Chapter 7

Fourier Series, Fourier Transform and Laplace Transform

7.1 Introduction

In the previous chapter, we discussed the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}, \quad -L < x < L, \tag{7.1}$$

for $f(x)$ in the interval $-L < x < L$ where

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{in\pi x/L} dx. \tag{7.2}$$

Previously, when sine and cosine series were discussed, we alluded to *Dirichlet's Theorem*, which tells us conditions under which Fourier series is a satisfactory representation of the original function $f(x)$. The full Dirichlet's Theorem is stated below.

7.2 Dirichlet Theorem

If $f(x)$ is a bounded and piecewise continuous function in $-L < x < L$, its Fourier series representation converges to $f(x)$ at each point x in the interval where $f(x)$ is continuous, and to the average of the left- and right-hand limits of $f(x)$ at those points where $f(x)$ is discontinuous. If $f(x)$ is periodic with period $2L$, the above statement applies throughout $-\infty < x < \infty$.

This theorem is easy to understand. The Fourier series, consisting of sines and cosines of period $2L$, has period $2L$. If $f(x)$ itself is also has the same period, then the Fourier series can be a good representation of $f(x)$ over the whole real axis $-\infty < x < \infty$. If on the other hand, $f(x)$ is either not periodic, or not defined beyond the interval $-L < x < L$, the Fourier series gives a good representation of $f(x)$ only in the stated interval. Beyond $-L < x < L$, the Fourier series is periodic and so simply repeats itself, but $f(x)$ may or may not be so. The above statements apply where $f(x)$ is continuous. When $f(x)$ takes a jump at a point, say x_0, the value of $f(x)$ at x_0 is not defined. The value which the Fourier seris of $f(x)$ converges to is the average of the value immediately to the left of x_0 and to the right of x_0, i.e. to $\lim_{\epsilon\to 0}\frac{1}{2}(f(x_0-\epsilon)+\frac{1}{2}f(x_0+\epsilon))$. We have already demonstrated this with the Fourier sine series. The same behavior applies to the full Fourier series.

7.3 Fourier integrals

Unless $f(x)$ is periodic, the Fourier series representation of $f(x)$ is an appropriate representation of $f(x)$ only over the interval $-L < x < L$. Question: Can we take $L \to \infty$, so as to obtain a good representation of $f(x)$ over the whole interval $-\infty < x < \infty$? The positive answer leads to the Fourier integral, and hence the Fourier transform, provided $f(x)$ is "integrable" over the whole domain.

We first rewrite (7.1) as a Riemann sum by letting

$$\Delta\omega = \pi/L \quad \text{and} \quad \omega_n = n\pi/L.$$

Thus, (7.1) becomes

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\Delta\omega F_n \cdot e^{-i\omega_n x}, \tag{7.3}$$

where

$$F_n \equiv (2Lc_n) = \int_{-L}^{L} f(x')e^{i\omega_n x'}dx'. \tag{7.4}$$

[We have changed the dummy variable in (7.4) from x to x', to avoid confusion later.]

In the limit $L \to \infty$, ω_n becomes ω, which takes on continuous values in $-\infty < \omega < \infty$. So $F_n \to F(\omega)$, where

$$F(\omega) = \int_{-\infty}^{\infty} f(x')e^{i\omega x'}dx', \tag{7.5}$$

and (7.3) becomes the integral:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}d\omega. \tag{7.6}$$

Substituting (7.5) into (7.6) leads to the *Fourier integral formula*:

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x')e^{i\omega x'}dx'\right]e^{-i\omega x}d\omega}\,. \tag{7.7}$$

The validity of the formula (7.7) is subject to the integrability of the function $f(x)$ in (7.5). Furthermore, the left-hand side of (7.7) must be modified at points of discontinuity of $f(x)$ to be the average of the left- and right-hand limits at the discontinuity, because the Fourier series, upon which the (7.7) is based, has this property.

The formula shows that the operation of integrating $f(x)e^{i\omega x}$ over all x is "reversible", by multiplying it by $e^{-i\omega x}$ and integrating over all ω. (7.7) allows us to define Fourier transforms and inverse transforms.

