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Abstract

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1 Nesterov's Accelerated Gradient

1.1 In preparations

{ass:smooth-plus-nonsmooth}

Assumption 1.1 (smooth add nonsmooth) The function $F = f + g$ where $f : \mathbb{R}^n \to \mathbb{R}$ is a L Lipschitz smooth and $\mu \geq 0$ strongly convex function. The function $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a closed convex proper function.

{ass:smooth-plus-nonsmooth-x}

Assumption 1.2 (admitting minimizers) Let $F = f + g$ and in addition assume that the set of minimizers $X^+ := \operatorname*{argmin}_x F(x)$ is non-empty.

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Definition 1.3 (Proximal gradient operator) *Suppose $F = f + g$ satisfies Assumption 1.1. Let $\beta > 0$. Then, we define the proximal gradient operator T_β as*

$$T_\beta(x|F) = \operatorname{argmin} z \left\{ g(z) + f(x) + \langle \nabla f(x), z - x \rangle + \frac{\beta}{2}\|z - x\|^2 \right\}.$$

Remark 1.4 If the function $g \equiv 0$, then it yields the gradient descent operator $T_\beta(x) = x - \beta^{-1}\nabla f(x)$. In the context where it's clear what the function $F = f + g$ is, we simply write $T_\beta(x)$ for short.

Definition 1.5 (Bregman Divergence) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a differentiable function. Then, for all the Bregman divergence $D_f : \mathbb{R}^n \times \operatorname{dom} \nabla f \to \mathbb{R}$ is defined as:*

$$D_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y \rangle.$$

Remark 1.6 If, f is $\mu \geq 0$ strongly convex and L Lipschitz smooth then, its Bregman
{def:rwapg-seq} Divergence has for all $x, y \in \mathbb{R}^n$: $\mu/2\|x - y\|^2 \leq D_f(x, y) \leq L/2\|x - y\|^2$.

Definition 1.7 (R-WAPG sequence) *Let $(L_k)_{k\geq 0}$ be a sequence such that $L_k > \mu$ for all k. Let $\alpha_0 \in (0, 1]$, $(\alpha_k)_{k\geq 1}$ has $\alpha_k \in (\mu/L_k, 1)$. Then define for all $k \geq 0$:*

$$\rho_k(1 - \alpha_{k+1})\alpha_k^2 = \alpha_{k+1}(\alpha_{k+1} - \mu/L_k).$$

Remark 1.8 When $\rho_k = 1$, the recursive relation between α_k, α_{k-1} is the same as the well known Nesterov's sequence used in algorithm such as FISTA and Nesterov's accelerated gradient. See Li and Wang [2] for more information.

{def:st-method} **Definition 1.9 (similar triangle representation of NAPG)**
Let $(\alpha_k)_{k\geq 0}$ be an R-WAPG sequence. Suppose that the base case $v_{-1}, x_{k-1} \in \mathbb{R}^n$ is given to initialize the algorithm. Then the algorithm produces the sequence of iterates $(y_k, x_k, v_k)_{k\geq 0}$ and auxiliary parameter sequence L_k, τ_k satisfying these inequalities:

$$\begin{aligned}
&\tau_k = L_k(1 - \alpha_k)(L_k\alpha_k - \mu)^{-1}, \\
&y_k = (1 + \tau_k)^{-1}v_{k-1} + \tau_k(1 + \tau_k)^{-1}x_{k-1}, \\
&D_f(x_k, y_k) \leq L_k/2\|x_k - y_k\|^2, \\
&x_k = T_{L_k}(y_k), \\
&v_k = x_{k-1} + \alpha_k^{-1}(x_k - x_{k-1}).
\end{aligned}$$

{thm:pg-ineq} The following theorems are critical in analyzing the behavior of algorithm in Definition 1.9.

Theorem 1.10 (proximal gradient inequality) *Let function F satisfies Assumption 1.1, so it's $\mu \geq 0$ strongly convex. For all $x \in \mathbb{R}^n$, define $x^+ = T_L(x)$, then there exists* *a $B \ge 0$ such that $D_f(x^+, x) \le B/2\|x^+ - x\|^2$. Then, for all $z \in \mathbb{R}^n$ it satisfies proximal gradient inequality at point x:*

$$
\begin{aligned}
0 &\le F(z) - F(x^+) - \frac{B}{2}\|z - x^+\|^2 + \frac{B-\mu}{2}\|z - x\|^2 \\
&= F(z) - F(x^+) - \langle B(x - x^+), z - x\rangle - \frac{\mu}{2}\|z - x\|^2 - \frac{B}{2}\|x - x^+\|^2.
\end{aligned}
$$

Since f is assumed to be L Lipschitz smooth, the above condition is true for all $x, y \in \mathbb{R}^n$ for all $B \ge L$.

Remark 1.11 The theorem is the same as in Nesterov's book [3, Theorem 2.2.13], but with the use of proximal gradient mapping and proximal gradient instead of project gradient hence making it equivalent to the theorem in Beck's book [1, Theorem 10.16]. The only generalization here is parameter B which made to accommodate algorithm that implements
{thm:jesen} Definition 1.9 with some line search routine.

Theorem 1.12 (Jensen's inequality) *Let $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a $\mu \ge 0$ strongly convex function. Then, it is equivalent to the following condition. For all $x, y \in \mathbb{R}^n$, $\lambda \in (0, 1)$ it satisfies the inequality*

$$
(\forall \lambda \in [0,1])\ F(\lambda x + (1-\lambda)y) \le \lambda F(x) + (1-\lambda)F(y) - \frac{\mu\lambda(1-\lambda)}{2}\|y - x\|^2.
$$

Remark 1.13 If x, y is out of $\operatorname{dom} F$, the inequality still work by convexity.

1.2 A compact argument for the convergence of NAPG with proximal gradient

Here, the abbreviation "NAPG" stands for "Nesterov Acceleration Proximal Gradient". It's made in acknowledgement of Algorithm 2.2.36 in Nesterov's book [3] and its extension known as FISTA in the literature. The following theorem provides a complete proof for the convergence rate of algorithms implementing Definition 1.9, which is equivalent to NAG, or NAPG. It made use of Definition 1.7 which accommodates a relaxed sequence compared to
{thm:onestep-napg-cnvg} the usual sequence that gives the optimal convergence rate.

Theorem 1.14 (one step convergence claim of NAPG) *Let $F = f + g$ satisfies Assumption 1.1 for some $L > \mu \ge 0$. Let the sequence $(\alpha_k)_{k\ge 0}$ be an R-WAPG sequence (Definition 1.7). Suppose that the iterates sequence $(x_k, y_k, v_k)_{k\ge 0}$ satisfy NAPG in similar triangle form (Definition 1.9) with initial guesses $v_{-1}, x_{-1} \in \mathbb{R}^n$. Then for all $k \ge 1$, the* *following inequality is true for all $\bar{x} \in \mathbb{R}^n$:*

$$
\begin{aligned}
& -F(\bar{x}) + F(x_k) + \frac{L_k \alpha_k^2}{2}\|\bar{x} - v_k\|^2 \\
& \le \max\left(1, \frac{L_k \rho_{k-1}}{L_{k-1}}\right)(1-\alpha_k)\left(F(x_{k-1}) - F(\bar{x}) + \frac{\alpha_{k-1}^2 L_{k-1}}{2}\|\bar{x} - v_{k-1}\|^2\right).
\end{aligned}
$$

If in addition, we choose $\alpha_0 = 1$, and let $x_{-1} = v_{-1}$, then a base case of the inequality is:

$$
F(x_0) - F(\bar{x}) + \frac{L_0}{2}\|\bar{x} - x_0\|^2 \le \frac{L_0 - \mu}{2}\|\bar{x} - v_{-1}\|^2.
$$

Proof. The proof is very intense algebraically hence before we step into it, we present the following intermediate results in advance to their proofs given at the end.

