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Abstract

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1 Nesterov's Accelerated Gradient

1.1 In preparations

{ass:smooth˙plus˙nonsmooth}

Assumption 1.1 (Smooth add nonsmooth) The function $F = f + g$ where $f : \mathbb{R}^n \to \mathbb{R}$ is a L Lipschitz smooth and $\mu \geq 0$ strongly convex function. The function $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a closed convex proper function.

Definition 1.2 (Proximal gradient operator) *Suppose $F = f + g$ satisfies Assumption 1.1. Let $\beta > 0$. Then, we define the proximal gradient operator T_β as*

$$T_\beta(x|F) = \text{argmin}\, z \left\{ g(z) + f(x) + \langle \nabla f(x), z - x \rangle + \frac{\beta}{2}\|z - x\|^2 \right\}.$$

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Remark 1.3 If the function $g \equiv 0$, then it yields the gradient descent operator $T_\beta(x) = x - \beta^{-1}\nabla f(x)$. In the context where it's clear what the function $F = f + g$ is, we simply write $T_\beta(x)$ for short.

Definition 1.4 (Bregman Divergence) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a differentiable function. Then, for all the Bregman divergence $D_f : \mathbb{R}^n \times \operatorname{dom}\nabla f \to \mathbb{R}$ is defined as:*

$$D_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y\rangle.$$

Remark 1.5 If, f is $\mu \geq 0$ strongy convex and L Lipschitz smooth then, its Bregman
{def:rwapg-seq} Divergence has for all $x, y \in \mathbb{R}^n$: $\mu/2\|x - y\|^2 \leq D_f(x, y) \leq L/2\|x - y\|^2$.

Definition 1.6 (R-WAPG sequence) *Let $(L_k)_{k\geq 0}$ be a sequence such that $L_k > \mu$ for all k. Let $\alpha_0 \in (0, 1]$, $(\alpha_k)_{k\geq 1}$ has $\alpha_k \in (\mu/L_k, 1)$. Then define for all $k \geq 0$:*

$$\rho_k(1 - \alpha_{k+1})\alpha_k^2 = \alpha_{k+1}(\alpha_{k+1} - \mu/L_k).$$

Remark 1.7 When $\rho_k = 1$, the recursive relation between α_k, α_{k-1} is the same as the well known Nesterov's sequence used in algorithm such as FISTA and Nesterov's accelerated gradient. See Li and Wang [2] for more information.

{def:st-method} **Definition 1.8 (similar triangle representation of NAPG)**
Let $(\alpha_k)_{k\geq 0}$ be an R-WAPG sequence. Suppose that the base case $v_{-1}, x_{k-1} \in \mathbb{R}^n$ is given to initialize the algorithm. Then the algorithm produces the sequence of iterates $(y_k, x_k, v_k)_{k\geq 0}$ and auxilary parameter sequence L_k, τ_k satisfying these inequalities:

$$\begin{aligned}
&\tau_k = L_k(1 - \alpha_k)(L_k\alpha_k - \mu)^{-1},\\
&y_k = (1 + \tau_k)^{-1}v_{k-1} + \tau_k(1 + \tau_k)^{-1}x_{k-1},\\
&D_f(x_k, y_k) \leq L_k/2\|x_k - y_k\|^2,\\
&x_k = T_{L_k}(y_k),\\
&v_k = x_{k-1} + \alpha_k^{-1}(x_k - x_{k-1}).
\end{aligned}$$

{thm:pg-ineq} The following theorems are critical in analyzing the behavior of algorithm in Definition 1.8.