7.4 Fourier transform and inverse transform

Let the Fourier transform of $f(x)$ be denoted by $\mathcal{F}[f(x)]$. We should use (7.5) for a definition of such an operation:

$$\boxed{F(\omega) = \mathcal{F}[f(x)] \equiv \int_{-\infty}^{\infty} f(x)e^{i\omega x}dx}\,. \tag{7.8}$$

Let $\mathcal{F}^{-1}$ be the inverse Fourier transform, which recovers the original function $f(x)$ from $F(\omega)$. (7.6) tells us that the inverse transform is given by

$$\boxed{f(x) = \mathcal{F}^{-1}[F(\omega)] \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}d\omega}\,. \tag{7.9}$$

Equations (7.8) and (7.9) form the transform pair. Not too many integrals can be "reversed". Take for example

$$\int_{-\infty}^{\infty} f(x)dx.$$

Once integrated, the information about $f(x)$ is lost and cannot be recovered. Therefore the relationships such as (7.8) and (7.9) are rather special and have wide application in solving PDEs.

Note that our definition of the Fourier transform and inverse transform is not unique. One could, as in some textbooks, put the factor $\frac{1}{2\pi}$ in (7.8) instead of in (7.9), or split it as $\frac{1}{\sqrt{2\pi}}$ in (7.8) and $\frac{1}{\sqrt{2\pi}}$ in (7.9). The only thing that matters is that in the Fourier integral formula there is the factor $\frac{1}{2\pi}$ when the two integrals are both carried out. Also, in (7.7), we can change ω to $-\omega$ without changing the form of (7.7). So, the Fourier transform in (7.8) can alternatively be defined with a negative sign in front of ω in the exponent, and the inverse transform in (7.9) with a positive sign in front of ω in the exponent. It does not matter to the final result as long as the transform and inverse transform have opposite signs in front of ω in their exponents.

7.5 Laplace transform and inverse transform

The Laplace transform is often used to transform a function of time, $f(t)$ for $t > 0$. It is defined as

$$\boxed{\mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}dt \equiv \tilde{f}(s)}\,. \tag{7.10}$$

Mathematically, it does not matter whether we denote our independent variable by the symbol t or x; nor does it matter whether we call t time and x space or vice versa. What does matter for Laplace transforms is the integration ranges only over a semi-infinite interval, $0 < t < \infty$. We do not consider what happens before $t = 0$. In fact, as we will see, we need to take $f(t) = 0$ for $t < 0$.

Functions which are zero for $t < 0$ are called one-sided functions. For one-sided $f(t)$, we see that the Laplace transform (7.10) is the same as the Fourier transform (7.8) if we replace x by t and ω by is. That is, from (7.8)

$$\begin{aligned} F(is) &= \int_{-\infty}^{\infty} f(t)e^{-st}dt = \int_0^{\infty} f(t)e^{-st}dt \\ &\equiv \tilde{f}(s). \end{aligned} \tag{7.11}$$

Since the Laplace transform is essentially the same as the Fourier transform, we can use the Fourier inverse transform (7.9) to recover $f(t)$ from its Laplace transform $\tilde{f}(s)$. Let

$$f(t) = \mathcal{L}^{-1}[\tilde{f}(s)].$$

Then the operation $\mathcal{L}^{-1}$, giving the inverse Laplace transform, must be defined by

$$f(t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\tilde{f}(s)e^{st}ds. \tag{7.12}$$

This is because

$$\begin{aligned}f(t)=\mathcal{F}^{-1}[F(\omega)]&=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{-i\omega t}d\omega\\&=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\tilde{f}(s)e^{st}ds\end{aligned}$$

through the change from ω to is. Since the inverse Laplace transform involves an integration in the complex plane, it is usually not discussed in elementary mathematics courses which deal with real integrals. Tables of Laplace transforms and inverse transforms are used instead. Our purpose here is simply to point out the connection between Fourier and Laplace transform, and the origin of both in Fourier series.

Note: The inverse Laplace transform formula in (7.12) was obtained from the Fourier integral formula, which applies only to integrable functions. For "nonintegrable", but one-sided $f(t)$, (7.12) should be modified, with the limits of integration changed to $\alpha-i\infty$ and $\alpha+i\infty$, where α is some positive real constant bounding the exponential growth of f allowed.

