1 GZ model

1.1 Mathematical model

The latent state transitions are

$$
\begin{aligned}
x_{a,t,i} &= \phi_a x_{a,t-1,i} + \boldsymbol{z}'_{a,t,i}\boldsymbol{\beta}_{\boldsymbol{Z_a}} + \varepsilon_{x_a} \ , \varepsilon_{x_a} \sim \mathcal{N}\left(0, \sigma^2_{x_a}\right) \ , \\
x_{b,t,i} &= \phi_b x_{b,t-1,i} + \boldsymbol{z}'_{b,t,i}\boldsymbol{\beta}_{\boldsymbol{Z_b}} + \varepsilon_{x_b} \ , \varepsilon_{x_b} \sim \mathcal{N}\left(0, \sigma^2_{x_b}\right) \ , \\
x_{p,t,i} &= \phi_p x_{p,t-1,i} + \boldsymbol{z}'_{p,t,i}\boldsymbol{\beta}_{\boldsymbol{Z_p}} + \varepsilon_{x_p} \ , \varepsilon_{x_p} \sim \mathcal{N}\left(0, \sigma^2_{x_p}\right) \ , \\
x_{q,t,i} &= \phi_q x_{q,t-1,i} + \boldsymbol{z}'_{q,t,i}\boldsymbol{\beta}_{\boldsymbol{Z_q}} + \varepsilon_{x_q} \ , \varepsilon_{x_q} \sim \mathcal{N}\left(0, \sigma^2_{x_q}\right) \ .
\end{aligned}
$$

The parameters of the state processes can be grouped as

$$
\begin{aligned}
\boldsymbol{\theta}_a &= \left(\phi_a, \boldsymbol{\beta}_{\boldsymbol{Z_a}}, \sigma^2_{x_a}\right) \ , \\
\boldsymbol{\theta}_b &= \left(\phi_b, \boldsymbol{\beta}_{\boldsymbol{Z_b}}, \sigma^2_{x_b}\right) \ , \\
\boldsymbol{\theta}_p &= \left(\phi_p, \boldsymbol{\beta}_{\boldsymbol{Z_p}}, \sigma^2_{x_p}\right) \ , \\
\boldsymbol{\theta}_q &= \left(\phi_q, \boldsymbol{\beta}_{\boldsymbol{Z_q}}, \sigma^2_{x_q}\right) \ .
\end{aligned}
$$

The measurement equation takes the general, highly nonlinear form, of

$$
\boldsymbol{y}_{t,i} = g\left(\exp\left(x_{a,t,i}\right), \exp\left(x_{b,t,i}\right), \exp\left(x_{p,t,i}\right), \exp\left(x_{q,t,i}\right)\right) \ .
$$

This implies $\forall \ t = 1, \ldots, T$ and $\forall \ i = 1, \ldots, N$

$$
\begin{aligned}
p_{\boldsymbol{\theta}_a}\left(x_{a,t,i} | x_{a,t-1,i}, \boldsymbol{z}'_{a,t,i}\right) &= \mathcal{N}\left(x_{a,t,i} \mid \phi_a x_{a,t-1,i} + \boldsymbol{z}'_{a,t,i}\boldsymbol{\beta}_{\boldsymbol{Z_a}}, \sigma^2_{x_a}\right) \ , \\
p_{\boldsymbol{\theta}_b}\left(x_{b,t,i} | x_{b,t-1,i}, \boldsymbol{z}'_{b,t,i}\right) &= \mathcal{N}\left(x_{b,t,i} \mid \phi_b x_{b,t-1,i} + \boldsymbol{z}'_{b,t,i}\boldsymbol{\beta}_{\boldsymbol{Z_b}}, \sigma^2_{x_b}\right) \ , \\
p_{\boldsymbol{\theta}_p}\left(x_{p,t,i} | x_{p,t-1,i}, \boldsymbol{z}'_{p,t,i}\right) &= \mathcal{N}\left(x_{p,t,i} \mid \phi_p x_{p,t-1,i} + \boldsymbol{z}'_{p,t,i}\boldsymbol{\beta}_{\boldsymbol{Z_p}}, \sigma^2_{x_p}\right) \ , \\
p_{\boldsymbol{\theta}_q}\left(x_{q,t,i} | x_{q,t-1,i}, \boldsymbol{z}'_{q,t,i}\right) &= \mathcal{N}\left(x_{q,t,i} \mid \phi_q x_{q,t-1,i} + \boldsymbol{z}'_{q,t,i}\boldsymbol{\beta}_{\boldsymbol{Z_q}}, \sigma^2_{x_q}\right) \ , \\
p\left(\boldsymbol{y}_{t,i} | x_{a,t,i}, x_{b,t,i}, x_{p,t,i}, x_{q,t,i}\right) &= \mathcal{MNL}\left(\boldsymbol{y}_{t,i} | \pi^{(1)}_{it}, \ldots, \pi^{(M_{it})}_{it}\right) \ ,
\end{aligned}
$$