For all $k \ge 0$, define $z_k = \alpha_k \bar{x} + (1-\alpha_k)x_{k-1}$.

(a) Theorem 1.10, with $z = z_k$, $k \ge 0$. We can use it because $F = f + g$ satisfies Assumption 1.1.

(b) Jensen's inequality (Theorem 1.12), with $z = z_k$, for $k \ge 0$. We can use it because F is $\mu \ge 0$ strongly convex.

(c) Definition 1.7 which has $\rho_k(1-\alpha_{k+1})\alpha_k^2 = \alpha_{k+1}(\alpha_{k+1} - \mu/L_k)$ for $k \ge 0$.

(d) The equality $z_k - y_k = (L_k - \mu)^{-1}((L_k\alpha_k - \mu)(\bar{x} - v_k) + \mu(1-\alpha_k)(\bar{x} - x_{k-1}))$ for all $k \ge 0$, it comes from Definition 1.9.

(e) The equality $z_k - x_k = \alpha_k(\bar{x} - v_k)$ for all $k \ge 0$ it comes from Definition 1.9.

(f) Using basic algebra, we have the following equality:

$$
(\forall k \ge 1)\ \frac{1}{2}\left(\frac{\mu^2(1-\alpha_k)^2}{L_k - \mu} - \mu\alpha_k(1-\alpha_k)\right) = \frac{(\alpha_k - 1)\mu(L_k\alpha_k - \mu)}{2(L_k - \mu)}.
$$

(g) Using (c), we have the following equality:

$$
(\forall k \ge 1)\ \frac{1}{2}\left(\frac{(L_k\alpha_k - \mu)^2}{L_k - \mu} - \alpha_{k-1}^2\rho_{k-1}L_k(1-\alpha_k)\right) = \frac{\mu(L_k\alpha_k - \mu)(\alpha_k - 1)}{2(L_k - \mu)}.
$$

(h) Definition 1.7 determine the inequality:

$$
(\forall k \ge 1)\ \frac{\mu(L_k\alpha_k - \mu)(\alpha_k - 1)}{2(L_k - \mu)} \le 0.
$$

With intermediate results (a) to (h), presented above, the proof of the theorem come smoothly from a chain of inequalities and equalities. The overall proof now follows. Start with the (a), the proximal gradient inequality it has:

$$
\begin{aligned}
0 &\le F(z_k) - F(x_k) - \frac{L_k}{2}\|z_k - x_k\|^2 + \frac{L_k - \mu}{2}\|z_k - y_k\|^2 \\
&\underset{\text{(b)}}{\le} \alpha_k F(\bar{x}) + (1-\alpha_k)F(x_{k-1}) - F(x_k) \\
&\quad - \frac{\mu\alpha_k(1-\alpha_k)}{2}\|\bar{x} - x_{k-1}\|^2 - \frac{L_k}{2}\|z_k - x_k\|^2 + \frac{L_k-\mu}{2}\|z_k - y_k\|^2.
\end{aligned}
$$

Using the chain of equality below:

$$
\begin{aligned}
& - \frac{\mu\alpha_k(1-\alpha_k)}{2}\|\bar{x} - x_{k-1}\|^2 + \frac{L_k-\mu}{2}\|z_k - y_k\|^2 \\
&\underset{\text{(d)}}{=} - \frac{\mu\alpha_k(1-\alpha_k)}{2}\|\bar{x} - x_{k-1}\|^2 \\
&\quad + \frac{L_k-\mu}{2}\left\|\frac{L_k\alpha_k - \mu}{L_k-\mu}(\bar{x} - v_{k-1}) + \frac{\mu(1-\alpha_k)}{L_k-\mu}(\bar{x} - x_{k-1})\right\|^2 \\
&= - \frac{\mu\alpha_k(1-\alpha_k)}{2}\|\bar{x} - x_{k-1}\|^2 \\
&\quad + \frac{(L_k\alpha_k-\mu)^2}{2(L_k-\mu)}\|\bar{x} - v_{k-1}\|^2 + \frac{\mu^2(1-\alpha_k)^2}{2(L_k-\mu)}\|\bar{x} - x_{k-1}\|^2 + \frac{(L_k\alpha_k-\mu)\mu(1-\alpha_k)}{L_k-\mu}\langle \bar{x} - x_{k-1}, \bar{x} - v_{k-1}\rangle \\
&= \left(\frac{\mu^2(1-\alpha_k)^2}{2(L_k-\mu)} - \frac{\mu\alpha_k(1-\alpha_k)}{2}\right)\|\bar{x} - x_{k-1}\|^2 + \left(\frac{(L_k\alpha_k-\mu)^2}{2(L_k-\mu)} - \frac{\alpha_{k-1}^2 L_k\rho_{k-1}(1-\alpha_k)}{2}\right)\|\bar{x} - v_{k-1}\|^2 \\
&\quad + \frac{\alpha_{k-1}^2 L_k\rho_{k-1}(1-\alpha_k)}{2}\|\bar{x} - v_{k-1}\|^2 + \frac{(L_k\alpha_k-\mu)\mu(1-\alpha_k)}{L_k-\mu}\langle \bar{x} - x_{k-1}, \bar{x} - v_{k-1}\rangle \\
&\underset{\text{(f)}}{=} \frac{(\alpha_k-1)\mu(L_k\alpha_k-\mu)}{2(L_k-\mu)}\|\bar{x} - x_{k-1}\|^2 + \left(\frac{(L_k\alpha_k-\mu)^2}{2(L_k-\mu)} - \frac{\alpha_{k-1}^2 L_k\rho_{k-1}(1-\alpha_k)}{2}\right)\|\bar{x} - v_{k-1}\|^2 \\
&\quad + \frac{\alpha_{k-1}^2 L_k\rho_{k-1}(1-\alpha_k)}{2}\|\bar{x} - v_{k-1}\|^2 + \frac{(L_k\alpha_k-\mu)\mu(1-\alpha_k)}{L_k-\mu}\langle \bar{x} - x_{k-1}, \bar{x} - v_{k-1}\rangle \\
&\underset{\text{(g)}}{=} \frac{(\alpha_k-1)\mu(L_k\alpha_k-\mu)}{2(L_k-\mu)}\|\bar{x} - x_{k-1}\|^2 + \frac{\mu(L_k\alpha_k-\mu)(\alpha_k-1)}{2(L_k-\mu)}\|\bar{x} - v_{k-1}\|^2 \\
&\quad + \frac{\alpha_{k-1}^2 L_k\rho_{k-1}(1-\alpha_k)}{2}\|\bar{x} - v_{k-1}\|^2 + \frac{(L_k\alpha_k-\mu)\mu(1-\alpha_k)}{L_k-\mu}\langle \bar{x} - x_{k-1}, \bar{x} - v_{k-1}\rangle \\
&= \frac{\alpha_{k-1}^2 L_k\rho_{k-1}(1-\alpha_k)}{2}\|\bar{x} - v_{k-1}\|^2 \\
&\quad \frac{(\alpha_k-1)\mu(L_k\alpha_k-\mu)}{2(L_k-\mu)}\left(\|\bar{x} - x_{k-1}\|^2 + \|\bar{x} - v_{k-1}\|^2 - 2\langle \bar{x} - x_{k-1}, \bar{x} - v_{k-1}\rangle\right) \\
&= \frac{\alpha_{k-1}^2 L_k\rho_{k-1}(1-\alpha_k)}{2}\|\bar{x} - v_{k-1}\|^2 + \frac{(\alpha_k-1)\mu(L_k\alpha_k-\mu)}{2(L_k-\mu)}\|x_{k-1} - v_{k-1}\|^2.
\end{aligned}
$$

