

Chapter 2

Physical Origins of Some PDEs

2.1 Introduction

In physical applications, PDEs are more ubiquitous than ODEs. This situation can be understood because physical quantities more often depend on space and time than on, say, time alone. A partial differential equation relates the variations of this physical quantity in time and in space. Of course, in mathematical abstraction, one does not need to assign the physical meaning of time to the symbol t , or space to the symbol x ; one is simply concerned with the variations of the unknown with respect to more than one independent variable as governed by a PDE.

We will be dealing with first and second order PDEs in this course. In this chapter we will discuss the physical origin of these equations. This chapter is suitable for assigned casual reading.

2.2 Conservation Laws:

Many physical laws can be expressed as a *conservation law* of the form

$$u_t + q_x = 0. \tag{2.1}$$

Here $u(x, t)$ is the “concentration” of something under consideration and q is its “flux” in the x -direction. If the quantity under consideration is nonconservative, we need to add a “source or sink” term to the right-hand side of (2.1). We will provide physical examples of such terms in a moment.

In more than one space dimensions, q would be a vector \mathbf{q} , and (2.1) would be replaced by

$$u_t + \nabla \cdot \mathbf{q} = 0$$

for the gradient vector ∇ . We will however not be concerned with more than one space dimension here; so you will not need to know vectors or divergence of a vector ($\nabla \cdot \mathbf{q}$).

If q in (2.1) is a function of u only and does not depend on its derivatives, then it can be written as

$$u_t + a(u)u_x = 0, \quad \text{where } a(u) \equiv \frac{dq}{du}. \quad (2.2)$$

(2.2) is a first order PDE. On the other hand, if the flux q depends on u_x , as is often the case for “down-gradient” fluxes, e.g.

$$q = -ku_x, \quad \text{for a constant } k,$$

(2.1) will become

$$u_t - ku_{xx} = 0, \quad (2.3)$$

which is a second-order PDE.

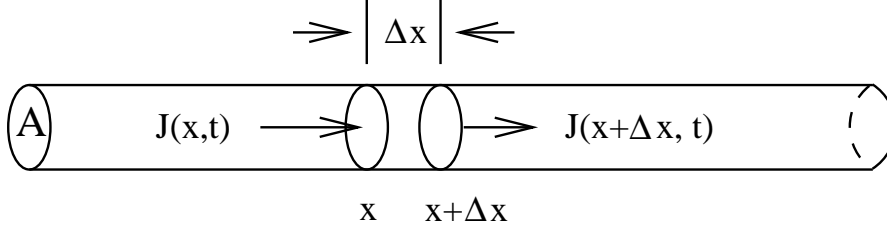
2.2.1 Diffusion of a tracer:

Let c be the concentration of a substance under consideration (in gm/volume). Let ρ be the density of the medium into which it is diffusing. For example, for the problem of diffusion of salt in water, c is the weight of salt per unit volume, and ρ is the weight of water per unit volume. Let $u = c/\rho$ be called the mass ratio of salt to water.

We now consider diffusion in one dimension, x . This can approximate the situation of diffusion in a thin tube of constant cross-sectional area A , with its axis oriented in the x -direction. The tube is filled with water and some salt is put into this tube initially. Since the tube is thin, we can ignore the diffusion of salt radially, assuming the salt quickly manages to diffuse radially to attain the same concentration throughout the cross-section. Its concentration varies mainly with x .

Consider the amount of salt between the sections x and $x + \Delta x$, a small distance apart. If Δx is so small that the variation of c with x can be ignored, that amount of salt contained is $cA\Delta x = \rho A\Delta x \cdot u$, since $A\Delta x$ is the volume of the section under consideration. Since the mass of salt is conserved, we can state that the time rate of change of salt in this volume,

$$\frac{\partial}{\partial t} \rho A \Delta x u,$$

Figure 2.1: Diffusion in a tube of cross-sectional area A .

is equal to the flux of salt into the volume at x , minus the flux of salt out of the volume at $x + \Delta x$, i.e.

$$[J(x, t)A - J(x + \Delta x, t)A],$$

where $J(x, t)$ is the flux at x . It is defined as the time rate of salt flowing across x per unit area.

