Chapter 4

Fourier Sine Series

4.1 Introduction

An important mathematical question raised by Joseph Fourier in 1807, arising from his practical work on heat conduction, is whether an arbitrary function $f(x)$ can be represented in the form of a "Fourier sine series":

$$\boxed{f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}, \quad 0 < x < L.} \tag{4.1}$$

A second question is: suppose we *can* indeed represent $f(x)$ by a Fourier sine series of the form (4.1), how do we calculate the "Fourier sine coefficients", a_n's?

4.2 Finding the Fourier coefficients

Let us deal with the second question first. Suppose (4.1) holds. We multiply both sides by $\sin \frac{m\pi x}{L}$, where m is any integer, and then integrate both sides from 0 to L. Thus,

$$\int_0^L f(x) \sin \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} a_n \int_0^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx, \tag{4.2}$$

where we have interchanged the order of integration and summation. Using the trigonometric identity:

$$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)],$$

the integral on the right-hand side of Eq. (4.2) can be evaluated:

$$\begin{aligned} I_{mn} \equiv \int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx &= \frac{1}{2}\int_0^L \cos\frac{(m-n)\pi x}{L}dx - \frac{1}{2}\int_0^L \cos\frac{(n+m)\pi x}{L}dx \\ &= \frac{1}{2}\frac{\sin((m-n)\pi x/L)}{(m-n)\pi/L}\Bigg|_0^L - \frac{1}{2}\frac{\sin((m+n)\pi x/L)}{(m+n)\pi/L}\Bigg|_0^L \end{aligned}$$

Thus, we obtain the so-called *orthogonality relationship* for sines,

$$I_{mn} = \frac{L}{2}\delta_{mn}, \tag{4.3}$$

where

$$\delta_{mn} \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}$$

[When $m = n$, we have

$$\begin{aligned} I_{mm} &= \int_0^L \left(\sin\frac{m\pi x}{L}\right)^2 dx = \frac{1}{2}\int_0^L \left(1 - \cos\frac{2m\pi x}{L}\right) dx \\ &= \frac{L}{2}.] \end{aligned}$$

Substituting (4.3) into (4.2), we find

$$\int_0^L f(x)\sin\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty} a_n \frac{L}{2}\delta_{mn} = \frac{L}{2}a_m.$$

So for any specified integer m,

$$a_m = \frac{2}{L}\int_0^L f(x)\sin\frac{m\pi x}{L}dx. \tag{4.4}$$

(4.4) gives:

$$\begin{aligned} a_1 &= \frac{2}{L}\int_0^L f(x)\sin\frac{\pi x}{L}dx \\ a_2 &= \frac{2}{L}\int_0^L f(x)\sin\frac{2\pi x}{L}dx \\ a_3 &= \frac{2}{L}\int_0^L f(x)\sin\frac{3\pi x}{L}dx \\ &\vdots \end{aligned}$$

In particular, we can write

$$\boxed{a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots} \tag{4.5}$$

and thus have completed our task of finding the Fourier sine series coefficients, a_n, in (4.1).

4.3 An Example:

Represent $f(x) = 100$ in the form of a Fourier sine series over the interval $0 < x < L$:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}, \quad 0 < x < L.$$

The Fourier coefficients, a_n, are given by

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \\ &= \frac{2}{L}\int_0^L 100\sin\frac{n\pi x}{L}dx \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{400}{n\pi} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

So we say that in $0 < x < L$, we have

$$100 = \frac{400}{\pi}\left[\frac{\sin(\pi x/L)}{1} + \frac{\sin(3\pi x/L)}{3} + \frac{\sin(5\pi x/L)}{5} + \ldots\right]. \tag{4.6}$$

(4.6) is rather strange; it says that a constant, 100, can be represented by a sum of sines. Let us see what we will get if we add up the sines in the right-hand side of (4.6). In Figure 4.1, we plot one term in the sum (i.e. $\frac{400}{\pi}\sin(\pi x/L)$). In Figure 4.2, we plot two terms, i.e. $\frac{400}{\pi}[\sin(\pi x/L) + \frac{1}{3}\sin(3\pi x/L)]$. In Figure 4.3, we plot 3 terms, etc. By the time we have included enough terms, we see that the right-hand side of (4.6) approaches the constant value of 100 in the interior of the interval, $0 < x < L$. (Near the edges $x = 0$ and $x = L$, the oscillations get increasingly confined to the edges, where the sum of sines tries very hard to approach 100 in the interior of the domain, $0 < x < L$, while being identically zero at $x = 0$ and $x = L$. A discontinuity is created at the edges. There is also the so-called Gibbs phenomenon present near the edges, where just within the boundaries, there is an overshoot of the true value of 100, by 18%.]

