Arbitrage-free SVI volatility surfaces

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Abstract

In this article, we show how to calibrate the widely-used SVI parameterization of the implied volatility surface in such a way as to guarantee the absence of static arbitrage. In particular, we exhibit a large class of arbitrage-free SVI volatility surfaces with a simple closed-form representation. We demonstrate the high quality of typical SVI fits with a numerical example using recent SPX options data.

1 Introduction

The *stochastic volatility inspired* or *SVI* parameterization of the implied volatility surface was originally devised at Merrill Lynch in 1999 and subsequently publicly disseminated in [11]. This parameterization has two key properties that have led to its subsequent popularity with practitioners:

- For a fixed time to expiry t, the implied Black-Scholes variance $\sigma_{\mathrm{BS}}^2(k,t)$ is linear in the log-strike k as $|k| \to \infty$ consistent with Roger Lee's moment formula [20].
- It is relatively easy to fit listed option prices whilst ensuring no calendar spread arbitrage.

The consistency of the SVI parameterization with arbitrage bounds for extreme strikes has also led to its use as an extrapolation formula [17]. Moreover, as shown in [13], the SVI parameterization is not arbitrary in the sense that the large-maturity limit of the Heston implied volatility smile is exactly SVI. However it is well-known that SVI smiles may be arbitrageable. Previous work has shown how to calibrate SVI to given implied volatility data (for example [23]). Other recent work [5] has been concerned with showing how to parameterize the volatility surface in such a way as to preclude dynamic

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arbitrage. There has been some work on arbitrage-free interpolation of implied volatilities or equivalently of option prices [1], [9], [14], [18]. Prior work has not successfully attempted to eliminate static arbitrage and indeed, efforts to find simple closed-form arbitrage-free parameterizations of the implied volatility surface are still widely considered to be futile.

In this article, we exhibit a large class of SVI volatility surfaces with a simple closed-form representation, for which absence of static arbitrage is guaranteed. Static arbitrage—as defined by Cox and Hobson [7]— corresponds to the existence of a non negative martingale on a filtered probability space such that European call option prices can be written as the expectation, under the risk-neutral measure, of a final payoff. This definition also implies (see [9]) that the corresponding total variance must be an increasing function of the maturity (absence of calendar spread arbitrage). Using some mathematics from the Renaissance, we show how to eliminate any calendar spread arbitrage resulting from a given set of SVI parameters. We also present a set of necessary conditions for the corresponding density to be non negative (absence of butterfly arbitrage), which corresponds—from the definition of static arbitrage—to call prices being decreasing and convex functions of the strike. We go on to use the existence of such arbitrage-free surfaces to devise a new algorithm for eliminating butterfly arbitrage should it occur. With both types of arbitrage eliminated, we achieve a volatility surface that typically calibrates well to given implied volatility data and is guaranteed free of static arbitrage.

In Section 2, we present various equivalent forms of the SVI parameterization. In Section 3, we show how to eliminate calendar spread arbitrage. In Section 4, we show how to eliminate butterfly arbitrage, or negative densities. In Section 5, we exhibit a large and useful class of SVI volatility surfaces that are guaranteed to be free of static arbitrage. In Section 6, we show how to calibrate SVI to observed option prices, avoiding both butterfly and calendar spread arbitrages. We further show how to interpolate and extrapolate in such a way as to guarantee the absence of static arbitrage. Finally, in Section 7, we summarize and conclude.

2 Notation and alternative formulations

In the foregoing, we consider a stock price process $(S_t)_{t\geq 0}$ with natural filtration $(\mathcal{F}_t)_{t\geq 0}$, and we define the forward price process $(F_t)_{t\geq 0}$ by $F_t := \mathbb{E}\left(S_t|\mathcal{F}_0\right)$. For any $k \in \mathbb{R}$ and $t > 0$, $C_{\mathrm{BS}}(k, \sigma^2 t)$ denotes the Black-Scholes price of a European Call option on S with strike $F_t \mathrm{e}^k$, maturity t and volatility $\sigma > 0$. We shall denote the Black-Scholes implied volatility by $\sigma_{\mathrm{BS}}(k,t)$, and define the total implied variance by

$$w(k,t) = \sigma^2_{\mathrm{BS}}(k,t)t.$$

The implied variance v shall be equivalently defined as $v(k,t) = \sigma^2_{\mathrm{BS}}(k,t) = w(k,t)/t$. We shall refer to the two-dimensional map $(k,t) \mapsto w(k,t)$ as the volatility surface, and for any fixed maturity $t > 0$, the function $k \mapsto w(k,t)$ will represent a slice. We propose below three different—yet equivalent—slice parameterizations of the total implied variance, and give the exact correspondence between them. For a given maturity slice, we shall use the notation $w(k,\chi)$ where χ represents a set of parameters, and drop the t-dependence.

The raw SVI parameterization

For a given parameter set $\chi_R = \{a, b, \rho, m, \sigma\}$, the *raw SVI parameterization* of total implied variance reads:

$$w(k;\chi_R) = a + b\left\{\rho\,(k-m) + \sqrt{(k-m)^2+\sigma^2}\right\}, \tag{2.1}$$

where $a \in \mathbb{R}$, $b \geq 0$, $|\rho| < 1$, $m \in \mathbb{R}$, $\sigma > 0$, and the obvious condition $a + b\,\sigma\,\sqrt{1-\rho^2} \geq 0$, which ensures that $w(k,\chi_R) \geq 0$ for all $k \in \mathbb{R}$. This condition indeed ensures that the minimum of the function $w(\cdot,\chi_R)$ is non-negative. It follows immediately that changes in the parameters have the following effects:

- Increasing a increases the general level of variance, a vertical translation of the smile;
- Increasing b increases the slopes of both the put and call wings, tightening the smile;
- Increasing ρ decreases (increases) the slope of the left(right) wing, a counter-clockwise rotation of the smile;
- Increasing m translates the smile to the right;
- Increasing σ reduces the at-the-money (ATM) curvature of the smile.

We exclude the trivial cases $\rho = 1$ and $\rho = -1$, where the volatility smile is respectively strictly increasing and decreasing. We also exclude the case $\sigma = 0$ which corresponds to a linear smile.

The natural SVI parameterization

For a given parameter set $\chi_N = \{\Delta, \mu, \rho, \omega, \zeta\}$, the *natural SVI parameterization* of total implied variance reads:

$$w(k;\chi_N) = \Delta + \frac{\omega}{2}\left\{1 + \zeta\rho\,(k-\mu) + \sqrt{\left(\zeta(k-\mu)+\rho\right)^2 + (1-\rho^2)}\right\}, \tag{2.2}$$

where $\omega \geq 0$, $\Delta \in \mathbb{R}$, $\mu \in \mathbb{R}$, $|\rho| < 1$ and $\zeta > 0$. It is straightforward to derive the following correspondence between the raw and natural SVI parameters:

$$(a, b, \rho, m, \sigma) = \left\{\Delta + \frac{\omega}{2}\left(1-\rho^2\right), \frac{\omega\zeta}{2}, \rho, \mu - \frac{\rho}{\zeta}, \frac{\sqrt{1-\rho^2}}{\zeta}\right\}, \tag{2.3}$$

and it is equally straightforward to derive the inverse transformation:

$$\{\Delta, \mu, \rho, \omega, \zeta\} = \left\{a - \frac{\omega}{2}\left(1-\rho^2\right), m + \frac{\rho\sigma}{\sqrt{1-\rho^2}}, \rho, \frac{2b\sigma}{\sqrt{1-\rho^2}}, \frac{\sqrt{1-\rho^2}}{\sigma}\right\}. \tag{2.4}$$

The SVI Jump-Wings (SVI-JW) parameterization

Neither the raw SVI nor the natural SVI parameterizations are intuitive to traders in the sense that a trader cannot be expected to carry around the typical value of these parameters in his head. Moreover, there is no reason to expect these parameters to be particularly stable. The *SVI-Jump-Wings (SVI-JW) parameterization* of the implied variance v (rather than the implied total variance w) was inspired by a similar parameterization attributed to Tim Klassen, then at Goldman Sachs. For a given time to expiry $t > 0$ and a parameter set $\chi_J = \{v_t, \psi_t, p_t, c_t, \widetilde{v}_t\}$ the SVI-JW parameters are defined from the raw SVI parameters as follows:

$$
\begin{aligned}
v_t &= \frac{a + b\left\{-\rho\, m + \sqrt{m^2+\sigma^2}\right\}}{t}, \\
\psi_t &= \frac{1}{\sqrt{w_t}}\frac{b}{2}\left(-\frac{m}{\sqrt{m^2+\sigma^2}} + \rho\right), \\
p_t &= \frac{1}{\sqrt{w_t}} b\,(1-\rho)\,, \\
c_t &= \frac{1}{\sqrt{w_t}} b\,(1+\rho)\,, \\
\widetilde{v}_t &= \left(a + b\,\sigma\,\sqrt{1-\rho^2}\right)/t,
\end{aligned}
\tag{2.5}
$$

with $w_t := v_t t$. The SVI-JW parameters have the following interpretations:

- v_t gives the ATM variance;
- ψ_t gives the ATM skew;
- p_t gives the slope of the left (put) wing;
- c_t gives the slope of the right (call) wing;
- $\widetilde{v}_t$ is the minimum implied variance.