Theorem 1.9 (proximal gradient inequality) *Let function F satisfies Assumption 1.1, so it's $\mu \geq 0$ strongly convex. For all $x \in \mathbb{R}^n$, define $x^+ = T_L(x)$, then there exists a $B \geq 0$ such that $D_f(x^+, x) \leq B/2\|x^+ - x\|^2$. Then, for all $z \in \mathbb{R}^n$ it satisfies proximal gradient inequality at point x:*

$$\begin{aligned}
0 &\leq F(z) - F(x^+) - \frac{B}{2}\|z - x^+\|^2 + \frac{B - \mu}{2}\|z - x\|^2\\
&= F(z) - F(x^+) - \langle B(x - x^+), z - x\rangle - \frac{\mu}{2}\|z - x\|^2 - \frac{B}{2}\|x - x^+\|^2.
\end{aligned}$$

Since f is assumed to be L Lispchitz smooth, the above condition is true for all $x, y \in \mathbb{R}^n$ for all $B \geq L$.

Remark 1.10 The theorem is the same as in Nesterov's book [3, Theorem 2.2.13], but with the use of proximal gradient mapping and proximal gradient instead of project gradient hence making it equivalent to the theorem in Beck's book [1, Theorem 10.16]. The only generalization here is parameter B which made to accomodate algorithm that implements Definition 1.8 with some line search routine.

{thm:jesen}

Theorem 1.11 (Jensen's inequality) *Let $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a $\mu \geq 0$ strongly convex function. Then, it is equivalent to the following condition. For all $x, y \in \mathbb{R}^n$, $\lambda \in (0,1)$ it satisfies the inequality*

$$(\forall \lambda \in [0,1])\ F(\lambda x + (1-\lambda)y) \leq \lambda F(x) + (1-\lambda)F(y) - \frac{\mu\lambda(1-\lambda)}{2}\|y - x\|^2.$$

Remark 1.12 If x, y is out of $\operatorname{dom} F$, the inequality still work by convexity.

1.2 A compact argument for the convergence of NAPG with proximal gradient

Here, the abbreviation "NAPG" stands for "Nesterov Acceleration Proximal Gradient". It's made in acknowledgement of Algorithm 2.2.36 in Nesterov's book [3] and its extension known as FISTA in the literature. The following theorem provide a complete proof for the convergence rate of algorithms implementing Definition 1.8, which is equivalent to NAG, or NAPG. It made use of Definition 1.6 which accomodates a relaxed sequence compared to the usual sequence that gives the optimal convergence rate.

{thm:onestep-napg-cnvg}

Theorem 1.13 (one step convergence claim of NAPG) *Let $F = f + g$ satisfies Assumption 1.1 for some $L > \mu \geq 0$. Let the sequence $(\alpha_k)_{k\geq 0}$ be an R-WAPG sequence (Definition 1.6). Suppose that the iterates sequence $(x_k, y_k, v_k)_{k\geq 0}$ satisfy NAPG in similar triangle form (Definition 1.8) with initial guesses $v_{-1}, x_{-1} \in \mathbb{R}^n$. Then for all $k \geq 1$, the following inequality is true for all $\bar{x} \in \mathbb{R}^n$:*

$$\begin{aligned}
&- F(\bar{x}) + F(x_k) + \frac{L_k\alpha_k^2}{2}\|\bar{x} - v_k\|^2 \\
&\leq \max\left(1, \frac{L_k\rho_{k-1}}{L_{k-1}}\right)(1-\alpha_k)\left(F(x_{k-1}) - F(\bar{x}) + \frac{\alpha_{k-1}^2 L_{k-1}}{2}\|\bar{x} - v_{k-1}\|^2\right).
\end{aligned}$$

If in addition, we choose $\alpha_0 = 1$, and let $x_{-1} = v_{-1}$, then a base case of the inequality is:

$$F(x_0) - F(\bar{x}) + \frac{L_0}{2}\|\bar{x} - x_0\|^2 \leq \frac{L_0 - \mu}{2}\|\bar{x} - v_{-1}\|^2.$$

Proof. The proof is very intense algebratically hence before we step into it, we present the following intermediate results in advance to their proofs given at the end.

For all $k \geq 0$, define $z_k = \alpha_k \bar{x} + (1-\alpha_k)x_{k-1}$.

(a) Theorem 1.9, with $z = z_k$, $k \geq 0$. We can use it because $F = f + g$ satisfies Assumption 1.1 .