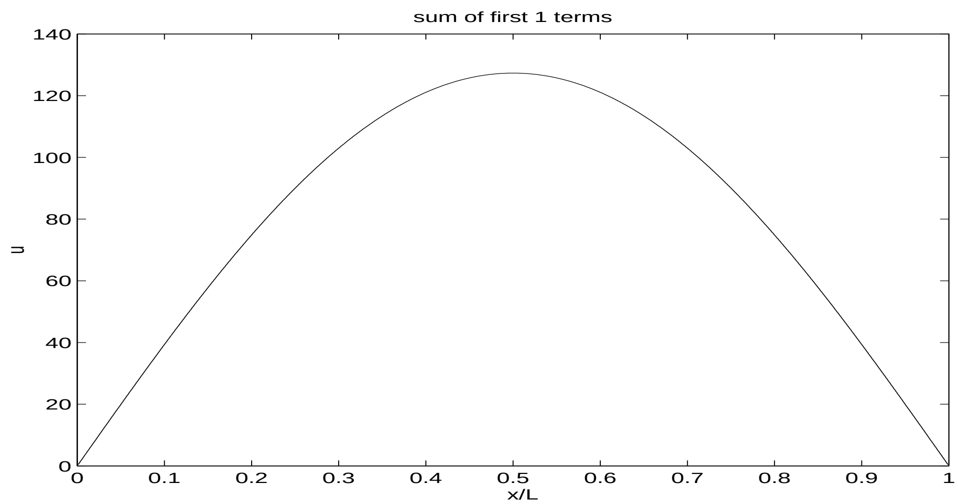

Figure 4.1: Plot of the first term in the Fourier sine expansion of 100.

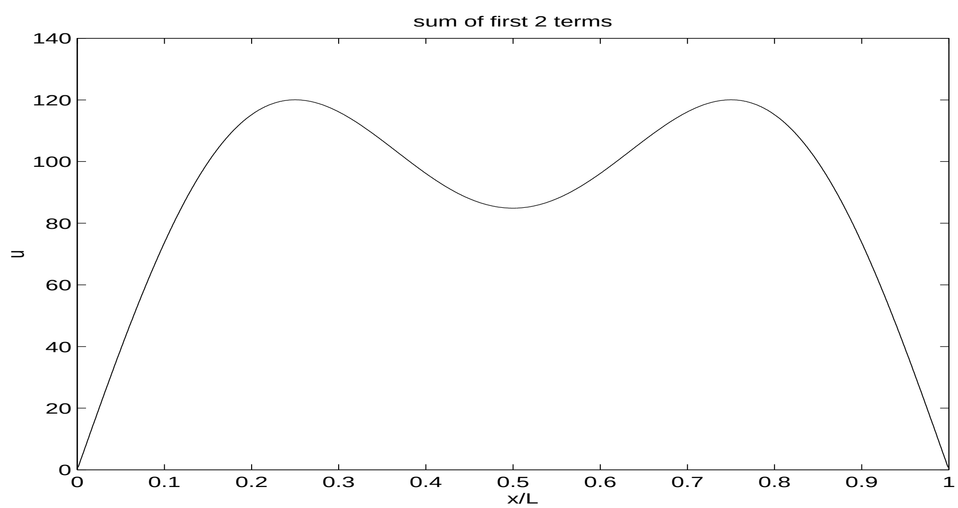

Figure 4.2: Plot of the sum of the first 2 terms in the Fourier sine expansion of 100.

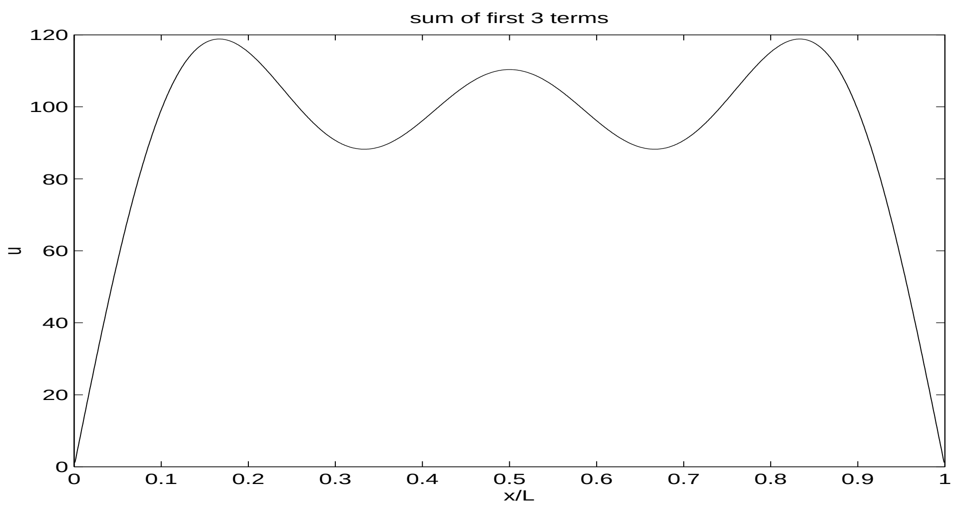

Figure 4.3: Plot of the sum of the first 3 terms in the Fourier sine expansion of 100.

