Proximal Gradient: Convergence, Implementations and Applications

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ToC

Sum of 2 Functions

Sum of 2

$$\min_{x} g(x) + h(x) \tag{1}$$

1. Throughout this presentation, we assume the objective of a function f is the sum of 2 functions.
2. We are interested in the paper: FISTA (Fast Iterative-Shrinkage Algorithm) by Beck and Teboulle [**?**].

1. When $h = \delta_Q$ with Q closed and convex with $Q \subseteq \text{ri} \circ \text{dom}(g)$, we use projected subgradient.
2. When g is **strongly smooth** and h is **closed convex proper** whose proximal oracle is easy to compute, we consider the use of FISTA.

Stuff to Go Over

What is FISTA

Simply put, the FISTA algorithm is the non-smooth analogy of gradient descent with Nesterov Momentum.

We will be going over these things in the presentations.

1. Derive the proximal gradient operator under standard convexity and regularity assumptions for the function g, h.
2. State one important lemma that arose during the proof for the proximal gradient method that is later useful for the proof of the FISTA.
3. Derive the FISTA algorithm and construct the sequence of the Nesterov Momentum during the proof using a template algorithm.
3. Numerical experiments results using Julia!

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Proximal Operator Definition

Definition (The Proximal Operator)

Let f be convex closed and proper, then the proximal operator parameterized by $\alpha > 0$ is a non-expansive mapping defined as:

$$\operatorname{prox}_{f,\alpha}(x) := \arg\min_{y} \left\{ f(y) + \frac{1}{2\alpha}\|y - x\|^2 \right\}.$$

Remark

When f is convex, closed, and proper,

Prox is the Resolvant of Subgradient

Lemma (Resolvant of the Subgradient)

When the function f is convex closed and proper, the $\text{prox}_{\alpha,f}$ can be viewed as the following operator $(I + \alpha\partial f)^{-1}$.

Proof.

Minimizer satisfies zero subgradient condition,

$$
\begin{aligned}
\mathbf{0} &\in \partial\left[f(y) + \frac{1}{2\alpha}\|y - x\|^2 \middle| y\right](y^+) \\
\mathbf{0} &\in \partial f(y^+) + \frac{1}{\alpha}(y^+ - x) \\
\frac{x}{\alpha} &\in (\partial f + \alpha^{-1}I)(y^+) \\
x &\in (\alpha\partial f + I)(y^+) \\
y &\in (\alpha\partial f + I)^{-1}(x).
\end{aligned}
$$

□

An Example of Prox

Definition (Soft Thresholding)

For some $x \in \mathbb{R}$, the proximal operator of the absolute value is:

$$\mathrm{prox}_{\lambda\|\cdot\|_1, t}(x) = \mathrm{sign}(x)\max(|x| - t\lambda, 0).$$

One could interpret the sign operator as projecting x onto the interval $[-1, 1]$ and the $\max(|x| - t\lambda, 0)$ as the distance of the point x to the interval $[-t\lambda, t\lambda]$.

Strong Smoothness

Definition (Strong Smoothness)

A differentiable function g is called strongly smooth with a constant α then it satisfies:

$$|g(y) - g(x) - \langle \nabla g(x), y - x\rangle| \le \frac{\alpha}{2}\|x - y\|^2 \quad \forall x, y \in \mathbb{E}. \tag{2}$$

Remark

The absolute value sign can be removed and replaced with $0 \le$ on the left when the function g is a convex function.

Equivalence of Strong Smoothness and Lipschitz Gradient

Theorem (Lipschitz Gradient Equivalence under Convexity)

Suppose g is differentiable on the entire of $\mathbb{E}$. It is closed convex proper. It is strongly smooth with parameter α if and only if the gradient ∇g is globally Lipschitz continuous with a parameter of α and g is closed and convex.

$$\|\nabla g(x) - \nabla g(y)\| \le \alpha \|x - y\| \quad \forall x, y \in \mathbb{E}$$

Proof.

