Greeks in a Non-Black-Scholes World

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Abstract

We describe the issues that arise when trying to define greeks, especially delta, gamma and vega when the world is not described by the Black-Scholes model. In that case, even the definitions to be used for vanilla options require some thought. Besides vanillas, we use cliquets [1] to illustrate some of the isssues. The different *delta types* and *vega types* one can define are implemented in the EQAL analytics library as generic operations on volsurfaces and available through the Excel API (as of EQAL 1.5 PROD).

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1 Introduction and Overview

One important thing to understand once one moves beyond the Black-Scholes world is that the definition of greeks becomes much more subtle. To be sure, to some extent this is just a "book-keeping" problem — the important question for the trader and risk manager is how to hedge a given product. However, if nothing else, some simplified measures of risk have to be reported internally and externally by a firm. It is therefore also important simply from a book-keeping perspective.

Ultimately what matters is that one knows how to hedge oneselves against any possible scenario of market data perturbations. In a Black-Scholes world, it is quite straighforward to look at, and hedge against, small movements of the underlier (delta, gamma), volatility (vega) as well as interest rates (rho) or dividend yields (rhoDiv). Various other higher-order and cross greeks can also be considered. Even with term-structure of volatility or rates it is relatively easy to obtain a sensible view of ones risk this way (via "bucketing").

Before we proceed we should make sure what we are *not* concerned about here. Namely, we are not discussing *model* risk, the fact that even if all vanilla prices were known one can not determine the "true" dynamic of the underlier. Infinitely many stochastic processes can be designed that match the vanilla prices but will produce different results for exotics, in general. Note that this even affects the price of an exotic, not just the greeks (although they can be even more model-dependent than prices). Instead, we here assume a model has been chosen and we know how to calibrate it to market data, so that each product has a well-defined theoretical price. We are not concerned about whether this is a "good" model, however that term may be defined; the issues we are concerned about exist for any model appropriate for the real world – that is, any model beyond term-structure Black-Scholes.

With these preliminaries out of the way, what *is* the issue we are discussing? Consider for example the delta of a vanilla call option. Recall that the "implied volatility" in terms of which market participants like to think, is purely a convenient way of reexpressing the market price of the option in question. The fact that implied vols, for a given term, are strike-dependent, proves that the Black-Scholes model is not applicable to the real world. An implied vol is, as someone cleverly put it, the "wrong number to put into the wrong formula to get the right price". Implied vols have no dynamic information, and in principle one has to recalculate implied vols with any tick in market data, eg with any change in the spot S of the underlier. Ignoring rates and dividends for the moment, we can think of the value of a vanilla of maturity T and strike K as parametrized in terms of the implied vol plugged into the Black-Scholes formula, ie as

$$V \;=\; V_{\rm BS}(T,K,S,\,\sigma\!=\!\sigma(T,K,S))\;. \tag{1}$$

Delta, the change of value with spot, is then given by

$$\begin{aligned}\Delta \;=\; \frac{dV}{dS} \;&=\; \frac{\partial V_{\rm BS}}{\partial S}(\ldots) \;+\; \frac{\partial V_{\rm BS}}{\partial \sigma}(\ldots)\cdot\frac{\partial \sigma}{\partial S}(\ldots) &(2)\\ &=\; \Delta_{\rm BS}(\ldots) \;+\; \upsilon_{\rm BS}\cdot\frac{\partial \sigma}{\partial S}(\ldots)\;. &(3)\end{aligned}$$

Here $\Delta_{\rm BS}$ and $\upsilon_{\rm BS}$ are what is often called the Black-Scholes delta and vega, respectively. We like to call them the *naive Black-Scholes* delta and vega. They are obtained by using the Black-Scholes formula for these greeks derived under the assumption of flat (or term-structure) vol, and simply plugging in the implied vol at the strike and term in question.

However, the actual change in value depends on the additional vega term in the above. It vanishes if and only if the implied vol at a given strike does not change when the spot changes. This is known as a *sticky-by-strike* (or *sticky-strike*) behavior of the vols. Whether this is true or not is an *empirical* question. Numerous studies (mostly unpublished) of this question have been performed by practitioners, and academics, over the years. Unfortunately, no concensus has emerged. As Leif Andersen recently [2] put it:

> "Recall that despite decades of observations and academic work, nobody can quite agree about how to hedge a simple SP500 call option in the presence of a volatility smile – banks have many different rules and models for smile moves and delta computation."