To show this, suppose $f(t)$ is not integrable because it grows as $t\to\infty$. Suppose that for some positive α the product

$$g(t)\equiv f(t)e^{-\alpha t}\to 0 \text{ as } t\to\infty$$

and is thus integrable. We proceed to find the Laplace transform of $g(t)$:

$$\begin{aligned}\widetilde{g}(s)\equiv\mathcal{L}[g(t)]&=\int_0^{\infty}f(t)e^{-\alpha t}e^{-st}d\tau\\&=\int_0^{\infty}f(t)e^{-(\alpha+s)t}dt.\end{aligned}$$

Thus if

$$\widetilde{f}(s)=\mathcal{L}[f(t)]=\int_0^{\infty}f(t)e^{-st}dt,$$

then

$$\widetilde{g}(s)=\widetilde{f}(\alpha+s).$$

The inverse of $\widetilde{g}(s)$ is

$$
\begin{aligned}
g(t) = \mathcal{L}^{-1}[\widetilde{g}(s)] &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \widetilde{g}(s)e^{st}ds \\
&= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \widetilde{f}(\alpha+s)e^{st}ds \\
&= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \widetilde{f}(s')e^{-\alpha t}e^{s't}ds'
\end{aligned}
$$

where we have made the substitution $s' = \alpha + s$. Since $g(t) \equiv f(t)e^{-\alpha t}$, we obtain, on cancelling out the $e^{-\alpha t}$ factor:

$$
f(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \widetilde{f}(s')e^{s't}ds'.
$$

This is a modified, and more general, formula for Laplace transform of a one-sided function $f(t)$, whether or not it decays as $t \to \infty$, as long as $f(t)e^{-\alpha t}$ is integrable.

$$
\begin{aligned}
\widetilde{f}(s) = \mathcal{L}[f(t)] &= \int_0^{\infty} f(t)e^{-st}dt, \ \ \mathrm{Re}\ s \geq \alpha \\
f(t) = \mathcal{L}^{-1}[\widetilde{f}(s)] &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \widetilde{f}(s)e^{st}ds.
\end{aligned}
$$

7.6 Parseval's Theorem

7.6.1 Parseval's Theorem

$$
\int_{-\infty}^{\infty} f(x)g(x)^{*}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G(\omega)^{*}d\omega, \tag{7.13}
$$

where

$$
F(\omega) = \mathcal{F}[f(x)] \quad \text{and} \quad G(\omega = \mathcal{F}[g(x)]
$$

Proof: since

$$
\begin{aligned}
g(x) = \mathcal{F}^{-1}[G(\omega)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{-i\omega x}d\omega \\
\int_{-\infty}^{\infty} f(x)g(x)^{*}dx &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty} G(\omega)^{*}e^{i\omega x}d\omega dx. \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{i\omega x}dxG(\omega)^{*}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G(\omega)^{*}d\omega.
\end{aligned}
$$

7.6.2 Placherel's Theorem

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \tag{7.14}$$

Proof: Let $g(x) = f(x)$ in Parseval's Theorem.

7.7 Cauchy-Schwarz inequality:

$$|\int_{-\infty}^{\infty} f(x)g(x)^* dx|^2 \leq \int_{-\infty}^{\infty} |f|^2 dx \cdot \int_{-\infty}^{\infty} |g(x)|^2 dx \tag{7.15}$$

This inequality is equivalent to the inner product:

$$(f, g) \equiv \int_{-\infty}^{\infty} f(x) g^*(x) dx \quad \text{between two "vectors" } f \text{ and } g$$

being no greater than the product of the magnitudes of f and g

$$\|f\| \equiv \left(\int_{-\infty}^{\infty} |f|^2 dx \right)^{1/2}, \quad \|g\| \equiv \left(\int_{-\infty}^{\infty} |g|^2 dx \right)^{1/2}.$$

That is

$$|f, g|^2 \leq \|f\|^2 \cdot \|g\|^2.$$

Proof: For all t,

$$0 \leq \|f + tg\|^2 = \|f\|^2 + |t|^2 \|g\|^2 + 2\,\mathrm{Re}\,\{t(f, g)\}$$

Take $t = -(f, g)^* / \|g\|^2$, $|t| = |(f, g)| / \|g\|^2 = r / \|g\|^2$, where we have written $(f, g) = re^{i\theta}$.