If smiles scaled perfectly as $1/\sqrt{w_t}$ (as is approximately the case empirically), these parameters would be constant, independent of the slice t. This makes it easy to extrapolate the SVI surface to expirations beyond the longest expiration in the data set. Also note that by definition, for any $t > 0$ we have

$$
\psi_t = \left.\frac{\partial \sigma_{\mathrm{BS}}(k,t)}{\partial k}\right|_{k=0}
$$

The choice of volatility skew as the skew measure rather than variance skew for example, reflects the empirical observation that volatility is roughly lognormally distributed. Specifically, following the lines of [12, Chapter 7], assume that the instantaneous variance process satisfies the SDE

$$
dv_t = \alpha(v_t)\, dt + \eta\sqrt{v_t}\,\beta(v_t)\, dZ_t, \quad \text{for all } t \geq 0
$$

where $\eta > 0$, $(Z_t)_{t\geq 0}$ is a standard Brownian motion and α and β two functions on $\mathbb{R}_+$ ensuring the existence of a unique strong solution to the SDE (see for instance [19] for exact conditions), then

$$\lim_{T\to 0} \frac{\partial \sigma_{\mathrm{BS}}(k,T)^2}{\partial k} = \beta(v),$$

and hence the limit of the at-the-money skew as the maturity tends to zero is constant and independent of the volatility level. In particular this implies that $\beta(v)$ behaves like $\sqrt{v}$ and therefore that the variance process is lognormal. This consistency of the SVI-JW parameterization with empirical volatility dynamics thus leads in practice to greater parameter stability over time. The following lemma provides the inverse representation of (2.5).

Lemma 2.1. *Assume that $m \neq 0$. For any $t > 0$, define the (t-dependent) quantities α and β by*

$$\beta := \rho - \frac{2\psi_t\sqrt{w_t}}{b} \quad \text{and} \quad \alpha := \mathrm{sign}(\beta)\sqrt{\frac{1}{\beta^2} - 1}.$$

Then, the raw SVI and SVI-JW parameters are related as follows:

$$\begin{aligned}
b &= \frac{\sqrt{w_t}}{2}\left(c_t + p_t\right),\\
\rho &= 1 - \frac{p_t\sqrt{w_t}}{b},\\
a &= \widetilde{v}_t t - b\sigma\sqrt{1-\rho^2},\\
m &= \frac{\left(v_t - \widetilde{v}_t\right)t}{b\left\{-\rho + \mathrm{sign}(\alpha)\sqrt{1+\alpha^2} - \alpha\sqrt{1-\rho^2}\right\}},\\
\sigma &= \alpha m.
\end{aligned}$$

If $m = 0$, then the formulae above for b, ρ and a still hold, but $\sigma = \left(v_t t - a\right)/b$.

Proof. The expressions for b, ρ and a follow directly from (2.5). Assume that $m \neq 0$ and let $\beta := \rho - 2\psi_t\sqrt{w_t}/b$ and $\alpha := \sigma/m \in \mathbb{R}$. Then the expressions in (2.5) give

$$\beta = \frac{\mathrm{sign}\left(\alpha\right)}{\sqrt{1+\alpha^2}},$$

which implies that

$$\alpha = \mathrm{sign}(\beta)\sqrt{\frac{1}{\beta^2} - 1} = \mathrm{sign}(m)\sqrt{\frac{2\left(m^2+\sigma^2\right)}{\sigma^2} - 1}.$$

Using (2.5), we also have

$$\frac{\left(v_t - \widetilde{v}_t\right)\, t}{b} = m\left\{-\rho + \mathrm{sign}(\alpha)\,\sqrt{1+\alpha^2} - \alpha\,\sqrt{1-\rho^2}\right\},$$

from which we deduce m in terms of α, and the expression of σ is recovered from the equality $\sigma = \alpha m$. The expression for σ in the case $m = 0$ is straightforward from (2.5). $\square$

3 Elimination of calendar spread arbitrage

Calendar spread arbitrage is usually expressed as the monotonicity of European call option prices with respect to the maturity (see for example [4] or [8]). Since our main focus here is on the implied volatility, we translate this definition into a property of the implied volatility. Indeed, assuming proportional dividends, we establish a necessary and sufficient condition for an implied volatility parameterization to be free of calendar spread arbitrage. This can also be found in [9] and [11] and we outline its proof for completeness.

Lemma 3.1. *If dividends are proportional to the stock price, the volatility surface w is free of calendar spread arbitrage if and only if*

$$\partial_t w(k,t) \geq 0, \quad \textit{for all } k \in \mathbb{R} \textit{ and } t > 0.$$

Proof. Let $(X_t)_{t\geq 0}$ be a martingale, $L \geq 0$ and $0 \leq t_1 < t_2$. Then the inequality

$$\mathbb{E}\left[(X_{t_2} - L)^+\right] \geq \mathbb{E}\left[(X_{t_1} - L)^+\right]$$

is standard. For any $i = 1, 2$, let C_i be options with strikes K_i and expirations t_i. Suppose that the two options have the same moneyness, i.e.

$$\frac{K_1}{F_{t_1}} = \frac{K_2}{F_{t_2}} =: \mathrm{e}^k$$

Then, if dividends are proportional, the process $(X_t)_{t\geq 0}$ defined by $X_t := S_t/F_t$ for all $t \geq 0$ is a martingale and

$$\frac{C_2}{K_2} = \mathrm{e}^{-k}\mathbb{E}\left[\left(X_{t_2} - \mathrm{e}^k\right)^+\right] \geq \mathrm{e}^{-k}\mathbb{E}\left[\left(X_{t_1} - \mathrm{e}^k\right)^+\right] = \frac{C_1}{K_1}$$

So, if dividends are proportional, keeping the moneyness constant, option prices are non-decreasing in time to expiration. The Black-Scholes formula for the non-discounted value of an option may be expressed in the form $C_{\mathrm{BS}}(k, w(k,t))$ with C_{BS} strictly increasing in its second argument. It follows that for fixed k, the function $w(k,\cdot)$ must be non-decreasing. □

Lemma 3.1 motivates the following definition.

Definition 3.1. *A volatility surface w is free of calendar spread arbitrage if*

$$\partial_T w(k,T) \geq 0, \quad \textit{for all } k \in \mathbb{R} \textit{ and } T > 0.$$

The following lemma establishes a necessary and sufficient condition for the absence of calendar spread arbitrage.

