Chapter 11

The Helmholtz equation in three dimensions

11.1 Introduction

Many classical equations arising from physical contexts contain the Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Some examples are:

The wave equation:

$$\frac{\partial^2}{\partial t^2} u = c^2 \nabla^2 u.$$

The Laplace equation:

$$\nabla^2 u = 0$$

The Schrödinger equation in quantum mechanics also involves this operator for the space derivatives:

$$i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2\mu} \nabla^2 \psi.$$

In previous chapters, we considered the one-dimensional problems, for which $\nabla^2 = \frac{\partial^2}{\partial x^2}$. Here we shall treat the problem in higher dimensions. Separating out the time dependence will result in the Helmoltz's eigenvalue problem for all the cases listed above. This problem is to solve:

$$\nabla^2 u = -\lambda^2 u \tag{11.1}$$

subject to appropriate homogeneous boundary conditions, and determine the eigenvalue λ^2 in the process. Laplace's equation is simply (11.1) with $\lambda^2 = 0$. It has the trivial solution unless the boundary conditions are nonhomogeneous.

11.2 An example: An electron in a box

Consider the case of an electron of mass μ contained in a cubic box with infinite potentials on all sides. This situation is described by the Schrödinger's equation

$$\begin{array}{lll} \text{PDE:} & i\hbar\dfrac{\partial}{\partial t}\psi = -\dfrac{\hbar^2}{2\mu}\nabla^2\psi, & 0 < x < L \\ & & 0 < y < L \\ & & 0 < z < L \\ \text{BC:} & \psi = 0 \text{ at } x = 0 \text{ and } x = L & \\ & \psi = 0 \text{ at } y = 0 \text{ and } y = L & \\ & \psi = 0 \text{ at } z = 0 \text{ and } z = L. & \end{array}$$

Separation of variables involves first assuming

$$\psi(x, y, z, t) = T(t)u(x, y, z).$$

This results:

$$\frac{i\hbar T'(t)}{T(t)} = -\frac{\hbar^2}{2\mu}\frac{\nabla^2 u}{u} = \text{const} \equiv E \tag{11.2}$$

From the solution of the $T(t)$ equation:

$$\frac{i\hbar T'(t)}{T(t)} = E,$$

$$T(t) = T(0)e^{-i(E/\hbar)t},$$

the separation constant E is interpreted as the energy of the electron (since $E/\hbar$ is the frequency and frequency times $\hbar$ is the energy in quantum mechanics).

From (11.2), the space dependence of ψ satisfies the Helmoltz equation:

$$\nabla^2 u = -\lambda^2 u, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

where $\lambda^2 \equiv 2\mu E/\hbar^2$. By determing the eigenvalue λ^2 in the solution process we will be able to determine the energy of the electron. It will turn out to be quantized.

Further separation of variables:

$$u(x,y,z) = X(x)Y(y)Z(y)$$

leads to

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = -\lambda^2.$$

We can argue that since

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} - \frac{Z''(z)}{Z(z)} - \lambda^2, \tag{11.3}$$

the left-hand side, which is a function of x only, can equal to the right-hand side, which is not a function of x, only if each side is equal to a constant. Setting that separation constant to $-a^2$, we will have, from (11.2)

$$\frac{X''(x)}{X(x)} = -a^2 \tag{11.4}$$

and

$$-\frac{Y''(y)}{Y(y)} - \frac{Z''(z)}{Z(z)} - \lambda^2 = -a^2 \tag{11.5}$$

Equation (11.4) is to be solved subject to the boundary condition

$$X(0) = 0, \quad X(L) = 0.$$

This yields

$$X(x) = X_n(x) \equiv \sin\frac{n\pi x}{L},$$

$$a = a_n \equiv \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots .$$

Equation (11.5) can similarly be re-arranged to

$$\frac{Y''(y)}{Y(y)} = -\frac{Z''(z)}{Z(z)} + a_n^2 - \lambda^2 = \text{const} \equiv -b^2 \tag{11.6}$$

