Chapter 12

Sturm-Liouville Theory

12.1 Introduction

Sturm-Liouville theory provides a general framework for considering eigenvalues and eigenfunctions for equations of the form

$$\frac{d}{dx}[p(x)\frac{d}{dx}y] + [\lambda r(x) - q(x)]y = 0.$$

As seen in the last chapter, this equation arises from separated partial differential equations in various coordinates. All the special functions studied in Chapter 10 are governed by this theory.

12.2 Regular and singular Sturm-Liouville problems

A Liouville differential equation has the general form

$$\frac{d}{dx}[p(x)\frac{d}{dx}y] + [\lambda r(x) - q(x)]y = 0 \tag{12.1}$$

where λ is a parameter (to be determined), $p(x)$, $r(x)$ and $q(x)$ are specified functions.

In a *regular Sturm-Liouville* problem, the domain should be a closed *finite* interval, $[a, b]$; $p(x)$, $p'(x)$, $q(x)$ and $r(x)$ are continuous in $[a, b]$; $p(x) \neq 0$, $r(x) \neq 0$ in $[a, b]$. (Without loss of generality we assume $p(x) > 0$, and, by changing the sign of λ if necessary, $r(x) > 0$.)

The boundary conditions in a regular Sturm-Liouville problem is of the form:

$$\begin{aligned} c_1 y(a) + c_2 y'(a) &= 0 \\ d_1 y(b) + d_2 y'(b) &= 0. \end{aligned} \tag{12.2}$$

At least one of c_1 and c_2, and at least one of d_1 and d_2, are nonzero.

A singular Sturm-Liouville problem is one consisting of (12.1) either on an infinite interval or on a finite interval but with at least one of the regularity properties not satisfied. Typically one or more of the coefficients in the equation (12.1) either goes to zero or ∞ at an end-point of the interval.

For the singular the problems we will consider here, we still have $p(x)$, $p'(x)$, $q(x)$ and $r(x)$ continuous on the open interval $a < x < b$.

$$p(x) > 0, \quad \text{and} \quad r(x) > 0 \text{ in } a < x < b,$$

but $p(x)$ and/or $r(x)$ may vanish $x = a$ or $x = b$. In this case, the boundary condition (12.2) may not be appropriate; it may be too restrictive. A typical boundary condition at a singular point could be that $y(x)$ be bounded there.

The solution to regular or singular Sturm-Liouville problem is the trivial solution

$$y(x) \equiv 0.$$

Nontrivial solutions exist only for some special values (eigenvalues) of λ.

Examples:

(a) *Fourier:*

$$\begin{gathered} y''(x) + \lambda y(x) = 0, \quad 0 \le y \le L \\ y(0) = 0, \quad y(L) = 0 \end{gathered}$$

This is a regular Sturm-Liouville problem.

(b) *Legendre:*

$$[(1 - x^2)y']' + \mu y = 0, \quad -1 < x < 1$$

$$y(x) \text{ bounded at } x = 1 \text{ and } x = -1.$$

This is a singular Sturm-Liouville system with $p(x) = 1 - x^2$, $r(x) = 1$.

$$p(x) = 0 \text{ at } x = \pm 1, \text{ the boundary end points.}$$

(c) *Bessel:*

$$x^2y'' + xy' + [\lambda x^2 - p^2]y = 0, \quad 0 < x < b$$

This can be put into the Liouville form by dividing the equation by x:

$$(xy')' + [\lambda x - \frac{p^2}{x}]y = 0, \quad 0 < x < b$$

Here $p(x) = x$, $r(x) = x$, $q(x) = p^2/x$. This is a singular system because $p(x) = 0$ at $x = 0$, one of the end points. The appropriate boundary conditions are

$$y(0) \text{ bounded, } d_1y(b) + d_2y'(b) = 0$$

(d) *Spherical Bessel:*

$$x^2y'' + 2xy' + [\lambda x^2 - n(n+1)]y = 0, \quad 0 < x < b$$

$$y(0) \text{ bounded, } d_1y(b) + d_2y'(b) = 0.$$

The differential equation can be put into the Liouville form:

$$(x^2y')' + [\lambda x^2 - n(n+1)]y = 0, \quad 0 < x < a$$

so

$$\begin{aligned} p(x) &= x^2, \quad p(0) = 0 \\ r(x) &= x^2, \quad r(0) = 0 \\ q(x) &= n(n+1). \end{aligned}$$

This is a singular Sturm-Liouville system because $p(0) = 0$, $r(0) = 0$.

(e) *Chebyshev:*

$$(1 - x^2)y'' - xy' + \lambda y = 0, \quad -1 < x < 1.$$

This can be put into the Liouville form by dividing the equation by $(1 - x^2)^{1/2}$

$$(1 - x^2)^{1/2}y'' - \frac{x}{(1 - x^2)^{1/2}}y' + \lambda\frac{1}{(1 - x^2)^{1/2}}y = 0$$

which is,

$$((1-x^2)^{1/2}y')' + \lambda\frac{1}{(1-x^2)^{1/2}}y = 0.$$

Therefore

$$p(x) = (1-x^2)^{1/2}, \quad r(x) = \frac{1}{(1-x^2)^{1/2}}, \ q(x) = 0.$$

The appropriate boundary condition for this singular system is:

$$y(x) \text{ bounded at } x = \pm 1.$$

12.3 Orthogonality Theorem

The eigenfunctions corresponding to different eigenvalues are orthogonal to each other with respect to the weight $r(x)$.