Lemma 3.2. *The raw SVI surface* (2.1) *is free of calendar spread arbitrage if a certain quartic polynomial (given in* (3.2) *below) has no real root.*

Proof. By definition, there is no calendar arbitrage if for any two dates $t_1 \neq t_2$, the corresponding slices $w(\cdot, t_1)$ and $w(\cdot, t_2)$ do not intersect. Let these two slices be characterised by the sets of parameters $\chi_1 := \{a_1, b_1, \sigma_1, \rho_1, m_1\}$ and $\chi_2 := \{a_2, b_2, \sigma_2, \rho_2, m_2\}$, and assume for convenience that $0 < t_1 < t_2$. We therefore need to determine the (real) roots of the equation $w(k, t_1) = w(k, t_2)$. The latter is equivalent to

$$a_1 + b_1 \left\{ \rho_1 (k - m_1) + \sqrt{(k - m_1)^2 + \sigma_1^2} \right\} = a_2 + b_2 \left\{ \rho_2 (k - m_2) + \sqrt{(k - m_2)^2 + \sigma_2^2} \right\}. \tag{3.1}$$

Leaving $\sqrt{(k - m_1)^2 + \sigma_1^2}$ on one side, squaring the equality and rearranging it leads to

$$2b_2 (\alpha + \beta k) \sqrt{(k - m_2)^2 + \sigma_2^2} = b_1^2 \left\{ (k - m_1)^2 + \sigma_1^2 \right\} - b_2^2 \left\{ (k - m_2)^2 + \sigma_2^2 \right\} - (\alpha + \beta k)^2,$$

where $\alpha := a_2 - a_1 + b_1 \rho_1 m_1 - b_2 \rho_2 m_2$ and $\beta := b_2 \rho_2 - b_1 \rho_1$. Squaring the last equation above gives a quartic polynomial equation of the form

$$\alpha_4 k^4 + \alpha_3 k^3 + \alpha_2 k^2 + \alpha_1 k + \alpha_0 = 0, \tag{3.2}$$

where each of the coefficients lengthy yet explicit expressions[1] in terms of the parameters $\{a_1, b_1, \rho_1, \sigma_1, m_1\}$ and $\{a_2, b_2, \rho_2, \sigma_2, m_2\}$. If this quartic polynomial has no real root, then the slices do not intersect and the lemma follows. Roots of a quartic polynomial are known in closed-form thanks to Ferrari and Cardano [2]. Thus there exist closed-form expressions in terms of χ_1 and χ_2 for the possible intersection points of the two SVI slices. □

Remark 3.1. *If the quartic polynomial* (3.2) *has one or more real roots, we need to check whether the latter are indeed solutions of the original problem* (3.1)*, or spurious solutions arising from the two squaring operations. The absence of real roots of the quartic polynomial is clearly a sufficient—but not necessary—condition.*

Remark 3.2. *By a careful study of the minima and the shapes of the two slices $w(\cdot, t_1)$ and $w(\cdot, t_2)$, it is possible to determine a set of conditions on the parameters ensuring no calendar spread arbitrage. However these conditions involve tedious combinations of the parameters and will hence not match the computational simplicity of the lemma.*

4 Butterfly arbitrage

In Section 3, we provided conditions under which a volatility surface parameterized using SVI could be guaranteed to be free of calendar spread arbitrage. We now consider a different type of arbitrage, namely butterfly arbitrage (Definition 4.1). This corresponds to the existence of a risk-neutral martingale measure and the classical definition of no static arbitrage, as developed in [10] or [7]. Sadly, it does not seem possible to find

[1] Explicit expressions for these coefficients can be found in the R-code posted on `http://faculty.baruch.cuny.edu/jgatheral`.

general conditions on the parameters that would eliminate butterfly arbitrage. In Section 5 however, we will introduce a class of SVI smiles for which the absence of butterfly arbitrage is guaranteed. In this section, we consider only one slice of the implied volatility surface, i.e. the map $k \mapsto w(k,t)$ for a given fixed maturity $t > 0$. For clarity we therefore drop—in this section only—the t-dependence of the smile and use the notation $w(k)$ instead.

Definition 4.1. *A slice is said to be free of butterfly arbitrage if the corresponding density is non-negative.*

Let us introduce the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(k) := \left(1 - \frac{kw'(k)}{2w(k)}\right)^2 - \frac{w'(k)^2}{4}\left(\frac{1}{w(k)} + \frac{1}{4}\right) + \frac{w''(k)}{2}. \tag{4.1}$$

This function will be the main ingredient in the determination of butterfly arbitrage as stated in the following lemma.

Lemma 4.1. *A slice is free of butterfly arbitrage if and only if $g(k) \geq 0$ for all $k \in \mathbb{R}$.*

Proof. It is well known that the probability density function p may be computed as

$$p(k) = \left.\frac{\partial^2 C_{\mathrm{BS}}}{\partial K^2}\right|_{K=F_t \mathrm{e}^k}, \quad \text{for any } k \in \mathbb{R}.$$

Explicit differentiation of the Black-Scholes formula then gives for any $k \in \mathbb{R}$,

$$p(k) = \frac{g(k)}{\sqrt{2\pi w(k)}} \exp\left(-\frac{d_2(k)^2}{2}\right),$$

where, according to the usual definition $d_2(k) := -k/\sqrt{w(k)} - \sqrt{w(k)}/2$. □

Example 4.1. *(From Axel Vogt on wilmott.com) Consider the raw SVI parameters:*

$$(a, b, m, \rho, \sigma) = (-0.0410, 0.1331, 0.3586, 0.3060, 0.4153), \tag{4.2}$$

with $t = 1$. These parameters give rise to the total variance smile w and the function g (defined in (4.1)) on Figure 1, where the negative density is clearly visible.

5 A SVI surface free of static arbitrage

We now introduce a class of SVI volatility surfaces as an extension of the natural parameterization (2.2). For any maturity $t \geq 0$, define the at-the-money (ATM) total variance $\theta_t := \sigma^2_{\mathrm{BS}}(0,t)t$. A simple no-arbitrage argument implies that $\theta_0 := \lim_{t\to 0}\theta_t = 0$.

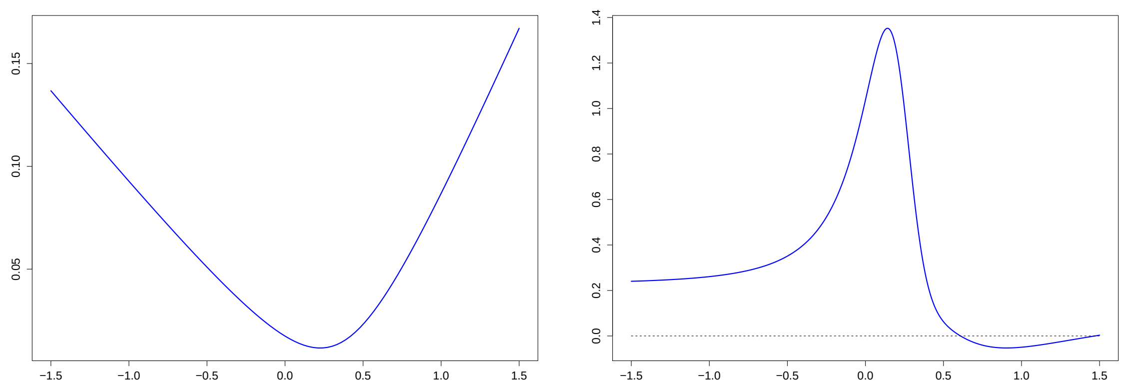

Figure 1: Plots of the total variance smile w (left) and the function g defined in (4.1) (right), using the parameters (4.2).

Consider now for any $t \geq 0$, the natural SVI volatility surface parameterization $\chi_N = \{0, 0, \rho, \theta_t, \varphi(\theta_t)\}$:

$$w(k, \theta_t) = \frac{\theta_t}{2} \left\{ 1 + \rho\varphi(\theta_t)k + \sqrt{(\varphi(\theta_t)k + \rho)^2 + (1 - \rho^2)} \right\} \tag{5.1}$$

with $\theta_t > 0$ for $t > 0$, and where φ is a smooth function from $(0, \infty)$ to $(0, \infty)$ such that the limit $\lim_{t \to 0} \theta_t \varphi(\theta_t)$ exists in $\mathbb{R}$. Note that this representation amounts to considering the volatility surface in terms of ATM variance time, instead of standard calendar time, similar in spirit to the stochastic subordination of [6]. The ATM total variance is $\theta_t = \sigma_{\mathrm{BS}}^2(0, t)\, t$ and the ATM volatility skew is given by

$$\partial_k \sigma_{\mathrm{BS}}(k, t)|_{k=0} = \frac{1}{2\sqrt{\theta_t t}} \partial_k w(k, \theta_t) \bigg|_{k=0} = \frac{\rho\sqrt{\theta_t}}{2\sqrt{t}} \varphi(\theta_t). \tag{5.2}$$

Furthermore the smile is symmetric around at-the-money if and only if $\rho = 0$. This is consistent with [3, Theorem 3.4] which states that in a standard stochastic volatility model, the smile is symmetric if and only if the correlation between the stock price and its instantaneous volatility is null. The following theorem gives precise sufficient conditions ensuring that the volatility surface (5.1) is free of calendar spread arbitrage (Lemma 3.1) and also matches the term structure of ATM volatility and the term structure of the ATM volatility skew.