Using line integral, we can prove Lipschitz gradient implies strong smoothness without convexity. The converse requires convexity and applying generalized Cauchy Inequality to (iv) in Theorem 5.8 for Beck's textbook [**?**]. □

A Major Assumption

Assumption (Convex Smooth Nonsmooth with Bounded Minimizers)

We will assume that $g : \mathbb{E} \mapsto \mathbb{R}$ *is* **strongly smooth** *with constant* L_g *and* $h : \mathbb{E} \mapsto \bar{\mathbb{R}}$ **is closed convex and proper**. *We define* $f := g + h$ *to be the summed function and* $ri \circ dom(g) \cap ri \circ dom(h) \neq \emptyset$. *We also assume that a set of minimizers exists for the function* f *and that the set is bounded. Denote the minimizer using* $\bar{x}$.

Envelope and Upper Bounding Functions

Upper Bounding Function

With assumption 1, we construct an upper bounding function at the point x evaluated at y for the function f, and it is given by:

$$g(x) + \nabla g(x)^T (y - x) + \frac{\beta}{2}\|y - x\|^2 + h(y) =: m_x(y|\beta) \quad \forall y \in \mathbb{E},$$

In brief, suppose we are at the point x of the iterations; we are minimizing the function $m_x(y|\beta)$ to obtain the next point for our iterations.

Theorem (Minimizer of the Envelope)

The minimizer for the envelope has a closed form, and it is $prox_{h,\beta^{-1}}(x + \beta^{-1}\nabla g(x))$, with assumption **??**.

The Prox Gradient Operator

Proof.

Minimizer of the Envelope We consider minimizing the envelope; zero is in the subgradient of the upper bounding function $m_x(y|\beta)$:

$$
\begin{aligned}
\mathbf{0} &\in \nabla g(x) + \beta(y - x) + \partial h(y) \\
\nabla g(x) + \beta x &\in \beta y + \partial h(y) \\
-\beta^{-1}\nabla g(x) + x &\in y + \beta^{-1}\partial h(y) \\
-\beta^{-1}\nabla g(x) + x &\in [I + \beta^{-1}\partial h](y) \\
\implies [I + \beta^{-1}\partial h]^{-1}(-\beta^{-1}\nabla g(x) + x) &\ni y,
\end{aligned}
$$

recall lemma **??**, it's the operator $\operatorname{prox}_{h,\beta^{-1}}(x + \beta^{-1}\nabla g(x))$. $\square$

Prox Step and the Proximal Gradient Algorithm

The Prox Step

For simplicity we will be calling the point $\text{prox}_{h,\beta^{-1}}(x + \beta^{-1}\nabla g(x))$ "the prox step", and we denote it as $\mathcal{P}^{g,h}_{\beta^{-1}}(x)$ when there is no ambiguity we simply use $\mathcal{P}x$.

The Proximal Gradient Method

Algorithm Proximal Gradient With Fixed Step-sizes

```
Input: $g, h$, smooth and nonsmooth, $L$ stepsize, $x^{(0)}$ an initial guess of solution.
for $k = 1, 2, \cdots, N$ do
        $x^{(k+1)} = \mathcal{P}^{g,h}_{L^{-1}} x^{(k)}$
    if $x^{(k+1)}, x^{(k)}$ close enough then
        Break
    end if
end for
```

The Proximal Gradient Method

1. (Proved in my report.) It converges *Monotonically* for stepsize L^{-1} such that $L \ge L_g$.
2. (Proved in my report.) It has a convergence rate of $\mathcal{O}(1/k)$ on the optimality gap $\Delta_k := f(x^{(k)}) - f(\bar{x})$ where $\bar{x}$ is one of the minimizers for f satisfying assumption **??**.
3. A line search routine can be applied, and the stepsize should satisfy the conditions: $m_x(\mathcal{P}x|L) \le f(x)$, it is possible by the Lipschitz Gradient property.

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The Accelerated Proximal Gradient Method

Momentum Template Method

Algorithm Template Proximal Gradient Method With Momentum

```
**Input:** $x^{(0)}, x^{(-1)}, L, h, g$; 2 initial guesses and stepsize L
$y^{(0)} = x^{(0)} + \theta_k(x^{(0)} - x^{(-1)})$
**for** $k = 1, \cdots, N$ **do**
    $x^{(k)} = \text{prox}_{h,L^{-1}}(y^{(k)} + L^{-1}\nabla g(y^{(k)})) =: \mathcal{P}^{g,h}_{L^{-1}}(y^{(k)})$
    $y^{(k+1)} = x^{(k)} + \theta_k(x^{(k)} - x^{(k-1)})$
**end for**
```