Solving

$$\frac{Y''(y)}{Y(y)} = -b^2$$

subject to the boundary condition

$$Y(0) = 0, \quad Y(L) = 0,$$

yields

$$\begin{aligned} Y(y) = Y_m(y) &\equiv \sin \frac{m\pi y}{L} \\ b = b_m &\equiv \frac{m\pi}{L}, \quad m = 1, 2, 3, \ldots. \end{aligned}$$

The last part in (11.6) to be solved is:

$$\frac{Z''(z)}{Z(z)} = -c^2, \quad Z(0) = 0, \quad Z(L) = 0$$

where we have written

$$c^2 \equiv \lambda^2 - (a_n^2 + b_m^2). \tag{11.7}$$

The solution is

$$\begin{aligned} Z(z) = Z_\ell(z) &\equiv \sin \frac{\ell\pi z}{L}, \\ c = c_\ell &\equiv \frac{\ell\pi}{L}, \quad \ell = 1, 2, 3, \ldots. \end{aligned}$$

Putting all the eigenvalues into (11.7), we find

$$\begin{aligned} \lambda^2 = \lambda_{nm\ell}^2 \equiv a_n^2 + b_m^2 + c_\ell^2 &= (n^2 + m^2 + \ell^2)\pi^2/L^2, \\ n = 1, 2, 3, \ldots, \quad m = 1, 2, 3, \ldots, &\quad \ell = 1, 2, 3, \ldots \end{aligned} \tag{11.8}$$

(11.8) is the desired eigenvalue for the Helmoltz equation. The eigenfunction is

$$u(x, y, z) = u_{nm\ell}(x, y, z) \equiv \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{L} \sin \frac{\ell\pi z}{L}.$$

For the original Schrödinger equation, the energy of the electron is found from $\lambda^2 = 2\mu E/\hbar^2$ to be

$$E = E_{nm\ell} \equiv \frac{\hbar^2\pi^2}{2\mu L^2}(n^2 + m^2 + \ell^2), \tag{11.9}$$

which is quantized because n, m and ℓ can only take an integer values.

The general solution is obtained by superposition

$$\psi(x, y, z, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{\ell=1}^{\infty} e^{-i(E_{nm\ell}/\hbar)t} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{L} \sin \frac{\ell\pi n}{L}. \tag{11.10}$$

Often however, it is the individual eigenfunction and eigenvalues which are discussed and displayed.

11.3 Sound waves in a rectangular cavity

The propagation of sound in a three-dimensional rectangular cavity is governed by

$$\frac{\partial^2}{\partial t^2}\psi = c^2\nabla^2\psi, \tag{11.11}$$

where ψ is the pressure fluctuation in the air caused by the sound wave, and c the speed of sound. Separation of variables:

$$\psi(x, y, z, t) = T(t)u(x, y, z)$$

leads to:

$$\frac{T''(t)}{c^2 T(t)} = \frac{\nabla^2 u}{u} = \text{const} \equiv -\lambda^2 \tag{11.12}$$

Again we arrive at the Helmoltz eigenvalue problem for the space dependences:

$$\nabla^2 u = -\lambda^2 u. \tag{11.13}$$

The physical interpretation of the eigenvalue λ can in this case be obtained from solving the time dependence part.

$$T''(t) = -c^2\lambda^2 T(t)$$

The solution is

$$T(t) = A\sin(c\lambda t) + B\cos(c\lambda t), \tag{11.14}$$

and so $\omega \equiv c\lambda$ has the physical interpretation of frequency of oscillation. This frequency is equal to the phase speed c times the "wavenumber" λ. The latter is determined by the geometry of the cavity Equation (11.11) is to be solved subject to the boundary conditions

$$\begin{aligned} u = 0 \text{ at } x &= 0 \text{ and } x = L_1 \\ y &= 0 \text{ and } y = L_2 \\ z &= 0 \text{ and } z = L_3, \end{aligned}$$

if the dimension of the cavity is L_1 by L_2 by L_3. The solution via the method of separation of variables is the same as in the previous section for a cubic box:

$$u(x, y, z) = X(x)Y(y)Z(z).$$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = -\lambda^2$$