(b) Jensen's inequality (Theorem 1.11), with $z = z_k$, for $k \geq 0$. We can use it because F is $\mu \geq 0$ strongly convex.

(c) Definition 1.6 which has $\rho_k(1-\alpha_{k+1})\alpha_k^2 = \alpha_{k+1}(\alpha_{k+1} - \mu/L_k)$ for $k \geq 0$.

(d) The equality $z_k - y_k = (L_k - \mu)^{-1}((L_k\alpha_k - \mu)(\bar{x} - v_k) + \mu(1-\alpha_k)(\bar{x} - x_{k-1}))$ for all $k \geq 0$.

(e) The equality $z_k - x_k = \alpha_k(\bar{x} - v_k)$ for all $k \geq 0$.

(f) The following equality:

$$(\forall k \geq 1)\ \frac{1}{2}\left(\frac{\mu^2(1-\alpha_k)^2}{L_k - \mu} - \mu\alpha_k(1-\alpha_k)\right) = \frac{(\alpha_k - 1)\mu(L_k\alpha_k - \mu)}{2(L_k - \mu)}.$$

(g) The following equality:

$$(\forall k \geq 1)\ \frac{1}{2}\left(\frac{(L_k\alpha_k - \mu)^2}{L_k - \mu} - \alpha_{k-1}^2\rho_{k-1}L_k(1-\alpha_k)\right) = \frac{\mu(L_k\alpha_k - \mu)(\alpha_k - 1)}{2(L_k - \mu)}.$$

(h) The equality:

$$(\forall k \geq 1)\ \frac{\mu(L_k\alpha_k - \mu)(\alpha_k - 1)}{2(L_k - \mu)} \leq 0.$$

With intermediate results (a) to (h), presented above, the proof of the theorem come smoothly from a chain of inequalities and equalities. The overall proof now follows. Start with the (a), the proximal gradient inequality it has:

$$\begin{aligned}
0 &\leq F(z_k) - F(x_k) - \frac{L_k}{2}\|z_k - x_k\|^2 + \frac{L_k - \mu}{2}\|z_k - y_k\|^2 \\
&\underset{\text{(b)}}{\leq} \alpha_k F(\bar{x}) + (1-\alpha_k)F(x_{k-1}) - F(x_k) \\
&\quad - \frac{\mu\alpha_k(1-\alpha_k)}{2}\|\bar{x} - x_{k-1}\|^2 - \frac{L_k}{2}\|z_k - x_k\|^2 + \frac{L_k - \mu}{2}\|z_k - y_k\|^2.
\end{aligned}$$

Using the chain of equality below:

$$
\begin{aligned}
& -\frac{\mu\alpha_k(1-\alpha_k)}{2}\|\bar{x}-x_{k-1}\|^2 - \frac{L_k}{2}\|z_k-x_k\|^2 + \frac{L_k-\mu}{2}\|z_k-y_k\|^2 \\
\underset{\text{(d)}}{=} & -\frac{\mu\alpha_k(1-\alpha_k)}{2}\|\bar{x}-x_{k-1}\|^2 - \frac{L_k}{2}\|z_k-x_k\|^2 \\
& + \frac{L_k-\mu}{2}\left\|\frac{L_k\alpha_k-\mu}{L_k-\mu}(\bar{x}-v_{k-1}) + \frac{\mu(1-\alpha_k)}{L_k-\mu}(\bar{x}-x_{k-1})\right\|^2 \\
= & -\frac{\mu\alpha_k(1-\alpha_k)}{2}\|\bar{x}-x_{k-1}\|^2 - \frac{L_k}{2}\|z_k-x_k\|^2 \\
& + \frac{(L_k\alpha_k-\mu)^2}{2(L_k-\mu)}\|\bar{x}-v_{k-1}\|^2 + \frac{\mu^2(1-\alpha_k)^2}{2(L_k-\mu)}\|\bar{x}-x_{k-1}\|^2 + \frac{(L_k\alpha_k-\mu)\mu(1-\alpha_k)}{L_k-\mu}\langle\bar{x}-x_{k-1},\bar{x}-v_{k-1}\rangle \\
= & \left(\frac{\mu^2(1-\alpha_k)^2}{2(L_k-\mu)} - \frac{\mu\alpha_k(1-\alpha_k)}{2}\right)\|\bar{x}-x_{k-1}\|^2 + \left(\frac{(L_k\alpha_k-\mu)^2}{2(L_k-\mu)} - \frac{\alpha_{k-1}^2L_k\rho_{k-1}(1-\alpha_k)}{2}\right)\|\bar{x}-v_{k-1}\|^2 \\
& + \frac{\alpha_{k-1}^2L_k\rho_{k-1}(1-\alpha_k)}{2}\|\bar{x}-v_{k-1}\|^2 - \frac{L_k}{2}\|z_k-x_k\|^2 + \frac{(L_k\alpha_k-\mu)\mu(1-\alpha_k)}{L_k-\mu}\langle\bar{x}-x_{k-1},\bar{x}-v_{k-1}\rangle \\
\underset{\text{(f)}}{=} & \frac{(\alpha_k-1)\mu(L_k\alpha_k-\mu)}{2(L_k-\mu)}\|\bar{x}-x_{k-1}\|^2 + \left(\frac{(L_k\alpha_k-\mu)^2}{2(L_k-\mu)} - \frac{\alpha_{k-1}^2L_k\rho_{k-1}(1-\alpha_k)}{2}\right)\|\bar{x}-v_{k-1}\|^2 \\
& + \frac{\alpha_{k-1}^2L_k\rho_{k-1}(1-\alpha_k)}{2}\|\bar{x}-v_{k-1}\|^2 - \frac{L_k}{2}\|z_k-x_k\|^2 + \frac{(L_k\alpha_k-\mu)\mu(1-\alpha_k)}{L_k-\mu}\langle\bar{x}-x_{k-1},\bar{x}-v_{k-1}\rangle \\
\underset{\text{(g)}}{=} & \frac{(\alpha_k-1)\mu(L_k\alpha_k-\mu)}{2(L_k-\mu)}\|\bar{x}-x_{k-1}\|^2 + \frac{\mu(L_k\alpha_k-\mu)(\alpha_k-1)}{2(L_k-\mu)}\|\bar{x}-v_{k-1}\|^2 \\
& + \frac{\alpha_{k-1}^2L_k\rho_{k-1}(1-\alpha_k)}{2}\|\bar{x}-v_{k-1}\|^2 - \frac{L_k}{2}\|z_k-x_k\|^2 + \frac{(L_k\alpha_k-\mu)\mu(1-\alpha_k)}{L_k-\mu}\langle\bar{x}-x_{k-1},\bar{x}-v_{k-1}\rangle \\
= & \frac{\alpha_{k-1}^2L_k\rho_{k-1}(1-\alpha_k)}{2}\|\bar{x}-v_{k-1}\|^2 - \frac{L_k}{2}\|z_k-x_k\|^2 \\
& \frac{(\alpha_k-1)\mu(L_k\alpha_k-\mu)}{2(L_k-\mu)}\left(\|\bar{x}-x_{k-1}\|^2 + \|\bar{x}-v_{k-1}\|^2 - 2\langle\bar{x}-x_{k-1},\bar{x}-v_{k-1}\rangle\right) \\
= & \frac{\alpha_{k-1}^2L_k\rho_{k-1}(1-\alpha_k)}{2}\|\bar{x}-v_{k-1}\|^2 - \frac{L_k}{2}\|z_k-x_k\|^2 + \frac{(\alpha_k-1)\mu(L_k\alpha_k-\mu)}{2(L_k-\mu)}\|x_{k-1}-v_{k-1}\|^2.
\end{aligned}
$$

