BSA: Homework Assignment 1

Ilia Ernston

November 2024

1 Calculus

1.1 Question 1

Let $f(x) = x^3 - 6x^2 + 9x - 4$.

1. Find $f'(x)$, the first derivative of $f(x)$.
2. Determine the critical points of $f(x)$ by solving $f'(x) = 0$.
3. Using the second derivative test, classify each critical point as a local maximum, minimum, or point of inflection.

Answer

1. $f'(x) = 3x^2 - 12x + 9$
2. $3x^2 - 12x + 9 = 0 \rightarrow 3(x^2 - 4x + 3) = 0 \rightarrow \left[\begin{array}{c} x = 1 \\ x = 3 \end{array}\right]$
3. $f''(x) = 6x - 12$
   $f''(x_1) = 6(1) - 12 = -6 < 0 \rightarrow x_1$ is local maximum
   $f''(x_2) = 6(3) - 12 = 6 > 0 \rightarrow x_2$ is local maximum

1.2 Question 2

Evaluate the following indefinite integral:

$$\int \frac{2x^3 - 5x + 3}{x^2} dx$$

Answer

$$\int \frac{2x^3 - 5x + 3}{x^2} dx =$$

$$= \int \frac{2x^3}{x^2} dx - \int \frac{5x}{x^2} dx + \int \frac{3}{x^2} dx = \int 2x dx - \int \frac{5}{x} dx + \int \frac{3}{x^2} dx =$$

$$= x^2 - 5 \ln |x| - \frac{3}{x} + C$$

1.3 Question 3

Compute the definite integral:

$$\int_0^3 (3x^2 - 4x + 2)dx$$

Then interpret the result as the area under the curve of $f(x) = 3x^2 - 4x + 2$ over the interval $[0, 3]$

Answer

$$\int_0^3 (3x^2 - 4x + 2)dx = \left[3\frac{x^3}{3} - 4\frac{x^2}{2} + 2x\right]_0^3 = (27 - 18 + 6) - (0) = 15$$

i.e. the area under the curve of $f(x) = 3x^2 - 4x + 2$ over the interval $[0, 3]$ equals 15.

1.4 Question 4

Let $g(x) = xe^x$.

1. Find $g'(x)$ using the product rule
2. Determine the integral $\int xe^x dx$ by using integration by part

Answer

1. $$g'(x) = \frac{d}{dx}(xe^x)$$

   applying the product rule:

   $$\frac{dx}{dx}(e^x) + \frac{d}{dx}(e^x)x = 1e^x + e^x x = e^x + e^x x$$

2. applying integration by parts to $xe^x dx$:

   $$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

1.5 Question 5

Suppose $h(x) = sin(x)e^{3x}$.

1. Calculate $h'(x)$.
2. Compute $\int sin(x)e^{3x} dx$ using integration by parts twice.

Answer

1. $$h'(x) = \frac{d}{dx} sin(x)e^{3x}$$

   applying the product rule:

   $$\frac{d}{dx}(\sin(x))e^{3x} + \frac{d}{dx}(e^{3x})\sin(x) = \cos(x)e^{3x} + e^{3x}3\sin(x)$$

2. applying integration by parts to $sin(x)e^{3x}dx$:

   $$\int sin(x)e^{3x}dx = -e^{3x}\cos(x) - \int -3e^{3x}\cos(x)dx = -e^{3x}\cos(x) - \left(-3\int e^{3x}\cos(x)dx\right)$$

   applying integration by pars to $e^{3x}\cos(x)dx$:

   $$\int e^{3x}\cos(x)dx = e^{3x}\sin(x) - \int e^{3x}3\sin(x)dx = e^{3x}\sin(x) - 3\int e^{3x}\sin(x)dx$$

   thus:

   $$\int e^{3x}\sin(x)dx = -e^{3x}\cos(x) + 3\left(e^{3x}\sin(x) - 3\int e^{3x}\sin(x)dx\right)$$

   isolate $e^{3x}\sin(x)\,dx$:

   $$\frac{3e^{3x}\sin(x)}{10} - \frac{e^{3x}\cos(x)}{10} = \frac{e^{3x}(3\sin(x) - \cos(x))}{10} + C$$

2 Linear Algebra

2.1 Question 6

Consider the following system of linear equations:

$$2x + 3y - z = 4$$

$$x - 2y + 4z = -3$$

$$3x + y + 2z = 5$$

1. Write the system in matrix form $AX = B$.
2. Find the determinant of A.
3. If the determinant is non-zero, use the inverse of A to solve for X.