where $\pi^{(k)}_{it} = \left(F_{\mathrm{GB2}}\left(c^{(k)}_{it}; x_{a,t,i}, x_{b,t,i}, x_{p,t,i}, x_{q,t,i}\right) - F_{\mathrm{GB2}}\left(c^{(k-1)}_{it}; x_{a,t,i}, x_{b,t,i}, x_{p,t,i}, x_{q,t,i}\right)\right)$. The income distribution function is a four-parameter GB2:

$$
\begin{aligned}
F(c^{(k)}_{it}; x_{a,t,i}, x_{b,t,i}, x_{p,t,i}, x_{q,t,i}) = B(d^{(k)}_{t,i}; x_{p,t,i}, x_{q,t,i}) &= \frac{\int_0^{d^{(k)}_{t,i}} t^{\exp(x_{p,t,i})-1}(1-t)^{\exp(x_{q,t,i})-1}dt}{\mathrm{B}\left(\exp\left(x_{p,t,i}\right), \exp\left(x_{p,t,i}\right)\right)} \ , \\
d^{(k)}_{t,i} &= \frac{(c^{(k)}_{it} / \exp\left(x_{b,t,i}\right))^{\exp(x_{a,t,i})}}{1 + (c^{(k)}_{it} / \exp\left(x_{b,t,i}\right))^{\exp(x_{a,t,i})}} \ .
\end{aligned}
$$

Finally, we need prior assumptions on $(\boldsymbol{\theta}_a, \boldsymbol{\theta}_b, \boldsymbol{\theta}_p, \boldsymbol{\theta}_q)$:

$$
\begin{aligned}
\sigma^2_{x_a} &\sim \mathcal{IG}\left(a_{x_a}, b_{x_a}\right) \ , a_{x_a} = b_{x_a} = 0.001 \ , \\
\sigma^2_{x_b} &\sim \mathcal{IG}\left(a_{x_b}, b_{x_b}\right) \ , a_{x_b} = b_{x_b} = 0.001 \ , \\
\sigma^2_{x_p} &\sim \mathcal{IG}\left(a_{x_p}, b_{x_p}\right) \ , a_{x_p} = b_{x_p} = 0.001 \ , \\
\sigma^2_{x_q} &\sim \mathcal{IG}\left(a_{x_q}, b_{x_q}\right) \ , a_{x_q} = b_{x_q} = 0.001 \ , \\
\left(\phi_a, \boldsymbol{\beta}_{\boldsymbol{Z_a}}\right) &\sim \mathcal{N}\left(0, \boldsymbol{I}_a\right) \ , \\
\left(\phi_b, \boldsymbol{\beta}_{\boldsymbol{Z_b}}\right) &\sim \mathcal{N}\left(0, \boldsymbol{I}_b\right) \ , \\
\left(\phi_p, \boldsymbol{\beta}_{\boldsymbol{Z_p}}\right) &\sim \mathcal{N}\left(0, \boldsymbol{I}_p\right) \ , \\
\left(\phi_q, \boldsymbol{\beta}_{\boldsymbol{Z_q}}\right) &\sim \mathcal{N}\left(0, \boldsymbol{I}_q\right) \ ,
\end{aligned}
$$

where $\boldsymbol{I}_a, \boldsymbol{I}_b, \boldsymbol{I}_p, \boldsymbol{I}_q$ are identity matrices of appropriate dimension. Alternatively, instead of $\boldsymbol{I}_a$, one can model the variances of the parameters explicitly i.e. using $\sigma^2_{\boldsymbol{\beta}_{\boldsymbol{Z}_a}, \phi_a} \times \boldsymbol{I}$ with an additional (hierarchical) $\mathcal{IG}\left(,\right)$-prior on $\sigma^2_{\boldsymbol{\beta}_{\boldsymbol{Z}_a}, \phi_a}$ (and similarly for $\boldsymbol{I}_b, \boldsymbol{I}_p, \boldsymbol{I}_q$).
Let $x_{2:T}$ be a generic state proces for some fixed $i = 1, \ldots, N$ i.e. either $\boldsymbol{x}_{a,2:T}, \boldsymbol{x}_{b,2:T}, \boldsymbol{x}_{p,2:T}$ or $\boldsymbol{x}_{q,2:T}$. Note that the model can be written in matrix form as:

$$
\begin{aligned}
x_{2:T} &= x_{1:T-1}\phi + \boldsymbol{z}_{2:T}\boldsymbol{\beta}_{\boldsymbol{Z}} + \boldsymbol{\varepsilon}_{x,2:T} \ , \\
x_{2:T} &= \boldsymbol{Z}_{2:T} \times \left(\phi, \boldsymbol{\beta}'_{\boldsymbol{Z}}\right)' + \boldsymbol{\varepsilon}_{x,2:T} \ ,
\end{aligned}
$$

where $\boldsymbol{Z}_{2:T}$ is a matrix containing as first column $x_{1:T-1}$ and the remaining K'regressors in $\boldsymbol{z}_{2:T}$. This makes it easier to calculate the Gibbs block for ϕ in the next sections.

1.2 Gibbs-Part: univariate (for one $i = 1, \dots, N$)

We derive $p(\boldsymbol{\theta}|x_{0:T}, \boldsymbol{y}_{1:T})$ for a particular $x_t \in \{x_{a,t,i}, x_{b,t,i}, x_{p,t,i}, x_{q,t,i}\}$ i.e. fixing the cross sectional unit i and picking one of the four GB2 parameters. The full probabilistic model with $\boldsymbol{\theta} = (\sigma_X^2, \phi, \boldsymbol{\beta_Z})$ can then be factorized according to

$$
\begin{aligned}
p(\boldsymbol{\theta}, x_{0:T}, \boldsymbol{y}_{1:T}) = p(\boldsymbol{y}_{1:T}|\boldsymbol{\theta}, x_{0:T})\, p(x_{0:T}, \boldsymbol{\theta}) &= \prod_{t=1}^{T} p(\boldsymbol{y}_t|x_t, \boldsymbol{\theta}) \prod_{t=1}^{T} p(x_t|x_{t-1}, \boldsymbol{\theta})\, p(x_0|\boldsymbol{\theta})\, p(\boldsymbol{\theta}) \ , \\
&= \prod_{t=1}^{T} (\boldsymbol{y}_t|\boldsymbol{\theta}, x_t) \\
&\times \frac{1}{(2\pi\sigma_X^2)^{T/2}} \prod_{t=1}^{T} \exp\left(-\frac{(x_t - \phi x_{t-1} - \boldsymbol{z}_t'\boldsymbol{\beta_Z})^2}{2\sigma_X^2}\right) \\
&\times p(\boldsymbol{\theta}) \ .
\end{aligned}
$$

Then, the conditional parameter distributions are conjugate and given as

$$
\begin{aligned}
p\left(\sigma_X^2|x_{0:T}, \boldsymbol{y}_{1:T}\right) &= \frac{1}{(2\pi\sigma_X^2)^{T/2}} \prod_{t=1}^{T} \exp\left(-\frac{(x_t - \phi x_{t-1} - \boldsymbol{z}_t'\boldsymbol{\beta_Z})^2}{2\sigma_X^2}\right) \\
&\times \frac{b_X^{a_X}}{\Gamma(a_X)} \left(\sigma_X^2\right)^{-a_X-1} \exp\left(-\frac{b_X}{\sigma_X^2}\right) \\
&\propto \left(\sigma_X^2\right)^{-(a_X+T/2)-1} \times \exp\left(-\frac{1}{\sigma_X^2}\left(b_X + \frac{\sum_{t=1}^{T}(x_t - \phi x_{t-1} - \boldsymbol{z}_t'\boldsymbol{\beta_Z})^2}{2}\right)\right) \ .
\end{aligned}
$$

With e.g. $a_X = b_X = 0.001$, we have

$$
\sigma_X^2 \sim \mathcal{IG}(a_X^*, b_X^*) \ , \ a_X^* = a_X + T/2 \ , \ b_X^* = b_X + \frac{\sum_{t=1}^{T}(x_t - \phi x_{t-1} - \boldsymbol{z}_t'\boldsymbol{\beta_Z})^2}{2} \ .
$$

