictionary whose associated BFS gives a greater value for the objective variable z?

Each feasible dictionary is associated with one and only one feasible point. This is the associated BFS obtained by setting all of the non-basic variables equal to zero. This is how we obtain (2.5). To change the feasible point identified in this way, we need to increase the value of one of the non-basic variables from its current value of zero. We cannot decrease the value of a non-basic variable since we wish to remain feasible, that is, we wish to keep all variables non-negative.

Note that the coefficient of each of the non-basic variables in the representation of the objective value z in (2.4) is positive. Hence, if we pick any one of these variables and increase its value from zero while leaving the remaining two at zero, we automatically increase the value of the objective variable z. Since the coefficient on x_1 in the representation of z is the greatest, we can increase z the fastest instantaneous rate by increasing x_1. But choosing the variable with the greatest coefficient is not required and may not yield the greatest increase in z. *Any non-basic variable with a positive coefficient in the representation of z can be used to increase the value of z.*

By how much can we increase x_1 and still remain feasible? For example, if we increase x_1 to 3 then (2.4) says that $x_4 = -1$, $x_5 = -1$, $x_6 = -1$ which is not feasible. So x_1 cannot be increased to 3. To see how much we can increase the value of x_1 we examine the equations in (2.4) one by one. Note that the first equation in the dictionary (2.4),

$$x_4 = 5 - 2x_1 - 3x_2 - x_3,$$

shows that x_4 remains non-negative as long as we do not increase the value of x_1 beyond $5/2$ (remember, x_2 and x_3 remain at the value zero). Similarly, using the second equation in the dictionary (2.4),

$$x_5 = 11 - 4x_1 - x_2 - 2x_3,$$

x_5 remains non-negative if $x_1 \leq 11/4$. Finally, the third equation in (2.4),

$$x_6 = 8 - 3x_1 - 4x_2 - 2x_3,$$

implies that x_6 remains non-negative if $x_1 \leq 8/3$. Therefore, we remain feasible, i.e. keep **all** variables non-negative, if our increase to the variable x_1 remains less than

$$\frac{5}{2} = \min\left\{\frac{5}{2}, \frac{11}{4}, \frac{8}{3}\right\}.$$

If we now increase the value of x_1 to $\frac{5}{2}$, then the value of x_4 is driven to zero. One way to think of this is that x_1 *enters the basis while* x_4 *leaves the basis*. Mechanically, we obtain the new dictionary having x_1 basic and x_4 non-basic by using the defining equation for x_4 in the current dictionary:

$$x_4 = 5 - 2x_1 - 3x_2 - x_3.$$

By moving x_1 to the left hand side of this equation and x_4 to the right, we get the new equation

$$2x_1 = 5 - x_4 - 3x_2 - x_3$$

or equivalently

$$x_1 = \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3.$$

The variable x_1 can now be *eliminated* from the remaining two equations in the dictionary by substituting in this equation for x_1 where it appears in these equations:

$$\begin{aligned}
x_1 &= \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \\
x_5 &= 11 - 4\left(\frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3\right) - x_2 - 2x_3 \\
&= 1 + 2x_4 + 5x_2 \\
x_6 &= 8 - 3\left(\frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3\right) - 4x_2 - 2x_3 \\
&= \frac{1}{2} + \frac{3}{2}x_4 + \frac{1}{2}x_2 - \frac{1}{2}x_3 \\
z &= 5\left(\frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3\right) + 4x_2 + 3x_3 \\
&= \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}x_3.
\end{aligned}$$

When this substitution is complete, we have the new dictionary and the new BFS:

$$\begin{aligned}
x_1 &= \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \\
x_5 &= 1 + 2x_4 + 5x_2 \\
x_6 &= \frac{1}{2} + \frac{3}{2}x_4 + \frac{1}{2}x_2 - \frac{1}{2}x_3 \\
z &= \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}x_3,
\end{aligned} \tag{2.6}$$

and the associated BFS is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1/2 \end{pmatrix} \qquad \text{with} \qquad z = \frac{25}{2}. \tag{2.7}$$

This process should seem very familiar to you. It is simply Gaussian elimination. As we know from our knowledge of linear systems of equations, Gaussian elimination can be performed in a matrix context with the aid of the augmented matrix (or, simplex tableau) (2.3). We return to this observation later to obtain a more efficient computational technique.

We now have a new dictionary (2.6) which identifies the basic feasible solution (BFS) (2.7) with associated objective value $z = \frac{25}{2}$. Can we improve on this BFS and obtain a higher objective value? Let's try the same trick again, and repeat the process we followed in going from the initial dictionary (2.4) to the new dictionary (2.6). Note that the coefficient of x_3 in the representation of z in the new dictionary (2.6) is positive. Hence if we increase the value of x_3 from zero, we will increase the value of z. By how much can we increase the value of x_3 and yet keep all the remaining variables non-negative? As before, we see that the first equation in the dictionary (2.6) combined with the need to keep x_1 non-negative implies that we cannot increase x_3 by more than $(5/2)/(1/2) = 5$. However, the second equation in (2.6) places no restriction on increasing x_3 since x_3 does not appear in this equation. Finally, the third equation in (2.6) combined with the need to keep x_6 non-negative implies that we cannot increase x_3 by more than $(1/2)/(1/2) = 1$. Therefore, in order to preserve the non-negativity of all variables, we can increase x_3 by at most

$$1 = \min\{5, 1\}.$$

When we do this x_6 is driven to zero, so x_3 enters the basis and x_6 leaves. More precisely, first move x_3 to the left hand side of the defining equation for x_6 in (2.6),

$$\frac{1}{2}x_3 = \frac{1}{2} + \frac{3}{2}x_4 + \frac{1}{2}x_2 - x_6,$$

or, equivalently,

$$x_3 = 1 + 3x_4 + x_2 - 2x_6,$$

then substitute this expression for x_3 into the remaining equations,

$$\begin{aligned} x_1 &= \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}[1 + 3x_4 + x_2 - 2x_6] \\ &= 2 - 2x_4 - 2x_2 + x_6 \\ x_5 &= 1 + 2x_4 + 5x_2 \\ z &= \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}[1 + 3x_4 + x_2 - 2x_6] \\ &= 13 - x_4 - 3x_2 - x_6\ , \end{aligned}$$

yielding the dictionary

$$\begin{array}{rl} x_3 = & 1 + 3x_4 + x_2 - 2x_6 \\ x_1 = & 2 - 2x_4 + 2x_2 + x_6 \\ x_5 = & 1 \qquad\quad + 2x_4 + 2x_2 \\ z = & 13 - x_4 - 3x_2 - x_6 \end{array} \tag{2.8}$$

which identifies the feasible solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{2.9}$$

having objective value $z = 13$.

Can we do better? If we try the same trick again on the dictionary (2.8) we find that we are stuck since all of the coefficients of the nonbasic variables in the objective row $z = 13 - x_4 - 3x_2 - x_6$, are non-positive. Hence, increasing any one of their values will not increase the value of the objective. So it seems as though either the method has failed or the associated BFS (2.9) is optimal for the LP (2.1). We claim that the BFS (2.9) is indeed optimal for the LP (2.1). However, we delay proving this fact until we have gathered enough tools for this purpose.

The process of moving from one feasible dictionary to the next is called a *simplex pivot*. The process of stringing together a sequence of simplex pivots in order to locate an optimal solution is called the *Simplex Algorithm*. The simplex algorithm is considered one of the ten most important algorithmic discoveries of the 20th century

(http://www.uta.edu/faculty/rcli/TopTen/topten.pdf).

The algorithm was discovered by George Dantzig (1914-2005) who is known as the father of linear programming. In 1984 Narendra Karmarkar published a paper describing a new approach to solving linear programs that was both numerically efficient and had *polynomial complexity*. This new class of methods are called *interior point* methods. These new methods have revolutionized the optimization field since their discovery, and they have led to efficient numerical methods for a wide variety of optimization problems well beyond the confines of linear programming. However, the simplex algorithm continues as an important numerical method for solving LPs, and for many specially structured LPs it is remains the most efficient algorithm.

2.2 Simplex Pivoting: Tableau Format (Augmented Matrix Format)

We now review the implementation of the simplex algorithm by applying Gaussian elimination to the augmented matrix (2.3), also known as the simplex tableau. For this problem, the initial simplex tableau is given by

$$
\left[\begin{array}{ccc|c} 0 & A & I & b \\ -1 & c & 0 & 0 \end{array}\right] = \begin{array}{c} \begin{array}{ccccccc} z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \left[\begin{array}{ccccccc|c} 0 & 2 & 3 & 1 & 1 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 \\ -1 & 5 & 4 & 3 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} . \tag{2.10}
$$

Each simplex pivot on a dictionary corresponds to one step of Gaussian elimination on the augmented matrix associated with the dictionary. For example, in the first simplex pivot, x_1 enters the basis and x_4 leaves the basis. That is, we use the first equation of the dictionary to rewrite x_1 as a function of the remaining variables, and then use this representation to eliminate x_1 form the remaining equations. In terms of the augmented matrix (2.10), this corresponds to first making the coefficient for x_1 in the first equation the number 1 by dividing this first equation through by 2. Then use this entry to eliminate the column under x_1, that is, make all other entries in this column zero (Gaussian elimination):

$$
\begin{array}{ccccccc|cccc}
 & \text{Pivot column} & & & & & & & \text{ratios} & & \\
 & \downarrow & & & & & & & \downarrow & & \\
z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & & & & \\
0 & 2 & 3 & 1 & 1 & 0 & 0 & 5 & 5/2 & \leftarrow & \text{Pivot row} \\
0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 & 11/4 & & \\
0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 & 8/3 & & \\
\hline
-1 & 5 & 4 & 3 & 0 & 0 & 0 & 0 & & & \\
\hline
\hline
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 & & & \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 & & & \\
0 & 0 & -1/2 & 1/2 & -3/2 & 0 & 1 & 1/2 & & & \\
\hline
-1 & 0 & -7/2 & 1/2 & -5/2 & 0 & 0 & -25/2 & & & \\
\hline
\end{array}
$$

In this illustration, we have placed a line above the cost row to delineate its special roll in the pivoting process. In addition, we have also added a column on the right hand side which contains the ratios that we computed in order to determine the pivot row. Recall that we must use the smallest ratio in order to keep all variables in the associated BFS non-negative. Note that we performed the exact same arithmetic operations but in the more efficient matrix format. The new augmented matrix,

$$
\begin{array}{c} \begin{array}{ccccccc} z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \left[\begin{array}{ccccccc|c} 0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 \\ 0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & -1/2 & 1/2 & -3/2 & 0 & 1 & 1/2 \\ -1 & 0 & -7/2 & 1/2 & -5/2 & 0 & 0 & -25/2 \end{array}\right] \end{array} , \tag{2.11}
$$

is the augmented matrix for the dictionary (2.6).

The initial augmented matrix (2.10) has basis x_4, x_5, and x_6. The columns associated with these variables in the initial tableau (2.10) are distinct columns of the identity matrix. Correspondingly, the basis for the second tableau is x_1, x_5, and x_6, and again this implies that the columns for these variables in the tableau (2.11) are distinct columns of the identity matrix. In tableau format, this will always be true of the basic variables, i.e., their associated columns are distinct columns of the identity matrix. To recover the BFS (basic feasible solution) associated with this tableau we first set the non-basic variables equal to zero (i.e. the variables not associated with columns of the identity matrix (except in very unusual circumstances)): $x_2 = 0$, $x_3 = 0$, and $x_4 = 0$. To find the value of the basic variables go to the column associated with that variable (for example, x_1 is in the second column), in that column find the row with the number 1 in it, then in that row go to the number to the right of the vertical bar (for x_1 this is the first row with the number to the right of the bar being 5/2). Then set this basic variable equal to that number ($x_1 = 5/2$). Repeating this for x_5 and x_6 we get $x_5 = 1$ and $x_6 = 1/2$. To get the corresponding value for z, look at the z row and observe that the corresponding linear equation is

$$-z - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4 = -\frac{25}{2},$$

but x_2, x_3, and x_4 are non-basic and so take the value zero giving $-z = -25/2$, or $z = 25/2$.

Of course this is all exactly the same information we obtained from the dictionary approach. The simplex, or augmented matrix approach is simply a more efficient computational procedure. For computational purposes, we use the tableau form of the simplex in to solve specific LPs. However, in order to understand the inner workings of the algorithm it is *essential* that you understand how to go back and forth between these two representations, i.e the dictionary representation and its corresponding simplex tableau (or, augmented matrix). Let us now continue with the second simplex pivot.

In every tableau we always reserve the bottom row for encoding the linear relationship between the objective variable z and the currently non-basic variables. For this reason we call this row the *objective row*, and to distinguish its special role, we place a line above it in the tableau (this is reminiscent of the way we place a vertical bar in an augmented matrix to distinguish the right hand side of a linear equation). In the objective row of the tableau (2.11),

$$[-1,\ 0,\ -7/2,\ 1/2,\ -5/2,\ 0,\ 0,\ |\ -25/2],$$

we see a positive coefficient, 1/2, in the 4th column. Hence the objective row coefficient for the non-basic variable x_3 in this tableau is 1/2. This indicates that if we increase the the value of x_3, we also increase the value of the objective z. This is not true for any of the other currently non-basic variables since their cost row coefficients are all non-positive. Thus, the only way to increase the value of z is to bring x_3 into the basis, or equivalently, pivot on the x_3 column which is the 4th column of the tableau. For this reason, we call the x_3 column the *pivot column*. Now if x_3 is to enter the basis, then which variable leaves? Just as with the dictionary representation, the variable that leaves the basis is that currently basic variable whose non-negativity places the greatest restriction on increasing the value of x_3. This restriction is computed as the smallest ratio of the right hand sides and the positive coefficients in the x_3 column:

$$1 = \min\{(5/2)/(1/2),\ (1/2)/(1/2)\}.$$

The ratios are only computed with the positive coefficients since a non-positive coefficient means that by increasing this variable we do not decrease the valued of the corresponding basic variable and so it is not a restricting equation. Since the minimum ratio in this instance is 1 and it comes from the third row, we find that the *pivot row* is the third row. Looking across the third row, we see that this row identifies x_6 as a basic variable since the x_6 column is a column of the identity with a 1 in the third row. Hence x_6 is the variable leaving the basis when x_3 enters. The intersection of the pivot column and the pivot row is called the *pivot*. In this instance it is the number $1/2$ which is the $(3,4)$ entry of the simplex tableau. Pivoting on this entry requires us to first make it 1 by multiplying this row through by 2, and then to apply Gaussian elimination to force all other entries in this column to zero:

$$\begin{array}{rrrrrrr|r c l}
 & & & \text{Pivot} & & & & & & \\
 & & & \text{column} & & & & & \text{ratios} & \\
 & & & \downarrow & & & & & & \\
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 & 5 & \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 & & \\
0 & 0 & -1/2 & 1/2 & -3/2 & 0 & 1 & 1/2 & 1 & \leftarrow \ \text{pivot row} \\
\hline
-1 & 0 & -7/2 & 1/2 & -5/2 & 0 & 0 & -25/2 & & \\
\hline
\hline
0 & 1 & 2 & 0 & 2 & 0 & -1 & 2 & & \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 & & \\
0 & 0 & -1 & 1 & -3 & 0 & 2 & 1 & & \\
\hline
-1 & 0 & -3 & 0 & -1 & 0 & -1 & -13 & &
\end{array}$$

This simplex tableau is said to be optimal since it is feasible (the associated BFS is non-negative) and the cost row coefficients for the variables are all non-positive. A BFS that is optimal is called and *optimal basic feasible solution*. The optimal BFS is obtained by setting the non-basic variables equal to zero and setting the basic variables equal to the value on the right hand side corresponding to the one in its column: $x_1 = 2,\ x_2 = 0,\ x_3 = 1,\ x_4 = 0,\ x_5 = 1,\ x_6 = 0$. The optimal objective value is obtained by taking the negative of the number in the lower right hand corner of the optimal tableau: $z = 13$.

We now recap the complete sequence of pivots in order to make a final observation that will help streamline the pivoting process: pivots are circled,

z	x_1	x_2	x_3	x_4	x_5	x_6	
0	2	3	1	1	0	0	5
0	4	1	2	0	1	0	11
0	3	4	2	0	0	1	8
-1	5	4	3	0	0	0	0
0	1	$3/2$	$1/2$	$1/2$	0	0	$5/2$
0	0	-5	0	-2	1	0	1
0	0	$-1/2$	$1/2$	$-3/2$	0	1	$1/2$
-1	0	$-7/2$	$1/2$	$-5/2$	0	0	$-25/2$
0	1	2	0	2	0	-1	2
0	0	-5	0	-2	1	0	1
0	0	-1	1	-3	0	2	1
-1	0	-3	0	-1	0	-1	-13

Observe from this sequence of pivots that the z column is never touched, that is, it remains the same in all tableaus. Essentially, it just serves as a place holder reminding us that in the linear equation for the cost row the coefficient of z is -1. Therefore, for the sake of expediency we will drop this column from our simplex computations in most settings, and simply re-insert it whenever instructive or convenient. However, *it is of great importance to always remember that it is there*! Indeed, we will make explicit and essential use of this column in order to arrive at a full understanding of the duality theory for linear programming. After removing this column, the above pivots take the following form:

x_1	x_2	x_3	x_4	x_5	x_6	
2	3	1	1	0	0	5
4	1	2	0	1	0	11
3	4	2	0	0	1	8
5	4	3	0	0	0	0
1	$3/2$	$1/2$	$1/2$	0	0	$5/2$
0	-5	0	-2	1	0	1
0	$-1/2$	$1/2$	$-3/2$	0	1	$1/2$
0	$-7/2$	$1/2$	$-5/2$	0	0	$-25/2$
1	2	0	2	0	-1	2
0	-5	0	-2	1	0	1
0	-1	1	-3	0	2	1
0	-3	0	-1	0	-1	-13

We close this section with a final example of simplex pivoting on a tableau giving only the essential details.

The LP

$$\begin{array}{lrcrcrcl} \text{maximize} & 3x_1 & + & 2x_2 & - & 4x_3 & & \\ & & & & & & & \\ \text{subject to} & x_1 & + & 4x_2 & & & \leq & 5 \\ & 2x_1 & + & 4x_2 & - & 2x_3 & \leq & 6 \\ & x_1 & + & x_2 & - & 2x_3 & \leq & 2 \\ & 0 & \leq & x_1, & x_2, & x_3 & & \end{array}$$

Simplex Iterations

x_1	x_2	x_3	x_4	x_5	x_6		ratios
1	4	0	1	0	0	5	5
2	4	-2	0	1	0	6	3
1	1	-2	0	0	1	2	2
3	2	-4	0	0	0	0	
0	3	2	1	0	-1	3	3/2
0	2	2	0	1	-2	2	1
1	1	-2	0	0	1	2	
0	-1	2	0	0	-3	-6	
0	1	0	1	-1	1	1	
0	1	1	0	1/2	-1	1	
1	3	0	0	1	-1	4	
0	-3	0	0	-1	-1	-8	

Optimal Solution

$$\begin{bmatrix} x_x \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \qquad \text{optimal value} \;=\; 8$$

This example illustrates a point that needs to be strongly emphasized. The pivot column in the second tableau is chosen to be the x_3 column since its objective row coefficient, "2", is the only positive entry in the objective row. Hence it is the only non-basic variable whose increase will increase the objective since the objective row in the dictionary is

$$z = 6 - x_2 + 2x_3 - 3x_6.$$

To continue pivoting, we now choose the pivot column by forming the ratios as shown, but we did not form the ratio associated with the entry "-2" in the pivot column. To see why, write out the row of the associated dictionary for the "-2" row. This gives

$$x_1 = 2 - x_2 + 2x_3 - x_6. \tag{2.12}$$

Since the pivot column is the third column, x_3 is the variable entering the basis. That is, on this pivot we increase the value of the currently nonbasic variable x_3 from zero to some positive number. The amount of increase is restricted by the need to keep all of the currently basic variables non-negative. This is why we form the ratios. For example, equation (2.12) above defines the basic variable x_1 in terms of the nonbasic variables x_2, x_3 and x_6. If we now increase the value of x_3 from zero, the value of x_1 increases as well. Consequently, this equation places no restriction on increasing the value of x_3. This is why we do not need to form a ratio for this row since this row places no restriction. *In general, any negative entry in the pivot column that is non-positive does not yield a restriction on the on the value of the incoming variable.* Thus, one does not need to compute the ratios associated with non-positive values.