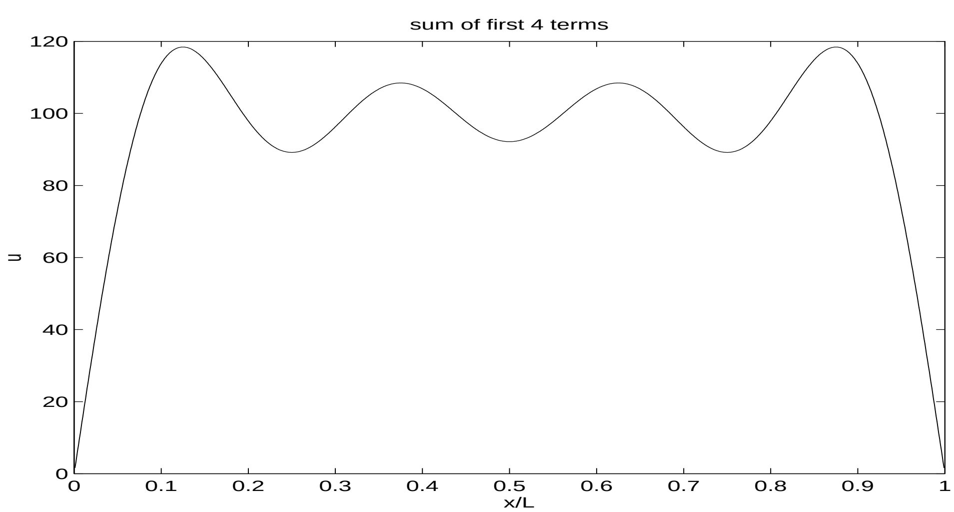

Figure 4.4: Plot of the sum of the first 100 terms in the Fourier sine expansion of 100.

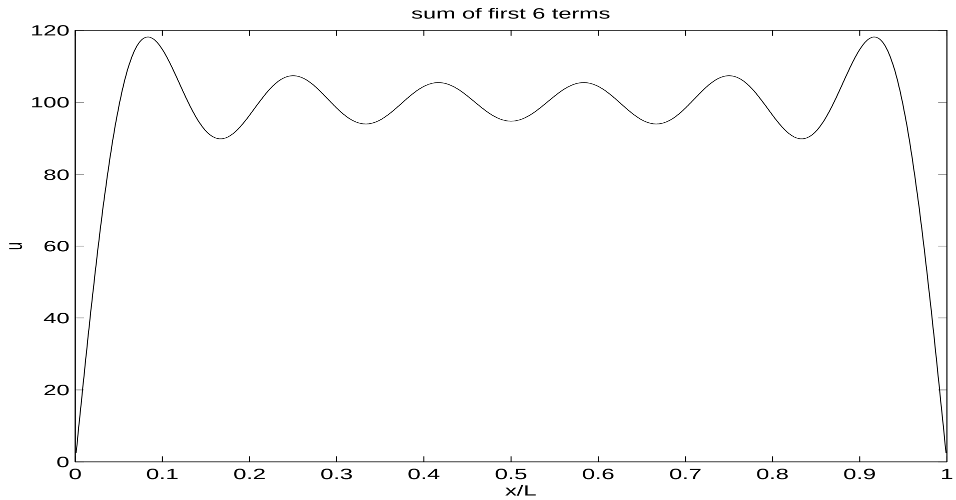

Figure 4.5: Plot of the sum of the first 6 terms in the Fourier sine expansion of 100.

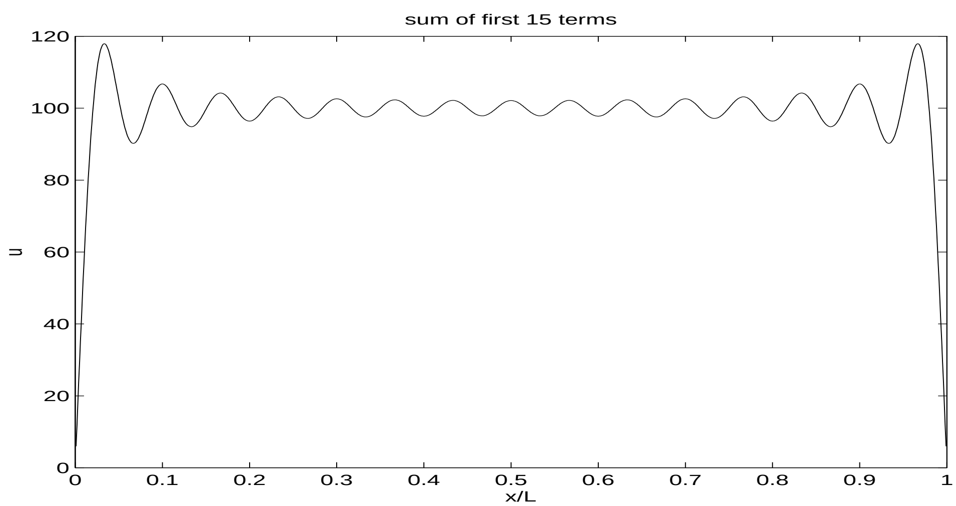

Figure 4.6: Plot of the sum of the first 15 terms in the Fourier sine expansion of 100.