$$\frac{X''(x)}{X(x)} = -a^2 : \quad X = X_n(x) \equiv \sin\frac{n\pi X}{L_1}, a = a_n \equiv \frac{n\pi}{L_1}, \quad n = 1, 2, 3, \ldots$$
$$\frac{Y''(y)}{Y(y)} = -b^2 : \quad Y = Y_m(y) \equiv \sin\frac{m\pi y}{L_1}, \quad b = b_m = \frac{m\pi}{L_2}, \quad m = 1, 2, 3, \ldots$$
$$\frac{Z''(z)}{Z(z)} = -(\lambda^2 - a^2 - b^2) : Z(z) = Z_\ell(z) \equiv \sin\frac{\ell\pi z}{L_3},$$
$$\sqrt{\lambda^2 - a^2 - b^2} = \frac{\ell\pi}{L_3}, \quad \ell = 1, 2, 3, \ldots$$

so the overall eigenvalue is determined:

$$\lambda^2 = \lambda^2_{nm\ell} \equiv \pi^2\left(\left(\frac{n}{L_1}\right)^2 + \left(\frac{m}{L_2}\right)^2 + \left(\frac{\ell}{L_3}\right)^2\right) \tag{11.15}$$

The frequency of the oscillation is “quantized”:

$$\omega \equiv c\lambda = c\pi\left(\left(\frac{n}{L_1}\right)^2 + \left(\frac{m}{L_2}\right)^2 + \left(\frac{\ell}{L_3}\right)^2\right)^{1/2} \tag{11.16}$$

The eigenfunction is

$$u_{nm\ell}(x, y, z) = \sin\frac{n\pi x}{L_1} \cdot \sin\frac{m\pi y}{L_2} \cdot \sin\frac{\ell\pi z}{L_3}. \tag{11.17}$$

A one-dimensional oscillator, such as the violin string, has “harmonic” frequencies:

$$\omega_n = n\omega_1, \quad \omega_1 \equiv \frac{c\pi}{L_1}, \quad n = 1, 2, 3, \ldots$$

That is, the higher frequencies are integer multiples of the fundamental frequency ω_1. Our human ear finds the superposition of harmonic frequencies pleasing. On the other hand, sounds from two-dimensional oscillators, such as drums, are not pleasing to the ear because their sounds are a superposition of incommensurable frequencies:

$$\omega_{nm} = c\pi((\frac{n}{L_1})^2 + (\frac{m}{L_2})^2)^{1/2}.$$

[One should have used the circular geometry for the drum head problem, however.]

11.4 Helmoltz eigenvalue problem in a cylinder

$$\nabla^2 u = -\lambda^2 u \tag{11.18}$$

In cylindrical coordinates we let z be measured along the length of the cylinder. $z = 0$ is one end and $z = L$ is the other end. Let r be the radial distance from the center of circular a cross-section of the cylinder and θ is the angle extended by this radius. In such a coordinate system, the Laplacian is given by

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}u + \frac{\partial^2}{\partial z^2}u.$$

Separation of variables

$$u(r, \theta, z) = R(r)\Theta(\theta)Z(z)$$

yields

$$\frac{Z''(z)}{Z(z)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} + \frac{\frac{1}{r}(rR'(r))'}{R} = -\lambda^2 \tag{11.19}$$

We have

$$\frac{Z''(z)}{Z(z)} = -b^2, \quad z(0) = 0,\ z(L) = 0$$

The solution is:

$$Z(z) = Z_\ell(z) \equiv \sin\frac{\ell\pi z}{L}, \quad \ell = 1, 2, 3, \ldots$$
$$b = b_\ell \equiv \frac{\ell\pi}{L}.$$

Equation (11.19) can be rewritten as

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = -\frac{r(rR'(r))'}{R} - (\lambda^2 - b^2)r^2 = \text{const} = -m^2. \tag{11.20}$$

The fact that Θ must be 2π-periodic demands the separation constant m be an integer:

$$\Theta = e^{im\theta}, \quad m = 0,\ \pm 1,\ \pm 2,\ \pm 3, \ldots \tag{11.21}$$

The R equation in (11.19) is:

$$r\frac{d}{dr}(r\frac{d}{dr}R) + [(\lambda^2 - b^2)r^2 - m^2]R = 0, \tag{11.22}$$

and is to be solved subject to the boundary conditions

$$R(0) \text{ bounded, and } R(a) = 0, \text{ where}$$

$r = a$ is the radius of the cylinder.