A final word of advise; when pivoting by hand, it is helpful to keep the tableaus vertically aligned in order to keep track of the arithmetic operations. This allows you to find errors quickly, and errors will occur. Lined paper helps to keep the rows straight. But the columns need to be straight as well. Many students find that it is easy to keep both the rows and columns straight if they do pivoting on graph paper having large boxes for the numbers.

2.3 Dictionaries: The General Case for LPs in Standard Form

Recall the following standard form for LPs:

$$\begin{array}{lll} \mathcal{P}: & \text{maximize} & c^T x \\ & \text{subject to} & Ax \le b \\ & & 0 \le x\ , \end{array}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$ and the inequalities $Ax \le b$ and $0 \le x$ are to be interpreted componentwise. We now provide a formal definition for a dictionary associated with an LP in standard form. Let

$$(D_I) \qquad \begin{array}{rl} x_{n+i} = & b_i - \sum_{j=1}^{n} a_{ij}x_j \\ z = & \sum_{j=1}^{n} c_j x_j \end{array}$$

be the defining system for the slack variables x_{n+i}, $i = 1, \cdots, n$ and the objective variable z. A dictionary for $\mathcal{P}$ is any system of the form

$$(D_B) \qquad \begin{array}{rll} x_i = & \widehat{b}_i - \sum_{j\in N} \widehat{a}_{ij}x_j & i \in B \\ z = & \widehat{z} + \sum_{j\in N} \widehat{c}_j x_j & \end{array}$$

where B and N are index sets contained in the set of integers $\{1, \dots, n+m\}$ satisfying

(1) B contains m elements,

(2) $B \cap N = \emptyset$

(3) $B \cup N = \{1, 2, \ldots, n + m\}$,

and such that the systems (D_I) and (D_B) have identical solution sets. The set $\{x_j : j \in B\}$ is said to be the basis associated with the dictionary (D_B) (we also refer to the index set B as the basis for the sake of simplicity), and the variables x_i, $i \in N$ are said to be the non-basic variables associated with this dictionary. The *basic solution* identified by this dictionary is

$$
\begin{aligned}
x_i &= \widehat{b}_i & i &\in B \\
x_j &= 0 & j &\in N.
\end{aligned}
\tag{2.13}
$$

The dictionary is said to be feasible if $0 \le \widehat{b}_i$ for $i \in B$. If the dictionary D_B is feasible, then the basic solution identified by the dictionary (2.13) is said to be a *basic feasible solution* (BFS) for the LP. A feasible dictionary and its associated BFS are said to be *optimal* if $\widehat{c}_j \le 0$ $j \in N$. At the end of this section, we show that optimal basic feasible solutions are optimal solutions to the linear program $\mathcal{P}$.

Simplex Pivoting by Matrix Multiplication

As we have seen simplex pivoting can either be performed on dictionaries or on the augmented matrices that encode the linear equations of a dictionary in matrix form. In matrix form, simplex pivoting reduces to our old friend, Gaussian elimination. In this section, we show that Gaussian elimination can be represented as a consequence of left multiplication by a specially designed matrix called a *Gaussian pivot matrix*.

Consider the vectors $e_j \in \mathbb{R}^n$, $j = 1, \ldots, n$, where each e_j is is defined to be the vector having a one in the jth position and zeros elsewhere. For example, in $\mathbb{R}^4$, we have

$$
e_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad e_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \text{and} \quad e_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.
$$

The set of vectors $\{e_1, e_2, \ldots, e_n\}$ in $\mathbb{R}^n$ is called the *standard unit coordinate basis* for $\mathbb{R}^n$. It is clearly a basis for $\mathbb{R}^n$ in the sense that these vectors are linearly independent and they span $\mathbb{R}^n$. They are called unit vectors since their magnitude is 1. Also observe that they form the columns of the $n \times n$ identity matrix $I_{n \times n}$, i.e

$$
I_{n \times n} = [e_1\, e_2\, \ldots\, e_n].
$$

Next consider a vector $v \in \mathbb{R}^m$ block decomposed as

$$
v = \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix}
$$

where $a \in \mathbb{R}^s$, $\alpha \in \mathbb{R}$, and $b \in \mathbb{R}^t$ with $m = s + 1 + t$. Assume that $\alpha \neq 0$. We wish to determine a matrix G such that

$$Gv = e_{s+1}.$$

We claim that the block matrix

$$G = \begin{bmatrix} I_{s\times s} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{t\times t} \end{bmatrix}$$

does the trick. Indeed,

$$Gv = \begin{bmatrix} I_{s\times s} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{t\times t} \end{bmatrix} \begin{pmatrix} a \\ \alpha \\ b \end{pmatrix} = \begin{bmatrix} a - a \\ \alpha^{-1}\alpha \\ -b + b \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e_{s+1}.$$

The matrix G is called a *Gaussian Pivot Matrix*. Note that G is invertible since

$$G^{-1} = \begin{bmatrix} I & a & 0 \\ 0 & \alpha & 0 \\ 0 & b & I \end{bmatrix},$$

and that for any vector of the form $w = \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}$ where $x \in \mathbb{R}^s$ $y \in \mathbb{R}^t$, we have

$$Gw = w.$$

The Gaussian pivot matrices perform precisely the operations required in order to execute a simplex pivot. That is, each simplex pivot can be realized as left multiplication of the simplex tableau by the appropriate Gaussian pivot matrix.

For example, consider the following initial feasible tableau:

$$\left[\begin{array}{cccccc|c} 1 & 4 & 2 & 1 & 0 & 0 & 11 \\ 3 & ② & 1 & 0 & 1 & 0 & 5 \\ 4 & 2 & 2 & 0 & 0 & 1 & 8 \\ \hline 4 & 5 & 3 & 0 & 0 & 0 & 0 \end{array}\right]$$

where the $(2, 2)$ element is chosen as the pivot element. In this case,

$$s = 1, \quad t = 2, \quad a = 4, \quad \alpha = 2, \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 5 \end{bmatrix},$$

and so the corresponding Gaussian pivot matrix is

$$G_1 = \begin{bmatrix} I_{1\times 1} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{2\times 2} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & \frac{-5}{2} & 0 & 1 \end{bmatrix}.$$

Multiplying the simplex on the left by G_1 gives

$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & \frac{-5}{2} & 0 & 1 \end{bmatrix} \left[\begin{array}{cccccc|c} 1 & 4 & 2 & 1 & 0 & 0 & 11 \\ 3 & 2 & 1 & 0 & 1 & 0 & 5 \\ 4 & 2 & 2 & 0 & 0 & 1 & 8 \\ \hline 4 & 5 & 3 & 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccccc|c} -5 & 0 & 0 & 1 & -2 & 0 & 1 \\ \frac{3}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{5}{2} \\ 1 & 0 & 1 & 0 & -1 & 1 & 3 \\ \hline -\frac{7}{2} & 0 & \frac{1}{2} & 0 & \frac{-5}{2} & 0 & \frac{-25}{2} \end{array}\right].$$

Repeating this process with the new pivot element in the $(3, 3)$ position yields the Gaussian pivot matrix

$$G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{2} & 1 \end{bmatrix},$$

and left multiplication by G_2 gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{2} & 1 \end{bmatrix} \left[\begin{array}{cccccc|c} -5 & 0 & 0 & 1 & -2 & 0 & 1 \\ \frac{3}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{5}{2} \\ 1 & 0 & 1 & 0 & -1 & 1 & 3 \\ \hline \frac{-7}{2} & 0 & \frac{1}{2} & 0 & \frac{-5}{2} & 0 & \frac{-25}{2} \end{array}\right] = \left[\begin{array}{cccccc|c} -5 & 0 & 0 & 1 & -2 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & \frac{-1}{2} & 1 \\ 1 & 0 & 1 & 0 & -1 & 1 & 3 \\ \hline -4 & 0 & 0 & 0 & \frac{-3}{2} & \frac{-1}{2} & -14 \end{array}\right]$$

yielding an optimal tableau.

If

$$T_0 := \begin{bmatrix} A & I & b \\ c^T & 0 & 0 \end{bmatrix} \tag{2.4}$$

is the initial tableau, then

$$G_2 G_1 T_0 = \left[\begin{array}{cccccc|c} -5 & 0 & 0 & 1 & -2 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & \frac{-1}{2} & 1 \\ 1 & 0 & 1 & 0 & -1 & 1 & 3 \\ \hline -4 & 0 & 0 & 0 & -2 & \frac{-1}{2} & -14 \end{array}\right]$$

That is, we would be able to go directly from the initial tableau to the optimal tableau if we knew the matrix

$$G = G_2 G_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & \frac{-5}{2} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & \frac{-1}{2} & 1 \end{bmatrix}$$

beforehand. Moreover, the matrix G is invertible since both G_1 and G_2 are invertible:

$$G^{-1} = G_1^{-1}G_2^{-1} = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 5 & 3 & 1 \end{bmatrix}$$

(you should check that $GG^{-1} = I$ by doing the multiplication by hand). In general, every sequence of simplex pivots has a representation as left multiplication by a single invertible matrix since since pivoting corresponds to left multiplication of the tableau by a Gaussian pivot matrix, and Gaussian pivot matrices are always invertible. We now examine the consequence of this observation more closely in the general case. In this discussion, it is essential that we include the column associated with the objective variable z which we have largely ignored up to this point.

Recall the initial simplex tableau, or augmented matrix associated with the system (D_I):

$$T_0 = \begin{bmatrix} 0 & A & I & b \\ -1 & c^T & 0 & 0 \end{bmatrix} .$$

Observe that we include the first column, i.e. the column associated with the objective variable z in the augmented matrix. Let the matrix

$$T_k = \begin{bmatrix} 0 & \widehat{A} & R & \widehat{b} \\ -1 & \widehat{c}^T & -\widehat{y}^T & \widehat{z} \end{bmatrix}$$

be another simplex tableau obtained from the initial tableau after a sequnce of k simplex pivots. The first column remains unchanged. since simplex pivots do not alter the first column. This is the reason why it does not appear in our hand computations. However, in this discussion, its presence and the fact that it remains unchanged by simplex pivoting is key! Since T_k is another simplex tableau the $m \times (n+m)$ matrix $[\widehat{A} \quad R]$ must posses among its columns the m columns of the $m \times m$ identity matrix. These columns of the identity matrix correspond to the basic variables associated with this tableau (except in the very unusual case when there are more than m columns of the identity present).

Our prior discussion on Gaussian pivot matrices tells us that T_k can be obtained from T_0 by multiplying T_0 on the left by some nonsingular $(m+1) \times (m+1)$ matrix G where G is the product of a sequence of Gaussian pivot matrices. In order to better understand the action of G on T_0 we need to decompose G into a block structure that is conformal with that of T_0:

$$G = \begin{bmatrix} M & u \\ v^T & \beta \end{bmatrix},$$

where $M \in \mathbb{R}^{m\times m}$, $u, v \in \mathbb{R}^m$, and $\beta \in \mathbb{R}$. Then

$$\begin{aligned}\begin{bmatrix} 0 & \widehat{A} & R & \widehat{b} \\ -1 & \widehat{c}^T & -\widehat{y}^T & \widehat{z} \end{bmatrix} &= T_k \\ &= GT_0 \\ &= \begin{bmatrix} M & u \\ v^T & \beta \end{bmatrix} \begin{bmatrix} 0 & A & I & b \\ -1 & c^T & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -u & MA + uc^T & M & Mb \\ -\beta & v^TA + \beta c^T & v^T & v^Tb \end{bmatrix} .\end{aligned}$$

By equating the blocks in the matrices on the far left and far right hand sides of this equation, we find from the first column that

$$u = 0 \qquad \text{and} \qquad \beta = 1 .$$

Here we see the key role played by our knowledge of the structure of the first column, i.e. the z or objective variable column. From the $(1, 3)$ and the $(2, 3)$ terms on the far left and right hand sides of (2.5), we also find that

$$M = R, \qquad \text{and} \qquad v = -\widehat{y} .$$

Putting all of this together gives the following representation of the k^{th} tableau T_k:

$$T_k = \begin{bmatrix} R & 0 \\ -\widehat{y}^T & 1 \end{bmatrix} \begin{bmatrix} 0 & A & I & b \\ -1 & c^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & RA & R & Rb \\ -1 & c^T - \widehat{y}^TA & -\widehat{y}^T & -\widehat{y}^Tb \end{bmatrix} , \tag{2.5}$$

where the matrix R is necessarily invertible since the matrix

$$G = \begin{bmatrix} R & 0 \\ -\widehat{y}^T & 1 \end{bmatrix}$$

is invertible (prove this!):

$$G^{-1} = \begin{bmatrix} R^{-1} & 0 \\ \widehat{y}^T R^{-1} & 1 \end{bmatrix}. \qquad \text{(check this out by computing the product } GG^{-1}\text{)}$$

The matrix R is called the *record* matrix for the tableau as it keeps track of almost all of the transformations required to obtain the new tableau. Again, the variables associated with the columns of the identity correspond to the basic variables. The tableau T_k is said to be *primal feasible*, or just feasible, if $\widehat{b} = Rb \geq 0$.

The beautiful structure revealed by equation (2.5) is perhaps the most important equation to be given in our discussion of linear programming. It is of central importance to linear programming duality theory and sensitivity analysis. *Its importance cannot be overstated.*

As a first step toward understanding the central significance of the equation (2.5), consider the case where the tableau T_k on the right hand side of (2.5) is optimal, i.e.

$$\begin{bmatrix} 0 & RA & R & Rb \\ -1 & c^T - \widehat{y}^TA & -\widehat{y}^T & -\widehat{y}^Tb \end{bmatrix} \tag{2.6}$$

is an optimal tableau for $\mathcal{P}$. Recall that (2.6) is optimal if and only if it is feasible, $Rb \geq 0$, and all of the variable coefficients in the objective row are non-positive,

$$A^T\widehat{y} \geq c \quad \text{and} \quad 0 \leq \widehat{y} \quad \text{with} \quad z = b^T\widehat{y}. \tag{2.7}$$

The BFS associated with this tableau, x, yields a vector $\widehat{x}$ with $\widehat{x}_j = x_j,\ j = 1, 2, \ldots, n$ such that

$$A\widehat{x} \leq b \quad \text{and} \quad 0 \leq \widehat{x} \quad \text{with} \quad z = c^T\widehat{x}.$$

The vector $\widehat{x}$ corresponds to the vector x with all of the slack variables removed. In particular, $\widehat{x}$ is feasible for $\mathcal{P}$ with objective value $c^T\widehat{x} = z = b^T\widehat{y}$. Now observe that the system (2.7) says that $\widehat{y}$ is feasible for the dual problem $\mathcal{D}$ with dual objective value $b^T\widehat{y}$. **This is absolutely amazing** since the Weak Duality Theorem now tells us that $\widehat{x}$ solves $\mathcal{P}$ and $\widehat{y}$ solves $\mathcal{D}$!!! That is, any optimal tableau similtaneously gives optimal solutions to *both* the primal and dual problems! We have just proved the following theorem.

Theorem 2.1 (Optimal Tableau Theorem) *Let x be the basic feasible solution for the tableau* (2.6)*. If* (2.6) *is an optimal tableau for the linear program $\mathcal{P}$, then $\widehat{y}$ is an optimal solution to the dual problem $\mathcal{D}$ and the vector $\widehat{x} \in \mathbb{R}^n$ given by $\widehat{x}_j = x_j,\ j = 1, 2, \ldots, n$ is an optimal solution to $\mathcal{P}$.*

Consequently, if the simplex algorithm works, in the sense that it arrives at an optimal tableau after a finite number of simplex pivots, then we have a method for solving all LPs! Unfortunately, the situation is not as simple as this. First, not every LP is feasible, so a solution obviously cannot exist. Second, even if an LP is feasible, it may be unbounded, so, again, a solution does not exist. Finally, even if a solution exists, we have no guarantee that the simplex algorithm can find it after a finite number of pivots, or for that matter an infinite number of pivots. To understand the relationship between linear programs and the simplex algorithm, we need a much deeper understanding of the algorithm itself.

4 Duality Theory

We now dive deeply into the duality theory of linear programming. As we will see, the solution to the dual problem is most often just as important as the solution to the primal, and in some cases more important. Recall from Section 1 that the dual to an LP in standard form

$$(\mathcal{P}) \qquad \begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b,\ 0 \leq x \end{array}$$

is the LP

$$(\mathcal{D}) \qquad \begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y \geq c,\ 0 \leq y. \end{array}$$

Since the problem $\mathcal{D}$ is a linear program, it too has a dual. The *duality* terminology suggests that the problems $\mathcal{P}$ and $\mathcal{D}$ come as a pair implying that the dual to $\mathcal{D}$ should be $\mathcal{P}$. This is indeed the case as we now show. Observe that by using standard techniques the dual problem can converted to standard form:

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y \geq c, \\ & 0 \leq y \end{array} \quad \overset{\substack{\text{standard} \\ \text{form}}}{\Longrightarrow} \quad \begin{array}{ll} -\text{maximize} & (-b)^T y \\ \text{subject to} & (-A^T) y \leq (-c), \\ & 0 \leq y. \end{array}$$

The problem on the right is in standard form so we can take its dual to get an LP which also can be written in standard form:

$$\begin{array}{ll} \text{minimize} & (-c)^T x \\ \text{subject to} & (-A^T)^T x \geq (-b),\ 0 \leq x \end{array} \quad \overset{\substack{\text{standard} \\ \text{form}}}{\Longrightarrow} \quad \begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b,\ 0 \leq x\ . \end{array}$$

Consequently, the dual of the dual is the primal.

Next recall that the primal-dual pair of LPs $\mathcal{P} - \mathcal{D}$ are related via the Weak Duality Theorem.

Theorem 4.1 (Weak Duality Theorem) *If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then*

$$c^T x \leq y^T A x \leq b^T y.$$

Thus, if $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is necessarily infeasible, and if $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is necessarily infeasible. Moreover, if $c^T \bar{x} = b^T \bar{y}$ with $\bar{x}$ feasible for $\mathcal{P}$ and $\bar{y}$ feasible for $\mathcal{D}$, then $\bar{x}$ must solve $\mathcal{P}$ and $\bar{y}$ must solve $\mathcal{D}$.

We now use The Weak Duality Theorem in conjunction with The Fundamental Theorem of Linear Programming to prove the *Strong Duality Theorem of Linear Programming*. The key ingredient in this proof is the general form for simplex tableaus derived at the end of Section 2 in (2.5).

Theorem 4.2 (The Strong Duality Theorem of Linear Programming) *If either $\mathcal{P}$ or $\mathcal{D}$ has a finite optimal value, then so does the other, the optimal values coincide, and optimal solutions to both $\mathcal{P}$ and $\mathcal{D}$ exist.*

Remark: This result states that the finiteness of the optimal value implies the existence of a solution. This is not always the case for nonlinear optimization problems. Indeed, consider the problem

$$\min_{x \in \mathsf{R}} e^x.$$

This problem has a finite optimal value, namely zero; however, this value is not attained by any point $x \in \mathbb{R}$. That is, it has a finite optimal value, but a solution does not exist. The existence of solutions when the optimal value is finite is one of the many special properties of linear programs.

Proof: Since the dual of the dual is the primal, we may as well assume that the primal has a finite optimal value. In this case, the Fundamental Theorem of Linear Programming says that an optimal basic feasible solution exists. By our formula for the general form of simplex tableaus (2.5), we know that there exists a nonsingular record matrix $R \in \mathbb{R}^{n \times n}$ and a vector $y \in \mathbb{R}^m$ such that the optimal tableau has the form

$$\begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} A & I & b \\ c^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} RA & R & Rb \\ c^T - y^T A & -y^T & -y^T b \end{bmatrix}.$$

Since this is an optimal tableau, we have

$$c - A^T y \leq 0, \qquad\qquad -y^T \leq 0$$

with $y^T b$ equal to optimal value in the primal problem. But then $A^T y \geq c$ and $0 \leq y$ so that y is feasible for the dual problem $\mathcal{D}$. In addition, the Weak Duality Theorem implies that

$$\begin{array}{rcllcl} b^T y & = & \text{maximize} & c^T x & \leq & b^T \widehat{y} \\ & & \text{subject to} & Ax \leq b,\ 0 \leq x & & \end{array}$$

for every vector $\widehat{y}$ that is feasible for $\mathcal{D}$. Therefore, y solves $\mathcal{D}$!!!! ■

This is an amazing fact! Our method for solving the primal problem $\mathcal{P}$, the simplex algorithm, simultaneously solves the dual problem $\mathcal{D}$! This fact is of enormous practical value when we study sensitivity analysis.