For $\boldsymbol{\beta}_{\boldsymbol{Z}}^* = (\phi, \boldsymbol{\beta}_{\boldsymbol{Z}}')'$ with a normal prior $\boldsymbol{\beta}_{\boldsymbol{Z}}^* \sim \mathcal{N}_{K+1}\left(\underline{\boldsymbol{\beta}_{\boldsymbol{Z}}^*}, \underline{\boldsymbol{\Omega}_Z}\right)$ and the previous $x_{2:T} = \boldsymbol{Z}_{2:T}\boldsymbol{\beta}_{\boldsymbol{Z}}^* + \boldsymbol{\varepsilon}_{x,2:T}$, we have

$$
\begin{aligned}
p\left(\boldsymbol{\beta}_{\boldsymbol{Z}}^*|x_{2:T}, \boldsymbol{Z}_{2:T}, \sigma_X^2\right) \propto \exp&\left\{-\frac{1}{2}(x_{2:T} - \boldsymbol{Z}_{2:T}\boldsymbol{\beta}_{\boldsymbol{Z}}^*)' \boldsymbol{\Omega}_{\varepsilon_{x,2:T}}^{-1} (x_{2:T} - \boldsymbol{Z}_{2:T}\boldsymbol{\beta}_{\boldsymbol{Z}}^*)\right\} \\
\times \exp&\left\{-\frac{1}{2}\left(\boldsymbol{\beta_Z} - \underline{\boldsymbol{\beta}_{\boldsymbol{Z}}^*}\right)' \underline{\boldsymbol{\Omega}_Z}^{-1} \left(\boldsymbol{\beta_Z} - \underline{\boldsymbol{\beta}_{\boldsymbol{Z}}^*}\right)\right\}
\end{aligned}
$$

Because we have

$$
\begin{aligned}
\left(x_{2:T} - \boldsymbol{Z}_{2:T}\boldsymbol{\beta}^*_{\boldsymbol{Z}}\right)' \boldsymbol{\Omega}^{-1}_{\varepsilon_x,2:T} \left(x_{2:T} - \boldsymbol{Z}_{2:T}\boldsymbol{\beta}^*_{\boldsymbol{Z}}\right) &= \boldsymbol{\beta}^{*\prime}_{\boldsymbol{Z}}\boldsymbol{Z}'_{2:T}\boldsymbol{\Omega}^{-1}_{\varepsilon_x,2:T}\boldsymbol{Z}_{2:T}\boldsymbol{\beta}^*_{\boldsymbol{Z}} - 2\boldsymbol{\beta}^{*\prime}_{\boldsymbol{Z}}\boldsymbol{Z}'_{2:T}\boldsymbol{\Omega}^{-1}_{\varepsilon_x,2:T}x_{2:T} \\
&\quad + x'_{2:T}\boldsymbol{\Omega}^{-1}_{\varepsilon_x,2:T}x_{2:T} \\
\left(\boldsymbol{\beta}^*_{\boldsymbol{Z}} - \underline{\boldsymbol{\beta}^*_{\boldsymbol{Z}}}\right)' \underline{\boldsymbol{\Omega}_Z}^{-1} \left(\boldsymbol{\beta}^*_{\boldsymbol{Z}} - \underline{\boldsymbol{\beta}^*_{\boldsymbol{Z}}}\right) &= \boldsymbol{\beta}^{*\prime}_{\boldsymbol{Z}}\underline{\boldsymbol{\Omega}_Z}^{-1}\boldsymbol{\beta}^*_{\boldsymbol{Z}} - 2\boldsymbol{\beta}^{*\prime}_{\boldsymbol{Z}}\underline{\boldsymbol{\Omega}_Z}^{-1}\underline{\boldsymbol{\beta}^*_{\boldsymbol{Z}}} \\
&\quad + \underline{\boldsymbol{\beta}^{*\prime}_{\boldsymbol{Z}}} * \underline{\boldsymbol{\Omega}_Z}^{-1}\underline{\boldsymbol{\beta}^*_{\boldsymbol{Z}}}
\end{aligned}
$$

we obtain

$$
\begin{aligned}
p\left(\boldsymbol{\beta}^*_{\boldsymbol{Z}} | x_{2:T}, \boldsymbol{Z}_{2:T}, \sigma^2_X\right) &= \mathcal{N}_{K+1}\left(\overline{\boldsymbol{\beta}_{\boldsymbol{Z}}}, \overline{\boldsymbol{\Omega}_Z}\right) \\
\overline{\boldsymbol{\Omega}_Z} &= \left[\boldsymbol{Z}'_{2:T}\boldsymbol{\Omega}^{-1}_{\varepsilon_x,2:T}\boldsymbol{Z}_{2:T} + \underline{\boldsymbol{\Omega}_Z}^{-1}\right]^{-1} \\
\overline{\boldsymbol{\beta}_{\boldsymbol{Z}}} &= \overline{\boldsymbol{\Omega}_Z} \times \left[\boldsymbol{Z}'_{2:T}\boldsymbol{\Omega}^{-1}_{\varepsilon_x,2:T}x_{2:T} + \underline{\boldsymbol{\Omega}_Z}^{-1}\boldsymbol{\beta}_{\boldsymbol{Z}}\right]
\end{aligned}
$$

