Overview of the Ibbotson Financial Simulation Engine (IFSE)

CONFIDENTIAL

Ibbotson Associates Methodology

Updated July, 2010

Overview

The Ibbotson Financial Simulation Engine (IFSE) forecasts returns for a large number of asset classes and currencies over a long time period. It is designed to be used with long-term asset allocation and wealth forecasting applications.

The IFSE combines a sophisticated econometric term structure model with a Truncated Lévy Flight simulation procedure.[1] The econometric models work to preserve serial correlations and mean reversion characteristics of asset class returns while the Truncated Lévy Flight simulation procedure helps preserve the forecasted cross-correlation structure of the different asset classes as well as the non-normal characteristics of returns, namely skewness and kurtosis. The end result is simulated asset class returns based on forward-looking capital market assumptions, with appropriate serial correlation, mean-reversion, and cross-correlation characteristics, coupled with fat tails that recognize that bad events happen far more often than the normal distribution would predict.

The IFSE is different from many, if not most, other simulation engines in that it is based on an explicit term structure model of interest rates. The term structure model provides the foundation on which returns for other asset classes are built. The parameters for the structural equations of the model are estimated from historical data. Model parameters can be updated and new asset classes can be added with relative ease.

The econometric portion of the IFSE models assets in a three-tier structure. Causation is understood to rise up through the tiers. The first tier comprises the fundamental driving assets and variables and is driven by the term structure model. Included in Tier 1 are U.S. short and long-term inflation rates, the return on U.S. government bonds and bills, and the return on high-grade U.S. corporate debt. Tier 2 comprises most other asset classes, including U.S. and non-U.S. equity classes, as well as currency exchange rates. Tier 3 includes alternative securities and individual holdings.

The market return dominates the equity asset class returns. The residual part of market return that is not explained by the first three factors (see page 14) is modeled using a Truncated Lévy Flight simulation procedure to account for the fat tails of the market. Empirically, it can be shown that the Truncated Lévy Flight distribution does an excellent job at modeling the non-normal characteristics of historical returns. Most notably, it has done an exceptional job at modeling downside risk – how often bad events occur and the magnitude of those bad events.

[1] For more information on the Truncated Lévy Flight distribution, see Xiong (2010)

The IFSE runs on the MatLab platform. MatLab is well suited for this purpose, as it is very fast and allows vector and matrix operations to be programmed directly. The later feature facilitates easier maintenance and development. The parameter estimation procedures are incorporated into an Excel spreadsheet and associated macros.

The IFSE produces two tables of output for each scenario of the simulation. Each table contains returns for every asset class and currency for every year out to the simulation horizon. One table presents total returns and the other table presents returns broken down into income and capital components. A typical simulation consists of 500 to 1000 scenarios. Successive simulations of the same type are collected in a single file. These files are the inputs to downstream applications.

We believe that the state-of-the-economics-art features incorporated in the IFSE will provide superior forecasting performance. The term structure model enables an explicit incorporation of the expectations that drive long-term interest rates. The IFSE yield curve is rebuilt every simulated month based on forecast forward rates at that time. The term structure approach also enables us to model documented aspects of inflation and interest rates such as serial correlation, heteroskedasticity, and mean reversion. Ibbotson is committed to the continued development of the IFSE and to maintaining leadership in this field.

Section I – Background on Term Structure Models

The IFSE is based on an equilibrium model of the term structure of interest rates and inflation. Interest rates and inflation are key factors in economic dynamics and the determination of equity returns. The timeseries behavior of these factors has long been of interest to financial economists. Within the last 10 to 15 years econometric models have been developed that have identified distinctive and statistically significant timeseries characteristics of both interest rates and inflation. Superior models of interest rates have also been developed that consistently describe the relationship between current and long-term rates. In contrast to the current state of research regarding equity returns, there is a considerable degree of consensus among economists as to the nature of these patterns. We believe that the quality of a long-term forecast can only be increased by the incorporation of these documented regularities. This section provides some background on term structure models and why we feel the equilibrium approach to term structure modeling is appropriate in long-term forecasting.

Term structure models come in two general forms: no arbitrage and equilibrium models. Equilibrium models begin with assumptions regarding the behavior of short-term interest rates, which is modeled using an equation that consists of a number of variables. [2] The variables (usually referred to as *state variables*) measure such things as long-run averages and the degree reversion. The assumption of *general equilibrium* is then imposed. This requires that the prices of goods in this economy are in equilibrium (supply equals demand) and that the market participants are economically rational. With this model in place we can then derive the pricing equation for pure discount bonds (bonds with no coupon payments). The resulting pricing equation is a function of state variables and is estimated from empirical data.

