236200 – Signal, Image, and Data Processing
Homework 2

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Part I
Theory

1 k-term best approximation in L^2

a.

We fix a finite family of orthonormal functions $\beta_1, \dots, \beta_n \in F$ s.t. $F = \mathrm{Vec}(\beta_1, \dots, \beta_n)$ and let $k \in \{1, \dots, n\}$.

(a)

Let $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ a set of k increasing integers between 1 and n.

The k-term approximation of f in F using $\mathrm{Vec}(\beta_{i_1}, \dots, \beta_{i_k})$ is the *projection* into this subspace (by Pythagoras' theorem), meaning

$$\hat{f} = \sum_{j=1}^{k} \langle f, \beta_{i_j} \rangle \beta_{i_j}$$

where we define $\langle f, g \rangle = \int_{-\infty}^{\infty} fg$. The squared error is

$$\mathrm{MSE}(\hat{f}) = \int_{-\infty}^{\infty} f^2 - \sum_{i=1}^{k} f_{i_j}^2, \quad f_{i_j} = \langle f, \beta_{i_j} \rangle.$$

(b)

The best approximation is given taking the k coefficients whose absolute values are the biggest, with their corresponding functions (β_{i_j}'s); They might be more than one approximation, in case two coefficients among the biggest k are the same. The associated MSE is

$$\mathrm{MSE}(\hat{f}) = \int_{-\infty}^{\infty} f^2 - \sum_{j=1}^{n} f_{i_j}^2, \quad f_{i_j} = \langle f, \beta_{i_j} \rangle.$$

where $\{f_{i_j}\}_{j=1}^{k}$ are the biggest k coefficients.

b.

We consider and fix two families of orthonormal functions $\beta_1, \ldots, \beta_n \in F$ and $\tilde{\beta}_1, \ldots, \tilde{\beta}_n \in F$.

(a)

Looking at the MSE, we can check by which family the approximation is better: if

$$\sum_{i=1}^{n} \langle f, \beta_i \rangle^2 > \sum_{i=1}^{n} \langle f, \tilde{\beta}_i \rangle^2$$

it is better to represent by the β family (since $\text{MSE}^{\beta} < \text{MSE}^{\tilde{\beta}}$), and if

$$\sum_{i=1}^{n} \langle f, \beta_i \rangle^2 < \sum_{i=1}^{n} \langle f, \tilde{\beta}_i \rangle^2$$

it is better to represent by the $\tilde{\beta}$ family (since $\text{MSE}^{\tilde{\beta}} < \text{MSE}^{\beta}$). Note that if

$$\sum_{i=1}^{n} \langle f, \beta_i \rangle^2 = \sum_{i=1}^{n} \langle f, \tilde{\beta}_i \rangle^2$$

no family if favored over the other (since $\text{MSE}^{\beta} = \text{MSE}^{\tilde{\beta}}$).

(b)

Again, each k-term approximation's MSE should be compared the following way in order to determine which family of functions to choose. (Also note that if for instance f is in the β family, then we prefer this family).

2 Haar matrix and Walsh-Hadamard matrix

We consider the signal

$$\phi(t) = a + b\cos(2\pi t) + c\cos^2(\pi t), \quad t \in [0, 1]$$

where $a, b, c \in \mathbb{R}$ are constants.

a.

(i)

The 4×4 Haar matrix:

$$\boldsymbol{H}_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix}.$$

We show that $\boldsymbol{H}_4$ is unitary:

$$\begin{aligned} \boldsymbol{H}_4^* \boldsymbol{H}_4 &= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \boldsymbol{I}_4. \end{aligned}$$

Hence $\boldsymbol{H}_4$ is unitary.

(ii)

Denote $\Delta_i = \left[\frac{i-1}{N}, \frac{i}{N}\right)$, $i = 1, 2, 3, 4$.

We obtain the Haar functions by:

$$\begin{pmatrix} \psi_1^H(t) \\ \psi_2^H(t) \\ \psi_3^H(t) \\ \psi_4^H(t) \end{pmatrix} = \mathbf{H}_4^* \begin{pmatrix} 2 \cdot \mathbf{1}_{\Delta_1}(t) \\ 2 \cdot \mathbf{1}_{\Delta_2}(t) \\ 2 \cdot \mathbf{1}_{\Delta_3}(t) \\ 2 \cdot \mathbf{1}_{\Delta_4}(t) \end{pmatrix}$$

The functions are:

$$\begin{aligned} \psi_1^H(t) &= \mathbf{1}_{\Delta_1}(t) + \mathbf{1}_{\Delta_2}(t) + \mathbf{1}_{\Delta_3}(t) + \mathbf{1}_{\Delta_4}(t) = \mathbf{1}_{[0,1)}(t) \\ \psi_2^H(t) &= \mathbf{1}_{\Delta_1}(t) + \mathbf{1}_{\Delta_2}(t) - \mathbf{1}_{\Delta_3}(t) - \mathbf{1}_{\Delta_4}(t) = \mathbf{1}_{[0,1/2)}(t) - \mathbf{1}_{[1/2,1)}(t) \\ \psi_3^H(t) &= \sqrt{2}\mathbf{1}_{\Delta_1}(t) - \sqrt{2}\mathbf{1}_{\Delta_2}(t) \\ \psi_4^H(t) &= \sqrt{2}\mathbf{1}_{\Delta_3}(t) - \sqrt{2}\mathbf{1}_{\Delta_4}(t) \end{aligned}$$

See Fig. 1.