To complete this sketch of the issues, let us briefly outline the additional complications that arise for volatility risk. For delta we may or may not come up with a good idea of how to estimate the correction term to the naive Black Scholes delta in (3), but at least we are talking about *one number*, the expected change of the value with respect to spot. For vol risk, there is however a whole implied volatility surface to consider.[1] While the price of a given vanilla option only depends on one point on this surface (all else being fixed), for any non-trivial exotic, and cliquets provide prime examples, the price depends on the whole surface in a generically very complicated manner. There is no way that the volatility risk, even if we are happy to define it as some kind of sensitivity to the implied volsurface of the underlier (which is problematic in itself, see the last footnote), can be captured in one number: Even though presumably there are only a limited number of "modes" that explain most of the variability of the volsurface of a given underlier, there is (a) certainly more than one mode that is relevant for some products, with cliquets again providing an example, (b) the shape and relative importance of these modes can change over time, and (c) be underlier-specific.

[1] And as we noted earlier, even the whole surface does not fully determine the dynamics of the underlier; even if one knew in which class of stochastic processes the dynamics of the underlier falls, one would presumably need exotics to calibrate the free parameters of the process. For example, in the context of cliquets, we know that "forward skew" (whatever this slightly fuzzy concept is taken to mean) is one of the aspects of the dynamics of the underlier that is not fully determined by just the implied volsurface.

2 Details

2.1 In an ideal world...

Before discussing what practical steps we can adopt to, over time, address these issues, it is helpful to consider what we would *like* to achieve in an *ideal* world:

1. Greeks should be intuitive and correspond to the major modes of how the underlying surface or curve tends to move. As discussed, for greeks like vega that means there will ultimately have to be more than one number to capture such risks. At the very least one would need one "vega" corresponding to the most likely up- and down-moves of the volsurface, another "skew-vega" corresponding to skew moves, perhaps a "smile-vega" along similar lines, and a "tilt-vega" corresponding to changes in term-structure of ATM vols. If one has a volsurface parametrization it becomes somewhat easier to define these different vegas precisely – though unless the parametrization includes a parametric form for the term-structure of the vol parameters there is always the question of what to assume for the moves of different terms relative to each other.

2. As mentioned earlier, for non-trival exotics there is the additional problems that the the implied volsurface does not fully determine the dynamics of the underlier. At least two approaches can be taken to address this. If for particular products one uses specific models – eg a forward skew model for cliquets – one might want to have specific additional vol risk measures. Eg in the case just mentioned, one might want to define a *forward skew vega* to measure sensitivity to changes in forward skew. Alternatively, one could use a true dynamic model (unlike the forward skew model) to price and hedge certain exotics. The simplest example might be a Heston model. In that case one could hedge with respect to the model parameters. The problem here is to come up with a model where everyone is comfortable that it truly captures all significant risks for a wide class of exotics relevant for our business.

3. Ideally one would like to have both a fine-grained and a high-level view of all these risks. For example, on the one hand one might want to take a look at ones vol risk sliced by underlier, term and strike – to the extent that this possible, ie really only for vanilla options. On the other hand, one would also like to able to sensibly aggregate vol risk across strikes, terms and even underlier. That however requires empirical knowledge of how, at least typically, vols at different strikes, terms and underliers move relative to each other. In the absence of detailed empirical studies one could in the short to medium term rely on trader/quant intuition for some of these questions. Even an approximate way of "normalizing" vega this way, would be presumably be better than none at all.

4. Greeks just correspond to infinitesimal versions of specific market data scenarios. Both on the level of a particular division, or on the firm level (eg for VaR), one will have to consider certain "big" aka disaster scenarios for specific deals or products, or, in fact, the whole portfolio of the division or firm. For the same reason, traders are often interested in seeing how their portfolios behave under "slides" of spot or vol over a significant range. Finally, one might have to perform some "big" adjustments to update stale market data (eg an old volsurface where at least the spot is out-of-date). Also, traders often want to look at "slides" of, say, spot or vol over wide ranges All such scenarios would be parametrized by the size of the change of the market data quantity in question. Ideally one would be using the same type of scenarios when changing a given quantity, eg spot, in all these cases. In particular the greeks would just correspond to infinitesimal versions of these scenarios.

5. Clearly, both more naive (but therefore more widely understood and used) and more sophisticated definitions of greeks like delta and vega(s) will presumably have to live side by side for the foreseeable future. If nothing else, than because one could for example imagine that firm-level management wants to get an aggregated view of certain risks across different divisions whose preferred way of viewing their risk exposures does not coincide.

