Chapter 8

Fourier Transform and Its Application to PDE

8.1 Introduction

We shall first practice taking the Fourier transform of some functions before applying it to solving PDEs in infinite domains.

8.2 Fourier transform of some simple functions

Example 1:

Take the Fourier transform of

$$f(x) = e^{-|x|}, \quad -\infty < x < \infty.$$

$$\begin{aligned}
F(\omega) = \mathcal{F}[e^{-|x||}] = \int_{-\infty}^{\infty} e^{-|x|} e^{i\omega x} dx \\
&= \int_{0}^{\infty} e^{-x+i\omega x} dx + \int_{-\infty}^{0} e^{x+i\omega x} dx \\
&= \frac{1}{-(1-i\omega)} e^{-(1-i\omega)x}|_0^\infty + \frac{1}{(1+i\omega)} e^{(1+i\omega)x}|_{-\infty}^0 \\
&= \frac{1}{1-i\omega} + \frac{1}{1+i\omega} = \frac{2}{1+\omega^2}.
\end{aligned}$$

Note its decay as $\omega \to \pm\infty$.

Example 2:

Take the Fourier transform of

$$f(x) = e^x, \quad -\infty < x < \infty.$$

$$\begin{aligned} F(\omega) = \mathcal{F}[e^x] &= \int_{-\infty}^{\infty} e^x e^{i\omega x} dx \\ &= \frac{1}{(1+i\omega)} e^{(1+i\omega)x}|_{-\infty}^{\infty}. \end{aligned}$$

The limit at $x = \infty$ blows up. We say the Fourier transform of e^x does not exist because the function $f(x) = e^x$ is not "integrable", i.e.

$$\int_{-\infty}^{\infty} |f(x)| dx = \int_{-\infty}^{\infty} e^x dx$$

does not have a finite value. [Draw a picture of e^x, and see that the area under the curve is infinitely large.]

Example 3:

Take the Fourier transform of

$$f(x) = e^{-x}, \quad -\infty < x < \infty.$$

This funciton is also not integrable, and so its Fourier transform does not exist. [Show this.]

Example 4:

Take the Fourier transform of

$$f(x) = e^{-x^2}, \quad -\infty < x < \infty.$$

The value of the function decreases rapidly when x is away from $x = 0$ in *both* the positive and negative x directions. There is a finite area under the curve e^{-x^2}, so this function is integrable. To find its Fourier transform, we need to perform the integral:

$$F(\omega) = \int_{-\infty}^{\infty} e^{-x^2} e^{i\omega x} dx.$$

You can either look up a table of integrals or by completing the square in the exponent to get

$$F(\omega) = \sqrt{\pi} e^{-\omega^2/4}.$$

[In case you are curious:

$$-x^2 + i\omega x = -(x - i\omega/2)^2 - \omega^2/4.$$

So $F(\omega) = e^{-\omega^2/4} \int_{-\infty}^{\infty} e^{-(x-i\omega)/2)^2} dx = e^{-\omega^2/4} \int_{-\infty}^{\infty} e^{-y^2} dy$, where we have made a change of variable $y = x - i\omega/2$ and also shifted the path of integration. The remaining integral is a standard one (Euler's integral) and is equal to $\sqrt{\pi}$. In general,

$$\int_{-\infty}^{\infty} e^{-(x+b)^2} dx = \sqrt{\pi}, \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-p^2x^2 \pm qx} dx = \frac{\sqrt{\pi}}{p} e^{\frac{q^2}{4p^2}}, \ p > 0.]$$

Example 5:

Take the inverse transform of

$$F(\omega) = \sqrt{\pi} e^{-\omega^2/4}, \quad -\infty < \omega < \infty.$$

$$\begin{aligned}
\mathcal{F}^{-1}[F(\omega)] &= \frac{\sqrt{\pi}}{2\pi} \int_{-\infty}^{\infty} e^{-\omega^2/4 - i\omega x} d\omega \\
&= \frac{\sqrt{\pi}}{2\pi} e^{-x^2} \int_{-\infty}^{\infty} e^{-(\omega/2 + ix)^2} d\omega \\
&= e^{-x^2}.
\end{aligned}$$

We have thus recovered the original function $f(x)$ in *Example 4.*

Example 6:

Take the inverse transform of

$$F(\omega) = \frac{2}{1+\omega^2}$$

$$\mathcal{F}^{-1}[F(\omega)] = \frac{2}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+\omega^2} e^{-i\omega x} d\omega.$$

This integral can be done very easily if you know complex variables and residue calculus. Otherwise, just rely on Tables of Integrals to tell you that

$$f(x) = e^{-|x|}.$$

We have thus recovered the $f(x)$ in *Example 1.*

8.3 Application to PDEs

The usual difficulty with PDEs is that the solution involves more than one independent variable. The transform method allows us to reduce one independent variable. We commonly try to transform the x-dependence through a Fourier transform, provided that the domain is infinite, i.e. $-\infty < x < \infty$. We sometimes use Laplace transform in t instead of or in addition to the Fourier transform in x, provided that $0 < t < \infty$.

Consider a function $u(x,t)$, with $-\infty < x < \infty$, $t > 0$. Let

$$\boxed{U(\omega,t) = \mathcal{F}[u(x,t)] = \int_{-\infty}^{\infty} u(x,t)e^{i\omega x}dx} \tag{8.1}$$

be the Fourier transform of $u(x,t)$ with respect to x. The original function $u(x,t)$ can then be recovered from the Fourier inverse transform:

$$\boxed{u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega,t)e^{-i\omega x}d\omega}\ . \tag{8.2}$$

[Note that in (8.1) and (8.2) t plays no role; it may be regarded as arbitrary.] This is very similar to our previous method of writing the solution in the form of an eigenfunction expansion when the domain is finite. With (8.2), taking derivatives with respect to x is now very easy:

$$\begin{aligned}
u_x(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (-i\omega)U(\omega,t)e^{-i\omega x}d\omega \\
u_{xx}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (-i\omega)^2U(\omega,t)e^{-i\omega x}d\omega && (8.3) \\
u_t(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} U_t(\omega,t)e^{-i\omega x}d\omega \\
u_{tt}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} U_{tt}(\omega,t)e^{-i\omega x}d\omega, && (8.4)
\end{aligned}$$

provided of course that these integrals exist. At this point, there is no need to worry about these mathematical issues of integrability because we don't even know what $U(\omega,t)$ is yet.

8.4 Examples

8.4.1. The wave equation in an infinite domain

$$\begin{aligned}
&\text{PDE:} \quad u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0 && (8.5)\\
&\text{BCs:} \quad u(x,t) \to 0 \quad \text{as} \quad x \to \pm\infty && (8.6)\\
&\text{ICs:} \quad u(x,0) = f(x), && (8.7)\\
&\qquad\quad u_t(x,0) = 0, \quad -\infty < x < \infty. && (8.8)
\end{aligned}$$

We assume the solution to be of the form of an integral (8.2) which we substitute into the PDE (8.4). This yields, using (8.3),

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} (U_{tt}(\omega,t) + c^2\omega^2 U(\omega,t))e^{-i\omega x} d\omega = 0,$$

which is the same as

$$\mathcal{F}^{-1}[U_{tt} + c^2\omega^2 U] = 0, \tag{8.9}$$

so (by taking $\mathcal{F}$ of (8.9)):

$$U_{tt} + c^2\omega^2 U = 0. \tag{8.10}$$

This is an ODE; the partial derivatives $\frac{\partial^2}{\partial x^2}$ have been converted to $(-i\omega)^2$, an algebraic multiplication. The ODE in t is to be solved subject to the following ICs:

$$u_t(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U_t(\omega,0)e^{-i\omega x} d\omega = 0,$$

and

$$u(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega,0)e^{-i\omega x} d\omega = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x} d\omega.$$

These imply:

$$U_t(\omega,0) = 0 \tag{8.11}$$

and

$$U(\omega,0) = F(\omega), \tag{8.12}$$

where the Fourier transform $F(\omega)$ of $f(x)$ is known if $f(x)$ is known.