In Detail:

1.3 Gibbs-Part: multivariate (full cross section $\forall i = 1, \ldots, N$)

We now consider vector valued processes stacked along the cross sectional dimension as e.g. $x_{t,1:N}, \boldsymbol{y}_{t,1:N}$. All the corresponding state transition and measurement equations factorize along the time dimension and given as

$$
\begin{aligned}
p_{\boldsymbol{\theta}_a}\left(x_{a,t,1:N} | x_{a,t-1,1:N}, \boldsymbol{z}_{a,t,1:N}\right) &= \mathcal{N}_{1:N}\left(x_{a,t,1:N} \mid \phi_a x_{a,t-1,1:N} + \boldsymbol{z}_{a,t,1:N}\boldsymbol{\beta}_{\boldsymbol{Z_a}}, \sigma^2_{x_a}\boldsymbol{I}_N\right) \ , \\
p_{\boldsymbol{\theta}_b}\left(x_{b,t,1:N} | x_{b,t-1,1:N}, \boldsymbol{z}_{b,t,1:N}\right) &= \mathcal{N}_{1:N}\left(x_{b,t,1:N} \mid \phi_b x_{b,t-1,1:N} + \boldsymbol{z}_{b,t,1:N}\boldsymbol{\beta}_{\boldsymbol{Z_b}}, \sigma^2_{x_b}\boldsymbol{I}_N\right) \ , \\
p_{\boldsymbol{\theta}_p}\left(x_{p,t,1:N} | x_{p,t-1,1:N}, \boldsymbol{z}_{p,t,1:N}\right) &= \mathcal{N}_{1:N}\left(x_{p,t,1:N} \mid \phi_p x_{p,t-1,1:N} + \boldsymbol{z}_{p,t,1:N}\boldsymbol{\beta}_{\boldsymbol{Z_p}}, \sigma^2_{x_p}\boldsymbol{I}_N\right) \ , \\
p_{\boldsymbol{\theta}_q}\left(x_{q,t,1:N} | x_{q,t-1,1:N}, \boldsymbol{z}_{q,t,1:N}\right) &= \mathcal{N}_{1:N}\left(x_{q,t,1:N} \mid \phi_q x_{q,t-1,1:N} + \boldsymbol{z}_{q,t,1:N}\boldsymbol{\beta}_{\boldsymbol{Z_q}}, \sigma^2_{x_q}\boldsymbol{I}_N\right) \ , \\
p\left(\boldsymbol{y}_{t,i} | x_{a,t,i}, x_{b,t,i}, x_{p,t,i}, x_{q,t,i}\right) &= \mathcal{MNL}\left(\boldsymbol{y}_{t,i} | \pi_{it}^{(1)}, \ldots, \pi_{it}^{(M_{it})}\right) \ ,
\end{aligned}
$$

with $\pi_{it}^{(k)} = \left(F_{\text{GB2}}\left(c_{it}^{(k)}; x_{a,t,i}, x_{b,t,i}, x_{p,t,i}, x_{q,t,i}\right) - F_{\text{GB2}}\left(c_{it}^{(k-1)}; x_{a,t,i}, x_{b,t,i}, x_{p,t,i}, x_{q,t,i}\right)\right)$ and income distribution function as a four-parameter GB2

$$
F(c_{it}^{(k)}; x_{a,t,i}, x_{b,t,i}, x_{p,t,i}, x_{q,t,i}) = B(d_{t,i}^{(k)}; x_{p,t,i}, x_{q,t,i}) = \frac{\int_0^{d_{t,i}^{(k)}} t^{\exp(x_{p,t,i})-1}(1-t)^{\exp(x_{q,t,i})-1} dt}{\mathrm{B}\left(\exp\left(x_{p,t,i}\right), \exp\left(x_{p,t,i}\right)\right)} \ ,
$$

$$
d_{t,i}^{(k)} = \frac{(c_{it}^{(k)} / \exp\left(x_{b,t,i}\right))^{\exp(x_{a,t,i})}}{1 + (c_{it}^{(k)} / \exp\left(x_{b,t,i}\right))^{\exp(x_{a,t,i})}} \ .
$$

