Chapter 6

Fourier Series

6.1 Introdution

We discussed the Fourier sine series in Chapter 4 and the Fourier cosine series in Chapter 5. Now, we shall combine the two to form a *periodic* Fourier series (or simply called the Fourier series). Before we do so however, we first try to motivate the need for such a series by looking for the eigenfunctions satisfying periodic boundary conditions.

6.2 Periodic Eigenfunctions

Consider heat conduction in a circular ring. Let us denote the circumference of the ring by $2L$. Denote any point on the ring by $x = 0$. Then the points $x = -L$ and $x = L$ are actually the same point. The problem is to be solved in the domain $-L < x < L$, subject to the boundary condition that the solution should be the same at $x = -L$ and $x = L$.

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx}, \quad -L < x < L, \quad t > 0 \tag{6.1}$$

$$\text{BCs:} \quad u(-L,t) = u(L,t) \tag{6.2}$$

$$u_x(-L,t) = u_x(L,t) \tag{6.3}$$

$$\text{IC:} \quad u(x,0) = f(x), \quad -L < x < L. \tag{6.4}$$

As we will show, that these boundary conditions are sufficient to define a periodic function. That is, we can look for a solution for all x, $-\infty < x < \infty$, with the condition that it repeats itself with period $2L$, i.e.

$$\boxed{u(x,t) = u(x+2L,t)}\,. \tag{6.5}$$

We will get the same result as (6.2) and (6.3).

We shall again use the method of separation of variables, and we first try

$$u(x,t) = T(t)X(x).$$

Substituting into the PDE yields

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2,$$

where $-\lambda^2$ is the separation constant. The eigenfunction $X(x) = X_n(x)$ is determined from

$$X''(x) + \lambda^2 X(x) = 0 \tag{6.6}$$
$$X(-L) = X(L), \quad X'(-L) = X'(L). \tag{6.7}$$

The boundary conditions for X comes from (6.2) and (6.3). The solution to (6.6) is

$$X(x) = a\sin\lambda x + b\cos\lambda x, \tag{6.8}$$

so the first boundary condition in (6.7) implies

$$-a\sin\lambda L + b\cos\lambda L = a\sin\lambda L + b\cos\lambda L,$$

i.e.

$$2a\sin\lambda L = 0. \tag{6.9}$$

The second boundary condition

$$\lambda a\cos\lambda L + \lambda b\sin\lambda L = \lambda a\cos\lambda L - \lambda b\sin\lambda L,$$

is

$$2b\lambda\sin\lambda L = 0. \tag{6.10}$$

Both (6.9) and (6.10) can be satisfied if $\sin\lambda L = 0$, or by taking

$$\lambda = n\pi/L \equiv \lambda_n, \quad n = 0, 1, 2, 3, \ldots. \tag{6.11}$$

So the eigenfunction corresponding to λ_n is

$$X(x) = a_n\sin\lambda_n x + b_n\cos\lambda_n x \equiv X_n(x), \quad n = 1, 2, 3, \ldots. \tag{6.12}$$

Superposition over all n then yields the general solution of the PDE in the form

$$u(x,t) = \sum_n T_n(t)X_n(x).$$

Thus:

$$\boxed{u(x,t) = \sum_{n=0}^{\infty}[a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}]e^{-(n\pi\alpha/L)^2 t}}\,. \tag{6.13}$$

To satisfy the IC, we need

$$\boxed{f(x) = \sum_{n=0}^{\infty}[a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}], \quad -L < x < L}\,. \tag{6.14}$$

The right-hand side of (6.14) is the Fourier series expansion of $f(x)$ in the domain $-L < x < L$. It is periodic with period $2L$ in $-\infty < x < \infty$.

6.3 Fourier Series

We now return to the mathematical problem of representing an arbitrary, piecewise continuous, function $f(x)$ in a Fourier series of period $2L$,

$$f(x) = \sum_{n=0}^{\infty}[a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}], \quad -L < x < L. \tag{6.15}$$

If $f(x)$ is itself periodic with period $2L$, then the representation is good for all x, $-\infty < x < \infty$. If $f(x)$ is not periodic outside the interval $-L < x < L$, or if $f(x)$ is not defined beyond this interval, the representation is good only in the restricted interval.

