Chapter 3

Separation of Variables

3.1 Introduction

The method of separation of variables is a standard technique for solving linear PDEs in finite domains. Fourier series arise naturally from this method of solution.

3.2 An example of heat conduction in a rod:

Consider the problem of a copper rod of thermal diffusivity α^2 and of length L with a known initial temperature $u(x,0) = f(x)$. For $t > 0$, the two ends of the rod are maintained at a constant temperature of $0°$ C. Find the temperature of the rod as a function of x and t.

The mathematical problem is specified by the partial differential equation (PDE) governing the heat conduction process, the boundary conditions (BCs), and the initial condition (IC):

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad t > 0 \tag{3.1}$$

$$\text{BCs:} \quad u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0 \tag{3.2}$$

$$\text{IC:} \quad u(x,0) = f(x), \quad 0 < x < L. \tag{3.3}$$

(It is understood that u is expressed in units of °C.)

The boundary conditions we have imposed here are of Dirichlet type. We prefer to first make them homogeneous. If the boundary values were instead

$$u(0,t) = T_1, \quad u(L,t) = T_2,$$

we would first try to make them zero by defining a new unknown $\tilde{u}(x,t)$:

$$\tilde{u}(x,t) = u(x,t) - \left[T_1 + \frac{x}{L}(T_2 - T_1)\right],$$

so that the new problem for $\tilde{u}(x,t)$ has homogeneous boundary conditions:

$$\begin{aligned}
&\text{PDE:} \quad \tilde{u}_t = \alpha^2 \tilde{u}_{xx}, \quad 0 < x < L, \quad t > 0\\
&\text{BCs:} \quad \tilde{u}(0,t) = 0, \quad \tilde{u}(L,t) = 0, \quad t > 0\\
&\text{IC:} \quad \tilde{u}(x,0) = f(x) - \left[T_1 + \frac{x}{L}(T_2 - T_1)\right] \equiv g(x), \quad 0 < x < L.
\end{aligned}$$

In the following section we will consider the system (3.1), (3.2), and (3.3), with the understanding that if the boundary conditions are not homogeneous, we can make them so with a redefinition of u and f.

3.3 Separation of variables:

- **Step 1:** We first assume the solution to the PDE (3.1) is of the "separable" form:

$$u(x,t) = T(t)X(x). \tag{3.4}$$

- **Step 2:** Substituting the assumed form (3.4) into Eq. (3.1) yields

$$\frac{d}{dt}T(t) \cdot X(x) = \alpha^2 T(t) \frac{d^2}{dx^2} X.$$

  We divide both sides of the equation by $\alpha^2 T(t) X(x)$ to get

$$\frac{\frac{d}{dt}T(t)}{\alpha^2 T(t)} = \frac{\frac{d^2}{dx^2}X(x)}{X(x)}. \tag{3.5}$$

  [Division by α^2 is not necessary, and will not make any difference to the procedure if this is not done.]

- **Step 3:** Notice that the left-hand side of Eq. (3.5) is a function of t only, while the right-hand side is a function of x only. The only way a function of t can be equal to a function of x is for each to equal to a constant. Let this *separation constant* be denoted by K.

  So (3.5) becomes

$$\frac{d}{dt}T(t)/\alpha^2 T(t) = \frac{d^2}{dx^2}X(x)/X(x) = K. \tag{3.6}$$

This is actually *two* ordinary differential equations:

$$\frac{d^2}{dx^2}X(x) = KX(x), \tag{3.7}$$

and

$$\frac{d}{dt}T(t) = \alpha^2 KT(t). \tag{3.8}$$

- **Step 4:** We know how to solve Eq. (3.7) from Chapter 1 if K is negative. Let us solve this case first and later do the K positive case.

  Let $K = -\lambda^2 < 0$, where λ^2 is some positive constant. Eq. (3.7) becomes the harmonic oscillator equation:

  $$\frac{d^2}{dx^2}X(x) + \lambda^2 X(x) = 0, \tag{3.9}$$

  whose solution is, from Chapter 1:

  $$X(x) = A\sin\lambda x + B\cos\lambda x. \tag{3.10}$$

  [We can alternatively use the complex notation and write $X(x) = ae^{i\lambda x} + be^{-i\lambda x}$. In the present case it is more convenient, for the purpose of applying boundary condition, to use the real solution (3.10).]