Theorem 5.1. *The surface* (5.1) *is free of calendar spread arbitrage if*

1. $\partial_t \theta_t \geq 0$, *for all* $t \geq 0$;
2. $\partial_\theta (\theta\varphi(\theta)) \geq 0$, *for all* $\theta > 0$;

3. $\partial_\theta \varphi(\theta) < 0$*, for all* $\theta > 0$*.*

This theorem means that the volatility surface (5.1) is free of calendar spread arbitrage if the skew in total variance terms is monotonically increasing in trading time and the skew in implied variance terms is monotonically decreasing in trading time. In practice, any reasonable skew term structure that a trader defines will have these properties.

Proof. Since the definition of calendar spread arbitrage does not depend on the log-moneyness k, there is no loss of generality in assuming k fixed. First note that $\partial_t w(k, \theta_t) = \partial_\theta w(k, \theta_t) \partial_t \theta_t$ so the volatility surface (5.1) is free of calendar spread arbitrage if $\partial_\theta w(k, \theta) \geq 0$ for all $\theta > 0$. To proceed, we compute, for any $\theta > 0$,

$$2\partial_\theta w(k, \theta) = 1 + \frac{1 + \rho x}{\sqrt{x^2 + 2\rho x + 1}} + \frac{\theta \varphi'(\theta) + \varphi(\theta)}{\varphi(\theta)} x \left\{ \frac{x + \rho}{\sqrt{x^2 + 2\rho x + 1}} + \rho \right\}, \tag{5.3}$$

with $x := k\,\varphi(\theta)$. Conditions 2 and 3 give

$$0 \leq \frac{\theta \varphi'(\theta) + \varphi(\theta)}{\varphi(\theta)} \leq 1, \quad \text{for any } \theta > 0.$$

Thus, if the last term in (5.3) is non-negative, we have for any $\theta > 0$,

$$\begin{aligned} 2\partial_\theta w(k, \theta) &\geq 1 + \frac{1 + \rho\, x}{\sqrt{x^2 + 2\rho x + 1}} = 1 + \frac{1 + \rho x}{\sqrt{(1 + \rho x)^2 + x^2\,(1 - \rho^2)}} \\ &\geq 1 - \frac{|1 + \rho x|}{\sqrt{(1 + \rho x)^2 + x^2\,(1 - \rho^2)}} \geq 0. \end{aligned}$$

If the last term in (5.3) is negative, we have for any $\theta > 0$,

$$\begin{aligned} 2\partial_\theta w(k, \theta) &\geq 1 + \frac{1 + \rho x}{\sqrt{x^2 + 2\rho x + 1}} + x \left\{ \frac{x + \rho}{\sqrt{x^2 + 2\rho x + 1}} + \rho \right\} \\ &= 1 + \sqrt{x^2 + 2\rho x + 1} + \rho x \geq \sqrt{(1 + \rho x)^2 + x^2\,(1 - \rho^2)} - |1 + \rho x| \geq 0. \end{aligned}$$

□

Remark 5.1. *The necessity of Condition 1 follows from imposing* $\partial_t w(0; \theta_t) \geq 0$*. Note that* (5.3) *implies*

$$\lim_{x \to \pm\infty} \frac{2}{x} \partial_\theta w \left(\frac{x}{\varphi(\theta)}, \theta \right) = \frac{\theta \varphi'(\theta) + \varphi(\theta)}{\varphi(\theta)} \left(\rho \pm 1 \right).$$

so $\partial_\theta w(k, \theta) \geq 0$ *(with* $x = k\varphi(\theta)$*) imposes the necessity of condition 2. That condition 3 is not necessary can be seen by setting* $\rho = 0$ *in* (5.3) *to give*

$$2\partial_\theta w(k, \theta) = 1 + \frac{1}{\sqrt{1 + x^2}} + \frac{\theta \varphi'(\theta) + \varphi(\theta)}{\varphi(\theta)} \frac{x^2}{\sqrt{1 + x^2}},$$

which is positive if condition 2 holds whether or not condition 3 also holds.

The following lemma is a straightforward consequence of (2.3) and (2.5).

Lemma 5.1. *The SVI-JW parameters associated with the volatility surface* (5.1) *are*

$$\begin{aligned}
v_t &= \theta_t/t, \\
\psi_t &= \frac{1}{2}\rho\sqrt{\theta_t}\,\varphi(\theta_t), \\
p_t &= \sqrt{\theta_t}\,\varphi(\theta_t)\,(1-\rho), \\
c_t &= \sqrt{\theta_t}\,\varphi(\theta_t)\,(1+\rho), \\
\widetilde{v}_t &= \frac{\theta_t}{t}\,(1-\rho^2).
\end{aligned}$$

We now give several examples of the implied volatility surface (5.1).

Example 5.1. A Heston-like parameterization
Consider the function φ defined as

$$\varphi(\theta) = \frac{1}{\lambda\theta}\left\{1 - \frac{1-\mathrm{e}^{-\lambda\theta}}{\lambda\theta}\right\},$$

with $\lambda > 0$. Then for all $\theta > 0$, we immediately obtain

$$\varphi'(\theta) = \frac{2-\mathrm{e}^{-\lambda\theta}(2+\lambda\theta)-\lambda\theta}{\lambda^2\theta^3} < 0, \quad \text{and} \quad \partial_\theta(\theta\varphi(\theta)) = \frac{\mathrm{e}^{-\lambda\theta}\left(\mathrm{e}^{\lambda\theta}-1-\lambda\theta\right)}{\lambda^2\theta^2} > 0,$$

so that this function φ is valid in the sense of Theorem 5.1. This function is consistent with the implied variance skew in the Heston model as shown in [12, Equation 3.19].

Example 5.2. Power-law parameterization
Consider $\varphi(\theta) = \eta\theta^{-\gamma}$ for some $\eta > 0$ and $0 < \gamma < 1$. Then the two inequalities $\varphi'(\theta) < 0$ and $\partial_\theta(\theta\varphi(\theta)) > 0$ hold for all $\theta > 0$. In particular if $\gamma = 1/2$ then Lemma 5.1 implies that the SVI-JW parameters ψ_t, p_t, and c_t associated with the volatility surface (5.1) *are constant and independent of the time to expiration t. Furthermore, Equation 5.2 implies that the ATM volatility skew is given by*

$$\partial_k\sigma_{\mathrm{BS}}(k,t)\big|_{k=0} = \frac{\rho\,\eta}{2\sqrt{t}}.$$

Remark 5.2. *With the parameterization* (5.1)*, since $\theta_0 = 0$, we have at time $t = 0$:*

$$w(k,\theta_0) = \frac{1}{2}\phi_0\,(\rho k + |k|), \quad \text{for any } k \in \mathbb{R}, \tag{5.4}$$

where $\phi_0 := \lim_{\theta\to 0}\theta\,\varphi(\theta)$. $\phi_0 = 0$ is characteristic of stochastic volatility models as in Example 5.1; $\phi_0 > 0$ as in Example 5.2 gives a V-shaped time zero smile which is characteristic of models with jumps and in particular, characteristic of empirically observed volatility surfaces.

We now determine conditions on the variable θ_t and the function φ in order to ensure that the volatility surface (5.1) is free of static arbitrage. The following definition is equivalent to the definition of static arbitrage for call options in the introduction (see [21]).

Definition 5.1. *An SVI volatility surface is free of static arbitrage if and only if the following conditions are satisfied:*

(i) it is free of calendar spread arbitrage;

(ii) each time slice is free of butterfly arbitrage;

In particular, absence of butterfly arbitrage ensures the existence of a (non negative) probability density, and absence of calendar spread arbitrage implies monotonicity of option prices with respect to the strike. The following theorem provides sufficient conditions for a volatility smile of the form (5.1) to be free of butterfly arbitrage and thereby free of static arbitrage.

Theorem 5.2. *The volatility surface* (5.1) *is free of butterfly arbitrage if the following conditions are satisfied for all* $\theta > 0$*:*

1. $\theta\varphi(\theta)\left(1+|\rho|\right) \leq 4$;
2. $\theta\varphi(\theta)^2\left(1+|\rho|\right) \leq 4$.