Consider two pairs of eigenfunctions and eigenvalues

$$(\phi_k, \lambda_k), \quad (\phi_j, \lambda_j),$$

where $y(x) = \phi_k(x)$ is the eigenfunction satisfying (12.1) and (12.2) with $\lambda = \lambda_k$; $y(x) = \phi_j(x)$ is the eigenfunction corresponding to $\lambda = \lambda_j$:

$$(p\phi_k')' + [\lambda_k r - q]\phi_k = 0 \tag{12.3}$$

$$(p\phi_j')' + [\lambda_j r - q]\phi_j = 0. \tag{12.4}$$

Multiply (12.3) by ϕ_j and (12.4) by ϕ_k, and subtract:

$$\phi_j(p\phi_k')' - \phi_k(p\phi_j)' = (\lambda_j - \lambda_k)r\phi_j\phi_k. \tag{12.5}$$

The left-hand side can be written as: $\frac{d}{dx}[\phi_j(p\phi_k') - \phi_k(p\phi_j')]$. Integrate both sides of (12.5) from a to b

$$(\lambda_j - \lambda_k)\int_a^b r(x)\phi_j(x)\phi_k(x)dx = [\phi_j(p\phi_k') - \phi_k(p\phi_j')]\Big|_a^b \tag{12.6}$$

The right-hand side of (12.6) vanishes either when the boundary conditions (12.2) are applied, or in the case of boundedness condition in a singular Sturm-Liouville problem when $p(x) = 0$ at the boundary. Thus

$$(\lambda_j - \lambda_k)\int_a^b r(x)\phi_j(x)\phi_k(x)dx = 0.$$

Hence:

$$\int_a^b r(x)\phi_j(x)\phi_k(x)dx = \begin{cases} 0 & \text{if } \lambda_j \neq \lambda_k \\ \int_a^b r(x)\phi_j^2(x)dx & \text{if } \lambda_j = \lambda_k. \end{cases} \tag{12.7}$$

12.4 Uniqueness of eigenfunctions

There is only one eigenfunction corresponding to an eigenvalue.

Let us suppose that this is not true, and that there are two different eigenfunctions $\phi_1(x)$ and $\phi_2(x)$ corresponding to the same eigenvalue λ. Then

$$\begin{aligned}(p\phi_1') + (\lambda r + q)\phi_1 &= 0 \\ (p\phi_2') + (\lambda r + q)\phi_2 &= 0\end{aligned}$$

multiply the first equation by ϕ_2 and the second equation by ϕ_1, and then subtract:

$$0 = \phi_2(p\phi_1') - \phi_1(p\phi_2')' = \frac{d}{dx}[p(\phi_2\phi_1' - \phi_1\phi_2')]$$

Integrating:

$$p(\phi_2\phi_1' - \phi_1\phi_2') = \text{ constant.}$$

The constant can be evaluated at one of the boundaries. If the boundary condition is of the regular Sturm-Liouville form of (12.2), then the constant is easily shown to be zero. It is also zero in the singular case where $p(x) = 0$ at the boundary point. It then follows that in the interior of the domain

$$\phi_1\frac{d\phi_2}{dx} - \phi_2\frac{d\phi_1}{dx} = 0,$$

which is

$$\frac{d}{dt}(\phi_2/\phi_1) = 0.$$

Hence

$$\phi_2(x) = c\phi_1(x);$$

they are the same eigenfunction. Note that this result fails for periodic boundary conditions.

12.5 All eigenvalues are real and positive

To show that all eigenvalues are real, we consider (12.1) and its complex conjugate:

$$[py'] + [\lambda r - q]y = 0 \tag{12.8}$$

$$[py^{*\prime}]' + [\lambda^* r - q]y^* = 0. \tag{12.9}$$

Using the same procedure as in section 3, we multiply (12.8) by y^* and (12.9) by y; then subtract and integrate:

$$(\lambda - \lambda^*) \int_a^b r|y|^2 dx = 0. \tag{12.10}$$

Since $\int_a^b r|y|^2 dx \neq 0$, we must have

$$\lambda = \lambda^*. \tag{12.11}$$

Thus λ is real.

All eigenvalues of the Sturm-Liouville system with real coefficients are real.

To show that λ is positive if $q(x) \geq 0$, we multiply (12.11) by y^* and integrate:

$$0 = \int_a^b y^*[py']'dx + \lambda \int_a^b r|y|^2 dx - \int_a^b q|y|^2 dx$$

This gives, after integrating by parts for the first term

$$\lambda = \int_a^b [p|y'|^2 + q|y|^2] dx \Big/ \int_a^b r|y|^2 dx \tag{12.12}$$

(12.11) implies that $\lambda \geq 0$ if q is nonnegative in (a, b). [The possibility of $\lambda = 0$ is allowed if $y'(x) \equiv 0$, such as the $n = 0$ case of $\phi_n(x) = \cos \frac{n\pi x}{L}$.]