There are two ways to find the coefficients a_n and b_n of the Fourier series. The standard way is to use the orthogonality conditions between sines and cosines. They are, for integers m and n:

$$\int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L} dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases} \tag{6.16}$$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L} dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \neq 0 \\ 2L & \text{if } m = n = 0 \end{cases} \tag{6.17}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L} \cos\frac{n\pi x}{L} dx = 0 \text{ for all } m \text{ and } n \tag{6.18}$$

Of the three orthogonality relations (6.16), (6.17) and (6.18), only the last one is really new. We derived the first two previously when the integration was over half the domain, from 0 to L. Since sines are odd functions of x and cosines are even function of x, the integrands in (6.16) and (6.17) are even in x. Thus,

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = 2\int_{0}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx$$
$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = 2\int_{0}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx.$$

(6.16) then follows from our previous results, namely (6.7), and (6.17) follows from (5.3). The last identity, (6.18) follows from the fact that the integrand in (6.18) is odd in x and so the integral over positive and negative values of x yields zero.

Using these orthogonality relations, we can now obtain the coefficients a_n and b_n in the following way. Multiply both sides of (6.15) by $\sin\frac{m\pi x}{L}$ and integrate with respect to x from $-L$ to L to get

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx &= \sum_{n=0}^{\infty}[a_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx \\ &\quad + b_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx] \\ &= a_m L.\end{aligned}$$

so

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx, \quad m = 0, 1, 2, 3, \ldots.$$

Multiplying (6.15) by $\cos\frac{m\pi x}{L}$ and integrating with respect to x from $-L$ to L yields, similarly,

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad m \neq 0$$
$$b_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx.$$

Summary: The Fourier series representation of a piecewise continuous function $f(x)$ in the interval $-L < x < L$ is given by

$$\boxed{f(x) = \sum_{n=0}^{\infty}[a_n\sin\frac{n\pi x}{L} + b_n\cos\frac{n\pi x}{L}], \quad -L < x < L}, \tag{6.19}$$

where

$$\boxed{\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx \\ b_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots . \end{aligned}}$$

Notes: (i) If $f(x)$ is an odd function of x, i.e.

$$f(-x) = -f(x), \quad -L < x < L,$$

then all the cosine coefficients b_n are zero, and the Fourier series (6.19) becomes a sine series:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}, \quad -L < x < L,$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

(ii) If $f(x)$ is an even function of x, i.e.

$$f(-x) = f(x),$$

then all the sine coefficients a_n are zero. The Fourier series (6.19) becomes a cosine series:

$$f(x) = \sum_{n=0}^{\infty} b_n \cos\frac{n\pi x}{L}, \quad -L < x < L,$$

where

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_{0}^{2} f(x)\cos\frac{n\pi x}{L}dx \\ b_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{0}^{L} f(x)dx. \end{aligned}$$

(iii) The above discussion suggests a second way for obtaining the coefficients of the Fourier series (6.19). Since any function $f(x)$ can be written as

$$f(x) = f_{sym}(x) + f_{anti}(x), \tag{6.20}$$

where $f_{sym}(x) \equiv \frac{1}{2}(f(x) + f(-x))$ is symmetric about $x = 0$, and

$$f_{anti}(x) \equiv \frac{1}{2}(f(x) - f(-x))$$

is antisymmetric about $x = 0$.

Now, the symmetric function can be represented by a cosine series in $-L < x < L$:

$$f_{sym}(x) = \sum_{n=0}^{\infty} b_n \cos\frac{n\pi x}{L}, \tag{6.21}$$

where, from (5.4),

$$b_n = \frac{2}{L}\int_0^L f_{sym}(x)\cos\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^{L} f_{sym}(x)\cos\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

and

$$b_0 = \frac{1}{L}\int_0^L f_{sym}(x)dx = \frac{1}{2L}\int_{-L}^{L} f_{sym}(x)dx = \frac{1}{2L}\int_{-L}^{L} f(x)dx.$$

Similarly, the antisymmetric function can be represented by a sine series in $-L < x < L$:

$$f_{anti}(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}, \tag{6.22}$$

where from (4.5),

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f_{anti}(x)\sin\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^{L} f_{anti}(x)\sin\frac{n\pi x}{L}dx \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx. \end{aligned}$$

Combining (6.21) and (6.22) into (6.20) then yields (6.19).