The inequality from previously simplifies and it has:

$$
\begin{aligned}
0 \le\; & \alpha_kF(\bar{x}) + (1-\alpha_k)F(x_{k-1}) - F(x_k) + \frac{\alpha_{k-1}^2L_k\rho_{k-1}(1-\alpha_k)}{2}\|\bar{x}-v_{k-1}\|^2 - \frac{L_k}{2}\|z_k-x_k\|^2 \\
& + \frac{(\alpha_k-1)\mu(L_k\alpha_k-\mu)}{2(L_k-\mu)}\|x_{k-1}-v_{k-1}\|^2 \\
\underset{\text{(h)}}{\le}\; & \alpha_kF(\bar{x}) + (1-\alpha_k)F(x_{k-1}) - F(x_k) \\
& + \frac{\alpha_{k-1}^2L_k\rho_{k-1}(1-\alpha_k)}{2}\|\bar{x}-v_{k-1}\|^2 - \frac{L_k}{2}\|z_k-x_k\|^2 \\
=\; & (1-\alpha_k)(F(x_{k-1}) - F(\bar{x})) + F(\bar{x}) - F(x_k) \\
& + \frac{\alpha_{k-1}^2L_k\rho_{k-1}(1-\alpha_k)}{2}\|\bar{x}-v_{k-1}\|^2 - \frac{L_k}{2}\|z_k-x_k\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\underset{\text{(e)}}{=} (1-\alpha_k)\left(F(x_{k-1}) - F(\bar{x}) + \frac{L_k\rho_{k-1}}{L_{k-1}}\frac{\alpha_{k-1}^2 L_{k-1}}{2}\|\bar{x} - v_{k-1}\|^2\right) \\
&\quad + F(\bar{x}) - F(x_k) - \frac{L_k\alpha_k^2}{2}\|\bar{x} - v_k\|^2 \\
&\le (1-\alpha_k)\left(F(x_{k-1}) - F(\bar{x}) + \max\left(1, \frac{L_k\rho_{k-1}}{L_{k-1}}\right)\frac{\alpha_{k-1}^2 L_{k-1}}{2}\|\bar{x} - v_{k-1}\|^2\right) \\
&\quad + F(\bar{x}) - F(x_k) - \frac{L_k\alpha_k^2}{2}\|\bar{x} - v_k\|^2 \\
&\le (1-\alpha_k)\left(\max\left(1, \frac{L_k\rho_{k-1}}{L_{k-1}}\right)(F(x_{k-1}) - F(\bar{x})) + \max\left(1, \frac{L_k\rho_{k-1}}{L_{k-1}}\right)\frac{\alpha_{k-1}^2 L_{k-1}}{2}\|\bar{x} - v_{k-1}\|^2\right) \\
&\quad + F(\bar{x}) - F(x_k) - \frac{L_k\alpha_k^2}{2}\|\bar{x} - v_k\|^2 \\
&= \max\left(1, \frac{L_k\rho_{k-1}}{L_{k-1}}\right)(1-\alpha_k)\left(F(x_{k-1}) - F(\bar{x}) + \frac{\alpha_{k-1}^2 L_{k-1}}{2}\|\bar{x} - v_{k-1}\|^2\right) \\
&\quad + F(\bar{x}) - F(x_k) - \frac{L_k\alpha_k^2}{2}\|\bar{x} - v_k\|^2.
\end{aligned}
$$

Finally for the base case, when $\alpha_0 = 1$, it has $y_0 = v_{-1} = x_{-1}$, and it makes $z_0 = \bar{x}$ therefore this makes the proximal gradient inequality into:

$$
\begin{aligned}
0 &\le F(z_0) - F(x_0) - \frac{L_0}{2}\|z_0 - x_0\|^2 + \frac{L_0 - \mu}{2}\|z_0 - y_0\|^2 \\
&= F(\bar{x}) - F(x_0) - \frac{L_0}{2}\|\bar{x} - x_0\|^2 + \frac{L_0 - \mu}{2}\|\bar{x} - v_{-1}\|^2.
\end{aligned}
$$

Going back to prove the intermediate results, the following will be useful. From Definition 1.8 it has for all $k \ge 0$

{thm:onestep-napg-cnvg-i}
$$
\tau_k = L_k(1-\alpha_k)(L_k\alpha_k - \mu)^{-1}. \tag{i}
$$