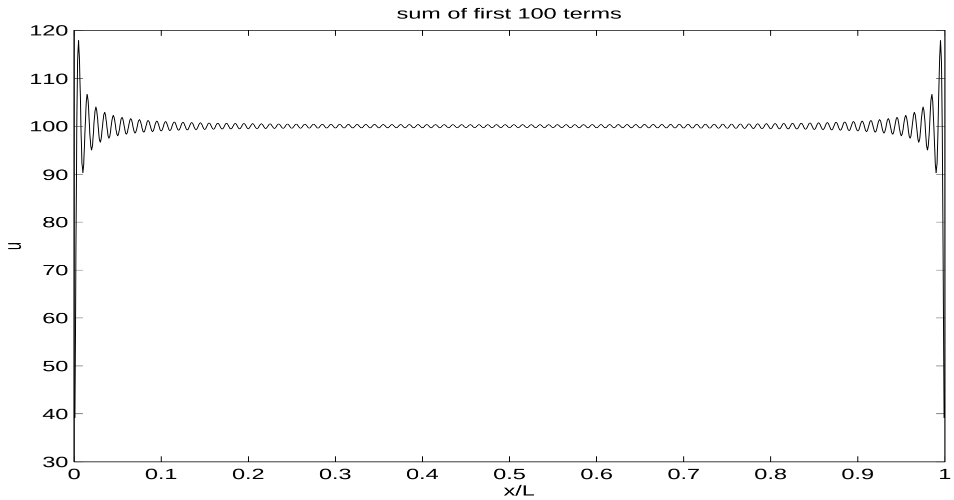

Figure 4.7: Plot of the sum of the first 100 terms in the Fourier sine expansion of 100.

4.4 Some comments:

What the example demonstrates is that the Fourier sine series can indeed represent $f(x)$ in the interval indicated. We can do this for other functions, and you will find that the Fourier sine series does a very good job in representing each of them. Actually, the most difficult function to represent by a Fourier sine series may be the one we have just done, $f(x) = \text{constant}$ in $0 < x < L$. This is because the sines all go to zero at $x = 0$ and $x = L$, but they have to add up to a nonzero constant slightly inside the boundaries. Many more terms in the sum are required to create this near discontinuity. For functions which are continuous and actually zero at the boundaries $x = 0$ and $x = L$, you will find that you do not need as many terms in the sum to give a good numerical representation of the original function.

It is not reasonable to expect that the sines can represent a function which blows up (i.e. attains infinite values) in the domain $0 < x < L$. Such unphysical functions are excluded in our consideration. The following mathematical result can be stated in a Theorem (a more general form is called Dirichlet's Theorem):

If $f(x)$ is a bounded function, which is continuous or piecewise continuous in a domain, the Fourier sine series representation of $f(x)$ converges to $f(x)$ for each point x in the domain where $f(x)$ is continuous. At those points where $f(x)$ jumps, the series converges to a value which is the average of the left- and right-hand limits of $f(x)$ at those points, where $f(x)$ is *discontinuous.*

[A piecewise continuous function is one which can take a finite number of finite jumps in the domain and be continuous elsewhere.]

4.5 A mathematical curiosity

If you are convinced that the function $f(x) = 100$ can be represented by a Fourier sine series in the interior of the domain:

$$100 = \frac{400}{\pi}\left(\sin\frac{\pi x}{L} + \frac{\sin\frac{3\pi x}{L}}{3} + \frac{\sin\frac{5\pi x}{L}}{5} + \ldots\right), \quad 0 < x < L,$$

then pick $x/L = \frac{1}{2}$ to get

$$100 = \frac{400}{\pi}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots)$$

or

$$\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots). \tag{4.7}$$

This relationship between π and the odd integers was discovered by Leibniz in 1673 by a different route.

You may try adding up the right-hand side of (4.7) and see how many terms are needed to approximate π to the accuracy you want.

4.6 Representing the cosine by sines

When Fourier presented his work on heat conduction and Fourier series to the Paris Academy in 1807, neither Laplace nor Lagrange would accept his use of Fourier series. In particular, Laplace could not accept the fact that $\cos x$ could be represented using a sum of sines. Let us see if Fourier was right.

Let us try to represent

$$f(x) = \cos x$$

in the interval $0 < x < \pi$ by a sine series

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx.$$

Fourier's formula for the coefficients gives

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx \; dx, \quad n = 1, 2, 3, \ldots.$$

Using the trigonometric identity

$$2\cos a \sin b = \sin(a+b) - \sin(a-b),$$

the integral can be evaluated to yield:

$$a_n = \begin{cases} \frac{4n}{\pi(n^2-1)}, & n = \text{even} \\ 0, & n = \text{odd}. \end{cases}$$

Thus, we find

$$\begin{aligned} \cos x &= \sum_{n\ even} \frac{4n}{\pi(n^2-1)} \sin nx, \quad 0 < x < \pi \\ &= \frac{8}{3\pi} \sin 2x + \frac{16}{15\pi} \sin 4x + \dots, \quad 0 < x < \pi. \end{aligned}$$

Try adding up as many terms as you can using a graphing calculator or computer. Does the sum approach $\cos x$ in the interval?

Comment: We can express a cosine in terms of a sine series only for half of its period. It is still true that a cosine cannot be expressed in terms of sines in its full period $-\pi < x < \pi$.

4.7 Application to the Heat Conduction Problem

PDE: $u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad t > 0$

BCs: $u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0$

IC: $u(x,0) = f(x), \quad 0 < x < L.$

In particular, we shall consider the case where $f(x) = 100$.

In Chapter 3, we found that the general solution to the PDE which satisfies the BCs is given by

$$u(x,t) = \sum_{n=1}^{\infty} T_n(0) e^{-\alpha^2 (\frac{n\pi}{L})t} \sin \frac{n\pi x}{L}, \quad 0 < x < L. \tag{4.8}$$

where the constants $T_n(0)$ are yet to be determined from the IC.

