Volatility Curve Parametrizations

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Abstract

We describe the various volatility curves currently in use and under discussion.

Version 1: Dec 14, 2005
Version 2: Dec 15, 2005, added remarks on baskets.
Version 3: Jan 11, 2006, added remark on fair/ATM vol ratio in JW,JW3 vs D12.

1 Introduction

This is a quick update on current discussions of implied volsurface parametrizations. For now they are always parametrized one term at a time, without imposing a (parametric!) term-structure on the parameters themselves (though we have some ideas there).

Some background and a more detailed description of the JW ("JumpWing") surface currently used for marking (index) vols and as input to exotic pricing models can be found in [1].

As a wish list for a good volatility parametrization one might write down:

- Fitted parameters should be stable from day to day.
- If possible the term-structure of the parameters (for a given surface) should be smooth (ideally flat, if that is possible).
- The parameters should be intuitive. Related to this, they ideally they would correspond to the main orthogonal "eigen-modes" of how volsurfaces move: overall vol level, skew, smile, etc. This should help when a trader has to make up a volsurface "on the fly". Also, one would hope that the parameters can easily and meaningfully be compared across terms, names, different historical regimes, etc.
- It should be possible to fit the implied vol curves implied by standard SVJ models reasonably well. This seems like an a priori good idea and certainly helps when calibrating such models.
- The local vols obtained from the fitted parametric volsurfaces should exist over as wide a range as possible, to avoid special purpose hacks when using local vols in practice.
- The last two items are related to the other obvious requirement one might have: any insights one can gain from general no-arbitrage constraints should be incorporated (see below).
- For basket options where the basket can be considered a single underlier it is useful to construct a volsurface using a phenomenological approach that combines a "naive" variance weighted average with a correlation skew; see [2] for a specific proposal. In such a situation it would be very helpful in practice if the so constructed basket volsurface conforms to the same vol parametrization used for the skew of the underliers going into the basket (at least within a few bps after refitting).

On the more technical level, for the vol curves currently under discussion we have converged on the following features for all them:

- They are all expressed in terms of normalized strike (denoted as z or NS depending on context):

$$z \;:=\; \frac{\log(K/F)}{\sigma_0\sqrt{T}} \quad , \tag{1}$$

  where $F = F(T)$ is the forward and $\sigma_0 = \sigma_0(T)$ the ATF vol of the term T in question. (Here T is in years, in an actual/365 convention.) Note that ATF corresponds to $z = 0$ for any term. The put wing corresponds to $z < 0$, the call wing to $z > 0$.

  Using normalized strike is also very useful when trying to come up with a framework for combining component volsurfaces into a basket volsurface with a correlation skew; see [2].

- For the curves discussed below, the first three parameters are always the ATF vol σ_0 as well as "skew" s and "smile" c, defined as the slope and curvature of the vol *shape* curve around ATF, via:

$$\sigma \;\equiv\; \sigma(z) \;=\; \sigma_0 \left(1 \,+\, s\,z \,+\, \tfrac{1}{2}c\,z^2 \,+\, \ldots\right) \tag{2}$$

  Note that s and c so defined are dimensionless, so they have some a priori chance of having weak term-structure and being easily comparable across different names, historical market regimes, etc.

Remember also that no-arbitrage implies that the worst way in which vols squared can diverge in the wings is like $\log(K/F)$. More precisely, it is easy to show that the following must hold for the implied vol at very small or large strikes:[1]

$$\sigma^2(T,K)\,T \;\leq\; C_\pm\,|\log(K/F)| \qquad \text{as} \qquad \log(K/F) \to \pm\infty \;, \tag{3}$$

for some non-negative constants $C_\pm$ which must be less or equal to 2. If there is a non-zero probability of default then the put-wing parameter $C_- = 2$ exactly; without default $C_\pm < 2$ strictly.

We should also mention that in any of the well-known models with stochastic volatility or jumps (Heston, Merton, Heston+Merton, etc), the above bounds are indeed saturated, ie there are non-zero parameters $C_\pm$ for which the above asymptotic behavior holds (in general, $C_\pm$ are not easily expressed in terms of the parameters of the underlying SVJ-type model, though).