The Secret Sauce

Lemma (Prox Step 2 Points)

With assumption **??***, and $\beta^{-1} > L_g$ still being our stepsize for algorithm* **??***, let $y \in \mathbb{E}$ and define $y^+ = \mathcal{P}_{\beta^{-1}}^{g,h}(y)$ we have for any $x \in \mathbb{E}$:*

$$f(x) - f(y^+) \geq \frac{\beta}{2}\|y^+ - y\|^2 + \beta\langle y - x, y^+ - y\rangle.$$

1. It is equivalent to the lemma 2.3 in the FISTA paper[**?**]
2. Proof for convergence with and without the momentum uses this lemma.

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Some Physical Quantities

1. $v^{(k)} = x^{(k)} - x^{(k-1)}$ is the velocity term.
2. $\bar{v}^{(k)} = \theta_k v^{(k)}$ is the weighed velocity term.
3. $e^{(k)} := x^{(k)} - \bar{x}$, where $\bar{x} \in \arg\min_x(f(x))$, where $\bar{x}$ is fixed.
4. $\Delta_k := f(x^{(k)}) - f(\bar{x})$ which represent the optimality gap at step k.

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Momentum Magic

Substitute $x = x^{(k)}, y = y^{(k+1)}$ to Prox Step 2 Points

$$
\begin{aligned}
f(x^{(k)}) - f \circ \mathcal{P}y^{(k+1)} &\geq \frac{L}{2}\|\mathcal{P}y^{(k+1)} - y^{(k+1)}\|^2 + L\langle y^{(k+1)} - x^{(k)}, \mathcal{P}y^{(k+1)} - y^{(k+1)}\rangle \\
[*1] \implies 2L^{-1}(\Delta_k - \Delta_{k+1}) &\geq \|x^{(k+1)} - y^{(k+1)}\|^2 + 2\langle x^{(k+1)} - y^{(k+1)}, y^{(k+1)} - x^{(k)}\rangle \\
[*2] \implies 2L^{-1}(\Delta_k - \Delta_{k+1}) &\geq \|v^{(k+1)} - \bar{v}^{(k)}\|^2 + 2\langle v^{(k+1)} - \bar{v}^{(k)}, \bar{v}^{(k)}\rangle \qquad (*)
\end{aligned}
$$

Substitute $x = \bar{x}, y = y^{(k+1)}$ to Prox Step 2 Points

$$
\begin{aligned}
-2L^{-1}\Delta_{k+1} &\geq \|x^{(k+1)} - y^{(k+1)}\|^2 + 2\langle y^{(k+1)} - \bar{x}, x^{(k+1)} - y^{(k+1)}\rangle \\
-2L^{-1}\Delta_{k+1} &\geq \|v^{(k+1)} - \bar{v}^{(k)}\|^2 + 2\langle v^{(k+1)} - \bar{v}^{(k)}, e^{(k)} + \bar{v}^{(k)}\rangle. \qquad (\star)
\end{aligned}
$$

Momentum Magic

Linear Combinations

Assume some linear combinations of the term $(*), (\star)$ with: $(t_k - 1) \ge 0$ for all k, then $(t_{k+1}-1)(*) + (\star)$ is:

$$\begin{aligned}
&2L^{-1}((t_{k+1}-1)\Delta_k - t_{k+1}\Delta_{k+1}) \\
&\ge t_{k+1}\|v^{(k+1)} - \bar{v}^{(k)}\|^2 + 2\langle t_{k+1}(v^{(k+1)} - \bar{v}^{(k)}), e^{(k)} + t_{k+1}\bar{v}^{(k)}\rangle, \quad (\star *)
\end{aligned}$$

1. No more monotonicity property.
2. The quantity on the right side bounds the weight differences of Δ_k, Δ_{k+1}.
3. **What if the expression can match the form of** $a_k - a_{k+1} \ge b_{k+1} - b_k \; \forall k \in \mathbb{N}$**?**

