Chapter 14

Nonhomogeneous Partial Differential Equations

14.1 Introduction

The method of eigenfunction expansions is used to solve nonhomogeneous partial differential equations.

Consider the following nonhomogeneous heat equation (with a given *heating* term $f(x,t)$) subject to the general boundary condition (which includes Dirichlet and Neumann as special cases):

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx} + f(x,t), \quad 0 < x < L, \quad t > 0 \tag{14.1}$$

$$\text{BCs:} \quad \alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0 \tag{14.2}$$

$$\alpha_2 u_x(L,t) + \beta_2 u(L,t) = 0$$

$$\text{IC:} \quad u(x,0) = \phi(x), \quad 0 < x < L. \tag{14.3}$$

14.2 Eigenfunction expansion

Step 1: Find the eigenfunction of the homogeneous problem. That is, first (drop $f(x,t)$ and) solve the following homogeneous problem:

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad t > 0 \tag{14.4}$$

$$\text{BCs:} \quad \alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0$$

$$\alpha_2 u_x(L,t) + \beta_2 u(L,t) = 0. \tag{14.5}$$

Write its solution in the form

$$u(x,t) = \sum_n T_n(t) X_n(x),$$

where the eigenfunctions, $X = X_n$ are determined by

$$\begin{cases} X''(x) + \lambda^2 x(x) = 0 \\ \alpha_1 X'(0) + \beta_1 X(0) = 0 \\ \alpha_2 X'(L) + \beta_2 X(L) = 0 \end{cases} \tag{14.6}$$

with the eigenvalues, $\lambda = \lambda_n$.
Do not work out $T_n(t)$ yet, since it will turn out that the $T_n(t)$ for the nonhomogeneous problem will be different than the $T_n(t)$ for the homogeneous problem.

Step 2: Expand the forcing term $f(x,t)$:

$$\boxed{f(x,t) = \sum_n f_n(t) X_n(x)}. \tag{14.7}$$

and expand the solution of the nonhomogeneous PDE in terms of these eigenfunction the same way:

$$\boxed{u(x,t) = \sum_n T_n(t) X_n(x)}. \tag{14.8}$$

Step 3: Substitute (10.7) and (10.8) into the PDE (10.1): Note that

$$\begin{aligned} u_t &= \sum_n T_n'(t) X_n(x) \\ u_{xx} &= \sum_n T_n(t) X_n''(x) = -\sum_n \lambda_n^2 T_n(t) X_n(x). \end{aligned}$$

Thus (10.1) becomes

$$\sum_n [T_n'(t) + \alpha^2 \lambda_n^2 T_n(t) - f_n(t)] X_n(x) = 0. \tag{14.9}$$

because X_n's are orthogonal, (10.9) implies that

$$T_n'(t) + \alpha^2 \lambda_n^2 T_n(t) - f_n(t) = 0. \tag{14.10}$$

Step 4: To satisfy the initial condition, we require

$$\boxed{u(x,0) = \phi(x) = \sum_n T_n(0)X_n(x)}, \tag{14.11}$$

yielding (see (9.22)):

$$T_n(0) = \frac{\int_0^L \phi(x)X_n(x)dx}{\int_0^L X_n^2(x)dx}. \tag{14.12}$$

Step 5: Solve the nonhomogeneous ODE (10.10) to get

$$\boxed{T_n(t) = T_n(0)e^{-\alpha^2\lambda_n^2 t} + \int_0^t e^{-\alpha^2\lambda_n^2(t-\tau)} f_n(\tau)d\tau}. \tag{14.13}$$

14.3 An example

Solve:

$$\begin{array}{ll} \text{PDE:} & u_t = \alpha^2 u_{xx} + \sin(3\pi x), \quad 0 < x < 1, \quad t > 0 \\ \text{BCs:} & u(0,t) = 0, \quad u(1,t) = 0 \\ \text{IC:} & u(x,0) = \sin(\pi x), 0 < x < 1. \end{array} \tag{14.14}$$