All eigenvalues of the Sturm-Liouville system with $q(x) \geq 0$ in (a, b) are nonnegative.

12.6 Eigenvalues are infinite in number

The eigenvalues of the Sturm-Liouville system are discrete, and *form an increasing sequence*:

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots$$

with $\lambda_j \to \infty$ as $j \to \infty$, provided that the domain is finite and $p(x) > 0$ and $r(x) > 0$ in $a < x < b$.

This will be shown in the next section.

12.7 Zeros of eigenfunctions

(Reference: Morse and Feshbach (1953))

Oscillation Theorem

The eigenfunction ϕ_k corresponding to the eigenvalue λ_k (which has been ordered: $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$, has $k-1$ zeros in (a, b).

Consider two solutions $\phi_{(1)}(x)$ and $\phi_{(2)}(x)$, satisfying the Liouville equation (1) with $\lambda = \lambda_{(1)}$ and $\lambda = \lambda_{(2)}$, respectively. $\lambda_{(1)}$ and $\lambda_{(2)}$ are not necessarily the first and second eigenvalues λ_1 and λ_2. As was done previously, we multiply the first equation by $\phi_{(2)}$, the second equation by $\phi_{(1)}$, and subtract:

$$\frac{d}{dx}[p(x)(\phi_{(2)}\frac{d}{dx}\phi_{(1)} - \phi_{(1)}\frac{d}{dx}\phi_{(2)})] = (\lambda_{(2)} - \lambda_{(1)})r(x)\phi_{(1)}\phi_{(2)} \qquad (12.13)$$

We integrate the above equation (12.13) from the left boundary $x = a$ to a point x in the interior

$$[p(x)(\phi_{(2)}(x)\frac{d}{dx}\phi_{(1)}(x) - \phi_{(1)}(x)\frac{d}{dx}\phi_{(2)}(x))] = (\lambda_{(2)} - \lambda_{(1)}) \int_a^x r(x)\phi_{(1)}\phi_{(2)}dx \qquad (12.14)$$

since $p(a)\ (\phi_{(2)}(a)\frac{d}{dx}\phi_{(1)}(a) - \phi_{(1)}(a)\frac{d}{dx}\phi_{(2)}(a)) = 0$, either because of the homogeneous boundary condition (12.2), or because $p(a) = 0$ in the singular case.

Choose $x = \xi_1 > a$ to be the smallest value of x for which $\phi_{(1)}(x) = 0$. That is

$$\phi_{(1)}(\xi_1) = 0$$

$$(p\phi_{(2)}\frac{d}{dx}\phi_{(1)})_{x=\xi_1} = (\lambda_{(2)} - \lambda_{(1)}) \int_a^{\xi_1} r(x)\phi_{(1)}\phi_{(2)}dx. \qquad (12.15)$$

Since ξ_1 is the smallest zero of $\phi_{(1)}(x)$, $\phi_{(1)}(x)$ does not change sign between $x = a$ and $x = \xi_1$. We therefore can take

$$\phi_{(1)}(x) > 0, \quad a < x < \xi_1,$$

and

$$\frac{d}{dx}\phi_{(1)}(\xi_1) < 0$$

since $\phi_{(1)}(x)$ goes from positive to zero at $x = \xi_1$.

If $\lambda_{(2)} > \lambda_{(1)}$, we want to show that $\phi_{(2)}(x)$ oscillates more rapidly by demonstrating that $\phi_{(2)}(x)$ has an extra zero in $a < x < \xi_1$ (than $\phi_{(1)}(x)$). If $\phi_{(2)}(x)$ did not go to zero in $a < x < \xi_1$, we can take it to be positive in this range. Then since $r(x) > 0$, the right-hand side of (12.15) is positive, while the left-hand side is negative. This contradiction demonstrates that $\phi_{(2)}(x)$ must go through a zero somewhere in $a < x < \xi_1$.

The same argument can be repeated for the range between the first zero ξ_1 of $\phi_{(1)}(x)$ and its second zero ξ_2, and show that $\phi_{(2)}(x)$ must have another zero in $\xi_1 < x < \xi_2$, and so on. Thus:

Sturm's First Comparison Theorem

The number of zeros of $y(x)$ increases as λ is increased.

For a low enough value of λ, there will be no zero of $y(x)$ inside $a < x < b$. Call this smallest λ value λ_1 and the corresponding $y(x)$, $\phi_1(x)$ (if the boundary condition at $x = b$ is satisfied). In other words, λ_1 is the lowest eigenvalue and $\phi_1(x)$ is the eigenfunction with the smallest number of zeros (in fact no zero).

We increase λ from λ_1 using a $y(x)$ which satisfies the boundary condition at $x = a$. At first it will not fit the boundary condition at $x = b$. Then when that boundary condition at $x = b$ is satisfied for higher value of $\lambda = \lambda_2$, the next eigenvalue, the eigenfunction $\phi_2(x)$ must have one more zero than $\phi_1(x)$.

As λ is increased further, the distance between the zeros of $y(x)$ decreases until, at the next eigenvalue λ_3, there is one more zero (than $\phi_2(x)$) inside $a < x < b$.