(iii)

The approximation is of the form:

$$\hat{\phi}(t) = \sum_{i=1}^{4} \phi_i \psi_i(t).$$

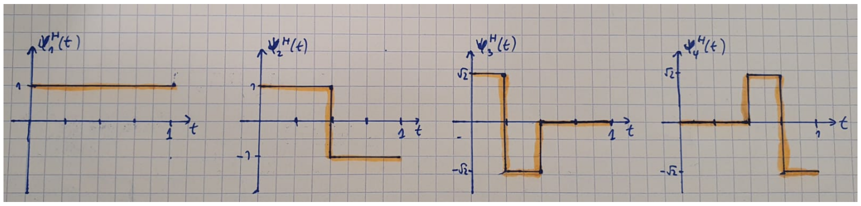

Figure 1: Set of Haar functions

The coefficients are ($i = 1, 2, 3, 4$):

$$\phi_i^H = \langle \phi(t), \psi_i^H(t) \rangle = \int_0^1 \phi(t)\psi_i^H(t)\,\mathrm{d}t.$$

Note that

$$\int \phi(t)\,\mathrm{d}t = at + \frac{\sin(2\pi t)}{2\pi} b + \left(\frac{1}{2}t + \frac{\sin(2\pi t)}{4\pi}\right) c.$$

Hence

1. $$\phi_1^H = \int_0^1 \phi(t)\,\mathrm{d}t = a + \frac{1}{2}c.$$

2. $$\phi_2^H = \int_0^{1/2} \phi(t)\,\mathrm{d}t - \int_{1/2}^1 \phi(t)\,\mathrm{d}t = 2\left(\frac{1}{2}a + \frac{1}{4}c\right) - \left(a + \frac{1}{2}c\right) = 0.$$

3. $$\begin{aligned}\phi_3^H &= \sqrt{2}\int_0^{1/4} \phi(t)\,\mathrm{d}t - \sqrt{2}\int_{1/4}^{1/2} \phi(t)\,\mathrm{d}t \\ &= \sqrt{2}\left[2\left(\frac{1}{4}a + \frac{b}{2\pi} + \left(\frac{1}{8} + \frac{1}{4\pi}\right)c\right) - \left(\frac{1}{2}a + \frac{1}{4}c\right)\right] \\ &= \frac{\sqrt{2}}{\pi}b + \frac{\sqrt{2}}{2\pi}c\end{aligned}$$

4. $$\begin{aligned}\phi_4^H &= \sqrt{2}\int_{1/2}^{3/4} \phi(t)\,\mathrm{d}t - \sqrt{2}\int_{3/4}^{1} \phi(t)\,\mathrm{d}t \\ &= \sqrt{2}\left[2\left(\frac{3}{4}a - \frac{1}{2\pi}b + \left(\frac{3}{8} - \frac{1}{4\pi}\right)c\right) - \left(\frac{1}{2}a + \frac{1}{4}c\right) - \left(a + \frac{1}{2}c\right)\right] \\ &= -\frac{\sqrt{2}}{\pi}b - \frac{\sqrt{2}}{2\pi}c\end{aligned}$$

Now, recall that the MSE is

$$\mathrm{MSE}(4) = \int_0^1 \phi^2(t)\,\mathrm{d}t - \sum_{i=1}^{n} \phi_i^2.$$

Now for the MSE. We calculate the energy of $\phi(t)$:

$$\begin{aligned}
\int_0^1 \phi^2(t)\,\mathrm{d}t &= \int_0^1 (a + b\cos(2\pi t) + c\cos^2(\pi t))^2\,\mathrm{d}t \\
&= a^2 \int_0^1 \mathrm{d}t + b^2 \int_0^1 \cos^2(2\pi t)\,\mathrm{d}t + c^2 \int_0^1 \cos^4(\pi t)\,\mathrm{d}t \\
&+ ab \int_0^1 \cos(2\pi t)\,\mathrm{d}t + ac \int_0^1 \cos^2(\pi t)\,\mathrm{d}t + bc \int_0^1 \cos(2\pi t)\cos^2(\pi t)\,\mathrm{d}t \\
&= a^2 + \frac{1}{2}b^2 + \frac{3}{8}c^2 + \frac{1}{2}ac + \frac{1}{4}bc
\end{aligned}$$

(all thanks to Wolfram Alpha for saving our time!). Also

$$\begin{aligned}
\sum_{i=1}^{4} (\phi_1^H)^2 &= \left(a + \frac{1}{2}c\right)^2 + 2 \cdot \frac{2}{\pi^2}\left(b + \frac{1}{2}c\right)^2 \\
&= a^2 + \frac{4}{\pi^2}b^2 + \frac{1}{2}c^2 + ac + \frac{4}{\pi^2}bc.
\end{aligned}$$