Answer

1.
$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}$$

where:
$$\mathbf{A} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \\ 3 & 1 & 2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

2.
$$\det(\mathbf{A}) = 2 \cdot \begin{vmatrix} -2 & 4 \\ 1 & 2 \end{vmatrix} - 3 \cdot \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} =$$
$$= 2(-2 * 2) - 2(4 * 1) - 3(1 * 2) + 3(4 * 3) - (1 * 1) + (-2 * 3) = 7$$

3. $\det(\mathbf{A}) = 7 \neq 0 \rightarrow \mathbf{A}$ is invertible and $X = \mathbf{A}^{-1}\mathbf{B}$.

$$\mathbf{A}^{-1} = \left[\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ 1 & -2 & 4 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1.14 & -1 & 1.43 & 1 & 0 & 0 \\ 1.43 & 1 & 1.29 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$X = \begin{bmatrix} 1.14 & -1 & 1.43 \\ 1.43 & 1 & 1.29 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5.57 \\ -3.71 \\ -4 \end{bmatrix}$$

2.2 Question 7

Given the matrices:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 4 & 1 \\ -2 & 5 \end{bmatrix}$$

1. Compute $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} - \mathbf{B}$
2. Find the product $\mathbf{AB}$ and $\mathbf{BA}$
3. Is $\mathbf{AB} = \mathbf{BA}$? Justify your answer.

Answer

1.
$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 2+4 & -1+1 \\ 0-2 & 3+5 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ -2 & 8 \end{bmatrix}$$
$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 2-4 & -1-1 \\ 0+2 & 3-5 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix}$$

2.

$$\mathbf{AB} = \begin{bmatrix} 2*4-1(-2) & 2*1-1*5 \\ 0*4+3(-2) & 0*1+3*5 \end{bmatrix} = \begin{bmatrix} 10 & -3 \\ -6 & 15 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} 8 & -1 \\ -4 & 17 \end{bmatrix}$$

3. Avobe results prove that $\mathbf{AB} \neq \mathbf{BA}$. This might be expected since matrix multiplication is non commutative in general case.

2.3 Question 8

Let $\mathbf{C}$ be a 3×3 matrix:

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 2 & 1 & 0 \end{bmatrix}$$

1. Find the transpose of $\mathbf{C}$, denoted $\mathbf{C}^T$
2. Calculate $\mathbf{C} + \mathbf{C}^T$
3. Determine if $\mathbf{C} + \mathbf{C}^T$ is symmetric.

Answer

1.

$$\mathbf{C}^T = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 1 \\ 3 & 4 & 0 \end{bmatrix}$$

2.

$$\mathbf{C} + \mathbf{C}^T = \begin{bmatrix} 1+1 & 2+0 & 3+2 \\ 0+2 & -1-1 & 4+1 \\ 2+3 & 1+4 & 0+0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 2 & -2 & 5 \\ 5 & 5 & 0 \end{bmatrix}$$

3. $\mathbf{C} + \mathbf{C}^T = (\mathbf{C} + \mathbf{C}^T)^T \rightarrow \mathbf{C} + \mathbf{C}^T$ is symmetric

2.4 Question 9

A matrix $\mathbf{D}$ is defined as:

$$\mathbf{D} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

1. Describe the type of matrix $\mathbf{D}$ and explain its properties.
2. If $\mathbf{D}$ is a diagonal matrix, find its determinant.
3. Explain the significance of a diagonal matrix in terms of linear transformations.

Answer

1. $\mathbf{D}$ is a diagonal matrix, as all its non-diagonal elements are 0.

   Main properties of diagonal matrices:

   - Adding or multiplying diagonal matrices results in another diagonal matrix
   - A diagonal matrix is invertible if all diagonal entries are nonzero such that $\mathbf{D}^{-1}$ is also diagonal with entries $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.
   - The diagonal entries are also the eigenvalues of a diagonal matrix.
   - The eigenvectors of diagonal matrix correspond to the standard basis vectors in the respective dimensions.
   - For any $n \in \mathbb{Z}$ $\mathbf{D}^n$ is diagonal, with diagonal entries a^n, b^n, c^n.

