section

1 Fundations

This sections focuses on important mathematical entities that are important for formulating, analyzing the Conjugate Gradient and the Lanczos Algorithm. Major parts of this sections cited from...

1.1 Projectors

There are 2 types of projector, an oblique Projector and Orthgonal Projector. An Orthogonal Projector is Hermitian and vice versa. A matrix P is called a projector if:

$$P^2 = P \tag{1.1.1}$$

This property is sometimes referred as idempotent. As a consequence, $\operatorname{ran}(I-P) = \operatorname{null}(P)$ and here is the proof:

Proof.

$$\forall x \in \mathbb{C}^n : P(I-P)x = \mathbf{0} \implies \operatorname{ran}(I-P) \subseteq \operatorname{null}(P) \tag{1.1.2}$$

$$\forall x \in \operatorname{null}(P) : Px = \mathbf{0} \implies (I-P)x = x \implies x \in \operatorname{ran}(I-P) \tag{1.1.3}$$

$$\implies \operatorname{ran}(I-P) = \operatorname{null}(P) \tag{1.1.4}$$

□

1.1.1 Orthogonal Projector

An orthogonal projector is a projector such that:

$$\operatorname{null}(P) \perp \operatorname{ran}(P) \tag{1.1.5}$$

This property is in fact, very special. A good example of an orthogonal projector would be the Householder Reflector Matrix. Or just any $\hat{u}\hat{u}^H$ where $\hat{u}$ is being an unitary vector. For convenience of proving, assume subspace $M = \operatorname{ran}(P)$. Consider the following lemma:

Lemma 1.

$$\operatorname{null}(P^H) = \operatorname{ran}(P)^\perp \tag{1.1.6}$$

$$\operatorname{null}(P) = \operatorname{ran}(P^H)^\perp \tag{1.1.7}$$

Using (1.1.4) and consider the proof:

Proof.

$$\langle P^H x, y\rangle = \langle x, Py\rangle \tag{1.1.8}$$

$$\forall x \in \operatorname{null}(P^H), y \in \mathbb{C}^n \tag{1.1.9}$$

$$\implies \langle P^H x, y\rangle = 0 = \langle x, Py\rangle \tag{1.1.10}$$

$$\implies \operatorname{null}(P^H) \perp \operatorname{ran}(P) \tag{1.1.11}$$

$$\forall y \in \operatorname{null}(P), x \in \mathbb{C}^n : \tag{1.1.12}$$

$$\langle x, Py\rangle = 0 = \langle P^H x, y\rangle \tag{1.1.13}$$

$$\implies \operatorname{ran}(P^H) \perp \operatorname{null}(P) \tag{1.1.14}$$

□

Proposition 1. A projector is orthogonal iff it's Hermitian.

Proof. $\Longleftarrow$ Assuming the matrix is Hermitian and it's a projector, then we wish to prove that it's an orthogonal projector. Let's recall:

$$\text{null}(P^H) = \text{ran}(P)^{\perp} \tag{1.1.15}$$

$$\text{null}(P) = \text{ran}(P^H)^{\perp} \tag{1.1.16}$$

Substituting $P^H = P$, we have $\text{null}(P) = \text{ran}(P)^{\perp}$, Which is the definition of Orthogonal Projector. Therefore, P is an orthogonal projector by the definition of the projector.

For the $\implies$ direction, we assume that P is an Orthogonal Projector, then we wish to show that it's also Hermitian. Observe that P^H is also a projector because $(P^H)^2 = (P^2)^H$. Then, using the definition of orthogonal projector:

□

1.2 Projectors and Norm Minimizations

An orthogonal projector always reduce the 2 norm of a vector. Given any subspace M, we can create a basis of vectors packing into the some matrix, say A, then P_M as a projector onto the basis M one example can be: $A(AA^T)^{-1}A^T$. Let's consider the claim:

$$\|P_M x\|^2 \le \|x\|^2 \tag{1.2.1}$$

Proof:

$$x = Px + (I - P)x \tag{1.2.2}$$

$$\|x\|^2 = \|Px\|^2 + \|(I - P)x\|^2 \tag{1.2.3}$$

$$\|x\|^2 \ge \|Px\|^2 \tag{1.2.4}$$

Using this property of the Orthogonal Projector, we consider the following minimizations problem:

$$\min_{x \in M} \|y - x\|_2^2 = \|y - P_M(y)\|_2^2 \tag{1.2.5}$$

Proof:

$$\|y - x\|_2^2 = \|y - P_M y + P_M y - x\|_2^2 \tag{1.2.6}$$

$$\|y - x\|_2^2 = \|y - P_M y\|_2^2 + \|P_M y - x\|_2^2 \tag{1.2.7}$$

$$\implies \|y - P_M y\|_2^2 \le \|y - x\|_2^2 \tag{1.2.8}$$

That concludes the proof. Observe that, $y - P_M y \perp M$ and $P_M y - x \in M$ because $P_M y, x \in M$, which allows us to split the norm of $y - x$ into 2 components. In addition using the fact that the projector is orthogonal. That concludes the proof.

1.3 Subspace Orthogonality Framework

1.4 Useful Theorems