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Modeling the Implied Volatility Surface

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Outline of this talk

- A compound Poisson model of stock trading

- The relationship between volatility and volume

- Clustering

- Correlation between volatility changes and log returns

- Stochastic volatility

- Dynamics of the volatility skew

- Similarities between stochastic volatility models

- Do stochastic volatility models fit option prices?

- Jumps

- The impact of large option trades

Stock trading as a compound Poisson process

- Consider a random time change from conventional calendar time to trading time such that the rate of arrival of stock trades in a given (transformed) time interval is a constant l .

- Intuitively, relative to real time, trading time flows faster when there is more activity in the stock and more slowly when there is less activity.

- Suppose that the (random) size n of a trade is independent of the level of activity in the stock.

- Assume further that each trade impacts the mid-log-price of the stock by an amount proportional to $\sqrt{n}$.

  - This is a standard assumption in the market microstructure literature

- Then the change in log-mid-price over some time interval is given by

$$\Delta x = \sum_{i=1}^{N} \text{sgn}(n_i)\sqrt{n_i}$$

- Note that both the number of trades N and the size of each trade n_i in a given time interval Δt are random.

Volatility and volume: a relationship

- The variance of this random sum of random variables is given by

$$Var\left[\Delta x\right] = E\left[N\right]Var\left[a\sqrt{n_i}\right] + Var\left[N\right]E\left[a\sqrt{n_i}\right]^2$$

$$= a^2 l\,\Delta t\,E\left[n_i\right]$$

- Rewriting this in terms of volatility, we obtain

$$Var\left[\Delta x\right] = s^2\Delta t = a^2 l\,\Delta t\,E\left[n_i\right]$$

- But $l\,\Delta t\,E\left[n_i\right]$ is just the expectation of the volume over the time interval Δt.

- The factor Δt cancels and transforming back to real time, we see that variance is directly proportional to volume in this simple model.

- Moreover, the distribution of returns *in trading time* is approximately Gaussian for large Δt.
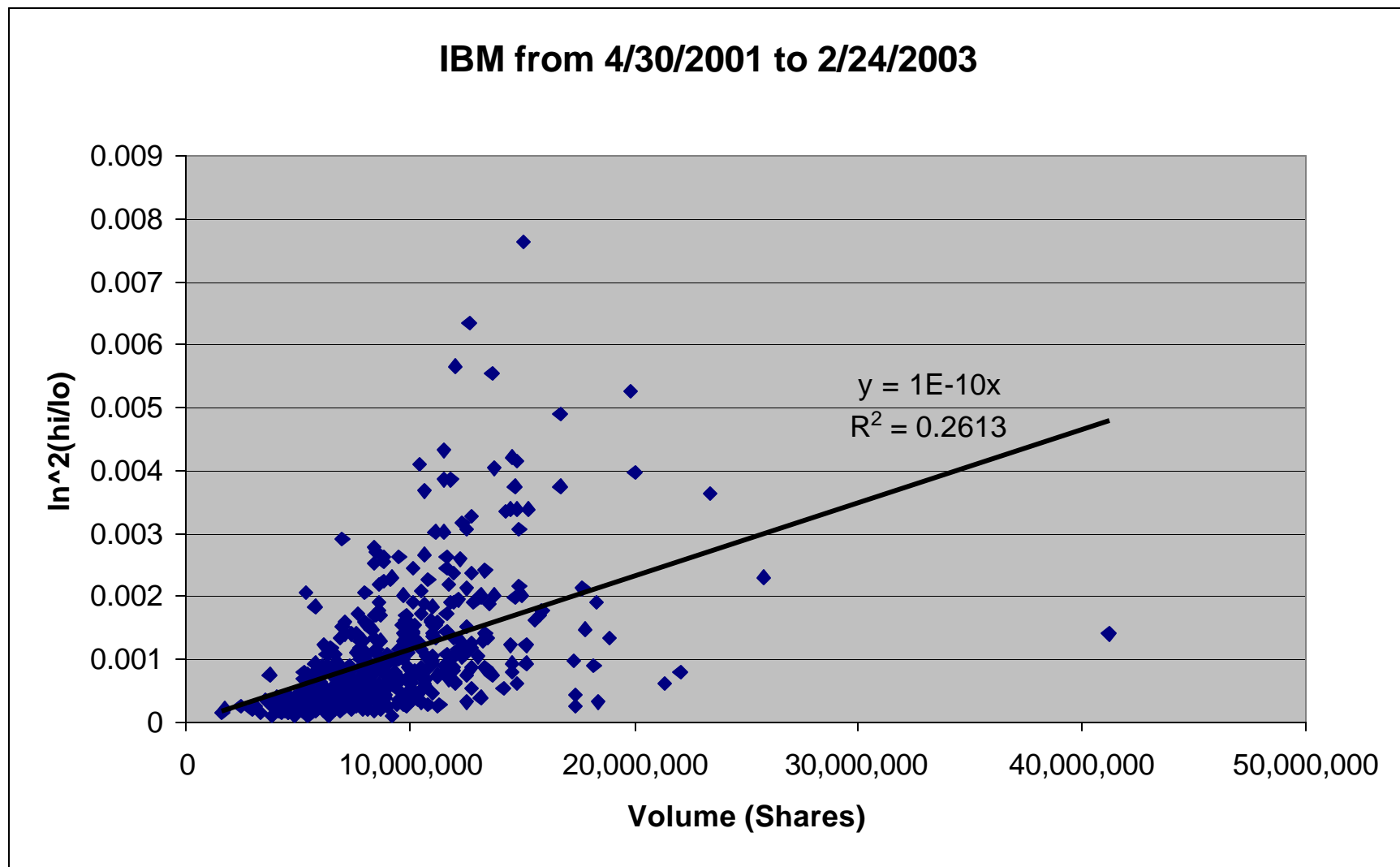
The $\sqrt{n}$ relationship

- A key assumption in our simple model is that market impact is proportional to the square root of the trade size $\sqrt{n}$. The following argument shows why this is plausible:

  - A market maker requires an excess return proportional to the risk of holding inventory.
  - Risk is proportional to $s\sqrt{T}$ where T is the holding period.
  - The holding period should be proportional to the size of the position.
  - So the required excess return must be proportional to $\sqrt{n}$.
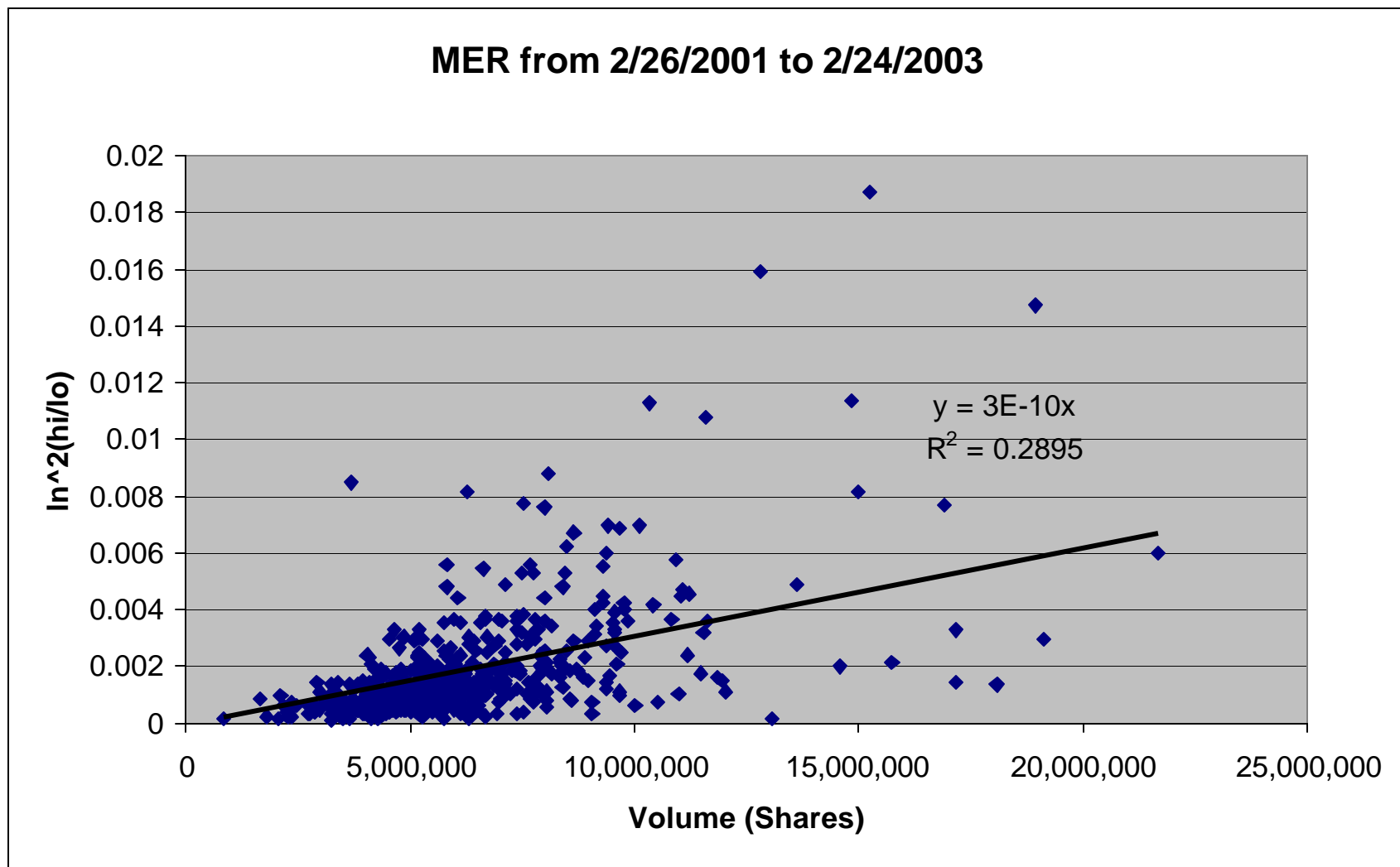
Average trade size is almost independent of activity

Counting log(number of trades) vs log(avg size of trades)

Empirical variance vs volume

IBM from 4/30/2001 to 2/24/2003

$y = 1E\text{-}10x$

$R^2 = 0.2613$

ln^2(hi/lo) vs Volume (Shares)

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Empirical variance vs volume