2. The determinant of a diagonal matrix is the product of its diagonal entries:

$$\det(\mathbf{D}) = a * b * c$$

3. - A diagonal matrix represents scaling along the coordinate axes.
   - If $\mathbf{A}$ is diagonalazible, it can be expressed as an eigendecomposition $\mathbf{PDP}^{-1}$ where $\mathbf{D}$ is diagonal matrix with the eigenvalues of $\mathbf{A}$ as diagonal entries.

2.5 Question 10

Let $\mathbf{E}$ be a 2×2 matrix defined as:

$$\mathbf{E} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$$

1. Find the eigenvalues of $\mathbf{E}$
2. For each eigenvalue, find the corresponding eigenvector.
3. Explain how the eigenvalues and eigenvectors can be used to understand transformations represented by $\mathbf{E}$

Answer

1. The eigenvalues λ can be found by solving the characteristic polynomial $\det(\mathbf{E} - \lambda\mathbf{I}) = 0$:

$$\begin{vmatrix} 3-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(1-\lambda) - (2)(4) = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$\lambda_1 = 5$$

$$\lambda_2 = -1$$

2. For given eigenvalue λ eigenvector $\mathbf{v}$ is found by substitution λ into $(\mathbf{E} - \lambda\mathbf{I})\mathbf{v} = 0$.

   Thus for $\lambda_1 = 5$:

$$(\mathbf{E} - 5\mathbf{I})\mathbf{v} = 0$$

$$\begin{bmatrix} 3-5 & 4 \\ 2 & 1-5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow -2x + 4y = 0 \rightarrow x = 2y$$

   then $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$;

   for $\lambda_2 = -1$:

$$(\mathbf{E} + 1\mathbf{I})\mathbf{v} = 0$$

$$\begin{bmatrix} 3+1 & 4 \\ 2 & 1+1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow 4x + 4y = 0 \rightarrow x = -y$$

   then $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

3. - The eigenvalues represent scaling factors of the transformation $\mathbf{E}$.
   - The eigenvectors represent the directions that remain unchanged (except for scaling) under the transformation $\mathbf{E}$.

3 Probability Theory

3.1 Question 11

In a population, 30% of individuals have a specific genetic marker. A test for this marker has a 95% accuracy rate for individuals with the marker and an 80% accuracy rate for those without the marker.,

1. Define the events and write out the given probabilities.

2. Using Bayes' theorem, calculate the probability that an individual actually has the marker given a positive test result.

Answer Let

M define that individual has the genetic marker;
$\neg M$ - individual does not have the genetic marker;
T^+ - test result is positive;
T^- - test result is negative.
Then:

$$P(M) = 0.3 \rightarrow P(\neg M) = 1 - 0.3 = 0.7$$

$$P(T^+|M) = 0.95 \rightarrow P(T^-|M) = 1 - 0.95 = 0.05$$

$$P(T^-|\neg M) = 0.8 \rightarrow P(T^+|\neg M) = 1 - 0.8 = 0.2$$

according to Bayes' theorem:

$$P(M|T^+) = \frac{P(T^+|M)P(M)}{P(T^+)}$$

then total probability of a positive test result:

$$P(T^+) = P(T^+|M)P(M) + P(T^+|\neg M)P(\neg M) =$$
$$= (0.95)(0.3) + (0.2)(0.7) = 0.425$$

substituting to initial formula:

$$P(M|T^+) = \frac{(0.95)(0.3)}{0.425} \approx 0.6706 = 67.06\%$$

3.2 Question 12

A die is rolled 12 times. Let X be the random variable representing the number of times a "5" appears.

1. What type of probability distribution does X follow?
2. Calculate the probability that "5" appears exactly 4 times.
3. Determine the expected number of times "5" will appear in 12 rolls.

Answer

1. (a) Number of trials is fixed;
   (b) Each trial is independent;
   (c) There are only two outcomes: success (i.e. rolling a "5") or failure;
   (d) The probability of success is constant for each trial.

   Thus $X \sim$ Binomial.