The inequality from previously simplifies, and it has:

$$
0 \le \alpha_k F(\bar{x}) + (1-\alpha_k)F(x_{k-1}) - F(x_k) + \frac{\alpha_{k-1}^2 L_k\rho_{k-1}(1-\alpha_k)}{2}\|\bar{x} - v_{k-1}\|^2 - \frac{L_k}{2}\|z_k - x_k\|^2
$$

$$
\begin{aligned}
&+ \frac{(\alpha_k - 1)\mu(L_k\alpha_k - \mu)}{2(L_k - \mu)}\|x_{k-1} - v_{k-1}\|^2 \\
&\underset{\text{(h)}}{\le} \alpha_k F(\bar{x}) + (1-\alpha_k)F(x_{k-1}) - F(x_k) \\
&\quad + \frac{\alpha_{k-1}^2 L_k \rho_{k-1}(1-\alpha_k)}{2}\|\bar{x} - v_{k-1}\|^2 - \frac{L_k}{2}\|z_k - x_k\|^2 \\
&= (1-\alpha_k)(F(x_{k-1}) - F(\bar{x})) + F(\bar{x}) - F(x_k) \\
&\quad + \frac{\alpha_{k-1}^2 L_k \rho_{k-1}(1-\alpha_k)}{2}\|\bar{x} - v_{k-1}\|^2 - \frac{L_k}{2}\|z_k - x_k\|^2 \\
&\underset{\text{(e)}}{=} (1-\alpha_k)\left(F(x_{k-1}) - F(\bar{x}) + \frac{L_k\rho_{k-1}}{L_{k-1}}\frac{\alpha_{k-1}^2 L_{k-1}}{2}\|\bar{x} - v_{k-1}\|^2\right) \\
&\quad + F(\bar{x}) - F(x_k) - \frac{L_k\alpha_k^2}{2}\|\bar{x} - v_k\|^2 \\
&\le (1-\alpha_k)\left(F(x_{k-1}) - F(\bar{x}) + \max\left(1, \frac{L_k\rho_{k-1}}{L_{k-1}}\right)\frac{\alpha_{k-1}^2 L_{k-1}}{2}\|\bar{x} - v_{k-1}\|^2\right) \\
&\quad + F(\bar{x}) - F(x_k) - \frac{L_k\alpha_k^2}{2}\|\bar{x} - v_k\|^2 \\
&\le (1-\alpha_k)\left(\max\left(1, \frac{L_k\rho_{k-1}}{L_{k-1}}\right)(F(x_{k-1}) - F(\bar{x})) + \max\left(1, \frac{L_k\rho_{k-1}}{L_{k-1}}\right)\frac{\alpha_{k-1}^2 L_{k-1}}{2}\|\bar{x} - v_{k-1}\|^2\right) \\
&\quad + F(\bar{x}) - F(x_k) - \frac{L_k\alpha_k^2}{2}\|\bar{x} - v_k\|^2 \\
&= \max\left(1, \frac{L_k\rho_{k-1}}{L_{k-1}}\right)(1-\alpha_k)\left(F(x_{k-1}) - F(\bar{x}) + \frac{\alpha_{k-1}^2 L_{k-1}}{2}\|\bar{x} - v_{k-1}\|^2\right) \\
&\quad + F(\bar{x}) - F(x_k) - \frac{L_k\alpha_k^2}{2}\|\bar{x} - v_k\|^2.
\end{aligned}
$$

Finally, for the base case, when $\alpha_0 = 1$, it has $y_0 = v_{-1} = x_{-1}$, and it makes $z_0 = \bar{x}$ therefore this makes the proximal gradient inequality into:

$$
\begin{aligned}
0 &\le F(z_0) - F(x_0) - \frac{L_0}{2}\|z_0 - x_0\|^2 + \frac{L_0 - \mu}{2}\|z_0 - y_0\|^2 \\
&= F(\bar{x}) - F(x_0) - \frac{L_0}{2}\|\bar{x} - x_0\|^2 + \frac{L_0 - \mu}{2}\|\bar{x} - v_{-1}\|^2.
\end{aligned}
$$

Going back to prove the intermediate results, the following will be useful. From Definition 1.9 it has for all $k \ge 0$

{thm:onestep-napg-cnvg-i}
$$\tau_k = L_k(1-\alpha_k)(L_k\alpha_k - \mu)^{-1}. \tag{i}$$

Then it has:

{thm:onestep-napg-cnvg-j}
$$(1+\tau_k)^{-1} \underset{\text{(i)}}{=} \left(1 + \frac{L_k(1-\alpha_k)}{L_k\alpha_k - \mu}\right)^{-1} = \left(\frac{L_k\alpha_k - \mu + L_k(1-\alpha_k)}{L_k\alpha_k - \mu}\right)^{-1} = \frac{L_k\alpha_k - \mu}{L_k - \mu}. \tag{j}$$

And also

{thm:onestep-napg-cnvg-k}
$$\tau_k(1+\tau_k)^{-1} \underset{(i),(j)}{=} \frac{L_k(1-\alpha_k)}{L_k\alpha_k - \mu}\frac{L_k\alpha_k - \mu}{L_k - \mu} = \frac{L_k(1-\alpha_k)}{L_k - \mu}. \tag{k}$$

Proof of (d) For all $k \geq 1$, from Definition 1.9 it has

$$\begin{aligned}
0 &= (1+\tau_k)^{-1}v_{k-1} + \tau_k(1+\tau_k)^{-1}x_{k-1} - y_k \\
&\underset{(k)}{=} (1+\tau_k)^{-1}v_{k-1} + \frac{L_k(1-\alpha_k)}{L_k - \mu}x_{k-1} - y_k \\
&= (1+\tau_k)^{-1}v_{k-1} + (1-\alpha_k)x_{k-1} \\
&\quad + \left(\frac{L_k(1-\alpha_k)}{L_k-\mu} - (1-\alpha_k)\right)x_{k-1} - y_k \\
&= (1+\tau_k)^{-1}v_{k-1} + (1-\alpha_k)x_{k-1} \\
&\quad + (1-\alpha_k)\left(\frac{L_k}{L_k-\mu} - 1\right)x_{k-1} - y_k \\
&= (1+\tau_k)^{-1}v_{k-1} + (1-\alpha_k)x_{k-1} + \frac{\mu(1-\alpha_k)}{L-\mu}x_{k-1} - y_k \\
\iff (1-\alpha_k)x_{k-1} - y_k &= -(1+\tau_k)^{-1}v_{k-1} - \frac{\mu(1-\alpha_k)}{L_k-\mu}x_{k-1}.
\end{aligned}$$