Proof. For ease of notation, we suppress the explicit dependence of θ and φ on t. By symmetry, it is enough to prove the theorem for $0 \leq \rho < 1$. We shall therefore assume so, and we define $z := \varphi k$. The function g defined in (4.1) reads

$$g(z) = \frac{f(z)}{64\left(z^2+2z\rho+1\right)^{3/2}},$$

where

$$f(z) := a - b\varphi^2\theta - \frac{c}{16}\varphi^2\theta^2,$$

and where a, b and c depend on z. In the following, we frequently use the inequality

$$z^2+2z\rho+1 = (z+\rho)^2 + 1 - \rho^2 \geq 0.$$

Computing the coefficient of $\varphi^2\theta^2$ in $f(z)$ explicitly gives

$$\begin{aligned} c &= \sqrt{z^2+2z\rho+1}\left\{\left(1+\rho^2\right)(z+\rho)^2 + 2\rho(z+\rho)\sqrt{z^2+2z\rho+1} + \left(1-\rho^2\right)\rho^2\right\} \\ &\geq \sqrt{z^2+2z\rho+1}\left\{\left(1+\rho^2\right)(z+\rho)^2 + 2\rho(z+\rho)^2 + \left(1-\rho^2\right)\rho^2\right\} \\ &= \sqrt{z^2+2z\rho+1}\left\{(1+\rho)^2(z+\rho)^2 + \left(1-\rho^2\right)\rho^2\right\} \geq 0. \end{aligned}$$

Thus if

$$0 \leq \theta\varphi \leq \frac{4}{1+\rho} \quad \text{and} \quad 0 \leq \theta\varphi^2 \leq \frac{4}{1+\rho},$$

we have

$$f(z) \geq \begin{cases} a - \dfrac{4\,b}{1+\rho} - \dfrac{c}{(1+\rho)^2} =: f_1(z), & \text{if } b \geq 0, \\ a - \dfrac{c}{(1+\rho)^2} =: f_2(z), & \text{if } b < 0. \end{cases}$$

It is then straightforward to verify that

$$\begin{aligned} \frac{2 f_1(z)}{(1+\rho)^2} &= \sqrt{z^2+2z\rho+1}\left\{z^2\rho - z(1-\rho)\rho + 2(1+\rho)\left(1-\rho^2\right)+\rho\right\} \\ &\quad + \rho\,(z+\rho)^2 + 3\rho\left(1-\rho^2\right) + 2\left(1-\rho^2\right) - z\rho\left(z^2+2z\rho+1\right), \end{aligned}$$

which is clearly positive for $z < 0$. To see that $f_1(z)$ is also positive when $z > 0$, we rewrite it as

$$\begin{aligned} &\frac{2\,f_1(z)}{(1+\rho)^2} \\ = \; &\left\{\sqrt{z^2+2z\rho+1}-(z+\rho)\right\}\left\{\rho\left(z-\frac{1-\rho}{2}\right)^2 + 2(1+\rho)\left(1-\rho^2\right)+\rho\left(1-\frac{(1-\rho)^2}{4}\right)\right\} \\ &+(1+\rho)\left\{z\left(2-\rho^2\right)+2\,(1+\rho)\left(1-\rho^2\right)+\rho\right\}. \end{aligned}$$

Consider now the function $f_2(z)$. It is straightforward to verify that

$$f_2(z) = -\frac{2z^3\rho}{(1+\rho)^2} + \left(z^2+2z\rho+1\right)^{3/2} + 2\left(z^2+2z\rho+1\right) + \sqrt{z^2+2z\rho+1}$$

which is positive by inspection if $z < 0$. To see that $f_2(z)$ is also positive when $z > 0$, we rewrite it as

$$\begin{aligned} f_2(z) &= z^3\,\frac{1+\rho^2}{(1+\rho)^2} + 3z^2\rho + 2\left(z^2+2z\rho+1\right) \\ &\quad + \left(z^2+2z\rho+1\right)\left\{\sqrt{z^2+2z\rho+1}-(z+\rho)\right\} \\ &\quad + \sqrt{z^2+2z\rho+1} + 2z\rho^2 + z + \rho. \end{aligned}$$

Thus $f(z) \geq 0$ in all cases and the theorem is proved. □

Remark 5.3. *A volatility smile of the form* (5.1) *is free of butterfly arbitrage if*

$$\sqrt{v_t\,t}\max\left(p_t,c_t\right) \leq 4, \quad \textit{and} \quad \left(p_t+c_t\right)\max\left(p_t,c_t\right) \leq 8,$$

hold for all $t > 0$. The proof follows from Lemma 5.1 by re-expressing Conditions 1 and 2 of Theorem 5.2 in terms of SVI-JW parameters.

The following lemma shows that Theorem 5.2 is almost if-and-only-if.

Lemma 5.2. *The volatility surface* (5.1) *is free of butterfly arbitrage only if*

$$\theta\varphi(\theta)\left(1+|\rho|\right) \le 4, \quad \textit{for all } \theta > 0.$$

Moreover, if $\theta\varphi(\theta)\left(1+|\rho|\right) = 4$*, the surface* (5.1) *is free of butterfly arbitrage only if*

$$\theta\varphi(\theta)^2\left(1+|\rho|\right) \le 4.$$

Thus Condition 1 of Theorem 5.2 is necessary and Condition 2 is tight.

Proof. Considering the surface (5.1) and the function g defined in (4.1), we have

$$g(k) = \begin{cases} \dfrac{16-\theta^2\varphi(\theta)^2\left(1+\rho\right)^2}{64} + \dfrac{4-\theta\varphi(\theta)^2\left(1+\rho\right)}{8\phi k} + \mathcal{O}\left(\dfrac{1}{k^2}\right), & \text{as } k \to +\infty, \\ \dfrac{16-\theta^2\varphi(\theta)^2\left(1-\rho\right)^2}{64} - \dfrac{4-\theta\varphi(\theta)^2\left(1-\rho\right)}{8\varphi(\theta) k} + \mathcal{O}\left(\dfrac{1}{k^2}\right), & \text{as } k \to -\infty. \end{cases}$$

The result follows by inspection. □

Remark 5.4. *The asymptotic behavior of the surface* (5.1) *as* $|k|$ *tends to infinity is*

$$w(k,\theta_t) = \frac{\left(1\pm\rho\right)\theta_t}{2}\varphi(\theta_t)\,|k| + \mathcal{O}(1), \quad \textit{for any } t > 0.$$

We thus observe that the condition $\theta\varphi(\theta)\left(1+|\rho|\right) \le 4$ *of Theorem 5.2 corresponds to the upper bound of* 2 *on the asymptotic slope established by Lee [20] and so again, Condition 1 of Theorem 5.2 is necessary.*

The following corollary follows directly from Theorems 5.1 and 5.2.

Corollary 5.1. *The surface* (5.1) *is free of static arbitrage if the following conditions are satisfied:*

1. $\partial_t\theta_t \ge 0$, *for all* $t > 0$
2. $\partial_\theta(\theta\varphi(\theta)) \ge 0$, *for all* $\theta > 0$;
3. $\partial_\theta\varphi(\theta) < 0$, *for all* $\theta > 0$;
4. $\theta\varphi(\theta)\left(1+|\rho|\right) \le 4$, *for all* $\theta > 0$;
5. $\theta\varphi(\theta)^2\left(1+|\rho|\right) \le 4$, *for all* $\theta > 0$.

Consider the function $\varphi(\theta) = \eta/\theta^\gamma$ with $\eta > 0$ from Example 5.2, then Condition 2 imposes $\gamma \le 1/2$. From Condition 4, such surfaces can be free of static arbitrage only up to some maximum expiry. Take for instance the simple case $\theta_t := \sigma^2 t$ for some $\sigma > 0$. Then the map $\psi : t \mapsto \theta_t\varphi(\theta_t)\left(1+|\rho|\right) - 4$ is clearly strictly increasing with $\psi(0) = -4$ and $\lim_{t\to\infty}\psi(t) = \infty$. Therefore there exists $t_0^* > 0$ such that $\psi(t) \le 0$ for $t \le t_0^*$. The map $\psi_2 : t \mapsto \theta_t\varphi(\theta_t)^2\left(1+|\rho|\right) - 4$ is

- strictly increasing if $\gamma \in (0, 1/2)$ with $\psi_2(0) = -4$ and $\lim_{t\to\infty} \psi(t) = +\infty$; there exists $t_1^* > 0$ such that $\psi_2(t) \leq 0$ for $t \leq t_1^*$.
- strictly decreasing if $\gamma \in (1/2, 1)$ with $\lim_{t\to 0} \psi_2(0) = +\infty$ and $\lim_{t\to\infty} \psi(t) = -4$; there exists $t_1^* > 0$ such that $\psi_2(t) \leq 0$ for $t \geq t_1^*$.
- constant if $\alpha = 1/2$ with $\psi_2 \equiv -4$.