6.4 Examples

6.4.1

(a) Represent $f(x) = 1$, as a Fourier sine series in $0 < x < L$. We let

$$\begin{aligned} f_s(x) &\equiv \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}, \\ a_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{n\pi}[1 - \cos n\pi]. \end{aligned}$$

The Fourier sine series representation, obtained by combining sines with coefficients determined above is an antisymmetric function and so looks like:

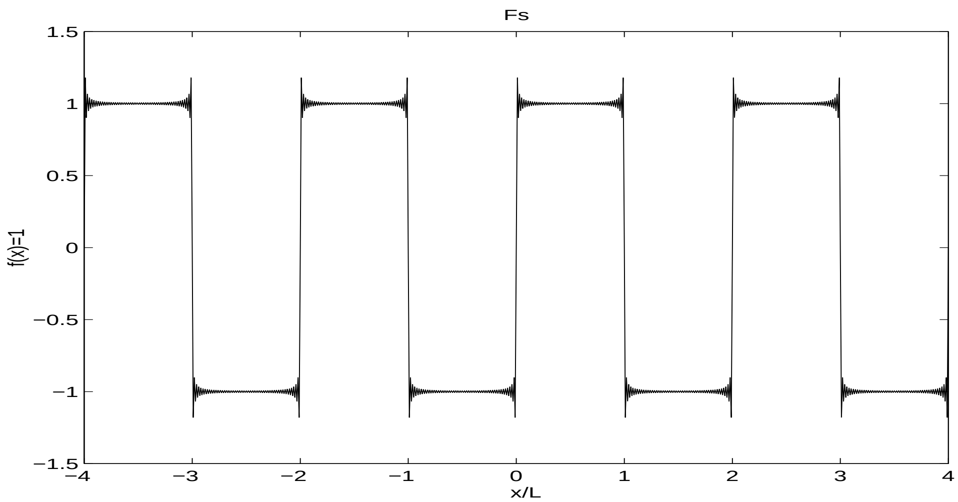

Figure 6.1 Fourier series representation of $f(x) = 1$, as a function of x/L; 50 terms used in the sum.

Thus, $f_s(x)$ looks like $f(x)$ only in the interval $0 < x < L$. It is antisymmetric about $x = 0$ and periodic with period $2L$.

(b) Represent $f(x) = 1$ as a Fourier cosine series in $0 < x < L$. We let

$$\begin{aligned}
f_c(x) &\equiv \sum_{n=0}^{\infty} b_n \cos\frac{n\pi x}{L}, \\
b_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx = 0, \quad n \neq 0 \\
b_0 &= \frac{1}{L}\int_0^L 1dx = 1.
\end{aligned}$$

$f_c(x) = b_0$ is actually a one-term cosine series. With $b_0 = 1$ it is a perfect representation of $f(x)$ in $0 < x < L$.

(c) Represent $f(x) = 1$ as a Fourier series in $-L < x < L$. We let

$$f_{sc}(x) \equiv \sum_{n=0}^{\infty}[a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}],$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = 0$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = 0$$
$$b_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = 1.$$

In this case $f_{sc}(x) = f_c(x) = f(x)$ in $-L < x < L$.

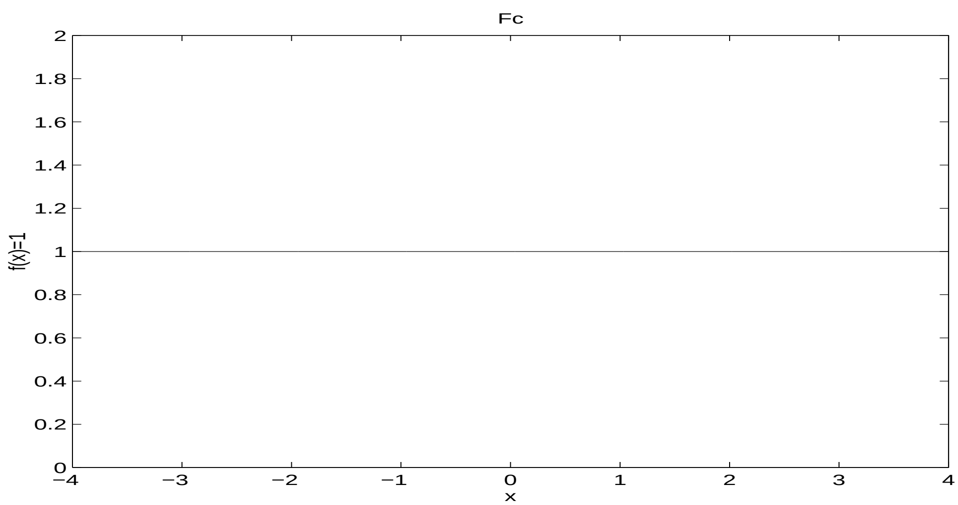

Figure 6.2 Fourier series representation of $f(x) = 1$.