Setting $t = 0$ in (4.8) and setting $u(x,0) = f(x)$, we arrive at

$$f(x) = \sum_{n=1}^{\infty} T_n(0) \sin \frac{n\pi x}{L}, \quad 0 < x < L, \tag{4.9}$$

Therefore $T_n(0)$ is the Fourier sine coefficient of $f(x)$ and is given by (4.4) as

$$T_n(0) = a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$$

For $f(x) = 100$, we know that

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{400}{n\pi} & \text{if } n \text{ is odd.} \end{cases}$$

Thus finally, the solution to the above PDE problem is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n e^{-\alpha^2(\frac{n\pi}{L})^2 t}\sin\frac{n\pi x}{L} \\ &= \sum_{k=1}^{N} \frac{400}{\pi}\frac{1}{(2k-1)} e^{-(2k-1)^2(t/t_e)}\sin\frac{(2k-1)\pi x}{L}, \quad 0 < x < L, \end{aligned} \tag{4.10}$$

where $t_e \equiv \left(\frac{L}{\alpha\pi}\right)^2$, and $N \to \infty$.

The solution in (4.10) is plotted in Figure 4.8 for different values of t/te. It turns out that unless t/te is very small, only a few terms are needed in the sum in (4.10). In Figure 4.9, we show that the solution can be represented to a high degree of accuracy by the first two terms:

$$u(x,t) \cong \frac{400}{\pi}e^{-t/te}\sin\frac{\pi n}{L} + \frac{400}{3\pi}e^{-9t/te}\sin\frac{3\pi x}{L}, \quad \text{for } t \gtrsim te.$$

Although we obtained this behavior already in Chapter 6, only now do we have the actual coefficients a_1 and a_3 etc calculated explicitly.

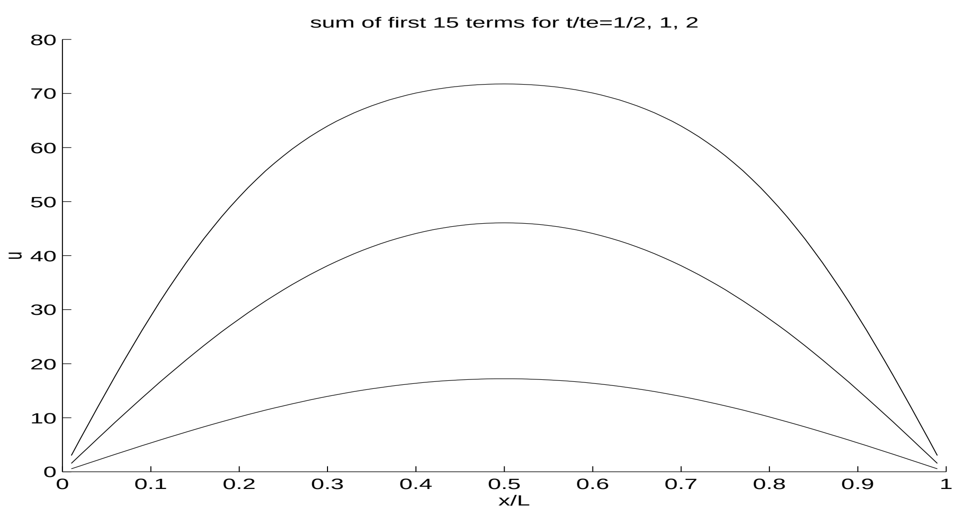

Figure 4.8: Plot of the sum of the first 15 terms in solution for $t/t_e = 1/2,\ 1,\ 2$.

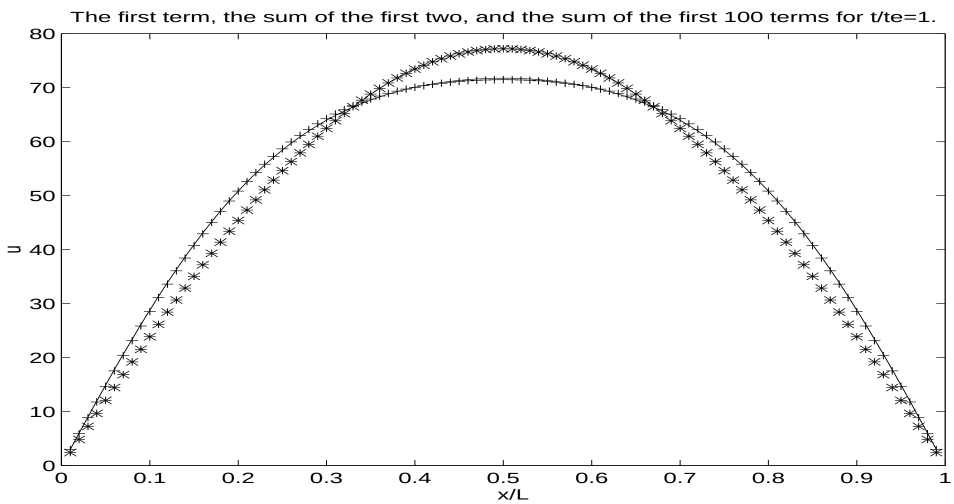

Figure 4.9: The first term, sum of the first two, and sum of the first 100 terms.