4.1 Complementary Slackness

The Strong Duality Theorem tells us that optimality is equivalent to equality in the Weak Duality Theorem. That is, x solves $\mathcal{P}$ and y solves $\mathcal{D}$ if and only if (x, y) is a $\mathcal{P} - \mathcal{D}$ feasible pair and

$$c^T x = y^T A x = b^T y.$$

We now carefully examine the consequences of this equivalence. Note that the equation $c^T x = y^T Ax$ implies that

$$0 = x^T(A^T y - c) = \sum_{j=1}^{n} x_j(\sum_{i=1}^{m} a_{ij}y_i - c_j). \tag{4.1}$$

In addition, primal and dual feasibility implies that

$$0 \leq x_j \qquad \text{and} \qquad 0 \leq \sum_{i=1}^{m} a_{ij}y_i - c_j \qquad \text{for } j = 1, \ldots, n,$$

respectively, and so

$$x_j(\sum_{i=1}^{m} a_{ij}y_i - c_j) \geq 0 \qquad \text{for} \qquad j = 1, \ldots, n.$$

Hence, the only way (4.1) can hold is if

$$x_j(\sum_{i=1}^{m} a_{ij}y_i - c_j) = 0 \qquad \text{for} \qquad j = 1, \ldots, n.$$

or equivalently,

$$x_j = 0 \qquad \text{or} \qquad \sum_{i=1}^{m} a_{ij}y_i = c_j \qquad \text{or both for } j = 1, \ldots, n. \tag{4.2}$$

Similarly, $y^T Ax = y^T b$ tells us that

$$0 = y^T(b - Ax) = \sum_{i=1}^{m} y_i(b_i - \sum_{j=1}^{n} a_{ij}x_j).$$

Again, and primal dual feasibility implies that

$$0 \leq y_i \qquad \text{and} \qquad 0 \leq b_i - \sum_{j=1}^{n} a_{ij}x_j \qquad \text{for } i = 1, \ldots, m,$$

respectively. Thus, we must have

$$y_i(b_i - \sum_{j=1}^{n} a_{ij}x_j) = 0 \qquad \text{for} \qquad j = 1, \ldots, n,$$

or equivalently,

$$y_i = 0 \qquad \text{or} \qquad \sum_{j=1}^{n} a_{ij}x_j = b_i \qquad \text{or both for } i = 1, \ldots, m. \tag{4.3}$$

The two observations (4.2) and (4.3) combine to yield the following theorem.

Theorem 4.8 (The Complementary Slackness Theorem) *The vector $x \in \mathsf{R}^n$ solves $\mathcal{P}$ and the vector $y \in \mathsf{R}^m$ solves $\mathcal{D}$ if and only if x is feasible for $\mathcal{P}$ and y is feasible for $\mathcal{D}$ and*

(i) either $0 = x_j$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$ or both for $j = 1, \ldots, n$, and

(ii) either $0 = y_i$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$ or both for $i = 1, \ldots, m$.

Proof: If x solves $\mathcal{P}$ and y solves $\mathcal{D}$, then by the Strong Duality Theorem we have equality in the Weak Duality Theorem. But we have just observed that this implies (4.2) and (4.3) which are equivalent to (i) and (ii) above.

Conversely, if (i) and (ii) are satisfied, then we get equality in the Weak Duality Theorem. Therefore, by Theorem 4.2, x solves $\mathcal{P}$ and y solves $\mathcal{D}$. ■

The Complementary Slackness Theorem can be used to develop a test of optimality for a putative solution to $\mathcal{P}$ (or $\mathcal{D}$). We state this test as a corollary.

Corollary 4.1 *The vector $x \in \mathbb{R}^n$ solves $\mathcal{P}$ if and only if x is feasible for $\mathcal{P}$ and there exists a vector $y \in \mathsf{R}^m$ feasible for $\mathcal{D}$ such that*

(i) for each $i \in \{1, 2, \ldots, m\}$, if $\sum_{j=1}^{n} a_{ij} x_j < b_i$, then $y_i = 0$, and

(ii) for each $j \in \{1, 2, \ldots, n\}$, if $0 < x_j$, then $\sum_{i=1}^{m} a_{ij} y_i = c_j$.

Proof: (i) and (ii) implies equality in the Weak Duality Theorem. The primal feasibility of x and the dual feasibility of y combined with Theorem 4.1 yield the result. ■

We now show how to apply this Corollary to test whether or not a given point solves an LP. Recall that all of the nonbasic variables in an optimal BFS take the value zero, and, if the BFS is nondegenerate, then all of the basic variables are nonzero. That is, m of the variables in the optimal BFS are nonzero since every BFS has m basic variables. Consequently, among the n original decision variables and the m slack variables, m variables are nonzero at a nondegenerate optimal BFS. That is, among the constraints

$$\begin{array}{ll} 0 \le x_j & j = 1, \ldots, n, \\ 0 \le x_{n+i} = c_i - \sum_{i \in N} a_{ij} x_j & i = 1, \ldots, m \end{array}$$

m of them are strict inequalities. If we now look back at Corollary 4.1, we see that every nondegenerate optimal basic feasible solution yields a total of m equations that an optimal dual solution y must satisfy. That is, Corollary 4.1 tells us that the m optimal dual variables

y_i satisfy m equations. Therefore, we can write an $m \times m$ system of equations to solve for y. We illustrate this by applying Corollary 4.1 to the following LP

$$
\begin{array}{lrcrcrcrcrcl}
\text{maximize} & 7x_1 & + & 6x_2 & + & 5x_3 & - & 2x_4 & + & 3x_5 & & \\
\text{subject to} & x_1 & + & 3x_2 & + & 5x_3 & - & 2x_4 & + & 2x_5 & \le & 4 \\
 & 4x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & + & x_5 & \le & 3 \\
 & 2x_1 & + & 4x_2 & + & 4x_3 & - & 2x_4 & + & 5x_5 & \le & 5 \\
 & 3x_1 & + & x_2 & + & 2x_3 & - & x_4 & - & 2x_5 & \le & 1 \\
 & & & & & \multicolumn{5}{c}{0 \le x_1, x_2, x_3, x_4, x_5.} & &
\end{array} \tag{4.10}
$$

Does the point

$$x^T = (x_1, x_2, x_3, x_4, x_5) = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)$$

solves this LP? Following Corollary 4.1, if x is optimal, then x must be feasible for (4.10) and there must exists a vector $y \in \mathbb{R}^4$ feasible for the dual LP to (4.10) and which satisfies the conditions given in items (i) and (ii) of the corollary. To check that x is primal feasible fist observe that x is componentwise positive. Next, by plugging x into the remaining constraints for (4.10) we see that equality is attained in each of the constraints except the third:

$$
\begin{array}{rcrcrcrcrcl}
(0) & + & 3\left(\frac{4}{3}\right) & + & 5\left(\frac{2}{3}\right) & - & 2\left(\frac{5}{3}\right) & + & 2(0) & = & 4 \\
4(0) & + & 2\left(\frac{4}{3}\right) & - & 2\left(\frac{2}{3}\right) & + & \left(\frac{5}{3}\right) & + & (0) & = & 3 \\
2(0) & + & 4\left(\frac{4}{3}\right) & + & 4\left(\frac{2}{3}\right) & - & 2\left(\frac{5}{3}\right) & + & 5(0) & < & 5 \\
3(0) & + & \left(\frac{4}{3}\right) & + & 2\left(\frac{2}{3}\right) & - & \left(\frac{5}{3}\right) & - & 2(0) & = & 1.
\end{array}
$$

Hence, x is primal feasible. Moreover, by item (i) of Corollary 4.1, we see that the vector $y \in \mathbb{R}^4$ that we seek must have

$$y_3 = 0 \tag{4.11}$$

due to the strict inequality in the associated primal constraint. Since $x_2 > 0$, $x_3 > 0$, and $x_4 > 0$, item (ii) of Corollary 4.1 implies that the vector y we are looking for must also satisfy the dual equalities

$$
\begin{array}{rcrcrcrcr}
3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & = & 6 \\
5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & = & 5 \\
-2y_1 & + & y_2 & - & 2y_3 & - & y_4 & = & -2.
\end{array} \tag{4.12}
$$

Putting (4.11) and (4.12) together, we see that y must satisfy

$$
\begin{bmatrix} 3 & 2 & 4 & 1 \\ 5 & -2 & 4 & 2 \\ -2 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ -2 \\ 0 \end{pmatrix},
$$

where the first three rows come from (4.12) and the last row comes from (4.11). We reduce the associated augmented system as follows:

$$
\begin{array}{rrrr|r l}
3 & 2 & 4 & 1 & 6 & \\
5 & -2 & 4 & 2 & 5 & \\
-2 & 1 & -2 & -1 & -2 & \\
0 & 0 & 1 & 0 & 0 & \\
\hline
3 & 2 & 0 & 1 & 6 & r_1 - 4r_4 \\
5 & -2 & 0 & 2 & 5 & r_2 - 4r_4 \\
-2 & 1 & 0 & -1 & -2 & r_3 + 2r_4 \\
0 & 0 & 1 & 0 & 0 & \\
\hline
1 & 3 & 0 & 0 & 4 & r_1 + r_3 \\
1 & 0 & 0 & 0 & 1 & r_2 + 2r_3 \\
-2 & 1 & 0 & -1 & -2 & \\
0 & 0 & 1 & 0 & 0 & \\
\hline
0 & 3 & 0 & 0 & 3 & r_1 - r_2 \\
1 & 0 & 0 & 0 & 1 & \\
0 & 1 & 0 & -1 & 0 & r_3 + 2r_2 \\
0 & 0 & 1 & 0 & 0 & \\
\hline
1 & 0 & 0 & 0 & 1 & r_2 \\
0 & 1 & 0 & 0 & 1 & \frac{1}{3}r_1 \\
0 & 0 & 1 & 0 & 0 & r_4 \\
0 & 0 & 0 & 1 & 1 & -r_3 + \frac{1}{3}r_1
\end{array}
$$

This gives $y^T = (1, 1, 0, 1)$ as the only possible vector y that can satisfy the requirements of (i) and (ii) in Corollary 4.1. It remains to check that this y is dual feasible, that is, we need check that y is feasible for the dual LP to (4.10):

$$
\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
 & 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
 & 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
 & -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
 & 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
 & & & 0 \leq y_1, y_2, y_3, y_4. & & & & & &
\end{array}
$$

Clearly, $0 \leq y$ and, by construction, the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality. Thus, it only remains to check the first and fifth inequalities:

$$
\begin{array}{rcrcrcrcl}
(1) & + & 4(1) & + & 2(0) & + & 3(1) & = & 8 \geq 7 \\
2(1) & + & (1) & + & 5(0) & - & 2(1) & = & 1 \not\geq 3.
\end{array}
$$

Therefore, y is not dual feasible. But as observed, this is the only possible vector y satisfying (i) and (ii) of Corollary (4.1), hence $x^T = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)$ cannot be a solution to the LP (4.10).

4.2 General Duality Theory

Thus far we have discussed duality theory as it pertains to LPs in standard form. Of course, one can always transform any LP into one in standard form and then apply the duality theory. However, from the perspective of applications, this is cumbersome since it obscures the meaning of the dual variables. It is very useful to be able to compute the dual of an LP without first converting to standard form. In this section we show how this can easily be done. For this, we still make use of a standard form, but now we choose one that is much more flexible:

$$\begin{array}{lllr}\mathcal{P} & \max & \sum_{j=1}^{n} c_j x_j & \\ & \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \le b_i & i \in I \\ & & \sum_{j=1}^{n} a_{ij} x_j = b_i & i \in E \\ & & 0 \le x_j & j \in R \quad . \end{array}$$

Here the index sets I, E, and R are such that

$$I \cap E = \emptyset,\ I \cup E = \{1, 2, \ldots, m\},\ \text{and } R \subset \{1, 2, \ldots, n\}.$$

We use the following primal-dual correspondences to compute the dual of an LP.

In the Dual	In the Primal
Restricted Variables	Inequality Constraints
Free Variables	Equality Constraints
Inequality Constraints	Restricted Variables
Equality Constraints	Free Variables

Using these rules we obtain the dual to $\mathcal{P}$:

$$\begin{array}{lllr}\mathcal{D} & \min & \sum_{i=1}^{m} b_i y_i & \\ & \text{subject to} & \sum_{i=1}^{m} a_{ij} y_i \ge c_j & j \in R \\ & & \sum_{i=1}^{m} a_{ij} y_i = c_j & j \in F \\ & & 0 \le y_i & i \in I \quad , \end{array}$$

where $F = \{1, 2, \ldots, n\} \setminus R$.

For example, the LP

$$\begin{array}{lll}\text{maximize} & x_1 - 2x_2 + 3x_3 & \\ \text{subject to} & 5x_1 + x_2 - 2x_3 & \le 8 \\ & -x_1 + 5x_2 + 8x_3 & = 10 \\ & x_1 \le 10,\ 0 \le x_3 & \end{array}$$

has dual

$$\begin{array}{lll}\text{minimize} & 8y_1 + 10y_2 + 10y_3 & \\ \text{subject to} & 5y_1 - y_2 + y_3 & = 1 \\ & y_1 + 5y_2 & = -2 \\ & -2y_1 + 8y_2 & \ge 3 \\ & 0 \le y_1,\ 0 \le y_3 & . \end{array}$$

The primal-dual pair $\mathcal{P}$ and $\mathcal{D}$ above are related by the following weak duality theorem.

Theorem 4.9 *[General Weak Duality Theorem]*
Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then

$$c^T x \le y^T A x \le b^T y.$$

Moreover, the following statements hold.

(i) If $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is infeasible.

(ii) If $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is infeasible.

(iii) If $\bar{x}$ is feasible for $\mathcal{P}$ and $\bar{y}$ is feasibe for $\mathcal{D}$ with $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is and optimal solution to $\mathcal{P}$ and $\bar{y}$ is an optimal solution to $\mathcal{D}$.

Proof: Suppose $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$. Then

$$\begin{aligned}
c^T x &= \sum_{j\in R} c_j x_j + \sum_{j\in F} c_j x_j \\
&\le \sum_{j\in R}(\sum_{i=1}^m a_{ij}y_i)x_j + \sum_{j\in F}(\sum_{i=1}^m a_{ij}y_i)x_j \\
&\qquad\qquad \text{(Since } c_j \le \textstyle\sum_{i=1}^m a_{ij}y_i \text{ and } x_j \ge 0 \text{ for } j \in R \\
&\qquad\qquad\quad \text{and } c_j = \textstyle\sum_{i=1}^m a_{ij}y_i \text{ for } j \in F.\text{)} \\
&= \sum_{i=1}^m \sum_{j=1}^n a_{ij}y_i x_j \\
&= y^T A x \\
&= \sum_{i\in I}(\sum_{j=1}^n a_{ij}x_j)y_i + \sum_{i\in E}(\sum_{j=1}^n a_{ij}x_j)y_i \\
&\le \sum_{i\in I} b_i y_i + \sum_{i\in E} b_i y_i \\
&\qquad\qquad \text{(Since } \textstyle\sum_{j=1}^n a_{ij}x_j \le b_i \text{ and } 0 \le y_i \text{ for } i \in I \\
&\qquad\qquad\quad \text{and } \textstyle\sum_{j=1}^n a_{ij}x_j = b_i \text{ for } i \in E. \\
&= \sum_{i=1}^m b_i y_i \\
&= b^T y\ .
\end{aligned}$$

■

4.3 The Dual Simplex Algorithm

Consider the linear program

$$\begin{array}{lll}\mathcal{P} & \text{maximize} & -4x_1 - 2x_2 - x_3 \\ & \text{subject to} & -x_1 - x_2 + 2x_3 \le -3 \\ & & -4x_1 - 2x_2 + x_3 \le -4 \\ & & x_1 + x_2 - 4x_3 \le 2 \\ & & 0 \le x_1, x_2, x_3 \ ,\end{array}$$

and its dual

$$\begin{array}{lll}\mathcal{D} & \text{minimize} & -3y_1 - 4y_2 + 2y_3 \\ & \text{subject to} & -y_1 - 4y_2 + y_3 \ge -4 \\ & & -y_1 - 2y_2 + y_3 \ge -2 \\ & & 2y_1 + y_2 - 4y_3 \ge -1 \\ & & 0 \le y_1, y_2, y_3 \ .\end{array}$$

Problem $\mathcal{P}$ does not have feasible origin, and so it appears that one must apply Phase I of the two phase simplex algorithm to obtain an initial basic feasible solution. On the other hand, the dual problem $\mathcal{D}$ does have feasible origin. Is it possible to apply the simplex algorithm to $\mathcal{D}$ and avoid Phase I altogether? Yes, however, we do it in a way that may at first seem odd. We *reverse* the usual simplex procedure by choosing a pivot row first, and then choosing the pivot column. The initial tableau for the problem $\mathcal{P}$ is

x_1	x_2	x_3	x_4	x_5	x_6	
-1	-1	2	1	0	0	-3
-4	-2	1	0	1	0	-4
1	1	-4	0	0	1	2
-4	-2	-1	0	0	0	0

A striking and important feature of this tableau is that every entry in the cost row is nonpositive! This is exactly what we are trying to achieve by our pivots in the simplex algorithm. This is a consequence of the fact that the dual problem $\mathcal{D}$ has feasible origin. Any tableau having this property we will call *dual feasible*. Unfortunately, the tableau is not feasible since some of the right hand sides are negative. Henceforth, we will say that such a tableau is not *primal feasible*. That is, instead of saying that a tableau (or dictionary) is feasible or infeasible in the usual sense, we will now say that the tableau is *primal feasible*, respectively, *primal infeasible*.

Observe that if a tableau is *both* primal and dual feasible, then it must be optimal, i.e. the basic feasible solution that it identifies is an optimal solution. We now describe an implementation of the simplex algorithm, called the *dual simplex algorithm*, that can be applied to tableaus that are dual feasible but not primal feasible. Essentially it is the simplex algorithm applied to the dual problem but using the tableau structure associated with the primal problem. The goal is to use simplex pivots to attain primal feasibility while maintaining dual feasibility.

Consider the tableau above. The right hand side coefficients are -3, -4, and 2. These correspond to the cost coefficients of the dual objective. Note that this tableau also identifies a basic feasible solution for the dual problem by setting the dual variable equal to the negative of the cost row coefficients associated with the slack variables:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

The dual variables are currently "nonbasic" and so their values are zero. Next note that by increasing the value of either y_1 or y_2 we decrease the value of the dual objective since the coefficients of these variables are -3 and -4. In the simplex algorithm terminology, we can pivot on either the first or second row to decrease the value of the dual objective. Let's choose the first row as our pivot row. How do we choose the pivot column? Similar to the primal simplex algorithm, we choose the pivot column to maintain dual feasibility. Let us have a look at what this means by examining the dual constraints:

$$\begin{array}{rrrl} -y_1 & -4y_2 & +y_3 & \geq -4 \\ -y_1 & -2y_2 & +y_3 & \geq -2 \\ 2y_1 & +y_2 & -4y_3 & \geq -1. \end{array} \tag{4.13}$$

If we plug $y = 0$ into these inequalities, we interpret the "dual slack" in the three equations as $r := A^T y - c = (4, 2, 1)^T$, while the dual variables are $y = (0, 0, 0)^T$. We call this the dual basic feasible solution associated with this dual feasible tableau, with the components of y being non-basic (having the value zero) and those of r basic. If we increase the value of y_1 from zero, we decrease the value of the dual objective since the y_1 coefficient in the dual objective is -3. The dual inequalities in (4.13) limit the amount by which we can increase y_1 and preserve dual feasibility, i.e. $r = A^T y - c \geq 0$ and $y \geq 0$. The first dual inequality in (4.13) limits the increase in y_1 to 4, the second limits the increase to 2, while the final inequality does not limit y_1 at all since the coefficient on y_1 is positive.