6.4.2

(a) Represent $f(x) = 1 + x$ as a Fourier sine series in $0 < x < 1$.

The sine series representation, $f_s(x)$, is plotted in Figure 6.3, for all x.

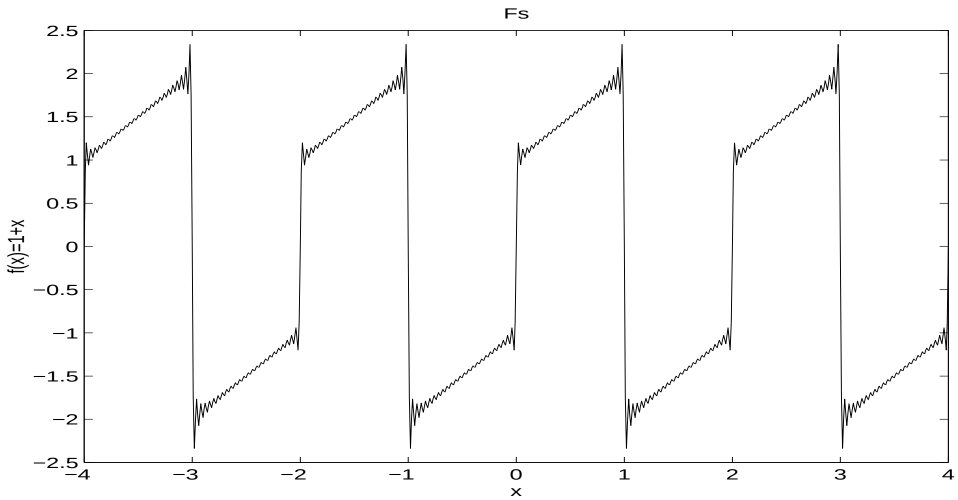

Figure 6.3 Fourier sine series representation of $f(x) = 1 + x$; 50 terms used in the sum.

(b) Represent $f(x) = 1 + x$ as a Fourier cosine series in $0 < x < 1$. The cosine series representation, $f_c(x)$, is plotted in Figure 6.4 for all x.

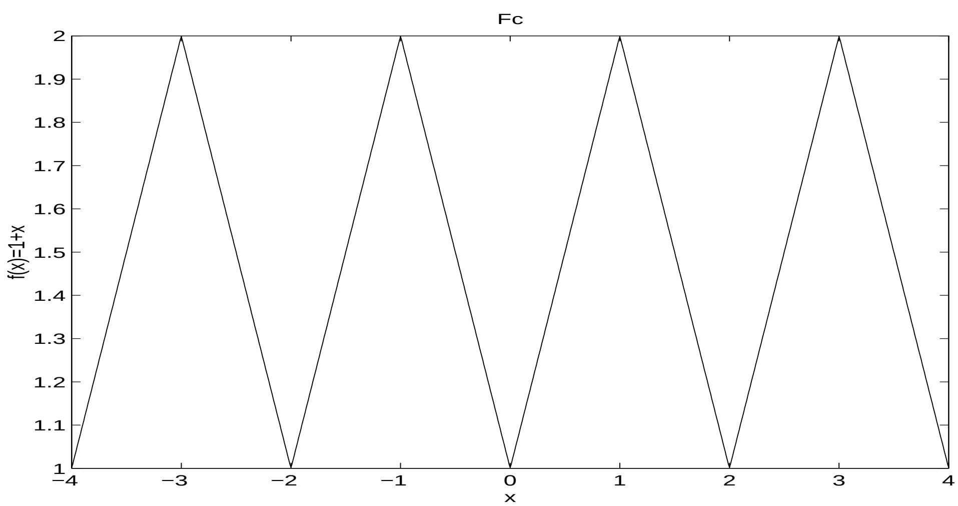

Figure 6.4 The Fourier cosine series representation of $f(x) = 1 + x$; using 50 terms in the sum.

(c) Represent $f(x) = 1 + x$ as a Fourier series in $-1 < x < 1$. $f_{sc}(x)$ is plotted in Figure 6.5 for all x. Notice that $f_{sc}(x)$ is different rom $f_c(x)$ or $f_s(x)$ beyond the interval $0 < x < 1$.