It is almost certainly true that no firm has ever achieved all or even most of the above goals, at this point (certainly not a young business like ours). But we have started taking steps in this direction and we now discuss what they are.

First of all, we have implemented various sets of scenarios as generic operations on volsurfaces. By generic we mean that these operations can be applied without the user having to know whether the underlying "voldata type" is a matrix or some kind of parametric form. The scenarios have a continuous parameter than can be big or small, allowing one to smoothly and consistently interpolate between the small bumps needed for greeks and big disaster, slide or stale market data adjustment scenarios. This realizes point 4 above.

Furthermore, for a given parameter to be bumped (say spot or volsurface), the scenario is parametrized by a discrete type. For spot bumps we will refer to this as the *delta type*, for vol(surface) bumps as the *vega type*. We have implemented both "naive" and at least some of the more "sophisticated" scenarios. New ones can easily be added without affecting the interfaces of the pricing or market data analytics in our library. This prepares us to satisfy point 5 above.

We now discuss the specifics relevant to different greeks.

2.2 Delta

We already mentioned the sticky-by-strike scenario, where vols are assumed to be the same *by strike* when the spot changes. We refer to this as the "StickyByK" delta type. The delta and gamma in this scenario are the same as what we like to call the naive Black-Scholes greeks. It does have some empirical support in that this is indeed how volatilities, at least around ATM, seem to behave in certain historical regimes (see eg [3]). However, one should realize that this "regime" can not possibly hold for extended periods, eg when an underlier increases significantly over some period: If it did hold in such a situation then the ATM volatilities would continuously be driven lower. This makes no sense: eventually ATM volatilities have to "equilibrate" — just because the SPX, say, is higher now than 10, 20 or 50 years ago is no reason that ATM volatilities should be lower (there might be other reasons, but not just the index level in itself). In the equity world the equilibration seems to be achieved through regime shifts, rather than though a more sensible persistent regime; see [3] for details.

Note that the FX options market behaves seemingly more rational in this respect, in that vols tend to be more sticky-by-delta. Note that this is very close to a sticky-by-normalized strike type of scenario (in practice it is much easier to work with normalized-strike than with delta in this context, so we prefer the former; cf. [4] for definitions and some discussion related to normalized strike). We will refer to this as "StickyByNS". This is a behavior that can be maintained forever.

Nevertheless, it seems that "most of the time" for short-term moves in the equity markets, at least the ATM vols of a given term behave sticky-by-strike. Many firms still use the resulting delta and gamma, at least for vanilla options. There are good practical reasons for this, related to the risk-management of vanilla options: This is the only *delta type*, as we will refer to the different possible regimes from now on, where, just like for the price of a vanilla, one only has to know the implied vol in question to calculate delta and gamma. For any other delta type one has to know $\partial\sigma/\partial S$ (which for most deltaTypes can be related to $\partial\sigma/\partial K$, which can be calculated from the implied volsurface). If the initial design of the risk management system did not take this into consideration, one might be faced with a lengthy and painful redesign. If one uses a vendor system there might be no hope of achieving this.

Some firms use a modified version of the sticky-by-strike rule, in which this behavior is only assumed to hold (for a given term), around ATMF (aka ATF = at-the-forward). The rest of the vol curve is determined by saying that the *shape* of the new curve is supposed to be *the same* as before the spot bump. To be precise, by this we mean that the shape is the same in normalized-strike space. We will refer to this behavior as "StickyATFrescaleByNS".

This assumption avoids the following problem: If, as is often the case for short term vol curves, there is a call wing[2] and one considers a significant up-bump of the spot, then, if one where to assume sticky-by-strike for all strikes, the vol curve around the new ATF (or ATM) point could have a very different slope from the old one. In fact, for big enough up-bump the new ATF slope could even have the opposite sign! It is quite unlikely that the market would behave like this.[3]

Even though such big up-bumps might be not relevant for delta and gamma, we would like to, in accordance with point 4 above, use the same kind of scenario for large as for small moves of the spot. There is at least some hope then that with such a setup the results of big spot slides or disaster scenarios are somewhat meaningful – note for example that, at least with our current cliquet pricer, if a large up-bump would lead to a large change in the ATF slope of short-term vol curves, the cliquet prices would change in a non-sensical manner (in fact we have seen exactly this happen).

Note also, that if one wants to do a "PnL explanation" on spot slides via delta and gamma one should use the same delta type in the spot slide as for the delta and gamma calculation.