Equation (11.22) can be put into the form of Bessel's equation by letting

$$x = \sqrt{\lambda^2 - b^2}r, \qquad y(x) = R(r)$$

to yield the standard form for Bessel's equation of order m:

$$x^2 \frac{d^2}{dx^2}y + x\frac{d}{dx}y + (x^2 - m^2)y = 0 \tag{11.23}$$

The solution can be obtained by Frobenius series expansion to be

$$y(x) = AJ_m(x) + BY_m(x).$$

$J_m(x)$ is the Bessel's function of the first kind, and has the series expansion of

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+m)!}(\frac{x}{2})^{2k+m}. \tag{11.24}$$

$Y_m(x)$ is the Bessel's function of the second kind. It has a log singularity at $x = 0$ and blows up at $x = 0$. We will not write out the series expansion for $Y_m(x)$ here.

The boundary condition that $R(0)$ be bounded requires that we set $B = 0$. Thus the required solution is:

$$R(r) = AJ_m(\sqrt{\lambda^2 - b^2}r). \tag{11.25}$$

To satisfy the second boundary condition, we require

$$J_m(\sqrt{\lambda^2 - b^2}a) = 0. \tag{11.26}$$

Since the Bessel function is similar to cosine (in fact for large $|z|$, $J_m(z) \cong \sqrt{\frac{2}{\pi z}}\cos(z - \frac{1}{2}m\pi - \frac{1}{4}\pi)$), it has an infinite number of zeros for each m. These zeros are tabulated, and can be denoted by z_{mn}, with

$$0 < z_{m1} < z_{m2} < z_{m3} < \dots$$

The first few zeros are listed in Table 1:

n	1	2	3	4
z_{on}	2.40483	5.52008	8.65373	11.7915
z_{1n}	3.83171	7.01559	10.1735	13.3237
z_{2n}	5.13562	8.41724	11.6198	14.796

Table: 1 The zeros of Bessel's function

Therefore the boundary condition (11.26) is satisfied by setting

$$\sqrt{\lambda^2 - b^2} = \frac{z_{mn}}{a},$$

resulting in the eigenvalue of the Helmoltz's system as:

$$\lambda^2 = \lambda^2_{nm\ell} = \frac{z^2_{mn}}{a^2} + \frac{\ell^2\pi^2}{L^2}, \ \ell = 1, 2, 3, \dots, \ \ n = 1, 2, 3, \dots, \ \ m = 0, \pm 1, \pm 2, \dots$$

The eigenfunction is

$$u_{mn\ell}(r, \theta, z) = J_m(z_{mn} r/a) e^{im\theta} \sin\frac{\ell \pi z}{L}.$$

For sound waves in a circular cylinder, the frequency of oscillation is given by (11.14) to be

$$\omega = \omega_{nm\ell} = c\lambda_{nm\ell} = c\{\frac{z^2_{mn}}{a^2} + \frac{\ell^2\pi^2}{L^2}\}^{1/2}.$$

11.5 Helmoltz's eigenvalue problem in a sphere

$$\nabla^2 u = -\lambda^2 u. \tag{11.27}$$

The Laplacian operator in spherical coordinates is givey by:

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r} u) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial}{\partial \theta} u) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\theta^2 \varphi^2} u,$$

where θ is the latitude (measured from the north poles) and φ is the longitude and r is the radius from the origin. [If you prefer to measure latitude relative to the equator, then all $\sin\theta$ in the above expression is changed to $\cos\theta$.]

