$X_1, \dots, X_n \sim iid \quad N(\mu, \sigma^2)$

identically
independently
distributed

Метод моментов

$\mathbb{E}[X_i] = \mu = \overline{X} \Rightarrow \hat{\mu} = \overline{X}$

$\mathbb{E}[X_i^2] = \mu^2 + \sigma^2 = \overline{X^2} \Rightarrow \hat{\sigma}^2 = \overline{X^2} - (\overline{X})^2$

Несмещенность

$$\mathbb{E}[\hat{\mu}] = \mathbb{E}[\overline{X}] = \mathbb{E}\left[\frac{1}{n}(X_1 + X_2 + \dots + X_n)\right] =$$

$$= \frac{1}{n}\Big(\underbrace{\mathbb{E}[X_1]}_{\mu} + \underbrace{\mathbb{E}[X_2]}_{\mu} + \dots + \underbrace{\mathbb{E}[X_n]}_{\mu}\Big) =$$

$$= \frac{1}{n} \cdot (n\mu) = \mu \Rightarrow \hat{\mu} = \overline{X} \text{ – несмещенная оценка}$$

$$\mathbb{E}[\hat{\sigma}^2] = \mathbb{E}[\overset{\text{I}}{\overline{X^2}} - \overset{\text{II}}{(\overline{X})^2}] = \mathbb{E}[\overline{X^2}] - \mathbb{E}[(\overline{X})^2]$$

I $\quad \mathbb{E}[\overline{X^2}] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i^2\right] = \frac{1}{n}\sum_{i=1}^{n} \underbrace{\mathbb{E}[X_i^2]}_{\mu^2+\sigma^2} = \frac{1}{n} \cdot n(\mu + \sigma^2) = \mu^2 + \sigma^2$

II $\quad \mathbb{E}[(\overline{X})^2] = \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right] = \mathbb{E}\left[\frac{1}{n^2}\left(\sum_{i=1}^{n} X_i\right)^2\right] =$

$$= \frac{1}{n^2}\mathbb{E}[(X_1 + X_2 + \dots X_n)^2] =$$

$$= \frac{1}{n^2}\mathbb{E}\Big[\underbrace{X_1^2}_{\mu^2+\sigma^2} + \underbrace{X_2^2}_{\mu^2+\sigma^2} + \dots \underbrace{X_n^2}_{\mu^2+\sigma^2} + 2\underbrace{X_1 X_2}_{\mu^2} + 2\underbrace{X_1 \cdot X_3}_{\mu^2} + \dots + 2\underbrace{X_{n-1} \cdot X_n}_{\mu^2}\Big] =$$

$\mathbb{E}[X_i \cdot X_j] = \mathbb{E}[X_i]\,\mathbb{E}[X_j] = \mu \cdot \mu = \mu^2$

если $X_i \perp\!\!\!\perp X_j,\ i \neq j$

независимы

$$C_n^k = C_n^2 = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)}{2}$$ – кол-во пар

$$= \frac{1}{n^{\cancel{2}}}\left(\cancel{n}(\mu^2+\sigma^2) + \cancel{2}\cdot\frac{\cancel{n}(n-1)}{\cancel{2}}\cdot\mu^2\right) =$$

$$= \frac{1}{n}\left(\cancel{\mu^2}+\sigma^2 + n\cdot\mu^2 - \cancel{\mu^2}\right) = \frac{\sigma^2}{n} + \mu^2$$

$$\mathbb{E}[\hat{\sigma}^2] = (\cancel{\mu^2}+\sigma^2) - \left(\frac{\sigma^2}{n} + \cancel{\mu^2}\right) = \sigma^2 - \frac{\sigma^2}{n} = \sigma^2\left(1-\frac{1}{n}\right) = \sigma^2\cdot\frac{n-1}{n} \neq \sigma^2$$

$\Rightarrow$ $\hat{\sigma}^2 = \overline{X^2} - (\overline{X})^2$ – смещенная оценка σ^2

$$\lim_{n\to\infty}\frac{n-1}{n} = 1 \Rightarrow \lim_{n\to\infty}\sigma^2\cdot\frac{n-1}{n} = \sigma^2$$

$\Rightarrow$ $\hat{\sigma}^2$ – асимптотически несмещенная

$$\mathbb{E}[\hat{\sigma}^2] = \frac{n-1}{n}\cdot\sigma^2$$

$$\hat{S}^2 = \hat{\sigma}^2\cdot\frac{n}{n-1} \Rightarrow \mathbb{E}[S^2] = \mathbb{E}\left[\hat{\sigma}^2\cdot\frac{n}{n-1}\right] = \frac{\cancel{n-1}}{\cancel{n}}\cdot\sigma^2\,\frac{\cancel{n}}{\cancel{n-1}} = \sigma^2$$