In no arbitrage models the underlying state variables are not estimated statistically. Within the no arbitrage framework an assumption is made about the stochastic process (or behavior) of short-term interest rates. The stochastic process contains the state variables. Actual bond prices are then used to derive possible future short term rates in such a way that current market prices are replicated by the model – hence no arbitrage is possible between the model and the market. No arbitrage models are estimated using numerical search routines.

---

[2] Examples of equilibrium models include Vasicek (1977), Brennan and Schwartz (1979), Cox, Ingersoll, and Ross (1985), Longstaff and Schwartz (1992), and Dybvig, Ingersoll, and Ross (1996). Examples of no arbitrage models include Ho and Lee (1986), Black, Derman, and Toy (1990), Black and Karisinski (1991), Heath, Jarrow, and Morton (1992), and Hull and White (1990).

Equilibrium models lend themselves to analytic or closed form solutions for the prices of discount bonds. They also allow the modeling of interest rates in such a way that distinctive features of the timeseries data, such as the degree of autocorrelation, are endogenous to the model. No arbitrage models allow the current term structure to be fit exactly and make no assumptions about whether the economy is in equilibrium or not. They also do not make any assumptions about investor behavior or asset pricing dynamics. No arbitrage models are preferred for short-term forecasts and contingent claim analyses, but are less desirable for long-term forecasting because the model changes continuously as market prices fluctuate. This would require simulations to be updated more frequently. Further, there would be less continuity between updated and the original forecasts. For these reasons Ibbotson chose the equilibrium approach.[3]

Section II – The Three Tiers of Asset Classes

The IFSE architecture is based on three tiers of asset classes and economic variables. The first tier is composed of basic economic variables (inflation) and assets (US government and high-grade corporate bonds) that are believed to drive the returns of the assets in the second tier. An alternative interpretation of the first tier variables is that they represent those characteristics of asset returns that are common to every asset class in Tier 2. In other words, those aspects of Tier 2 asset returns that are not explained by the Tier 1 assets are unique to those assets. For example, the portion of a Tier 2 asset (say Large Cap Equity) that is not explained by the Tier 1 variables (Inflation and Bonds) is unique to that asset. Similarly, the Tier 1 and 2 assets are used to explain the returns for the Tier 3 assets. Again, the returns of those Tier 3 assets not explained by Tier 1 and 2 assets are assumed to be unique to that asset class or security. As we progress from Tier 1 to Tier 3 we move from macro-type assets and variables to more specific and unique assets and securities. So interest rates (Tier 1) help explain the returns of an equity index (Tier 2) which in turn may be a factor in the returns of an individual equity security (Tier 3).

The three tiers and the current asset classes that comprise them are:

- **Tier 1:**
  - Inflation
  - U.S. Government Bonds
  - U.S. High-grade Corp. Bonds
  - U.S. Municipal Bonds

[3] An overview of term structure models can be found in Campbell, Lo, and MacKinlay (1997), and Rebonato (1996).

- **Tier 2:**
  - U.S. Large Cap Equity
  - U.S. Small Cap Equity
  - European Equity in U.S. $
  - Pacific Equity in U.S. $
  - Canadian Equity in U.S. $
  - Emerging Market Equity in U.S. $
  - European Long Term Government Bonds in U.S. $
  - Pacific Long Term Government Bonds in U.S. $
  - Canadian Long Term Government Bonds in U.S. $
  - Emerging Market Bonds in U.S. $
  - REITS
  - Commodities
  - Euro Currency basket (Pounds and Euro)
  - Pacific Currency basket (Australian Dollar and Japanese Yen)
  - Canadian Currency
  - Canadian Inflation

- **Tier 3:**
  - Real Estate
  - Private Equity
  - Hedge Funds
  - Individual Large Cap U.S. Equity
  - Individual Small Cap U.S. Equity
  - U.S. High Yield Bonds

The difference between the second and third tier is that the third tier asset classes are thought of either as alternative assets or individual holdings. In addition, Tier 3 asset classes may contain less reliable data and this could affect our estimation process. This explains the presence of high yield bonds in the third tier.

Section III – Tier 1 and the Equilibrium Term Structure Model

Tier 1 variables are modeled using a four variable equilibrium term structure model. Specifically, inflation is modeled using two factors in a similar manner as in Brennan and Schwartz (1979) and Longstaff and Schwartz (1992). The first two factors are interpreted as a short and long-term inflation factors. Since nominal interest rates are the sum of inflation and real rates the third factor represents the short-term real interest rate.

Assuming statistical independence, these first three factors are then added together to model the return for a default-free discount U.S. Government bond. Finally, the fourth factor is the spread premium over default-free discount bonds obtained from accepting the added default risk of holding high-grade corporate bonds. High-grade corporate bond returns are then modeled by adding the four independent factors together. The independence assumption is critical for the model to be tractable.[4]

The presentation here uses the discrete time version of the models as found in Backus, Foresi, and Telmer (2000). Most presentations such as the referenced studies are in continuous time. Our approach allows the model to be more easily estimated because all marketable securities are in discrete time.