To summarize the present discussion, there are a number of assumptions one can make about how a volsurface behaves when the spot of the underlier changes. We have implemented the more obvious scenarios discussed above (and a few others) in a generic manner in the volsurface analytics of the EQAL library. On the Excel level we exposed a function `Eq.VolData.Adjust.To.New.Spot` that takes a delta type as input.

As discussed, as far as short-term moves of the volsurface are concerned, the delta type that makes the most sense, in accordance with points 1 and 4 above, is StickyATFrescaleByNS. For vanilla options the difference between StickyByK and StickyATFrescaleByNS is relatively small, in fact they coincide around ATF. StickyByNS can, however, be quite different, the more so the larger the skew/smile. We illustrate this in figure 1. The reader can explore these issues independently with the addins provided in Excel.

2.3 Vega

We already discussed the additional issues that arise for vega, compared to delta and gamma, Namely, that to really capture vol-related risks one needs a whole "family of vegas" – there is no way that one number "vega", however one wants to define it, could be used to explain the PnL of

[2] To be precise, by this we mean the following: For the given term there is a minimum volatility at a finite strike somewhere on the call side of the vol curve, so that after the minimum the vols increase with increasing strike.

[3] We should point that even the StickyATFrescaleByNS scenario can be criticized: if in the same situation with a call wing, we consider an even bigger up-bump of the spot, beyond the strike where the vol is lowest, this could lead to a new ATF vol that is higher than the old one (or at least higher than the one for a smaller up-bump), which makes no sense. One could prevent this by not allowing the new ATF vol to be higher than the minimum vol if one goes beyond it. We will implement this improvement of the StickyATFrescaleByNS scenario at some point.