When $\gamma \in (0, 1/2)$, the surface is guaranteed to be free of static arbitrage only for $t \leq t_0^* \wedge t_1^*$. For $\gamma \in (1/2, 1)$, this remains true only for $t \in (0, t_0^*) \cap (t_1^*, \infty)$ (which may be empty). When $\gamma = 1/2$, static arbitrage cannot occur for $t \leq t_0^*$. However, the behavior for large θ can be easily modified so as to ensure that the entire surface is free of static arbitrage. For example, the choice

$$\varphi(\theta) = \frac{\eta}{\theta^\gamma (1+\theta)^{1-\gamma}} \tag{5.5}$$

gives a surface that is completely free of static arbitrage provided that $\eta (1 + |\rho|) \leq 2$.

Remark 5.5. *In the Heston-like parameterization of Example 5.1, note that*

$$\lim_{\theta\to+\infty} \theta\varphi(\theta) (1 + |\rho|) = \frac{1 + |\rho|}{\lambda}.$$

Therefore Condition 1 of Theorem 5.2 imposes $\lambda \geq (1 + |\rho|) /4$.

We can expand the class of volatility surfaces that are guaranteed to be free of static arbitrage by adding a suitable time-dependent function.

Theorem 5.3. *Let the volatility surface* (5.1) *satisfy the conditions of Corollary 5.1. If* $\alpha_t \geq 0$ *and* $\partial_t\alpha_t \geq 0$*, for all* $t > 0$*, then the volatility surface* $w_\alpha(k, \theta_t) := w(k, \theta_t) + \alpha_t$ *is free of static arbitrage.*

Proof. From Corollary 5.1, $w(k, \theta_t)$ is free of static arbitrage. It follows immediately that $w_\alpha(k, \theta_t) := w(k, \theta_t) + \alpha_t$ is free of calendar spread arbitrage if $\partial_t\alpha_t \geq 0$ and $\alpha_t \geq 0$.

We now show that w_α is also free of butterfly arbitrage. For clarity, since butterfly arbitrage does not depend on the time parameter t, we shall use the simplified notation $w(k) := w(k, \theta_t)$, and likewise $w_\alpha(k) := w_\alpha(k, \theta_t)$. Similarly, in view of (4.1), we shall define the function $g_\alpha(k)$, where the function w is replaced by w_α. We consider the case $\rho < 0$ since the case $\rho > 0$ follows by symmetry, and the result is obvious when $\rho = 0$. Let us consider the function $G_\alpha : \mathbb{R} \to \mathbb{R}$ defined by

$$G_\alpha(k) := g(k) - g_\alpha(k), \quad \text{for all } k \in \mathbb{R},$$

and let $k^* := -2\rho/\phi(\theta_t) > 0$ be the unique solution to the equation $w'(k) = 0$. We can compute explicitly the following:

$$G_\alpha(k) = \frac{w'(k)}{4} \left(\frac{1}{w_\alpha(k)} - \frac{1}{w(k)}\right) \left(4k + w'(k) - w'(k)k^2 \left(\frac{1}{w_\alpha(k)} + \frac{1}{w(k)}\right)\right),$$

which implies

$$\partial_\alpha G_\alpha(k) = -\frac{w'(k)}{4}\frac{2k\left[2w_\alpha(k) - kw'(k)\right] + w_\alpha(k)w'(k)}{w_\alpha(k)^3}.$$

It suffices to prove $\partial_\alpha G_\alpha(k) < 0$ for then we have the inequality $g_\alpha(k) > g(k) \geq 0$ and there is no butterfly arbitrage.

First consider the case $k > k^*$, so that $w'(k) > 0$. Recall that a continuously differentiable function f is convex on the interval (a, b) if and only if $f(x) - f(y) \geq f'(x)(x - y)$ for all $(x, y) \in (a, b)$. Setting $x = k$ and $y = 0$, we conclude that $2w_\alpha(k) - kw'(k) > 0$ since $w_\alpha(0) \geq 0$. It follows that $\partial_\alpha G_\alpha(k) < 0$ for any $k > k^*$.

For any $k < 0$, we always have $w'(k) < 0$, the inequality $2w_\alpha(k) - kw'(k) > 0$ follows by convexity as above, and hence $\partial_\alpha G_\alpha(k) < 0$ for any $k < 0$.

Let us finally consider the case $k \in (0, k^*)$. We prove here that $g_\alpha(k) \geq g(0)$ for all such k. Since the latter is non negative by absence of static arbitrage, the result follows. Note first that the inequalities $k > 0$ and $w'(k) < 0$ imply that

$$\left(1 - \frac{kw'(k)}{2w_\alpha(k)}\right)^2 \geq 1 = \left.\left(1 - \frac{kw'(k)}{2w(k)}\right)^2\right|_{k=0}.$$

Since w_α is strictly decreasing, positive, and convex on the interval $k \in (0, k^*)$, we obtain

$$-\frac{w'(k)^2}{4}\left(\frac{1}{w_\alpha(k)} + \frac{1}{4}\right) \geq -\frac{w'(0)^2}{4}\left(\frac{1}{w_\alpha(0)} + \frac{1}{4}\right) \geq -\frac{w'(0)^2}{4}\left(\frac{1}{w(0)} + \frac{1}{4}\right),$$

where the second inequality follows from the strict monotonicity of $\alpha \mapsto w_\alpha$. Finally, a straightforward analysis shows that the function $k \mapsto w''(k)$ is strictly increasing on the interval $(0, k^*/2)$ and strictly decreasing on $(k^*/2, k^*)$. The easy computation $w''(0) = w''(k^*)$ implies that $w''(k) \geq w''(0)$ on $(0, k^*)$, and hence concludes the proof.

□

Remark 5.6. *Given a set of expirations $0 < t_1 < \ldots < t_n$ ($n \geq 1$) and at-the-money total variances $0 < \theta_{t_1} < \ldots < \theta_{t_n}$, Corollary 5.1 gives us the freedom to match three features of one smile (level, skew, and curvature say) but only two features of all the other smiles (level and skew say), subject of course to the given smiles being themselves arbitrage-free. Theorem 5.3 may allow us to match an additional feature of each smile through α_t.*

6 Numerics and calibration methodology

6.1 How to eliminate butterfly arbitrage

In Section 5, we showed how to define a volatility smile that is free of butterfly arbitrage. This smile is completely defined given three observables. The ATM volatility and ATM skew are obvious choices for two of them. The most obvious choice for the third observable in equity markets would be the asymptotic slope for k negative and in FX markets and interest rate markets, perhaps the ATM curvature of the smile might be more appropriate.

In view of Lemma 5.1, supposing we choose to fix the SVI-JW parameters v_t, ψ_t and p_t of a given SVI smile, we may guarantee a smile with no butterfly arbitrage by choosing the remaining parameters c_t' and $\widetilde{v}_t'$ as

$$c_t' = p_t + 2\,\psi_t, \quad \text{and} \quad \widetilde{v}_t' = v_t\,\frac{4 p_t c_t'}{(p_t + c_t')^2}.$$

In other words, given a smile defined in terms of its SVI-JW parameters, we are guaranteed to be able to eliminate butterfly arbitrage by changing the call wing c_t and the minimum variance $\widetilde{v}_t$, both parameters that are hard to calibrate with available quotes in equity options markets.