  The constants A and B are (presumably) to be determined from the boundary conditions, which are, from (3.2) and (3.4):

  $$X(0) = 0, \quad X(L) = 0. \tag{3.11}$$

  From (3.10), we have

  $$X(0) = B,$$

  so the first boundary conditions demands that $B = 0$. Thus,

  $$X(x) = A\sin\lambda x, \tag{3.12}$$

  $$X(L) = A\sin\lambda L.$$

  The second boundary condition, $X(L) = 0$, then implies either

  $$A = 0 \quad \text{or} \quad \sin\lambda L = 0.$$

  The first possibility, $A = 0$, gives a trivial solution

  $$X(x) \equiv 0.$$

For nontrivial solutions, we need $\sin \lambda L = 0$, yielding the *eigenvalue*:

$$\lambda = \frac{n\pi}{L} \equiv \lambda_n, \quad n = 1, 2, 3, 4, 5, \dots \ . \tag{3.13}$$

[The nagative integer values of n doesnot give different solutions from the positive values because $A \sin(-\lambda_n x) = A' \sin(\lambda_n x)$, where $A' = -A$.]

The corresponding *eigenfunction* (from (3.12) and (3.13)) is:

$$X(x) = \sin \lambda_n x \equiv X_n(x). \tag{3.14}$$

[Note that we have set the arbitrary constant A to 1 in (3.14), without loss of generality, because we can always absorb a different A in $T(t)$. It is the *product*, $u(x,t) = T(t)X(x)$, that matters.]

It is easy to show that for $K > 0$, the solution to (3.7) is

$$X(x) = Ae^{\sqrt{K}x} + Be^{-\sqrt{K}x}.$$

$X(0) = 0$ implies

$$A + B = 0, \quad \text{or} \quad A = -B.$$

$X(L) = 0$ implies

$$Ae^{\sqrt{K}x} + Be^{-\sqrt{K}x} = 0,$$

or

$$B\left(-e^{\sqrt{K}L} + e^{-\sqrt{K}L}\right) = 0.$$

Since $K > 0$, $e^{\sqrt{K}L} > e^{-\sqrt{K}L}$, B should be zero. Thus $A = -B$ is also zero, leading to a trivial solution $X(x)$.

For $K = 0$, the ordinary differential equation for $X(x)$ becomes

$$\frac{d^2}{dx^2}X = 0.$$

Its solution is

$$X(x) = Ax + B.$$

Applying the boundary condition $X(0) = 0$ leads to $B = 0$. Applying $X(L) = 0$ then implies that $A = 0$. This again leads to the trivial solution $X(x)$.

Thus we conclude that for nontrivial solutions, K can only be negative, and furthermore K can only equal the following discrete values

$$K = -\lambda_n^2 = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, 4, \dots .$$

- **Step 5:** The $T(t)$ equation (3.8) now becomes

$$\frac{d}{dt}T = -\alpha^2 \lambda_n^2 T(t). \tag{3.15}$$

We denote the solution for each value of n:

$$T(t) = T_n(t) = T_n(0)e^{-\alpha^2 \lambda_n^2 t}. \tag{3.16}$$

- **Step 6:** We have in fact found an infinite number of solutions to the PDE (3.1), each satisfying the boundary conditions (3.2). They are of the form

$$\begin{aligned} u_n(x,t) &\equiv T_n(t)X_n(x) \\ &= T_n(0)e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L}, \end{aligned} \tag{3.17}$$

each corresponding to a value of n, $n = 1, 2, 3, 4, \ldots$.

In (3.17), the $T_n(0)$'s are arbitrary constants, (presumably) to be determined from the initial condition (3.3). However, this is only feasible if the initial condition (3.3) is such a sine function. For example, if the initial condition is

$$u(x,0) = \sin\frac{\pi x}{L}, \tag{3.18}$$

we should then pick $n = 1$, and $T_1(0) = 1$. This leads to the solution

$$u(x,t) = u_1(x,t) = e^{-\alpha^2\left(\frac{\pi}{L}\right)^2 t} \sin\frac{\pi x}{L}. \tag{3.19}$$

You should now verify that (3.19) satisfies the PDE (3.18), and is therefore *the* solution we are looking for in this particular special case.