Our first factor is short-term inflation. We assume that the log of the short term inflation, π_t, is described by the following equation:[5]

$$\pi_{t+1} = (1-\varphi_\pi)\pi + \varphi_\pi \pi_t + \pi_t^{\frac{1}{2}} \nu_{t+1}, \qquad (1)$$

where ν_t is a normally distributed variable that has a constant variance σ_ν^2;
π is the long-run average value of the short term inflation factor;
φ_π is the mean reversion coefficient; and
π_t is the value of short-term inflation at time t.

Deviations from the average value are initiated by the realized values of ν_t, the mean zero residual term, also known as a temporary shock. As long as φ_π is less than one in absolute value, π_t should approach π at a rate depending on φ_π. The closer φ_π is to unity, the slower the expected approach to π. Alternatively, when φ_π is zero the mean or average for the short-term inflation is the long run average of the short-term inflation. The square root of π_t multiplies the shock ν_{t+1} in order to model observed *heteroskedasticity* in the inflation process. A timeseries process is heteroskedastic if the variance of the shock component is not constant over time. The specification chosen here models the variability of inflation as proportional to the level of inflation, π_t. This behavior is consistent with the observation that

---

[4] Regarding the independence assumption see Cox, Ingersoll, and Ross (1985) and Dybvig (1989). For further development of the theoretical and empirical properties of equilibrium models refer to Longstaff (1989, 1992), Dybvig, Ingersoll, and Ross (1996), Constantinides (1992), Brown and Dybvig (1986).

[5] We are using log returns so we can add the log inflation directly to the log real rate to obtain the log nominal rate for default free zero coupon bonds (e.g., U.S. Government bonds).

inflation has demonstrated increased volatility when it has been at historically high levels than if the error term was assumed to have a constant level of volatility.[6]

Our second factor is long-term inflation. This factor represents the possibility that there are long lasting structural inflationary shocks that die out very slowly or over the long term. For instance if the Federal Reserve is seen as changing its long-term monetary policy then market participants might infer that the long-term average rate of inflation has shifted. The first factor could be thought of as describing the effects of temporary shocks on inflation, such as short-term swings in consumer demand, while the second factor represents effects from permanent shocks. This long-term inflation factor will be denoted μ_t, and the equation driving its movement is similar to the one above, except for the simpler assumption of a homoskedastic error process, that is one where the variance of the shock u_{t+1} is constant over time:

$$\mu_{t+1} = (1-\varphi_\mu)\mu + \varphi_\mu \mu_t + u_{t+1}, \qquad (2)$$

where u_{t+1} is a normally distributed variable that has a constant variance σ^2_u;
μ is the long-run average value of the long-term inflation factor;
φ_μ is the mean reversion coefficient for long-term inflation; and
μ_t is the value of long term inflation at time t.

The state variables have the same interpretation as in equation (1). The behavior of the long-term inflation suggested by equation (2) is very different from the short-term inflation factor of equation (1). Specifically, the model's value of φ_μ will be much closer to unity than that of φ_π allowing equation (2) to be more accommodative of a permanent shock (such as a shift in monetary policy). The homoskedastic property allows μ_t to become negative and so care must be taken when simulating equation (2) so that this property does not overwhelm the simulation results.

Our third factor is the real short-term risk free rate. We assume that the short-term real rate follows the same kind of process as equation (1):

$$x_{t+1} = (1-\phi)x + \phi x_t + x_t^{\frac{1}{2}}\varepsilon_{t+1}, \qquad (3)$$

where ε_{t+1} is a normally distributed variable that has a constant variance σ^2_ε;

[6] Cox, Ingersoll, and Ross (1985) popularized the continuous time version of this model. Pearson and Sun (1994) extend this version to include models that allow the variance of the state variables to be linear in the level of the state variables, rather than proportional to it.

x is the long-run average value of the short-term real risk free interest rate;
ϕ is the mean reversion coefficient of the real short-term risk free rate; and
x_t is the value of the real short-term risk free rate at time t.

Equation (3) is interpreted in the same way as equation (1). The behavior of short-term rates will exhibit heteroskedasticity and mean reversion toward its long-term average rate x with a speed of ϕ. Like long term inflation we expect ϕ to be close to 1.0. The ε_{t+1} is normally distributed with a constant volatility, σ_ε.