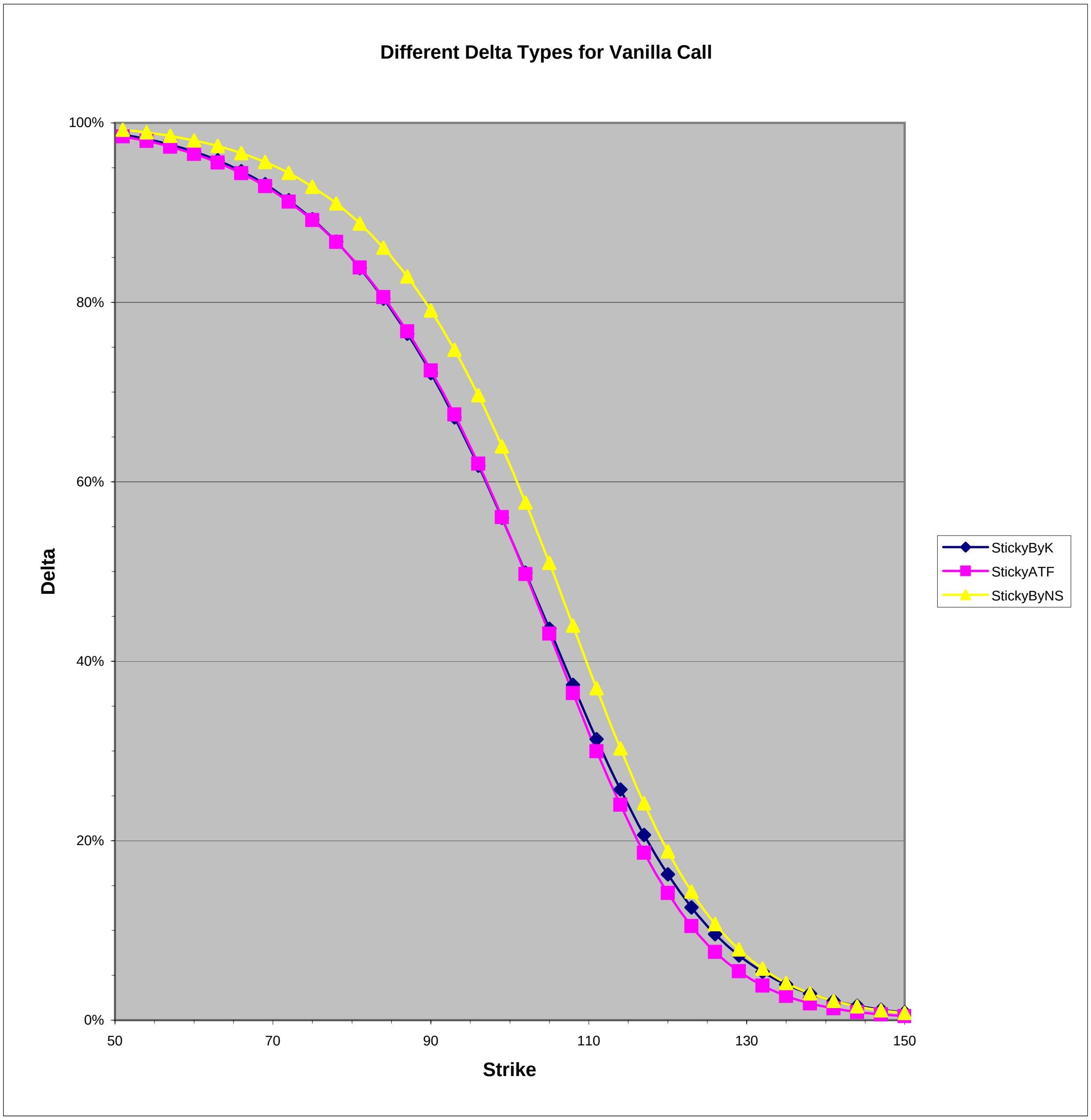

Figure 1: Delta as a function of strike for a vanilla call. We show (from top to bottom, as ordered at strike 120, say) the deltaTypes StickyByNS, StickyByK (aka naive Black-Scholes), and StickyATFrescaleByNS. The vanilla has a maturity of $T = 1$, spot $S = 100$, and the rates are zero. The vol curve is taken to be a JW3 parametrization [5] with $\sigma_0 = 0.2$, $s = -0.2$, $c = 0.04$.

a cliquet position, or any other non-trivial exotic, under generic daily volsurface moves.

Detailed steps towards a "multi-vega world" will have to wait a bit. For now we have implemented and exposed two *vega types*: "ConstShiftByK" and "PropShiftByNS". The former corresponds to shifting all vols of a given term by the same absolute amount, say 1%, *for every strike*.[4] The latter corresponds to shifting the ATF vols of a given term by a user-specified absolute amount, but the wings by corresponding *proportional* amounts. So the put wing, for example, would go up more than the ATF vol, such that the ratio of vols (by normalized strike) of the old and the new vol curve is the same for all values of normalized strike.

The Excel addin `Eq.VolData.Bump.Vol` allows one to apply these scenarios on a given volsurface in a universal manner. In the core code it is used, for example, to calculate the vega of a cliquet. For now, we think it is preferable to use the ConstShiftByK vega type for cliquets, as it then becomes consistent with the vega calculated for vanillas and variance swaps.

2.4 Other Greeks

For other greeks like rho, rhoDiv, theta, exactly the same issue arises: what are we supposed to assume about the behavior of the volsurface when we perturb rates or move the reference date forward? This decision tends to be not quite as urgent here, as the numerical effect is smaller and less relevant for hedging our books. Currently most of our pricers assume that vols are StickyByNS when rates move. Details are postponed for future discussions.

3 Conclusion

As is clear from the above, even if one ignores the question of what model(s) is(are) appropriate to capture the risk of various classes of exotic options, moving to a consistent set of greeks in the risk management system is a long-term project, that even affects the vanilla risk measures. We have outlined what kind of steps we have to take implement this project.

For reasons indicated above, the delta and gamma of cliquets are the first that we are considering to change to be of delta type StickyATFrescaleByNS. To be sure, the delta and gamma of vanilla and variance swaps for example (where we use StickyByK), will then for a while not be quite consistent with the cliquet ones. However, for variance swaps consistency with our current vanilla greeks is more important, and for vanillas the difference between StickyByK and StickyATFrescaleByNS are relatively small. We therefore think this is an acceptable compromise for an interim period.

Finally, we should point out that the designers of most vendor risk management systems are not even "aware" that these issues related to greeks *exist* (the same is true for many other issues).

References

[1] T. Klassen (2004), *Cliquet Pricing with the Forward Skew Model*, Wachovia, updated June 20, 2006.

[2] L. Andersen (2004), *Synthetic CDO pricing and portfolio risk management*, Quant Congress, New York, November 2004.

[4] Note that this is *not* quite the same as shifting all vols of a given term by the same amount for, say, fixed normalized strike. The reason is that shifting the ATF vol changes the relation between strike and normalized strike.

[3] E. Derman (1999), *Regimes of Volatility*, Goldman Sachs Research Note.

[4] T. Klassen (2003), *Equity Volatility Surfaces*, Wachovia.

[5] T. Klassen (2005), *Volatility Curve Parametrizations*, Wachovia.