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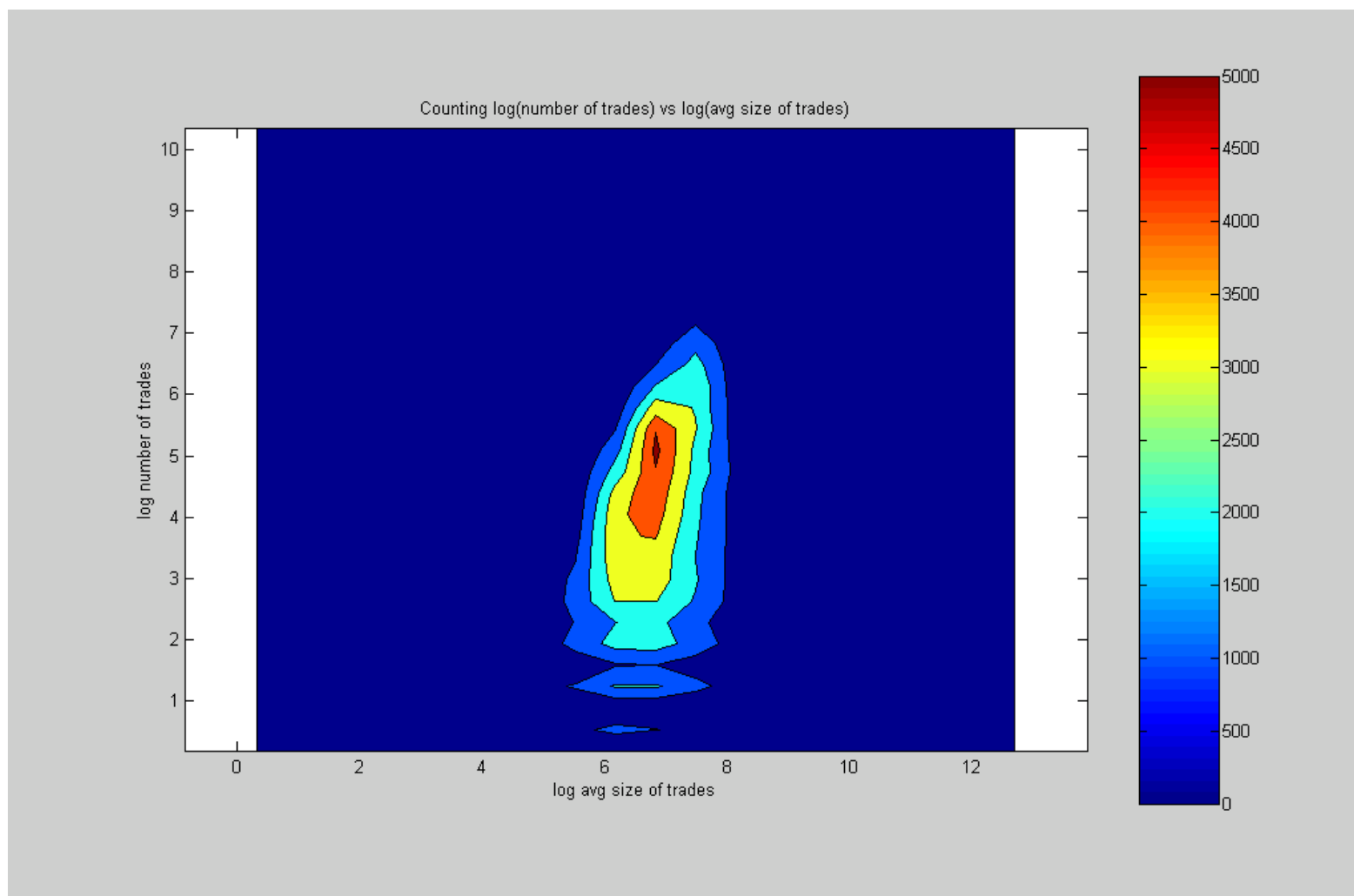
Implications of our simple model

- Our simple but realistic model has the following modeling implications
  - Log returns are roughly Gaussian with constant variance *in trading time* (sometimes call intrinsic time) defined in terms of transaction volume
  - Trading time is the inverse of variance
  - When we transform from trading time to real time, variance appears to be random
- It is natural to model the log stock price as a diffusion process subordinated to another random process which is really trading volume in our model - stochastic volatility
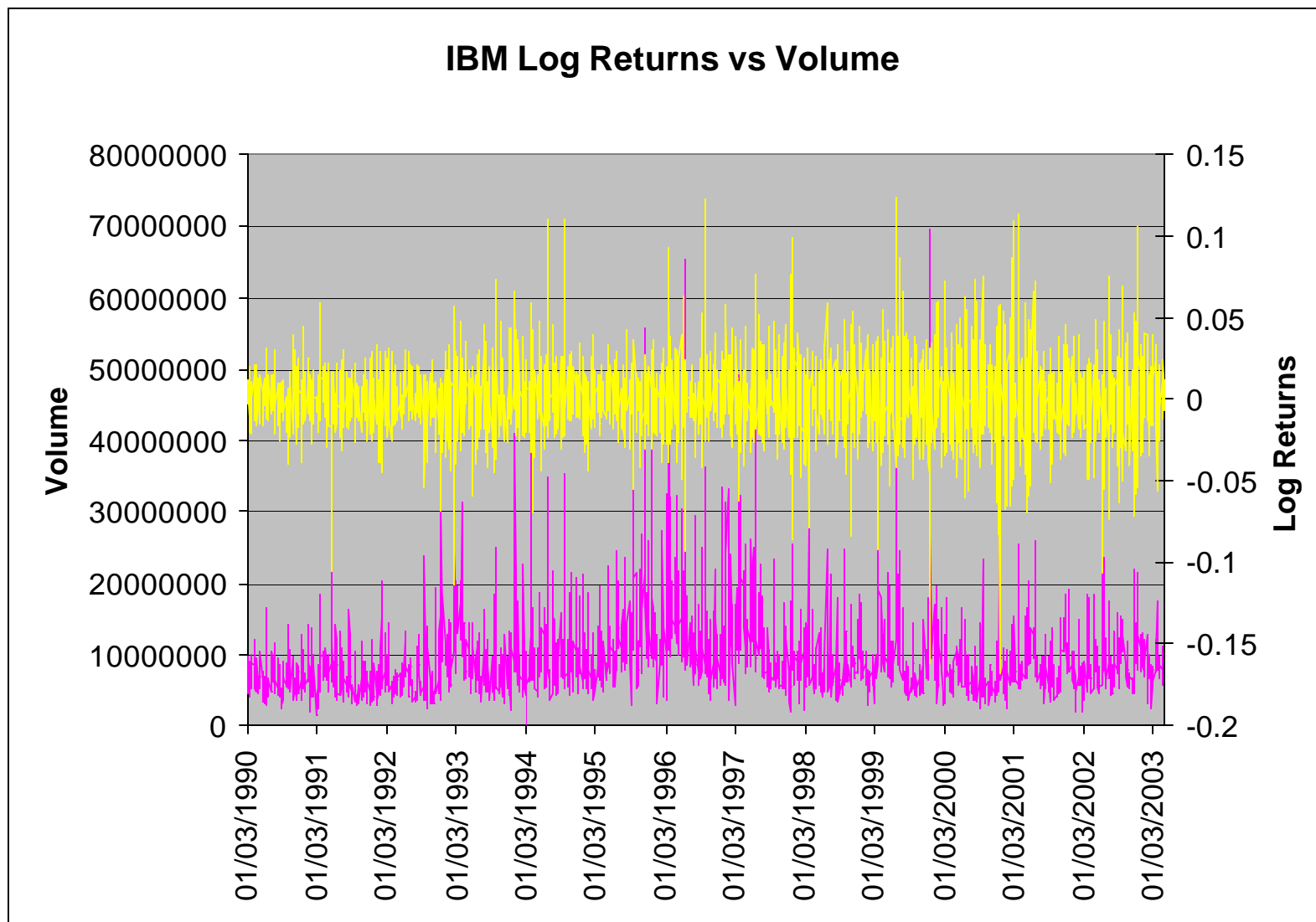- What form should the volatility process take?

Mean reversion of volatility: an economic argument

- There is a simple economic argument which justifies the mean reversion of volatility (the same argument that is used to justify the mean reversion of interest rates):

  - Consider the distribution of the volatility of IBM in one hundred years time say. If volatility were not mean-reverting (*i.e.* if the distribution of volatility were not stable), the probability of the volatility of IBM being between 1% and 100% would be rather low. Since we believe that it is overwhelmingly likely that the volatility of IBM would in fact lie in that range, we deduce that volatility must be mean-reverting.
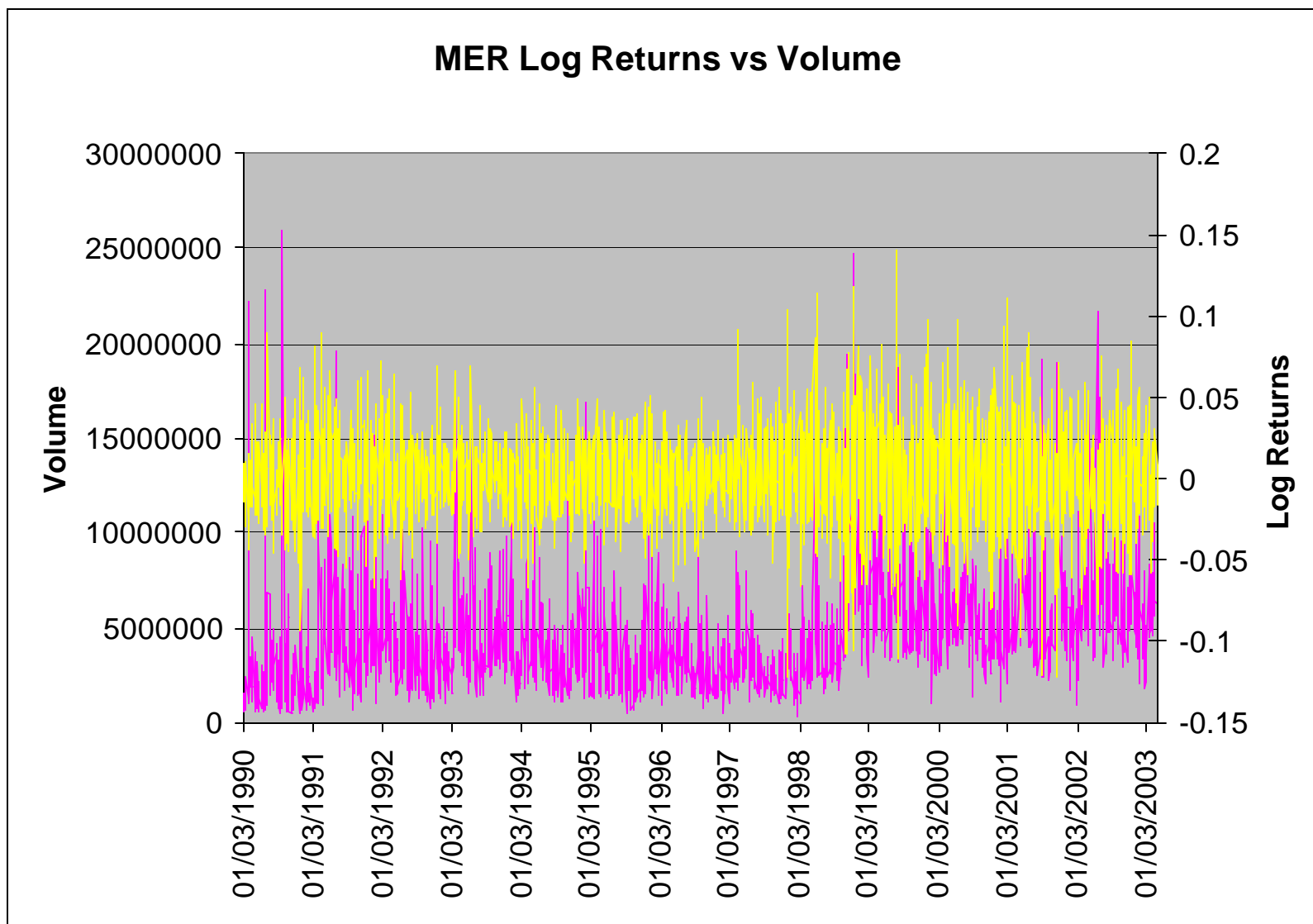
Empirical volatility term structure observations

- Short-dated implied volatilities move more than long-dated implied volatilities

- The term structure of implied volatility has the form of exponential decay to a long-term level

- The shape and dynamics of the volatility term structure imply that volatility must mean-revert *i.e.* that volatility changes are auto-correlated

- The following slides show that this is also true empirically.