Recall the definition for z_k at the start of the proof and, use the above results it yields:

$$\begin{aligned}
z_k - y_k &= \alpha_k\bar{x} + (1-\alpha_k)x_{k-1} - y_k \\
&= \alpha_k\bar{x} - (1+\tau_k)^{-1}v_{k-1} - \frac{\mu(1-\alpha_k)}{L_k-\mu}x_{k-1} \\
&\underset{(j)}{=} \alpha_k\bar{x} - \frac{L_k\alpha_k-\mu}{L_k-\mu}v_{k-1} - \frac{\mu(1-\alpha_k)}{L_k-\mu}x_{k-1} \\
&= \frac{L_k\alpha_k-\mu}{L_k-\mu}(\bar{x} - v_{k-1}) + \left(\alpha_k - \frac{L_k\alpha_k-\mu}{L_k-\mu}\right)\bar{x} - \frac{\mu(1-\alpha_k)}{L_k-\mu}x_{k-1} \\
&= \frac{L_k\alpha_k-\mu}{L_k-\mu}(\bar{x} - v_{k-1}) + \frac{\alpha_kL_k - \alpha_k\mu - L_k\alpha_k + \mu}{L_k-\mu}\bar{x} - \frac{\mu(1-\alpha_k)}{L_k-\mu}x_{k-1} \\
&= \frac{L_k\alpha_k-\mu}{L_k-\mu}(\bar{x} - v_{k-1}) + \frac{\mu(1-\alpha_k)}{L_k-\mu}(\bar{x} - x_{k-1}).
\end{aligned}$$

Proof of (e). The proof is direct using the equality with x_k in Definition 1.9.

$$\begin{aligned}
z_k - x_k &= \alpha_k\bar{x} + (1-\alpha_k)x_{k-1} - x_k \\
&= \alpha_k\bar{x} + x_{k1} - x_k - \alpha_kx_{k-1} \\
&= \alpha_k(\bar{x} - \alpha_k^{-1}(x_k - x_{k-1}) - x_{k-1}) \\
&= \alpha_k(\bar{x} - v_k).
\end{aligned}$$

Proof of (f). The proof is direct and it has:

$$
\begin{aligned}
\frac{\mu^2(1-\alpha_k)^2}{2(L_k-\mu)} - \frac{\mu\alpha_k(1-\alpha_k)}{2} &= \frac{1}{2(L_k-\mu)}\left(\mu^2(1-\alpha_k)^2 - (L_k-\mu)\mu\alpha_k(1-\alpha_k)\right) \\
&= \frac{1-\alpha_k}{2(L_k-\mu)}\left(\mu^2 - \mu^2\alpha_k - (L_k\mu\alpha_k - \mu^2\alpha_k)\right) \\
&= \frac{1-\alpha_k}{2(L_k-\mu)}\left(\mu^2 - L_k\mu\alpha_k\right) \\
&= \frac{(1-\alpha_k)\mu(\mu - L_k\alpha_k)}{2(L_k-\mu)} \\
&= \frac{(\alpha_k-1)\mu(L_k\alpha_k-\mu)}{2(L_k-\mu)}.
\end{aligned}
$$

Proof of (g) The proof is direct:

$$
\begin{aligned}
\frac{(L_k\alpha_k-\mu)^2}{2(L_k-\mu)} - \frac{\alpha_{k-1}^2 L_k\rho_{k-1}(1-\alpha_k)}{2} &\underset{\text{(c)}}{=} \frac{(L\alpha_k-\mu)^2}{2(L_k-\mu)} - \frac{L_k\alpha_k(\alpha_k-\mu/L_k)}{2} \\
&= \frac{1}{2(L_k-\mu)}\left((L_k\alpha_k-\mu)^2 - (L_k-\mu)L_k\alpha_k(\alpha_k-\mu/L_k)\right) \\
&= \frac{1}{2(L_k-\mu)}\left((L_k\alpha_k-\mu)^2 - (L_k-\mu)\alpha_k(L_k\alpha_k-\mu)\right) \\
&= \frac{L_k\alpha_k-\mu}{2(L_k-\mu)}\left(L_k\alpha_k-\mu-(L-\mu)\alpha_k\right) \\
&= \frac{L_k\alpha_k-\mu}{2(L_k-\mu)}\left(\mu\alpha_k-\mu\right) \\
&= \frac{(L\alpha_k-\mu)\mu(\alpha_k-1)}{2(L_k-\mu)}.
\end{aligned}
$$

Proof of (h). For all $k \geq 1$, by (c), the definition of the R-WAPG sequence, $\alpha_k \in (\mu/L_k, 1)$, then it has $L_k\alpha_k \in (\mu, L_k)$, so $L_k\alpha_k - \mu > 0$, and $\alpha_k - 1 < 0$. Finally, we have $L_k \geq \mu$, therefore, the fraction is negative. ■

1.3 stochastic accelerated proximal gradient

The following assumption about the objective function is fundamental in incremental gradient method for Machine Learning, data science other similar tasks.

{ass:sum-of-many}

Assumption 1.15 (sum of many) Define $F := g + (1/n)\sum_{i=1}^{n} f_i$, assume that $f_i : \mathbb{R}^n \to \mathbb{R}$ are all $K^{(i)}$ smooth and $\mu^{(i)} \geq 0$ strongly convex function such that $K^{(i)} > \mu^{(i)}$ and, g :

$\mathbb{R}^n \to \overline{\mathbb{R}}$ is a closed convex proper function. Consequently, the function f cane be written as $F = g + f$ where $f = (1/n)\sum_{i=1}^{n} f_i$ and, it satisfies Assumption 1.1 with $L = \max_{i=1,\dots,n} K^{(i)}$ and $\mu = (1/n)\sum_{i=1}^{n} \mu^{(i)}$.

This assumption is stronger than Assumption 1.1. The interpolation hypothesis from Machine Learning stated that the model has the capacity to perfect fit all the observed data.

{ass:interp-hypothesis}

Assumption 1.16 (interpolation hypothesis) Suppose that $F := f + (1/n)\sum_{i=1}^{n} f_i$ satisfying Assumption 1.15. In addition, assuming that it has $0 = \inf_x F(x)$ and, there exists some $\bar{x} \in \mathbb{R}^n$ such that for all $i = 1, \dots, n$ it satisfies $0 = f_i(\bar{x})$. Obviously, all such $\bar{x}$ forms the set of minimizers of F.

