# Measurability of Convex Sets

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September 1, 2023

**Abstract**

What is Lebesgue measurable sets in $\mathbb{R}^n$? How to show that convex sets are measurable in $\mathbb{R}^n$? We answer these two questions in this report.

## 1 Introduction, Preliminaries

We list two theorems in [2, UC Davis notes], which is useful for establishing Lebesgue Measurable sets, how to prove sets are Lebesgue measurable, and the fact that open sets in $\mathbb{R}^n$ are alwas measurable. To start, take it for granted for the definitions and properties of outer measure $\mu^*$ for $\mathbb{R}^n, n \in \mathbb{N}$.

**Definition 1** (Lebesgue Measurable Sets)**.** $A \subseteq \mathbb{R}^n$ is Lebesgue Measurable if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C)$ for all subsets $E \subseteq \mathbb{R}^n$.

We may refer to Lebesgue measurable sets simply as measurable sets in the coming parts of the report.

**Remark 1.0.1.** There are many equivalent defintions. This is the Caratheodory Criterion for Lebesgue Measurable sets[2, definition 2.8]. They form a $\sigma$-algebra.

Next, rectangles (or boxes in $\mathbb{R}^n$) are measurable in $\mathbb{R}^n$, see [2, proposition 2.11], so are their finite or countably infinite unions. Finite union of boxes are called elementary sets according to Tao[4, 1.1.1]. Limits of elementary sets produces Jordan measurable sets. Open sets in $\mathbb{R}^n$ is the union of countable many rectangles [1, theorem 1.4][2, proposition 2.20], therefore open sets are measurable by the fact that open sets are countable unions of rectangles.

The Lebesgue measure of open sets are their outer measure [2, definition 2.10]. Next, sets that have an outer measure of zero is also a measurable sets in $\mathbb{R}^n$. We prove this here. It's a neat proof using the properties of outer measure and the Charatheodory criterion.

**Theorem 1** (Zero outer Measure sets Are Lebesgue Measurable)**.** Let $A \subseteq \mathbb{R}^n$, If $\mu^*(E) = 0$ then the set $E$ is a measurable sets.

*Proof.* Outer measure preserve partial order of $\subseteq$ then $\mu^*(A \cap E) \le \mu^*(E) = 0$. Also observe that

$$\begin{aligned} A \cap E^C \subseteq A &\implies m^*(A) \ge m^*(A \cap E^C) \\ &\implies m^*(A) \ge 0 + m^*(A \cap E^C) \\ &\iff m^*(A) \ge m^*(A \cap E) + m^*(A \cap E^C). \end{aligned}$$

The $\leq$ direction is already asserted by the sub-additivity of outer measure, combing it with the above results we have equality and hence, the Caratheodory condition. Therefore, sets with a zero outer measure are Lebesgue Measurable. $\square$

To show that a set is Lebesgue Measurable, we make use of the measurability of an open sets. We list [2, theorem 2.24] here without prove.

**Theorem 2** (Equivalent Characterizations of Lebesgue Measurability using Open Sets)**.** A subset $A \subseteq \mathbb{R}^b$ is Lebesgue measurable if and only if for every $\epsilon > 0$ thre is an open set $G \supset A$ such that

$$\mu^*(G \setminus A) < \epsilon.$$

Finally, we state a theorem for the measure of sets after its linear transform.

**Theorem 3** (Measure over Linear Transform)**.** Let $\mathbb{E} \subseteq$ be Lebesgue measurable, let $T : \mathbb{R}^d \mapsto \mathbb{R}^d$ be a linear mapping then the set $T(E)$ is Lebesgue measurable with measure $m(T(E)) = |\det(T)|m(E)$.

We skip the proof because it's requiring completing a lot of other proofs in the exercise of Tao's Book. We given an over view in what I think it's the right approach in proving the theorem. Using Tao's Introduction to measure theory[4], the full proof would require completing exercise 1.1.10, to show that a Unit Euclidean ball is Jordan Measurable. Then use the hint for exercise 1.1.11 to show that if $E \subseteq \mathbb{R}^n$ is an elementary set, then $T(E)$ where $T : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a measurable set with measure $|\det(T)|m(E)$. Then use the this result to complete exercise 1.2.21 to show that for Lebesgue measurable sets and a mapping $T : \mathbb{R}^d \mapsto \mathbb{R}^d$, we have $m(T(E)) = |\det(T)|m(E)$.

# 2 Convex Sets in $\mathbb{R}^n$

Before we start, we define relative interior/boundary, affine span of a set.

**Definition 2** (The Affine Span of a Set)**.** Let $Q \subseteq \mathbb{R}$, then the affine span of the set $Q$ denoted as $\text{aff}(Q)$ is the intersection of all affine space containing $Q$.