Volatility and Volume Clustering

IBM Log Returns vs Volume

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Volatility and Volume Clustering


MER Log Returns vs Volume

Correlation between volatility changes and log returns

- The empirical fact that implied volatility is a decreasing function of strike price indicates that volatility changes must be negatively correlated with log returns.

Implied Volatility vs Strike

June 2002 options as of 4/24/2002
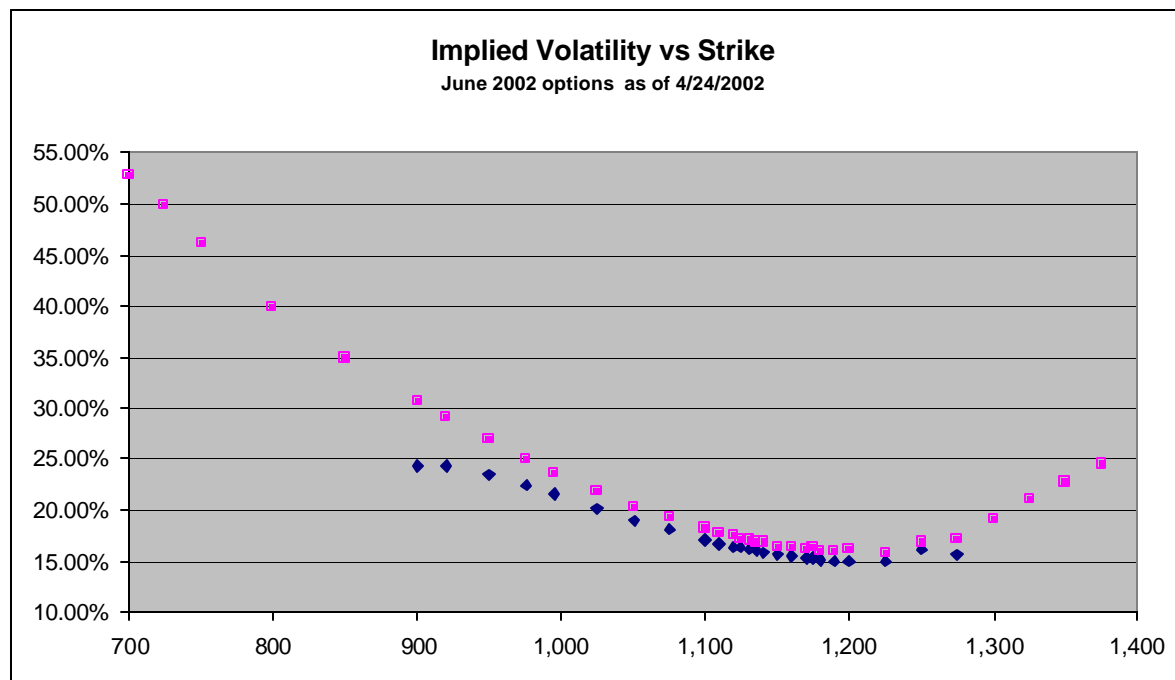
- The following slide shows that volatility changes really are anti-correlated with stock price changes

SPX from 1/1/1990 to 2/24/2003

Correlation of vol changes with log returns

$y = -1.1066x + 0.0003$

$R^2 = 0.6218$

Volatility Change

Log Return
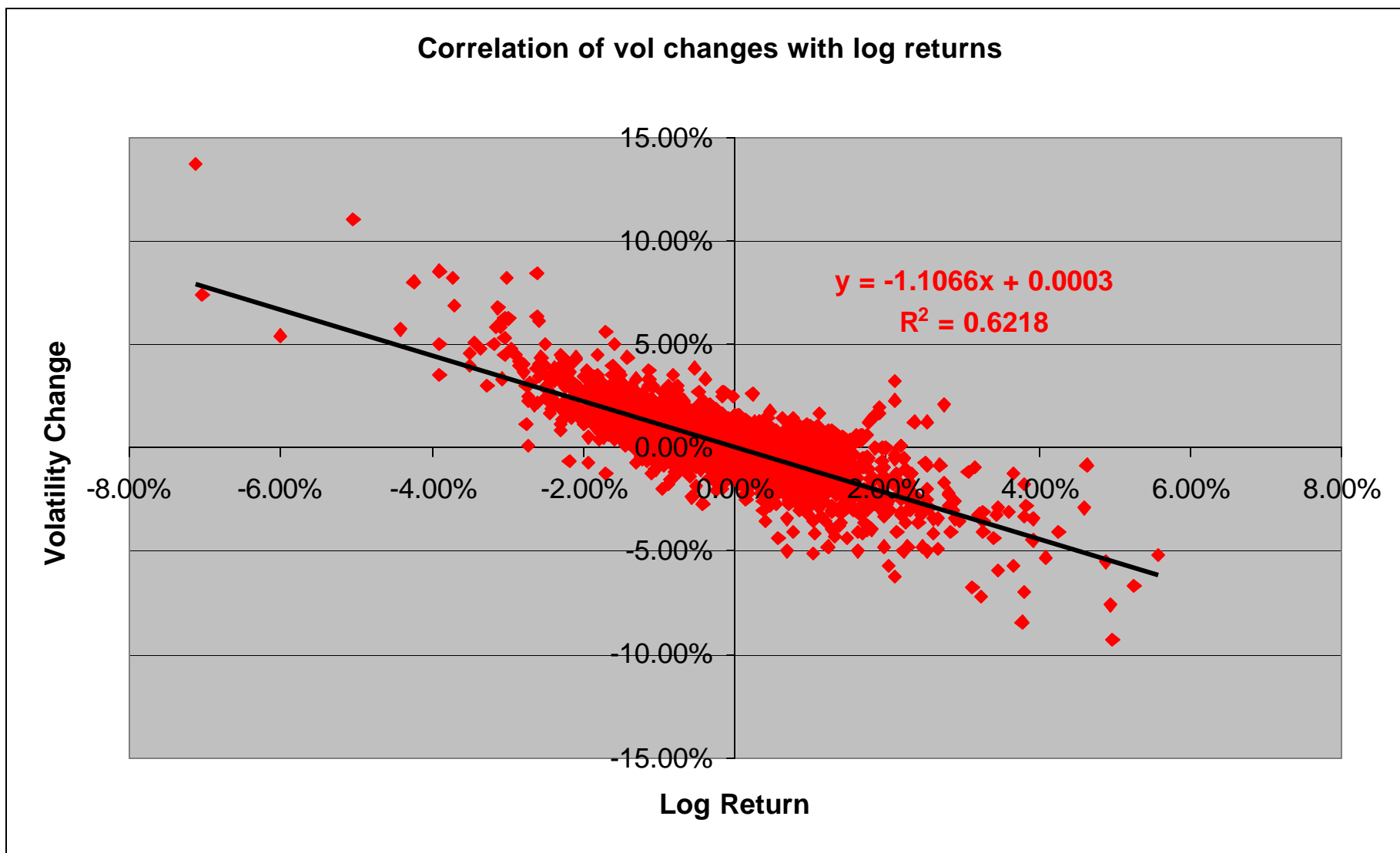
A generic stochastic volatility model

- We are now in a position to write down a generic stochastic volatility model consistent with our observations. Let x denote the log stock price and v denote its variance. Then

$$dx = \boldsymbol{m}dt + \sqrt{v}\, dZ_1$$

$$dv = \boldsymbol{a}\left(v\right) + \boldsymbol{h}\, v^{\boldsymbol{b}}\, \sqrt{v}\, dZ_2$$

with $\left\langle dZ_1, dZ_2 \right\rangle = \boldsymbol{r}\, dt$.

- $\boldsymbol{a}\left(v\right)$ is a mean-reversion term, $\boldsymbol{r}$ is the correlation between volatility moves and stock price moves and $\boldsymbol{h}$ is called "volatility of volatility".

- $\boldsymbol{b} = 0$ gives the Heston model

 $\boldsymbol{b} = \frac{1}{2}$ gives Wiggins' lognormal model

Dynamics of the volatility skew

- So far, we have ascertained that the volatility process must be mean-reverting and that volatility moves are anti-correlated with log returns.

- Can we say anything about the diffusion coefficient?