Then it has:

{thm:onestep-napg-cnvg-j}
$$
(1+\tau_k)^{-1} \underset{\text{(i)}}{=} \left(1 + \frac{L_k(1-\alpha_k)}{L_k\alpha_k - \mu}\right)^{-1} = \left(\frac{L_k\alpha_k - \mu + L_k(1-\alpha_k)}{L_k\alpha_k - \mu}\right)^{-1} = \frac{L_k\alpha_k - \mu}{L_k - \mu}. \tag{j}
$$

And also

{thm:onestep-napg-cnvg-k}
$$
\tau_k(1+\tau_k)^{-1} \underset{\text{(i),(j)}}{=} \frac{L_k(1-\alpha_k)}{L_k\alpha_k - \mu}\frac{L_k\alpha_k - \mu}{L_k - \mu} = \frac{L_k(1-\alpha_k)}{L_k - \mu}. \tag{k}
$$

Proof of (d) For all $k \ge 1$, from Definition 1.8 it has

$$
\begin{aligned}
0 &= (1+\tau_k)^{-1}v_{k-1} + \tau_k(1+\tau_k)^{-1}x_{k-1} - y_k \\
&\underset{(k)}{=} (1+\tau_k)^{-1}v_{k-1} + \frac{L_k(1-\alpha_k)}{L_k-\mu}x_{k-1} - y_k \\
&= (1+\tau_k)^{-1}v_{k-1} + (1-\alpha_k)x_{k-1} \\
&\quad + \left(\frac{L_k(1-\alpha_k)}{L_k-\mu} - (1-\alpha_k)\right)x_{k-1} - y_k \\
&= (1+\tau_k)^{-1}v_{k-1} + (1-\alpha_k)x_{k-1} \\
&\quad + (1-\alpha_k)\left(\frac{L_k}{L_k-\mu} - 1\right)x_{k-1} - y_k \\
&= (1+\tau_k)^{-1}v_{k-1} + (1-\alpha_k)x_{k-1} + \frac{\mu(1-\alpha_k)}{L-\mu}x_{k-1} - y_k
\end{aligned}
$$

$$
\iff (1-\alpha_k)x_{k-1} - y_k = -(1+\tau_k)^{-1}v_{k-1} - \frac{\mu(1-\alpha_k)}{L_k-\mu}x_{k-1}.
$$

Recall the definition for z_k at the start of the proof and, use the above results it yields:

$$
\begin{aligned}
z_k - y_k &= \alpha_k\bar{x} + (1-\alpha_k)x_{k-1} - y_k \\
&= \alpha_k\bar{x} - (1+\tau_k)^{-1}v_{k-1} - \frac{\mu(1-\alpha_k)}{L_k-\mu}x_{k-1} \\
&\underset{(j)}{=} \alpha_k\bar{x} - \frac{L_k\alpha_k-\mu}{L_k-\mu}v_{k-1} - \frac{\mu(1-\alpha_k)}{L_k-\mu}x_{k-1} \\
&= \frac{L_k\alpha_k-\mu}{L_k-\mu}(\bar{x}-v_{k-1}) + \left(\alpha_k - \frac{L_k\alpha_k-\mu}{L_k-\mu}\right)\bar{x} - \frac{\mu(1-\alpha_k)}{L_k-\mu}x_{k-1} \\
&= \frac{L_k\alpha_k-\mu}{L_k-\mu}(\bar{x}-v_{k-1}) + \frac{\alpha_kL_k - \alpha_k\mu - L_k\alpha_k + \mu}{L_k-\mu}\bar{x} - \frac{\mu(1-\alpha_k)}{L_k-\mu}x_{k-1} \\
&= \frac{L_k\alpha_k-\mu}{L_k-\mu}(\bar{x}-v_{k-1}) + \frac{\mu(1-\alpha_k)}{L_k-\mu}(\bar{x}-x_{k-1}).
\end{aligned}
$$

Proof of (e). The proof is direct using the equality with x_k in Definition 1.8.