The eigenfunctions and eigenvalues of the homogeneous PDE are

$$X(x) = X_n(x) = \sin \lambda_n x$$
$$\lambda = \lambda_n = n\pi, \quad n = 1, 2, 3, \ldots$$

We will therefore use a sine series expansion of the solution:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin n\pi x, \quad 0 < x < 1,$$

and the forcing term:

$$f(x,t) = \sin 3\pi x = \sum_{n=1}^{\infty} f_n \sin n\pi x, \quad 0 < x < 1.$$

The latter means simply $f_n = 0$ except $f_3 = 1$. Substituting the expansions into the PDE, we have

$$T_n'(t) - \alpha^2(n\pi)^2 T_n = f_n, \quad n = 1, 2, 3, \ldots.$$

For $n \neq 3$, this is

$$T_n'(t) - \alpha^2(n\pi)^2 T_n(t) = 0,$$

so

$$T_n(t) = T_n(0)e^{-\alpha^2 n^2 \pi^2 t}, \quad n \neq 3.$$

For $n = 3$:

$$T_3'(t) - 9\pi^2\alpha^2 T_3(t) = 1.$$

The solution is:

$$T_3(t) = T_3(0)e^{-9\pi^2\alpha^2 t} + \frac{1}{(3\pi\alpha)^2}[1 - e^{-9\pi^2\alpha^2 t}].$$

To satisfy the initial condition, we require

$$\sin \pi x = \sum_{n=1}^{\infty} T_n(0) \sin n\pi x, \quad 0 < x < 1,$$

so we take $T_n(0) = 0$ except $T_1(0) = 1$. Thus,

$$\begin{aligned}
T_1(t) &= e^{-\alpha^2\pi^2 t} \\
T_2(t) &= 0 \\
T_3(t) &= \frac{1}{(3\pi\alpha)^2}[1 - e^{-(3\pi\alpha)^2 t}] \\
T_4(t) &= 0 \\
&\vdots
\end{aligned}$$

The final solution is the two-term expansion

$$u(x,t) = e^{-(\alpha\pi)^2 t} \sin \pi x + \frac{1}{(3\pi\alpha)^2}[1 - e^{-(3\pi x)^2 t}] \sin(3\pi x).$$

Comments

For this simple problem where the forcing term $f(x,t)$ is a function of x only, there exists an alternative, perhaps simpler method. We write the solution as the sum of two parts, a steady state solution, $u_{\text{steady}}(x)$, and a transient solution, $u_{\text{transient}}(x,t)$. The steady state solution is to satisfy the steady state PDE, i.e. (10.14) without the time derivative term:

$$0 = \alpha^2 \frac{d^2}{dx^2} u_{\text{steady}} + \sin(3\pi x).$$

This yields

$$u_{\text{steady}}(x) = \frac{1}{(3\pi\alpha)^2}\sin(3\pi x).$$

The transient solution is found by substituting

$$u(x,t) = u_{\text{steady}}(x) + u_{\text{transient}}(x,t)$$

into the original PDE, (10.14). Thus $u_{\text{transient}}$ now satisfies a *homogeneous* PDE:

$$\begin{array}{ll} \text{PDE:} & \dfrac{\partial}{\partial t}u_{\text{transient}} = \alpha^2\dfrac{\partial^2}{\partial x^2}u_{\text{transient}} \\ \text{BC:} & u_{\text{transient}}(0,t) = u_{\text{transient}}(1,t) = 0 \\ \text{IC:} & u_{\text{transient}}(x,0) = \sin(\pi x) - u_{\text{steady}}(x). \end{array}$$

The solution to this system is

$$u_{\text{transient}}(x,t) = e^{-(\alpha\pi)^2 t}\sin(\pi x) - \frac{1}{(3\pi\alpha)^2}e^{-(3\pi\alpha)^2 t}\sin(3\pi x).$$