Thus the equation for the conservation of salt is:

$$\rho A \Delta x \frac{\partial}{\partial t} u = A [J(x, t) - J(x + \Delta x, t)]. \quad (2.4)$$

Dividing both sides by $A \Delta x$, we get

$$\rho \frac{\partial}{\partial t} u = - \frac{J(x + \Delta x, t) - J(x, t)}{\Delta x}. \quad (2.5)$$

We now take the limit as $\Delta x \rightarrow 0$, since the smaller Δx is, the better our previous approximation of assuming c to be constant between x and Δx is. The right-hand side of Equation (2.5) becomes,

$$- \lim_{\Delta x \rightarrow 0} \frac{J(x + \Delta x, t) - J(x, t)}{\Delta x} = - \frac{\partial J}{\partial x}(x, t).$$

Equation (2.5) is now in the form of a “conservation law”, Equation (2.1), if we write $q \equiv J/\rho$:

$$\frac{\partial}{\partial t} u = - \frac{\partial}{\partial x} q. \quad (2.6)$$

To relate the flux q to u , we use *Fick's law of diffusion*, which is obtained from experimental descriptions. It says:

The flux (of salt) is proportional to the negative gradient (of salt) (since salt always diffuses from a high concentration region to a low concentration region). Mathematically, we write this law as

$$J(x, t) = -k \frac{\partial}{\partial x} c = -k \rho \frac{\partial}{\partial x} u, \quad (2.7)$$

where the proportionality constant, k , is the coefficient of diffusivity for salt. Equation (2.6) finally becomes:

$$\boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}}. \quad (2.8)$$

It is a second-order PDE, because its highest derivative, $\frac{\partial^2 u}{\partial x^2}$, is second order. Equation (2.8) is called the *diffusion equation*. With a different value for k it can be applied to the problem of diffusion of pollutants in air (with ρ being the density of air and c the concentration of the pollutant). In biology and ecology, c could be the population of a species.

2.2.2 Advection of a tracer:

The diffusion of a tracer, say salt in water discussed in 2.2.1, is a macroscopic result of small-scale random molecular motion of salt and water molecules. This is a slow process. Advection, on the other hand, is a faster process if it occurs. If a pollutant of concentration c is put into a river whose water is flowing with speed V , the flux of that pollutant, defined as the time rate of the pollutant flowing across x per unit area, is

$$J(x, t) = V \cdot c, \quad (2.9)$$

because the pollutant is carried by the water with speed V across x . Letting $u = c/\rho$, and $q = J/\rho$, we have, from (2.6)

$$\frac{\partial}{\partial t} u = - \frac{\partial}{\partial x} q$$

or

$$\boxed{\frac{\partial}{\partial t} u + V \frac{\partial}{\partial x} u = 0}. \quad (2.10)$$

Equation (2.10) is a first-order PDE. It describes the advection of a tracer by a medium with velocity V .

If the time scales of advection and diffusion are comparable, we should include them both. In that case, the flux becomes

$$J(x, t) = V \cdot c - k \frac{\partial}{\partial x} c, \quad (2.11)$$

and the governing PDE becomes:

$$\boxed{\frac{\partial}{\partial t} u + V \frac{\partial}{\partial x} u = k \frac{\partial^2}{\partial x^2} u}. \quad (2.12)$$

Equation (2.9) is called the *advection-diffusion equation*.

2.2.3 Nonlinear advection:

The advection equation, (2.10), is *linear* if V does not depend on the unknown u . Otherwise it is *nonlinear*. A linear PDE is one where the unknown and its derivatives appear linearly, i.e. not multiplied by itself or its partial derivatives.

Nonlinear advection arises if the quantity u , whose conservation is being considered, is also related to the advecting velocity V . This is the case, for example, when we are considering the conservation of momentum ρu of the fluid, where ρ is the density of the fluid and u is the fluid's velocity in the x -direction. Since the momentum is advected by the fluid velocity u , the conservation law will look something like:

$$\rho \left[\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u \right] = \text{the forces acting.}$$

The simplest such equation is the nonlinear advection equation:

$$\boxed{\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u = 0} . \quad (2.13)$$

If nonconservative forces, such as molecular diffusion (so-called viscosity) is included, one obtains the nonlinear counterpart to (2.12):

$$\boxed{\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u = \nu \frac{\partial^2}{\partial x^2} u} . \quad (2.14)$$

Equation (2.14) is the famous *Burgers' equation*. Burgers' equation has been studied by mathematicians as a prototype model of the balance between nonlinear advection and viscosity in fluid flows (where ν is the coefficient of viscosity), although the equations governing real fluid flows are more complicated. [Burgers' equation does not include, among other things, the pressure forces.]