The general solution to the ODE (8.10) is

$$U(\omega,t) = A(\omega)\sin(c\omega t) + B(\omega)\cos(c\omega t).$$

The ICs (8.10) and (8.11) can be used to determine the constants A and B to be $B(\omega) = F(\omega)$ and $A(\omega) = 0$. Thus,

$$U(\omega, t) = F(\omega)\cos(c\omega t). \tag{8.13}$$

We recover $u(x,t)$ by subsituting (8.13) back into (8.2).

$$\begin{aligned} u(x,t) = \mathcal{F}^{-1}[U(\omega,t)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega,t)e^{-i\omega x}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\cos(c\omega t)e^{-i\omega x}d\omega. \end{aligned} \tag{8.14}$$

Typically one cannot perform the integral explicitly unless $F(\omega)$ is known. In the particular case of the wave equation however, progress can be made by noting that

$$\cos(c\omega t) = \frac{1}{2}(e^{ic\omega t} + e^{-ic\omega t}),$$

and so (8.14) can be rewritten as

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2}F(\omega)e^{-i\omega(x-ct)}d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2}F(\omega)e^{-i\omega(x+ct)}d\omega \\ &= \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct), \end{aligned} \tag{8.15}$$

since

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}d\omega,$$

so

$$f(x-ct) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega(x-ct)}d\omega$$

and

$$f(x+ct) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega(x+ct)}d\omega.$$

The physical intepretation of the solution (8.15) to the wave equation (8.4) is that an initial displacement of $f(x)$ will split into two shapes for $t > 0$, each with half the amplitude of the original shape, one propagates to the left and one propagates to the right, both with speed c. The quantity c is therefore called the wave speed.

8.4.2. Diffusion equation in an infinite domain:

$$\text{PDE:}\quad u_t = \alpha^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0 \tag{8.16}$$

$$\text{BCs:}\quad u(x,t) \to 0 \quad \text{as} \quad x \to \pm\infty, \tag{8.17}$$

$$\text{IC:}\quad u(x,0) = f(x), \quad -\infty < x < \infty. \tag{8.18}$$

We assume a solution of the form of an integral (8.2) and substitute it into the PDE (8.16). This yields, using (8.3)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(U_t + \alpha^2\omega^2 U)e^{-i\omega x}d\omega = 0,$$

which implies

$$U_t + \alpha^2\omega^2 U = 0. \tag{8.19}$$

The ODE (8.19) is solved subject to the IC

$$U(\omega, 0) = F(\omega), \tag{8.20}$$

which is obtained by taking the Fourier transform of (8.18).

The solution is

$$U(\omega,t) = A(\omega)e^{-\alpha^2\omega^2 t} = F(\omega)e^{-\alpha^2\omega^2 t}. \tag{8.21}$$

The final solution is obtained by substituting (8.21) into (8.2)

$$\boxed{u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-\alpha^2\omega^2 t - i\omega x}d\omega.} \tag{8.22}$$

For the special case of

$$f(x) = ae^{-(x/L)^2}, \quad -\infty < x < \infty,$$

we know from Section 8.2 that

$$F(\omega) = \mathcal{F}[f(x)] = aL\sqrt{\pi}e^{-(L\omega)^2/4}.$$

Then

$$u(x,t) = \frac{aL}{2\pi}\int_{-\infty}^{\infty}\sqrt{\pi}e^{-(L\omega)^2/4-\alpha^2\omega^2 t - i\omega x}d\omega,$$

can be evaluated by completing squares

$$\begin{aligned} u(x,t) &= \frac{aL}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(\alpha^2 t + \frac{L^2}{4})\omega^2 - i\omega x}d\omega \\ &= \frac{aL}{\sqrt{4\alpha^2 t + L^2}}e^{-x^2/(4\alpha^2 t + L^2)}. \end{aligned} \tag{8.23}$$

The physical interpretation of the solution (8.23) is that an initial concentration near $x = 0$ an initial with width of approximately $2L$ spreads out into a wider and wider region while its amplitude at $x = 0$ decreases monotonically to zero. This is a typical behavior of solutions to the diffusion/heat equation. The underlying physical process reduces gradients and spreads any initial concentration/heat to wider regions.

