Proximal Gradient: Convergence, Implementations and Applications

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ToC

Sum of 2 Functions

$$\min_{x} g(x) + h(x) \tag{1}$$

Through out the presentation we assume that the objective of some kind of function f can be interpreted as the sum of 2 functions. The paper we will be focusing on: FISTA (Fast Iterative-Shrinkage Algorithm) by Beck and Teboulle.

1. When $h = \delta_Q$ with Q closed and convex with $Q \subseteq \mathrm{ri} \circ \mathrm{dom}(h)$, we use projected subgradient.
2. When g is **strongly smooth** and h is **closed convex proper** whose proximal oracle is easy to compute, we consider the use of FISTA.
3. BIG Numerical Experiments!

Stuff to Go Over

What is FISTA

Simply speaking, the FISTA algorithm is the non-smooth analogy of gradient descend with Nesterov Momentum.

We will be going over these things in the presentations.

1. Derive the proximal gradient operator under standard convexity and regularity assumptions for the function g, h.
2. State one important lemma that arised during the proof for the proximal gradient method that is later useful for the proof for the FISTA.
3. Derive the FISTA algorithm's convergence rate and construct the sequence of the Nesterov Momentum during the proof using a template algorithm.

Proximal Operator Definition

Definition

The Proximal Operator Let f be convex closed and proper, then the proximal operator paramaterized by $\alpha > 0$ is a non-expansive mapping defined as:

$$\operatorname{prox}_{f,\alpha}(x) := \arg\min_{y} \left\{ f(y) + \frac{1}{2\alpha}\|y - x\|^2 \right\}.$$

Prox is the Resolvant of Subgradient

Lemma (Resolvant of the Subgradient)

When the function f is convex closed and proper, the $\text{prox}_{\alpha,f}$ can be viewed as the following operator $(I + \alpha\partial f)^{-1}$.

Proof.

$$
\begin{aligned}
\mathbf{0} &\in \partial\left[f(y) + \frac{1}{2\alpha}\|y - x\|^2 \middle| y\right](y^+) \\
\mathbf{0} &\in \partial f(y^+) + \frac{1}{\alpha}(y^+ - x) \\
\frac{x}{\alpha} &\in (\partial f + \alpha^{-1} I)(y^+) \\
x &\in (\alpha\partial f + I)(y^+) \\
y &\in (\alpha\partial f + I)^{-1}(x).
\end{aligned}
$$

$\square$

An Example of Prox

Definition (Soft Thresholding)

For some $x \in \mathbb{R}$, the proximal operator of it's absolute value is given as:

$$\text{prox}_{\lambda\|\cdot\|_1, t}(x) = \text{sign}(x)\max(|x| - t\lambda, 0).$$

One could interpret the sign operator as projecting x onto the interval $[-1, 1]$ and the $\max(|x| - t\lambda, 0)$ as the distance of the point x to the interval $[-t\lambda, t\lambda]$.

Strong Smoothness

Definition (Strong Smoothness)

A differentiable function g is called strongly smooth with a constant α then it satisfies:

$$|g(y) - g(x) - \langle \nabla g(x), y - x \rangle| \le \frac{\alpha}{2}\|x - y\|^2 \quad \forall x, y \in \mathbb{E}. \tag{2}$$

Remark

The absolute value sign can be removed and replaced with $0 \le$ on the left when the function g is a convex function.

Equivalence of Strong Smoothness and Lipschitz Gradient

Theorem (Lipschitz Gradient Equivalence under Convexity)

Suppose g is differentiable on the entire of $\mathbb{E}$. It is closed convex proper. It is strongly smooth with parameter α if and only if the gradient ∇g is globally Lipschitz continuous with a parameter of α and g is closed and convex.

$$\|\nabla g(x) - \nabla g(y)\| \leq \alpha \|x - y\| \quad \forall x, y \in \mathbb{E}$$

A Major Assumption

Assumption (Convex Smooth Nonsmooth with Bounded Minimizers)

We will assume that $g : \mathbb{E} \mapsto \mathbb{R}$ *is* **strongly smooth** *with constant* L_g *and* $h : \mathbb{E} \mapsto \bar{\mathbb{R}}$ **is closed convex and proper**. *We define* $f := g + h$ *to be the summed function and* $ri \circ dom(g) \cap ri \circ dom(h) \neq \emptyset$. *We also assume that a set of minimizers exists for the function* f *and that the set is bounded. Denote the minimizer using* $\bar{x}$.

Envelope and Upper Bounding Functions

Upper Bounding Function

With assumption 1, we construct an upper bounding functions at the point x evaluated at y for the function f and it's given by:

$$g(x) + \nabla g(x)^T(y - x) + \frac{\beta}{2}\|y - x\|^2 + h(y) =: m_x(y|\beta) \quad \forall y \in \mathbb{E},$$

Minimizers wrt to y

References