Finally, our fourth factor represents credit spread between high-grade corporate debt and default free government debt. Not surprisingly, we chose the same functional form as both equations 1 and 3 for the spread, s:

$$s_{t+1} = (1-\theta)s + \theta s_t + s_t^{\frac{1}{2}}\eta_{t+1}, \qquad (4)$$

where η_{t+1} is a normally distributed variable that has a constant variance σ^2_η;
s is the long-run average value of the spread;
θ is the mean reversion coefficient of the credit spread; and
s_t is the value of the credit spread at time t.

Equation (4) is interpreted in the same way as equations (1) and (3). The behavior of the spread will exhibit heteroskedasticity and mean reversion toward a long-term rate s with a speed of θ. The η_{t+1} is normally distributed with a constant volatility, σ_η.

Equation (2) models inflation in such a way that allows the long-term behavior of inflation to change only when permanent macroeconomic shocks take place. The long-term inflation can also become negative which it has historically. Equation (1) models inflation such that the short-term inflation tends to revert to its long run average but can temporarily move away from the long run average as short-term random macroeconomic factors shock it. The empirical evidence supports this behavior. The short-term real rate and the spread behave similarly to short run inflation. Empirically, these factors have exhibited behavior that they tend to a long run averages but have periods where they move away from these averages only to return over a period of time. Moreover, they tend to be more volatile during times when they are at unusually high levels compared to when they are at low levels. Combining these we have created a model that both intuitively and empirically makes sense.

Table 1 presents the estimated empirical values of the state variables from equations (1) through (4). The estimation is described in Section IIIa. All of the mean reversion coefficients (φ_π, φ_μ, ϕ, and θ) are relatively close to one which means that each variable tends to move away from its long run average and then revert slowly back after a period of time. As expected φ_π and θ are less than φ_μ and ϕ. The volatility term for the short run (σ_v) inflation and the corporate spread (σ_η) are close to one another and are exponentially larger than the volatility of both the long run inflation term (σ_u) and the real risk free rate (σ_ε). This indicates that the short-term inflation and the spread jump around much more than the long-term inflation and the real risk free rate. This result matches historical behavior of these variables.

Section IIIa – Putting the Factors Together

The previous section introduced the four factors that we will use to completely model the Tier 1 assets. This section explains how we use these four factors to accomplish that objective.

Before we can combine equations (1) – (4) we need to make a few additional assumptions. First, as already stated, we assume that all the random shock terms in the four factor equations are independent. Specifically, we assume that v_{t+1}, u_{t+1}, ε_{t+1}, and η_{t+1} are all independent. The pairwise correlation between any two of these terms is assumed to be zero. This assumption, made by Cox, Ingersoll, and Ross (1985) and Dybvig (1989), allows us add them together as needed to model the Tier 1 assets. The second assumption is that the economy is in a general rational expectations equilibrium.[7]

The IFSE is driven by the theoretical default-free and high-grade corporate yield curves. To derive these yields we start with the economic variables of equations (1)-(4), π_t, μ_t, x_t, s_t. Following the same methodology as developed in Campbell, Lo, and MacKinlay (1997) and especially in Backus, Foresi, and Telmer (2000) we can write the discount bond yield for a T-period bond as:

$$Y_{Tt} * T = -P_{Tt} = A_T + B_T z_t, \qquad (5)$$

where z_t is a generic proxy for variables that follow processes similar to equations 1-4. In other words, z_t is either π_t, μ_t, x_t, and s_t. Because equations (1), (3), and (4) are distinctly different from equation (2) the expressions for A_T and B_T will be different. For equations (1), (3), and (4), A_T and B_T are of the form:

$$\begin{aligned} A_T &= A_{T-1} + d_z B_{T-1}, \\ B_T &= a_z + B_{T-1}(b_z + c_z B_{T-1}\ ), \end{aligned} \qquad (6)$$

[7] See Campbell, Lo and MacKinlay (1997) for a further development of the second assumption.

For equation (2) A_T and B_T are of the form:

$$A_T = A_{T-1} + B_{T-1}(1-\phi)\mu_z - \frac{\sigma^2}{2}(\beta_z + B_{T-1})^2,$$
$$B_T = 1 + \phi_z B_{T-1}, \tag{6'}$$

Unfortunately, there is little intuition behind equations 5 and 6; the equations are largely an algebraic result from the mathematics underlying the model.

We see from equation (6) that the log yield of a T-period bond is found recursively since it is a function of T-1 variables. That is, we can express equation (6) for a 1-period bond and substitute that into the expression for a 2-period bond. The result is that we now have an expression for a 2-period bond in terms of only the state variables. Repeat this process over and over until we have an expression for a T-period bond in terms of only the state variables. The resulting yields and prices are then found by recursively applying these equations. The multivariate case is simply repeating the above for each variable and adding the resulting terms together (due to the independence assumption). By solving the problem in this manner we create the yield curve because in order to get the yield at T we need to have all the yields at every time step between 0 and T-1.