2.2.4 Heat conduction in a rod:

Consider a long cylindrical rod, insulated at the curved sides so that heat can flow in or out only through the ends. The rod is thin, so that the radial variation of temperature u can be ignored. We consider the variation of u with respect to x , measured along the axis of the cylinder, and time t , i.e.

$$u = u(x, t)$$

The time rate of change of heat energy in a small volume $A\Delta x$, situated between sections x and $x + \Delta x$, is

$$c\rho A\Delta x \frac{\partial u}{\partial t},$$

where c is the specific heat and ρ the density of the material of the rod (e.g. copper). This should be equal to the net flux of heat into the volume, i.e.

$$c\rho A\Delta x \frac{\partial u}{\partial t} = J(x, t)A - J(x + \Delta x, t)A. \quad (2.15)$$

where $J(x, t)$ is the flux of heat into the positive x direction across the section at x . Taking the limit as $\Delta x \rightarrow 0$, Equation (2.15) becomes

$$\frac{\partial u}{\partial t} = -\frac{\partial q}{\partial x}, \quad \text{where } q \equiv J/c\rho, \quad (2.16)$$

Equation (2.16) is in the form of a conservation law, (2.1).

To express the flux J in terms of the temperature u , we use Fourier's law of heat conduction: If there are differences in temperature in a conducting medium, the heat energy would flow from the hotter to the colder region. Therefore there is "down gradient" heat flux, proportional to the negative of the temperature gradient, i.e.

$$q(x, t) = -\alpha^2 \frac{\partial}{\partial x} u. \quad (2.17)$$

Here α^2 is the coefficient of thermal diffusivity. For a uniform α^2 , Equation (2.7) becomes

$$\boxed{\frac{\partial}{\partial t} u = \alpha^2 \frac{\partial^2}{\partial x^2} u}. \quad (2.18)$$

Equation (2.9) is called the *heat equation*. It has the same form as the diffusion equation derived earlier.

2.2.5 Ubiquity of the Diffusion Equation

We have seen that the same diffusion equation arises from seemingly unrelated phenomena, from diffusion of salt and pollutants to conduction of heat. The ubiquity of the diffusion equation arises from (a) the ubiquity of the conservation law (2.1) and (b) the *Fickian flux-gradient relationship* (2.7):

$$J \propto -\frac{\partial c}{\partial x},$$

which is simply a statement that diffusion tends to transport matter or heat from high to low levels. This experimental law is a reflection of the underlying random molecular motion. Consider an imaginary interface at x separating water with no salt to the left and water with salt to the right. Suppose on average it is as likely, through their random motion, for molecules from the right to cross x to the left as it is for molecules from the left to go to the right. Since some of the molecules at the right are salt molecules, after a while some salt molecules have moved to the left while the water molecules from the left have taken their place at the right. The macroscopic result is that there is a *flux* of salt from the high salt concentration region to the low concentration region.

Similarly, let us consider the region of high temperature as consisting of molecules of higher vibrational energy. The energy can be transferred to other molecules upon collision. Thus, although it is equally likely for an energetic molecule to move to the left as it is for it to move to the right, when it moves to the left it has a higher probability of hitting a less energetic molecule, and transferring “heat” in the process. Therefore, macroscopically, heat flows from hot to cold regions.

Diverse phenomena in biology, ranging from animal and insect dispersal to the spread of diseases, can be modeled approximately by a Fickian flux-gradient relationship, and hence can be described by some sort of diffusion equation. (see Okubo (1980): *Diffusion and Ecological Problems: Mathematical Models*, Springer).

2.3 Random Walk

2.3.1 A drunken sailor

Consider the problem of predicting where a (microscopic) “drunken sailor” will be at time t if he walks randomly from an initial position at $x = 0$. For simplicity let us consider here the one dimensional problem where he is constrained to walk straight along a narrow alley (assuming that the sailor can indeed walk straight!). He takes a step of size Δx in a time interval Δt , and it is equally likely for him to take that step in either direction.