- The following four slides give a sense of the dynamics of the volatility surface

- We see that as volatility increases
  - so does volatility of volatility
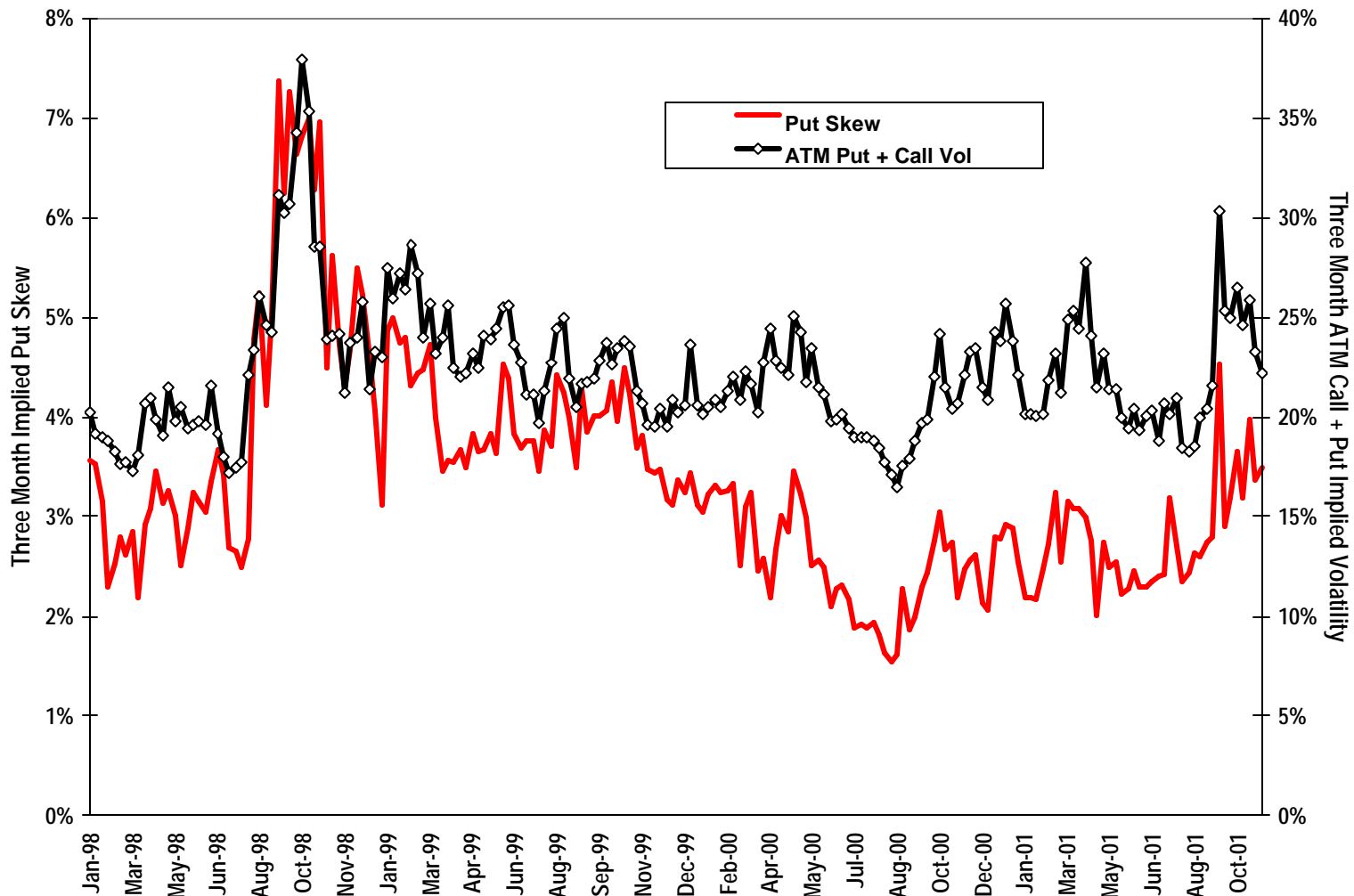 - and so does the volatility skew

Historical SPX implied volatility
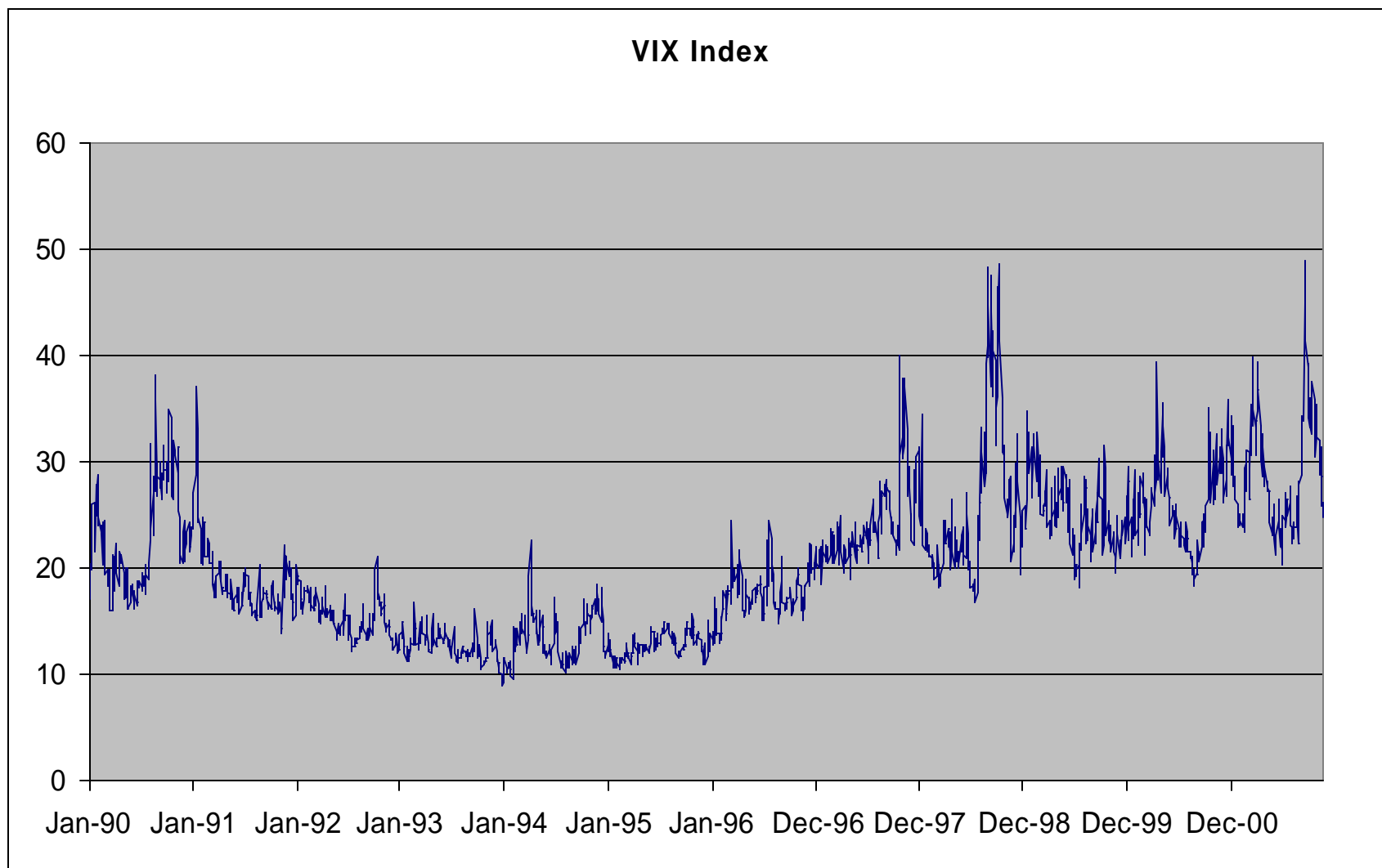
VIX Index

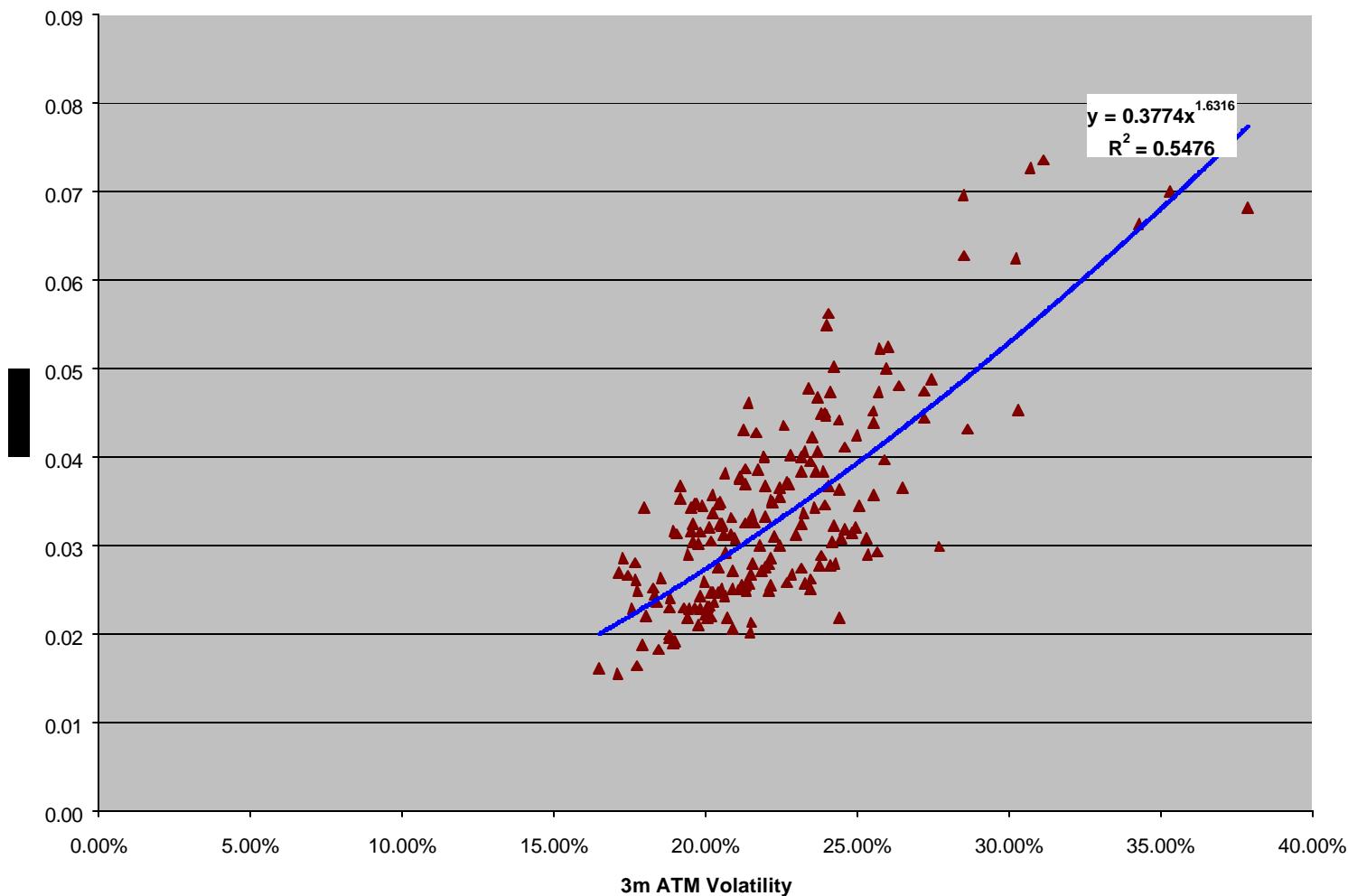# Regression of VIX volatility vs VIX level

