Lorum ipsum

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Abstract

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2010 Mathematics Subject Classification: Primary 47H05, 52A41, 90C25; Secondary 15A09, 26A51, 26B25, 26E60, 47H09, 47A63. **Keywords:**

```julia
"""
A doubly stochastic chain sampler that uses wrapped guassian distributions on both directions in 2D
with a fixed variance.
"""
function wrapped_gaussian_sampler_2d(
    state::Vector{T},
    lower_left::Tuple{Real, Real},
    upper_right::Tuple{Real, Real};
    sigma::Real=1
) where {T<:Real}
    lower = [lower_left[1], lower_left[2]]
    upper = [upper_right[1], upper_right[2]]
    function loop_back(x, l, u)  # assert periodic conditions on the rectangle.
        return mod(x, u - l) + l
    end
    N = Normal(0, sigma)
    return loop_back.(state + rand(N, 2), lower, upper)
end
```

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1 Schur's Cone and Its Polar

Example 1.1 Define

$$Q = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{k} x_i \geq 0,\ k = 1, \cdots, n-1,\ x_1 + \cdots + x_n = 0 \right\}. \tag{1}$$

Then

$$\begin{aligned} Q^\circ &= \{ y \in \mathbb{R}^n | \langle y, x \rangle \geq 0 \ \forall x \in Q \} & (2) \\ &= \{ y \in \mathbb{R}^n | y_1 \geq y_2 \geq \cdots \geq y_n \}. & (3) \end{aligned}$$

Proof. We have $Q = H(\mathbb{R}_+^{n-1})$, where

$$H = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & & \\ & & \ddots & 1 \\ & & & -1 \end{bmatrix}, \tag{4}$$

H is a bi-diagonal matrix with zeros on the diagonal and -1 on its sub-diagonals. Indeed, for $x = (x_1, \cdots, x_{n-1}) \in \mathbb{R}_+^{n-1}$, we have

$$\begin{aligned} y &= Hx & (5) \\ &= \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{bmatrix} x_2 + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -1 \end{bmatrix} x_{n-1} & (6) \\ &= \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ -x_{n-2} + x_{n-1} \\ -x_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}, & (7) \end{aligned}$$

where $y_1 \geq 0, y_2 + y_1 \geq 0, \cdots, y_{n-1} + \cdots + y_1 \geq 0$, and $y_1 + \cdots + y_n = 0$. So $y \in Q$. Hence $H(\mathbb{R}_+^{n-1}) \subseteq Q$. Similarly, one can write for $y \in Q$, $\exists x \in \mathbb{R}_+^{n-1}$ with $y = Hx$. In fact, $x = y_1, x_2 = y_1 + y_2, \cdots, x_{n-1} = y_1 + \cdots + y_{n-1}, -x_{n-1} = y_n$ because $y_1 + \cdots + y_n = 0$. In particular, $x \in \mathbb{R}_+^{n-1}$.

Note that $y \in Q^\circ \Leftrightarrow y^\intercal a_i \geq 0$ for $i = 1, \cdots, n-1$. Where

$$a_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0, \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0, \end{bmatrix}, \cdots, a_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{bmatrix}. \tag{8}$$

Now $y^\intercal a_i \geq 0 \Leftrightarrow y_i - y_{i+1} \geq 0$, for $i = 1, \cdots, n-1$. Hence $Q^\circ = \{y \in \mathbb{R}^n | y_1 \geq y_2 \geq \cdots \geq y_n\}$.

■

2 Lorentz Cone and its Polar

Example 2.1

$$L = \left\{(\xi, t) \in \mathbb{R}^n \;\middle|\; \|\xi\| \leq t, \xi \in \mathbb{R}^{n-1}\right\}. \tag{9}$$

Then the polar cone of L is $L^\circ = \{(x, s) \mid \langle (x, s), (\xi, t) \rangle \geq 0\}\} = L$.

Proof.

Let $(x, s) \in L^\circ$. Then $\langle (x, s), (\xi, t) \rangle \geq 0$, $\forall (\xi, t) \in L$, i. e. $\langle x, \xi \rangle + st \geq 0$.

For $t > 0$, we have $\langle x, \xi/t \rangle \geq -s$, i.e. $\langle x, -\xi/t \rangle \leq s$. Using $\|\xi\| \leq t$ and by taking supremum over $\|\xi/t\| \leq 1$, we obtain $\|x\| \leq s$.

Hence $L^\circ = L$. ■

Corollary 2.2 *Let $A \in \mathbb{R}^{n \times n}$ be positive definite. Define*

$$L_A = \left\{(\xi, t) \in \mathbb{R}^n \;\middle|\; \sqrt{\langle \xi, A\xi \rangle} \leq t\right\}.$$

Then $(L_A)^\circ = \{(\xi, t) \mid \sqrt{\langle \xi, A^{-1}\xi \rangle} \leq t\}$.

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