We apply the method of separation of variables again:

$$u(r, \theta, \varphi) = R(r) Y(\theta, \varphi)$$

and substitute into (11.27):

$$\frac{\frac{1}{r^2}\frac{d}{dr}(r^2 \frac{d}{dr} R)}{R} + \frac{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \frac{\partial}{\partial\theta} Y) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\varphi^2} Y}{Y} = -\lambda^2,$$

to get:

$$\frac{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \frac{\partial}{\partial\theta} Y) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} Y}{Y} = -\lambda^2 r^2 - \frac{\frac{d}{dr}(r^2 \frac{d}{dr} R)}{R}.$$

Since the left-hand side is a function of the angles only and the right-hand side is a function of radius only, they can equal to each other only if they each equal to a constant. We set this separation constant to $-\eta$. Thus

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}Y)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}Y+\eta Y=0 \tag{11.28}$$

and

$$R''+\frac{2}{r}R'-(\frac{\eta}{r^2}-\lambda^2)R=0 \tag{11.29}$$

11.5.1 The spherical harmonics

Equation (11.28) is the *spherical harmonic equation.* $Y(\theta,\varphi)$ can further be separated into:

$$Y(\theta,\varphi)=\Theta(\theta)\Phi(\varphi).$$

(11.28) becomes

$$\frac{\sin\theta(\sin\theta\Theta')'+\eta\sin^2\theta\Theta}{\Theta}=-\frac{\Phi''}{\Phi}=\text{const}\equiv\alpha^2. \tag{11.30}$$

Thus

$$\Phi''(\varphi)+\alpha^2\Phi(\varphi)=0,\qquad 0\le\varphi\le 2\pi$$

subject to the periodic boundary condition

$$\Phi(\varphi+2\pi)=\Phi(\varphi).$$

The solution to the equation is:

$$\Phi(\varphi)=Ae^{i\alpha\varphi}+Be^{-i\alpha\varphi}$$

To satisfy the periodic boundary, α must be an integer, $m=0,\ 1,\ 2,\ 3,\ \ldots$:

$$\begin{aligned}\alpha&=m\equiv\alpha_m\\ \Phi(\varphi)=\Phi_m(\varphi)&=A_me^{im\varphi}+B_me^{-im\varphi},\quad m=0,1,2,\ldots\\ &=A_me^{im\varphi},\quad m=0,\quad \pm 1,\quad \pm 2,\quad \pm 3,\ldots,\end{aligned} \tag{11.31}$$

where we have defined A_m for negative m to be $B_{|m|}$.

Equation (11.30) becomes

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d}{d\theta}\Theta)+(\eta-\frac{m^2}{\sin^2\theta})\Theta=0,\qquad 0\le\theta\le\pi$$

With a change in varaible:

$$x = \cos\theta, \quad dx = -\sin\theta d\theta, \quad \frac{d}{d\theta} = -\sin\theta\frac{d}{dx}, \quad \sin^2\theta = 1 - \cos^2\theta = 1 - x^2$$

$$\frac{d}{dx}[(1-x^2)\frac{d}{dx}\Theta] + (\eta - \frac{m^2}{1-x^2})\Theta = 0, \quad -1 \le x \le 1 \tag{11.32}$$

Equation (11.32) is the associated Legendre equation. It is regular at $x = 0$, but has a regular singular point at $x = \pm 1$. The solutions which are bounded at $x = \pm 1$ are the Associated Legendre functions, $P_n^m(x)$. The eigenvalues are $\eta = \eta_n \equiv n(n+1)$, $n = 0, 1, 2, 3, \ldots$. For η not equal to these discrete values, the solution blows up at $x = \pm 1$, the north and south poles.

$$\begin{aligned} \Theta &= \Theta_n = P_n^m(x), \qquad n = 0, 1, 2, 3, \ldots \\ \eta &= n(n+1), \qquad m = -n, -n+1, \ldots, n-1, n. \end{aligned}$$

The spherical harmonics are defined by

$$Y(\theta,\varphi) = Y_{nm}(\theta,\varphi) = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} P_n^m(\cos\theta)e^{im\varphi}, \qquad -n \le m < n \tag{11.33}$$

It is the eigenfunction to equation (11.28) corresponding to the eigenvalue

$$\eta = n(n+1), \qquad n = 0, 1, 2, 3, \ldots$$

satifying 2π-periodic boundary condition in the φ-direction, and boundedness condition at $\theta = 0$ and $\theta = \pi$.