We order the sequence of eigenfunctions, ϕ_1, ϕ_2, ϕ_3, ..., such that the corresponding eigenvalues are in increasing order:

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \lambda_{n+1} < \ldots$$

and $\phi_{n+1}(x)$ has one more zero than $\phi_n(x)$. $\phi_n(x)$ has $n - 1$ zeros in the interior.

We next want to show that the eigenvalues are *discrete*, i.e. the difference $(\lambda_{n+1} - \lambda_n)$ is finite, if $(b - a)$ is finite.

Take λ to be a value intermediate between λ_n and λ_{n+1}, and $\phi(x)$ to be the corresponding solution of (12.1) satisfying the boundary condition at $x = a$, but not at $x = b$ (otherwise it would have been an eigenfunction). Using the previous result (12.15), we now have:

$$(\lambda - \lambda_n) \int_a^b r(x)\phi_n(x)\phi(x)dx = p(b)(\phi(b)\phi_n'(b) - \phi_n(b)\phi'(b)) \qquad (12.16)$$

Since $\phi(x)$ is not an eigenfunction, the right-hand side is not zero. It remains nonzero even as $n \to \infty$.

Thus unless $\int_a^b r(x)\phi_n(x)\phi(x)dx$ is infinite (which is possible if $b-a$ is infinite), the difference $\lambda - \lambda_n$ is finite. Since $\lambda_{n+1} > \lambda$, we have the result that the difference $(\lambda_{n+1} - \lambda_n)$ cannot be infinitesimal, if $(b-a)$ is not infinite, no matter how large n is.

It follows that the sequence

$$\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n, \lambda_{n+1}, \ldots$$

can have no upper bound but must continue to $+\infty$.

Note that the possibility exists for a continuous distribution of λ only if $(b-a)$ is infinite. Nevertheless, if the solution decay sufficiently rapidly as $x \to \infty$ such that $\int_a^b r\phi_n\phi dx$ is finite, the eigenvalues are still discrete even if the domain is infinite.

12.8 Variational Principle

We define the *Rayleigh quotient* for any piecewise continuous function $\psi(x)$ by

$$\Omega(\psi) \equiv \int_a^b (p\psi'^2 + q\psi^2)dx \Big/ \int_a^b r\psi^2 dx \geq 0 \tag{12.17}$$

The function $\psi(x)$ does not have to satisfy (12.1), but we will assume that it satisfies the same boundary conditions as the eigenfunctions $\phi_k(x)$. Since $\Omega(\psi)$ is nonnegative, it must have a greatest lower bound. That minimum turns out to be λ_1, the lowest eigenvalue, i.e.

$$\lambda_1 = \Omega(\phi_1) = \min \Omega(\psi).$$

To show this, suppose $\Omega(\psi)$ is the minimum for some ψ, then

$$\Omega(\psi(x) + \alpha g(x)) \geq \Omega(\psi(x)) \equiv \mu$$

for any constant α and any continuous differentiable $g(x)$ which vanishes at $x = a$ and $x = b$. Or

$$\left.\frac{\partial\Omega(\psi + \alpha g)}{\partial\alpha}\right|_{\alpha=0} = 0$$

This derivative is

$$0 = \frac{2\int_a^b (p\psi' g' + q\psi g)dx}{\int_a^b r\psi^2 dx} - \frac{2\int_a^b r\psi g dx \int_a^b (p\psi'^2 + q\psi^2)dx}{(\int_a^b r\psi^2 dx)^2},$$

which implies

$$\int_a^b [p\psi' g' + q\psi g - \mu r\psi g]dx = 0.$$

We integrate by parts the above equation to obtain:

$$-\int_a^b g[(p\psi')' + (\mu r - q)\psi]dx = 0.$$

Since this is true for every admisssible function $g(x)$, the expression in the brackets must be zero, i.e.

$$[(p\psi')' + (\mu r - q)\psi] = 0$$

This is the same as the Liouville equation (12.1). So μ must be the eigenvalue λ_1, and ψ must be the eigenfunction $\phi_1(x)$.

By the same procedure we can show that λ_2 is the minimum of $\Omega(\psi)$ under the additional constraint that ψ be orthogonal to the first eigenfunction ϕ_1, i.e.

$$\int_a^b r\psi\phi_1 dx = 0.$$

The minimizing function ψ is ϕ_2. Continuing this way we can show that

$$\lambda_k = \min \Omega(\psi) \tag{12.18}$$

subject to the constraint that ψ be orthogonal to ϕ_1, ϕ_2, ..., ϕ_{k-1}.

12.9 The eigenfunctions are complete

A set of eigenfunctions, $\{\phi_k\}$, is *complete* if a series (a linear superposition of them) can provide an accurate representation of any piecewise continuous function $f(x)$ in the domain where they are defined. By "accurate" representation we mean that the least square-error between $f(x)$ and its representation goes to zero if the number of terms in the series goes to infinity.