However, as all state transitions share the same structure, we derive them for a particular $x_{t,1:N} \in \{x_{a,t,1:N}, x_{b,t,1:N}, x_{p,t,1:N}, x_{q,t,1:N}\}$. Now, to obtain $p\left(\boldsymbol{\theta} | x_{0:T,1:N}, \boldsymbol{y}_{0:T,1:N}\right)$, consider the full probabilistic model with $\boldsymbol{\theta} = (\sigma_X^2, \phi, \boldsymbol{\beta}_{\boldsymbol{Z}})$ as

$$
\begin{aligned}
p\left(\boldsymbol{\theta}, x_{0:T,1:N}, \boldsymbol{y}_{0:T,1:N}\right) &= p\left(\boldsymbol{y}_{0:T,1:N} | \boldsymbol{\theta}, x_{0:T,1:N}\right) p\left(x_{0:T,1:N}, \boldsymbol{\theta}\right) \\
&= \prod_{t=1}^{T} p\left(\boldsymbol{y}_{t,1:N} | x_{t,1:N}, \boldsymbol{\theta}\right) \prod_{t=1}^{T} p\left(x_{t,1:N} | x_{t-1,1:N}, \boldsymbol{\theta}\right) p\left(x_{0,1:N} | \boldsymbol{\theta}\right) p\left(\boldsymbol{\theta}\right) \\
&= \prod_{t=1}^{T} p\left(\boldsymbol{y}_{t,1:N} | x_{t,1:N}\right) \times \prod_{t=1}^{T} \frac{1}{(2\pi)^{N/2}\left(\det\left(\sigma_X^2 \boldsymbol{I}_N\right)\right)^{1/2}} \\
&\times \prod_{t=1}^{T} \exp\left(-\frac{1}{2\sigma_X^2}\left(x_{t,1:N} - \phi x_{t-1,1:N} - \boldsymbol{z}_{t,1:N}\boldsymbol{\beta}_{\boldsymbol{Z}}\right)'\left(x_{t,1:N} - \phi x_{t-1,1:N} - \boldsymbol{z}_{t,1:N}\boldsymbol{\beta}_{\boldsymbol{Z}}\right)\right) \\
&\times p\left(\boldsymbol{\theta}\right) \\
&= \prod_{t=1}^{T} p\left(\boldsymbol{y}_{t,1:N} | x_{t,1:N}\right) \times \left(2\pi\sigma_X^2\right)^{-NT/2} \\
&\times \exp\left(-\frac{1}{2\sigma_X^2}\sum_{t=1}^{T}\left(x_{t,1:N} - \boldsymbol{\mu}_{x,t}\right)'\left(x_{t,1:N} - \boldsymbol{\mu}_{x,t}\right)\right) \\
&\times p\left(\boldsymbol{\theta}\right) \ , \ \boldsymbol{\mu}_{x,t} = \phi x_{t-1,1:N} + \boldsymbol{z}_{t,1:N}\boldsymbol{\beta}_{\boldsymbol{Z}} \ .
\end{aligned}
$$

Then, the conditional parameter distributions are conjugate and given as

$$p\left(\sigma_X^2|x_{0:T,1:N}\right) = \left(2\pi\sigma_X^2\right)^{-NT/2} \times \exp\left(-\frac{1}{2\sigma_X^2}\sum_{t=1}^{T}\left(x_{t,1:N}-\boldsymbol{\mu}_{x,t}\right)'\left(x_{t,1:N}-\boldsymbol{\mu}_{x,t}\right)\right)$$
$$\times \frac{\underline{b_X}^{\underline{a_X}}}{\Gamma\left(\underline{a_X}\right)}\left(\sigma_X^2\right)^{-\underline{a_X}-1}\exp\left(-\frac{\underline{b_X}}{\sigma_X^2}\right)$$

$$\propto \left(\sigma_X^2\right)^{-(\underline{a_X}+NT/2)-1} \times \exp\left(-\frac{1}{\sigma_X^2}\left(\underline{b_X}+\frac{\sum_{t=1}^{T}\left(x_{t,1:N}-\boldsymbol{\mu}_{x,t}\right)'\left(x_{t,1:N}-\boldsymbol{\mu}_{x,t}\right)}{2}\right)\right) .$$

With e.g. $\underline{a_X} = \underline{b_X} = 0.001$, we have

$$\sigma_X^2|x_{0:T,1:N} \sim \mathcal{IG}\left(\overline{a_X}, b_X^*\right)\ ,\ \overline{a_X} = \underline{a_X} + NT/2\ ,\ \overline{b_X} = \underline{b_X} + \frac{\sum_{t=1}^{T}\left(x_{t,1:N}-\boldsymbol{\mu}_{x,t}\right)'\left(x_{t,1:N}-\boldsymbol{\mu}_{x,t}\right)}{2} .$$