Let $p(x, t)$ be the probability that we will find him at x in time t . In a previous time step $t - \Delta t$, he could be either at the neighboring locations $x - \Delta x$ or at $x + \Delta x$, with equal probability. Therefore we have

$$p(x, t) = \frac{1}{2}p(x - \Delta x, t - \Delta t) + \frac{1}{2}p(x + \Delta x, t - \Delta t). \quad (2.19)$$

(2.19) is simply a statement that the sailor could have arrived at x from either $x - \Delta x$ by taking a forward step (or lurch), or from $x + \Delta x$ by stepping backward, with equal likelihood. If we take Δx and Δt to be small in some sense, we can expand the right-hand side of (2.17) in a Taylor series about its two variables:

$$\begin{aligned}
 p(x - \Delta x, t - \Delta t) &\simeq p(x, t - \Delta t) + \frac{\partial p(x, t - \Delta t)}{\partial x}(-\Delta x) \\
 &\quad + \frac{1}{2} \frac{\partial^2 p(x, t - \Delta t)}{\partial x^2}(\Delta x)^2 + \dots \\
 &\simeq p(x, t) + \frac{\partial p(x, t)}{\partial t}(-\Delta t) + \frac{1}{2} \frac{\partial^2 p(x, t)}{\partial t^2}(-\Delta t)^2 \\
 &\quad + \frac{\partial p(x, t)}{\partial x}(-\Delta x) + \frac{\partial^2 p(x, t)}{\partial x \partial t}(-\Delta t)(-\Delta x) + \dots \\
 &\quad + \frac{1}{2} \frac{\partial^2 p(x, t)}{\partial x^2}(-\Delta x)^2 + \dots
 \end{aligned}$$

Similarly the Taylor series expansion for $p(x + \Delta x, t - \Delta t)$ is the same as for $p(x - \Delta x, t - \Delta t)$ except with $-\Delta x$ replaced by Δx . Therefore (2.19) becomes:

$$p(x, t) = p(x, t) - \frac{\partial}{\partial t} p(x, t) \Delta t + \frac{1}{2} \frac{\partial^2}{\partial t^2} p(x, t) (\Delta t)^2 + \frac{1}{2} \frac{\partial^2}{\partial x^2} p(x, t) (\Delta x)^2 + \dots$$

Upon taking the limit $\Delta t \rightarrow 0$, $\Delta x \rightarrow 0$, we find

$$\boxed{\frac{\partial}{\partial t} u(x, t) = D \frac{\partial^2}{\partial x^2} u(x, t)}, \quad (2.20)$$

where $u(x, t) \equiv p(x, t)/\Delta x$ is the *probability density*, which is finite no matter how small Δx is, and $D \equiv \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta t \rightarrow 0}} \frac{(\Delta x)^2}{2\Delta t}$ is assumed to exist.

Equation (2.20) has the same form as the diffusion equation. Although we used the drunken sailor as an example, the derivation we have just outlined applies better to an ensemble of microscopic particles in random motion, where it makes more sense to take $\Delta x \rightarrow 0$, $\Delta t \rightarrow 0$, and to treat x and t as continuous variables.

2.3.2 Price of stocks as a random walk

In an efficient market, the price for a share of a company stock is an amount for which as many people wish to buy as to sell. Those wishing to buy are probably expecting the stock price to rise in the near future, while those

willing to sell at that price are probably anticipating the stock to drop in price. In such a market, insider trading is curbed. Any information known at time t about the company which might impact its stock price is already reflected in the price at that time. It is equally likely for the price of that stock to rise a given amount Δx at a future time $t + \Delta t$ as it is to fall by the same amount. Consequently, the expectation of a change in price of a stock in a future time should behave like a random walk, and therefore should be governed by the same diffusion equation (2.20), with however a different value for the “diffusion coefficient” D depending on the “volatility” of each stock.

2.4 The Wave Equation

Let us consider as an example the problem of a vibrating (guitar) string. The string is stretched lengthwise with uniform tension T . To fix ideas, let us say that in its equilibrium position the string lies horizontally (in the x -direction), and we consider a vertical displacement $u(x, t)$ from this equilibrium position. We assume that these displacements are small, as compared to the equilibrium length of the string.

We consider a small section of the string between x and $x + \Delta x$. See Figure 2.2.

We apply Newton’s law of motion:

$$ma = F$$

(mass times acceleration balancing force), to the vertical motion of this small section of the string. Its mass m is $\rho A \Delta x$, where ρ is the density of the material of the string and A its cross-sectional area. The acceleration in the vertical direction is

$$a = \frac{\partial^2}{\partial t^2} u.$$

The force should be the vertical component of the tension, plus other forces such as gravity and air friction.