This process of computing the largest possible increase in a dual variable while maintaining dual feasibility corresponds the similar process in the primal simplex algorithm where we computed ratios and chose the minimum ratio. In the dual simplex algorithm we again must compute ratios, but this time it is the ratios of the negative entries in the pivot row with the corresponding cost row entries:

ratios for the first two columns are 4 and 2

$$\begin{array}{|rrrrrr|r|l} -1 & \boxed{-1} & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\ -4 & -2 & 1 & 0 & 1 & 0 & -4 & \\ 1 & 1 & -4 & 0 & 0 & 1 & 2 & \\ \hline -4 & -2 & -1 & 0 & 0 & 0 & 0 & \end{array}$$

The smallest ratio is 2 so the pivot column is column 2 in the tableau, and the pivot element is therefore the (1,2) entry of the tableau. Note that this process of choosing the pivot is the reverse of how the pivot is chosen in the primal simplex algorithm. In the dual simplex algorithm we first choose a pivot row, then compute ratios to determine the pivot column which identifies the pivot. We now successive apply this process to the above tableau until optimality is achieved.

-1	$\boxed{-1}$	2	1	0	0	-3	$\leftarrow$ pivot row
-4	-2	1	0	1	0	-4	
1	1	-4	0	0	1	2	
-4	-2	-1	0	0	0	0	
1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	$\boxed{-2}$	1	0	1	-1	$\leftarrow$ pivot row
-2	0	-5	-2	0	0	6	
1	1	0	-2	0	-1	4	
-2	0	0	-7/2	1	-3/2	7/2	
0	0	1	-1/2	0	-1/2	1/2	
-2	0	0	-9/2	0	-5/2	17/2	optimal

Therefore, the optimal solutions to $\mathcal{P}$ and $\mathcal{D}$ are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 1/2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 9/2 \\ 0 \\ 5/2 \end{pmatrix},$$

respectively, with optimal value $z = -17/2$.

Next consider the LP

$$\begin{array}{lll} \mathcal{P} & \text{maximize} & -4x_1 - 2x_2 - x_3 \\ & \text{subject to} & -x_1 - x_2 + 2x_3 \leq -3 \\ & & -4x_1 - 2x_2 + x_3 \leq -4 \\ & & x_1 + x_2 - x_3 \leq 2 \\ & & 0 \leq x_1, x_2, x_3 \ . \end{array}$$

This LP differs from the previous LP only in the x_3 coefficient of the third linear inequality. Let's apply the dual simplex algorithm to this LP.

-1	$\boxed{-1}$	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-1	0	0	1	2	
-4	-2	-1	0	0	0	0	

1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	1	1	0	1	-1	← pivot row
-2	0	-5	-2	0	0	6	

The first dual simplex pivot is given above. Repeating this process again, we see that there is only one candidate for the pivot row in our dual simplex pivoting strategy. What do we do now? It seems as though we are stuck since there are no negative entries in the third row with which to compute ratios to determine the pivot column. What does this mean? Recall that we chose the pivot row because the negative entry in the right hand side implies that we can decrease the value of the dual objective by bring the dual variable y_3 into the dual basis. The ratios are computed to preserve dual feasibility. In this problem, the fact that there are no negative entries in the pivot row implies that we can increase the value of y_3 as much as we want without violating dual feasibility. That is, the dual problem is unbounded below, and so, by the weak duality theorem, the primal problem must be infeasible!

We will make extensive use of the dual simplex algorithm in our discussion of sensitivity analysis in linear programming.

1 LP Geometry

We now briefly turn to a discussion of LP geometry extending the geometric ideas developed in Section 1 for 2 dimensional LPs to n dimensions. In this regard, the key geometric idea is the notion of a hyperplane.

Definition 1.1 *A hyperplane in* $\mathbb{R}^n$ *is any set of the form*

$$H(a, \beta) = \{x : a^T x = \beta\}$$

where $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}$, *and* $a \neq 0$.

We have the following important fact whose proof we leave as an exercise for the reader.

Fact 1.2 $H \subset \mathbb{R}^n$ *is a hyperplane if and only if the set*

$$H - x_0 = \{x - x_0 : x \in H\}$$

where $x_0 \in H$ *is a subspace of* $\mathbb{R}^n$ *of dimension* $(n-1)$.

Every hyperplane $H(a, \beta)$ generates two closed half spaces:

$$H_+(a, \beta) = \{x \in \mathbb{R}^n : a^T x \geq \beta\}$$

and

$$H_-(a, \beta) = \{x \in \mathbb{R}^n : a^T x \leq \beta\}.$$

Note that the constraint region for a linear program is the intersection of finitely many closed half spaces: setting

$$H_j = \{x : e_j^T x \geq 0\} \qquad \text{for } j = 1, \ldots, n$$

and

$$H_{n+i} = \{x : \sum_{j=1}^{n} a_{ij} x_j \leq b_i\} \qquad \text{for } i = 1, \ldots, m$$

we have

$$\{x : Ax \leq b, 0 \leq x\} = \bigcap_{i=1}^{n+m} H_i.$$

Any set that can be represented in this way is called a *convex polyhedron.*

Definition 1.3 *Any subset of* $\mathbb{R}^n$ *that can be represented as the intersection of finitely many closed half spaces is called a convex polyhedron.*

Therefore, a linear programming is simply the problem of either maximizing or minimizing a linear function over a convex polyhedron. We now develop some of the underlying geometry of convex polyhedra.

Fact 1.4 *Given any two points in* $\mathbb{R}^n$*, say* x *and* y*, the line segment connecting them is given by*

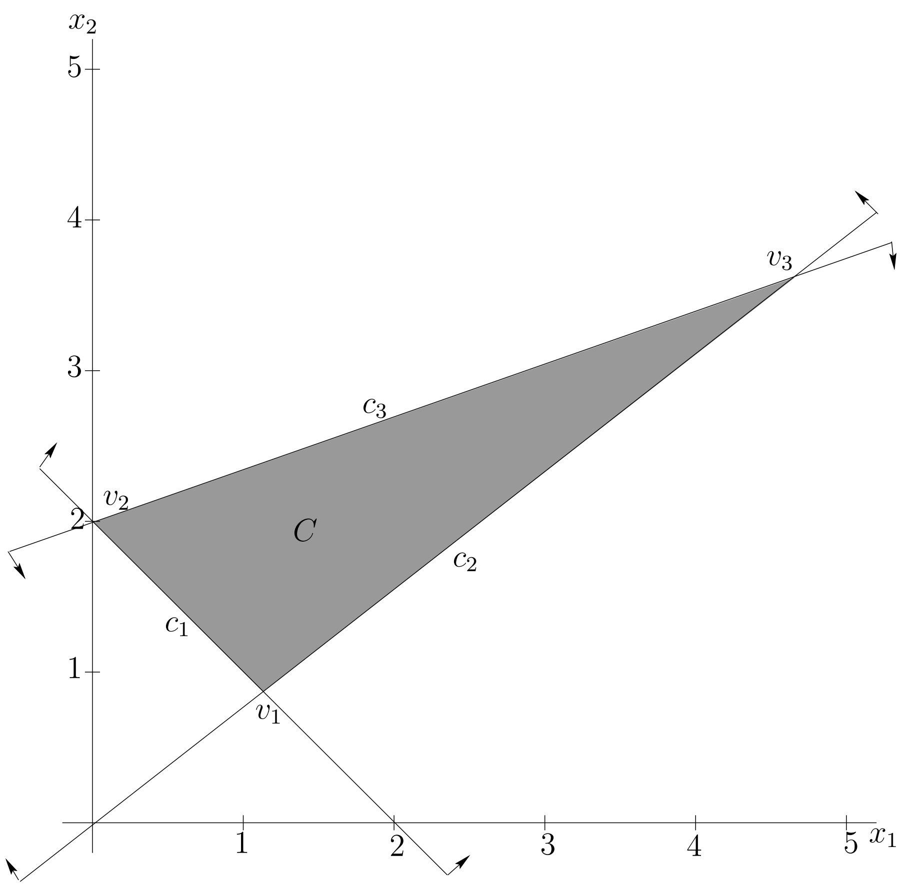
$$[x, y] = \{(1-\lambda)x + \lambda y : 0 \le \lambda \le 1\}.$$

Definition 1.5 *A subset* $C \in \mathbb{R}^n$ *is said to be convex if* $[x, y] \subset C$ *whenever* $x, y \in C$.

Fact 1.6 *A convex polyhedron is a convex set.*

We now consider the notion of vertex, or corner point, for convex polyhedra in $\mathbb{R}^2$. For this, consider the polyhedron $C \subset \mathbb{R}^2$ defined by the constraints

$$\begin{aligned} c_1 &: \quad -x_1 - x_2 \le -2 \\ c_2 &: \quad 3x_1 - 4x_2 \le 0 \\ c_3 &: \quad -x_1 + 3x_2 \le 6. \end{aligned} \tag{1.6}$$



The vertices are $v_1 = \left(\frac{8}{7}, \frac{6}{7}\right)$, $v_2 = (0, 2)$, and $v_3 = \left(\frac{24}{5}, \frac{18}{5}\right)$. One of our goals in this section is to discover an intrinsic geometric property of these vertices that generalizes to n dimensions and simultaneously captures our intuitive notion of what a vertex is. For this we examine our notion of convexity which is based on line segments. Is there a way to use line segments to make precise our notion of vertex?

Consider any of the vertices in the polyhedron C defined by (1.7). Note that any line segment in C that contains one of these vertices must have the vertex as one of its end points. Vertices are the only points that have this property. In addition, this property easily generalizes to convex polyhedra in $\mathbb{R}^n$. This is the rigorous mathematical formulation for our notion of vertex that we seek. It is simple, has intuitive appeal, and yields the correct objects in dimensions 2 and 3.

Definition 1.7 *Let C be a convex polyhedron. We say that $x \in C$ is a vertex of C if whenever $x \in [u, v]$ for some $u, v \in C$, it must be the case that either $x = u$ or $x = v$.*

This definition says that a point is a vertex if and only if whenever that point is a member of a line segment contained in the polyhedron, then it must be one of the end points of the line segment. In particular, this implies that vertices must lie in the boundary of the set and the set must somehow make a corner at that point. Our next result gives an important and useful characterization of the vertices of convex polyhedra.

Theorem 1.8 (Fundamental Representation Theorem for Vertices) *A point x in the convex polyhedron $C = \{x \in \mathbb{R}^s \,|\, Tx \le g\}$, where $T = (t_{ij})_{s \times n}$ and $g \in \mathbb{R}^s$, is a vertex of this polyhedron if and only if there exist an index set $\mathcal{I} \subset \{1, \dots, s\}$ with such that x is the unique solution to the system of equations*

$$\sum_{j=1}^{n} t_{ij} x_j = g_i \quad i \in \mathcal{I}. \tag{1.7}$$

Moreover, if x is a vertex, then one can take $|\mathcal{I}| = n$ in (1.7), where $|\mathcal{I}|$ denotes the number of elements in $\mathcal{I}$.

Proof: We first prove that if there exist an index set $\mathcal{I} \subset \{1, \dots, s\}$ such that $x = \bar{x}$ is the unique solution to the system of equations (1.7), then $\bar{x}$ is a vertex of the polyhedron C. We do this by proving the contraposition, that is, we assume that $\bar{x} \in C$ is not a vertex and show that it cannot be the unique solution to any system of the form (1.7) with $\mathcal{I} \subset \{1, 2, \dots, s\}$.

If $\bar{x}$ is not a vertex of C, then there exist $u, v \in C$ and $0 < \lambda < 1$ such that $\bar{x} = (1 - \lambda)u + \lambda v$. Let $\mathbb{A}(x)$ denote the set of *active indices* at x:

$$\mathbb{A}(x) = \left\{ i \;\middle|\; \sum_{j=1}^{n} t_{ij} x_j = g_i \right\}.$$

For every $i \in \mathbb{A}(\bar{x})$

$$\sum_{j=1}^{n} t_{ij} \bar{x}_j = g_i, \; \sum_{j=1}^{n} t_{ij} u_j \le g_i, \; \text{and} \; \sum_{j=1}^{n} t_{ij} v_j \le g_i. \tag{1.8}$$

Therefore,

$$\begin{aligned} 0 &= g_i - \sum_{j=1}^n t_{ij}\bar{x}_j \\ &= (1-\lambda)g_i + \lambda g_i - \sum_{j=1}^n t_{ij}((1-\lambda)u + \lambda v) \\ &= (1-\lambda)\left[g_i - \sum_{j=1}^n t_{ij}u_j\right] \ + \ \lambda\left[g_i - \sum_{j=1}^n t_{ij}v_j\right] \\ &\geq 0 \ . \end{aligned}$$

Hence,

$$0 = (1-\lambda)\left[g_i - \sum_{j=1}^n t_{ij}u_j\right] \ + \ \lambda\left[g_i - \sum_{j=1}^n t_{ij}v_j\right]$$

which implies that

$$g_i = \sum_{j=1}^n t_{ij}u_j \text{ and } g_i = \sum_{j=1}^n t_{ij}v_j$$

since both $\left[g_i - \sum_{j=1}^n t_{ij}u_j\right]$ and $\left[g_i - \sum_{j=1}^n t_{ij}v_j\right]$ are non-negative. That is, $\mathbb{A}(\bar{x}) \subset \mathbb{A}(u) \cap \mathbb{A}(v)$. Now if $\mathcal{I} \subset \{1, 2, \ldots, s\}$ is such that (1.7) holds at $x = \bar{x}$, then $\mathcal{I} \subset \mathbb{A}(\bar{x})$. But then (1.7) must also hold for $x = u$ and $x = v$ since $\mathbb{A}(\bar{x}) \subset \mathbb{A}(u) \cap \mathbb{A}(v)$. Therefore, $\bar{x}$ is not a unique solution to (1.7) for any choice of $\mathcal{I} \subset \{1, 2, \ldots, s\}$.

Let $\bar{x} \in C$. We now show that if $\bar{x}$ is a vertex of C, then there exist an index set $\mathcal{I} \subset \{1, \ldots, s\}$ such that $x = \bar{x}$ is the unique solution to the system of equations (1.7). Again we establish this by contraposition, that is, we assume that if $\bar{x} \in C$ is such that, for every index set $\mathcal{I} \subset \{1, 2, \ldots, s\}$ for which $x = \bar{x}$ satisfies the system (1.7) there exists $w \in \mathbb{R}^n$ with $w \neq \bar{x}$ such that (1.7) holds with $x = w$, then $\bar{x}$ cannot be a vertex of C . To this end take $\mathcal{I} = \mathbb{A}(\bar{x})$ and let $w \in \mathbb{R}^n$ with $w \neq \bar{x}$ be such that (1.7) holds with $x = w$ and $\mathcal{I} = \mathbb{A}(\bar{x})$, and set $u = w - \bar{x}$. Since $\bar{x} \in C$, we know that

$$\sum_{j=1}^n t_{ij}\bar{x}_j < g_i \quad \forall\ i \in \{1, 2, \ldots, s\} \setminus \mathbb{A}(\bar{x}) \ .$$

Hence, by continuity, there exists $\tau \in (0, 1]$ such that

$$\text{(1.9)} \qquad \sum_{j=1}^n t_{ij}(\bar{x}_j + tu_j) < g_i \quad \forall\ i \in \{1, 2, \ldots, s\} \setminus \mathbb{A}(\bar{x}) \text{ and } |t| \leq \bar{\tau}.$$

Also note that

$$\sum_{j=1}^n t_{ij}(\bar{x}_j \pm \tau u_j) = (\sum_{j=1}^n t_{ij}\bar{x}_j) \pm \tau(\sum_{j=1}^n t_{ij}w_j \ - \ \sum_{j=1}^n t_{ij}\bar{x}_j) = g_i \pm \tau(g_i - g_i) = g_i \ \forall\ i \in \mathbb{A}(\bar{x}).$$

Combining these equivalences with (1.9) we find that $\bar{x}+\tau u$ and $\bar{x}-\tau u$ are both in C. Since $x = \frac{1}{2}(x+\tau u) + \frac{1}{2}(x-\tau u)$ and $\tau u \neq 0$, $\bar{x}$ cannot be a vertex of C.

It remains to prove the final statement of the theorem. Let $\bar{x}$ be a vertex of C and let $\mathcal{I} \subset \{1, 2, \ldots, s\}$ be such that $\bar{x}$ is the unique solution to the system (1.7). First note that since the system (1.7) is consistent and its solution unique, we must have $|\mathcal{I}| \geq n$; otherwise, there are infinitely many solutions since the system has a non-trivial null space when $n > |\mathcal{I}|$. So we may as well assume that $|\mathcal{I}| > n$. Let $\mathcal{J} \subset \mathcal{I}$ be such that the vectors $t_{i\cdot} = (t_{i1}, t_{i2}, \ldots, t_{in})^T$, $i \in \mathcal{J}$ is a maximally linearly independent subset of the set of vectors $t_{i\cdot} = (t_{i1}, t_{i2}, \ldots, t_{in})^T$, $i \in \mathcal{I}$. That is, the vectors $t_{i\cdot}$ $i \in \mathcal{J}$ form a basis for the subspace spanned by the vectors $t_{i\cdot}$, $i \in \mathcal{I}$. Clearly, $|\mathcal{J}| \leq n$ since these vectors reside in $\mathbb{R}^n$ and are linearly independent. Moreover, each of the vectors $t_{r\cdot}$ for $r \in \mathcal{I} \setminus \mathcal{J}$ can be written as a linear combination of the vectors $t_{i\cdot}$ for $i \in \mathcal{J}$;

$$t_{r\cdot} = \sum_{i \in J} \lambda_{ri} t_{i\cdot}, \ r \in \mathcal{I} \setminus \mathcal{J}.$$

Therefore,

$$g_r = t_{r\cdot}^T \bar{x} = \sum_{i \in \mathcal{J}} \lambda_{ri} t_{i\cdot}^T \bar{x} = \sum_{i \in \mathcal{J}} \lambda_{ri} g_i, \quad r \in \mathcal{I} \setminus \mathcal{J},$$

which implies that any solution to the system

$$t_{i\cdot}^T x = g_i, \ i \in \mathcal{J} \tag{1.10}$$

is necessarily a solution to the larger system (1.7). But then the smaller system (1.10) must have $\bar{x}$ as its unique solution; otherwise, the system (1.7) has more than one solution. Finally, since the set of solutions to (1.10) is unique and $|\mathcal{J}| \leq n$, we must in fact have $|\mathcal{J}| = n$ which completes the proof. ∎

We now apply this result to obtain a characterization of the vertices for the constraint region of an LP in standard form.

Corollary 1.1 *A point x in the convex polyhedron described by the system of inequalities*

$$Ax \leq b \qquad \text{and} \qquad 0 \leq x,$$

where $A = (a_{ij})_{m \times n}$, is a vertex of this polyhedron if and only if there exist index sets $\mathcal{I} \subset \{1, \ldots, m\}$ and $\mathcal{J} \in \{1, \ldots, n\}$ with $|\mathcal{I}| + |\mathcal{J}| = n$ such that x is the unique solution to the system of equations

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &= b_i \qquad i \in \mathcal{I}, \qquad \text{and} \\ x_j &= 0 \qquad j \in \mathcal{J}. \end{aligned} \tag{1.9}$$

Proof: Take

$$T = \begin{bmatrix} A \\ -I \end{bmatrix} \quad \text{and} \quad g \begin{bmatrix} b \\ 0 \end{bmatrix}$$

in the previous theorem. ■

Recall that the symbols $|\mathcal{I}|$ and $|\mathcal{J}|$ denote the number of elements in the sets $\mathcal{I}$ and $\mathcal{J}$, respectively. The constraint hyperplanes associated with these indices are necessarily a subset of the set of *active* hyperplanes at the solution to (1.9).

Theorem 1.1 is an elementary yet powerful result in the study of convex polyhedra. We make strong use of it in our study of the geometric properties of the simplex algorithm. As a first observation, recall from Math 308 that the coefficient matrix for the system (1.9) is necessarily non-singular if this $n \times n$ system has a unique solution. How do we interpret this system geometrically, and why does Theorem 1.1 make intuitive sense?