Example 6.1. *Consider again the arbitrageable smile from Example 4.1. The corresponding SVI-JW parameters read*

$$(v_t, \psi_t, p_t, c_t, \widetilde{v}_t) = (0.01742625, -0.1752111, 0.6997381, 1.316798, 0.0116249)\,.$$

We know then that choosing $(c_t, \widetilde{v}_t) = (0.3493158, 0.01548182)$ *gives a smile free of butterfly arbitrage. It follows that there must exist some pair of parameters* $\{c_t, \widetilde{v}_t\}$ *with* $c_t \in (0.349, 1.317)$ *and* $\widetilde{v}_t \in (0.0116, 0.0155)$ *such that the new smile is free of butterfly arbitrage and is as close as possible to the original one in some sense. In this particular case, choosing the objective function as the sum of squared option price differences plus a large penalty for butterfly arbitrage, we arrive at the following "optimal" choices of the call wing and minimum variance parameters that still ensure no butterfly arbitrage:*

$$(c_t, \widetilde{v}_t) = (0.8564763, 0.0116249)\,.$$

Note that the optimizer has left $\widetilde{v}_t$ *unchanged but has decreased the call wing. The resulting smiles and plots of the function* g *are shown in Figure 2.*

Remark 6.1. *The additional flexibility potentially afforded to us through the parameter* α_t *of Theorem 5.3 sadly does not help us with the Vogt smile of Example 6.1. For* α_t *to help, we must have* $\alpha_t > 0$*; it is straightforward to verify that this translates to the condition* $v_t\,(1-\rho^2) < \tilde{v}_t$ *which is violated in the Vogt case.*

6.2 Calibration of SVI parameters to implied volatility data

There are many possible ways of defining an objective function, the minimization of which would permit us to calibrate SVI to observed implied volatilities. Whichever calibration strategy we choose, we need an efficient fitting algorithm and a good choice of initial guess. The approach we will present here involves taking a square-root fit as the initial guess. We then fit SVI slice-by-slice with a heavy penalty for calendar spread arbitrage

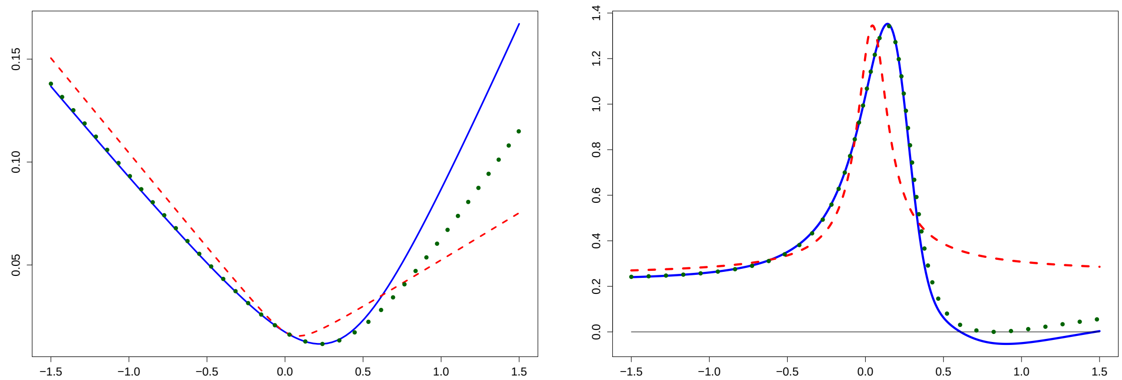

Figure 2: Plots of the total variance smile (left) and the function g defined in (4.1) (right), using the parameters (4.2). The graphs corresponding to the original Vogt parameters is solid, to the guaranteed butterfly-arbitrage-free parameters dashed, and to the "optimal" choice of parameters dotted.

(i.e. crossed lines on a total variance plot). Consider two SVI slices with parameters χ_1 and χ_2 where $t_2 > t_1$. We first compute the points k_i ($i = 1, \ldots, n$) with $n \leq 4$ at which the slices cross, sorting them in increasing order. If $n > 0$, we define the points $\widetilde{k}_i$ as

$$\begin{aligned} \widetilde{k}_1 &:= k_1 - 1, \\ \widetilde{k}_i &:= \frac{1}{2}(k_{i-1} + k_i), \quad \text{if } 2 \leq i \leq n, \\ \widetilde{k}_{n+1} &:= k_n + 1. \end{aligned}$$

For each of the $n+1$ points $\widetilde{k}_i$, we compute the amounts c_i by which the slices cross:

$$c_i = \max\left[0, w(\widetilde{k}_i, \chi_1) - w(\widetilde{k}_i, \chi_2)\right].$$

Definition 6.1. *The* crossedness *of two SVI slices is defined as the maximum of the* c_i ($i = 1, \ldots, n$). *If* $n = 0$, *the crossedness is null.*

An example SVI calibration recipe

- Given mid implied volatilities $\sigma_{ij} = \sigma_{\rm BS}(k_i, t_j)$, compute mid option prices using the Black-Scholes formula.
- Fit the square-root SVI surface by minimizing sum of squared distances between the fitted prices and the mid option prices. This is now the initial guess.
- Starting with the square-root SVI initial guess, change SVI parameters slice-by slice so as to minimize the sum of squared distances between the fitted prices and the mid option prices with a big penalty for crossing either the previous slice or the next slice (as quantified by the crossedness from Definition 6.1).

There are obviously many possible variations on this recipe. The objective function may be changed and when finally working to optimize the fit slice-by-slice, one can work from the shortest expiration to the longest expiration or in the reverse order. In practice, working forward or in reverse seems to make little difference. Changing the objective function on the other hand will make some difference especially for very short expirations.

6.3 Interpolation and extrapolation of calibrated slices

Suppose we follow the above recipe above to fit SVI to options with a discrete set of expiries. In particular, each of the resulting SVI smiles will be free of butterfly arbitrage. It's not immediately obvious that we can interpolate these smiles in such a way as to ensure the absence of static arbitrage in the interpolated surface. The following lemma shows that it is possible to achieve this.

Lemma 6.1. *Given two SVI smiles $w(k, t_1)$ and $w(k, t_2)$ with $t_1 < t_2$ where the two smiles are free of butterfly arbitrage and such that $w(k, \tau_2) \geq w(k, \tau_1)$ for all k, there exists an interpolation such that the interpolated volatility surface is free of static arbitrage for $t_1 < t < t_2$.*

Proof. Given the two SVI smiles $w(k, t_1)$ and $w(k, t_2)$, we may compute the (undiscounted) prices $C(F_i, K_i, t_i) =: C_i$ of European calls with expirations t_i $(i = 1, 2)$ using the Black-Scholes formula. In particular, since the two smiles are free of butterfly arbitrage,

$$\frac{\partial^2 C_i}{\partial K^2} \geq 0, \qquad \text{for } i = 1, 2.$$

Consider any monotonic interpolation θ_t of the at-the-money total variance $w(0, t)$. Let $K_i = F_i \mathrm{e}^k$ and $K_t = F_t \mathrm{e}^k$. Then for any $t_1 < t < t_2$, define the price $C_t = C(F_t, K_t, t)$ of a European call option to be

$$\frac{C_t}{K_t} = \alpha_t \frac{C_1}{K_1} + (1 - \alpha_t) \frac{C_2}{K_2}, \tag{6.1}$$

where for any $t \in (t_1, t_2)$, we define

$$\alpha_t := \frac{\sqrt{\theta_{t_2}} - \sqrt{\theta_t}}{\sqrt{\theta_{t_2}} - \sqrt{\theta_{t_1}}} \in [0, 1]. \tag{6.2}$$

By construction, for fixed k, the inequality

$$\frac{\partial}{\partial \tau} \frac{C_t}{K_t} \geq 0$$

holds so that there is no calendar spread arbitrage. Also, because of the square-roots in the definition (6.2), the at-the-money interpolated option price will be almost perfectly consistent with the chosen total variance interpolation θ_t. Moreover, if the two smiles

$w(k,t_1)$ and $w(k,t_2)$ are free of butterfly arbitrage, we have $\partial_{K,K}C(k,t) \geq 0$. To see this, first note that because all the options have the same moneyness, the identity (6.1) is equivalent to

$$\frac{C_t}{F_t} = \alpha_t \frac{C_1}{F_1} + (1-\alpha_t)\,\frac{C_2}{F_2}. \tag{6.3}$$

Then note that the ratio $C(F,K,t)/F$ is a function of F and K only through the log-moneyness k. Also, for $K = K_t, K_1, K_2$, we have

$$K^2 \frac{\partial^2 f}{\partial K^2} = \frac{\partial^2 f}{\partial k^2} - \frac{\partial f}{\partial k}.$$