2 Sequences

Lemma

2 Bounded Sequences Consider the sequences $a_k, b_k \ge 0$ *for* $k \in \mathbb{N}$ *with* $a_1 + b_1 \le c$. *Inductively the two sequences satisfy* $a_k - a_{k+1} \le b_{k+1} - b_k$, *which describes a sequence with oscillations bounded by the difference of another sequence. Consider the telescoping sum:*

$$\begin{aligned} a_k - a_{k+1} &\ge b_{k+1} - b_k \quad \forall k \in \mathbb{N} \\ \implies -\sum_{k=1}^{N} a_{k+1} - a_k &\ge \sum_{k=1}^{N} b_{k+1} - b_k \\ -(a_{N+1} - a_1) &\ge b_{N+1} - b_1 \\ c \ge a_1 + b_1 &\ge b_{N+1} + a_{N+1} \\ \implies c \ge a_{N+1}. \end{aligned}$$

Form Match

We can Match the Template to That Form

Yes, it does, and it is:

$$\begin{aligned} & 2L^{-1}((t_{k+1}^2 - t_{k+1})\Delta_k - t_{k+1}^2\Delta_{k+1}) \\ & \geq t_{k+1}^2\|v^{(k+1)} - \bar{v}^{(k)}\|^2 + 2\langle t_{k+1}^2(v^{(k+1)} - \bar{v}^{(k)}), e^{(k)} + t_{k+1}\bar{v}^{(k)}\rangle \\ & = \|t_{k+1}(v^{(k+1)} - \bar{v}^{(k)})\|^2 + 2\langle t_{k+1}^2(v^{(k+1)} - \bar{v}^{(k)}), e^{(k)} + t_{k+1}\bar{v}^{(k)}\rangle \\ & = \|t_{k+1}v^{(k+1)} - t_{k+1}\bar{v}^{(k)} + e^{(k)} + t_{k+1}\bar{v}^{(k)}\|^2 - \|e^{(k)} - t_{k+1}\bar{v}^{(k)}\|^2 \\ & = \|t_{k+1}v^{(k+1)} + e^{(k)}\|^2 - \|e^{(k)} - t_{k+1}\bar{v}^{(k)}\|^2 \\ [1] \implies & = \|t_{k+1}v^{(k+1)} + e^{(k)}\|^2 - \|v^{(k)} + e^{(k-1)} + t_{k+1}\bar{v}^{(k)}\|^2 \\ & = \|t_{k+1}v^{(k+1)} + e^{(k)}\|^2 - \|e^{(k-1)} + (t_{k+1}\theta_k + 1)v^{(k)}\|^2, \end{aligned} \tag{$\star\star$}$$

To match, we need:

$$\begin{cases} t_{k+1}^2 - t_{k+1} = t_k^2, \\ t_k = t_{k+1}\theta_k + 1. \end{cases} \tag{$\star**$}$$

Nesterov Momentum Sequences

Nesterov Momentum Sequences

The Nesterov momentum sequence solves $(\star * *)$! This is the sequence:

$$
\begin{aligned}
t_k &= \frac{1 + \sqrt{1 + 4t_k^2}}{2}, \\
\theta_k &= \frac{t_k - 1}{t_{k+1}}, \qquad (\star \star *)
\end{aligned}
$$

1. It has the property that $t_k \geq (k+1)/2$.
2. I continued from here to prove the $\mathcal{O}(1/k^2)$ convergence rate of FISTA. Read my report on that one.

SIMPLE LASSO

The Lasso Problem

Lasso minimizes the 2-norm objective with one norm penalty.

$$\min_{x}\left\{\frac{1}{2}\|Ax-b\|_2^2+\lambda\|x\|_1\right\}$$

And the prox for $\|\cdot\|_1$ is given by:

$$(\text{prox}_{\lambda\|\cdot\|,t}(x))_i = \text{sign}(x_i)\max(|x_i|-t\lambda, 0),$$

For our experiment:

1. $A = \text{diag}(\text{linsapce}(0, 2, 128))$.
2. Vector b is the diagonal of A and every odd index is changed into $\epsilon \sim N(0, 10^{-3})$.