To answer these questions, let us return to the convex polyhedron C defined by (1.7). In this case, the dimension n is 2. Observe that each vertex is located at the intersection of precisely two of the bounding constraint lines. Thus, each vertex can be represented as the unique solution to a 2×2 system of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2, \end{aligned}$$

where the coefficient matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is non-singular. For the set C above, we have the following:

(a) The vertex $v_1 = (\frac{8}{7}, \frac{6}{7})$ is given as the solution to the system

$$\begin{aligned} -x_1 - x_2 &= -2 \\ 3x_1 - 4x_2 &= 0, \end{aligned}$$

(b) The vertex $v_2 = (0, 2)$ is given as the solution to the system

$$\begin{aligned} -x_1 - x_2 &= -2 \\ -x_1 + 3x_2 &= 6, \end{aligned}$$

and

(c) The vertex $v_3 = \left(\frac{24}{5}, \frac{18}{5}\right)$ is given as the solution to the system

$$\begin{aligned} 3x_1 - 4x_2 &= 0 \\ -x_1 + 3x_2 &= 6. \end{aligned}$$

Theorem 1.1 indicates that any subsystem of the form (1.9) for which the associated coefficient matrix is non-singular, has as its solution a vertex of the polyhedron

$$Ax \leq b, \qquad 0 \leq x \tag{1.10}$$

if this solution is in the polyhedron. We now connect these ideas to the operation of the simplex algorithm.

The system (1.10) describes the constraint region for an LP in standard form. It can be expressed componentwise by

$$\begin{array}{rcll} \sum_{j=1}^{n} a_{ij}x_j & \leq & b_i & \qquad i = 1, \dots, m \\ 0 & \leq & x_j & \qquad j = 1, \dots, n. \end{array}$$

The associated slack variables are defined by the equations

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij}x_j \qquad i = 1, \dots, m. \tag{1.11}$$

Let $\bar{x} = (\bar{x}_1, \dots, \bar{x}_{n+m})$ be any solution to the system (1.11) and set $\widehat{x} = (\bar{x}_1, \dots, \bar{x}_n)$ ($\widehat{x}$ gives values for the decision variables associated with the underlying LP). Note that if for some $j \in \mathcal{J} \subset \{1, \dots, n\}$ we have $\bar{x}_j = 0$, then the hyperplane

$$H_j = \{x \in \mathbb{R}^n : e_j^T x = 0\}$$

is *active* at $\widehat{x}$, i.e., $\widehat{x} \in H_j$. Similarly, if for some $i \in \mathcal{I} \subset \{1, 2, \dots, m\}$ we have $\bar{x}_{n+i} = 0$, then the hyperplane

$$H_{n+i} = \{x \in \mathbb{R}^n : \sum_{j=1}^{n} a_{ij}x_j = b_i\}$$

is *active* at $\widehat{x}$, i.e., $\widehat{x} \in H_{n+i}$. Next suppose that $\bar{x}$ is a basic feasible solution for the LP

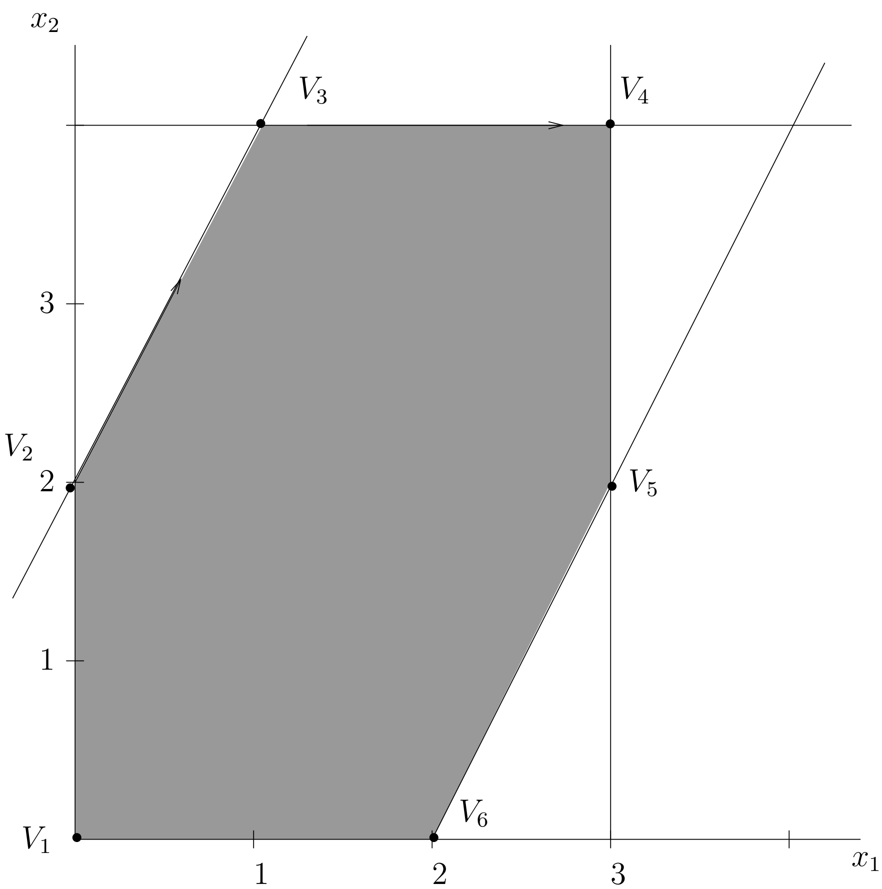
$$(\mathcal{P}) \qquad \begin{array}{l} \max c^T x \\ \text{subject to } Ax \leq b, 0 \leq x. \end{array}$$

Then it must be the case that n of the components $\bar{x}_k$, $k \in \{1, 2, \dots, n+m\}$ are assigned to the value zero since every dictionary has m basic and n non-basic variables. That is, every basic feasible solution is in the polyhedron defined by (1.10) and is the unique solution to a system of the form (1.9). But then, by Theorem 1.1, basic feasible solutions correspond precisely to the vertices of the polyhedron defining the constraint region for the LP $\mathcal{P}$!! This amazing geometric fact implies that the simplex algorithm proceeds by moving from vertex to adjacent vertex of the polyhedron given by (1.10). This is the essential underlying geometry of the simplex algorithm for linear programming!

By way of illustration, let us observe this behavior for the LP

$$
\text{(1.12)} \qquad \begin{array}{lll}
\text{maximize} & 3x_1 + 4x_2 & \\
\text{subject to} & -2x_1 + x_2 & \leq 2 \\
 & 2x_1 - x_2 & \leq 4 \\
 & 0 \leq x_1 \leq 3, & 0 \leq x_2 \leq 4.
\end{array}
$$

The constraint region for this LP is graphed on the next page.



The simplex algorithm yields the following pivots:

-2	$\boxed{1}$	1	0	0	0	2	vertex
2	-1	0	1	0	0	4	$V_1 = (0, 0)$
1	0	0	0	1	0	3	
0	1	0	0	0	1	4	
3	4	0	0	0	0	0	
-2	1	1	0	0	0	2	vertex
0	0	1	1	0	0	6	$V_2 = (0, 2)$
1	0	0	0	1	0	3	
$\boxed{2}$	0	-1	0	0	1	2	
11	0	-4	0	0	0	-8	
0	1	0	0	0	1	4	vertex
0	0	1	1	0	0	6	$V_3 = (1, 4)$
0	0	$\boxed{\frac{1}{2}}$	0	1	$-\frac{1}{2}$	2	
1	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	1	
0	0	$\frac{3}{2}$	0	0	$\frac{-11}{2}$	-19	
0	1	0	0	0	1	4	vertex
0	0	0	1	-2	1	2	$V_4 = (3, 4)$
0	0	1	0	2	-1	4	
1	0	0	0	1	0	3	
0	0	0	0	-3	-4	-25	

The Geometry of Degeneracy

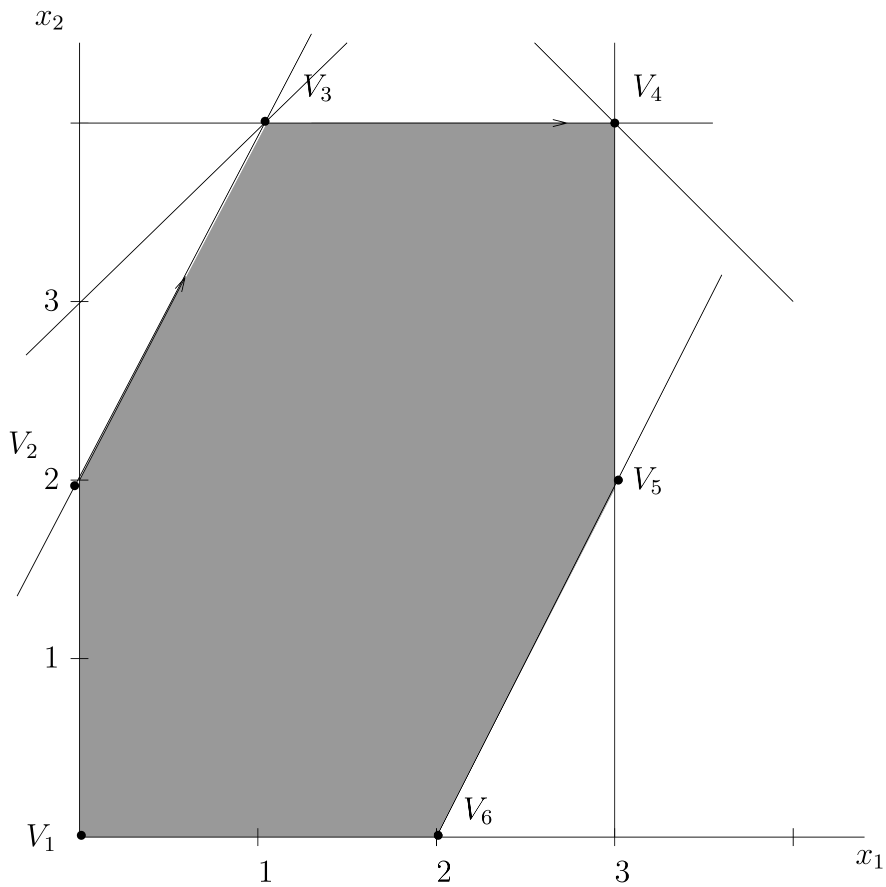
We now give a geometric interpretation of degeneracy in linear programming. Recall that a basic feasible solution, or vertex, is said to be degenerate if one or more of the basic variables is assigned the value zero. In the notation of (1.11) this implies that more than n of the hyperplanes H_k, $k = 1, 2, \ldots, n+m$ are active at this vertex. By way of illustration, suppose we add the constraints

$$-x_1 + x_2 \leq 3$$

and

$$x_1 + x_2 \leq 7$$

to the system of constraints in the LP (1.12). The picture of the constraint region now looks as follows:



Notice that there are redundant constraints at both of the vertices V_3 and V_4. Therefore, as we pivot we should observe that the tableaus associated with these vertices are degenerate.

-2	1	1	0	0	0	0	0	2	vertex
2	-1	0	1	0	0	0	0	4	$V_1 = (0, 0)$
-1	1	0	0	1	0	0	0	3	
1	1	0	0	0	1	0	0	7	
1	0	0	0	0	0	1	0	3	
0	1	0	0	0	0	0	1	4	
3	4	0	0	0	0	0	0	0	
-2	1	1	0	0	0	0	0	2	vertex
0	0	1	1	0	0	0	0	6	$V_2 = (0, 2)$
1	0	-1	0	1	0	0	0	1	
3	0	-1	0	0	1	0	0	5	
1	0	0	0	0	0	1	0	3	
2	0	-1	0	0	0	0	1	2	
11	0	-4	0	0	0	0	0	-8	
0	1	-1	0	2	0	0	0	4	vertex
0	0	1	1	0	0	0	0	6	$V_3 = (1, 4)$
1	0	-1	0	1	0	0	0	1	
0	0	2	0	-3	1	0	0	2	
0	0	1	0	-1	0	1	0	2	
0	0	1	0	-2	0	0	1	0	degenerate
0	0	7	0	-11	0	0	0	-19	
0	1	0	0	0	0	0	1	4	vertex
0	0	0	1	2	0	0	1	6	$V_3 = (1, 4)$
1	0	0	0	-1	0	0	1	1	
0	0	0	0	1	1	0	-2	2	
0	0	0	0	1	0	1	-1	2	
0	0	1	0	-2	0	0	1	0	degenerate
0	0	0	0	3	0	0	-7	-19	
0	1	0	0	0	0	0	1	4	vertex
0	0	0	1	0	-2	0	5	2	$V_4 = (3, 4)$
1	0	0	0	0	1	0	-1	3	
0	0	0	0	1	1	0	-2	2	optimal
0	0	0	0	0	-1	1	1	0	degenerate
0	0	1	0	0	2	0	-3	4	
0	0	0	0	0	-3	0	-1	-25	

In this way we see that a degenerate pivot arises when we represent the same vertex as the intersection point of a different subset of n active hyperplanes. Cycling implies that we are cycling between different representations of the same vertex. In the example given above, the third pivot is a degenerate pivot. In the third tableau, we represent the vertex

$V_3 = (1,4)$ as the intersection point of the hyperplanes

$$\begin{aligned} -2x_1 + x_2 &= 2 \qquad &&(\text{since } x_3 = 0)\\ \text{and} \qquad -x_1 + x_2 &= 3. &&(\text{since } x_5 = 0) \end{aligned}$$

The third pivot brings us to the 4th tableau where the vertex $V_3 = (1,4)$ is now represented as the intersection of the hyperplanes

$$\begin{aligned} -x_1 + x_2 &= 3 \qquad &&(\text{since } x_5 = 0)\\ \text{and} \qquad x_2 &= 4 &&(\text{since } x_8 = 0). \end{aligned}$$

Observe that the final tableau is both optimal and degenerate. Just for the fun of it let's try pivoting on the only negative entry in the 5th row of this tableau (we choose the 5th row since this is the row that exhibits the degeneracy). Pivoting we obtain the following tableau.

$$\begin{array}{cccccccc|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4\\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 3 & 2\\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3\\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 2\\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & -1 & 4\\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -2 & -3 & -25 \end{array}$$

Observe that this tableau is also optimal, but it provides us with a different set of optimal dual variables. In general, a degenerate optimal tableau implies that the dual problem has infinitely many optimal solutions.

FACT: If an LP has an optimal tableau that is degenerate, then the dual LP has infinitely many optimal solutions.

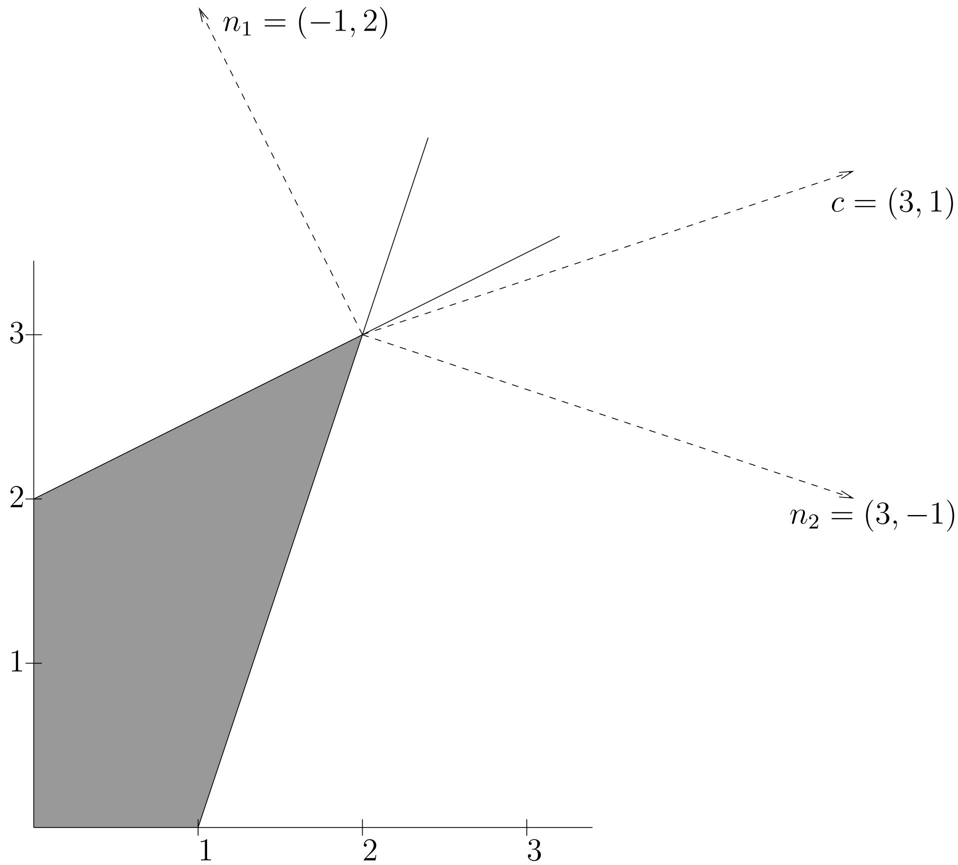
We will arrive at an understanding of why this fact is true after we examine the geometry of duality.

The Geometry of Duality

Consider the linear program

$$\begin{array}{lll} \text{maximize} & 3x_1 + x_2 & \\ \text{subject to} & -x_1 + 2x_2 & \leq 4\\ & 3x_1 - x_2 & \leq 3\\ & 0 \leq x_1, & x_2. \end{array} \tag{1.13}$$

This LP is solved graphically below.



The solution is $x = (2, 3)$. In the picture, the vector $n_1 = (-1, 2)$ is the normal to the hyperplane

$$-x_1 + 2x_2 = 4,$$

the vector $n_2 = (3, -1)$ is the normal to the hyperplane

$$3x_1 - x_2 = 3,$$

and the vector $c = (3, 1)$ is the objective normal. Geometrically, the vector c lies *between* the vectors n_1 and n_2. That is to say, the vector c can be represented as a non-negative linear combination of n_1 and n_2: there exist $y_1 \geq 0$ and $y_2 \geq 0$ such that

$$c = y_1 n_1 + y_2 n_2,$$

or equivalently,

$$\begin{aligned} \begin{pmatrix} 3 \\ 1 \end{pmatrix} &= y_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + y_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} \\ &= \begin{bmatrix} -1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \end{aligned}$$

Solving for (y_1, y_2) we have

$$\begin{array}{rr|r}
-1 & 3 & 3 \\
2 & -1 & 1 \\
\hline
1 & -3 & -3 \\
0 & 5 & 7 \\
\hline
1 & -3 & -3 \\
0 & 1 & \frac{7}{5} \\
\hline
1 & 0 & \frac{6}{5} \\
0 & 1 & \frac{7}{5}
\end{array}$$

or $y_1 = \frac{6}{5}$, $y_2 = \frac{7}{5}$. I claim that the vector $y = (\frac{6}{5}, \frac{7}{5})$ is the optimal solution to the dual! Indeed, this result follows from the complementary slackness theorem and gives another way to recover the solution to the dual from the solution to the primal, or equivalently, to check whether a point that is feasible for the primal is optimal for the primal.