When integrated over the surface of a sphere (at any radius), the spherical harmonic are orthogonal (since $e^{im\varphi}$, and P_n^m are orthogonal):

$$\int_0^{2\pi}\int_0^{\pi} Y_{nm}(\theta,\varphi)Y^*_{n'm'}(\theta,\varphi)\sin\theta d\theta d\varphi = \begin{cases} 0 & \text{if } n \ne n' \text{ or } m \ne m' \\ 1 & \text{if } n = n' \text{ and } m = m' \end{cases} \tag{11.34}$$

11.5.2 The spherical Bessel equation

Equation (11.29) in the radial direction is, with $\eta = n(n+1)$:

$$r^2R''(r) + 2rR'(r) + [\lambda^2r^2 - n(n+1)]R(r) = 0 \tag{11.35}$$

This is to be solved subject to the boundary condition: $R(0)$ bounded, and $R(a) = 0$, where $r = a$ is the radius of the sphere. Equation (11.35) is called the *spherical Bessel equation*. Its relation to the Bessel equation of order p is revealed through the transformation:

$$R(r) = r^{-1/2}y(x), \quad x \equiv \lambda r$$

$$x^2y''(x) + xy'(x) + (x^2 - p^2)y(x) = 0 \tag{11.36}$$

with $p = n + \frac{1}{2}$. The solution to (11.36) is J_p and Y_p (or J_p and J_{-p}).

The solution $R(r)$ which is bounded at $r = 0$ is the spherical Bessel function of the first kind:

$$R(r) = j_n(x) \equiv (\frac{\pi}{2x})^{1/2} J_{n+\frac{1}{2}}(x), \quad n = 0, 1, 2, 3, \ldots \tag{11.37}$$

To satisfy the boundary condition at $r = a$, we require

$$0 = R(a) = j_n(\lambda a) = (\frac{\pi}{2\lambda a})^{1/2} J_{n+\frac{1}{2}}(\lambda a).$$

This determines the eigenvalue λ is

$$\lambda = \lambda_{nk} = \frac{z_{n+\frac{1}{2},k}}{a}, \quad k = 1, 2, 3, \ldots \tag{11.38}$$

where z_{pk} is the kth zero of $J_p(z)$.

It turns out (as we will prove in a minute) that the spherical Bessel functions are related to the trigonometric functions through

$$j_n(x) = x^n(-\frac{1}{x}\frac{d}{dx})^n(\frac{\sin x}{x}). \tag{11.39}$$

So

$$\begin{aligned} j_0(x) &= \frac{\sin x}{x} \\ j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x} \\ j_2(x) &= (\frac{3}{x^3} - \frac{1}{x})\sin x - \frac{3}{x^2}\cos x, \ldots \end{aligned}$$

The first few zeros, $z_{n+\frac{1}{2},k}$ are listed below

k	1	2	3	4
$z_{1/2,k}$	3.14159	6.283185	9.424778	12.566370
$z_{3/2,k}$	4.493409	7.725252	10.904122	14.066194
$z_{5/21,k}$	5.763459	9.095011	12.322941	15.514603
$z_{7/21,k}$	6.987932	10.417119	13.698023	16.923621

Table 2: Zeros of the spherical Bessel functions

Exercise: For the symmetric case, $n = 0$, find the eigenvalue and eigenfunction of (11.35) satisfying $R(a) = 0$ and $R(0)$ bounded.

Write $x = \lambda r$ and

$$R(r) = x^{-1}w(x)$$

From (11.35) for the case of $n = 0$:

$$x^2 \frac{d^2}{dx^2}R + 2x\frac{d}{dx}R + x^2 R = 0$$

and

$$\begin{aligned}\frac{d}{dx}R &= -x^2 w(x) + x^{-1}w'(x)\\ \frac{d^2}{dx^2}R &= 2x^{-3}w(x) - 2x^{-2}w'(x) + x^{-1}w''(x),\end{aligned}$$

we have:

$$\begin{aligned}0 &= x^2 \frac{d^2}{dx^2}R + 2x\frac{d}{dx}R + x^2 R\\ &= x\{w''(x) + w(x)\}\end{aligned}$$