For $\boldsymbol{\beta}_{\boldsymbol{Z}}^* = (\phi, \boldsymbol{\beta}_{\boldsymbol{Z}}')'$ with a normal prior $\boldsymbol{\beta}_{\boldsymbol{Z}}^* \sim \mathcal{N}_{K+1}\left(\underline{\boldsymbol{\beta}_{\boldsymbol{Z}}^*}, \underline{\boldsymbol{\Omega}_Z}\right)$ and $x_{t,1:N} = \phi x_{t-1,1:N} + \boldsymbol{z}_{t,1:N}\boldsymbol{\beta}_{\boldsymbol{Z}} + \boldsymbol{\varepsilon}_{x_{t,1:N}} = \boldsymbol{Z}_{t,1:N}\boldsymbol{\beta}_{\boldsymbol{Z}}^* + \boldsymbol{\varepsilon}_{x_{t,1:N}}$, we have

$$p\left(\boldsymbol{\beta}_{\boldsymbol{Z}}^*|x_{0:T,1:N}, \boldsymbol{Z}_{0:T,1:N}, \sigma_X^2\right) \propto \exp\left\{-\frac{1}{2}\sum_{t=1}^{T}\left(x_{t,1:N}-\boldsymbol{Z}_{t,1:N}\boldsymbol{\beta}_{\boldsymbol{Z}}^*\right)'\boldsymbol{\Omega}_{\boldsymbol{\varepsilon}_{x_{t,1:N}}}^{-1}\left(x_{t,1:N}-\boldsymbol{Z}_{t,1:N}\boldsymbol{\beta}_{\boldsymbol{Z}}^*\right)\right\}$$
$$\times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}_{\boldsymbol{Z}}^*-\underline{\boldsymbol{\beta}_{\boldsymbol{Z}}^*}\right)'\underline{\boldsymbol{\Omega}_Z}^{-1}\left(\boldsymbol{\beta}_{\boldsymbol{Z}}^*-\underline{\boldsymbol{\beta}_{\boldsymbol{Z}}^*}\right)\right\}\ ,$$

which can as a whole expression be written as

$$\exp\left\{-\frac{1}{2}\left[\sum_{t=1}^{T}\left(x_{t,1:N}-\boldsymbol{Z}_{t,1:N}\boldsymbol{\beta}_{\boldsymbol{Z}}^*\right)'\boldsymbol{\Omega}_{\boldsymbol{\varepsilon}_{x_{t,1:N}}}^{-1}\left(x_{t,1:N}-\boldsymbol{Z}_{t,1:N}\boldsymbol{\beta}_{\boldsymbol{Z}}^*\right)+\left(\boldsymbol{\beta}_{\boldsymbol{Z}}^*-\underline{\boldsymbol{\beta}_{\boldsymbol{Z}}^*}\right)'\underline{\boldsymbol{\Omega}_Z}^{-1}\left(\boldsymbol{\beta}_{\boldsymbol{Z}}^*-\underline{\boldsymbol{\beta}_{\boldsymbol{Z}}^*}\right)\right]\right\}$$

Because we have for every $t = 1, \ldots, T$

$$\left(x_{t,1:N}-\boldsymbol{Z}_{t,1:N}\boldsymbol{\beta}_{\boldsymbol{Z}}^*\right)'\boldsymbol{\Omega}_{\boldsymbol{\varepsilon}_{x_{t,1:N}}}^{-1}\left(x_{t,1:N}-\boldsymbol{Z}_{t,1:N}\boldsymbol{\beta}_{\boldsymbol{Z}}^*\right) = \boldsymbol{\beta}_{\boldsymbol{Z}}^{*\prime}\boldsymbol{Z}_{t,1:N}'\boldsymbol{\Omega}_{\boldsymbol{\varepsilon}_{x_{t,1:N}}}^{-1}\boldsymbol{Z}_{t,1:N}\boldsymbol{\beta}_{\boldsymbol{Z}}^* - 2\boldsymbol{\beta}_{\boldsymbol{Z}}^{*\prime}\boldsymbol{Z}_{t,1:N}'\boldsymbol{\Omega}_{\boldsymbol{\varepsilon}_{x_{t,1:N}}}^{-1}x_{t,1:N}$$
$$+ x_{t,1:N}'\boldsymbol{\Omega}_{\boldsymbol{\varepsilon}_{x_{t,1:N}}}^{-1}x_{t,1:N}$$