Let's expand $f(x)$ in an *eigenfunction expansion* of the form

$$f(x) \sim \sum_{k=1}^{\infty} a_k\phi_k(x) \equiv \tilde{f}(x), \tag{12.19}$$

where

$$a_k = \int_a^b f(x)\phi_k(x)r(x)dx \Big/ \int_a^b \phi_k^2(x)r(x)dx \tag{12.20}$$

is called the *generalized Fourier coefficient.*

(12.20) is obtained from (12.19) by multiplying it by $\phi_j(k)r(x)$, integrating from a to b, and making use of the orthogonality condition $\phi_k(x)$ with respect to weight $r(x)$. The series $\tilde{f}(x)$ is the "representation" of $f(x)$.

Consider a function $\psi(x)$ defined by

$$\psi(x) \equiv f(x) - \sum_{n=1}^{k-1} a_n \phi_n \tag{12.21}$$

$\psi(x)$ is "orthogonal" to the eigenfunctions $\phi_1,\ \phi_2,\ \ldots,\ \phi_{k-1}$ because, for $j = 1,\ 2,\ \ldots,\ k-1$:

$$\begin{aligned}\int_a^b r\psi\phi_j dx &= \int_a^b r(f - \sum_{n=1}^{k-1} a_n\phi_n)\phi_j dx \\ &= \int_a^b rf\phi_j dx - a_j \int_a^b r\phi_j^2 dx = 0\end{aligned}$$

due to the definition of a_j by (12.20). We further make $\psi(x)$ satisfy the boundary conditions by enforcing the boundary condition on $f(x)$ at the two ends of the domain. $f(x)$ can remain arbitrary in $a < x < b$.

From (12.18), we have

$$\lambda_k \le \Omega(\psi)$$

From the definition of $\Omega(\psi)$ in (12.17), we have

$$\begin{aligned}\int_a^b r\psi^2 dx &\le \frac{1}{\lambda_k}\int_a^b [p\psi'^2 + q\psi^2]dx \\ &= \frac{1}{\lambda_k}\{\int_a^b (pf'^2 + qf^2)dx - 2\sum_{n=1}^{k-1} a_n \int_a^b (pf'\phi_n' + qf\phi_n)dx \\ &\quad + \sum_{n=1}^{k-1}\sum_{m=1}^{k-1} a_n a_m \int_a^b (p\phi_n'\phi_m' + q\phi_n\phi_m)dx\}.\end{aligned}$$

Integrating by parts and using (12.1)

$$\begin{aligned}\int_a^b (p\phi_n'\phi_m' + q\phi_n\phi_m)dx &= \lambda_n \int_a^b r\phi_n\phi_m dx \\ &= \begin{cases} 0 & \text{if } m \ne n \\ \lambda_n \int_a^b r\phi_n^2 dx & \text{if } m = n. \end{cases}\end{aligned}$$

$$\int_a^b (pf'\phi_n' + qf\phi_n)dx = \int_a^b f[-(p\phi_n)' + q\phi_n]dx$$
$$= \lambda_n \int_a^b rf\phi_n dx = \lambda_n a_n \int_a^b r\phi_n^2 dx$$

Thus

$$0 \le \int_a^b r\psi^2 dx \le \frac{1}{\lambda_k}\{\int_a^b (pf'^2 + qf^2)dx - \sum_{n=1}^{k-1} \lambda_n a_n^2 \int_a^b r\phi_n^2 dx\}$$
$$\le \frac{1}{\lambda_k}\int_a^b (pf'^2 + qf^2)dx \tag{12.22}$$

Since we have shown in section 12.7 that $\lambda_k \to \infty$ as $k \to \infty$, the right-hand side $\to 0$ as $k \to \infty$.

The least-square mean error for a representation of $f(x)$ by a partial sum of $k-1$ terms is defined by

$$\epsilon_{k-1} \equiv \int_a^b r(f - \sum_{n=1}^{k-1} a_n\phi_n)^2 dx$$
$$= \int_a^b r\psi^2 dx$$

by our definition of ψ in (12.21). The result in (12.22) then implies

$$\lim_{k\to\infty} \epsilon_k \to 0.$$

In other words, the generalized Fourier series representation ($\tilde{f}(x)$) of $f(x)$ converges to $f(x)$ in the mean. The set of eigenfunctions $\{\phi_k\}$ is then said to be *complete*.

12.10 Examples

(a) Fourier sine:

$$y''(x) + \lambda y(x) = 0, \quad 0 < y < L,$$
$$y(0) = 0, \quad y(L) = 0$$

This is a regular Sturm-Liouville system with $p(x) = 1$, $r(x) = 1$ and $q(x) = 0$.

The eigenfunctions are

$$y(x) = \phi_n(x) \equiv \sin \frac{n\pi x}{L},$$

corresponding to the eigenvalues

$$\lambda = \lambda_n \equiv (\frac{n\pi}{L})^2, \quad n = 1, 2, 3, \dots$$

They are infinite in number, discrete, real and positive. They can be ordered as

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots$$

with $\lambda_n \to \infty$ as $n \to \infty$.

The orthogonality condition is

$$\frac{2}{L}\int_0^L \sin\frac{m\pi x}{L}\sin\frac{m\pi x}{L}dx = \delta_{mn}.$$

An orbitrary piecewise continuous function $f(x)$ can be represented by a Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}, \quad 0 < x < L$$

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx,$$

and the representation will converge to $f(x)$ in the mean.