{def:snapg-v2}

Definition 1.17 (SNAPG-V2) *Let F satisfies Assumption 1.15. Let $(I_k)_{k\ge 0}$ be a list of i.i.d random variables uniformly sampled from set $\{0, 1, 2, \cdots, n\}$. Initialize $v_{-1} = x_{-1}, \alpha_0 = 1$. The SNAPG generates the sequence $(y_k, x_k, v_k)_{k\ge 0}$ such that for all $k \ge 0$ they satisfy:*

$$
\begin{aligned}
&(L_{k-1}/L_k)(1-\alpha_k)\alpha_{k-1}^2 = \alpha_k\left(\alpha_k - \mu/L_k\right),\\
&\tau_k = L_k(1-\alpha_k)\left(L_k\alpha_k - \mu^{(I_k)}\right)^{-1},\\
&y_k = (1+\tau_k)^{-1}v_{k-1} + \tau_k(1+\tau_k)^{-1}x_{k-1},\\
&x_k = T_{L_k}(y_k|F_{I_k}) \text{ s.t: } D_f(x_k, y_k) \le L_k/2\|y_k - x_k\|^2,\\
&v_k = x_{k-1} + \alpha_k^{-1}(x_k - x_{k-1}).
\end{aligned}
$$

{lemma:snapg-v2-seq-range}

Lemma 1.18 (range of the momentum sequence in SNAPG-V2)

{thm:snapg2-one-step}

Theorem 1.19 (SNAPG-V2 one step convergence) *Let F satisfies assumption 1.16. Suppose that an algorithm satisfying Definition 1.17 takes this F, for all $k \ge 1$, it has the following inequality*

$$
\begin{aligned}
&\mathbb{E}_k\left[F_{I_k}(x_k)\right] - F(\bar{x}) + \mathbb{E}_k\left[\frac{L_k\alpha_k^2}{2}\|\bar{x} - v_k\|^2\right]\\
&\le (1-\alpha_k)\left(\mathbb{E}_k\left[F_{I_k}(x_{k-1})\right] - F(\bar{x}) + \mathbb{E}_k\left[\frac{\alpha_{k-1}^2 L_{k-1}}{2}\right]\|v_{k-1} - \bar{x}\|^2\right)\\
&\quad + \mathbb{E}_k\left[\frac{(\alpha_k - 1)\mu^{(I_k)}\left(L_k\alpha_k - \mu^{(I_k)}\right)}{2\left(L_k - \mu^{(I_k)}\right)}\right]\|x_{k-1} - v_{k-1}\|^2.
\end{aligned}
$$

Proof. Let's suppose that $I_k = i$ and, for all $k \ge 0$ let $z_k = \alpha_k\bar{x} + (1-\alpha_k)x_{k-1}$ where $\bar{x}$ is a minimizer of F. With following intermediate results the proof can be built easily. Results (d)-(g) is showed at the end.

(a) We can use proximal gradient inequality from Theorem 1.10 with $z = z_k$ because each F_i is K_i Lipschitz smooth and, $\mu^{(i)}$ strongly convex with $K_i \ge \mu^{(i)}$.

(b) We can use Jensen's inequality of Theorem 1.12 with $z = z_k$ on F_i.

(c) The sequence $(\alpha_k)_{k\geq 0}$ has $(L_{k-1}/L_k)(1-\alpha_k)\alpha_{k-1}^2 = \alpha_k\,(\alpha_k - \mu/L_k)$. It is a special case of Definition 1.7 with $\rho_{k-1} = L_{k-1}/L_k$.

(d) From Definition 1.17 it has the following equality

$$(\forall k \geq 1)\; z_k - y_k = \frac{L_k\alpha_k - \mu^{(i)}}{L_k - \mu^{(i)}}(\bar{x} - v_{k-1}) + \frac{\mu^{(i)}(1-\alpha_k)}{L_k - \mu^{(i)}}(\bar{x} - x_{k-1}).$$

(e) From Definition 1.17 it has: $(\forall k \geq 1)\; z_k - x_k = \alpha_k(\bar{x} - v_k)$.

(f) Using direct algebra, we have for all $k \geq 1$:

$$\frac{\left(\mu^{(i)}\right)^2 (1-\alpha_k)^2}{2(L_k - \mu^{(i)})} - \frac{\mu^{(i)}\alpha_k(1-\alpha_k)}{2} = \frac{(\alpha_k - 1)\mu^{(i)}\left(L_k\alpha_k - \mu^{(i)}\right)}{2\left(L_k - \mu^{(i)}\right)}.$$

(g) Using (c), we have for all $k \geq 1$:

$$\frac{\left(L_k\alpha_k - \mu^{(i)}\right)^2}{2(L_k - \mu^{(i)})} - \frac{\alpha_{k-1}^2 L_{k-1}(1-\alpha_k)}{2} = \frac{\left(L_k\alpha_k - \mu^{(i)}\right)\mu^{(i)}\left(\alpha_k - 1\right)}{2(L_k - \mu^{(i)})} + \frac{\alpha_k(\mu - \mu^{(i)})}{2}.$$

(h) Because we assumed interpolation hypothesis in Assumption 1.16, it has $\mathbb{E}[F_{I_k}(\bar{x})] = F(\bar{x})$ for all $\bar{x}$ that is a minimizer of F.

For all $k \geq 1$, starting with (a) we have:

{ineq:snapg2-one-step-chain1}

$$\begin{aligned}
0 &\leq F_i(z_k) - F_i(x_k) - \frac{L_k}{2}\|z_k - x_k\|^2 + \frac{L_k - \mu^{(i)}}{2}\|z_k - y_k\|^2 \\
&\underset{\text{(b)}}{\leq} \alpha_k F_i(\bar{x}) + (1-\alpha_k)F_i(x_{k-1}) - F_i(x_k) \\
&\quad - \frac{\mu^{(i)}\alpha_k(1-\alpha_k)}{2}\|\bar{x} - x_{k-1}\|^2 - \frac{L_k}{2}\|z_k - x_k\|^2 + \frac{L_k - \mu^{(i)}}{2}\|z_k - y_k\|^2.
\end{aligned} \tag{1.1}$$

And we have the following chain of equalities:

$$\begin{aligned}
&- \frac{\mu^{(i)}\alpha_k(1-\alpha_k)}{2}\|\bar{x} - x_{k-1}\|^2 + \frac{L_k - \mu^{(i)}}{2}\|z_k - y_k\|^2 \\
&\underset{\text{(d)}}{=} -\frac{\mu^{(i)}\alpha_k(1-\alpha_k)}{2}\|\bar{x} - x_{k-1}\|^2 \\
&\quad + \frac{L_k - \mu^{(i)}}{2}\left\|\frac{L_k\alpha_k - \mu^{(i)}}{L_k - \mu^{(i)}}(\bar{x} - v_{k-1}) + \frac{\mu^{(i)}(1-\alpha_k)}{L_k - \mu^{(i)}}(\bar{x} - x_{k-1})\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\mu^{(i)}\alpha_k(1-\alpha_k)}{2}\|\bar{x}-x_{k-1}\|^2 \\
&\quad + \frac{(L_k\alpha_k-\mu^{(i)})^2}{2(L_k-\mu^{(i)})}\|\bar{x}-v_{k-1}\|^2 + \frac{(\mu^{(i)})^2(1-\alpha_k)^2}{2(L_k-\mu^{(i)})}\|\bar{x}-x_{k-1}\|^2 \\
&\quad + \frac{(L_k\alpha_k-\mu^{(i)})\mu^{(i)}(1-\alpha_k)}{(L_k-\mu^{(i)})}\langle \bar{x}-v_{k-1}, \bar{x}-x_{k-1}\rangle \\
&= \left(\frac{(\mu^{(i)})^2(1-\alpha_k)^2}{2(L_k-\mu^{(i)})} - \frac{\mu^{(i)}\alpha_k(1-\alpha_k)}{2}\right)\|\bar{x}-x_{k-1}\|^2 \\
&\quad + \left(\frac{(L_k\alpha_k-\mu^{(i)})^2}{2(L_k-\mu^{(i)})} - \frac{\alpha_{k-1}^2L_{k-1}(1-\alpha_k)}{2}\right)\|\bar{x}-v_{k-1}\|^2 + \frac{\alpha_{k-1}^2L_{k-1}(1-\alpha_k)}{2}\|\bar{x}-v_{k-1}\|^2 \\
&\quad + \frac{(L_k\alpha_k-\mu^{(i)})\mu^{(i)}(1-\alpha_k)}{(L_k-\mu^{(i)})}\langle \bar{x}-v_{k-1}, \bar{x}-x_{k-1}\rangle \\
&\underset{\text{(f)}}{=} \frac{(\alpha_k-1)\mu^{(i)}\left(L_k\alpha_k-\mu^{(i)}\right)}{2\left(L_k-\mu^{(i)}\right)}\|\bar{x}-x_{k-1}\|^2 \\
&\quad + \left(\frac{(L_k\alpha_k-\mu^{(i)})^2}{2(L_k-\mu^{(i)})} - \frac{\alpha_{k-1}^2L_{k-1}(1-\alpha_k)}{2}\right)\|\bar{x}-v_{k-1}\|^2 + \frac{\alpha_{k-1}^2L_{k-1}(1-\alpha_k)}{2}\|\bar{x}-v_{k-1}\|^2 \\
&\quad + \frac{(L_k\alpha_k-\mu^{(i)})\mu^{(i)}(1-\alpha_k)}{(L_k-\mu^{(i)})}\langle \bar{x}-v_{k-1}, \bar{x}-x_{k-1}\rangle \\
&\underset{\text{(g)}}{=} \frac{(\alpha_k-1)\mu^{(i)}\left(L_k\alpha_k-\mu^{(i)}\right)}{2\left(L_k-\mu^{(i)}\right)}\|\bar{x}-x_{k-1}\|^2 \\
&\quad + \left(\frac{\left(L_k\alpha_k-\mu^{(i)}\right)\mu^{(i)}\left(\alpha_k-1\right)}{2(L_k-\mu^{(i)})} + \frac{\alpha_k(\mu-\mu^{(i)})}{2}\right)\|\bar{x}-v_{k-1}\|^2 \\
&\quad + \frac{\alpha_{k-1}^2L_{k-1}(1-\alpha_k)}{2}\|\bar{x}-v_{k-1}\|^2 + \frac{(L_k\alpha_k-\mu^{(i)})\mu^{(i)}(1-\alpha_k)}{(L_k-\mu^{(i)})}\langle \bar{x}-v_{k-1}, \bar{x}-x_{k-1}\rangle \\
&= \frac{(\alpha_k-1)\mu^{(i)}\left(L_k\alpha_k-\mu^{(i)}\right)}{2\left(L_k-\mu^{(i)}\right)}\left(\|\bar{x}-x_{k-1}\|^2 + \|\bar{x}-v_{k-1}\|^2 - 2\langle \bar{x}-v_{k-1}, \bar{x}-x_{k-1}\rangle\right) \\
&\quad + \frac{\alpha_k(\mu-\mu^{(i)})}{2}\|\bar{x}-v_{k-1}\|^2 + \frac{\alpha_{k-1}^2L_{k-1}(1-\alpha_k)}{2}\|\bar{x}-v_{k-1}\|^2 \\
&= \frac{(\alpha_k-1)\mu^{(i)}\left(L_k\alpha_k-\mu^{(i)}\right)}{2\left(L_k-\mu^{(i)}\right)}\|x_{k-1}-v_{k-1}\|^2 \\
&\quad + \frac{\alpha_k(\mu-\mu^{(i)})}{2}\|\bar{x}-v_{k-1}\|^2 + \frac{\alpha_{k-1}^2L_{k-1}(1-\alpha_k)}{2}\|\bar{x}-v_{k-1}\|^2.
\end{aligned}
$$

Substituting the above back to the tail of Inequality (1.1) it gives:

$$
0 \le \alpha_kF_i(\bar{x}) + (1-\alpha_k)F_i(x_{k-1}) - F_i(x_k)
$$

$$
\begin{aligned}
&- \frac{L_k}{2}\|z_k - x_k\|^2 + \frac{(\alpha_k - 1)\mu^{(i)}\left(L_k\alpha_k - \mu^{(i)}\right)}{2\left(L_k - \mu^{(i)}\right)}\|x_{k-1} - v_{k-1}\|^2 \\
&+ \frac{\alpha_k(\mu - \mu^{(i)})}{2}\|\bar{x} - v_{k-1}\|^2 + \frac{\alpha_{k-1}^2 L_{k-1}(1-\alpha_k)}{2}\|\bar{x} - v_{k-1}\|^2 \\
\underset{\text{(e)}}{=}\ & \alpha_k F_i(\bar{x}) + (1-\alpha_k)F_i(x_{k-1}) - F_i(x_k) \\
&- \frac{L_k\alpha_k^2}{2}\|\bar{x} - v_k\|^2 + \frac{(\alpha_k - 1)\mu^{(i)}\left(L_k\alpha_k - \mu^{(i)}\right)}{2\left(L_k - \mu^{(i)}\right)}\|x_{k-1} - v_{k-1}\|^2 \\
&+ \frac{\alpha_k(\mu - \mu^{(i)})}{2}\|\bar{x} - v_{k-1}\|^2 + \frac{\alpha_{k-1}^2 L_{k-1}(1-\alpha_k)}{2}\|\bar{x} - v_{k-1}\|^2 \\
=\ & (\alpha_k - 1)F_i(\bar{x}) + (1-\alpha_k)F_i(x_{k-1}) - F_i(x_k) + F_i(\bar{x}) \\
&- \frac{L_k\alpha_k^2}{2}\|\bar{x} - v_k\|^2 + \frac{(\alpha_k - 1)\mu^{(i)}\left(L_k\alpha_k - \mu^{(i)}\right)}{2\left(L_k - \mu^{(i)}\right)}\|x_{k-1} - v_{k-1}\|^2 \\
&+ \frac{\alpha_k(\mu - \mu^{(i)})}{2}\|\bar{x} - v_{k-1}\|^2 + \frac{\alpha_{k-1}^2 L_{k-1}(1-\alpha_k)}{2}\|\bar{x} - v_{k-1}\|^2 \\
=\ & (1-\alpha_k)\left(F_i(x_{k-1}) - F_i(\bar{x}) + \frac{\alpha_{k-1}^2 L_{k-1}}{2}\|v_{k-1} - \bar{x}\|^2\right) \\
&- \left(F_i(x_k) - F_i(\bar{x}) + \frac{L_k\alpha_k^2}{2}\|\bar{x} - v_k\|^2\right) \\
&+ \frac{\alpha_k(\mu - \mu^{(i)})}{2}\|\bar{x} - v_{k-1}\|^2 + \frac{(\alpha_k - 1)\mu^{(i)}\left(L_k\alpha_k - \mu^{(i)}\right)}{2\left(L_k - \mu^{(i)}\right)}\|x_{k-1} - v_{k-1}\|^2.
\end{aligned}
$$

Recall that $i = I_k$ is the random variable from Definition 1.17. Rearranging the last expression in the above equality chain can be conveniently written as