- **Step 7:** To satisfy more general initial conditions, we need to construct a more general solution. We do so by adding up all possible component solutions in (3.17). [This is referred to as the *principle of superposition.*]

We write:

$$\begin{aligned} u(x,t) &= u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots \\ &= \sum_{n=1}^{\infty} u_n(x,t) \qquad (3.20) \\ &= \sum_{n=1}^{\infty} T_n(0)e^{-(\alpha\pi/L)^2 t} \sin\frac{n\pi x}{L}. \qquad (3.21) \end{aligned}$$

You should check that the sum in (3.20) satisfies the PDE (3.1) and the boundary condition (3.2), presuming the infinite series converges.

- **Step 8:** We now use the more general solution (3.21) to satisfy the initial condition (3.3). To satisfy

$$u(x,0) = f(x), \quad 0 < x < L.$$

we require

$$f(x) = \sum_{n=1}^{\infty} T_n(0) \sin \frac{n\pi x}{L}, \quad 0 < x < L, \tag{3.22}$$

The remaining task is to evaluate $T_n(0)$ given $f(x)$.

- **Step 9:** If we can express a function $f(x)$ in terms of what is now known as a *Fourier sine series*:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}, \quad 0 < x < L, \tag{3.23}$$

then we can equate each term in (3.22) and (3.23) and find that

$$T_n(0) = a_n, \quad n = 1, 2, 3, \ldots$$

and the problem is then completely solved.

The question is, can we always represent an arbitrary function $f(x)$ in the form of a Fourier sine series (3.23)? We will leave this issue to the next chapter, where Fourier series will be discussed.

The French scientist Joseph Fourier faced these questions when he studied the heat conduction problem, much as we presented it here. Fourier claimed in 1807, when he presented his paper on heat conduction to the Paris Academy, that an arbitrary function $f(x)$ could indeed be expressed as a sum of sines in the form of (3.23). There was not much mathematical rigor in Fourier's arguments; he was probably motivated by his physical understanding of the heat conduction problem for which the general solution should be expressible in the form of (3.21). Setting $t = 0$ in this solution would then seen to "require" the initial arbitrary temperature distribution to be expressible as a sum of sines, as in (3.22). This assertion of Fourier's was ridiculed by the mathematician Lagrange at the time. We now know of course that Fourier was right: Any physically reasonable function $f(x)$ can be written in the form of a sum of sines (or cosines for that matter).

Assuming that (3.23) is true and the series converges, we can obtain the coefficient a_n of the Fourier sine series of $f(x)$ in the following manner.

Multiply both sides of Eq. (3.23) by $\sin \frac{m\pi x}{L}$, where m is any integer, and integrate over the domain:

$$\int_0^L f(x) \sin \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} a_n \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx. \qquad (3.24)$$

Note that in (3.24) we have switched the order of integration and summation. This is allowable if the series is uniformly convergent. In the next chapter, we will derive the so-called *orthogonality relationship* of sines which states

$$\frac{2}{L} \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \delta mn \equiv \begin{cases} 1 & if \quad m = n \\ 0 & if \quad m \neq n. \end{cases} \qquad (3.25)$$

Substituting (3.25) into (3.24), we find that only one term remains in the infinite sum on the right-hand side:

$$\sum_{n=1}^{\infty} a_n \frac{L}{2} \delta mn = \frac{L}{2} a_m. \qquad (3.26)$$

Equating (3.26) to the left-hand side of (3.24), we obtain:

$$a_m = \frac{2}{L} \int_0^L f(x) \sin \frac{m\pi x}{L} dx, \quad \text{where } m = 1, 2, 3, 4, \ldots. \qquad (3.27)$$

Since m is an arbitrary index, we can use any other symbol, including n. Thus (3.27) is the same as

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \ldots. \qquad (3.28)$$

Either (3.27) or (3.28) can be used to generate the coefficients a_1, a_2, a_3, a_4, ...

- **Step 10:** Finally, we have the solution which satisfies the PDE, the BCs, and the IC:

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(\alpha n\pi/L)^2 t} \sin \frac{n\pi x}{L}, \quad 0 < x < L$$

$$\text{where} \quad a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx. \qquad (3.29)$$

You should try to verify *a posteri* that (3.29) does indeed satisfy (3.1), (3.2), and (3.3).