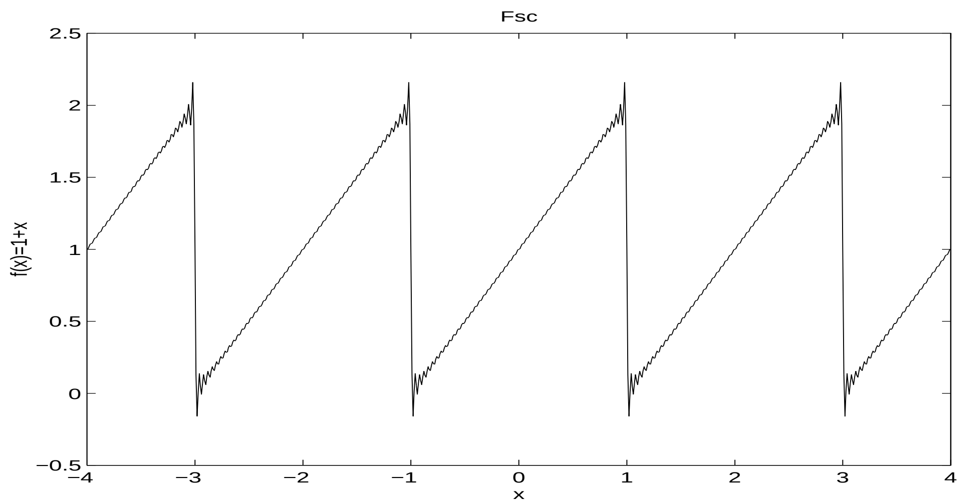

Figure 6.5 Fourier series representation of $f(x) = 1 + x$; 50 terms used in the sum.

6.4.3

(a) Represent $f(x) = e^x$ as a Fourier sine series in $0 < x < 1$. $f_s(x)$ is plotted for all x in Figure 6.6

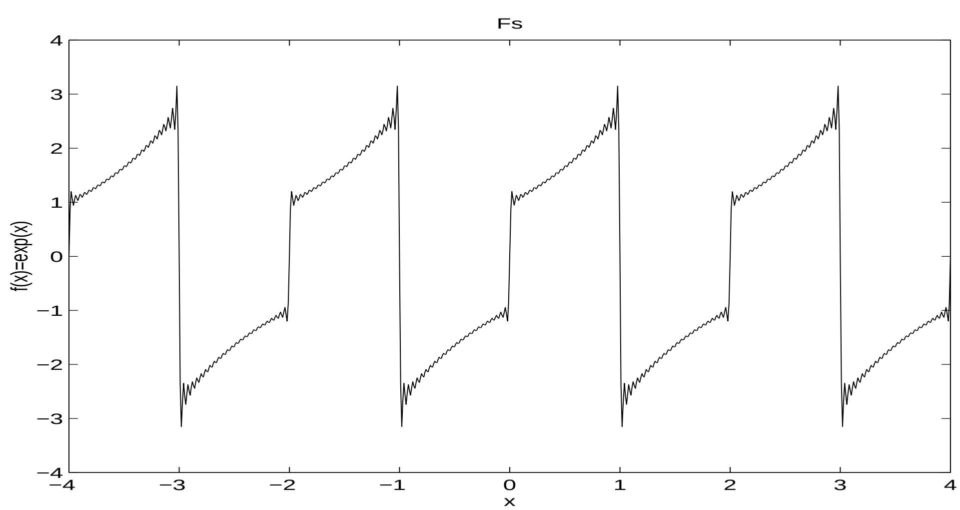

Figure 6.6 Fourier sine series representation of $f(x) = e^x$; 50 terms used in the sum.

(b) Represent $f(x) = e^x$ as a Fourier cosine series in $0 < x < 1$. $f_c(x)$ is plotted for all x in Figure 6.7