$$\begin{aligned}
\mathrm{Re}\,\{t(f, g)\} &= -\,\mathrm{Re}\,\{\frac{|(f, g)|^2}{\|g\|^2}\} = -r^2 / \|g\|^2 \\
0 &\leq \|f\|^2 + r^2 / \|g\|^2 - 2r^2 \|g\|^2 \\
&= \|f\|^2 - r^2 / \|g\|^2 \\
&= \{\|f\|^2 \|g\|^2 - |(f, g)]|^2\} / \|g\|^2
\end{aligned}$$

This if $\|g\|^2 \neq 0$, we have

$$|(f, g)|^2 \leq \|f\|^2 \cdot \|g\|^2.$$

7.8 The Uncertainty Principle

7.8.1 Mathematical Uncertainty Principle

Let

$$\Delta_a f(x) \equiv \int_{-\infty}^{\infty} (x-a)^2 |f(x)|^2 dx \Big/ \int_{-\infty}^{\infty} |f(x)|^2 dx$$

be the uncertainty of $f(x)$ about a point $x = a$, and

$$\Delta_\alpha F(\omega) \equiv \int_{-\infty}^{\infty} (\omega-\alpha)^2 |F(\omega)|^2 d\omega \Big/ \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

be the "uncertainty" of $F(\omega)$ about a wavenumber $\omega = \alpha$.

The mathematical uncertainty principle is:

$$\Delta_a f(x) \cdot \Delta_\alpha F(\omega) \geq \frac{1}{4}. \tag{7.16}$$

Proof: Let's first do it for $a = 0$, $\alpha = 0$. The more general case of $a \neq 0$, $\alpha \neq 0$ can be obtained by shifting the axes.

Consider

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega \\ f'(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} -i\omega F(\omega) e^{-i\omega x} d\omega \end{aligned}$$

That is, $-i\omega F(\omega)$ is the Fourier transform of $f'(x)$.

Using $f'(x)$ in Plancherel's Theorem then yields

$$\begin{aligned} \int_{-\infty}^{\infty} |f'(x)|^2 dx &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\omega F(\omega)|^2 d\omega \\ &= \Delta_\alpha F(\omega) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \\ &= \Delta_\alpha F(\omega) \cdot \int_{-\infty}^{\infty} |f(x)|^2 dx. \end{aligned} \tag{7.17}$$

Plancherel's Theorem has been used in the last step.

Now consider the following integration by parts:

$$\int_{-\infty}^{\infty} f'(x) [x f(x)]^* dx$$

$$= x|f(x)|^2 \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} [|f(x)|^2 + xf(x)f'(x)^*]dx$$

Assuming that $x|f(x)|^2 \to 0$ as $x \to \pm\infty$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} |f(x)|^2 dx &= -\int_{-\infty}^{\infty} \{xf(x) \cdot [f'(x)]^* + f'(x)[xf(x)]^*\}dx \\ &= -2\int_{-\infty}^{\infty} \mathrm{Re}\,\{xf(x) \cdot [f'(x)]^*\}dx\end{aligned}$$

Since Re $z \le |z|$, we have

$$\begin{aligned}\left(\left|\int_{-\infty}^{\infty} |f(x)|^2 dx\right|\right)^2 &= 4\left(\left|\int_{-\infty}^{\infty} \mathrm{Re}\,\{xf(x) \cdot [f'(x)]^*\}dx\right|\right)^2 \\ &\le 4\left(\left|\int_{-\infty}^{\infty} |xf(x)||f'(x)|dx\right|\right)^2 \\ &\le 4\int_{-\infty}^{\infty} |xf(x)|^2 dx \cdot \int_{-\infty}^{\infty} |f'(x)|^2 dx \qquad (7.18)\end{aligned}$$

using the Cauchy-Schwarz inequality in the last step.

This inequality (7.18) is

$$\frac{\int_{-\infty}^{\infty} x^2|f(x)|^2 dx}{\int_{-\infty}^{\infty} |f(x)|^2 dx} \cdot \frac{\int_{-\infty}^{\infty} |f'(x)|^2 dx}{\int_{-\infty}^{\infty} |f(x)|^2 dx} \ge \frac{1}{4} \qquad (7.19)$$

Thus, using (7.17)

$$\Delta_a f(x) \cdot \Delta_\alpha F(\omega) \ge \frac{1}{4}. \qquad (7.20)$$

Although this is proved only for $a = 0$ and $\alpha = 0$, it is easy to show that it remains true for $a \ne 0$, $\alpha \ne 0$, simply by shifting the x axis by a and the ω axis by α.