8.5 The "drunken sailor" problem

In Chapter 2, the "drunken sailor" problem was derived. Here we shall solve it using Fourier transform:

$$\text{PDE:} \qquad \frac{\partial}{\partial t}u = D\frac{\partial^2}{\partial x^2}u, \quad -\infty < x < \infty, \ t > 0 \tag{8.24}$$

$$\text{BC:} \qquad u(x,t) = 0 \text{ as } x \to \pm\infty \tag{8.25}$$

$$\text{IC:} \qquad u(x,0) = f(x) = \delta(x), \quad -\infty < x < \infty. \tag{8.26}$$

By taking Fourier transform in x, we found, in (8.22), that the solution can be written as

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-D\omega^2 t - i\omega x}d\omega, \tag{8.27}$$

where $F(\omega)$ is the Fourier transform of the initial distribution $u(x,0) = f(x)$. With $f(x) = \delta(x)$,

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(x)e^{i\omega x}dx \\ &= \int_{-\infty}^{\infty} \delta(x)e^{i\omega x}dx = e^{i\omega 0} = 1. \end{aligned}$$

Thus

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-D\omega^2 t - i\omega x} \\ &= (4\pi Dt)^{-1/2}\exp\{-\frac{x^2}{4Dt}\}. \end{aligned} \tag{8.28}$$

Also, since the delta function can be obtained from the limit

$$\lim_{L\to 0}\frac{1}{\sqrt{\pi}L}e^{-(x/L)^2} = \delta(x)$$

one can obtain the result in (8.28) by taking the limit of $L \to 0$ in (8.23), with $a = \frac{1}{\sqrt{\pi}L}$.

8.6 Laplace transform solution of the same problem (optional)

The "drunken sailor" problem was solved in 8.5 using the Fourier transform in x. The same problem can also be solved using Laplace transform in t. The second approach is more difficult, because the transformed equation is a second-order ordinary differential equation, while the first approach (of using Fourier transform on the independent variable with the highest derivatives) yields a simpler first-order ordinary differential equation

$$\begin{aligned}
\text{PDE:} &\quad \frac{\partial}{\partial t}u = D\frac{\partial^2}{\partial x^2}u, \quad -\infty < x < \infty, \quad 0 < t < \infty \\
\text{BC:} &\quad u(x,t) = 0 \text{ as } x \to \pm\infty \\
\text{IC:} &\quad u(x,0) = f(x) = \delta(x), \quad -\infty < x < \infty.
\end{aligned}$$

Let $\tilde{u}(x,s)$ be the Laplace transform of the solution:

$$\tilde{u}(x,s) = \int_0^\infty e^{-st}u(x,t)dt.$$

Taking the Laplace transform of the PDE yields

$$\mathcal{L}\left\{\frac{\partial}{\partial t}u\right\} = D\tilde{u}_{xx}.$$

Since

$$\begin{aligned}
\mathcal{L}\left\{\frac{\partial}{\partial t}u\right\} &= \int_0^\infty \frac{\partial}{\partial t}ue^{-st}dt = ue^{-st}\Big|_0^\infty + s\int_0^\infty ue^{-st}dt \\
&= s\tilde{u} - u(x,0),
\end{aligned}$$

we have the following ordinary differential equation to solve:

$$\tilde{u}_{xx} - \frac{s}{D}\tilde{u} = -\frac{1}{D}\delta(x).$$

For $x > 0$:

$$\tilde{u}_{xx} - \frac{s}{D}\tilde{u} = 0.$$

The solution which vanishes as $x \to \infty$ is:

$$\tilde{u}(x,s) = A(s)\exp\{-(s/D)^{1/2}x\}, \quad \text{Re } s^{1/2} > 0.$$

For $x < 0$:

$$\tilde{u}_{xx} - \frac{s}{D}\tilde{u} = 0.$$

The solution which vanishes as $x \to -\infty$ is:

$$\tilde{u}(x,s) = B(s)\exp\{(s/D)^{1/2}x\}, \quad \text{Re } s^{1/2} > 0.$$

Assuming that $\tilde{u}$ is continuous at $x = 0$ implies that $A = B$. Another matching condition at $x = 0$ is obtained by integrating the ordinary differential equation across $x = 0$:

$$\int_{0^-}^{0^+} (\tilde{u}_{xx} - \frac{s}{D}\tilde{u})dx = \int_{0^-}^{0^+} -\frac{1}{D}\delta(x)dx.$$

$$\tilde{u}_x|_{0^-}^{0^+} = -\frac{1}{D},$$

since $\int_{0^-}^{0^+}\delta(x)dx = 1$, and $\int_{0^-}^{0^+}\tilde{u}dx = 0$ if $\tilde{u}$ is finite. Since $\tilde{u}_x$ at 0^+ is $-(s/D)^{1/2}A$, and $\tilde{u}_x$ at 0^- is $(s/D)^{1/2}B$ we have

$$2(s/D)^{1/2}A = -\frac{1}{D}.$$

Thus we find:

$$\tilde{u}(x,s) = \frac{1}{2(Ds)^{1/2}}\exp\left\{-(\frac{s}{D})^{1/2}|x|\right\}.$$