and prior $\left(\boldsymbol{\beta}^*_{\boldsymbol{Z}} - \underline{\boldsymbol{\beta}^*_{\boldsymbol{Z}}}\right)' \underline{\boldsymbol{\Omega}_Z}^{-1} \left(\boldsymbol{\beta}^*_{\boldsymbol{Z}} - \underline{\boldsymbol{\beta}^*_{\boldsymbol{Z}}}\right) = \boldsymbol{\beta}^{*\prime}_{\boldsymbol{Z}} \underline{\boldsymbol{\Omega}_Z}^{-1} \boldsymbol{\beta}^*_{\boldsymbol{Z}} - 2\boldsymbol{\beta}^{*\prime}_{\boldsymbol{Z}} \underline{\boldsymbol{\Omega}_Z}^{-1} \underline{\boldsymbol{\beta}^*_{\boldsymbol{Z}}} + \underline{\boldsymbol{\beta}^{*\prime}_{\boldsymbol{Z}}} * \underline{\boldsymbol{\Omega}_Z}^{-1} \underline{\boldsymbol{\beta}^*_{\boldsymbol{Z}}}$ we obtain

$$
\begin{aligned}
p\left(\boldsymbol{\beta}^*_{\boldsymbol{Z}} | x_{0:T,1:N}, \boldsymbol{Z}_{0:T,1:N}, \sigma^2_X\right) = \mathcal{N}_{K+1}\left(\overline{\boldsymbol{\beta}_{\boldsymbol{Z}}}, \overline{\boldsymbol{\Omega}_Z}\right) \\
\overline{\boldsymbol{\Omega}_Z} &= \left[\sum_{t=1}^{T} \boldsymbol{Z}'_{t,1:N} \boldsymbol{\Omega}^{-1}_{\varepsilon_{x_{t,1:N}}} \boldsymbol{Z}_{t,1:N} + \underline{\boldsymbol{\Omega}_Z}^{-1}\right]^{-1} \\
\overline{\boldsymbol{\beta}_{\boldsymbol{Z}}} &= \overline{\boldsymbol{\Omega}_Z} \times \left[\sum_{t=1}^{T} \boldsymbol{Z}'_{t,1:N} \boldsymbol{\Omega}^{-1}_{\varepsilon_{x_{t,1:N}}} x_{t,1:N} + \underline{\boldsymbol{\Omega}_Z}^{-1} \boldsymbol{\beta}_{\boldsymbol{Z}}\right]
\end{aligned}
$$

In Detail:

In Detail:

ALGORITHM: *Conditional BPF*

START **I. Initiliaze** $(t = 0)$ **:**

For $i = 1, \ldots, N$:

1. Sample $x_0^i \sim p(x_0)$
3. Set $w_0^i = \frac{1}{N}$
2. Set $x_0^N = x_0^{\mathcal{R}}$ (conditioning)

II. For $t = 1$ to T:

For $i = 1, \ldots, N$:

1. Draw $a_t^i \sim \mathcal{C}\left(\{w_t^i\}_{j=1}^N\right)$
2. Sample $x_t^i \sim q(x_t|x_{t-1}^{a_t^i}, \boldsymbol{\theta}) \underbrace{=}_{\text{BPF}} p(x_t|x_{t-1}^{a_t^i}, \boldsymbol{\theta})$

For $i = N$:

3. Set $x_t^N = x_t^{\mathcal{R}}$ (conditioning)
4. Sample $a_t^N \in \{1, \ldots, N\}$ with probability

$$\mathbb{P}\left(a_t^N = i \propto w_{t-1}^i p\left(x_t^{\mathcal{R}}|x_{t-1}^i, \boldsymbol{\theta}\right)\right) \ , \quad \text{(AS-step)}$$

For $i = 1, \ldots, N$:

5. Set $\tilde{w}_t^i = p\left(y_t|x_t^i\right)$
6. Normalize weights $w_t^i = \frac{\tilde{w}_t^i}{\sum_{t=1}^T \tilde{w}_t^i}$

III. For t=T:

1. Draw $b \sim \mathcal{C}\left(\{w_T^i\}_{i=1}^N\right)$ and compute $x_{0:T}^b$

END *Output* $x_{0:T}^{\mathcal{R}} = x_{0:T}^b$

ALGORITHM: *PGAS with conditional BPF*

START **Initiliaze (m=1):**

1. Set $x_{0:T}[1]$ and $\boldsymbol{\theta}[1]$ arbitrarily

For $m = 1, \ldots, N$:

2. Draw $\boldsymbol{\theta}[m] \sim p\left(\boldsymbol{\theta}|x_{0:T}[m-1], x_{0:T}^{\mathcal{R}}\right)$
3. Draw $x_{0:T}[m] \sim \kappa_{N,\boldsymbol{\theta}[m]}\left(x_{0:T}[m-1], x_{0:T}^{\mathcal{R}}\right)$

END *Output* $\boldsymbol{\theta}[1:m]$ and $x_{0:T}[1:m]$