Theorem 1.14 (Geometric Duality Theorem) *Consider the LP*

$$(\mathcal{P}) \qquad \begin{array}{ll} \textit{maximize} & c^T x \\ \textit{subject to} & Ax \leq b, 0 \leq x. \end{array}$$

where A is an $m \times n$ matrix. Given a vector $\bar{x}$ that is feasible for $\mathcal{P}$, define

$$\mathcal{Z}(\bar{x}) = \{j \in \{1, 2, \ldots, n\} : \bar{x}_j = 0\} \text{ and } \mathcal{E}(\bar{x}) = \{i \in \{1, \ldots, m\} : \sum_{j=1}^{n} a_{ij}\bar{x}_j = b_i\}.$$

The indices $\mathcal{Z}(\bar{x})$ and $\mathcal{E}(\bar{x})$ are the active *indices at $\bar{x}$ and correspond to the active hyperplanes at $\bar{x}$. Then $\bar{x}$ solves $\mathcal{P}$ if and only if there exist non-negative scalars r_j, $j \in \mathcal{Z}(\bar{x})$ and y_i, $i \in \mathcal{E}(\bar{x})$ such that*

$$c = -\sum_{j \in \mathcal{Z}(\bar{x})} r_j e_j + \sum_{i \in \mathcal{E}(\bar{x})} y_i a_{i\bullet} \tag{1.14}$$

where for each $i = 1, \ldots, m$, $a_{i\bullet} = (a_{i1}, a_{i2}, \ldots, a_{in})^T$ is the ith column of the matrix A^T, and, for each $j = 1, \ldots, n$, e_j is the jth unit coordinate vector. Moreover, the vector $\bar{y} \in \mathbb{R}^m$ given by

$$\bar{y}_i = \begin{cases} y_i & \textit{for } i \in \mathcal{E}(\bar{x}) \\ 0 & \textit{otherwise} \end{cases}, \tag{1.15}$$

solves the dual problem

$$(\mathcal{D}) \qquad \begin{array}{ll} \textit{maximize} & b^T x \\ \textit{subject to} & A^T y \geq c, 0 \leq y. \end{array}$$

PROOF: Let us first suppose that $\bar{x}$ solves $\mathcal{P}$. Then there is a $\bar{y} \in \mathbb{R}^n$ solving the dual $\mathcal{D}$ with $c^T\bar{x} = \bar{y}^T A\bar{x} = b^T\bar{y}$ by the Strong Duality Theorem. We need only show that there exist $r_j,\ j \in \mathcal{Z}(\bar{x})$ such that (1.14) and (1.15) hold. The Complementary Slackness Theorem implies that

$$\bar{y}_i = 0 \text{ for } i \in \{1, 2, \ldots, m\} \setminus \mathcal{E}(\bar{x}) \tag{1.16}$$

and

$$\sum_{i=1}^{m} \bar{y}_i a_{ij} = c_j \text{ for } j \in \{1, \ldots, n\} \setminus \mathcal{Z}(\bar{x}). \tag{1.17}$$

Note that (1.16) implies that $\bar{y}$ satisfies (1.15). Define $r = A^T\bar{y} - c$. Since $\bar{y}$ is dual feasible we have both $r \geq 0$ and $\bar{y} \geq 0$. Moreover, by (1.17), $r_j = 0$ for $j \in \{1, \ldots, n\} \setminus \mathcal{Z}(\bar{x})$, while

$$r_j = \sum_{i=1}^{n} \bar{y}_i a_{ij} - c_j \geq 0 \text{ for } j \in \mathcal{Z}(\bar{x}),$$

or equivalently,

$$c_j = -r_j + \sum_{i=1}^{m} \bar{y}_i a_{ij} \text{ for } j \in \mathcal{Z}(\bar{x}). \tag{1.18}$$

Combining (1.18) with (1.17) and (1.16) gives

$$c = -\sum_{j \in \mathcal{Z}(\bar{x})} r_j e_j \ + \ A^T\bar{y} = -\sum_{j \in \mathcal{Z}(\bar{x})} r_j e_j \ + \sum_{i \in \mathcal{E}(\bar{x})} \bar{y}_i a_{i\bullet},$$

so that (1.14) and (1.15) are satisfied with $\bar{y}$ solving $\mathcal{D}$.

Next suppose that $\bar{x}$ is feasible for $\mathcal{P}$ with $r_j,\ j \in \mathcal{Z}(\bar{x})$ and $\bar{y}_i,\ i \in \mathcal{E}(\bar{x})$ non-negative and satisfying (1.14). We must show that $\bar{x}$ solves $\mathcal{P}$ and $\bar{y}$ solves $\mathcal{D}$. Let $\bar{y} \in \mathbb{R}^m$ be such that its components are given by the $\bar{y}_i$'s for $i \in \mathcal{E}(\bar{x})$ and by (1.15) otherwise. Then the non-negativity of the r_j's in (1.14) imply that

$$A^T\bar{y} = \sum_{i \in \mathcal{E}(\bar{x})} \bar{y}_i a_{i\bullet} \geq -\sum_{j \in \mathcal{Z}(\bar{x})} r_j e_j \ + \sum_{i \in \mathcal{E}(\bar{x})} \bar{y}_i a_{i\bullet} = c,$$

so that $\bar{y}$ is feasible for $\mathcal{D}$. Moreover,

$$c^T\bar{x} = -\sum_{j \in \mathcal{Z}(\bar{x})} r_j e_j^T \bar{x} \ + \sum_{i \in \mathcal{E}(\bar{x})} \bar{y}_i a_{i\bullet}^T \bar{x} = \bar{y}^T A\bar{x} = \bar{y}^T b,$$

where the final equality follows from the definition of the vector $\bar{y}$ and the index set $\mathcal{E}(\bar{x})$. Hence, by the Weak Duality Theorem $\bar{x}$ solves $\mathcal{P}$ and $\bar{y}$ solves $\mathcal{D}$ as required. ■

REMARK: As is apparent from the proof, the Geometric Duality Theorem is nearly equivalent to the complementary Slackness Theorem even though it provides a superficially different test for optimality.

We now illustrate how to apply this result with an example. Consider the LP

$$
\begin{array}{llrrrrl}
 & \text{maximize} & x_1 & +x_2 & -x_3 & +2x_4 & \\
 & \text{subject to} & x_1 & +3x_2 & -2x_3 & +4x_4 & \le -3 \\
(1.15) & & & 4x_2 & -2x_3 & +3x_4 & \le 1 \\
 & & & -x_2 & +x_3 & -x_4 & \le 2 \\
 & & -x_1 & -x_2 & +2x_3 & -x_5 & \le 4 \\
 & & 0 \le & x_1, & x_2, & x_3, & x_4 \quad .
\end{array}
$$

Does the vector $\bar{x} = (1, 0, 2, 0)^T$ solve this LP? If it does, then according to Theorem 1.14 we must be able to construct the solution to the dual of (1.15) by representing the objective vector $c = (1, 1, -1, 2)^T$ as a non-negative linear combination of the outer normals to the active hyperplanes at $\bar{x}$. Since the active hyperplanes are

$$
\begin{array}{rcrcrcrcr}
x_1 & + & 3x_2 & - & 2x_3 & + & 4x_4 & = & -3 \\
 & - & x_2 & + & x_3 & - & x_4 & = & 2 \\
 & - & x_2 & & & & & = & 0 \\
 & & & & & - & x_4 & = & 0.
\end{array}
$$

This means that $y_2 = y_4 = 0$ and y_1 and y_3 are obtained by solving

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -1 & -1 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & -1 & 0 & -1 \end{bmatrix}
\begin{bmatrix} y_1 \\ y_3 \\ r_2 \\ r_4 \end{bmatrix}
= \begin{pmatrix} 1 \\ 1 \\ -1 \\ 2 \end{pmatrix}.
$$

Row reducing, we get

$$
\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 1 \\
3 & -1 & -1 & 0 & 1 \\
-2 & 1 & 0 & 0 & -1 \\
4 & -1 & 0 & -1 & 2 \\
\hline
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 2 \\
0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 2
\end{array}.
$$

Therefore, $y_1 = 1$ and $y_3 = 1$. We now check to see if the vector $\bar{y} = (1, 0, 1, 0)$ does indeed solve the dual to (1.15);

$$
\begin{array}{llrcrcrcrcr}
 & \text{minimize} & -3y_1 & + & y_2 & + & 2y_3 & + & 4y_4 & & \\
 & \text{subject to} & y_1 & & & & & - & y4 & \ge & 1 \\
(1.16) & & 3y_1 & + & 4y_2 & - & y_3 & - & y_4 & \ge & 1 \\
 & & -2y_1 & - & 2y_2 & + & y_3 & + & 2y_4 & \ge & -1 \\
 & & 4y_1 & + & 3y_2 & - & y_3 & - & y_4 & \ge & 2 \\
 & & & & \multicolumn{5}{c}{0 \le y_1, y_2, y_3, y_4.} & &
\end{array}
$$

Clearly, $\bar{y}$ is feasible for (1.16). In addition,

$$b^T \bar{y} = -1 = c^T \bar{x}.$$

Therefore, $\bar{y}$ solves (1.16) and $\bar{x}$ solves (1.15) by the Weak Duality Theorem.

6 Sensitivity Analysis

In this section we study general questions involving the sensitivity of the solution to an LP under changes to its input data. As it turns out LP solutions can be extremely sensitive to such changes and this has very important practical consequences for the use of LP technology in applications. Let us now look at a simple example to illustrate this fact.

Consider the scenario where we be believe the federal reserve board is set to decrease the prime rate at its meeting the following morning. If this happens then bond yields will go up. In this environment, you have calculated that for every dollar that you invest today in bonds will give a return of a half percent tomorrow so, as a bond trader, you decide to invest in lots of bonds today. But to do this you will need to borrow money on margin. For the 24 hours that you intend to borrow the money you will need to place a reserve with the exchange that is un-invested, and then you can borrow up to 100 times this reserve. Regardless of how much you borrow, the exchange requires that you pay them back 10% of your reserve tomorrow. To add an extra margin of safety you will limit the sum of your reserve and one hundreth of what you borrow to be less than 200,000 dollars. Model the problem of determining how much money should be put on reserve and how much money should be borrowed to maximize your return on this 24 hour bond investment.

To model this problem, let R denote your reserve in \$10,000 units and let B denote the amount you borrow in the same units. Due to the way you must pay for the loan (i.e. it depends on the reserve, not what you borrow), your goal is to

$$\text{maximize } 0.005B - 0.1R \ .$$

Your borrowing constraint is

$$B \leq 100R \ ,$$

and your safety constraint is

$$\frac{B}{100} + R \leq 20 \ .$$

The full LP model is

$$\begin{array}{ll} \text{maximize} & 0.005B - 0.1R \\ \text{subject to} & B - 100R \leq 0 \\ & 0.01B + R \leq 20 \\ & 0 \leq B,\ R \ . \end{array}$$

We conjecture that the solution occurs at the intersection of the two nontrivial constraint lines. We check this by applying the geometric duality theorem, i.e., we solve the system

$$\begin{pmatrix} 0.005 \\ -0.1 \end{pmatrix} = y_1 \begin{pmatrix} 1 \\ -100 \end{pmatrix} + y_2 \begin{pmatrix} 0.01 \\ 1 \end{pmatrix}$$

which gives $(y_1, y_2) = (0.003,\ 0.2)$. Since the solution is non-negative, the solution does occur at the intersection of the two nontrivial constraint lines giving $(B, R) = (1000,\ 10)$

with dual solution $(y_1, y_2) = (0.003,\ 0.2)$ and optimal value 4, or equivalently a profit of \$40,000 on a \$100,000 investment (the cost of the reserve).

But suppose that somehow your projections are wrong, and the Fed left rates alone and bond yields dropped by half a percent rather than increase by half a percent. In this scenario you would have lost $\$60,000$ on the \$100,000 investment. That is, the difference between a rise of the interest rate by half a percent to a drop in the interest rate by half a percent is one hundred thousand dollars. Clearly, this is a very risky investment opportunity. In this environment the downside risks must be fully understood before an investment is made. Doing this kind of analysis is called *sensitivity analysis*. We will look at some techniques for sensitivity analysis in this section. All of our discussion will be motivated by examples.

In practice, performing sensitivity analysis on solutions to LPs is absolutely essential. One should never report a solution to an LP without the accompanying sensitivity analysis. This is because all of the numbers defining the LP are almost always subject to error. The errors may be modeling errors, statistical errors, or data entry errors. Such errors can lead to catastrophically bad optimal solutions to the LP. Sensitivity analysis techniques provide tools for detecting and avoiding bad solutions.

6.1 Break-even Prices and Reduced Costs

The first type of sensitivity problem we consider concerns variations or *perturbations* to the objective coefficients. For this we consider the following LP problem.

SILICON CHIP CORPORATION

A Silicon Valley firm specializes in making four types of silicon chips for personal computers. Each chip must go through four stages of processing before completion. First the basic silicon wafers are manufactured, second the wafers are laser etched with a micro circuit, next the circuit is laminated onto the chip, and finally the chip is tested and packaged for shipping. The production manager desires to maximize profits during the next month. During the next 30 days she has enough raw material to produce 4000 silicon wafers. Moreover, she has 600 hours of etching time, 900 hours of lamination time, and 700 hours of testing time. Taking into account depreciated capital investment, maintenance costs, and the cost of labor, each raw silicon wafer is worth \$1, each hour of etching time costs \$40, each hour of lamination time costs \$60, and each hour of inspection time costs \$10. The production manager has formulated her problem as a profit maximization linear program with the following initial tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
raw wafers	100	100	100	100	1	0	0	0	4000
etching	10	10	20	20	0	1	0	0	600
lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	2000	3000	5000	4000	0	0	0	0	0

where x_1, x_2, x_3, x_4 represent the number of 100 chip batches of the four types of chips and the objective row coefficients for these variables correspond to dollars profit per 100 chip batch. After solving by the Simplex Algorithm, the final tableau is:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
0.5	0	0	1	.015	0	$-.1$	.05	5
-1500	0	0	0	-5	0	-100	-50	$-145,000$

Thus the optimal production schedule is $(x_1, x_2, x_3, x_4) = (0,\ 25,\ 10,\ 5)$. In this solution we see that type 1 chip is not efficient to produce.

The first problem we address is to determine the sale price at which it is efficient to produce type 1 chip. That is, what sale price p for which it is not efficient to produce type 1 chip below this sale price, but it is efficient to produce above this sale price? This is called the *breakeven sale price* of type 1 chip. As a first step let us compute the current sale price of type 1 chip. From the objective row we see that each 100 type 1 chip batch has a profit of \$2000. The cost of production of each 100 unit batch of type 1 chip is given by

$$\text{chip cost} + \text{etching cost} + \text{lamination cost} + \text{inspection cost},$$

where

$$\begin{aligned}
\text{chip cost} &= \text{no. chips} \times \text{cost per chip} = 100 \times 1 = 100 \\
\text{etching cost} &= \text{no. hours} \times \text{cost per hour} = 10 \times 40 = 400 \\
\text{lamination cost} &= \text{no. hours} \times \text{cost per hour} = 20 \times 60 = 1200 \\
\text{inspection cost} &= \text{no. hours} \times \text{cost per hour} = 20 \times 10 = 200\ .
\end{aligned}$$

Hence the costs per batch of 100 type 1 chips is \$1900. Therefore, the sale price of each batch of 100 type 1 chips is \$2000 + \$1900 = \$3900, or equivalently, \$39 per chip.

Since we do not produce type 1 chip in our optimal production mix, the breakeven sale price must be greater than \$39 per chip. Let θ denote the amount by which we need to increase the current sale price of type 1 chip so that it enters the optimal production mix. With this change to the sale price of type 1 chip the initial tableau for the LP becomes

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
raw wafers	100	100	100	100	1	0	0	0	4000
etching	10	10	20	20	0	1	0	0	600
lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	$2000+\theta$	3000	5000	4000	0	0	0	0	0

.

Next let us suppose that we repeat on this tableau all of the pivots that led to the previously optimal tableau given above. What will the new tableau look like? That is, how does θ appear in this new tableau? This question is easily answered by recalling our general observations on simplex pivoting as left multiplication of an augmented matrix by a sequence of Gaussian elimination matrices.

Recall that given a problem in standard form,

$$\begin{array}{lll} \mathcal{P} & \text{maximize} & c^T x \\ & \text{subject to} & Ax \le b, \ 0 \le x \ , \end{array}$$

the initial tableau is an augmented matrix whose block form is given by

$$\begin{bmatrix} A & I & b \\ c^T & 0 & 0 \end{bmatrix} . \tag{6.1}$$

Pivoting to an optimal tableau corresponds to left multiplication by a matrix of the form

$$G = \begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix}$$

where the nonsingular matrix R is called the *record matrix* and where the block form of G is conformal with that of the initial tableau. Hence the optimal tableau has the form

$$\begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} A & I & b \\ c^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} RA & R & Rb \\ (c - A^T y)^T & -y^T & -b^T y \end{bmatrix} , \tag{6.2}$$

where $0 \le y, \ A^T y \ge c$, and the optimal value is $b^T y$. Now changing the value of one (or more) of the objective coefficients c corresponds to replacing c by a vector of the form $c + \Delta c$. The corresponding new initial tableau is

$$\begin{bmatrix} A & I & b \\ (c + \Delta c)^T & 0 & 0 \end{bmatrix} . \tag{6.3}$$

Performing the same simplex pivots on this tableau as before simply corresponds to left multiplication by the matrix G given above. This yields the simplex tableau

$$\begin{bmatrix} RA & R & Rb \\ (c + \Delta c - A^T y)^T & -y^T & -b^T y \end{bmatrix} = \begin{bmatrix} RA & R & Rb \\ \Delta c^T + (c - A^T y)^T & -y^T & -b^T y \end{bmatrix} . \tag{6.4}$$

That is, we just add Δc to the objective row in the old optimal tableau. Observe that this matrix may or may not be a simplex tableau since some of the basic variable cost row coefficients in $c - A^T y$ (which are zero) may be non-zero in $\Delta c + (c - A^T y)$. To completely determine the effect of the perturbation Δc one must first use Gaussian elimination to return the basic variabl coefficients in $\Delta c + (c - A^T y)$ to zero. After returning (6.4) to a simplex tableau, the resulting tableau is optimal if it is dual feasible, that is, if all of the objective row coefficients are non-positive. These non-positivity conditions place restrictions on how large the entries of Δc can be before one must pivot to obtain the new optimal tableau.

Let us apply these observations to the Silicon Chip Corp. problem and the question of determining the breakeven sale price of type 1 chip. In this case the expression Δc takes the form $\Delta c = \theta e_1$, where e_1 is the first unit coordinate vector. Plugging this into (6.4) gives

$$(c - A^T y) + \Delta c = \begin{pmatrix} -1500 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \theta \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \theta - 1500 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, the perturbed tableau (6.4) remains optimal if and only if $\theta \leq 1500$. That is, as soon as θ increases beyond 1500, type 1 chip enters the optimal production mix, and for $\theta = 1500$ we obtain multiple optimal solutions where type 1 chip may be in the optimal production mix if we so choose. The number 1500 appearing in the optimal objective row is called the *reduced cost* for type 1 chip. In general, the negative of the objective row coefficient for decision variables in the optimal tableau are the reduced costs of these variables. The reduced cost of a decision variable is the precise amount by which one must increase its objective row coefficient in order for it to be included in the optimal solution. Therefore, for nonbasic variables one can compute breakeven sale prices by simply reading off the reduced costs from the optimal tableau. In the case of the type 1 chip in the Silicon Chip Corp. problem above, this gives a breakeven sale price of

$$\begin{aligned} \text{breakeven price} &= \text{current price} + \text{reduced cost} \\ &= \$39 + \$15 = \$54 \ . \end{aligned}$$

With the long winded derivation of breakeven prices behind us, let us now consider a more intuitive and simpler explanation. One way to determine the breakeven sale price, is to determine by how much our profit is reduced if we produce one batch of these chips. Recall that the objective row coefficients in the optimal tableau correspond to the following expression for the objective variable z:

$$z = 145000 - 1500x_1 - 5x_5 - 100x_7 - 50x_8.$$

Hence, if we make one batch of type 1 chip, we reduce our optimal value by \$1500. Thus, to recoup this loss we must charge \$1500 more for these chips yielding a breakeven sale price of $\$39 + \$15 = \$54$ per chip.

6.2 Range Analysis for Objective Coefficients

Range analysis is a tool for understanding the effects of both objective coefficient variations as well as resource availability variations. In this section we examine objective coefficient variations. In the previous section we studied the idea of break-even sale prices. These prices are associated with activities that do not play a role in the currently optimal production schedule. In computing a breakeven price one needs to determine the change in the associated objective coefficient that make it efficient to introduce this activity into the optimal production mix, or equivalently, to determine the smallest change in the objective coefficient of this currently nonbasic decision variable that requires one to bring it into the basis in order to maintain optimality. A related question that can be asked of any of the objective coefficients is *what is the range of variation of a given objective coefficient that preserves the current basis as optimal*? The answer to this question is an interval, possibly unbounded, on the real line within which a given objective coefficient can vary but these variations do not effect the currently optimal basis.

For example, consider the objective coefficient on type 1 chip analyzed in the previous section. The range on this objective coefficient is $(-\infty, 3500]$ since within this range one need not change the basis to preserve optimality. Note that for any nonbasic decision variable x_i the range at optimality is given by $(-\infty, c_i + r_i]$ where r_i is the reduced cost of this decision variable in the optimal tableau.

How does one compute the range of a basic decision variable? That is, if an activity is currently optimal, what is the range of variations in its objective coefficient within which the optimal basis does not change. The answer to this question is again easily derived by considering the effect of an arbitrary perturbation to the objective vector c. For this we again consider the perturbation Δc to c and the associated initial tableau (6.3). This tableau yields the perturbed optimal tableau (6.4). When computing the range for the objective coefficient of a optimal basic variable x_i one sets $\Delta c = \theta e_i$.

