Chapter 5

Fourier Cosine Series

5.1 Introduction

In the previous chapter, function $f(x)$ were represented by a series of sines. It is also possible to express the same function alternatively in a series of cosines, in the form

$$f(x) = \sum_{n=0}^{\infty} b_n \cos \frac{n\pi x}{L}, \quad 0 < x < L. \tag{5.1}$$

Here the summation starts from $n = 0$, because $\cos 0 = 1$ is not zero. We will delay the motivation for wanting to write $f(x)$ in this form until later. Here we discuss only *how* to find the Fourier cosine series coefficients b_n assuming that $f(x)$ can be represented in the form of (5.1).

5.2 Finding the Fourier coefficients

We multiply both sides of (5.1) by $\cos \frac{m\pi x}{L}$, where m is any integer, $m = 0$, 1, 2, 3, ..., and integrate from 0 to L:

$$\int_0^L f(x) \cos \frac{m\pi x}{L} dx = \sum_{n=0}^{\infty} b_n \int_0^L \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx. \tag{5.2}$$

There is *an orthogonality* condition for the cosines which can be written as

$$I_{mn} \equiv \int_0^L \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{2} & \text{if } m = n \neq 0 \\ L & \text{if } m = n = 0. \end{cases} \tag{5.3}$$

To show this, we note the trigonometric identity

$$\cos a \cos b = \frac{1}{2}\cos(a-b) + \frac{1}{2}\cos(a+b),$$

so

$$\begin{aligned}
\int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx &= \frac{1}{2}\int_0^L \cos\frac{(m-n)\pi x}{L}dx + \frac{1}{2}\int_0^L \cos\frac{(m+n)\pi x}{L}dx \\
&= \left.\frac{\sin\frac{(m-n)\pi x}{L}}{2(m-n)\pi/L}\right|_0^L + \left.\frac{\sin\frac{(m+n)\pi x}{L}}{2(m+n)\pi/2}\right|_0^L \\
&= 0 \quad \text{if } m \neq n.
\end{aligned}$$

When $n = m \neq 0$

$$\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L} = \frac{1}{2}\left(1 + \cos\frac{2m\pi x}{L}\right).$$

The integral from 0 to L of the first term, $\frac{1}{2}$, is $L/2$, while the integral of the second term, $-\frac{1}{2}\cos\frac{2m\pi x}{L}$, is zero. When $m = n = 0$,

$$\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L} = 1,$$

so its integral from 0 to L is L. Thus we have derived the identity in (5.3).

Substituting (5.3) into (5.2) then yields

$$\int_0^L f(x)\cos\frac{m\pi x}{L}dx = b_m I_{mn} = \begin{cases} b_0 L & \text{if } m = 0 \\ b_m \frac{L}{2} & \text{if } m \neq 0. \end{cases}$$

Thus we have the Fourier cosine series representation for $f(x)$ in the form

$$\boxed{f(x) = \sum_{n=0}^{\infty} b_n \cos\frac{n\pi x}{L}, \quad 0 < x < L},$$

where,

$$\begin{aligned}
b_0 &= \frac{1}{L}\int_0^L f(x)dx \\
b_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, 4, \ldots.
\end{aligned} \tag{5.4}$$

5.3 Application to PDE with Neumann Boundary Conditions

Consider heat conduction in a rod of length L whose initial temperature is given as

$$\boxed{\text{IC:} \quad u(x,0) = f(x), \quad 0 < x < L}. \tag{5.5}$$

Find the evolution of $u(x,t)$ for $t > 0$ if the ends of the rod are insulated, i.e.

$$\boxed{\text{BCs:} \quad u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0}. \tag{5.6}$$

Assume heat conduction is governed by the heat equation:

$$\boxed{\text{PDE:} \quad u_t = \alpha^2 u_{xx}, \quad 0 < x < L}. \tag{5.7}$$

The usual method of separation of variables will lead us to the solution in the form:

$$u(x,t) = \sum_n T_n(t) X_n(x), \tag{5.8}$$

where the "eigenfunction", $X_n(x)$, satisfies

$$\frac{d^2}{dx^2} X_n(x) + \lambda_n^2 X_n(x) = 0. \tag{5.9}$$

The only difference between this case and the previous one in Chapter 3, section 2, is the boundary conditions. Here the Neumann condition. (5.6) implies

$$\frac{d}{dx} X_n(0) = 0, \quad \frac{d}{dx} X_n(L) = 0. \tag{5.10}$$

Nontrivial solutions to (5.9) and (5.10) are

$$X_n(x) = \cos \lambda_n x \tag{5.11}$$

provided

$$\lambda_n = \frac{n\pi}{L}, \quad n = 0, 1, 2, 3, \ldots.$$

The $T_n(t)$ satisfies, as in section 3.2:

$$\frac{d}{dt} T_n(t) + \alpha^2 \lambda_n^2 T_n(t) = 0. \tag{5.12}$$

So

$$T_n(t) = T_n(0) e^{-\alpha^2 \lambda_n^2 t}. \tag{5.13}$$

The general solution, satisfying the PDE and the BCs, is

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} T_n(0) e^{-\alpha^2 \lambda_n^2 t} \cos \frac{n\pi x}{L}.} \tag{5.14}$$

To satisfy the IC, we require, at $t = 0$,

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} T_n(0) \cos \frac{n\pi x}{L}, \quad 0 < x < L. \tag{5.15}$$

(5.15) implies that the constants, $T_n(0)$'s, are the Fourier cosine coefficients for $f(x)$. Thus

$$\boxed{\begin{aligned} T_n(0) = b_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \ldots \\ T_0(0) = b_0 &= \frac{1}{L} \int_0^L f(x) dx. \end{aligned}}$$

The problem is now completely solved, assuming $f(x)$ is given.

An Example:

Solve

PDE: $u_t = u_{xx}$, $0 < x < 1$, $t > 0$
BCs: $u_x(0,t) = 0$, $u_x(1,t) = 0$, $t > 0$
IC: $u(x,0) = x$, $0 < x < 1$.

Since the boundary conditions are homogeneous Neumann, try a cosine series expansion of the solution

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t) X_n(x), \tag{5.16}$$

where $X_n(x) = \cos n\pi x$. Substituting the assumed form (7.16) into the PDE yields:

$$\frac{d}{dt} T_n(t) = -(n\pi)^2 T_n(t), \quad n = 0, 1, 2, 3, \ldots. \tag{5.17}$$

The solution of (5.17) is

$$T_n(t) = T_n(0) e^{-(n\pi)^2 t}.$$

Therefore,

$$u(x,t) = \sum_{n=0}^{\infty} T_n(0) e^{-(n\pi)^2 t} \cos n\pi x, \quad 0 < x < 1.$$

To satisfy the IC, we require

$$x = \sum_{n=0}^{\infty} T_n(0) \cos n\pi x, \quad 0 < x < 1.$$

So the $T_n(0)$'s are the Fourier cosine coefficients of the function x, and thus

$$T_n(0) = \int_0^1 x dx = \frac{1}{2}$$

$$T_n(0) = 2\int_0^1 x \cos n\pi x dx = \begin{cases} 0 & \text{if } n = \text{even} \\ -\frac{4}{(n\pi)^2} & \text{if } n = \text{odd}. \end{cases}$$

Finally,

$$u(x,t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} e^{-(n\pi)^2 t} \cos n\pi x.$$