The JW, JW3, D12 curves discussed below are integrated in our volsurface infra-structure in the EQAL library, meaning that vols can easily be accessed in terms of any strike type (K, K/S, NS, delta, etc), and local vols, implied densities etc are all available for "free". They also can be used in "sticky-strike" form. Finally, we have routines to fit vol matrix data to any of these curves, see the add-in `Eq.VolData.Fit`.

2 JumpWing (JW/JWSS) Curve

We refer to [1] for details (at some point that report should be updated with all we have since learned). Here we just add a few remarks.

[1] We emphasize that this comes from the simple constraint that a put of a lower strike should be worth less than that of a higher strike for a given term, similar for the call wing, using very small or very large strikes, respectively.

- In the JWSS form of this parametrization the first three of its five parameters vol0, skew, smile, AC, AP have exactly the same meaning as the parameters σ_0, s, c introduced above. AC and AP are simply related to the parameters $C_\pm$ introduced in (3).
- The JW curve is the simplest (and most intuitive) curve we know that can fit the implied vols coming from SVJ-type models over a wide range of terms and strikes (say $z = -3 \ldots 3$) within a few bps.
- When combining basket component skews of JW form in a suitable basket volsurface framework, eg as in [2], the basket volsurfaces can be refitted to JW within a few bps or less.

3 JW3

The JW3 curve is the simplest 3-parameter curve (we have thought of) that satisfies the no-arbitrage constraint (3). It is defined by

$$\sigma^2 \;=\; \sigma_0^2 \left(\tfrac{1}{2}(1+\tilde{s}z) \;+\; \sqrt{\tfrac{1}{4}(1+\tilde{s}z)^2 + \tfrac{1}{2}\tilde{c}z^2} \right) . \tag{4}$$

Taking the square root and expanding around small z one obtains

$$\sigma \;=\; \sigma_0 \left(1 \;+\; \tfrac{1}{2}\tilde{s}\, z \;+\; \tfrac{1}{4}\,(\tilde{c} - \tfrac{1}{2}\tilde{s}^2)\, z^2 \;+\; \ldots \right) . \tag{5}$$

The relationship of $\tilde{s}, \tilde{c}$ — which are the dimensionless slope and curvature of the ATF expansion of vol *squared* — to the earlier defined skew s and smile c is therefore

$$s \;=\; \tfrac{1}{2}\tilde{s}, \qquad c \;=\; \tfrac{1}{2}\tilde{c} - \tfrac{1}{4}\tilde{s}^2 . \tag{6}$$

Note that as long as $\tilde{c} > 0$ the JW3 curve can never go negative, no matter how negative the skew s is.

For reference we give the wing asymptotics for large normalized strike $|z|$:

$$\sigma^2 \;=\; \sigma_0^2\, |z| \left(\sqrt{\tfrac{1}{4}\tilde{s}^2 + \tfrac{1}{2}\tilde{c}} \;\pm\; \tfrac{1}{2}\tilde{s} \right) \qquad \text{as} \qquad z \to \pm\infty . \tag{7}$$

For comparison with the JW/JWSS curve [1], the terms in parentheses are the analog of the parameters AC, AP, except that here they are not independent parameters, but instead determined in terms of the slope and curvature around ATF. Note, however, that one can *not exactly* match a JW3 curve to a special case of the JW curve; the functional forms *are* different. But one can fit any JW3 curve to a JW curve within a few bps over a very wide range of strikes. Although this is perhaps a slight theoretical blemish that we might want to address in the future (eg by finding 4- or 5-parameter curves that contain JW3 as a special case), it is not really a significant practical problem for now.

As fits of real-world data (either our own or Totem's) have shown, the JW3 curve leads to robust and sensible volsurfaces, as shown eg by the fact that local vols exist over a very large strike range (down to 10% or even 1% and less of current spot). For short terms the fits are not as good as for JW, as expected from the fact that it has 2 parameters less, but it might well be the "best" 3-parameter vol curve one can come up with — at least I'm not aware of any other one with nearly as good properties.