$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2$ – (обычная (смещенная)) дисперсия

$$\hat{S}^2 \; \frac{n}{n-1}\hat{\sigma}^2 = \frac{\cancel{n}}{n-1}\,\frac{1}{\cancel{n}}\sum_{i=1}^{n}(X_i-\overline{X})^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2$$

– несмещенная выборочная дисперсия

ddof = delta degrees of freedom

в Python

Состоятельность

если $\mathbb{P}(|\hat{\theta} - \theta| > \varepsilon) \xrightarrow[n \to \infty]{} 0$,

то говорят, что $\hat{\theta} \xrightarrow[n \to \infty]{\mathbb{P}} \theta$

{ $\hat{\theta}$ сходится по вероятности }

Неравенство Чебышева:

$$\mathbb{P}(|X - \mathbb{E}[X]| > \varepsilon) \leq \frac{var[X]}{\varepsilon^2}$$

если $\mathbb{P}(|\hat{\theta} - \theta| > \varepsilon) \leq \frac{var(\hat{\theta})}{\varepsilon^2} \xrightarrow[n \to \infty]{} 0$, то $\hat{\theta}$ - состоятельная

$$\hat{\mu} = \bar{X}$$

$$var(\hat{\mu}) = var(\bar{X}) = var\left(\frac{1}{n}(X_1 + X_2 + \ldots + X_n)\right) =$$

$$= \frac{1}{n^2}\left(\underbrace{var(X_1)}_{\sigma^2} + \underbrace{var(X_2)}_{\sigma^2} + \ldots + \underbrace{var(X_n)}_{\sigma^2}\right) =$$

$$= \frac{1}{n^{\cancel{2}}}(\cancel{n} \cdot \sigma^2) = \frac{\sigma^2}{n}$$

$$\mathbb{P}(|\hat{\mu} - \mu| > \varepsilon) \leq \frac{var(\hat{\mu})}{\varepsilon^2} = \frac{\sigma^2}{n \cdot \varepsilon^2} \xrightarrow[n \to \infty]{} 0 \Rightarrow \hat{\mu} = \bar{X} -$$

– сост. оценка

Эффективность

$$\operatorname{var}(\hat{\theta}) \geqslant \frac{1}{n \cdot \mathcal{I}(\theta)}$$

$$\mathcal{I}(\theta) = \mathbb{E}\left[\left(\frac{\partial \ln f(x,\theta)}{\partial \theta}\right)^2\right] = -\mathbb{E}\left[\frac{\partial^2 \ln f(x,\theta)}{\partial \theta^2}\right]$$

$$f(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

$$\ln f(x,\mu,\sigma^2) = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}(x-\mu)^2$$

$$\ln(a \cdot b) = \ln(a) + \ln(b)$$
$$\ln(a^c) = c \cdot \ln(a)$$
$$\ln(a^c \cdot b^d) = c \cdot \ln(a) + d \cdot \ln(b)$$

$$\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}, \quad \sqrt{x} = x^{\frac{1}{2}}$$
$$\ln(e^a) = a$$
$$\exp(\ln(a)) = a$$

$$\frac{\partial}{\partial \mu} \ln f(x,\mu,\sigma^2) = \frac{-1}{2\sigma^2} \cdot 2 \cdot (x-\mu)^{\overbrace{2-1}^{=1}} \cdot (-1) = \frac{1}{\sigma^2}(x-\mu) = \frac{x}{\sigma^2} - \frac{\mu}{\sigma^2}$$

$$\frac{\partial^2}{\partial \mu^2} \ln f(x,\mu,\sigma^2) = -\frac{1}{\sigma^2}$$

$$\mathcal{I}(\mu) = -\mathbb{E}\left[-\frac{1}{\sigma^2}\right] = \frac{1}{\sigma^2}$$

$$\operatorname{var}(\mu) \geqslant \frac{1}{n \cdot \mathcal{I}(\mu)} = \frac{1}{n \cdot \frac{1}{\sigma^2}} = \boxed{\frac{\sigma^2}{n}} \equiv \operatorname{var}(\hat{\mu})$$

совпала с дисперсией выше

$\Rightarrow \hat{\mu} = \bar{x}$ – эффективная оценка в классе всех несмещенных

(у $\hat{\mu}$ – самая низкая дисперсия)