The solution for $w(x)$ is

$$w(x) = A\sin x + B\cos x$$

Therefore

$$R(r) = x^{-1}w(x) = A\frac{\sin x}{x} + B\frac{\cos x}{x}.$$

The solution which is bounded at $x = 0$ is constructed by setting $B = 0$:

$$R(r) = A\frac{\sin x}{x} = Aj_0(x).$$

To satisfy the boundary condition $R(a) = 0$, we require

$$0 = R(a) = A\frac{\sin(\lambda a)}{\lambda a},$$

implying

$$\lambda = \lambda_k = k\pi/a, \qquad k = 1, 2, 3, \ldots$$

Finally, the eigenfunctions are

$$R(r) = R_k(r) = \frac{\sin(\lambda_k r)}{(\lambda_k r)},$$

corresponding to the eigenvalue:

$$\lambda = \lambda_k = k\pi/a, \quad k = 1, 2, 3, \ldots$$

Exercise: Show that the spherical Bessel functions of zeroth order $j_0(\lambda_k r)$ is orthogonal with respect to weight r^2, specifically:

$$I_{k\ell} \equiv \int_0^a j_0(\lambda_k r) j_0(\lambda_\ell r) r^2 dr = \begin{cases} 0 & \text{if } k \neq \ell \\ \frac{a^3}{2\pi^2 k^2} & \text{if } k = \ell \end{cases}$$

where

$$\lambda_k = k\pi/a, \quad k = 1, 2, 3, \ldots$$

Since

$$j_0(\lambda_k r) = \frac{\sin(\lambda_k r)}{\lambda_k r}$$

$$\begin{aligned} I_{k\ell} &= \int_0^a \frac{\sin(\lambda_k r)\sin(\lambda_\ell r)}{\lambda_k \lambda_\ell} dr \\ &= \int_0^a \frac{\sin(k\pi r/a)\sin(\ell\pi r/a) dr}{\lambda_k \lambda_\ell} \\ &= \frac{a/2}{\lambda_k \lambda_\ell} \delta_{k\ell}, \end{aligned}$$

since

$$\frac{2}{a}\int_0^a \sin(k\pi r/a)\sin(\ell\pi r/a) dr = \delta_{k\ell},$$

from the orthogonality of the sines, previously established.

Next we turn to the general case of (11.35):

$$\frac{d}{dx}(x^2 \frac{d}{dx} R) + [x^2 - n(n+1)]R = 0, \tag{11.40}$$

(where we have written $x \equiv \lambda r$) and want to show that (11.39):

$$R = j_n(x) = x^n(-\frac{1}{x}\frac{d}{dx})^n(\frac{\sin x}{x}) \tag{11.41}$$

satisfies it. You may want to do so by substituting (11.41) into (11.40), but the differentiation is tedious. An easier alternative is to use the identity for Bessel functions

$$\frac{d}{dx}[x^{-p} J_p(x)] = -x^{-p} J_{p+1}(x)$$

by making use of the connection

$$j_n(x) \equiv (\frac{\pi}{2x})^{1/2} J_{n+\frac{1}{2}}(x), \quad n = 0, 1, 2, \ldots$$

For $p = n + 1/2$, the identity is

$$\frac{d}{dx}[x^{-n} \frac{1}{x^{1/2}} J_{n+\frac{1}{2}}] = -x^{-n} \frac{1}{x^{1/2}} J_{(n+1)+1/2}(x)$$

and so is

$$\frac{d}{dx}[x^{-n} j_n(x)] = -x^{-n} j_{n+1}(x).$$

Or:

$$j_n(x) = x^n(-\frac{1}{x}\frac{d}{dx})[x^{-n+1} j_{n-1}(x)].$$

For $n = 1$, this is

$$j_1 = x^1(-\frac{1}{x}\frac{d}{dx}) j_0(x) = x^1(-\frac{1}{x}\frac{d}{dx})(\frac{\sin x}{x})$$

For $n = 2$ and so on:

$$\begin{aligned}
j_2 &= x^2(-\frac{1}{x}\frac{d}{dx})[x^{-1} j_1(x)] = x^2(-\frac{1}{x}\frac{d}{dx})^2(\frac{\sin x}{x}) \\
j_3 &= x^3(-\frac{1}{x}\frac{d}{dx})^3(\frac{\sin x}{x}) \\
&\vdots \\
j_n &= x^n(-\frac{1}{x}\frac{d}{dx})^n(\frac{\sin x}{x}).
\end{aligned}$$