$$y = 0.0954x^{1.339}$$
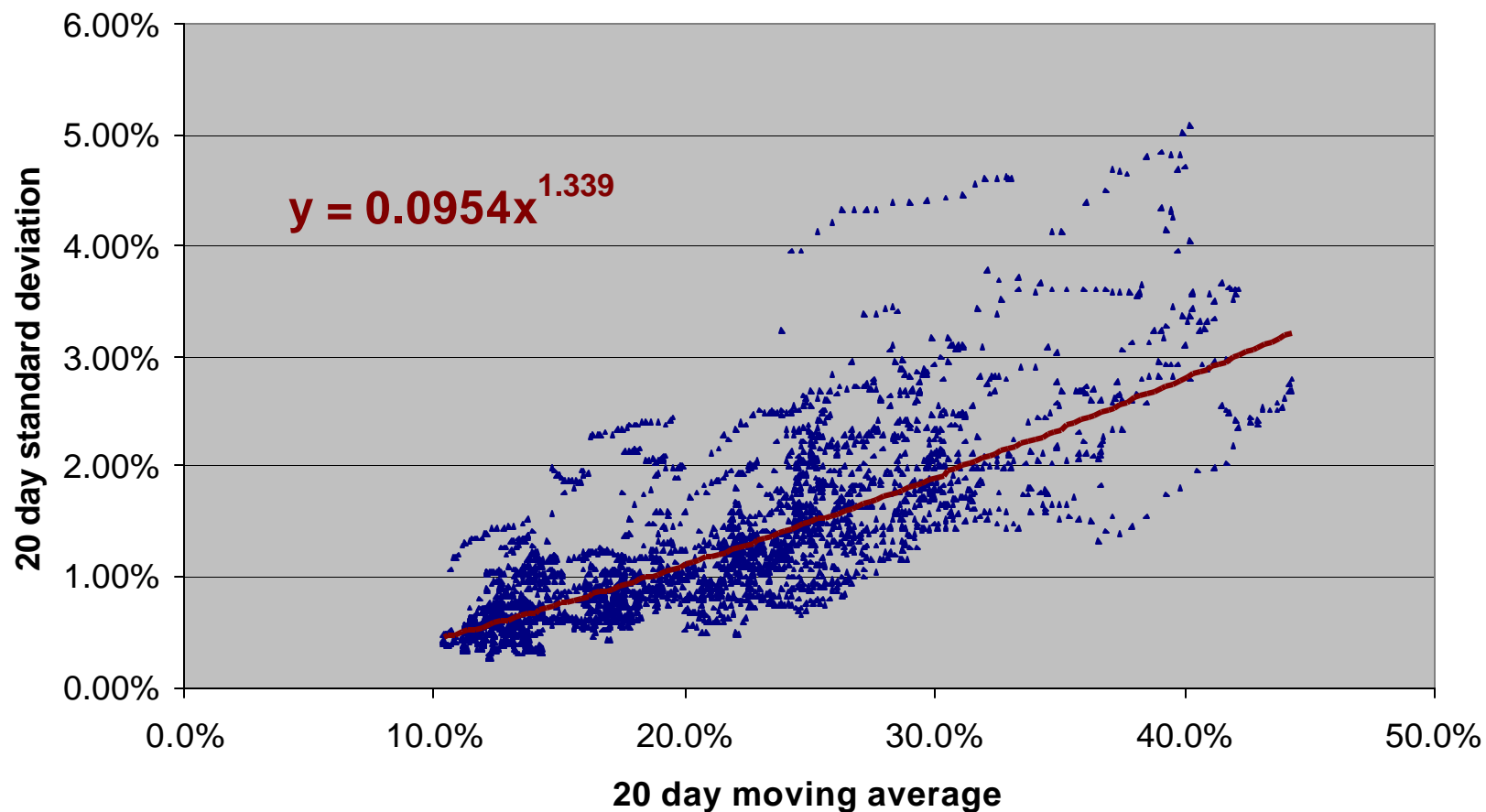
SPX implied volatility skew vs implied volatility level

Note that skew is defined to be the difference in volatility for ± 0.25 delta

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Regression of skew vs volatility level

$$y = 0.3774x^{1.6316}$$
$$R^2 = 0.5476$$

3m ATM Volatility

Note that skew is defined to be the difference in volatility for ± 0.25 delta

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Similarities between stochastic volatility models

- All stochastic volatility models generate volatility surfaces with approximately the same shape
- The Heston model $dv = -\boldsymbol{l}\left(v - \bar{v}\right)dt + \boldsymbol{h}\sqrt{v}\,dZ$ has an implied volatility term structure that looks to leading order like

$$\boldsymbol{s}_{BS}\left(x,T\right)^2 \approx \bar{v} + \left(v - \bar{v}\right)\frac{(1 - e^{-lT})}{lT}$$

It's easy to see that this shape should not depend very much on the particular choice of model.

- Also, Gatheral (2002) shows that the term structure of the volatility skew has the following approximate behavior for all stochastic volatility models of the generic form $dv = \boldsymbol{a}\left(v\right)dt + \boldsymbol{h}\,\boldsymbol{b}\left(v\right)\sqrt{v}\,dZ$

$$\frac{\partial}{\partial x}\boldsymbol{s}_{BS}\left(x,T\right)^2 \approx \frac{\boldsymbol{rh}\,\boldsymbol{b}\left(v\right)}{\boldsymbol{l}'T}\left\{1 - \frac{(1 - e^{-l'T})}{\boldsymbol{l}'T}\right\}$$

with $\boldsymbol{l}' = \boldsymbol{l} - \boldsymbol{rh}\,\boldsymbol{b}\left(v\right)/2$

Implications for the volatility process

- In our generic stochastic volatility model parameterization

$$\Delta v \approx h \, v^b \, \sqrt{v} \, \sqrt{\Delta t} \, Z$$

- Transforming to volatility $s = \sqrt{v}$ gives

$$\Delta s \approx \frac{h}{2} s^{2b} \sqrt{\Delta t} \, Z$$

- $b = \frac{1}{2}$ gives the lognormal model of Wiggins (1987) and $b = 1$ gives the 3/2 model studied in detail by Lewis (2000).

- All four graphs conclusively reject the Heston model which predicts that volatility of volatility is constant, independent of volatility level.
  - This is very intuitive; vols should move around more if the volatility level is 100% than if it is 10%

- The regression of VIX volatility vs VIX level gives $b \approx 0.67$ for the SPX

- To interpret the result of the regression of skew vs volatility level, we need to do some more work...

Interpreting the regression of skew vs volatility

- Recall that if the variance satisfies the SDE

$$dv \sim v^b \sqrt{v}\, dZ$$

at-the-money variance skew should satisfy

$$\left.\frac{\partial v}{\partial k}\right|_{k=0} \propto v^b$$

- Delta is of the form $N(d_1)$ with $\quad d_1 = -\dfrac{k}{s\sqrt{T}} + \dfrac{s\sqrt{T}}{2}$

- Then, if the regression of volatility skew vs volatility is of the form

$$\left.\frac{\partial s_{BS}}{\partial d}\right|_{k=0} \propto s^g$$

we must also have

$$\left.\frac{\partial v}{\partial k}\right|_{k=0} = 2s_{BS}\frac{\partial s_{BS}}{\partial d}\left.\frac{\partial d}{\partial k}\right|_{k=0} \propto s^g$$

- Referring back to the regression result, we get $b = g/2 \approx 0.82$. The two estimates are not so far apart! Both are between lognormal and 3/2.

Do stochastic volatility models fit option prices?

- Once again, we note that the shape of the implied volatility surface generated by a stochastic volatility model does not strongly depend on the particular choice of model.

- Given this observation, do stochastic volatility models fit the implied volatility surface? The answer is "more or less". Moreover, fitted parameters are reasonably stable over time.

- For very short expirations however, stochastic volatility models certainly don't fit as the next slide will demonstrate.

Short Expirations

- Here's a graph of the SPX volatility skew on 17-Sep-02 (just before expiration) and various possible fits of the volatility skew formula:
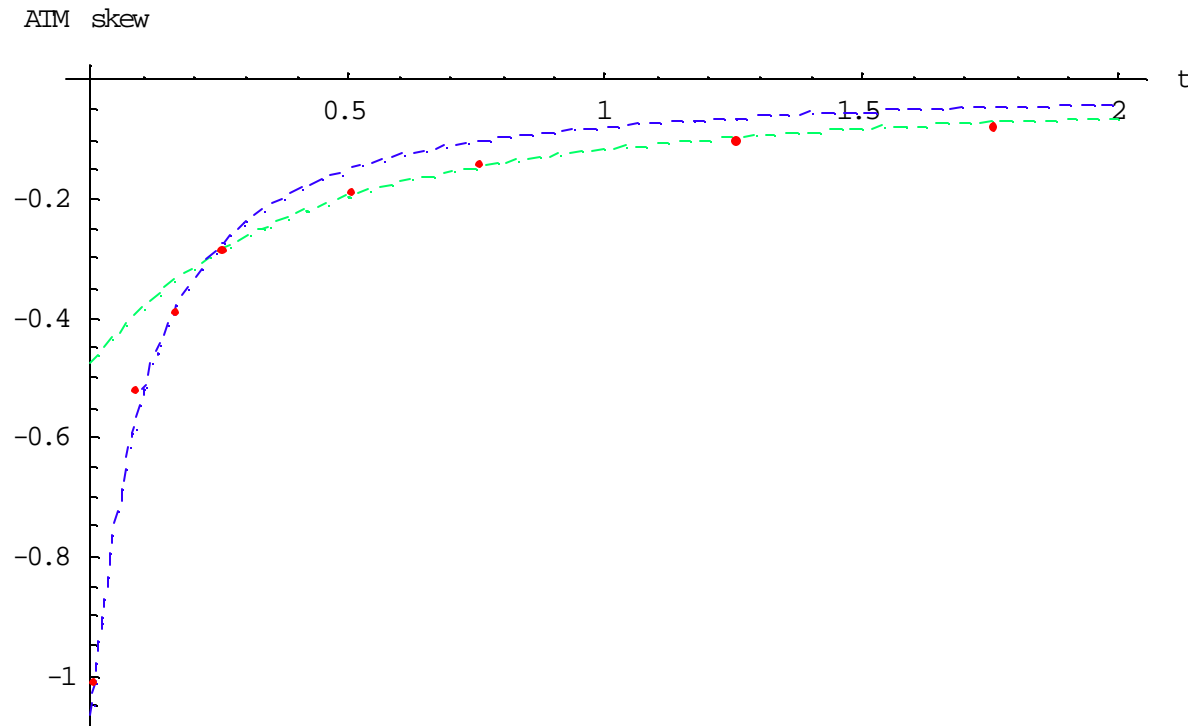
- We see that the form of the fitting function is too rigid to fit the observed skews.

- Jumps could explain the short-dated skew!

More reasons to add jumps

- The statistical (historical) distribution of stock returns and the option implied distribution have quite different shapes.