7.8.2 Quantum Mechanical Uncertainty Principle

This quantum mechanical uncertainty principle is due to Heisenberg. It states that there is a limit to how presicely one can determine a particle's position and momentum simultaneously. We will see that it is actually a mathematical result, with quantum mechanical interpretation added.

In quantum mechanics, the probability density of finding a particle in the interval between x and $x + dx$ is

$$|\psi(x,t)|^2,$$

where $\psi(x,t)$ is the wavefunction. Thus

$$\int_{-\infty}^{\infty} |\psi(x,t)|^2 dx = 1, \tag{7.21}$$

since the particle has to be located somewhere.

The uncertainty of finding a particle at $x = a$ is defined as

$$\begin{aligned} \Delta_{\text{position}} &\equiv \int_{-\infty}^{\infty} (x-a)^2 |\psi(x,t)|^2 dx \Big/ \int_{-\infty}^{\infty} |\psi(x,t)|^2 dx \\ &= \int_{-\infty}^{\infty} (x-a)^2 |\psi(x,t)|^2 dx. \end{aligned} \tag{7.22}$$

If you know for certain that the particle is at $x = 0$, then $|\psi(x,t)| = \delta(x)$, and Δ_{position} would be zero.

In quantum mechanics, the momentum of a particle is related to the wavenumber ω through the de Broglie formula:

$$p = \omega\hbar. \tag{7.23}$$

Let $\Psi(\omega,t)$ be the Fourier transform of $\psi(x,t)$, and consider:

$$\Delta_{\alpha/\hbar}\Psi(\omega,t) \equiv \int_{-\infty}^{\infty} (\omega - \alpha/\hbar)^2 |\Psi(\omega,t)|^2 d\omega \Big/ \int_{-\infty}^{\infty} |\Psi(\omega,t)|^2 d\omega$$

Through a change in variable: $\omega = p/\hbar$, this becomes

$$\Delta_{\alpha/\hbar}\Psi(\omega,t) = \frac{1}{\hbar^3} \int_{-\infty}^{\infty} (p-\alpha)^2 |\Psi(\frac{p}{\hbar},t)|^2 dp \Big/ \int_{-\infty}^{\infty} |\Psi(\omega,t)|^2 d\omega \tag{7.24}$$

Let the uncertainty in momentum about $p = \alpha$ be defined by:

$$\Delta_{\text{momentum}} \equiv \int_{-\infty}^{\infty} (p-\alpha^2) |\Psi(\frac{p}{\hbar},t)|^2 dp \Big/ \int_{-\infty}^{\infty} |\Psi(\frac{p}{\hbar},t)|^2 dp$$

Here $\Psi(\frac{p}{\hbar},t)$ is the unscaled probability density wave function for momentum p. Since

$$\begin{aligned} \int_{-\infty}^{\infty} |\Psi(\frac{p}{\hbar},t)|^2 dp &= \hbar \int_{-\infty}^{\infty} |\Psi(\frac{p}{\hbar},t)|^2 d(p/\hbar) \\ &= \hbar \int_{-\infty}^{\infty} |\Psi(\omega,t)|^2 d\omega, \end{aligned}$$

we have:

$$\Delta_{\text{momentum}} = \frac{1}{\hbar} \int_{-\infty}^{\infty} (p-\alpha)^2 |\Psi(\frac{p}{\hbar}, t)|^2 dp \Big/ \int_{-\infty}^{\infty} |\Psi(\omega, t)|^2 d\omega \tag{7.25}$$

And so from (7.24):

$$\Delta_{\alpha/\hbar} \Psi(\omega, t) = \frac{1}{\hbar^2} \Delta_{\text{momentum}} \tag{7.26}$$

Substituting (7.22) and (7.26) into the mathematical uncertainty principle (7.20), we arrive at Heisenberg's uncertainty principle:

$$\Delta_{\text{position}} \cdot \Delta_{\text{momentum}} \geq \frac{\hbar^2}{4} \tag{7.27}$$

To decrease the uncertainty of a particle's position about some point a causes an increase in the uncertainty of the momentum of the particle about all values α, and vice versa.