$$
\begin{aligned}
z_k - x_k &= \alpha_k\bar{x} + (1-\alpha_k)x_{k-1} - x_k \\
&= \alpha_k\bar{x} + x_{k1} - x_k - \alpha_kx_{k-1} \\
&= \alpha_k(\bar{x} - \alpha_k^{-1}(x_k - x_{k-1}) - x_{k-1}) \\
&= \alpha_k(\bar{x} - v_k).
\end{aligned}
$$

Proof of (f). The proof is direct and it has:

$$\begin{aligned}\frac{\mu^2(1-\alpha_k)^2}{2(L_k-\mu)} - \frac{\mu\alpha_k(1-\alpha_k)}{2} &= \frac{1}{2(L_k-\mu)}\left(\mu^2(1-\alpha_k)^2 - (L_k-\mu)\mu\alpha_k(1-\alpha_k)\right) \\ &= \frac{1-\alpha_k}{2(L_k-\mu)}\left(\mu^2 - \mu^2\alpha_k - (L_k\mu\alpha_k - \mu^2\alpha_k)\right) \\ &= \frac{1-\alpha_k}{2(L_k-\mu)}\left(\mu^2 - L_k\mu\alpha_k\right) \\ &= \frac{(1-\alpha_k)\mu(\mu - L_k\alpha_k)}{2(L_k-\mu)} \\ &= \frac{(\alpha_k-1)\mu(L_k\alpha_k-\mu)}{2(L_k-\mu)}.\end{aligned}$$

Proof of (g) The proof is direct:

$$\begin{aligned}\frac{(L_k\alpha_k-\mu)^2}{2(L_k-\mu)} - \frac{\alpha_{k-1}^2 L\rho_{k-1}(1-\alpha_k)}{2} &\underset{\text{(c)}}{=} \frac{(L\alpha_k-\mu)^2}{2(L_k-\mu)} - \frac{L_k\alpha_k(\alpha_k-\mu/L_k)}{2} \\ &= \frac{1}{2(L_k-\mu)}\left((L_k\alpha_k-\mu)^2 - (L_k-\mu)L_k\alpha_k(\alpha_k-\mu/L_k)\right) \\ &= \frac{1}{2(L_k-\mu)}\left((L_k\alpha_k-\mu)^2 - (L_k-\mu)\alpha_k(L_k\alpha_k-\mu)\right) \\ &= \frac{L_k\alpha_k-\mu}{2(L_k-\mu)}\left(L_k\alpha_k - \mu - (L-\mu)\alpha_k\right) \\ &= \frac{L_k\alpha_k-\mu}{2(L_k-\mu)}\left(\mu\alpha_k-\mu\right) \\ &= \frac{(L\alpha_k-\mu)\mu(\alpha_k-1)}{2(L_k-\mu)}.\end{aligned}$$

Proof of (h). For all $k \geq 1$, by (c), the definition of the R-WAPG sequence, $\alpha_k \in (\mu/L_k, 1)$, then it has $L_k\alpha_k \in (\mu, L_k)$, so $L_k\alpha_k - \mu > 0$, and $\alpha_k - 1 < 0$. Finally, we have $L_k \geq \mu$, therefore, the fraction is negative. ■

1.3 stochastic accelerated proximal gradient

The following assumption about the objective function is fundamental in incremental gradient method for Machine Learning, data science other similar tasks.

{ass:sum-of-many}

Assumption 1.14 (sum of many) Define $F := g + (1/n)\sum_{i=1}^{n} f_i$, assume that $f_i : \mathbb{R}^n \to \mathbb{R}$ are all K_i smooth and $\mu^{(i)} \geq 0$ strongly convex function and $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a closed convex proper function. Consequently, the function f cane be written as $F = g + f$ where $f = (1/n)\sum_{i=1}^n f_i$ and it satisfies Assumption 1.1 with $L = \max_{i=1,\ldots,n} K_i$ and $\mu = (1/n)\sum_{i=1}^n \mu^{(i)}$.

{def:snapg} **Definition 1.15 (SNAPG)**

References

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