- The size of the volatility of volatility parameter estimated from fits of stochastic volatility models to option prices is too high to be consistent with empirical observations

SV versus SVJJ

- Duffie, Pan and Singleton fitted a stochastic volatility (SV) model and a double-jump stochastic volatility model (SVJJ) to November 2, 1993 SPX options data. Their results were:

	SV	SVJJ	
r	-0.70	-0.82	
$\bar{v}$	0.019	0.008	
l	6.21	3.46	
h	0.61	0.14	← Vol of vol is 4.5 times greater in the SV model!
l_J		0.47	
m_J		-0.10	
s_J		0.0001	
m_v		0.05	
r_J		-0.38	
$\sqrt{v_0}$	10.1%	8.7%	

- Note the unreasonable size of the volatility of volatility parameter h in SV!

Volatility skew from the characteristic function

- We can compute the volatility skew directly if we know the characteristic function $f_T(u) = E[e^{iux_T}]$.

- The volatility skew is given by the formula (Gatheral (2002)):

$$\left.\frac{\partial s_{BS}}{\partial k}\right|_{k=0} = -e^{-s_{BS}^2 T/8} \sqrt{\frac{2}{p}} \frac{1}{\sqrt{T}} \int_0^\infty du \frac{u \, \text{Im}[f_T(u-i/2)]}{u^2 + 1/4}$$

- The Heston characteristic function is given by

$$f_T^{SV}(u) = \exp\left\{C(u,T)\overline{v} + D(u,T)v\right\}$$

where $C(u,T)$ and $D(u,T)$ are the familiar Heston coefficients with parameters l, h, r.

- Adding a jump in the stock price (SVJ) gives the characteristic function

$$f_T^{SVJ}(u) = f_T^{SV}(u)f_T^{J}(u)$$

with $f_T^{J}(u) = \exp\left\{-iu \, l_J T\left(e^{a+d^2/2}-1\right) + l_J T\left(e^{iau-u^2d^2/2}-1\right)\right\}$

SVJJ

- Adding a simultaneous jump in the volatility gives the characteristic function (Matytsin (2000)):

$$f_T^{SVJ}(u) = f_T^{SV}(u) f_T^{J}(u) f_T^{J_V}(u)$$

with $$f_T^{J_V}(u) = \exp\left\{\bar{v} \mathbf{1}_J T \left(e^{ia\,u - u^2 d^2/2} I(u) - 1\right)\right\}$$

where $$I(u) = \frac{1}{T} \int_0^T dt\, e^{g_V\, D(u,T)} = -\frac{2 g_V}{p_+ p_-} \int_0^{-g_V\, D(u,T)} \frac{e^{-z} dz}{(1 + z/p_+)(1 + z/p_-)}$$

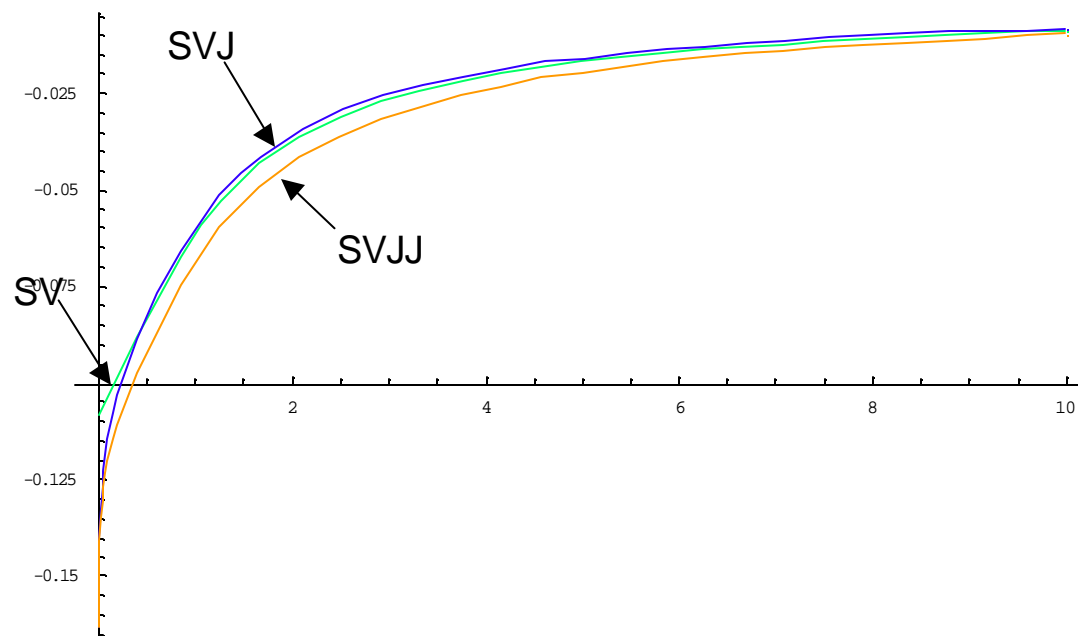with $$p_\pm = \frac{g_V}{h^2}\{b - rh\,iu \pm d\} \quad \text{(usual Heston notation)}$$

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Comparing ATM skews from different models

- With parameters

  $l \circledR 2.03, r \circledR - 0.57, h \circledR 0.38, v \circledR 0.1, \dot{v} \circledR 0.04, l_J \circledR 0.59, a \circledR - 0.05, d \circledR 0.07, g_v \circledR 0.1$
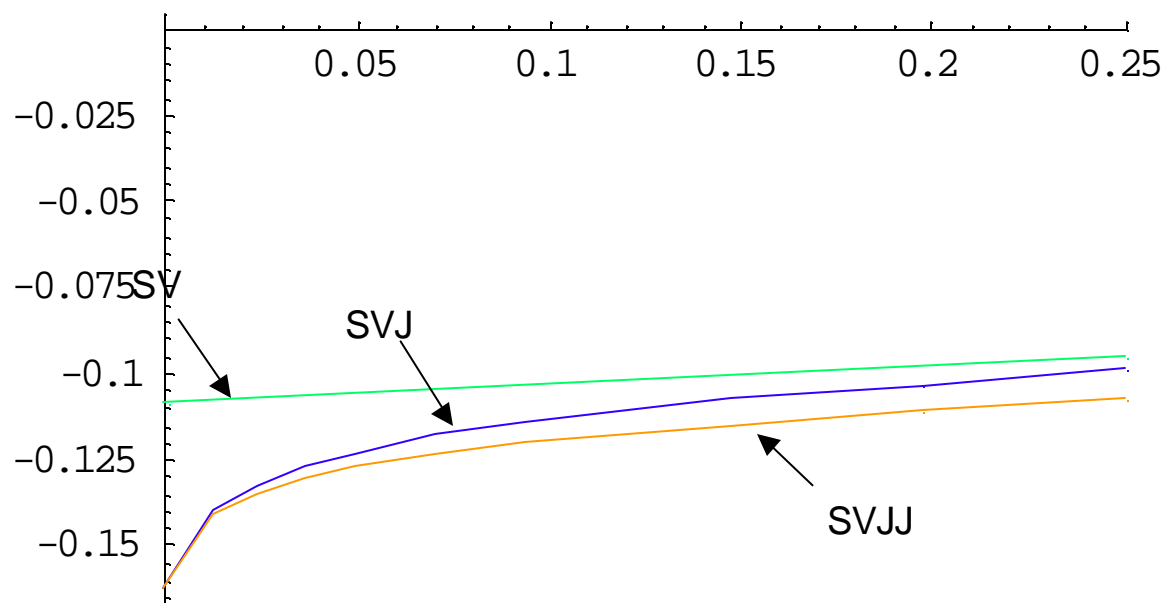
  we get the following plots of ATM variance skew vs time to expiration

Short expiration detail

- SV and SVJ skews essentially differ only for very short expirations

Estimating Volatility of Volatility

- Recall that variance in the Heston model follows the SDE (dropping the drift term)

$$dv \approx h\sqrt{v}\, dZ$$

- Converting this to volatility terms gives

$$2s\, ds \approx hs\, dZ$$

- So

$$ds \approx \frac{h}{2} dZ$$

- $h \approx 0.61$ implies that annualized SPX volatility moves around $\frac{h}{2}\sqrt{\Delta t} \approx 2\%$ per day.

- A more typical fit of Heston to SPX implied volatilities would give $h \approx 0.80$ implying a daily move of around 2.5 annualized volatility points per day.

- Historically, the daily move in short-dated implied volatility is around 1.5 volatility points as shown in the following graph.
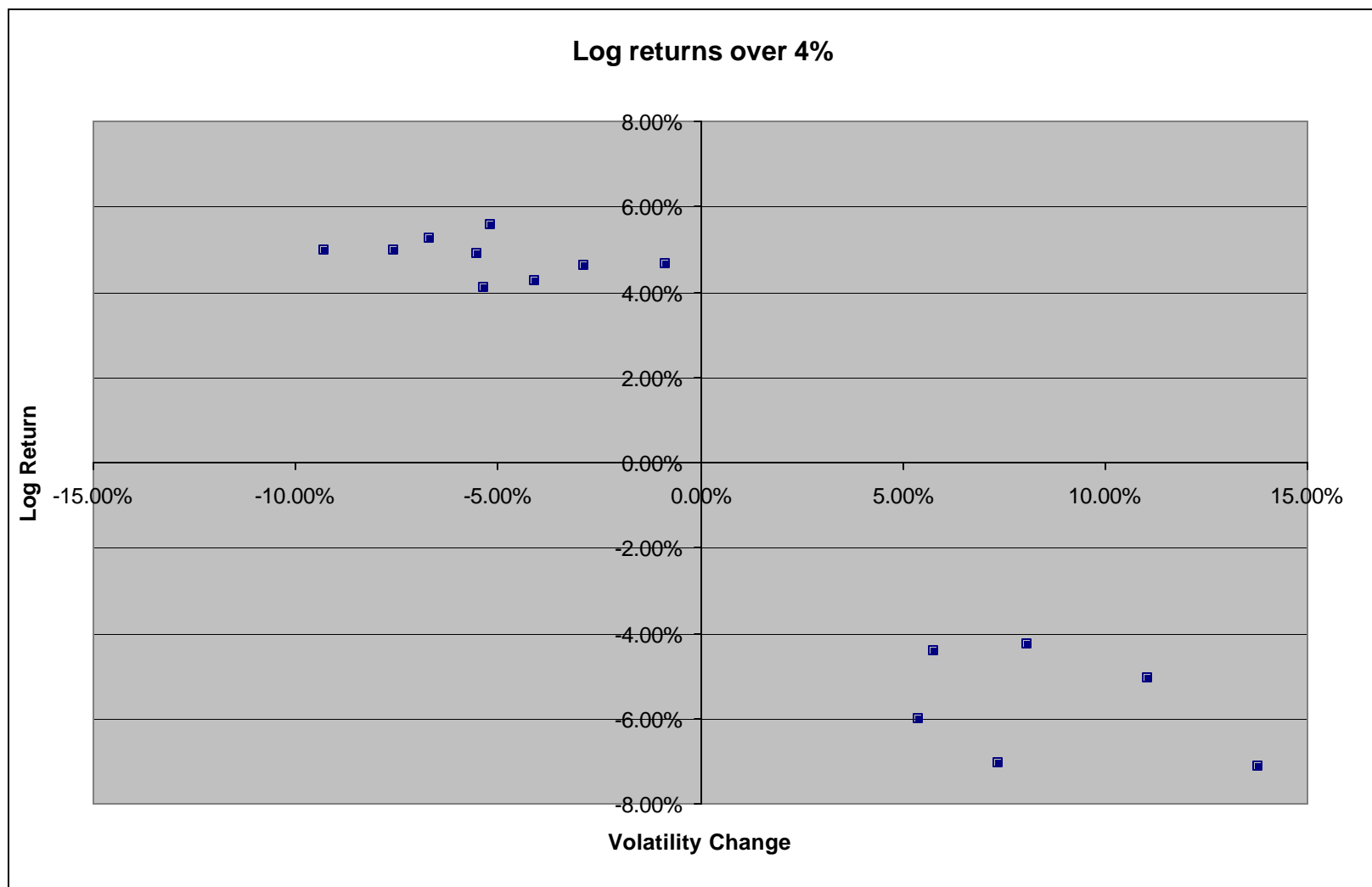
Empirical volatility of volatility