Fourier cosine:

$$y''(x) + \lambda y(x) = 0, \quad 0 < y < L$$
$$y'(0) = 0, \quad y'(L) = 0$$

Same as Fourier sine except the eigenfunctions are

$$y(x) = \phi_n(x) \equiv \cos\frac{n\pi x}{L},$$

corresponding to the eigenvalues

$$\lambda = \lambda_n = (\frac{n\pi}{L})^2, \quad n = 0, 1, 2, 3, \dots$$

The orthogonality condition is

$$\frac{2}{L}\int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = \begin{cases}\delta_{mn} & \text{if } m \neq 0\\ 2 & \text{if } m = n = 0.\end{cases}$$

An arbitrary piecewise continuous function $f(x)$ can be expanded in a Fourier cosine series

$$f(x) = \sum_{n=0}^{\infty} b_n \cos\frac{n\pi x}{L}, \quad 0 < x < L$$

where

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots \\ b_0 &= \frac{1}{L}\int_0^L f(x)dx. \end{aligned}$$

(b) Legendre:

$$[(1-x^2)y']' + \mu y = 0, \quad -1 < x < 1$$
$$y \text{ bounded at } x = \pm 1$$

This is a singular Sturm-Liouville system

$$\begin{aligned} &p(x) = 1 - x^2 > 0 \quad \text{in } -1 < x < 1 \\ &p(x) = 0 \quad \text{at } x = \pm 1 \\ &r(x) = 1 \\ &q(x) = 0 \end{aligned}$$

The eigenvalues are

$$\mu = \mu_n \equiv n(n+1), \quad n = 0, 1, 2, 3, \ldots$$

They are infinite in number, real, nonnegative, discrete and tend to infinity as $n \to \infty$.

The eigenfunctions are

$$\begin{aligned} &y(x) = \phi_n(x) \equiv P_n(x) \\ &P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} \\ &P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x, \quad \ldots \end{aligned}$$

They are othogonal with respect to weight 1:

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n. \end{cases}$$

An arbitrary piecewise continuous function $f(x)$ can be expanded in a Legendre series:

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x), \quad -1 < x < 1$$

where

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)P_n(x)dx.$$

The series will converge to $f(x)$ in the mean.

Associated Legendre

$$[(1-x^2)y']' + [\mu - \frac{m^2}{1-x^2}]y = 0, \quad -1 < x < 1$$
$$y \text{ bounded at } x = \pm 1.$$

This is a singular Sturm-Liouville system with

$$\begin{aligned} p(x) &= 1 - x^2 > 0 \text{ in } -1 < x < 1 \\ p(x) &= 0 \text{ at } x = \pm 1 \\ r(x) &= 1 \\ q(x) &= \frac{m^2}{1-x^2}. \end{aligned}$$

The eigenvalues are

$$\mu = \mu_n = n(n+1), \quad n = 0, 1, 2, 3, \ldots.$$

(discrete, real, nonnegative, infinite in number, tending to infinity as $n \to \infty$).

The corresponding eigenfunctions are

$$y(x) = \phi_n(x) \equiv P_n^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x).$$

They are orthogonal with respect to weight 1:

$$\int_{-1}^{1} P_n^m(x) P_{n'}^m(x)dx = \frac{2(n+m)!}{(2n+1)(n-m)!}\delta_{nn}.$$

A piecewise continues function $f(x)$ can be expanded in a series of associated Legendre functions:

$$f(x) = \sum_{n=0}^{\infty} a_n P_n^m(x), \qquad -1 < x < 1$$
$$a_n = \frac{(2n+1)(n-m)!}{2(n+m)!} \int_{-1}^{1} f(x) P_n^m(x) dx,$$

and the representation will converge to $f(x)$ in the mean.

(c) Bessel

$$(xy')' + [\lambda x - \frac{m^2}{x}]y = 0, \quad 0 < x < a$$
$$y \text{ bounded at } x = 0, \quad y(a) = 0.$$

This is a singular Sturm-Liouville system with

$$\begin{aligned} p(x) &= x > 0, \quad 0 < x < a, \text{ but } p(x) = 0 \text{ at } x = 0 \\ r(x) &= x > 0, \quad 0 < x < a \\ q(x) &= m^2/x. \end{aligned}$$

The eigenfunctions are

$$y(x) = \phi_j(x) \equiv J_m(\sqrt{\lambda_j} x),$$

where the eigenvalues λ_j's are determined implicitly from the zeros of the Bessel function

$$\lambda_j = (z_{mj}/a)^2,$$

where

$$J_m(z_{mj}) = 0, \quad \text{and } z_{mj} \text{ is the } j\text{th root of } J_m(z) = 0.$$

The eigenvalues are discrete, real, positive, infinite in number, and tend to infinity as $j \to \infty$. The eigenfunctions are orthogonal with respect to weight x.

$$\frac{2}{a^2} \int_0^a J_m(\sqrt{\lambda_j} x) J_m(\sqrt{\lambda_k} x) x dx = J_{m+1}^2(z_{mj}) \delta_{jk}.$$

An arbitrary piecewise continuous function $f(x)$ can be expanded in a Bessel series as:

$$f(x) = \sum_{j=1}^{\infty} a_j J_m(z_{mj} x/a), \quad 0 < x < a$$

where

$$a_j = \frac{2}{a^2}\int_0^a f(x)J_m(z_{mj}x/a)xdx/J_{m+1}^2(z_{mj}).$$

and the representation will converge to $f(x)$ in the mean.