3.4 Physical interpretation of the solution:

The general solution (3.29) to the heat conduction problem, although complicated, has some simple physical interpretations.

It can be rewritten as

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t/t_e} \sin \frac{n\pi x}{L},$$

where $t_e \equiv (L/(\pi\alpha))^2$ is about an hour for a copper rod of length 2m (with $\alpha^2 = 1.16$ cm^2/s). The initial temperature distribution

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}$$

can be quite complicated and can consist of many sine modes. As time goes on, however, the small scales (i.e. the higher n sine modes) in the sum for $u(x,t)$ decay much faster than the larger scale modes. If a_1 and a_n are nonzero, we can look at the ratio

$$\frac{a_n e^{-n^2 t/t_e}}{a_1 e^{-t/t_e}}$$

of the coefficients of the nth mode and the first mode.

At $t = t_e$, about one hour later, the ratio gets smaller and smaller for increasing n because

$$\begin{aligned} e^{-n^2}/e^{-1} \quad &= 0.05 \quad \text{for } n = 2 \\ &3 \times 10^{-4} \quad \text{for } n = 3 \\ &3 \times 10^{-7} \quad \text{for } n = 4. \end{aligned}$$

Therefore, for most practical purposes, the full solution is dominated by the first term, the lowest sine mode:

$$u(x,t) \simeq a_1 e^{-t/t_e} \sin \frac{\pi x}{L} \quad \text{for } t \gtrsim t_e.$$

Eventually, even this lowest mode decays to zero, as the rod approaches a uniform zero temperature consistent with the temperature specified at the boundaries.

For small times, smaller scale modes are significant, if these were present in the initial temperature distribution. However, the smaller scales decay faster than the larger scales. We see a gradual *smoothing* of the solution. This is a common property of diffusion and heat conduction, which always tends to smooth out gradients that are present.

3.5 A vibrating string problem:

Consider a vibrating (guitar) string of length L—the length of the vibrating part being determined by where the player presses on the string. Let $c^2 = T/\rho$, where T is the tension on the string, which can be adjusted by the player, and ρ is the density of its material.

$$\text{PDE:} \quad u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0 \tag{3.30}$$

$$\text{BCs:} \quad u(0,t) = 0, \quad u(L,t) = 0 \tag{3.31}$$

$$\text{IC:} \quad u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L. \tag{3.32}$$

A few words are in order about the initial conditions. Mathematically, since there is a second derivative in t in the governing PDE, there should be two initial conditions to completely specify the problem. We will see this as we proceed further with the solution of the problem and discover that there will be undetermined constants in the solution if we don't prescribe a second initial condition. This is unlike the case of the heat equation, which needs only a first order derivative in t be given.

In (3.32), $f(x)$ is the shape of the initial displacement and $g(x)$ is the shape of the initial velocity. We will see that the intensity of the sound and the spectrum of frequencies of sound generated depend on both initial conditions.

We again use the method of separation of variables to solve this problem. Since we have already discussed the ten steps of this method in detail in the previous section, there is no need to repeat every detail here again. You can follow this template in doing your homework and exam problems.

We first assume that the solution can be written in the separable form:

$$u(x,t) = T(t)X(x)$$

anticipating that we will ultimately superpose such solutions to satisfy initial conditions.

Substituting into the PDE yields

$$\frac{1}{c^2}\frac{d^2T}{dt^2}/T = \frac{d^2X}{dx^2}/X = -\lambda^2,$$

where $-\lambda^2$ is the separation constant. Solving the ordinary differential equation for X:

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0,$$

subject to the boundary conditions:

$$X(0) = 0, \quad X(L) = 0,$$

yields the eigenvalues:

$$\lambda = \lambda_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

and the corresponding eigenfunction:

$$X(x) = X_n(x) = \sin \lambda_n x$$

(again scaling out any multiplicative factor)

The T-equation:

$$\frac{d^2T}{dt^2} = -c^2\lambda^2 T$$

can be solved for each value of $\lambda = \lambda_n$ to yield:

$$T(t) = T_n(t) = A_n \sin(\omega_n t) + B_n \cos(\omega_n t),$$

where $\omega_n = c\lambda_n = cn\pi/L$ is the *frequency* of oscillation and the constants A_n and B_n remain arbitrary.