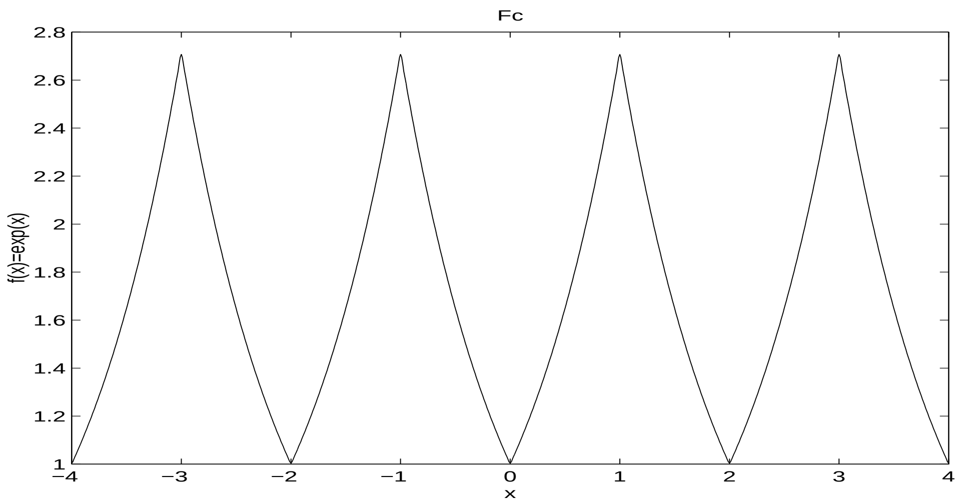

Figure 6.7 Fourier cosine series representation of $f(x) = e^x$; 50 terms used in the sum.

(c) Represent $f(x) = e^x$ as a Fourier series in $-1 < x < 1$. $f_{sc}(x)$ is plotted for all x in Figure 6.8.

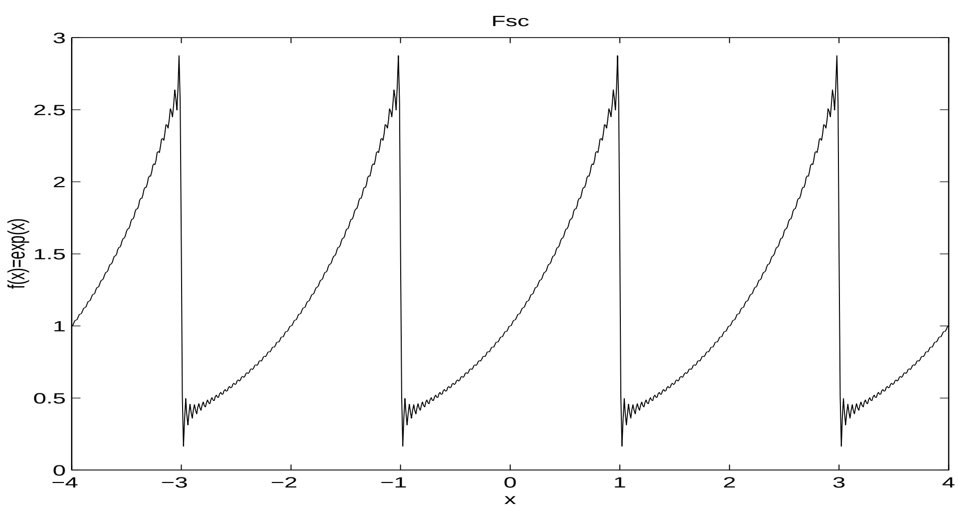

Figure 6.8 Fourier series representation of $f(x) = e^x$; 50 terms used in the sum.

6.5 Complex Fourier series

In this section we will discuss other forms of Fourier series (6.19).

In (6.19), the Fourier coefficients are

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots \\
b_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx.
\end{aligned}
$$

If we change n to $-n$ in the above definition for a_n and b_n, we will find

$$a_{-n} = -a_n, \quad b_{-n} = b_n.$$

Therefore we can rewrite the sum in (6.19) as

$$f(x) = \frac{1}{2}\sum_{n=-\infty}^{\infty}[a_n \sin\frac{n\pi x}{L} + b_n\cos\frac{n\pi x}{L}], \quad -L < x < L, \tag{6.23}$$

where now, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx,
\end{aligned}
$$

[Note that b_0 defined this way is twice as big as the definition obtained in (6.19). This is an advantage, since b_0 now has the same form as the rest of the b_n's.]

The form (6.23) is equivalent to (6.19) but is sometimes preferred because the coefficients are easier to remember.