(d) Spherical Bessel

$$(x^2y')' + [\lambda x^2 - n(n+1)]y = 0, \quad 0 < x < a$$

$$y(0) \text{ bounded and } y(a) = 0.$$

This is a singular Sturm-Liouville system with

$$\begin{aligned}
&p(x) = x^2 > 0, \quad 0 < x < a, \quad p(x) = 0 \text{ at } x = 0.\\
&r(x) = r^2 > 0, \quad 0 < x < a\\
&q(x) = n(n+1)
\end{aligned}$$

The eigenfunction is

$$y(x) = \phi_j(x) = j_n(\sqrt{\lambda_j}x),$$

where

$$\sqrt{\lambda_j} = z_{nj}/a, \quad j = 1, 2, 3, \ldots.$$

with z_{nj} being the jth positive root to $j_n(z) = 0$

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots.$$

The eigenfunctions are orthogonal with respect to weight x^2:

$$\frac{2}{a^3}\int_0^a j_n(z_{nj}x/a)j_n(z_{nk}x/a)x^2dx = j_{n+1}^2(z_{nj})\delta_{jk}$$

An arbitrary piecewise continuous function $f(x)$ can be expanded in a spherical Bessel series as:

$$f(x) = \sum_{j=1}^{\infty} a_j j_n(z_{nj}x/a), \quad 0 < x < a$$

where

$$a_j = \frac{2}{a^3}\int_0^a f(x)j_p(z_{nj}/a)x^2dx/j_{n+1}^2(z_{nj}),$$

and the representation will converge to $f(x)$ in the mean.

(e) Chebyshev

$$[(1-x^2)^{1/2}y']' + [\lambda\frac{1}{(1-x^2)^{1/2}}]y = 0, \quad -1 < x < 1$$
$$y \quad \text{bounded at} \quad x = \pm 1.$$

The eigenfunction is
$$y(x) = \phi_n(x) \equiv T_n(x),$$
where the Chebyshev polynomiial of the first kind is defined by
$$T_n(x) = \cos(n\cos^{-1}x).$$

The corresponding eigenvalue is
$$\lambda = \lambda_n \equiv n^2, \quad n = 0, 1, 2, 3, \ldots$$

The eigenfunctions are orthogonal with respect to weight $r(x) = (1-x^2)^{-1/2}$:
$$\frac{2}{\pi}\int_{-1}^{1} T_n(x)T_{n'}(x)(1-x^2)^{-1/2}dx = \begin{cases} \delta_{nn}, & n \neq 0 \\ 2, & n = n' = 0 \end{cases}$$

Any arbitrary piecewise continuous function $f(x)$ can be expanded in a Chebyshov series:
$$f(x) = \sum_{n=0}^{\infty} a_n T_n(x), \quad -1 < x < 1$$

where
$$a_n = \frac{2}{\pi}\int_{-1}^{1} f(x)T_n(x)((1-x^2)^{-1/2}dx, \quad n > 0$$
$$a_0 = \frac{1}{\pi}\int_{-1}^{1} f(x)(1-x^2)^{-1/2}dx.$$

The appretation will converge to $f(x)$ in the mean.

(f) Hermite

$$y'' + (\lambda - x^2)y = 0, \quad -\infty < x < \infty$$
$$y \quad \text{bounded as } x \to \pm\infty.$$

This is a singular Sturm-Liouville system because the domain is infinite.

The eigenfunctions are

$$y(x) = \phi_n(x) \equiv e^{-x^2/2} H_n(x),$$

where $H_n(x)$ is the Hermite polynomial of order n.

The eigenvalues are discrete:

$$\lambda = \lambda_n \equiv (2n+1), \quad n = 0, 1, 2, \ldots,$$

despite the fact that the domain is infinite. This is because the eigenfunctions decrease so rapidly as $x \to \pm\infty$ that $\int_{-\infty}^{\infty} \phi_n(x)^2 dx$ is finite.

The orthogonality condition is

$$\frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_{n'}(x) dx = \delta_{nn}.$$

An arbitrary piecewise continuous function $f(x)$ can be expanded in a Hermite series provided that the integrals are finite (i.e. $f(x)e^{-x^2/2}H_n(x)$ is integrable).

$$f(x) = \sum_{n=0}^{\infty} a_n e^{-x^2/2} H_n(x), \quad -\infty < x < \infty$$

where

$$a_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} f(x) e^{-x^2/2} H_n(x) dx.$$

The representation will converge to $f(x)$ in the mean.

12.11 Eigenfunction expansion

The method of eigenfunction expansion is useful in solving non-homogeneous partial differential equations. This will be discussed in a later chapter. Here we demonstrate the concept using ordinary differential equaitons.