Equations (5) and (6) are estimated using a numerical search routine. We do not use econometric timeseries methods due to the variables' instabilities and sensitivities to model parameterization. We calibrate the model so that the parameters chosen minimize a distance function. For example, the parameters for the π_t process are chosen to match the historical average, standard deviation, and serial correlation of realized inflation as measured by the CPI-U series. [8] The strength of this process is that we are able to preserve certain properties of inflation and interest rates that have held in the past and are judged to be likely to hold in the future without getting bogged down in the econometric problems associated with complicated time series models.

Using the Coleman-Fisher-Ibbotson synthetic bond data for our historical yield information we estimate the all the parameters in equations (5) and (6) in such a way that our model is as close to the historical values as possible. This ensures that the yields generated by our model match historical yields as best as possible. We are now using the actual data to estimate our

[8] The average is measured from January 1926 to May 2010, while the standard deviation and serial correlation were measured from September 1975 to May 2010.

parameters (the state variables) ensuring that the model is both behaviorally and empirically valid. The history we use covers the period from December 1940 to January 2007. A simple Newton-Raphson search routine is used to estimate all of the state parameters. Specifically we want to minimize the deviation from historical values.[9] Once the parameters are estimated we can then simulate the yields for zero coupon bonds (both default free government bonds and high-grade corporate bonds). With these estimates we will then be able to simulate all the Tier 2 assets as well.

For the credit premium, we fit the difference between the default-free zero coupon yields at 60 and 120 months to maturity and the yields on the Lehman Brothers intermediate and long-term corporate bond series.

Some idea of the magnitude of the parameters can be obtained by reference to Table 1.

Section IIIb – Using the Model to Generate Coupon Bond Returns

Section IIIa described the method of creating the simulated discount yield curve. However, we need to create return series for coupon bonds. Coupon bonds are needed because zero coupon corporate bonds do not exist and government strip securities exhibit peculiar characteristics. Consequently, equations (5) and (6) are used to create returns for coupon bonds.

We must generate returns from coupon bonds, or more correctly, constant maturity portfolios of coupon bonds. To accomplish this, we make the assumption that the maturity of each bond portfolio is kept constant. At the end of a period, a bond is purchased and then held to the next period, at which time the interest payment is received. Then this bond is sold and the capital gain is realized. Finally next period's bond is bought, starting the process over again. For the purpose of the simulation, we assume that the bond in the model makes monthly interest payments.

We now have two simulated return series: default free coupon bonds and high-grade corporate bonds. We will use these in the next section to describe the return series of the Tier 2 assets.

[9] For more on the Newton-Raphson routine refer to Hoffman (1992).

Section IV – Using the Term Structure Model of Tier 1 to Model Tier 2 Assets

This section describes the modeling of Tier 2 assets building on the values of Tier 1 timeseries. The equilibrium model and the bond returns from the model are now used to create four "factors" that are used to explain the Tier 2 asset returns. The factors are:

1. Inflation (F^1);
2. Total return on a 20-year government bond (F^2);
3. The return premium of long-terms (LT) corporate bonds over LT government bonds (F^3); and
4. The portion of the return on the world market portfolio that is not explained by the first three factors (F^4).

The first three factors are derived directly from our equilibrium model. The development of factor four will be explained below. Factors 1-3 are found using the process described in the previous section for 20-year bonds.

The fourth factor, the portion of the return on the world market portfolio that is not explained by the first three factors, is defined in the following manner. The excess return (total return less the 30-day Treasury rate) of the world market portfolio is assumed to follow the following ordinary least squares (OLS) regression equation:

$$R_t^M = \alpha^M + \gamma^1 F_t^1 + \gamma^2 F_t^2 + \gamma^3 F_t^3 + e_t. \qquad (7)$$

The results of the estimation of equation (7) are presented in Table 2. The intuition here is that the residuals to this regression capture the unique aspects of the "world" portfolio. That is, the portion of returns from the world portfolio that is not explained by the first three factors is considered to be the unique risk attributed to the world portfolio.

The construction the construction of the world market factor, F^4, is then:

$$F_t^4 = \alpha^M + e_t. \qquad (8)$$

Where e_t is a Truncated Lévy Flight (TLF) process, which has a mean of zero, a standard deviation equal to that measured from the regression, and is assumed to be independent of all other random shocks in the model (ν, u, ε, η, e, ω, and ξ). The TLF is introduced to account for the fat tails (Xiong, 2010a, 2010b). In the simulation, we adjust the value used for α^M in order that the expected premium of the world market basket over US long-term government bonds in the simulation matches the observed value from January 1970 to May 2010.