(6.23) can be further written more compactly using the complex notation:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}, \quad -L < x < L, \tag{6.24}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}dx.$$

To show this, we use Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Changing θ to $-\theta$ gives

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

Adding yields

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}).$$

Subtracting yields

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}).$$

We now also rewrite $\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$ in terms of $e^{\pm in\pi x/L}$. (6.23) becomes

$$\begin{aligned} f(x) &= \frac{1}{2}\sum_{n=-\infty}^{\infty}[b_n\frac{1}{2}(e^{in\pi x/L} + e^{-in\pi x/L}) \\ &\quad + a_n\frac{1}{2i}(e^{in\pi x/L} - e^{-in\pi x/L})] \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{1}{2}(b_n - ia_n)e^{in\pi x/L} + \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{1}{2}(b_n + ia_n)e^{-in\pi x/L}. \end{aligned}$$

In the first sum we change n to $-n$, which is permitted since n is a dummy variable. We then see that the first sum is exactly the same as the second sum. Thus,

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}, \quad -L < x < L}, \tag{6.25}$$

where

$$c_n = \frac{1}{2}(b_n + ia_n) = \frac{1}{2}\left\{\frac{1}{L}\int_{-L}^{L} f(x)(\cos\frac{n\pi x}{L} + i\sin\frac{n\pi x}{L})dx\right\}.$$

Thus,

$$\boxed{c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}dx, \quad n = 0,\ \pm 1,\ \pm 2, \ldots}. \tag{6.26}$$

The complex form (6.25) appears to be the most convenient to use. (6.26) can also be obtained directly from (6.25), by multiplying it by $e^{im\pi x/L}$, integrating from $-L$ to L, and utilizing the orthogonality relation:

$$\frac{1}{2L}\int_{-L}^{L} e^{im\pi x/L}e^{-in\pi x/L}dx = \delta_{mn}.$$

This relation can be obtained by direct integration (the integrand is just $e^{i(m-n)\pi x/L}$.

6.6 Example, Laplace's equation in a circular disk

Consider again the solution of Laplace's equation in a region bounded by a circle of radius a:

$$\begin{aligned} \text{PDE:} \quad & \nabla^2 u = 0, \quad 0 \le r < a \\ \text{BC:} \quad & u(r,\theta) = f(\theta) \text{ for } r = a \end{aligned}$$

and is periodic in θ with period 2π. In polar coordinates, the Laplace operator is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

where θ is the angle and r is the radius.

Since $f(\theta)$ is periodic with period 2π, we can expand it in a periodic Fourier series:

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{-in\theta},$$

with c_n given by (from 6.26)

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{in\theta} d\theta.$$

Since the solution $u(r,\theta)$ should also be periodic with period 2π, it too can be expanded in a periodic Fourier series:

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} R_n(r) e^{-in\theta}.$$

Substituting this assumed form for u into the PDE yields:

$$R_n''(r) + \frac{1}{r}R_n'(r) - \frac{n^2}{r^2}R_n(r) = 0.$$

This ODE belongs to the "equi-dimensional type" and the solution is of the form r^b. Substituting this assumed form into the ODE we find that $b = \pm n$. Thus the solution is

$$R_n(r) = \alpha_n r^n + \beta_n r^{-n}$$

In order that $R_n(r)$ be finite at $r = 0$, we set $\beta_n = 0$ for $n > 0$ and set $\alpha_n = 0$ for $n < 0$. Also to satisfy the BC at $r = a$, we want $R_n(a) = c_n$. Thus

$$R_n(r) = c_n(r/a)^{|n|}, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Finally the solution to the PDE, satisfying all BCs, is:

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} c_n (r/a)^{|n|} e^{-in\theta}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{in\theta} d\theta.$$

As an example, suppose the specified boundary value is

$$u(a,\theta) = f(\theta) = \sin^2\theta = [\frac{1}{2i}(e^{i\theta} - e^{-i\theta})]^2$$

Then

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2\theta \cdot e^{in\theta} = -\frac{1}{4}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2i\theta}e^{in\theta} - \frac{1}{4}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-2i\theta}e^{in\theta}d\theta + \frac{1}{2} \\ &= -\frac{1}{4}\delta_{2n} - \frac{1}{4}\delta_{-2n} + \frac{1}{2}\delta_{0n}. \end{aligned}$$

Thus

$$c_2 = -\frac{1}{4}, \quad c_{-2} = -\frac{1}{4}, c_0 = \frac{1}{2}, \quad \text{and} \quad c_n = 0 \quad \text{for other } n\text{'s}.$$

Finally:

$$u(r,\theta) = \frac{1}{2} - \frac{1}{2}(r/a)^2 \cos 2\theta.$$

An alternative method would be to use the real form of the Fourier series, which turns out to be a little easier.