Results

The plot of Δ_k:

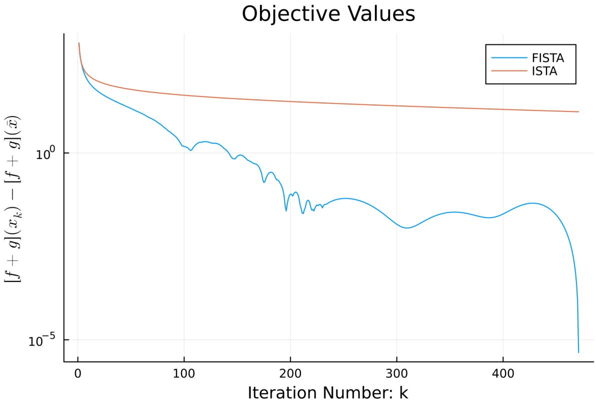
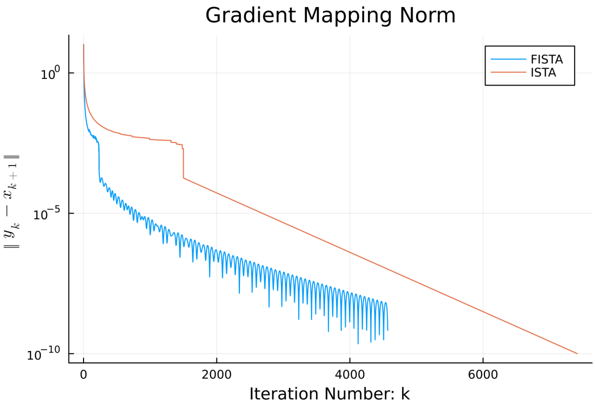


Figure: The left is the objective value of the function during all iterations.

Results

The plot of $\|y^{(k)} - x^{(k+1)}\|_\infty$:



Experiment Setup

Given an image that is convoluted by a Guassian kernel with some guassian noise, we want to recover the image, given the parameters for convolutions.

1. Guassian blur with a discrete 15 by 15 kernel is a linear transform represented by a sparse matrix A in the computer.
2. When an image is 500 by 500 with 3 color channels, A is 750000×750000.
3. Let the noise be on all normalized colors values with $N(0, 10^{-2})$
4. We let $\lambda = \alpha \times (3 \times 500^2)^{-1}$.
5. Implemented in Julia, and the code is too long to be shown here.

The Blurred Image

We consider blurring the image of a pink unicorn that I own.

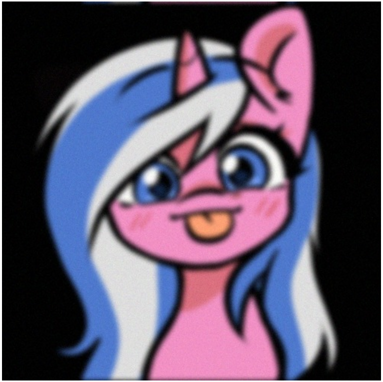


Figure: The image is blurred by the Gaussian Blurred matrix A with a tiny amount of noise on the level of 2×10^{-2} that is barely observable. Zoom in to observe the tiny amount of Gaussian noise on top of the blur.

Results

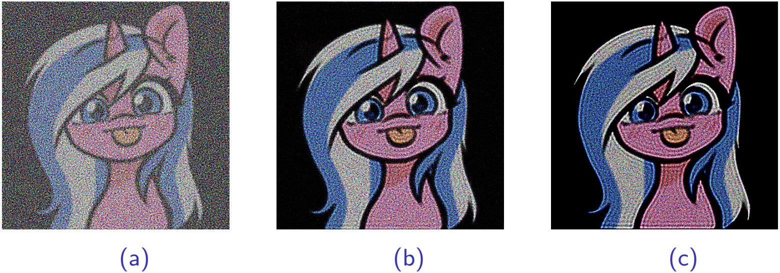


Figure: (a) $\alpha = 0$, without any one norm penalty, is not robust to the additional noise. (b) $\alpha = 0.01$, there is a tiny amount of λ. (c) $\alpha = 0.1$, it is more penalty compared to (a).

Contributions

The Paper's Contribution

Beck's contribution involves proving that the Nesterov accelerations work for the proximal gradient method. Beck popularized the use of momentum in the broader context to improve the convergence of algorithms such as the proximal gradient method.

My Contribution

I wrote the same proof in a different way for Proximal gradient in both the accelerated case and the unaccelerated case. I avoided using the assumption of Nesterov Momentum term and instead used a template method and the idea of form match to derive the sequence in the middle of the convergence proof.

References