This interpretation of the residuals is used throughout the remainder of this section and so is worth emphasizing. The residuals of the regressions that follow are assumed to represent the *unique* risk of the particular asset class. In other words, the residuals represent the risk associated with a particular asset that can not be explained by the right hand side variables in the regression and so are interpreted as the unique risk of that specific asset class. This is an important assumption because the entire simulation for the asset classes will depend heavily on it. The variance-covariance matrix created by the residuals from the regressions is used to define the simulated interaction between the asset class returns.

There are now four factors available to help explain a portion of the returns to the other asset classes in the model. We proceed to regress the monthly returns of each second tier asset class on these four factors and a constant. Assume that for each asset class i in the second tier, the excess return (the total return less the 30-day Treasury bill return) is generated by:

$$R_t^i = a^i + b^{i1}F_t^1 + b^{i2}F_t^2 + b^{i3}F_t^3 + b^{i4}F_t^4 + \omega_t^i, \qquad (9)$$

where ω_t^i is the residual for asset class i at time t. The residuals for all the assets are then used to create a variance-covariance matrix that represents the covariance of each Tier 2 asset class with all other Tier 2 assets. We denote this symmetric matrix as Ω. Finally, note that equation (9) does not include income return from the asset classes. We will address how to generate income returns below.

Note that equation (9) has two applications within our model. The first was just described. We use historical values to estimate the regression. Once that procedure is completed we then use equation (9) within the simulation. For each simulated yield curve we simulate the four factors and then use them in equation (9) to simulate the returns for the Tier 2 assets.

As an example of the application of equation (9), Table 3 shows the results of its estimation for large-cap U.S. equities. Also, Table 4 shows the estimation of parameters for REITS. The approach is similar for other Tier 2 asset classes.

As already detailed, the results from the regressions using equation (9) are used within the simulation. Specifically, we make use of the b^i terms estimated from the regressions, as well as the matrix Ω that was obtained from the residuals of these regressions. Some of the intercepts, the a^i terms, can be adjusted in the spirit of reasonability. For example, the regression-estimated value of a^i is used in the case of Canadian equity. Setting the value to zero for this asset class would have implied out performance that was deemed unlikely.

This describes how the simulation operates to create paths of Tier 1 and Tier 2 timeseries. To summarize, at each time increment (1 month) we use random numbers drawn from different mean zero constant variance normal distribution to "shock" each of equations (1) through (4). The variance for the distributions is defined by the individual shock terms in these equations (ν, u, ε, η). From that simulation we can generate both government and corporate bond yields as defined in equations (5) and (6). This process is repeated for every simulated month along the yield curve – currently 240 times for a 20-year horizon. For this single "path" we have one completely simulated yield curve. Equation (9) (the mean return) and Ω (the variance-covariance from which random numbers are drawn) is then used to simulate return paths for each Tier 2 asset class at every point (each monthly increment) along the yield curve. The resulting return paths for each Tier 1 and 2 timeseries creates a matrix; the matrix columns represent each asset and the rows represent a monthly time increment along each asset path. This process is iterated a large number of times (typically 500-1000) to generate a representative number of possible return series for the asset classes. This produces a large data set from which asset allocation and terminal wealth analysis can be performed.

Section V – Tier 3 Asset Classes

The next step in the simulation is to create a model for the Tier 3 assets from the simulated Tier 1 and Tier 2 timeseries. The method is almost identical to that used for Tier 2 except that we now include the residuals from relevant Tier 2 asset classes in the regressions and impose some restrictions on their residuals. Equation (10) presents the typical regression equation for the Tier 3 assets. The excess return for each Tier 3 asset is assumed to follow:

$$R_t^i = K^i + \sum_{i=1}^{4} \lambda^i F_t^i + o^i \cdot \omega_t + \xi_t^i, \qquad (10)$$

where the residuals, ξ^i ,are uncorrelated across other Tier 3 assets and with the residuals from equation (9) and the shocks of equations (1) through (4), that is ν, u, ε, η, e, and ω. That is,

the variance-covariance matrix for all Tier 3 assets is a diagonal matrix;
K^i is the intercept term from the regression for Tier 3 asset i;
λ^i are the regression coefficients for each of the four factors (F^i) for each Tier 3 asset i;
o^i are the regression coefficients for each Tier 2 residual term (ω_t) used in the regression to describe Tier 3 asset i.

Some Tier 3 asset classes may include none or more than one of the Tier 2 regression residuals. We would not expect many of the Tier 3 assets to have a relationship with all of the Tier 2 assets. For example, the estimated total return process of hedge fund includes residuals from large cap and small cap as regressors but the high yield bond has no Tier 2 residuals in its regression.

The regression results for the Tier 3 assets are interpreted in the same way as for the Tier 2 assets. They are also implemented in the same manner as the Tier 2 regression within the simulation.