$u(x,t)$ is then found via the inverse Laplace transform:

$$u(x,t) = \frac{1}{2\pi i}\int_L \frac{1}{2(Ds)^{1/2}}\exp\{st - (s/D)^{1/2}|x|\}ds,$$

where L is the vertical line in the complex-s plane to the right of all singularities. There is a branch point at $s = 0$. The branch with Re $s^{1/2} > 0$ is defined by $s = re^{i\theta}$, $s^{1/2} = r^{1/2}e^{i\theta/2}$, and so $\cos(\theta/2) > 0$. Thus $\theta/2$ should be between $-\pi/2$ and $\pi/2$, and so

$$-\pi < \theta < \pi.$$

This implies a branch cut from $s = 0$ along the negative real-s axis. A closed contour C can be constructed as

$$C = L + C_R + C_2 + C_3 + C_4.$$

There is no singularity inside C and so $\oint_C ds = 0$. $\int_{C_R} ds \to 0$ by Jordan's lemma as the radius of the semicircle goes to infinity. The integral over the small circle C_3 vanishes as its radius goes to zero. On the upper horizontal line C_2, $s = re^{i\pi^-}$, and so

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_2} &= \frac{1}{2\pi i}\int_{\infty}^{0} e^{-rt-i(r/D)^{1/2}|x|}\frac{e^{i\pi}dr}{2i(Dr)^{1/2}}\\ &= -\frac{1}{4\pi D^{1/2}}\int_0^\infty \frac{e^{-rt-i(r/D)^{1/2}|x|}}{r^{1/2}}dr.\end{aligned}$$

On the lower horizontal line C_4, $s = re^{-i\pi^+}$, and so:

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_4} &= \frac{1}{2\pi i}\int_{0}^{\infty} e^{-rt+i(r/D)^{1/2}|x|}\frac{e^{-i\pi}dr}{-2i(Dr)^{1/2}}\\ &= -\frac{1}{4\pi D^{1/2}}\int_0^\infty \frac{e^{-rt+i(r/D)^{1/2}|x|}}{r^{1/2}}dr.\end{aligned}$$

Since

$$\begin{aligned}u(x,t) &= \frac{1}{2\pi i}\int_L = \frac{1}{2\pi i}\left\{\oint_C - \int_{C_R} - \int_{C_2} - \int_{C_3} - \int_{C_4}\right\}\\ &= -\frac{1}{2\pi i}\left\{\int_{C_2} + \int_{C_4}\right\}\\ &= \frac{1}{4\pi D^{1/2}}\int_0^\infty \frac{dr}{r^{1/2}}e^{-rt}\left\{e^{i(r/D)^{1/2}|x|} + e^{-i(r/D)^{1/2}|x|}\right\}\end{aligned}$$

By setting $y = r^{1/2}$, this becomes

$$\begin{aligned}u(x,t) &= \frac{1}{4\pi D^{1/2}}\int_0^\infty 2dye^{-y^2t}\{e^{i(|x|/D^{1/2})y} + e^{-i(|x|/D^{1/2})y}\}\\ &= \frac{1}{2\pi D^{1/2}}\int_{-\infty}^{\infty} e^{-ty^2+i\frac{|x|}{D^{1/2}}y}dy\\ &= \frac{1}{(4\pi Dt)^{1/2}}\exp\{-\frac{x^2}{4Dt}\}.\end{aligned}$$

This is the same as previously found using Fourier transform in x.

8.7 Wave equation in 3-D (optional)

$$\text{PDE:} \frac{\partial^2}{\partial t^2}u = c^2\nabla^2 u,\ \nabla^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$
$$\text{BC:} u \to 0 \text{ as } x^2 + y^2 + z^2 \to \infty$$
$$\text{IC:} u(\mathbf{x}, t) = u_0(r)$$
$$u_t(\mathbf{x}, t) = 0$$

where $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. The initial u is assumed to have radial symmetry about the origin, and hence is a function of r only.