To demonsrate the construction of $\text{aff}(Q)$, let $x_0 \in Q$, then the line $\{x\} + \text{span}(x_1)$ for any $x_1 \neq x_0 \in Q$ will be an affine subspace that $Q$ contains. Then choose $x_2 \in Q, x_2 \notin \{x_0\} + \text{span}(x_1)$, then choose $x_3 \in Q, x_3 \notin \{x_0\} + \text{span}(x_1, x_2)$, repeat the process until the whole set $Q \subseteq \{x_0\} + \text{span}(x_1, \cdots, x_k)$, then $\text{aff}(Q) = \{x_0\} + \text{span}(x_1, \cdots, x_k)$. In finite dimension, the process terminates with $k \leq n$.

**Lemma 2.0.1** (Accessibility Lemma)**.** Let $C \subseteq \mathbb{R}^n$ be convex, then for all $x_1 \in C^\circ, x_0 \in \overline{C}$, the point $x_\lambda = \lambda x_1 + (1 - \lambda)x_0$ for all $\lambda \in (0, 1]$ will be in $C^\circ$.

This is theorem 6.1, proved in Rockafellar's convex book[3]. Additionally one can define the relative boundary to be: $\text{ribd}(C) = \overline{C} \setminus \text{ri}(C)$.

**Theorem 4** (Convex Sets are Measurable in Finite Euclidean Space)**.** $C \subseteq \mathbb{R}^n$ is a measurable sets when $C$ is convex.

Comming subsection attempts to prove the theorem. The following proof of the theorem came from my own investigation of the math stack exchange discussion.

## 2.1 Convex Sets Are Measurable

We had all the ingredients, we now summarize the idea of the proof. Firstly, the relative interior of a set is always measurable.

**Proposition 2.1** (Relative Interior of Convex is Measurable)**.** Let $C \subseteq \mathbb{R}^n$ be convex then its relative interior is measurable.

*Proof.* WLOG choose $\mathbf{0} \in C$, this can be done by translating any non-empty convex set by $C - \{x_0\}$ with $x_0 \in C$. If $C$ empty, then this is trivial and it has zero measure. Next, using affine span we have $C - \{\mathbf{0}\} \subseteq \text{span}(x_1, \cdots, x_m)$, with $m \leq n$. Then we pack the vector into columns of a matrix $A = [x_1 \; \cdots \; x_m]$. Then we have the equality $\text{ri}(C) = A(A^TA)^{-1}A^T\text{ri}(C)$. Using theorem 3, we have

$$\mu(\text{ri}(C)) = |\det(A^\dagger)|\mu(\text{ri}(C)), \text{where: } A^\dagger = A(A^TA)^{-1}A^T.$$

Observe that if $|\det(A^T)| = 0$, then it's trivial and the set has zero measure, (We note that in Tao's measure theory[4, prolouge], they construct on the augmented real has $0 * (+\infty) = 0$) else, $\text{aff}(Q) = \mathbb{R}^n$ in which case we have $C^\circ = \text{ri}C$ (Direct from the definition of relative interior.), hence the relative interior would be an open set, hence $\mu(\text{ri}(C)) = \mu(C^\circ)$ and an open set is measurable. □

Using this, we can state and prove that every convex sets in $\mathbb{R}^n$ is measurable.

**Proposition 2.2** (Convex Sets are Measurable)**.** Let $C \subseteq \mathbb{R}^n$ be convex, then $C$ is a Lebesgue Measurable set.

*Proof.* WLOG let $\mathbf{0} \in \text{ri}(C)$, let $x_1 \in \overline{C}$ by accessibility lemma, let $x_\lambda := \mathbf{0} + (1-\lambda)x_1$ then for all $\lambda \in (0, 1]$, $x_\lambda \in \text{ri}(C)$. So $x_1 \in (1-\lambda)^{-1}\text{ri}(C)$. $x_1$ arbitrary then $\overline{C} \subseteq (1-\lambda)^{-1}\text{ri}(C), \forall \lambda \in (0, 1]$. From the definiton of relative boundary:

$$\begin{aligned}
&\text{ribd}(C) = \overline{C} \setminus \text{ri}(C) \\
\implies &\text{ribd}(C) \subseteq (1-\lambda)^{-1}\text{ri}(C) \setminus \text{ri}(C) \\
&\mu \circ \text{ribd}(C) \leq \mu((1-\lambda)^{-1}\text{ri}(C)) - \mu \circ \text{ri}(C) \\
&\qquad = (1-\lambda)^{-n}\mu \circ \text{ri}(C) - \mu \circ \text{ri}(C) \\
&\qquad = \mu \circ \text{ri}(C) \lim_{\lambda \searrow 0}((1-\lambda)^{-n} - 1) \\
&\qquad = 0.
\end{aligned}$$

Zero outer measure set is Lebesgue Measurable by theorem 1, therefore $\text{ribd}(C)$ is measurable. Since a convex set is the disjoin union of its relative boundary, and relative interior, we have $C = \text{ri}(C) \sqcup \text{ribd}(C)$, the measure of two disjoint measurable sets is still a measurable set. □

# References


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