Using this method for individual stocks using information specific to that security is possible, however, is not implemented here because of the difficulty involved. Instead, we use a slightly different approach. For individual large and small cap equity we assume:

1. That the values of a^i and b^i are equal to those of the corresponding asset class for individual large and small cap equities.

2. That the value of o^i is one for the corresponding asset class and zero otherwise. That is, we only use the residual from the corresponding asset class and no others.

3. The residual has a standard deviation that will force the total volatility of the individual stock to be equal to the average volatility of individual stocks of that size.

These levels were determined in an independent study and rounded to annual levels of 40% for large cap and 80% for small cap individual stocks. Doing this reduces the number of equations needed in the simulation and is employed in an identical manner as described above.

Section V – Income Returns

The income returns for Tier 2 and 3 asset classes are assumed to be functions of their previous period's income return and the long-term government bond yield.[10] We develop the model for income returns by running the following autoregressions for each asset class:

[10] Currency, Canadian Inflation, Hedge Funds, and Private Equity have zero income returns in the model.

$$IR_{t+1}^{i} - Y_{t+1}^{240} = k^{i} + \psi^{i}\left(IR_{t}^{i} - Y_{t}^{240}\right) + \zeta_{t+1}^{i}, \qquad (11)$$

where ζ^i is the residual from this regression and ζ^i and ζ^j are independent for $i \neq j$. ζ^i is also independent of ν, u, ε, η, e, ω, and ξ.
k^i is the intercept term from the regression for Tier 3 asset i;
ψ^i are the regression coefficients for each $(IR_t^i - Y_t^{240})$;
IR_t^i is the current income of asset class i; and
Y_t^{240} is the current yield of a 20-year US government bond.

We measure the standard deviation of the residual term σ_{ζ^i} and use this in the simulation along with the assumption that ζ^i and ζ^j are independent for $i \neq j$. Since the above equation does not rule out negative values of income returns, we make a modification in the simulation model to ensure only positive income returns.

To summarize, equations (1) through (4) are used to generate return series and income series for the Tier 1 assets and variables. They are also used to generate the return series for the Tier 2 assets through equation (9) along with the Ω matrix. These return series are simulated using random draws from five independent normal distributions. Tier 3 asset returns are simulated by using equation (10) and a fifth normally distributed random draw to model ξ. The individual stocks within Tier 3 are dealt with in a unique way as described above but are simulated in the same way as the rest of the Tier 3 assets. Finally, the asset class income returns are simulated using equation (11) and a sixth normally distributed random draw to model ζ. We now have excess return series and income return series for all assets in the model across any period time at any frequency we desire. This relationship between the tiers is illustrated in Chart 1.

Section VII – Conclusion

This overview has described the essential features of the IFSE. The IFSE is based on an equilibrium term structure model of the yield behavior of government and corporate bonds. This model is used as a key input to modeling the other asset classes. The IFSE endogenously generates documented and significant patterns in interest rates and inflation such as autoregression, heteroskewdasticity, and mean reversion. For equity asset classes, the fat tails are captured by the TLF model.

The design of the IFSE allows for easy updating of model parameters, and for maintenance and further development. We believe that using the IFSE will provide superior portfolio and terminal wealth analyses, thus enabling provision of superior investment advice and long-term planning services.

Section VIII – Appendix

Chart 1 – IFSE Flow Chart

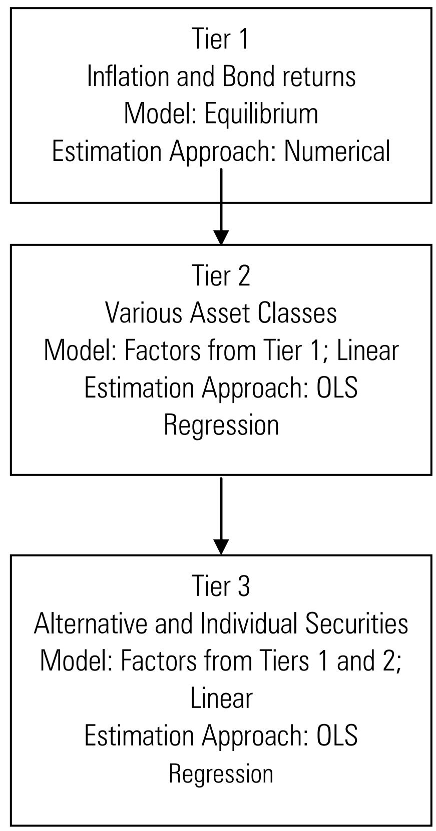

Table 1 –Tier 1 Model Parameters

Process	Parameter	Value
Short Inflation	π	0.002
	φ_{π}	0.55
	σ_{v}	0.04
	β_{π}	0
Long Inflation	μ	0.0006
	φ_{μ}	0.993
	σ_{u}	0.006
	β_{μ}	0.24
Short Rate	x	0.00075
	ϕ	0.876
	σ_{ε}	0.01
	β_{x}	-6.52
Credit spread	s	0.00008
	θ	0.97
	σ_{η}	0.008
	β_{s}	-4

ibbotson

Table 2 – Market Factor Regression

Parameter	Estimated Value	Standard Error	t-Statistic
α^M	0.00637	0.003	2.155
γ^1	-0.355	0.549	-0.647
γ^2	0.49	0.072	6.792
γ^3	1.02	0.155	6.543