The net vertical component of tension is

$$\begin{aligned} & T \sin \theta_2 - T \sin \theta_1 \\ \cong & T[\theta_2 - \theta_1] \\ \cong & T[u_x(x + \Delta x, t) - u_x(x, t)], \end{aligned}$$

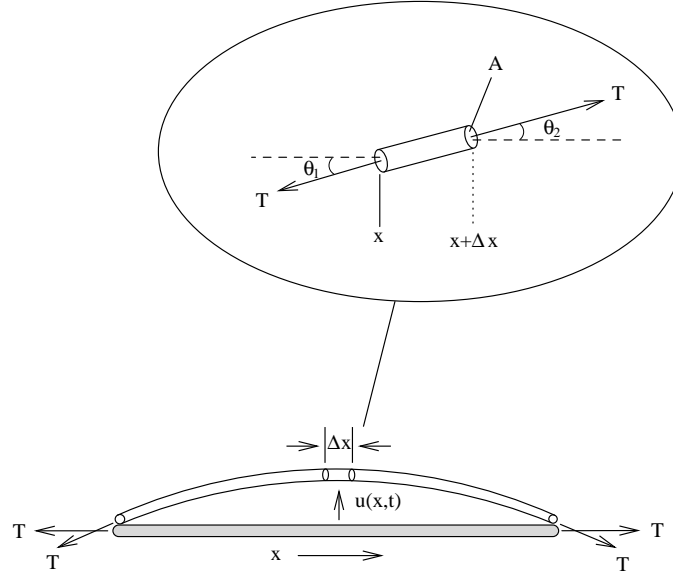


Figure 2.2: A stretched elastic string

assuming that the angles θ_1 and θ_2 are small. Putting these all together, we have

$$\rho A \Delta x \frac{\partial^2 u}{\partial t^2} = T A [u_x(x + \Delta x, t) - u_x(x, t)] + \rho A \Delta x \cdot f \quad (2.21)$$

where f represents all additional force per unit mass. Equation (2.21) is

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \frac{1}{\Delta x} [u_x(x + \Delta x, t) - u_x(x, t)] + f,$$

which becomes, as $\Delta x \rightarrow 0$:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + f, \quad (2.22)$$

where $c^2 \equiv T/\rho$.

The additional force f could represent gravity, in which case $f = -g$ ($g = 980 \text{ cm/s}^2$), or a frictional force of the form: $f = -\gamma u_t$, where γ is a damping coefficient. In most of the examples to be considered, f will be ignored, and we will be dealing with the simple homogeneous wave equation

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}. \quad (2.23)$$

The wave equation is also quite ubiquitous. Equation (2.23) also governs the propagation of sound waves in the atmosphere, with c changed to the speed of sound. It also governs water waves travelling on the surface of shallow water, with c replaced by \sqrt{gh} and h being the depth of the water.

2.5 Multiple Dimensions

Although our discussions have so far used one dimensional examples, extensions to two or three spatial dimensions are straightforward. These are indicated below:

1-D diffusion equation:

$$\frac{\partial}{\partial t}u = k \frac{\partial^2}{\partial x^2}u.$$

3-D diffusion equation:

$$\boxed{\frac{\partial}{\partial t}u = k \nabla^2 u}, \quad (2.24)$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator.

1-D wave equation:

$$\frac{\partial^2}{\partial t^2}u = c^2 \frac{\partial^2}{\partial x^2}u.$$

3-D wave equation:

$$\boxed{\frac{\partial^2}{\partial t^2}u = c^2 \nabla^2 u}. \quad (2.25)$$

Essentially, in multi-dimensions, the Laplacian operator replaces the $\frac{\partial^2}{\partial x^2}$ term of the one-dimensional problem.

At steady state the heat equation in multi-dimensions is

$$\boxed{\nabla^2 u = 0}. \quad (2.26)$$

Equation (2.24) is called Laplace's equation. Its solution gives the steady state temperature distribution in multi-dimensions, for example, Laplace's equation also governs the distribution of electrostatic potential and the velocity potential in ideal fluid flows.