20 Day Standard Deviation of VIX Changes
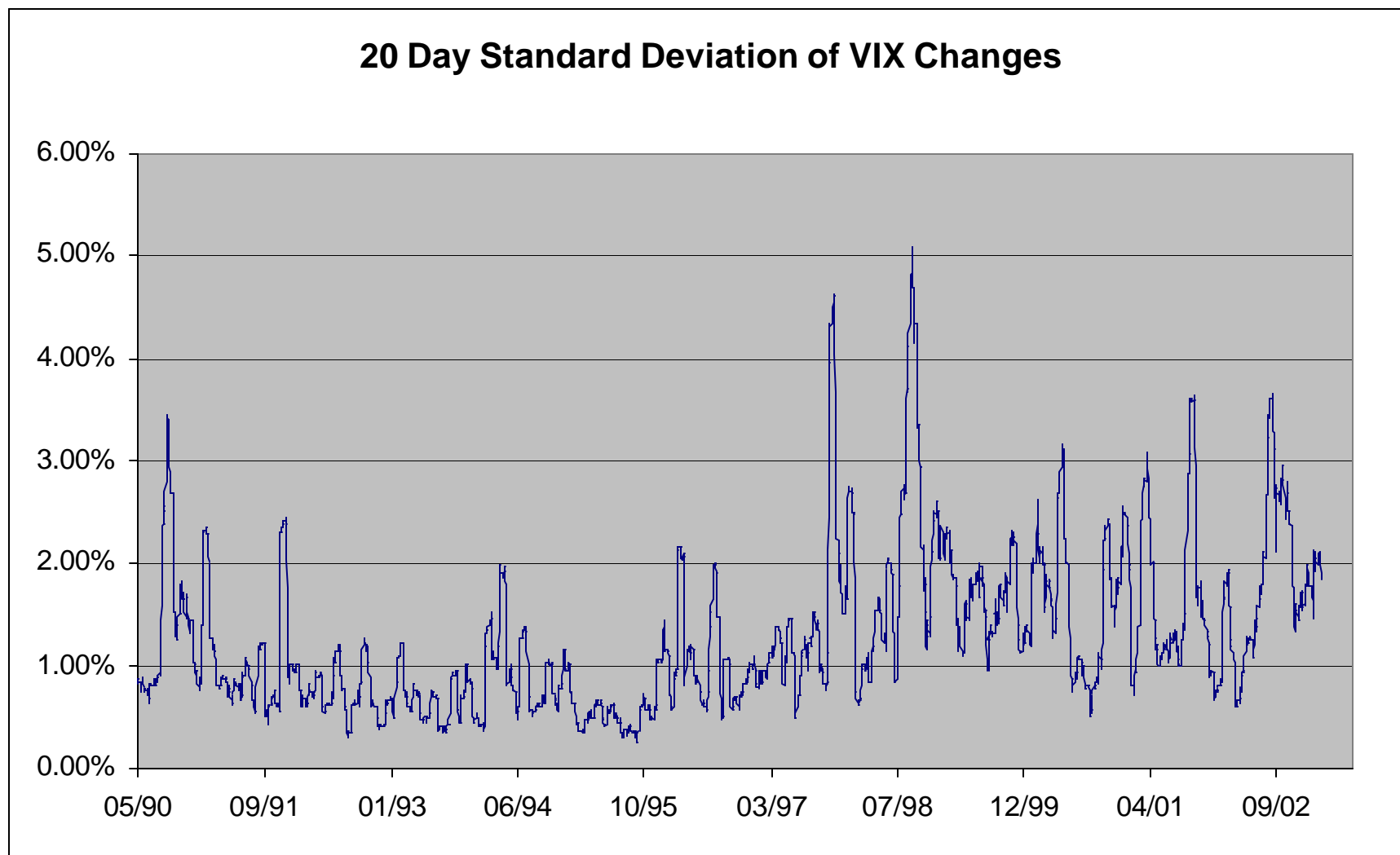
Problems for diffusion models

- If the underlying stochastic process for the stock is a diffusion, we should be able to get from the statistical measure to the risk neutral measure using Girsanov's Theorem

  - This change of measure preserves volatility of volatility.

- However, historical volatility of volatility is significantly lower than the SV fitted parameter.

- Finally, in the few days prior to SPX expirations, out-of-the money option prices are completely inconsistent with the diffusion assumption

  - For example a 5 cent bid for a contract 10 standard deviations out-of-the-money.

Simple jump diffusion models don't work either

- Although jumps may be necessary to explain very short dated volatility skews, introducing jumps introduces more parameters and this is not necessarily a good thing. For example, Bakshi, Cao, and Chen (1997 and 2000) find that adding jumps to the Heston model has little effect on pricing or hedging longer-dated options and actually worsens hedging performance for short expirations (probably through overfitting).

- Different authors estimate wildly different jump parameters for simple jump diffusion models.

SPX large moves from 1/1/1990 to 2/24/2003

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SPX large moves from 1/1/1990 to 2/24/2003

Vol changes over 6%
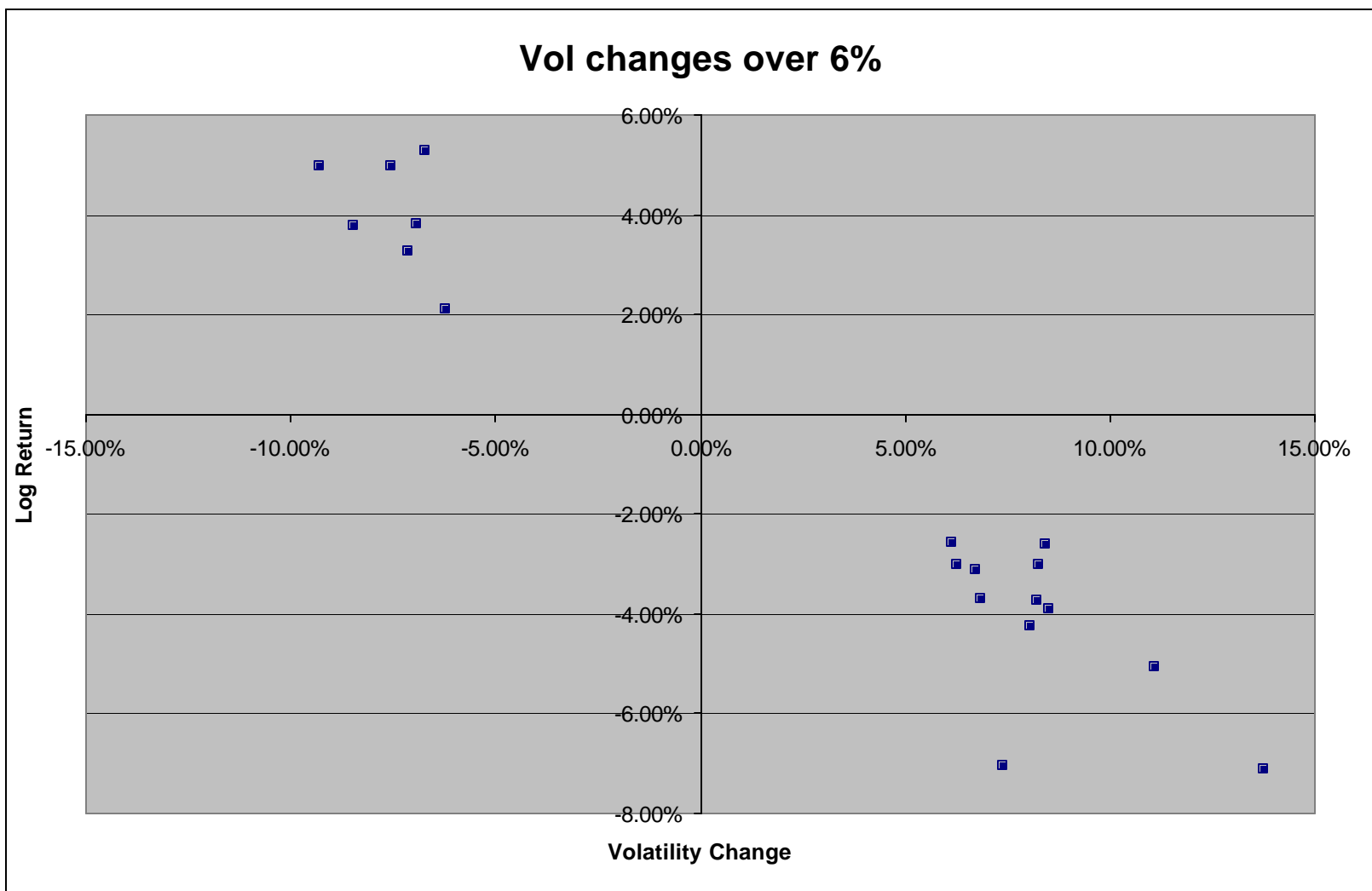
Empirical jump observations

- A large move in the SPX index is invariably accompanied by a large move in volatility
  - Volatility changes and log returns have opposite sign
- This is consistent with clustering
  - If there is a large move, more large moves follow *i.e.* volatility must jump

- We conclude that any jumps must be double jumps!

What about pure jump models?

- Dilip Madan and co-authors have written extensively on pure jump models with stable increments
  - The latest versions of these models involve subordinating a pure jump process to the integral of a CIR process – trading time again.
- Pure jump models are more aesthetically pleasing than SVJJ
  - The split between jumps and and diffusion is somewhat *ad hoc* in SVJJ
- However, large jumps in the stock price don't force an increase in implied volatilities.