The completeness theorem for eigenfunctions for the Sturm-Liouville system allows an arbitrary function $f(x)$ to be represented by an infinite sum of the eigenfunctions. Since any set of Sturm-Liouville eigenfunctions will do, we have the choice of picking a particular set to suit the problem at hand. This will be illustrated in the following examples.

Consider the simple nonhomogeneous boundary value problem

$$y'' + \lambda y = 1, \quad 0 < x < L$$

$$y'(0) = 0, \quad y'(L) = 0.$$

We know that the set of eigenfunctions

$$\phi_n(x) = \cos\frac{n\pi x}{L}, \quad 0 < x < L, \quad n = 0, 1, 2, 3, \ldots$$

is complete, and so an arbitrary function, including the unknown $y(x)$, can be expanded in a cosine series

$$y(x) = \sum_{n=0}^{\infty} b_n \cos\frac{n\pi x}{L}, \quad 0 < x < L$$

The choice of cosines series has the advantage that the boundary conditions at $x = 0$ and $x = L$ are automatically satisfied. Substituting the assumed cosine series into the differential equation, we get:

$$\sum_{n=0}^{\infty}[-(\frac{n\pi}{L})^2 + \lambda]b_n \cos\frac{n\pi x}{L} = \sum_{n=0}^{\infty} f_n \cos\frac{n\pi x}{L}$$

where we have also expanded the forcing term, 1 in this case, in a cosine series. So $f_0 = 1$, $f_n = 0$, $n > 0$.

Equating the coefficients of cosines, we get

$$[\lambda - (\frac{n\pi}{L})^2]b_n = f_n, \quad n = 0, 1, 2, 3, \ldots$$

If $\lambda \neq (\frac{n\pi}{L})^2$, $n = 0, 1, 2, 3, \ldots$, the coefficients b_n can be determined as

$$b_n = f_n/[\lambda - (\frac{n\pi}{L})^2],$$

and so

$$b_n = 0, \quad \text{for } n = 1, 2, 3, \ldots$$
$$b_0 = f_1/\lambda = 1/\lambda$$

The solution is then simply

$$y(x) = 1/\lambda.$$

[One could have guessed this solution by trying $y(x) = \text{constant}$.] If however, λ is equal to one of the nonzero eigenvalues, say $(\frac{\pi}{L})^2$, then b_1 does not need to be zero, and can be arbitrary.

$$y(x) = 1/\lambda + b_1 \cos\frac{\pi x}{L}.$$

[If $\lambda = 0$, then it is easier to go back to the equation:

$$y'' = 1$$

and integrate twice to yield

$$y(x) = \frac{x^2}{2} + Ax + B$$

$A = 0$ to satisfy the boundary condition at $x = 0$. The boundary condition at $x = L$ cannot be satisfied. Therefore there is no solution to this problem. The problem was actually not well posed.]

Now consider the problem:

$$[(1 - x^2)y']' + \lambda y = 1, \qquad -1 < x < 1.$$

We can still use the cosine series to represent the unknown:

$$y(x) = \sum_{n=0}^{\infty} b_n \cos\frac{n\pi x}{L},$$

but because of the nonuniform coefficients $(1 - x^2)$, the first term in the differential equation becomes very complicated when expressed in terms of b_n. A better way is to represent $y(x)$ in terms of Legendre polynomials $P_n(x)$:

$$y(x) = \sum_{n=0}^{\infty} a_n P_n(x), \qquad -1 < x < 1$$

$$[(1 - x^2)y']' = \sum_{n=0}^{\infty} a_n \frac{d}{dx}[(1 - x^2)\frac{d}{dx}P_n] = \sum_{n=0}^{\infty} -a_n n(n+1)P_n(x),$$

since $P_n(x)$ satisfies

$$\frac{d}{dx}[(1 - x^2)\frac{d}{dx}P_n] + n(n+1)P_n = 0.$$

The boundary conditions are also satisfied.

The differential equation implies:

$$\sum_{n=0}^{\infty} [\lambda - n(n+1)]a_n P_n(x) = 1$$

Since $P_0(x) = 1$, we have

$$a_0 = 1/\lambda$$
$$a_n = 0, \quad n = 1, 2, 3, \ldots$$

if $\lambda \neq m(m+1)$, $m = 0, 1, 2, 3, \ldots$.

If $\lambda = m(m+1)$ for some integer $m \neq 0$, the solution becomes

$$y(x) = 1/\lambda + b_m P_m(x),$$

for arbitrary constant b_m.

[If $\lambda = 0$, the equation is

$$[(1 - x^2)y']' = 1$$

Integrate once

$$(1 - x^2)y' = x + A$$
$$y' = \frac{x}{1 - x^2} + \frac{B}{1 - x^2}$$

Integrate again:

$$y(x) = \ell n[(1 - x^2)^{-1/2}] + B\ell n[\frac{(1 + x)^{1/2}}{(1 - x)^{1/2}}] + C$$

$B = -1$ to satisfy one of the boundary conditions. y bounded at $x = 1$.

$$y(x) = \ell n[\frac{1}{1 + x}] + C.$$

The boundary condition at $x = -1$ cannot be satisfied. There is therefore no solution for $\lambda = 0$.]