2.6 Types of second-order PDEs

There are three types of second-order PDEs, whose solutions have distinctly different behaviors. These are *parabolic*, *hyperbolic*, and *elliptic* PDEs. Parabolic equations are diffusion like, while hyperbolic equations are typified by the wave equation. Laplace's equation belongs to the category of elliptic PDEs. Namely:

| | |
|---|-----------------|
| $\frac{\partial}{\partial t}u = k \frac{\partial^2}{\partial x^2}u :$ | Parabolic type |
| $\frac{\partial^2}{\partial t^2}u = c^2 \frac{\partial^2}{\partial x^2}u :$ | Hyperbolic type |
| $\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u = 0 :$ | Elliptic type |

In general a second-order linear PDE in two independent variables (denoted by x and y) can be written in the form (with u_{xx} denoting $\frac{\partial^2}{\partial x^2}u$ etc.):

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

where A, B, C, D, E, F and G are given functions of x and y . This equation is said to be *parabolic* if $B^2 - 4AC = 0$, *hyperbolic* if $B^2 - 4AC > 0$, and *elliptic* if $B^2 - 4AC < 0$.

The equation is homogeneous if $G \equiv 0$.

2.7 Boundary Conditions

The mathematical specification of a problem involving a partial differential equation in space x and time t is incomplete without the imposition of *boundary conditions* and *initial conditions*.

Three different types of boundary conditions for second order PDEs:

A. Dirichlet conditions, where the value of the unknown u is specified at the spatial boundaries.

For example, suppose we want to solve for $u(x, t)$ for $0 < x < L$, the boundaries of the domain are at $x = 0$ and $x = L$. We specify $u = T_1$ at $x = 0$ and $u = T_2$ at $x = L$, this is a Dirichlet boundary condition, because here the values of the unknown u is specified at the boundaries $x = 0$ and

$x = L$. If the values specified at the boundaries are zero, then we call this the *homogeneous* Dirichlet boundary condition. So

$$\boxed{u(0, t) = 0, \quad u(L, t) = 0}$$

is a homogeneous Dirichlet boundary condition for a problem in $0 < x < L$.

For the heat conduction problem discussed in 2.2.3, this boundary condition is equivalent to specifying the value of the temperatures at the two ends of the rod.

B. Neumann Condition, where the value of the normal derivative of the unknown is specified at the boundaries. [The normal derivative is the derivative normal to the boundary. In a one-dimensional space x , u_x is the normal derivative of u .] For the above mentioned domain,

$$\boxed{u_x(0, t) = b_1, \quad u_x(L, t) = b_2}$$

is a Neumann boundary condition if b_1 and b_2 are specified. If $b_1 = 0$ and $b_2 = 0$, we then have a homogeneous Neumann boundary condition.

For the heat conduction problem, the Neumann condition is equivalent to specifying the heat fluxes at the two ends of the rod. A zero flux represents the fact that the ends of the rod are insulated.

C. Robin condition, which involves the specification of a linear combination of u and its normal derivative. An example is

$$\boxed{\begin{aligned} ku_x(0, t) &= h[u(0, t) - b_1] \\ ku_x(L, t) &= h[u(L, t) - b_2] \end{aligned}}.$$

For the heat conduction problem, the above condition describes the heat fluxes at the ends of the rod as a difference of the rod temperature and the temperature b_1 and b_2 of the ambient medium with which the rod is in contact.

2.8 Initial Conditions

For the diffusion or heat equation, the PDE governs the diffusion of a tracer or the conduction of heat, whereas the boundary conditions tell us what is happening at the boundaries to affect the solution inside the domain of interest. The initial condition tells us the state from which the solution evolves. Without it, the mathematical specification of the problem is incomplete.

Physically, we understand that two identical conducting rods with the same boundary conditions can evolve differently if they start with different initial temperatures. A completely specified problem can be, for example:

$$\begin{aligned}\text{PDE: } & u_t = ku_{xx}, \quad 0 < x < L, \quad t > 0 \\ \text{BC: } & u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0 \\ \text{IC: } & u(x, 0) = f(x), \quad 0 < x < L, \quad \text{where } f(x) \text{ is given.}\end{aligned}$$

For the wave equation, which has a second-order derivative in time, we need two initial conditions. An example of a completely specified problem is:

$$\begin{aligned}\text{PDE: } & u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0 \\ \text{BC: } & u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0 \\ \text{IC: } & u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 < x < L,\end{aligned}$$

where $f(x)$ and $g(x)$ prescribed.

For the vibrating string problem, $f(x)$ represents the initial position of the string, while $g(x)$ represents the initial velocity with which the string is plucked. Both are needed to completely specify the problem. Physically, we understand that the note emitted by a vibrating guitar string is different if we gently displace it and then let go (with $g = 0$) than if we displace the string with a sudden pull.