The instantaneous volatility impact of option trades

- Recall our simple market price impact model

$$\Delta x = s \sqrt{x}$$

  where x is the number of days' volume represented by the trade.

- We can always either buy an option or replicate it by delta hedging until expiration: in equilibrium, implied volatilities should reflect this

- If we delta hedge, each rebalancing trade will move the price against us by

$$s \sqrt{\frac{\Delta d}{ADV}} = s \sqrt{\frac{n \, \Gamma \, \Delta S}{ADV}} = s \sqrt{x \, \Gamma} \sqrt{\Delta S}$$

- In the spirit of Leland (1985), we obtain the shifted expected instantaneous volatility

$$\hat{s} \approx s \left\{ 1 \pm 2 \sqrt{\frac{2}{p}} \sqrt{\frac{s \, S \, \Gamma \, x}{\sqrt{d \, t}}} \right\}$$
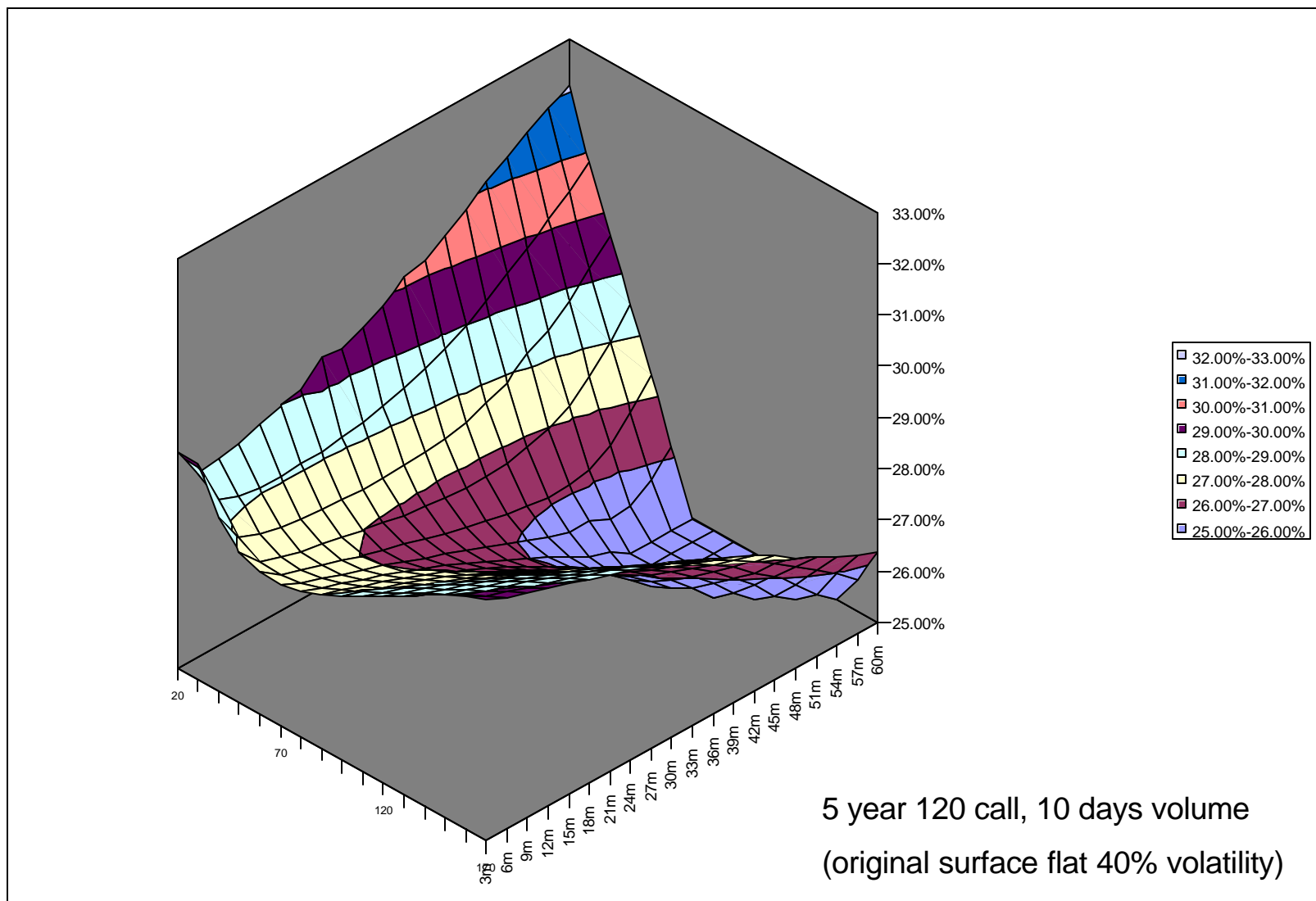
- What does this mean for the implied volatility?

The implied volatility impact of an option trade

■ To get implied volatility from instantaneous volatilities, integrate local variance along the most probable path (see Gatheral (2002))

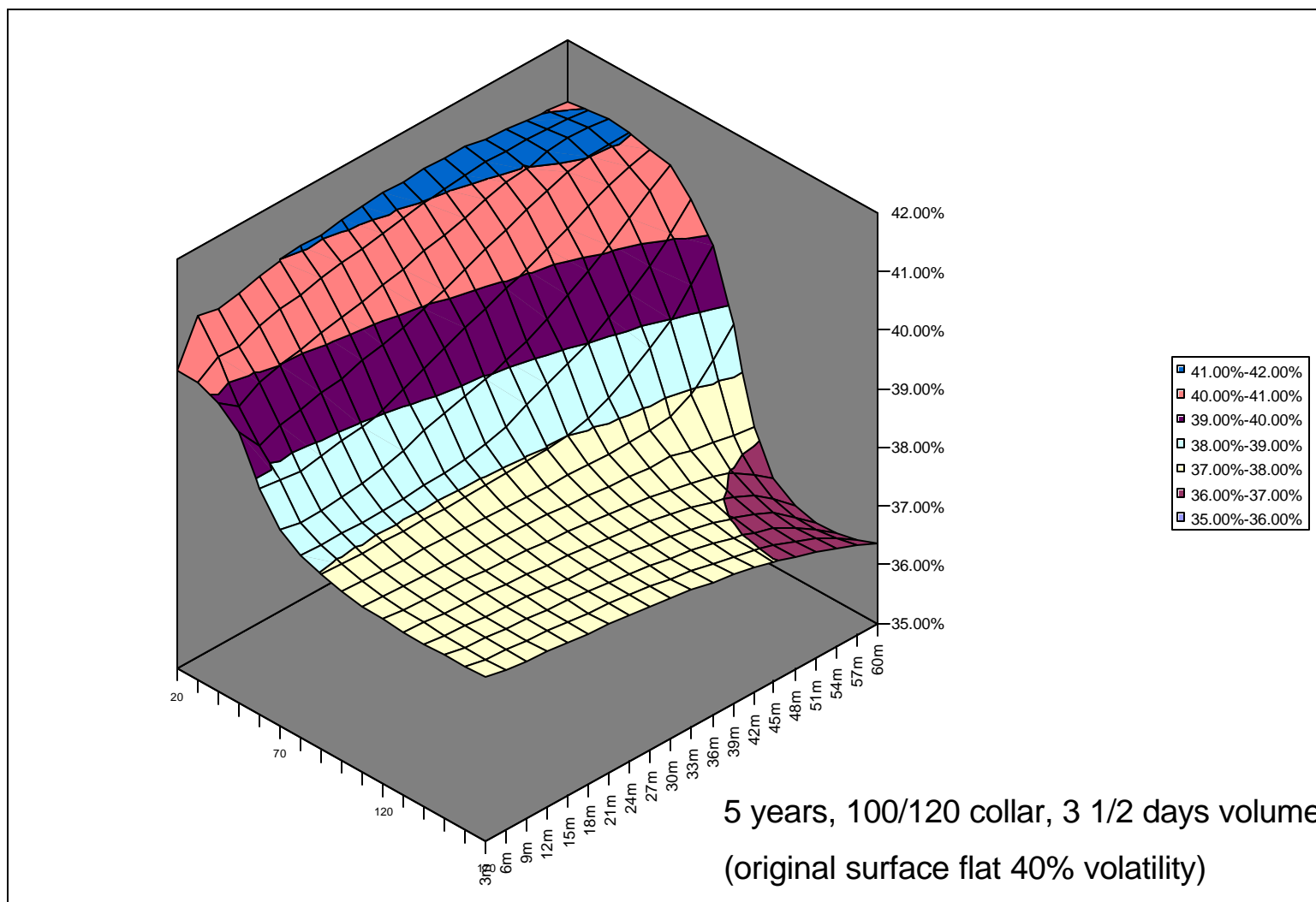$$s_{BS}^{2}(K,T) \approx \frac{1}{T}\int_{0}^{T} s^{2}(\tilde{S}_{t},t)\,dt$$

with $\tilde{S}_{t} \approx S_{0}\left(\dfrac{K}{S_{0}}\right)^{t/T}$.

The volatility impact of a 5 year 120 strike call

5 year 120 call, 10 days volume
(original surface flat 40% volatility)

The volatility impact of a 5 year 100/120 collar trade

5 years, 100/120 collar, 3 1/2 days volume

(original surface flat 40% volatility)

Legend:
- 41.00%-42.00%
- 40.00%-41.00%
- 39.00%-40.00%
- 38.00%-39.00%
- 37.00%-38.00%
- 36.00%-37.00%
- 35.00%-36.00%

Liquidity and the volatility surface

- We see that the shape of the implied volatility surface should reflect the structure of open delta-hedged option positions.

- In particular, if delta hedgers are structurally short puts and long calls, the skew will increase relative to a hypothetical market with no frictions.

  - Part of what we interpret as volatility of volatility when we fit stochastic volatility models to the market can be ascribed to liquidity effects.

How option prices reflect the behavior of stock prices

- Short-dated implied vol. more volatile than long-dated implied vol.

- Significant at-the-money skew

- Skew depends on volatility level

- Extreme short-dated implied volatility skews

- High implied volatility of volatility

- Clustering - mean reversion of volatility

- Anti-correlation of volatility moves and log returns

- Volatility of volatility increases with volatility level

- Jumps

- Stock volatility depends on the strikes and expirations of open delta-hedged options positions.

Conclusions

- Far from being *ad hoc*, stochastic volatility models are natural continuous time extensions of simple but realistic discrete-time models of stock trading.

- Stylized features of log returns can be related to empirically observed features of implied volatility surfaces.

- By carefully examining the various stylized features of option prices and log returns, we are led to reject all models except SVJJ

- Investor risk preferences and liquidity effects also affect the observed volatility skew so SV-type models may be misspecified and fitted parameters unreasonable.

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