4.8 Exercises

1. Let $f(x)$ be given by

$$f(x) = \begin{cases} x, & 0 < x < L/2 \\ (L - x), & L/2 < x < L. \end{cases}$$

Represent $f(x)$ by a Fourier sine series of the form

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}, \quad 0 < x < L.$$

(a) Find a_n, $n = 1, 2, 3, \ldots$.

(b) Retain the first N terms as an approximation

$$f(x) \cong f_N(x) \equiv \sum_{n=1}^{N} a_n \sin \frac{n\pi x}{L}, \quad 0 < x < L.$$

Using a graphing calculator or a computer, plot $f_N(x)$ as a function of x/L for $N = 1$, 3, 5, 31, and (optional) 101.

[You will notice that $a_n = 0$ for even n's, so the actual number of nonzero terms in the sum is halved.]

2. The solution to

$$\begin{array}{ll} \text{PDE:} & u_t = \alpha^2 u_{xx}, \quad 0 < x < L \\ \text{BCs:} & u(0,t) = 0, \quad u(L,t) = 0 \\ \text{IC:} & u(x,0) = f(x), \quad \text{where } f(x) \text{ is given in problem 1,} \end{array}$$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2(t/t_e)} \sin\frac{n\pi x}{L}, \quad 0 < x < L,$$

where $t_e = (L/\pi\alpha)^2$.

Again replace

$$\sum_{n=1}^{\infty} \quad \text{by} \quad \sum_{n=1}^{N} \quad \text{as an approximation.}$$

Plot out the solution as a function of x/L for $t = \frac{1}{2}t_e$, t_e and $2t_e$. Use a large enough N so that your solution does not change noticeably when N is increased. You will find that you need only a few terms in the sum to get an accurate solution.

4.9 Solutions

Problem 1, figures 4.10-4.14

(a) We know that $f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L)$. Thus $a_n = 2/L\int_0^L f(x)\sin(n\pi x/L)dx$ which upon integration yields;

$a_n = (4L/(n\pi)^2)\sin(n\pi/2)$.

$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L)$ where a_n is given above.

(b) see figures 4.10-4.14

Problem 2, figures 4.15-4.17

$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2(t/t_e)} \sin(n\pi x/L)$ where a_n is given above.

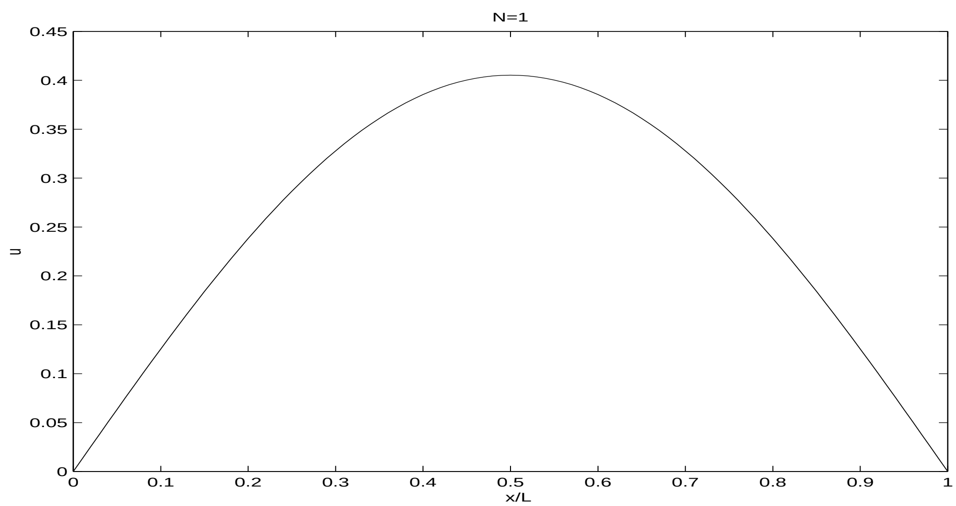

Figure 4.10: Sum of the first 1 terms in the Fourier sine expansion of x(L-x), L=1.

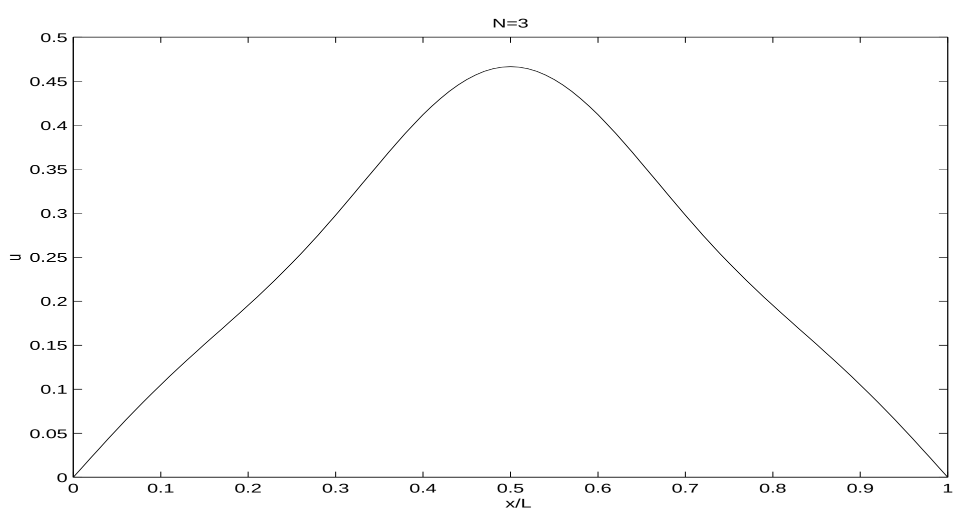

Figure 4.11: Sum of the terms up to N=3 in the Fourier sine expansion of x(L-x), L=1.