Applying this to (6.3), we obtain

$$\frac{K_\tau^2}{F_t}\frac{\partial^2 C_t}{\partial K_t^2} = \alpha_t \frac{K_1^2}{F_1}\frac{\partial^2 C_1}{\partial K_1^2} + (1-\alpha_t)\,\frac{K_2^2}{F_2}\frac{\partial^2 C_2}{\partial K^2}.$$

All the terms on the rhs are non-negative, so the lhs must also be non-negative. We conclude that there is no butterfly arbitrage in the interpolated slice and thus that there is no static arbitrage. The interpolated volatility surface may be retrieved by inversion of the Black-Scholes formula. □

We could conceive of a myriad of algorithms for extrapolating the volatility surface. For example, one way to extrapolate a given set of $n \geq 1$ (arbitrage-free) volatility smiles with expirations $0 < t_1 < \ldots < t_n$ would be as follows: At time $t_0 = 0$, the value of a call option is just the intrinsic value. We may then interpolate between t_0 and t_1 using the algorithm presented in the proof of Lemma 6.1, thereby guaranteeing no static arbitrage. For extrapolation beyond the final slice, we suggest to first recalibrate the final slice using the simple SVI form (5.1). Then fix a monotonic increasing extrapolation of θ_t (asymptotically linear in time would seem to be reasonable) and extrapolate the smile for $t > t_n$ according to

$$w(k,\theta_t) = w(k,\theta_{t_n}) + \theta_t - \theta_{t_n},$$

which is free of static arbitrage if $w(k,\theta_{t_n})$ is free of butterfly arbitrage by Theorem 5.3.

6.4 A calibration example

We take SPX option quotes as of 3pm on 15-Sep-2011 (the day before triple-witching) and compute implied volatilities for all 14 expirations. The result of fitting square-root SVI is shown in Figure 3. The result of fitting SVI following the recipe provided in Section 6.2 is shown in Figure 4. With the sole exception of the first expiration (options expiring at the market open on the following morning), the fit quality is almost perfect.

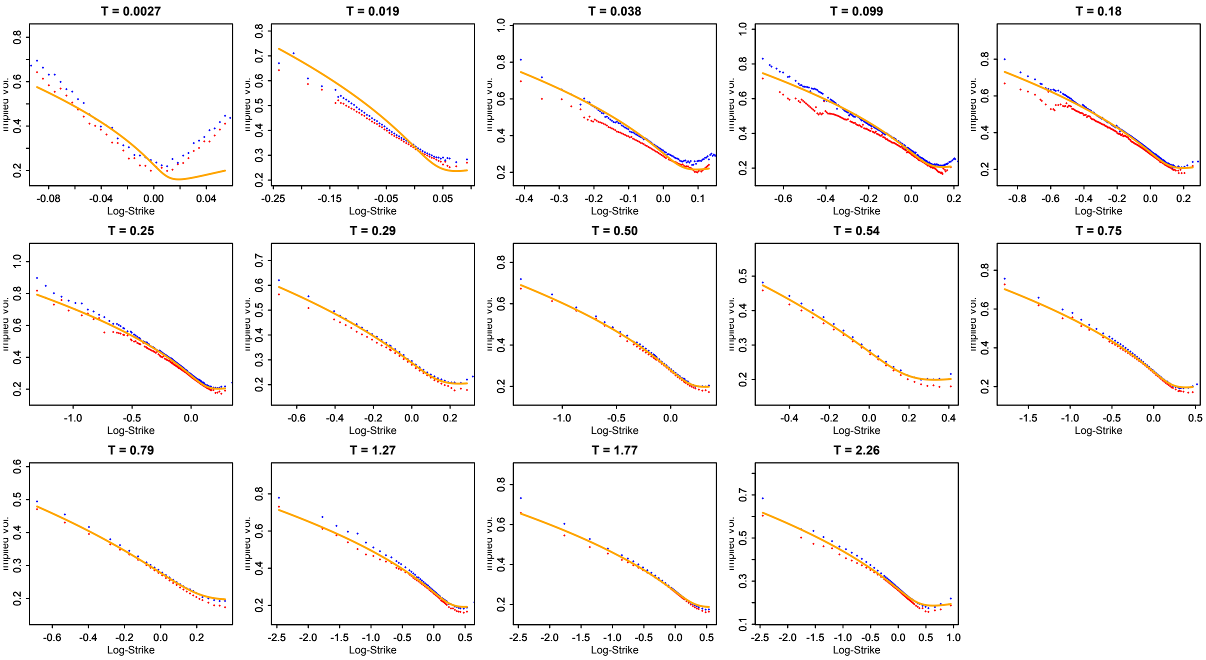

Figure 3: Red dots are bid implied volatilities; blue dots are offered implied volatilities; the orange solid line is the square-root SVI fit

7 Summary and conclusion

We have found and described a large class of arbitrage-free SVI volatility surfaces with a simple closed-form representation. Taking advantage of the existence of such surfaces, we showed how to eliminate both calendar spread and butterfly arbitrages when calibrating SVI to implied volatility data. We have also demonstrated the high quality of typical SVI fits with a numerical example using recent SPX options data.

The potential applications of this work to modeling the dynamics of the implied volatility surface are left for future research.

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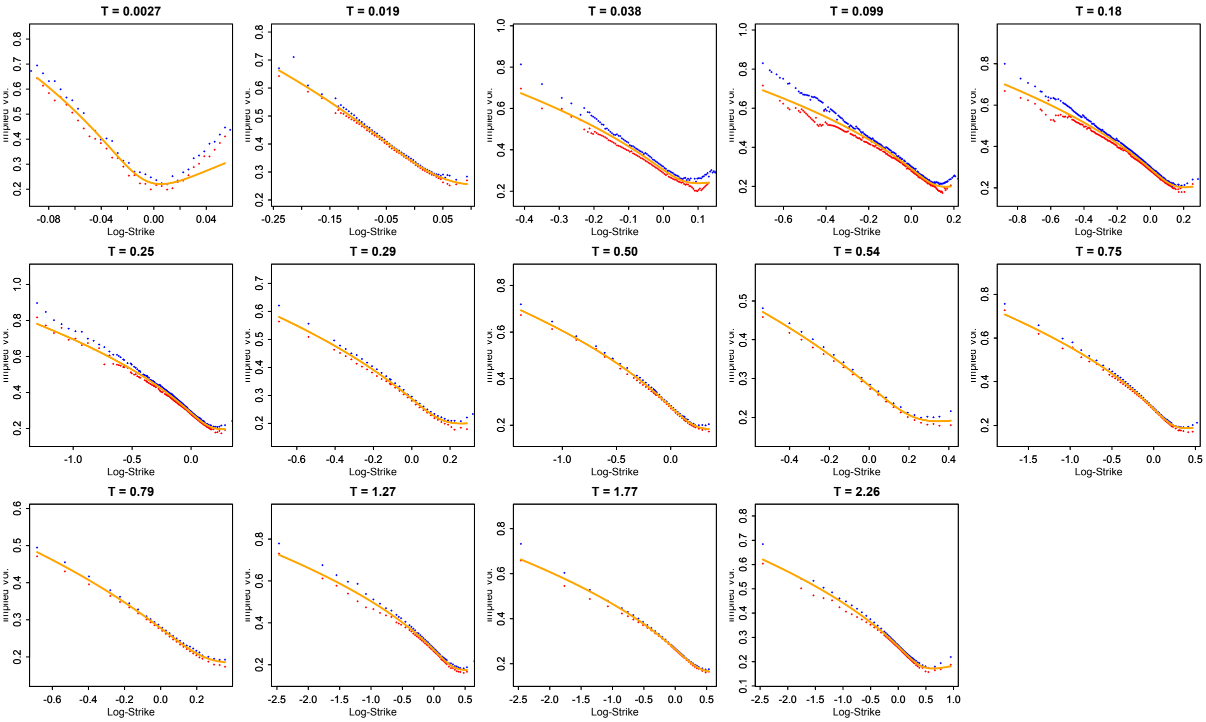

Figure 4: Red dots are bid implied volatilities; blue dots are offered implied volatilities; the orange solid line is the SVI fit following recipe of Section 6.2

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