{ineq:snapg2-one-step-presult1}

$$
\begin{aligned}
&\left(F_{I_k}(x_k) - F_{I_k}(\bar{x}) + \frac{L_k\alpha_k^2}{2}\|\bar{x} - v_k\|^2\right) \\
&\le (1-\alpha_k)\left(F_{I_k}(x_{k-1}) - F_{I_k}(\bar{x}) + \frac{\alpha_{k-1}^2 L_{k-1}}{2}\|v_{k-1} - \bar{x}\|^2\right) \\
&\quad + \frac{\alpha_k(\mu - \mu^{(I_k)})}{2}\|\bar{x} - v_{k-1}\|^2 + \frac{(\alpha_k - 1)\mu^{(I_k)}\left(L_k\alpha_k - \mu^{(I_k)}\right)}{2\left(L_k - \mu^{(I_k)}\right)}\|x_{k-1} - v_{k-1}\|^2.
\end{aligned}
\tag{1.2}
$$

Recall $\mathbb{E}_k$ denotes the conditional expectation on $I_0, I_1, \ldots, I_{k-1}$. Taking the conditional expectation on the LHS of the (1.2) yields:

$$
\begin{aligned}
&\mathbb{E}_k\left[F_{I_k}(x_k) - F_{I_k}(\bar{x}) + \frac{L_k\alpha_k^2}{2}\|\bar{x} - v_k\|^2\right] \\
&\underset{\text{(h)}}{=} \mathbb{E}_k\left[F_{I_k}(x_k)\right] - F(\bar{x}) + \mathbb{E}_k\left[\frac{L_k\alpha_k^2}{2}\|\bar{x} - v_k\|^2\right].
\end{aligned}
$$

On the RHS of (1.2), using the linearity property while taking the conditional expectation yields:

$$
\begin{aligned}
&\mathbb{E}_k\left[(1-\alpha_k)\left(F_{I_k}(x_{k-1}) - F_{I_k}(\bar{x}) + \frac{\alpha_{k-1}^2 L_{k-1}}{2}\|v_{k-1}-\bar{x}\|^2\right)\right] \\
&\quad + \mathbb{E}_k\left[\frac{\alpha_k(\mu - \mu^{(I_k)})}{2}\|\bar{x}-v_{k-1}\|^2\right] + \mathbb{E}_k\left[\frac{(\alpha_k-1)\mu^{(I_k)}\left(L_k\alpha_k - \mu^{(I_k)}\right)}{2\left(L_k - \mu^{(I_k)}\right)}\|x_{k-1}-v_{k-1}\|^2\right] \\
&\underset{(1)}{=} (1-\alpha_k)\left(\mathbb{E}_k\left[F_{I_k}(x_{k-1})\right] - \mathbb{E}_k[F_{I_k}(\bar{x})] + \mathbb{E}_k\left[\frac{\alpha_{k-1}^2 L_{k-1}}{2}\right]\|v_{k-1}-\bar{x}\|^2\right) \\
&\quad + \mathbb{E}_k\left[\frac{\alpha_k(\mu - \mu^{(I_k)})}{2}\right]\|\bar{x}-v_{k-1}\|^2 + \mathbb{E}_k\left[\frac{(\alpha_k-1)\mu^{(I_k)}\left(L_k\alpha_k - \mu^{(I_k)}\right)}{2\left(L_k - \mu^{(I_k)}\right)}\right]\|x_{k-1}-v_{k-1}\|^2. \\
&\underset{\text{(h)}}{=} (1-\alpha_k)\left(\mathbb{E}_k\left[F_{I_k}(x_{k-1})\right] - F(\bar{x}) + \mathbb{E}_k\left[\frac{\alpha_{k-1}^2 L_{k-1}}{2}\right]\|v_{k-1}-\bar{x}\|^2\right) \\
&\quad + \mathbb{E}_k\left[\frac{\alpha_k(\mu - \mu^{(I_k)})}{2}\right]\|\bar{x}-v_{k-1}\|^2 + \mathbb{E}_k\left[\frac{(\alpha_k-1)\mu^{(I_k)}\left(L_k\alpha_k - \mu^{(I_k)}\right)}{2\left(L_k - \mu^{(I_k)}\right)}\right]\|x_{k-1}-v_{k-1}\|^2 \\
&\underset{(2)}{=} (1-\alpha_k)\left(\mathbb{E}_k\left[F_{I_k}(x_{k-1})\right] - F(\bar{x}) + \mathbb{E}_k\left[\frac{\alpha_{k-1}^2 L_{k-1}}{2}\right]\|v_{k-1}-\bar{x}\|^2\right) \\
&\quad + \mathbb{E}_k\left[\frac{(\alpha_k-1)\mu^{(I_k)}\left(L_k\alpha_k - \mu^{(I_k)}\right)}{2\left(L_k - \mu^{(I_k)}\right)}\right]\|x_{k-1}-v_{k-1}\|^2
\end{aligned}
$$

We note that at label (1), we used the fact that α_k is a constant and, x_{k-1}, v_{k-1} only depends on random variable $I_0, I_1, \ldots, I_{k-1}$ hence it falls out of the conditional expectation $\mathbb{E}_k$. At label (2), we used assumption (Assumption 1.15) that the averages of all the $\mu^{(I_k)}$ on each F_{I_k} equals to μ hence, the expectation evaluates to zero by linearity of the expected value operator.

Combing the above results on the expectation of RHS, and LHS of (1.2), we have the one step inequaity in expectation:

$$
\begin{aligned}
&\mathbb{E}_k\left[F_{I_k}(x_k)\right] - F(\bar{x}) + \mathbb{E}_k\left[\frac{L_k\alpha_k^2}{2}\|\bar{x}-v_k\|^2\right] \\
&\le (1-\alpha_k)\left(\mathbb{E}_k\left[F_{I_k}(x_{k-1})\right] - F(\bar{x}) + \mathbb{E}_k\left[\frac{\alpha_{k-1}^2 L_{k-1}}{2}\right]\|v_{k-1}-\bar{x}\|^2\right) \\
&\quad + \mathbb{E}_k\left[\frac{(\alpha_k-1)\mu^{(I_k)}\left(L_k\alpha_k - \mu^{(I_k)}\right)}{2\left(L_k - \mu^{(I_k)}\right)}\right]\|x_{k-1}-v_{k-1}\|^2.
\end{aligned}
$$

Proof of (d). From Definition 1.17, it has

$$(1+\tau_k)^{-1} = \left(1 + \frac{L_k(1-\alpha_k)}{L_k\alpha_k - \mu^{(i)}}\right)^{-1} = \left(\frac{L_k\alpha_k - \mu^{(i)} + L_k(1-\alpha_k)}{L_k\alpha_k - \mu^{(i)}}\right)^{-1} = \frac{L_k\alpha_k - \mu^{(i)}}{L_k - \mu^{(i)}}.$$

Therefore, for all $k \geq 0$ y_k has

$$\begin{aligned}
0 &= (1+\tau_k)^{-1}v_{k-1} + \tau_k(1+\tau_k)^{-1}x_{k-1} - y_k \\
&= \frac{L_k\alpha_k - \mu^{(i)}}{L_k - \mu^{(i)}}\left(v_{k-1} + \frac{L_k(1-\alpha_k)}{L_k\alpha_k - \mu^{(i)}}x_{k-1}\right) - y_k \\
&= \frac{L_k\alpha_k - \mu^{(i)}}{L_k - \mu^{(i)}}v_{k-1} + \frac{L_k(1-\alpha_k)}{L_k - \mu^{(i)}}x_{k-1} - y_k \\
&= \frac{L_k\alpha_k - \mu^{(i)}}{L_k - \mu^{(i)}}v_{k-1} + (1-\alpha_k)x_{k-1} + \left(\frac{L_k(1-\alpha_k)}{L_k - \mu^{(i)}} - (1-\alpha_k)\right)x_{k-1} - y_k \\
&= \frac{L_k\alpha_k - \mu^{(i)}}{L_k - \mu^{(i)}}v_{k-1} + (1-\alpha_k)x_{k-1} + (1-\alpha_k)\left(\frac{L_k - L_k + \mu^{(i)}}{L_k - \mu^{(i)}}\right)x_{k-1} - y_k \\
&= \frac{L_k\alpha_k - \mu^{(i)}}{L_k - \mu^{(i)}}v_{k-1} + (1-\alpha_k)x_{k-1} + \frac{\mu^{(i)}(1-\alpha_k)}{L_k - \mu^{(i)}}x_{k-1} - y_k.
\end{aligned}$$