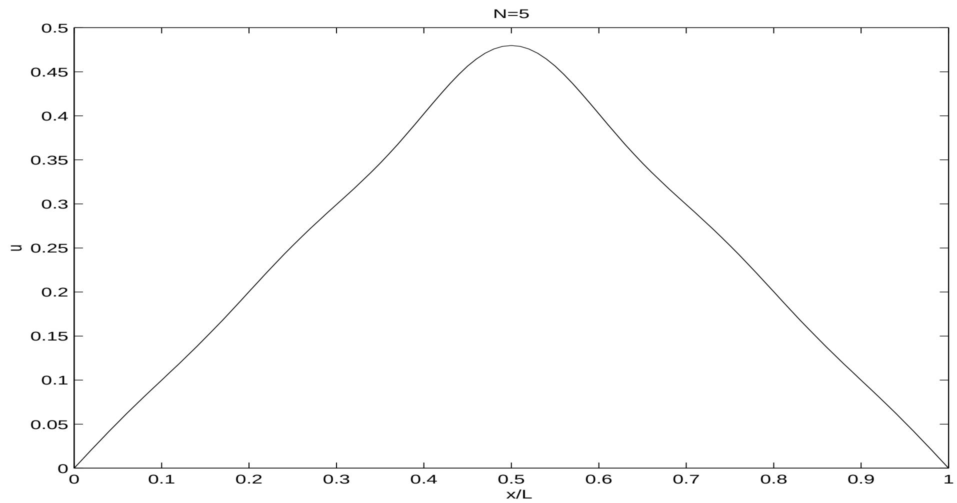

Figure 4.12: Sum of the terms up to N=5 in the Fourier sine expansion of x(L-x), L=1.

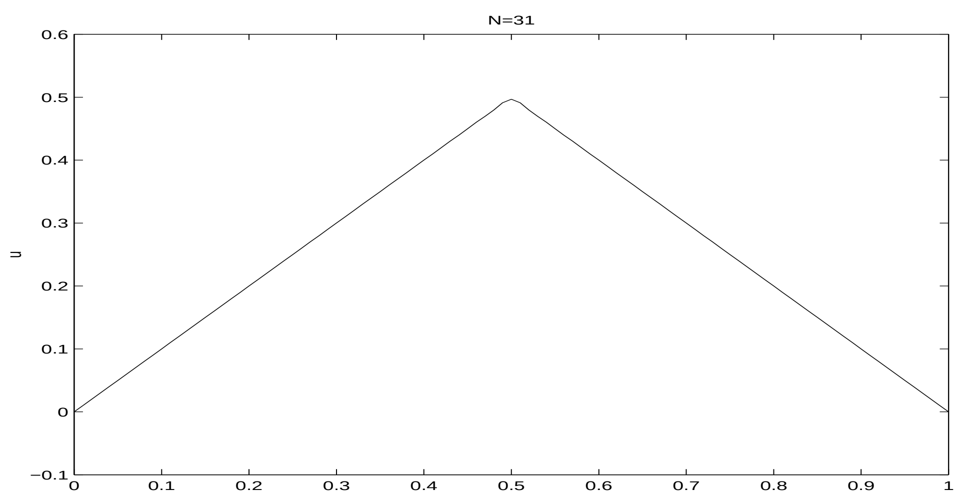

Figure 4.13: Sum of the terms up to N=31 in the Fourier sine expansion of x(L-x), L=1.

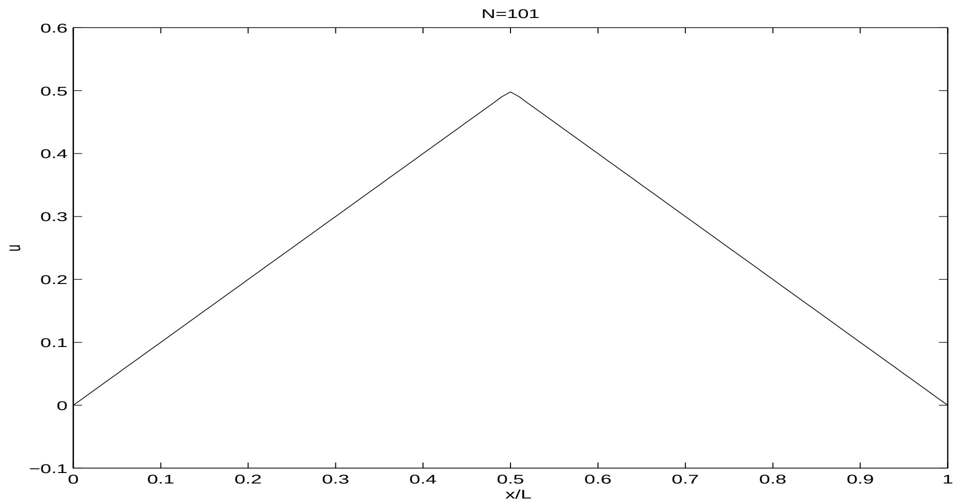

Figure 4.14: Sum of the terms up to N=101 in the Fourier sine expansion of x(L-x), L=1.