2. If "5" appears exactly 4 times, than: $n = 12, p = \frac{1}{6}, k = 4$.
   Since probability is binomial, probability that "5" appears exactly 4 times is defined as:
   $$P(X = k) = \left(\begin{array}{c} n \\ k \end{array} \right) p^k (1-p)^{n-k}$$
   $$P(X = 4) = \left(\begin{array}{c} 12 \\ 4 \end{array} \right) \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^8 = \frac{12!}{4!(12-4)!} \left(\frac{1}{1296}\right) \left(\frac{390625}{1679616}\right) \approx$$
   $$\approx 0.0888 = 8.88\%$$

3. The expected value $\mathbb{E}[X]$ for a binomial random variable is given by:

$$\mathbb{E}[X] = np$$

Substituting parameters from given distribution:

$$\mathbb{E}[X] = 12\frac{1}{6} = 2$$

3.3 Question 13

A bookstore averages 3 customer visits per hour. Assume the number of customer visits per hour follows a Poisson distribution.

1. Calculate the probability that exactly 5 customers visit in a given hour.

2. Find the probability that at least 1 customer visits in a given hour.

Answer Given number of customers follows Poisson distribution, the probability mass function is given by:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

where λ is mean number of customers per hour, k is the number of customer visits.

1. For $\lambda = 3$ and $k = 5$:

$$P(X = 5) = \frac{3^5 e^{-3}}{5!} \approx \frac{243 * 0.0498}{120} \approx 0.1009 = 10.09\%$$

2. Probability of at least 1 customer is given by $P(X \geq 1)$. Since $k \in \mathbb{N}$, complement of $P(X \geq 1)$ is $P(X = 0)$.

$$P(X = 0) = \frac{3^0 e^{-3}}{0!} = \frac{1 * e^{-3}}{1} = e^{-3} \approx 0.0498$$

$$P(X \geq 1) = 1 - P(X = 0) = 0.9502 = 95.02\%$$

3.4 Question 14

Assume a continuous random variable Y follows a normal distribution with mean $\mu = 50$ and standard deviation $\sigma = 5$.

1. Calculate the probability that Y takes a value between 45 and 55.

2. Find the value y such that $P(Y \leq y) = 0.85$.

Answer

1. Probability can be evaluated for the standardized value Z, given by:

$$Z = \frac{Y - \mu}{\sigma}$$

For $Y = 45$ (given Y is normally distributed):

$$Z_45 = \frac{45 - 50}{5} = -1$$

For $Y = 55$:

$$Z_55 = \frac{55 - 50}{5} = 1$$

From standard normal tables:

$$P(Z \leq 1) \approx 0.8413, P(Z \leq -1) \approx 0.1587$$

$$P(Z_45 \leq Y \leq Z_55) = P(Z \leq 1) - P(Z \leq -1) = 0.8413 - 0.1587 =$$
$$= 0.6826 = 68.26\%$$

2. We nned to evaluate y for

$$P(Y \leq y) = 0.85$$

From the standard normal distribution table for $Y = 0.85$ $Z \approx 1.036$.

Since $Y = \mu + Z * \sigma$:

$$y = 50 + (1.036)(5) = 50 + 5.18 = 55.18$$

3.5 Question 15

A company's product quality control process involves two independent tests. Each test has a 98% probability of detecting a defect if it is present.

1. What is the probability that both tests detect a defect when it is present?
2. What is the probability that at least one test detects a defect when it is present?
3. If the company requires both tests to detect a defect before rejecting a product, calculate the probability that a defective product will be rejected.

Answer

1. Let:

   $P(A) = 0.98$ be the probability that the first test detects a defect;

   $P(B) = 0.98$ be the probability that the second test detects a defect.

   Since the tests are independent, the probability that both tests detect a defect is given by:

   $$P(A \cap B) = P(A) * P(B) = 0.98 * 0.98 = 0.9604 = 96.04\%$$

2. The complement of "at least one test detects a defect" is "neither test detects a defect". The probability that a test fails to detect a defect is:

   $$P(\neg A) = P(\neg B) = 1 - P(A) = 1 - P(B) = 1 - 0.98 = 0.02$$

   The probability that neither test detects a defect is:

   $$P(\neg A \cap \neg B) = P(\neg A) * P(\neg B) = 0.02 * 0.02 = 0.0004$$

   Thus, the probability that at least one test detects a defect is:

   $$P(\neg A \cup \neg B) = 1 - P(\neg A \cup \neg B) = 1 - 0.0004 = 0.9996 = 99.96\%$$

3. The company requires both tests to detect a defect before rejecting a product. This is the same as the probability that both tests detect a defect:

   $$P(A \cap B) = 0.9604 = 96.04\%$$