4 D12

The D12 vol σ was originally defined by the following implicit equation

$$\sigma \;=\; \tilde{\sigma_0}\left(1 \;+\; sz \;+\; \tilde{c}\,\frac{d_1 d_2}{\sigma}\right) , \tag{8}$$

in terms of the three parameters $\tilde{\sigma_0}, s, \tilde{c}$. This means, as we will see in a moment, that to calculate the D12 vol at a given term and strike one has to solve a quartic equation.

We used the tilded parameters $\tilde{\sigma_0}$ and $\tilde{c}$ in the above because it will turn out to be convenient to redefine the vol and curvature parameters. To see this use

$$d_1 d_2 \;=\; \frac{1}{\sigma^2 T}\left(y^2 - \tfrac{1}{4}\sigma^4 T^2\right) \qquad \text{where} \qquad y := \log(K/F) , \tag{9}$$

to rewrite (8), after multiplying by $\sigma^3 T$, as

$$\sigma^4\, T\, (1 + \tfrac{1}{4}\tilde{c}\tilde{\sigma_0} T) \;-\; \sigma^3\, T\, \tilde{\sigma_0}(1 + sz) \;-\; \tilde{c}\,\tilde{\sigma_0}\, y^2 \;=\; 0 . \tag{10}$$

If we introduce a new vol parameter

$$\sigma_0 \;:=\; \frac{\tilde{\sigma_0}}{1 + \frac{1}{4}\tilde{c}\tilde{\sigma_0} T} \tag{11}$$

we can rewrite this as

$$\sigma^4 \;-\; \sigma^3 \sigma_0 (1 + sz) \;-\; \tilde{c}\,\sigma_0^3\, z^2 \;=\; 0 . \tag{12}$$

The parameter $\tilde{c}$ in the above is dimensionful, but if we define $c := 2\tilde{c}/\sigma_0$ we get a quartic equation for the D12 vol

$$\sigma^4 \;-\; \sigma^3\, \sigma_0\, (1 + sz) \;-\; \tfrac{1}{2}\, c\, \sigma_0^4\, z^2 \;=\; 0 \tag{13}$$

that has one dimensionful parameter, σ_0, and two dimensionless ones, s and c. In fact, these parameter names were chosen such that they have exactly the same meaning as in equation (2). We leave a proof of this as an easy exercise to the reader.

Because we then have "universality" of the meaning of the first three parameters across the JW, JW3 and D12 curves, we adopt (13) as our *definition* of the D12 vol curve.

From eqn (13) it is easy to derive the put and call wing asymptotics of the D12 curve, namely (assuming skew $s < 0$)

$$\sigma \;=\; \sigma_0\, |s|\, |z| \qquad \text{for} \quad z \to -\infty \tag{14}$$

and

$$\sigma \;=\; \sigma_0 \left(\frac{c\, z}{2|s|}\right)^{1/3} \qquad \text{for} \quad z \to +\infty , \tag{15}$$

respectively. Note that the behavior in the put wing *violates* the bound (3) imposed by no-arbitrage; the D12 vol is, asymptotically at least, *too large* in the put wing. When fitting implied vol data to the D12 curve, this shows up as a failure of the local vol derived from the D12 curve for modestly small strikes (eg 50% of spot) to exist. More intuitively, perhaps, the implied density will go negative for sufficiently small strikes. We have never seen this happen with the JW or JW3 curves.

Another area in which the different behavior of D12 in the put wing is significant is for the ratio of fair vol (in the sense of a variance swap) and ATMF vol. Namely, for maturities of a few years upwards this ratio will be significantly higher for D12 than for JW or JW3 (with the latter two agreeing very well). This is true even in cases where around ATMF the D12 curve fits market vols (slightly) better than JW3. This might be an issue in marking variance swaps with the D12 curve.

References

[1] T. Klassen (2003), *Equity Volatility Surfaces*, ELPD Wachovia.

[2] T. Klassen (2004), *Basket Volatility Surfaces*, ELPD Wachovia.