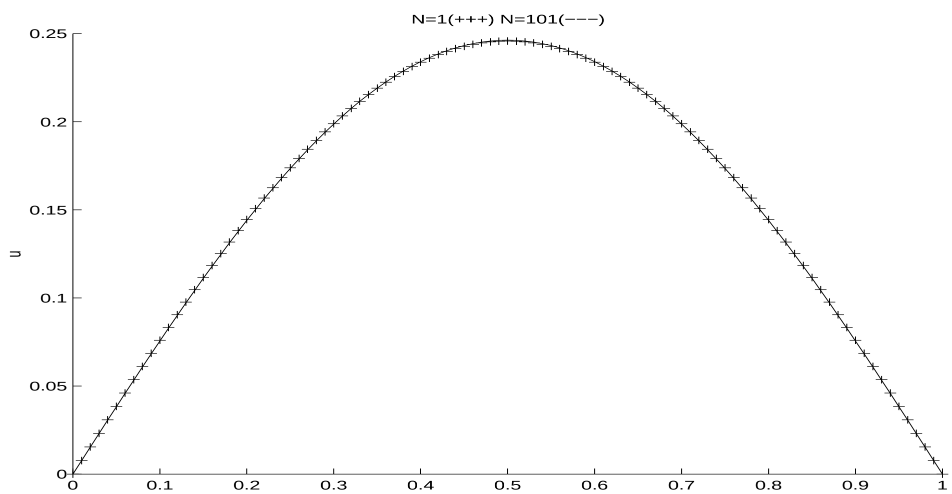

Figure 4.15: Sum of the terms for N=1 and N=101 in u(x,t) for t/te=1/2.

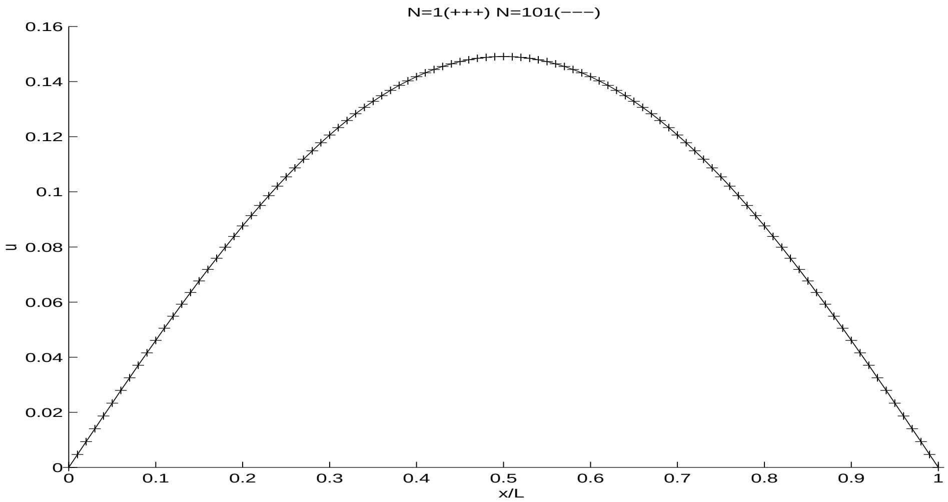

Figure 4.16: Sum of the terms for N=1 and N=101 in u(x,t) for t/te=1.

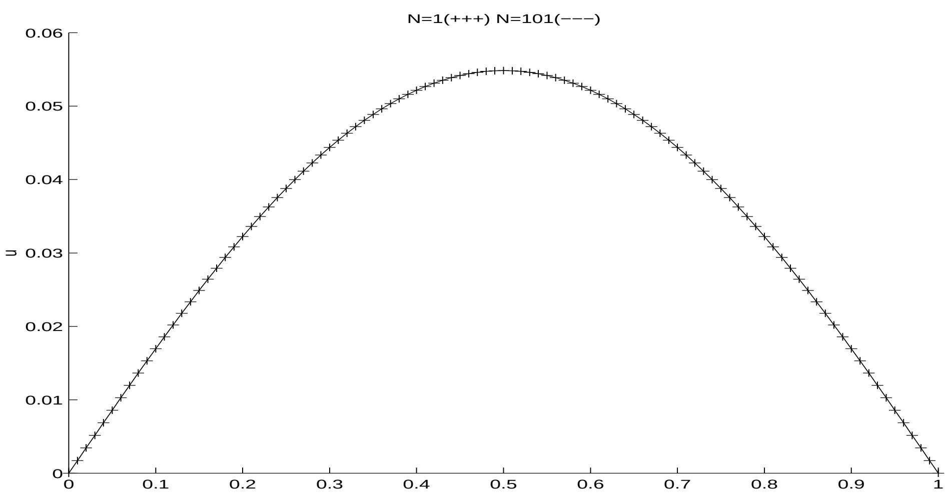

Figure 4.17: Sum of the terms for N=1 and N=101 in u(x,t) for t/te=2.