For the Laplace equation, there is no need for initial conditions. Only boundary conditions are needed. Physically, Laplace's equation determines the temperature distribution at steady state and so this temperature should depend on the conditions at the boundaries only.

If we nevertheless insist on treating one of the independent variables (say y) in the Laplace equation as time-like, and specify "initial conditions" of the form

$$\begin{aligned}u(x, 0) &= f(x), \\ u_y(x, 0) &= g(x),\end{aligned}$$

the problem in many cases becomes unphysical ("ill-posed"). An example of an ill-posed problem is:

$$\begin{aligned}\text{PDE: } & \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0, \quad 0 < x < L, \quad y > 0 \\ \text{BC: } & u(0, y) = 0, \quad u(L, y) = 0 \\ \text{IC: } & u(x, 0) = f(x), \quad u_y(x, 0) = g(x), \quad 0 < x < L.\end{aligned}$$

2.9 Exercises

1. Classify the following PDEs and their associated boundary conditions:

(a) PDE: $u_t - u_{xx} = \sin x$, $0 < x < \pi$, $t > 0$

BC: $u(0, t) = 0$, $u(\pi, t) = 0$, $t > 0$

(b) PDE: $u_{tt} - a(x)^2 u_{xx} = 0$, $t > 0$, $0 < x < L$ where $a(x)$ is a real and nonzero given function.

BC: $u_x(0, t) = u_x(L, t) = 0$.

2. Consider the following heat equation

$$u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \quad \alpha = \text{constant.}$$

for a rod of length L whose ends are maintained at temperature $u = 0$, i.e.

$$u(x, t) = 0 \text{ at } x = 0 \text{ and at } x = L.$$

Initially, at $t = 0$, the temperature distribution is given by

$$u(x, 0) = \sin\left(\frac{\pi x}{L}\right), \quad 0 < x < L.$$

Solve for $u(x, t)$ for $t > 0$. [Hint: Assume that the solution can be written in the “separable” form:

$$u(x, t) = T(t) \sin \frac{\pi x}{L}.$$

Find $T(t)$ by substituting it into the PDE. Make sure that the initial condition and the boundary conditions are also satisfied.]

3. Same as problem # 1, except the initial condition is

$$u(x, 0) = \sin \frac{\pi x}{L} + \frac{1}{4} \sin \frac{2\pi x}{L}.$$

[Hint: Assume $u(x, t) = T_1(t) \sin \frac{\pi x}{L} + T_2(t) \sin \frac{2\pi x}{L}$.]

2.10 Solutions

1. (a) PDE: *parabolic*, nonhomogeneous, linear, second order, two independent variables

BC: Homogeneous Dirichlet.

- (b) PDE: *Hyperbolic*, homogeneous, linear, second order, two independent variables.

BC: Homogeneous Neumann.

2. Solve the heat equation,

$$u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad t > 0$$

with boundary conditions, $u(0, t) = u(L, t) = 0$ and initial condition $u(x, 0) = \sin(\pi x/L)$.

Assume that the solution can be written in the “separable” form: $u(x, t) = T(t) \sin(\pi x/L)$.

Substitute into the PDE to find: $dT/dt + (\alpha\pi/L)^2 T = 0$.

This is a first order ODE with general solution: $T(t) = ce^{-(\alpha\pi/L)^2 t}$.

Now we use the initial condition to find: $c = 1$. Thus we have the solution:

$$u(x, t) = e^{-(\alpha\pi/L)^2 t} \sin(\pi x/L).$$

3. This problem is the same as problem 2 except that we now have the initial condition:

$$u(x, 0) = \sin(\pi x/L) + \sin(2\pi x/L)/4.$$

Following the hint we assume that the solution has the form $u(x, t) = u_1(x, t) + u_2(x, t)$ where $u_1(x, t) = T_1(t) \sin(\pi x/L)$ and $u_2(x, t) = T_2(t) \sin(2\pi x/L)$.

We can see that $u_1(x, t)$ is the solution we found in problem 2 and that in finding the solution $u_2(x, t)$ we do exactly as in problem 2, where the general solution becomes $T_2(t) = c_2 e^{-(2\alpha\pi/L)^2 t}$ with the constant $c_2 = 1/4$.

Thus we have, after adding u_1 and u_2 , the solution to problem 3 as:

$$u(x, t) = e^{-(\alpha\pi/L)^2 t} \sin(\pi x/L) + e^{-(2\alpha\pi/L)^2 t} \sin(2\pi x/L)/4.$$