R^2 :	0.123
Observations:	**480**
Period:	1/70-5/10
Periodicity:	Monthly

Table 3 – US Large-cap Equity Regression

Parameter	Estimated Value	Standard Error	t-Statistic
a	-0.00367	0.00037	-9.89
b^1	-0.662	0.0732	-9.05
b^2	0.492	0.0092	53.25
b^3	0.994	0.021	47.47
b^4	0.982	0.006	159.56

R^2 :	0.987
Observations:	**372**
Period:	1/79-5/10
Periodicity:	Monthly

Table 4 – NAREIT Regression

Parameter	Estimated Value	Standard Error	t-Statistic
a	-0.00043	0.0028	-0.155
b^1	-0.0028	0.51	-0.0055
b^2	0.4036	0.068	5.886
b^3	1.18	0.152	7.76
b^4	0.58	0.044	13.22

R^2 : 0.352

Observations: 456
Period: 1/72-5/10
Periodicity: Monthly

References

Backus, D., S. Foresi, and C. Telmer, 2000, "Discrete-Time Models of Bond Pricing," in *Advanced Fixed-Income Valuation Tools*, Jegadeesh and Tuckman, Ed., Wiley.

Black, F., and P. Karasinski, 1991, "Bond and Option Pricing when Short Rates are Lognormal," *Financial Analyst Journal*, July-August, 52-59.

Black, F., E. Derman, and W. Toy, 1990, "A One Factor Model of Interest Rates and Its Application to the Treasury Bond Options," *Financial Analyst Journal*, January-February, 33-39.

Brennan, M., and E. Schwartz, 1979, "A Continuous Time Approach to the Pricing of Bonds," *Journal of Banking and Finance*, 3, 133-155.

Brown, S., and P. Dybvig, 1986, "The Empirical Implications of the Cox, Ingersoll, and Ross Theory of the Term Structure of Interest Rates," *Journal of Finance*, 41, 617-632.

Campbell, J., 1986, "Bond and Stock Returns in a Simple Exchange Model," *Quarterly Journal of Economics*, 101, 785-803.

Campbell, J., A. Lo, and A. C. Mackinlay, 1997, *The Econometrics if Financial Markets*, Princeton University Press.

Constantinides, G., 1992, "A Theory of the Nominal Term Structure of Interest Rates," *Review of Financial Studies*, 5, 531-552.

Cox, J., J. Ingersoll, and S. Ross, 1985, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53, 385-408.

Dybvig, P., J. Ingersoll, and S. Ross, 1996, "Long Forward and Zero Coupon Rates Can Never Fall," *Journal of Business*, 69, 1-25.

Heath, D., R. Jarrow, and A. Morton, 1992, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims," *Econometrica*, 60, 77-105.

Ho, T., and S. Lee, 1986, "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance*, 41, 1011-1029.

Hoffman, J., 1992, *Numerical Methods for Engineers and Scientists*, Wiley.

Hull, J., and A. White, 1990, "Pricing Interest Rate Derivatives," *Review of Financial Studies*, 3, 573592.

Longstaff, F., 1989, "A Non-linear General Equilibrium Model of the Term Structure of Interest Rates," *Journal of Financial Economics*, 23, 195-224.

___________, 1992, "Multiple Equilibria and Term Structure Models," *Journal of Financial Economics*, 32, 333-344.

Pearson, N., and T. Sun, 1994, "Exploiting the Conditional Density in Estimating the Term Structure: An Application to the Cox, Ingersoll, and Ross Model," *Journal of Finance*, 49, 1279-1304.

Rebonato, R., 1996, *Interest-Rate Option Models*, Wiley.

Sun, T., 1992, "Real and Nominal Interest Rates: A Discrete-Time Model and Its Continuous Time Limit," *Review of Financial Studies*, 5, 581-611.

Vasicek, O., 1977, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5, 177-188.

Xiong, J.X. (2010a). "Nailing Down Risk." Morningstar Advisor Magazine, February / March, 2010, 24 – 27.

Xiong, J.X. (2010b). "Using Truncated Lévy Flight to Estimate Downside Risk." Journal of Risk Management in Financial Institutions, Vol. 3, No. 3, 2010 231-242.