We construct the general solution by superimposing all possible solutions of the form $T_n(t)X_n(x)$, to yield:

$$\boxed{\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} T_n(t)X_n(x) \\ &= \sum_{n=1}^{\infty} [A_n \sin(\omega_n t) + B_n \cos(\omega_n t)] \sin \frac{n\pi x}{L} \end{aligned}}\ . \tag{3.33}$$

To satisfy the initial conditions (3.32), we require

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad 0 < x < L, \tag{3.34}$$

and

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \frac{cn\pi}{L} A_n \sin \frac{n\pi x}{L}, \quad 0 < x < L. \tag{3.35}$$

(3.34) is a Fourier sine series for the initial displacement $f(x)$, and so (see (3.29))

$$\boxed{B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx}\ . \tag{3.36}$$

(3.35) is a Fourier sine series for the initial velocity $g(x)$, and so

$$\boxed{A_n = \frac{2}{\pi n c}\int_0^L g(x)\sin\frac{n\pi x}{L}dx}\,. \tag{3.37}$$

The solution (3.33) is now completely specified. Convergence of the series solution remains to be considered.

Physical Interpretation:

Unlike the solution of the heat/diffusion equation, the solution to the wave equation does not decay in time. Instead there are *standing waves* set up between the two ends of the string. The gravest (fundamental) standing-mode, $\sin\frac{\pi x}{L}$, oscillates with a frequency $\omega_1 = c\lambda_1$, and the nth standing-mode, $\sin\frac{n\pi x}{L}$, oscillates with a frequency

$$\omega_n = c\lambda_n = \frac{n\pi c}{L} = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}}.$$

i.e. n times the fundamental frequency w_1.

This property, that all frequencies generated by a vibrating string are integer multiples of the fundamental frequency, is what makes the sound of a violin or guitar pleasing to the human ear. This property is a consequence of the one space dimensionality of the vibrating string, and is not shared by two dimensional vibrating membranes (such as a drumhead), where the higher-order frequencies are not integer multiples of the fundamental one.

The frequencies produced by a vibrating string

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}}$$

depend on a few physical parameters, as revealed by the solution. The higher the tension on the string, the higher the frequency; the denser the string material, the lower the frequency; and the longer the length of the vibrating part of the string, the lower the frequency. The latter part is controlled by the guitarist's placement of his (or her) finger when he clamps down on the string.

3.6 Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2\mu}\nabla^2\psi + V\psi$$

where $\hbar$ is Planck's constant ($\hbar = 1.054 \times 10^{-34} J \cdot s$), μ is the mass of the particle under consideration, and V is the potential energy (time independent) of the force field ($F = -\nabla V$). ψ is the wave function; $|\psi|^2$ gives the probability density of finding the particle in a particular location. Consider the following simple example:

1-D Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2}\psi + V(x)\psi, \qquad -\infty < x < \infty.$$

Suppose $V(x) = 0$ if $0 < x < L$, but ∞ elsewhere. Therefore the particle cannot be located anywhere other than in $0 < x < L$. The problem simplifies to

$$\begin{aligned} &\text{PDE:} \quad i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2}\psi \qquad 0 < x < L \\ &\text{BC:} \quad \psi(0,t) = 0, \quad \psi(L,t) = 0 \end{aligned}$$

Separation of variables:

$$\begin{aligned} \psi(x,t) &= u(x)T(t) \\ \frac{i\hbar T'(t)}{T(t)} &= -\frac{\hbar^2}{2\mu}\frac{u''(x)}{u(x)} = \text{const} \equiv E \\ T'(t) &= -iET(t)/\hbar \\ u''(x) &= -E2\mu/\hbar^2 u(x). \end{aligned}$$

Since

$$T(t) = T(0)e^{-iEt/\hbar}$$

$E/\hbar$ is interpreted as the frequency of oscillation, ω. In quantum mechanics, ω times $\hbar$ is the energy of the oscillator. This is the reason the symbol E was used ($E = \omega\hbar$) as the separation constant. It is to be determined from the boundary value problem as an eigenvalue.

$$\begin{cases} u''(x) = -2\mu E/\hbar^2 u(x) \\ u(0) = 0, \quad u(L) = 0 \end{cases}$$