For example, in the Silicon Chip Corp. problem the decision variable x_3 associated with type 3 chips is in the optimal basis. For what range of variations in $c_3 = 5000$ does the current optimal basis $\{x_2, x_3, x_4, x_6\}$ remain optimal? Setting $\Delta c = \theta e_3$ in (6.4) we get the perturbed tableau

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
0.5	0	0	1	.015	0	$-.1$	.05	5
-1500	0	θ	0	-5	0	-100	-50	$-145,000$

.

This augmented matrix is no longer a simplex tableau since the objective row coefficient of one of the basic variables, namely x_3, is not zero. To convert this to a proper simplex tableau we must eliminate θ from the objective row entry under x_3. Multiplying the fourth row by $-\theta$ and adding to the objective row gives the tableau

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
0.5	0	0	1	.015	0	$-.1$	.05	5
-1500	0	0	0	$-5 + 0.02\theta$	0	$-100 - 0.1\theta$	-50	$-145,000 - 10\theta$

.

For this tableau to remain optimal it must be both primal and dual feasible. Obviously primal feasibility is not an issue, but dual feasibility is due to the presence of θ in the objective row. For dual feasibility to be preserved the entries in the objective row must remain nonpositive; otherwise, a primal simplex pivot must be taken which will alter the currently optimal basis. That is, to preserve the current basis as optimal, we must have

$$\begin{array}{rcr} -5 + 0.02\theta \leq 0, & \text{or equivalently,} & \theta \leq 250 \\ -100 - 0.1\theta \leq 0, & \text{or equivalently,} & -1000 \leq \theta \end{array} .$$

Thus, the range of θ that preserves the current basis as optimal is

$$-1000 \leq \theta \leq 250,$$

and the corresponding range for c_3 that preserves the current basis as optimal is

$$4000 \leq c_3 \leq 5250.$$

Similarly, we can consider the range of the objective coefficient for type 4 chips. For this we simply multiply the fourth row by $-\theta$ and add it to the objective row to get the new objective row

$$\begin{array}{cccccccc|c} -1500 - 0.5\theta & 0 & 0 & 0 & -5 - 0.015\theta & 0 & -100 + 0.1\theta & -50 - 0.05\theta & -145,000 - 5\theta \end{array} .$$

Again, to preserve dual feasibility we must have

$$\begin{array}{rcr} -1500 - 0.5\theta \leq 0, & \text{or equivalently,} & -3000 \leq \theta \\ -5 - 0.015\theta \leq 0, & \text{or equivalently,} & -333.\bar{3} \leq \theta \\ -100 + 0.1\theta \leq 0, & \text{or equivalently,} & \theta \leq 1000 \\ -50 - 0.05\theta \leq 0, & \text{or equivalently,} & -1000 \leq \theta \end{array} .$$

Thus, the range of θ that preserves the current basis as optimal is

$$-333.\bar{3} \leq \theta \leq 1000,$$

and the corresponding range for c_4 that preserves the current basis as optimal is

$$3666.\bar{6} \leq c_4 \leq 5000.$$

6.3 Resource Variations, Marginal Values, and Range Analysis

We now consider questions concerning the effect of resource variations on the optimal solution. We begin with a concrete instance of such a problem in the case of the Silicon Chip Corp. problem above.

Suppose we wish to purchase more silicon wafers this month. Before doing so, we need to answer three obvious questions.

a) How many should we purchase?

b) What is the most that we should pay for them?

c) After the purchase, what is the new optimal production schedule?

The technique we develop for answering these questions is similar to the technique used to determine objective coefficient ranges.. We begin by introducing a variable θ for the number of silicon wafers that will be purchased, and then determine how this variable appears in the tableau after using the same simplex pivots encoded in the matrix G given above. In this case the new initial tableau looks like

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
raw wafers	100	100	100	100	1	0	0	0	$4000+\theta$
etching	10	10	20	20	0	1	0	0	600
lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	2000	3000	5000	4000	0	0	0	0	0

In the general case, we are discussing perturbations, or variations, to the resource vector b in the LP $\mathcal{P}$ in standard form given above. If we let Δb denote this variation, then the associated initial tableau is

$$\begin{bmatrix} A & I & b+\Delta b \\ c^T & 0 & 0 \end{bmatrix} .$$

Again, multiplying on the left by the matrix G gives

$$\begin{bmatrix} RA & R & Rb+R\Delta b \\ (c-A^Ty)^T & -y^T & -y^Tb-y^T\Delta b \end{bmatrix} .$$

This time the terms $R\Delta b$ and $y^T\Delta b$ encode the complete change to the optimal tableau by introducing the perturbation Δb. Clearly this new tableau is dual feasible and so it remains optimal as long as it remains primal feasible. That is, the new tableau is optimal as long as $0 \le Rb + R\Delta b$, or equivalently,

$$-Rb \le R\Delta b \;. \tag{6.5}$$

These inequalities place restrictions on the values Δb may take and still preserve the optimality of the tableau. If (6.5) holds, then the new optimal value is $y^Tb + y^T\Delta b$. That is, the rate of change in the optimal value is given by the vector y, the solution to the dual LP.

In the case of the Silicon Chip Corp. problem where we are interested in varying the number of silicon wafers available, we have $\Delta b = \theta e_1$ and the matrix R and vector y are given by

$$R = \begin{bmatrix} .015 & 0 & 0 & -.05 \\ -.05 & 1 & 0 & -.5 \\ -.02 & 0 & .1 & 0 \\ .015 & 0 & -.1 & .05 \end{bmatrix} \quad \text{and} \quad y = \begin{pmatrix} 5 \\ 0 \\ 100 \\ 50 \end{pmatrix} .$$

Therefore, the inequality (6.5) takes the form

$$-\begin{pmatrix} 25 \\ 50 \\ 10 \\ 5 \end{pmatrix} \le \theta \begin{pmatrix} 0.015 \\ -0.05 \\ -0.02 \\ 0.015 \end{pmatrix},$$

or equivalently,

$$\begin{array}{rclcrcr} -25 & \le & .015\theta & \text{implies} & \theta & \ge & -5000/3 \\ -50 & \le & -.05\theta & \text{implies} & \theta & \le & 1000 \\ -10 & \le & -.02\theta & \text{implies} & \theta & \le & 500 \\ -5 & \le & .015\theta & \text{implies} & \theta & \ge & -1000/3\ , \end{array}$$

which reduces to the simple inequality

$$-\frac{1000}{3} \le \theta \le 500.$$

This is the interval on which we may vary θ and not change the optimal basis. This interval is called the *range of the raw chip resource in the optimal solution.* If the variation θ stays within this interval, then the optimal solution is given by

$$\begin{pmatrix} x_2 \\ x_6 \\ x_3 \\ x_4 \end{pmatrix} = Rb + R\Delta b = \begin{pmatrix} 25 + .015\theta \\ 50 - .05\theta \\ 10 - .02\theta \\ 5 + .015\theta \end{pmatrix}$$

with optimal value

$$y^T b + y^T \Delta b = 145000 + 5\theta.$$

Observe from the expression for the optimal value that the profit increases by \$5 for every new silicon wafer that we get (up to 500 wafers). That is, if we pay less than \$5 over current costs for new wafers, then our profit increases. The dual value 5 is called the *shadow price*, or *marginal value*, for the raw silicon wafer resource. It represents the increased value of this resource due to the production process. It tells us the rate at which the optimal value increases due to increases in this resource. For example, we know that we currently pay \$1 per wafer. If another vendor offers wafers to us for \$2.50 per wafer, then we should buy them since our unit increase in profit with this purchase price is \$5 − \$1.5 = \$3.5 since \$2.5 is \$1.5 greater than the \$1 we now pay. So in answer to the questions we started out this discussion with, it seems that we should purchase 500 raw wafers at a purchase price of no more than \$5 + \$1 = \$6 dollars per wafer. With this purchase the new optimal production schedule is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 25 + .015\theta \\ 10 - .02\theta \\ 5 + .015\theta \end{pmatrix}_{\theta=500} = \begin{pmatrix} 0 \\ 32.5 \\ 0 \\ 12.5 \end{pmatrix}.$$

Is this all of the wafers we should purchase? The answer to this question is not immediately obvious. This is because all we know about the range of values $-\frac{1000}{3} \leq \theta \leq 500$ is that if we move θ beyond these range boundaries, then the optimal basis will change. In particular, moving θ above 500 will introduce a negative entry in the third row of the simplex tableau. But the tableau will remain dual feasible. Hence to determine then new optimal solution for θ above 500 we must perform a dual simplex pivot in the third row. We can formally perform this pivot with the variable θ staying in the tableau:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	$25 + .015\theta$
-5	0	0	0	$-.05$	1	0	$-.5$	$50 - .05\theta$
0	0	1	0	$-.02$	0	.1	0	$10 - .02\theta$
0.5	0	0	1	.015	0	$-.1$	.05	$5 + .015\theta$
-1500	0	0	0	-5	0	-100	-50	$-145,000 - 5\theta$

.

0.5	1	.75	0	0	0	.075	$-.05$	32.5
-5	0	-2.5	0	0	1	$-.25$	$-.5$	25
0	0	-50	0	1	0	-5	0	$-500 + \theta$
0.5	0	.75	1	0	0	$-.025$	.05	12.5
-1500	0	-250	0	0	0	-125	-50	-147500

Observe that for $\theta > 500$ we have pivoted to an optimal tableau with the slack for raw silicon wafers basic. Hence we cannot use any more wafers and their shadow price has fallen to zero. The new optimal solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 32.5 \\ 0 \\ 12.5 \end{pmatrix},$$

and this solution persists regardless of how many more raw silicon wafers we get. The final conclusion is that we should only buy 500 raw wafers at a price less than 6 per wafer.

Let us briefly review the range analysis for the right hand side coefficients. In the range analysis for a right hand side coefficient the goal is to determine the range of variation in a particular right hand side coefficient b_i within which the optimal basis does not change. In this regard it is very similar to objective coefficient range analysis, but in this case we add θ times the column associated with the slack variable s_i (or x_{n+i}) to the right hand side coefficients in the optimal tableau and then determine the variations in θ that preserve the primal feasibility of this tableau.

In the discussion above we computed the range for b_1, or the raw wafer resource. Let us now do a range analysis on b_2 the etching time resource. Note that this resource is slack in the optimal tableau since it is in the basis. Regardless, the new right hand side resulting from the perturbation $\Delta b = \theta e_2$ to b is

$$Rb + R\Delta b = \begin{pmatrix} 25 \\ 50+\theta \\ 10 \\ 5 \end{pmatrix} .$$

To preserve primal feasibility we only require $0 \leq 50+\theta$, or equivalently, $-50 \leq \theta$. Therefore, the range for b_2 is

$$[550, \ +\infty) .$$

The upper bound of $+\infty$ make sense since etching time is already slack so any more etching time won't make any difference. The lower bound of 550 implies that if etching time drops to 550 or less, then this constraint will be binding in the optimal tableau.

Similarly, we can compute the range for the lamination and testing time resources. For example, to compute the range of the lamination time resource we simply add θ times the x_7 column to the optimal right hand side to get

$$Rb + R\Delta b = \begin{pmatrix} 25 \\ 50 \\ 10+0.1\theta \\ 5-0.1\theta \end{pmatrix} .$$

To preserve primal feasibility we must have

$$\begin{array}{rcc} 0 \leq 10+0.1\theta, & \text{or equivalently,} & -100 \leq \theta \\ 0 \leq 5-0.1\theta, & \text{or equivalently,} & \theta \leq 50 \end{array} .$$

Therefore, the range on b_3 is

$$800 \leq b_3 \leq 950 .$$

6.4 Pricing Out New Products

Next we consider the problem of adding a new product to our product line. In the context of the Silicon Chip Corp. problem, we consider a new chip that requires ten hours each of etching, lamination, and testing time per 100 chip batch. If it can be sold for \$ 33.10 per chip, we would like to know the answer to the following two questions:

(a) Is it efficient to produce?

(b) If it efficient to produce, what is the new production schedule?

We analyze this problem in the same way that we analyzed the two previous problems. That is, we first determine how this new chip effects the original initial tableau, and then see how the original pivoting process effects the new initial tableau by multiplying this new tableau on the left by the matrix G.

In the context of a new product, the original initial tableau is altered by the addition of a new column:

$$\begin{bmatrix} a_{\text{new}} & A & I & b \\ c_{\text{new}} & c^T & 0 & 0 \end{bmatrix} .$$

Again, multiplying on the left by the matrix G gives

$$\begin{bmatrix} Ra_{\text{new}} & RA & R & Rb \\ c_{\text{new}} - a_{\text{new}}^T y & (c - A^T y)^T & -y^T & -y^T b \end{bmatrix} . \tag{6.6}$$

The expression $(c_{\text{new}} - a_{\text{new}}^T y)$ determines whether this new tableau is optimal or not. The act of forming this expression is called *pricing out* the new product. If this number is nonpositive, then the new product does not price out, and we do not produce it since in this case the new tableau is optimal with the new product nonbasic. If, on the other hand, $(c_{\text{new}} - a_{\text{new}}^T y) > 0$, then we say that the new product does price out and it should be introduced into the optimal production mix. The new optimal production mix is found by applying the standard primal simplex algorithm to the tableau (6.6) since this tableau is primal feasible but not dual feasible.

Let us return to the Silicon Chip Corp. problem and the new chip under consideration. In this case we have

$$a_{\text{new}} = \begin{pmatrix} 100 \\ 10 \\ 10 \\ 10 \end{pmatrix} .$$

We also need to compute c_{new}. The stated sale price or revenue for each 100 chip batch of the new chip is \$3310. We need to subtract from this number the cost of producing each 100 chip batch. Recall from the Silicon Chip Corp. problem statement that each raw silicon wafer is worth \$1, each hour of etching time costs \$40, each hour of lamination time costs \$60, and each hour of inspection time costs \$10. Therefore, the cost of producing each 100 chip batch of these new chips is

$$\begin{array}{rl} 100 & \text{(cost of the raw wafers)} \\ +10 \times 40 & \text{(cost of etching time)} \\ +10 \times 60 & \text{(cost of lamination time)} \\ +10 \times 10 & \text{(cost of testing time)} \\ \hline 1200 & \text{(total cost)} . \end{array}$$

Hence the profit on each 100 chip batch of these new chips is \$3310 − \$1200 = \$2110, or \$21.10 per chip, and so $c_{\text{new}} = 2110$. Pricing out the new chip gives

$$c_{\text{new}} - a_{\text{new}}^T y = 2110 - \begin{pmatrix} 100 \\ 10 \\ 10 \\ 10 \end{pmatrix}^T \begin{pmatrix} 5 \\ 0 \\ 100 \\ 50 \end{pmatrix} = 2110 - 2000 = 110 ,$$

which is positive, and so this chip prices out. The new column in the tableau associated with this chip is

$$\begin{pmatrix} Ra_{\text{new}} \\ c_{\text{new}} - a_{\text{new}}^T y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 110 \end{pmatrix} .$$

Pivoting on the new tableau yields

$$\begin{array}{rrrrrrrrr|r}
x_{\text{new}} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & b \\
1 & 0.5 & 1 & 0 & 0 & .015 & 0 & 0 & -.05 & 25 \\
0 & -5 & 0 & 0 & 0 & -.05 & 1 & 0 & -.5 & 50 \\
-1 & 0 & 0 & 1 & 0 & -.02 & 0 & .1 & 0 & 10 \\
1 & 0.5 & 0 & 0 & 1 & .015 & 0 & -.1 & .05 & 5 \\
\hline
110 & -1500 & 0 & 0 & 0 & -5 & 0 & -100 & -50 & -145,000 \\
\hline
0 & 0 & 1 & 0 & -1 & 0 & 0 & .1 & -.1 & 20 \\
0 & -5 & 0 & 0 & 0 & -.05 & 1 & 0 & -.5 & 50 \\
0 & .5 & 0 & 1 & 1 & -.005 & 0 & 0 & .05 & 15 \\
1 & 0.5 & 0 & 0 & 1 & .015 & 0 & -.1 & .05 & 5 \\
\hline
0 & -1555 & 0 & 0 & -110 & -6.65 & 0 & -88.9 & -55.5 & -145550
\end{array} .$$

The new optimal solution is

$$\begin{pmatrix} x_{\text{new}} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 20 \\ 15 \\ 0 \end{pmatrix} .$$

6.5 Fundamental Theorem on Sensitivity Analysis

The purpose of this section is to state and prove a theorem that captures much of the flavor of the results on sensitivity analysis that we have seen in this section. While there are many possible results one might choose to present, the theorem we give is a stepping stone to the more advanced theory of *Lagrangian duality*. This result focuses on variations in resource availability. We presented this result in Section 1 of these notes on 2-dimensional LPs.

Let $A \in \mathsf{R}^{m\times n}$, $b \in \mathsf{R}^m$, and $c \in \mathsf{R}^n$. This data defines an LP in standard form by

$$\begin{array}{lll} \mathcal{P} & \text{maximize} & c^T x \\ & \text{subject to} & Ax \le b,\ 0 \le x \ . \end{array}$$

We associate $\mathcal{P}$ the *optimal value function* $V : \mathsf{R}^m \mapsto \mathsf{R} \cup \{\pm\infty\}$ defined by

$$\begin{array}{rll} V(u) = & \text{maximize} & c^T x \\ & \text{subject to} & Ax \le b + u,\ 0 \le x \end{array}$$

for all $u \in \mathsf{R}^m$. Let

$$\mathcal{F}(u) = \{x \in \mathsf{R}^n \mid Ax \le b + u,\ 0 \le x\}$$

denote the feasible region for the LP associated with value $V(u)$. If $\mathcal{F}(u) = \emptyset$ for some $u \in \mathsf{R}^m$, we define $V(u) = -\infty$.

Theorem 6.1 (Fundamental Theorem on Sensitivity Analysis) *If $\mathcal{P}$ is primal nondegenerate, i.e. the optimal value is finite and no basic variable in any optimal tableau takes the value zero, then the dual solution y^* is unique and there is an $\epsilon > 0$ such that*

$$V(u) = b^T y^* + u^T y^* \quad \textit{whenever}\ \ |u_i| \le \epsilon,\ i = 1, \ldots, m\ .$$

Thus, in particular, the optimal value function V is differentiable at $u = 0$ with $\nabla V(0) = y^*$.

Proof: Let

$$\begin{bmatrix} RA & R & Rb \\ (c - A^T y^*)^T & -(y^*)^T & -b^T y^* \end{bmatrix}$$

be any optimal tableau for $\mathcal{P}$. Primal nondegeneracy implies that every component of the vector Rb is strictly positive. If there is another dual optimal solution $\tilde{y}$ associated with another tableau, then we can pivot to it using simplex pivots. All of these simplex pivots must be degenerate since the optimal value cannot change. But degenerate pivots can only be performed if the tableau is degenerate, i.e. there is an index i such that $(Rb)_i = 0$. But then the basic variable associated with $(Rb)_i$ must take the value zero contradicting the hypothesis that Rb is a strictly positive vector. Hence the only possible optimal tableau is the one given. The only other way to have multiple dual solutions is if there is an unbounded ray of optimal solutions emanating from the optimal solution identified by the unique optimal tableau. For this to occur, there must be a row in the optimal tableau such that any positive multiple of that row can be added to the objective row without changing the optimal value. Again, this can only occur if some $(Rb)_i$ is zero leading to the same contradiction. Therefore, primal nondegeneracy implies the uniqueness of the dual solution y^*.