We apply Fourier transform to each space dimension by letting

$$\begin{aligned} U(\boldsymbol{\lambda}, t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(\mathbf{x}, t)e^{i(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)}dx_1 dx_2 dx_3 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(\mathbf{x}, t)e^{i\boldsymbol{\lambda}\cdot\mathbf{x}}d\mathbf{x}. \end{aligned}$$

It is understood that

$$d\mathbf{x} = dx_1 dx_2 dx_3, \mathbf{x} = (x_1, x_2, x_3), \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3).$$

If we take the 3-D transform of the PDE we will get

$$\frac{\partial^2}{\partial t^2}U = -c^2\lambda^2 U,$$

where $\lambda^2 \equiv \lambda_1^2 + \lambda_2^2 + \lambda_3^2$. The solution to the ODE is

$$U(\boldsymbol{\lambda}, t) = A(\boldsymbol{\lambda})\cos c\lambda t + B(\boldsymbol{\lambda})\sin c\lambda t.$$

Applying the IC, we find $B(\boldsymbol{\lambda}) = 0$ and $A(\boldsymbol{\lambda}) = U(\boldsymbol{\lambda}, 0)$, where

$$\begin{aligned} U(\boldsymbol{\lambda}, 0) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(r)e^{i\boldsymbol{\lambda}\cdot\mathbf{x}}d\mathbf{x} \\ &= \int_0^{\infty} dr \int_0^{\pi} r^2 \sin\theta d\theta \int_0^{2\pi} d\varphi u_0(r)e^{i\lambda r\cos\theta} \end{aligned}$$

in spherical coordinates. [We have oriented the coordinate systems so that θ is the angle the vector $\mathbf{x}$ makes relative to a (fixed) vector $\boldsymbol{\lambda}$.]

$$\begin{aligned} U(\boldsymbol{\lambda}, 0) &= 2\pi\int_0^{\infty} drr^2 u_0(r)\int_0^{\pi} d(-\cos\theta)e^{i\lambda r\cos\theta} \\ &= 2\pi\int_0^{\infty} drr^2 u_0(r)(e^{i\lambda r\cos\theta}/(-i\lambda r))\Big|_0^{\pi} \\ &= 4\pi\int_0^{\infty} u_0(r)\frac{\sin\lambda r}{\lambda}rdr \equiv U_0(\lambda), \end{aligned}$$

which is a function of the magnitude of $\boldsymbol{\lambda}$ only. Thus

$$U(\boldsymbol{\lambda}, t) = U_0(\lambda)\cos c\lambda t.$$

The inverse Fourier transform is

$$\begin{aligned} u(\mathbf{x}, t) &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_0(\lambda)\cos c\lambda t e^{-i\boldsymbol{\lambda}\cdot\mathbf{x}} d\boldsymbol{\lambda} \\ &= \frac{1}{(2\pi)^3}\int_0^{\infty} d\lambda \int_0^{\pi} 2\pi \sin\theta \lambda^2 U_0(\lambda)\cos(c\lambda t) e^{-i\lambda r\cos\theta} d\theta \\ &= \frac{2}{(2\pi)^2}\int_0^{\infty} \lambda d\lambda U_0(\lambda)\cos(c\lambda t)\sin\lambda r/r \end{aligned}$$

Since $\sin\lambda r(\cos c\lambda t = \frac{1}{2}\sin\lambda(r-ct) + \frac{1}{2}\sin\lambda(r+ct)$ and

$$\begin{aligned} u(\mathbf{x}, 0) = u_0(r) &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_0(\lambda) e^{-i\boldsymbol{\lambda}\cdot\mathbf{x}} d\boldsymbol{\lambda} \\ &= \frac{2}{(2\pi)^2}\int_0^{\infty} \lambda d\lambda U_0(\lambda)\sin\lambda r/r, \end{aligned}$$

we have

$$\begin{aligned} ru(\mathbf{x}, t) &= \frac{1}{(2\pi)^2}\int_0^{\infty} \lambda d\lambda U_0(\lambda)\sin\lambda(r-ct) \\ &\quad + \frac{1}{(2\pi)^2}\int_0^{\infty} \lambda d\lambda U_0(\lambda)\sin\lambda(r+ct) \\ &= \frac{1}{2}(r-ct)u_0(r-ct) + \frac{1}{2}(r+ct)u_0(r+ct). \end{aligned}$$