Solution:

$$u(x) = A \sin\sqrt{\frac{2\mu E}{\hbar^2}}x + B\cos\sqrt{\frac{2\mu E}{\hbar^2}}x$$

$u(0) = 0$ implies $B = 0$. $u(L) = 0$ implies

$$A \sin\sqrt{\frac{2\mu E}{\hbar^2}}L = 0.$$

Therefore $\sqrt{2\mu E}L/\hbar = n\pi$, $n = 1, 2, 3, \ldots$, which is:

$$E = E_n = \frac{n^2\pi^2\hbar^2}{2\mu L^2}, \qquad n = 1,\ 2,\ \ldots.$$

The energy is quantized! The solution is:

$$\psi(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} e^{-iE_n t/\hbar}, \quad 0 < x < L, \quad 0 \text{ elsewhere}.$$

a_n can be found from the initial condition, but this step is often not done. Conceptually the more important result is the quantization of the eigenvalues and hence the quantization of the energy.

3.7 Exercises

1. Solve the following heat equation:

$$\begin{aligned}
&\text{PDE:} && u_t = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0\\
&\text{BCs:} && u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0\\
&\text{IC:} && u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 < x < L,
\end{aligned}$$

   where the a_n's are known constants.

2. (a) Solve the following wave equation for a guitar string of density ρ under tension T:

$$\begin{aligned}
&\text{PDE:} && u_{tt} = \left(\frac{T}{\rho}\right) u_{xx}, \quad 0 < x < L, \quad t > 0\\
&\text{BCs:} && u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0\\
&\text{IC:} && u(x,0) = a_1 \sin\frac{\pi x}{L}, \quad 0 < x < L\\
& && u_t(x,0) = 0, \quad 0 < x < L.
\end{aligned}$$

   (b) What is the frequency of oscillation of the string?

3.8 Solutions

1. Solve the following heat equation:

$u_t \;=\; \alpha^2 u_{xx}, \quad 0 < x < L, \quad t > 0$

with initial condition $u(x,0) \;= \sum_{n=0}^{\infty} a_n \sin(n\pi x/L)$,

and boundary conditions $u(0,t) = 0$, and $u(L,t) = 0$.

We use separation of variables. Since we have homogeneous Dirichlet BCs we use:

$u(x,t) \;= \sum_{n=0}^{\infty} T_n(t) \sin(n\pi x/L)$.

Plugging into the PDE, we are left with a first order ODE in time which has the solution:

$T_n(t) \;=\; A_n e^{-(n\pi/L)^2 \alpha^2 t}$.

Using the initial condition we find:

$\sum_{n=0}^{\infty} A_n \sin(n\pi x/L) = \sum_{n=0}^{\infty} a_n \sin(n\pi x/L)$.

Thus $A_n = a_n$ and

$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-(n\pi/L)^2 \alpha^2 t} \sin(n\pi x/L)$.

2. Solve the following vibrating string problem:

$u_{tt} \;=\; (T/\rho) u_{xx}$, $(T/\rho) =$ constant,

with initial conditions $u(x,0) \;= a_1 \sin(\pi x/L)$, $u_t(x,0) \;= 0$,

and boundary conditions $u(0,t) = 0$, and $u(L,t) = 0$.

We use separation of variables. Since we have homogeneous Dirichlet BC we find:

$u(x,t) \;= \sum_{n=0}^{\infty} T_n(t) \sin(n\pi x/L)$.

Plugging into the PDE we obtain a second order ODE in time, which has the solution:

$T_n(t) \;=\; A_n \sin(\omega_n t) + B_n \cos(\omega_n t)$, where $\omega_n = (n\pi/L)(T/\rho)^{1/2}$.

Using the initial conditions we find:

$\sum_{n=0}^{\infty} A_n \sin(n\pi x/L) = 0$ and $\sum_{n=0}^{\infty} B_n \sin(n\pi x/L) = a_1 \sin(\pi x/L)$.

Thus we have that: $A_n = 0$ for all n, $B_1 = a_1$ and $B_n = 0$ for $n \neq 1$, leaving us with:

$u(x,t) = a_1 \cos((\pi/L)(T/\rho)^{1/2} t) \sin(\pi x/L)$.

The frequency of vibration is $(\pi/L)(T/\rho)^{1/2}$.