Next let $0 < \delta < \min\{(Rb)_i \mid i = 1, \ldots, m\}$. Due to the continuity of the mapping $u \to Ru$, there is an $\epsilon > 0$ such that $|(Ru)_i| \le \delta\ i = 1, \ldots m$ whenever $|u_j| \le \epsilon\ j = 1, \ldots, n$. Hence, if we perturb b by u, then

$$R(b + u) = Rb + Ru \ge Rb - \delta \mathsf{e} > 0$$

whenever $|u_j| \le \epsilon\ j = 1, \ldots, n$, where e is the vector of all ones. Therefore, if we perturb b by u in the optimal tableau with $|u_j| \le \epsilon\ j = 1, \ldots, n$, we get the tableau

$$\begin{bmatrix} RA & R & Rb + Ru \\ (c - A^T y^*)^T & -(y^*)^T & -b^T y^* - b^T u \end{bmatrix}$$

which is still both primal and dual feasible, hence optimal with optimal value $V(u) = b^T y^* + b^T u$ proving the theorem. ■

Introduction to Game Theory
Matrix Games and Lagrangian Duality

1. Introduction

In this section we study only *finite, two person, zero-sum, matrix games*. We introduce the basics by studying a Canadian drinking game. One and two dollar coins are very popular in Canada. The one dollar coin, introduced in 1987, is adorned with the picture of the common loon an aquatic bird found throughout Canada. Quickly the one dollar coin was nicknamed the *loonie*. The two dollar coin was issued in 1996. At the time there was a national competition for its naming with *Nanuq* the winning name. But the popular name for the coin quickly became the portmanteau *toonie*. The game is played with loonies and toonies, and, so I suppose, should be called *loonie-toonie*. Actually, this is a version of an ancient game called *Morra* which dates back to at least Roman times and most probably much earlier. Regardless, the rules are as follows: each player chooses either the loonie or the toonie and places the single coin in their closed right hand with the choice hidden from their opponent. Each player then guesses the play of the other. If only one guesses correctly, then the other player pays to the correct guesser the sum of the coins in both their hands. If both guess incorrectly or both correctly, then there is no payoff. This is an example of a *zero-sum* game since in each case, what one player loses the other player gains. This is not always the case. For example, in a casino, the house always takes a commission from the winner. That is, the winner makes less than the loser loses.

We now model the game of Morra mathematically. The first step is to define the *payoff matrix*. Designate one of the players as the *column player* and the other the *row player*. The payoff matrix consists of the payoff to the column player based on the strategy employed by both players in a given round of play. The strategies for either player are the same and they consist of a pair of decisions. The first is the choice of coin to hide, and the second is the guess for the opponents hidden coin. We denote these decisions by (i, j) with $i = 1, 2$ and $j = 1, 2$. For example, the strategy $(2, 1)$ is to hide the toonie in your fist and to guess your opponent is hiding a loonie. The payoff matrix P to the column player is given by

$$\begin{array}{c|cccc} & (1,1) & (1,2) & (2,1) & (2,2) \\ \hline (1,1) & 0 & -2 & 3 & 0 \\ (1,2) & 2 & 0 & 0 & -3 \\ (2,1) & -3 & 0 & 0 & 4 \\ (2,2) & 0 & 3 & -4 & 0 \end{array} \; .$$

For example, if the row player plays strategy $(2, 2)$ while the column player uses strategy $(2, 1)$, then the column player must pay the row player \$4.

The elements of P are the payoffs for the use of a *pure* strategy. But this game is played over and over again. So it is advisable for the column player to use a different pure strategy on each play. How should these strategies be chosen? One possibility is for the column player to decide on a long run frequency of play for each strategy, or equivalently, to decide on a probability of play for each strategy on each play. This is called a *mixed* strategy which can be represented as a vector of probabilities in R^4:

$$0 \le x \quad \text{and} \quad \mathsf{e}^T x = 1, \tag{1}$$

where e always represents the vector of all ones of the appropriate dimension, in this case $\mathsf{e} = (1, 1, 1, 1)^T$. Given a particular mixed strategy, one can easily compute the expected payoff to the column player for each choice of pure strategy by the row player. For example, if the row player chooses pure strategy $(1,1)$, then the expected payoff to the column player is

$$0 \cdot x_1 - 2 \cdot x_2 + 3 \cdot x_3 + 0 \cdot x_4 = \sum_{j=1}^{4} P_{1j} x_j \ .$$

Now given that the column player will use a mixed strategy, what mixed strategy should be chosen? One choice is the strategy that maximizes the column player's minimum expected payoff over the range of the row player's pure strategies. This strategy can be found by solving the optimization problem

$$\max_{0 \le x,\ \mathsf{e}^T x = 1} \ \min_{i=1,2,3,4} \sum_{j=1}^{4} P_{ij} x_i \ . \tag{2}$$

Note that this problem is equivalent to the linear program

$$\begin{array}{lll} \mathcal{C} & \text{maximize} & \gamma \\ & \text{subject to} & \gamma \mathsf{e} \le Px, \\ & & \mathsf{e}^T x = 1 \\ & & 0 \le x. \end{array}$$

On the flip side, the row player can also chose a mixed strategy of play, $0 \le y,\ \mathsf{e}^T y = 1$. In this case, the expected payoff to the column player when the column player uses the pure strategy $(2,1)$ is

$$3 \cdot y_1 + 0 \cdot y_2 + 0 \cdot y_3 - 4 \cdot y_4 = \sum_{i=1}^{4} P_{i3} y_i \ .$$

How should the row player decide on their strategy? One approach is for the row player to minimize the maximum expected payoff to the column player:

$$\min_{0 \le y,\ \mathsf{e}^T y = 1} \ \max_{j=1,2,3,4} \sum_{i=1}^{4} P_{ij} y_i. \tag{3}$$

This problem is equivalent to the linear program

$$\begin{array}{lll} \mathcal{R} & \text{minimize} & \eta \\ & \text{subject to} & P^T y \le \eta \mathsf{e}, \\ & & \mathsf{e}^T y = 1 \\ & & 0 \le y. \end{array}$$

Both the column player's problem $\mathcal{C}$ and the row player's problem $\mathcal{R}$ are linear programming problems. Let us pause for a moment to consider their dual linear programs. We begin with the column player's problem by putting it into our general standard form so that we can immediately write down its dual LP. Rewriting we have

$$\begin{array}{llll} \mathcal{C} & \text{maximize} & \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} \gamma \\ x \end{pmatrix} & \\ & & & \\ & \text{subject to} & \begin{bmatrix} 0 & \mathsf{e}^T \end{bmatrix} \begin{pmatrix} \gamma \\ x \end{pmatrix} = 1 & (\tau) \\ & & & \\ & & \begin{bmatrix} \mathsf{e} & -P \end{bmatrix} \begin{pmatrix} \gamma \\ x \end{pmatrix} \le 0 & (y) \\ & & 0 \le x & . \end{array}$$

The dual problem becomes

$$\begin{array}{ll} \text{minimize} & \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} \tau \\ y \end{pmatrix} \\ \text{subject to} & \begin{bmatrix} 0 & \mathsf{e}^T \end{bmatrix} \begin{pmatrix} \tau \\ y \end{pmatrix} = 1 \\ & \\ & \begin{bmatrix} \mathsf{e} & -P^T \end{bmatrix} \begin{pmatrix} \tau \\ y \end{pmatrix} \geq 0 \\ & 0 \leq y. \end{array}$$

Rewriting this dual, we have the LP

$$\begin{array}{ll} \text{minimize} & \tau \\ \text{subject to} & P^T y \leq \tau \mathsf{e} \\ & \mathsf{e}^T y = 1 \\ & 0 \leq y \ . \end{array}$$

But this is just the row player's problem $\mathcal{R}$! That is, the row player's and the column players problems are dual to each other! Also, observe that the feasible regions for both the primal and dual problems are always nonempty (why?) and bounded in the variables x and y, respectively (why?), and so the optimal values of both are necessarily bounded (why?). Hence, by the Strong Duality Theorem solutions to both the primal and dual problems exist with the optimal values coinciding.

Let us now apply this structure to the loonie-toonie game. One way to compute an optimal solution is to guess what it is and then use the Weak Duality Theorem to verify. In this game neither the column player nor the row player has a clear advantage, so it is reasonable to guess that their optimal strategies should be the same giving the same optimal value of zero. One guess that corresponds to this intuition is

$$\bar{\gamma} = 0, \quad \bar{x} = \begin{pmatrix} 0 \\ 3/5 \\ 2/5 \\ 0 \end{pmatrix} \quad \text{and} \quad \bar{\eta} = 0, \quad \bar{y} = \begin{pmatrix} 0 \\ 4/7 \\ 3/7 \\ 0 \end{pmatrix},$$

with

$$P = \begin{bmatrix} 0 & -2 & 3 & 0 \\ 2 & 0 & 0 & -3 \\ -3 & 0 & 0 & 4 \\ 0 & 3 & -4 & 0 \end{bmatrix}.$$

Observe that

$$P\bar{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/7 \end{pmatrix} \quad \text{and} \quad P^T\bar{y} = \begin{pmatrix} -1/7 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

so $(\bar{\gamma}, \bar{x})$ is primal feasible while $(\bar{\eta}, \bar{y})$ is dual feasible and their optimal values coincide at zero. Therefore, by the Weak Duality Theorem, they are optimal for their respective problems. Note that although the game seems to be structured in a way that it makes no difference who the column and row players are, the optimal solutions given above are not the same strategy. Is there an issue yet to be resolved? What can be said about the strategies

$$\bar{\gamma} = 0, \quad \bar{x} = \begin{pmatrix} 0 \\ 4/7 \\ 3/7 \\ 0 \end{pmatrix} \quad \text{and} \quad \bar{\eta} = 0, \quad \bar{y} = \begin{pmatrix} 0 \\ 3/5 \\ 2/5 \\ 0 \end{pmatrix}?$$

Can more be said about the structure of the solution sets for both the primal and dual games in this case?

Any matrix $P \in \mathrm{R}^{m\times n}$ can define a matrix game of the form $\mathcal{C}$ having dual $\mathcal{R}$. These games are always feasible with bounded feasible region. Therefore, by the Strong Duality Theorem, both $\mathcal{C}$ and $\mathcal{R}$ always have optimal solutions with a common optimal value. This common optimal value is called the value of the game. If the value of the game is zero, then it is said to be a *fair game* since neither the column or row player has an advantage. Games such as Morra are said to be symmetric since their payoff matrix is skew symmetric, i.e., $P^T = -P$. Symmetric games are always fair (why?). A pair of strategies for the column and row players are said to constitute a *Nash equilibrium* if prior knowledge of the mixed strategy of ones opponent has no effect on ones own choice of strategy. Is the case for the strategies provided by our minimax approach to the game of Morra?

2. Equilibria and Minimax Problems

We can change the way we represent matrix games by making a simple observation about how the maximum or minimum of a finite set of objects can be represented. If $\{= a_1, a_2, \ldots, a_N\}$ is any collection of real numbers, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)^T$ is any N-dimensional probability vector, then

$$\min\{a_1, a_2, \ldots, a_N\} \;\le\; \sum_{i=1}^{N} \lambda_i a_i \;\le\; \max\{a_1, a_2, \ldots, a_N\}.$$

That is, the expected value, or average, of the a_i's in any discrete probability distribution always lies between the minimum and the maximum values of the a_i's. Consequently,

$$\min\{a_1, a_2, \ldots, a_N\} = \min_{0\le y, \mathrm{e}^T y = 1} y^T a \quad \text{and} \quad \max\{a_1, a_2, \ldots, a_N\} = \max_{0\le y, \mathrm{e}^T y = 1} y^T a.$$

By applying this observation to (2) and (3), we obtain the following representations for $\mathcal{C}$ and $\mathcal{R}$, respectively:

$$\max_{0\le x, \mathrm{e}^T x = 1} \; \min_{0\le y, \mathrm{e}^T y = 1} \; y^T P x \;, \tag{4}$$

and

$$\min_{0\le y, \mathrm{e}^T y = 1} \; \max_{0\le x, \mathrm{e}^T x = 1} \; y^T P x \;. \tag{5}$$

This implies that the difference between the column player's problem and the row player's problem is simply reversing the order in which the min andmax are taken. The fact that the optimal values in (4) and (5) coincide is an instance of what is known as a *Minimax Theorem*. Such theorems play an important role in several areas of application, particularly in economics and game theory. The first big theorem of this type was proven by John von Neumann of which the following theorem is an elementary special case.

Theorem 1. *Given any matrix $P \in \mathrm{R}^{m\times n}$, one has*

$$\max_{0\le x, \mathrm{e}^T x = 1} \; \min_{0\le y, \mathrm{e}^T y = 1} \; y^T P x \;=\; \min_{0\le y, \mathrm{e}^T y = 1} \; \max_{0\le x, \mathrm{e}^T x = 1} \; y^T P x \;.$$

3. Lagrangian Duality

A general minimax problem can be obtained from any function $L\colon \mathrm{R}^n \times \mathrm{R}^m \mapsto \mathrm{R}$ and two sets $X \subset \mathrm{R}^n$ and $Y \subset \mathrm{R}^m$ and writing the two problems

$$\max_{x\in X}\ \min_{y\in Y}\ L(x,y) \qquad \text{and} \qquad \min_{y\in Y}\ \max_{x\in X}\ L(x,y).$$

In the case of matrix games, we have $L(x,y) = y^T Px$. Returning to the general case, define the function $p\colon \mathrm{R}^n \mapsto \mathrm{R}\cup\{-\infty\}$ and $d\colon \mathrm{R}^m \mapsto \mathrm{R}\cup\{+\infty\}$ by

$$p(x) := \min_{y\in Y} L(x,y) \qquad \text{and} \qquad d(y) := \max_{x\in X} L(x,y)\ .$$

We call p the primal objective function and d the dual objective, and we call the problem

$$\mathcal{P} \qquad \max_{x\in X} p(x)$$

the *Primal Problem* and

$$\mathcal{D} \qquad \min_{y\in Y} d(y)$$

the *Dual Problem*. Note that for every pair $(\bar{x},\bar{y}) \in X \times Y$,

$$p(\bar{x}) = \min_{y\in Y} L(\bar{x},y) \le L(\bar{x},\bar{y}) \le \max_{x\in X} L(x,\bar{y}) = d(\bar{y}). \tag{6}$$

The inequality (6) is called the *Weak Duality Theorem* for minimax problems of this type.

Theorem 2 (Weak Duality for Minimax)**.** *Let L, p, and d, be as defined above. Then for every $(x,y) \in X \times Y$,*

$$p(x) \le d(y). \tag{7}$$

Moreover, if $(\bar{x},\bar{y})$ are such that $p(\bar{x}) = d(\bar{y})$, then $\bar{x}$ solves $\mathcal{P}$ and $\bar{y}$ solves $\mathcal{D}$.

We call a point $(\bar{x},\bar{y}) \in X \times Y$ a *saddle point* for L, if

$$L(x,\bar{y}) \le L(\bar{x},\bar{y}) \le L(\bar{x},y) \quad \forall\ (x,y) \in X \times Y. \tag{8}$$

Theorem 3 (Saddle Point Theorem)**.** *Let L, p, and d, be as defined above.*

(i) *If $(\bar{x},\bar{y})$ is a saddle point for L, the $\bar{x}$ solves $\mathcal{P}$ and $\bar{y}$ solves $\mathcal{D}$ with the optimal value in both $\mathcal{P}$ and $\mathcal{D}$ equal to the saddle point value $L(\bar{x},\bar{y})$.*

(ii) *If $\bar{x}$ solves $\mathcal{P}$ and $\bar{y}$ solves $\mathcal{D}$ with the optimal values coinciding, then $(\bar{x},\bar{y})$ is a saddle point for L.*

Proof. (i) Suppose $(\bar{x},\bar{y})$ is a saddle point for L. Let $\epsilon > 0$ and choose $(x_\epsilon, y_\epsilon) \in X \times Y$ so that

$$d(\bar{y}) - \epsilon \le L(x_\epsilon,\bar{y}) \qquad \text{and} \qquad L(\bar{x},y_\epsilon) \le p(\bar{x}) + \epsilon.$$

By combining this with (8), we obtain

$$d(\bar{y}) - \epsilon \le L(\bar{x},\bar{y}) \le p(\bar{x}) + \epsilon.$$

Since this holds for all $\epsilon > 0$, we have $d(\bar{y}) \le L(\bar{x},\bar{y}) \le p(\bar{x})$. But then, by the Weak Duality Theorem, $p(\bar{x}) \le d(\bar{y}) \le L(\bar{x},\bar{y}) \le p(\bar{x}) \le d(\bar{y})$ which, again by the Weak Duality Theorem, proves the result. □

3.1. **Linear Programming Duality.** Let $A \in \mathrm{R}^{m\times n}$, $b \in \mathrm{R}^m$, and $c \in \mathrm{R}^n$, and define

$$L(x,y) := c^T x + y^T b - y^T Ax,$$

with $X := \mathrm{R}^n_+$ and $Y = \mathrm{R}^m_+$. Then

$$\begin{aligned} p(x) &= \min_{0\le y} L(x,y) = \min_{0\le y} c^T x + y^T(b - Ax) \\ &= c^T x + \min_{0\le y} y^T(b-Ax) \\ &= c^T x + \begin{cases} 0 & , Ax \le b, \\ -\infty & , \text{else.} \end{cases} \end{aligned}$$

and

$$\begin{aligned} d(y) &= \max_{0\le x} L(x,y) = \max_{0\le x} y^T b + (c - A^T y)^T x \\ &= y^T b + \max_{0\le x}(c - A^T y)^T x \\ &= y^T b + \begin{cases} 0 & , A^T y \ge c, \\ +\infty & , \text{else.} \end{cases} \end{aligned}$$

Therefore, the primal problem has the form

$$\mathcal{P} \qquad \max_{0\le x} p(x) \;=\; \begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \le b,\ 0 \le x, \end{array}$$

while the dual problem takes the form

$$\mathcal{D} \qquad \min_{0\le y} d(y) \;=\; \begin{array}{ll} \min & b^T y \\ \text{s.t.} & A^T y \ge c,\ 0 \le y. \end{array}$$

In this case, the function L is called the *Lagrangian*, and this development is an instance of *Lagrangian duality*. Observe that if $\bar{x}$ solves $\mathcal{P}$ and $\bar{y}$ solves $\mathcal{D}$, then the Saddle Point Theorem tells us that $p(\bar{x}) = L(\bar{x},\bar{y}) = d(\bar{y})$, or equivalently,

$$c^T\bar{x} = c^T\bar{x} + b^T\bar{y} - \bar{y}^T A\bar{x} = b^T\bar{y},$$

or equivalently,

$$\bar{y}^T(b - A\bar{x}) = 0 \qquad \text{and} \qquad \bar{x}^T(c - A^T\bar{y}) = 0,$$

which is just the Complementary Slackness Theorem.

3.2. **Convex Quadratic Programming Duality.** One can also apply the Lagrangian Duality Theory in the context of Convex Quadratic Programming. To see how this is done let $Q \in \mathrm{R}^{n\times n}$ be symmetric and positive definite, and let $c \in \mathrm{R}^n$. Consider the convex quadratic program

$$\mathcal{D} \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T Qx + c^T x \\ \text{subject to} & Ax \le b,\ 0 \le x\ . \end{array}$$

The Lagrangian is given by

$$L(x,y,v) = \frac{1}{2}x^T Qx + c^T x + y^T(A^T x - b) - v^T x \quad \text{where } 0 \le y,\ 0 \le v.$$

The dual objective function is

$$g(y,v) = \min_{x\in\mathrm{R}^n} L(x,y,v)\ .$$

The goal is to obtain a closed form expression for g with the variable x removed by using the first-order optimality condition $0 = \nabla_x L(x, y, v)$. This optimality condition completely identifies the solution since L is convex in x. We have

$$0 = \nabla_x L(x, y, v) = Qx + c + A^T y - v.$$

Since Q is invertible, we have

$$x = Q^{-1}(v - A^T y - c).$$

Plugging this expression for x into $L(x, y, v)$ gives

$$\begin{aligned} g(y, v) &= L(Q^{-1}(v - A^T y - c), y, v) \\ &= \frac{1}{2}(v - A^T y - c)^T Q^{-1}(v - A^T y - c) \\ &\qquad + c^T Q^{-1}(v - A^T y - c) + y^T(AQ^{-1}(v - A^T y - c) - b) - v^T Q^{-1}(v - A^T y - c) \\ &= \frac{1}{2}(v - A^T y - c)^T Q^{-1}(v - A^T y - c) - (v - A^T y - c)^T Q^{-1}(v - A^T y - c) - b^T y \\ &= -\frac{1}{2}(v - A^T y - c)^T Q^{-1}(v - A^T y - c) - b^T y \ . \end{aligned}$$

Hence the dual problem is

$$\begin{array}{ll} \text{maximize} & -\frac{1}{2}(v - A^T y - c)^T Q^{-1}(v - A^T y - c) - b^T y \\ \text{subject to} & 0 \le y, \ 0 \le v \ . \end{array}$$

Moreover, $(\bar{y}, \bar{v})$ solve the dual problem if an only if $\bar{x} = Q^{-1}(\bar{v} - A^T \bar{y} - c)$ solves the primal problem with the primal and dual optimal values coinciding.

All of this is just a glimpse into what is possible!