Therefore, we establish the equality

$$(1-\alpha_k)x_{k-1} - y_k = -\frac{L_k\alpha_k - \mu^{(i)}}{L_k - \mu^{(i)}}v_{k-1} - \frac{\mu^{(i)}(1-\alpha_k)}{L_k - \mu^{(i)}}x_{k-1}.$$

Using the above, we have

$$\begin{aligned}
z_k - y_k &= \alpha_k\bar{x} + (1-\alpha_k)x_{k-1} - y_k \\
&= \alpha_k\bar{x} - \frac{L_k\alpha_k - \mu^{(i)}}{L_k - \mu^{(i)}}v_{k-1} - \frac{\mu^{(i)}(1-\alpha_k)}{L_k - \mu^{(i)}}x_{k-1} \\
&= \frac{L_k\alpha_k - \mu^{(i)}}{L_k - \mu^{(i)}}(\bar{x} - v_{k-1}) + \left(\alpha_k - \frac{L_k\alpha_k - \mu^{(i)}}{L_k - \mu^{(i)}}\right)\bar{x} - \frac{\mu^{(i)}(1-\alpha_k)}{L_k - \mu^{(i)}}x_{k-1} \\
&= \frac{L_k\alpha_k - \mu^{(i)}}{L_k - \mu^{(i)}}(\bar{x} - v_{k-1}) + \left(\frac{\alpha_k L_k - \alpha_k\mu^{(i)} - L_k\alpha_k + \mu^{(i)}}{L_k - \mu^{(i)}}\right)\bar{x} - \frac{\mu^{(i)}(1-\alpha_k)}{L_k - \mu^{(i)}}x_{k-1} \\
&= \frac{L_k\alpha_k - \mu^{(i)}}{L_k - \mu^{(i)}}(\bar{x} - v_{k-1}) + \frac{\mu^{(i)}(1-\alpha_k)}{L_k - \mu^{(i)}}\bar{x} - \frac{\mu^{(i)}(1-\alpha_k)}{L_k - \mu^{(i)}}x_{k-1} \\
&= \frac{L_k\alpha_k - \mu^{(i)}}{L_k - \mu^{(i)}}(\bar{x} - v_{k-1}) + \frac{\mu^{(i)}(1-\alpha_k)}{L_k - \mu^{(i)}}(\bar{x} - x_{k-1}).
\end{aligned}$$

Proof of (e). From Definition 1.17 it has directly:

$$\begin{aligned} z_k - x_k &= \alpha_k \bar{x} + (1-\alpha_k)x_{k-1} - x_k \\ &= \alpha_k \bar{x} + x_{k1} - x_k - \alpha_k x_{k-1} \\ &= \alpha_k(\bar{x} - \alpha_k^{-1}(x_k - x_{k-1}) - x_{k-1}) \\ &= \alpha_k(\bar{x} - v_k). \end{aligned}$$

Proof of (f). The proof is direct algebra and, it has:

$$\begin{aligned} &\frac{\left(\mu^{(i)}\right)^2(1-\alpha_k)^2}{2(L_k-\mu^{(i)})} - \frac{\mu^{(i)}\alpha_k(1-\alpha_k)}{2} \\ &= \frac{1}{2\left(L_k-\mu^{(i)}\right)}\left(\left(\mu^{(i)}\right)^2(1-\alpha_k)^2 - \left(L_k-\mu^{(i)}\right)\mu^{(i)}\alpha_k(1-\alpha_k)\right) \\ &= \frac{1-\alpha_k}{2\left(L_k-\mu^{(i)}\right)}\left(\left(\mu^{(i)}\right)^2 - \left(\mu^{(i)}\right)^2\alpha_k - \left(L_k\mu^{(i)}\alpha_k - \left(\mu^{(i)}\right)^2\alpha_k\right)\right) \\ &= \frac{1-\alpha_k}{2(L_k-\mu)}\left(\left(\mu^{(i)}\right)^2 - L_k\left(\mu^{(i)}\right)\alpha_k\right) \\ &= \frac{(1-\alpha_k)\mu^{(i)}\left(\mu^{(i)} - L_k\alpha_k\right)}{2\left(L_k-\mu^{(i)}\right)} \\ &= \frac{(\alpha_k-1)\mu^{(i)}\left(L_k\alpha_k - \mu^{(i)}\right)}{2\left(L_k-\mu^{(i)}\right)}. \end{aligned}$$

Proof of (g). From the property of the sequence stated in item (c), we have:

$$\begin{aligned} &\frac{\left(L_k\alpha_k-\mu^{(i)}\right)^2}{2(L_k-\mu^{(i)})} - \frac{\alpha_{k-1}^2 L_{k-1}(1-\alpha_k)}{2} \\ &= \frac{\left(L_k\alpha_k-\mu^{(i)}\right)^2}{2(L_k-\mu^{(i)})} - \frac{L_k\alpha_k(\alpha_k-\mu/L_k)}{2} \\ &= \frac{\left(L_k\alpha_k-\mu^{(i)}\right)^2}{2(L_k-\mu^{(i)})} - \frac{L_k\alpha_k(\alpha_k-\mu^{(i)}/L_k)}{2} + \frac{L_k\alpha_k(\alpha_k-\mu^{(i)}/L_k)}{2} - \frac{L_k\alpha_k(\alpha_k-\mu/L_k)}{2} \\ &= \frac{\left(L_k\alpha_k-\mu^{(i)}\right)^2}{2(L_k-\mu^{(i)})} - \frac{\alpha_k\left(L_k\alpha_k-\mu^{(i)}\right)}{2} + \frac{L_k\alpha_k}{2}\frac{\left(\mu-\mu^{(i)}\right)}{L_k} \\ &= \frac{L_k\alpha_k-\mu^{(i)}}{2(L_k-\mu^{(i)})}\left(L_k\alpha_k-\mu^{(i)} - \left(L_k-\mu^{(i)}\right)\alpha_k\right) + \frac{\alpha_k(\mu-\mu^{(i)})}{2} \\ &= \frac{L_k\alpha_k-\mu^{(i)}}{2(L_k-\mu^{(i)})}\left(\mu^{(i)}\alpha_k-\mu^{(i)}\right) + \frac{\alpha_k(\mu-\mu^{(i)})}{2} \\ &= \frac{\left(L_k\alpha_k-\mu^{(i)}\right)\mu^{(i)}(\alpha_k-1)}{2(L_k-\mu^{(i)})} + \frac{\alpha_k(\mu-\mu^{(i)})}{2}. \end{aligned}$$

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