Then

$$\mathrm{MSE}(4) \approx 0.095b^2 - 0.125c^2 - 0.5ac - 0.155bc.$$

(iv)

We got:

$$\phi_1^H = a + 0.5c, \quad \phi_2^H = 0,$$

$$\phi_3^H = \frac{\sqrt{2}}{\pi}b + \frac{\sqrt{2}}{2\pi} \approx 0.45b + 0.22c, \quad \phi_4^H = -\frac{\sqrt{2}}{\pi}b - \frac{\sqrt{2}}{2\pi} \approx -0.45b - 0.22c.$$

If $a \geq b \geq 0$ and $c \geq 0$ we have

$$|\phi_1^H| > |\phi_3^H| = |\phi_4^H| > |\phi_2^H| = 0$$

(note that $|\phi_1^H| = \phi_1^H$, $|\phi_3^H| = \phi_3^H$ and $|\phi_4^H| = -\phi_4^H$). Therefore:

- The 1-term approximation of ϕ:

$$\hat{\phi}_1(t) = \left(a + \frac{1}{2}c\right)\psi_1(t) = \left(a + \frac{1}{2}c\right)\mathbf{1}_{[0,1]}(t).$$

- The 2-term approximation of ϕ:

$$\begin{aligned}\hat{\phi}_2(t) &= \left(a + \frac{1}{2}c\right)\psi_1(t) + \left(\frac{\sqrt{2}}{\pi}b + \frac{\sqrt{2}}{2\pi}c\right)\psi_3(t) \\ &= \left(a + \frac{2}{\pi}b + \left(\frac{1}{2} + \frac{1}{\pi}\right)c\right)\mathbf{1}_{\Delta_1} + \left(a - \frac{2}{\pi}b + \left(\frac{1}{2} - \frac{1}{\pi}\right)c\right)\mathbf{1}_{\Delta_2} \\ &+ \left(a + \frac{1}{2}c\right)\mathbf{1}_{[\frac{1}{2},1]}.\end{aligned}$$

(note we could take $\hat{\phi}_2(t) = \left(a + \frac{1}{2}c\right)\psi_1(t) - \left(\frac{\sqrt{2}}{\pi}b + \frac{\sqrt{2}}{2\pi}c\right)\psi_4(t)$).

- The 3-term approximation of ϕ:

$$\begin{aligned}\hat{\phi}_3(t) &= \hat{\phi}_2(t) + \left(-\frac{\sqrt{2}}{\pi}b - \frac{\sqrt{2}}{2\pi}c\right)\psi_4^H(t) \\ &= \left(a + \frac{2}{\pi}b + \left(\frac{1}{2} + \frac{1}{\pi}\right)c\right)\mathbf{1}_{\Delta_1} + \left(a - \frac{2}{\pi}b + \left(\frac{1}{2} - \frac{1}{\pi}\right)c\right)\mathbf{1}_{\Delta_2} \\ &+ \left(a - \frac{2}{\pi}b + \left(\frac{1}{2} - \frac{1}{\pi}\right)c\right)\mathbf{1}_{\Delta_3} + \left(a + \frac{\sqrt{2}}{\pi}b + \left(\frac{1}{2} + \frac{1}{\pi}\right)c\right)\mathbf{1}_{\Delta_4}.\end{aligned}$$

- The 4-term approximation of ϕ: $\hat{\phi}_4(t) = \hat{\phi}_3(t)$ since $\phi_2^H = 0$.

(v)

Now we assume that $a = \frac{1}{\pi}, b = 1$ and $c = \frac{3}{2}$:

1. $\phi_1^H = \frac{1}{\pi} + \frac{3}{4} \approx 1.068$.
2. $\phi_2^H = 0$.
3. $\phi_3^H = \frac{\sqrt{2}}{\pi} + \frac{3\sqrt{2}}{4\pi} \approx 0.788$.
4. $\phi_4^H = -\frac{\sqrt{2}}{\pi} - \frac{3\sqrt{2}}{4\pi} \approx -0.788$.

Hence:

1. 1-term approximation:

$$\hat{\phi}_1(t) = 1.068\psi_1^H(t) = 1.068\mathbf{1}_{[0,1]}.$$

2. 2-term approximation:

$$\begin{aligned}\hat{\phi}_2(t) &= \hat{\phi}_1(t) + 0.788\psi_3^H = 1.068\mathbf{1}_{[0,1]} + 0.788(\sqrt{2}\mathbf{1}_{\Delta_1}(t) - \sqrt{2}\mathbf{1}_{\Delta_2}(t)) \\ &= 2.182\mathbf{1}_{\Delta_1}(t) - 0.046\mathbf{1}_{\Delta_2}(t) + 1.068\mathbf{1}_{[\frac{1}{2},1]}.\end{aligned}$$

3. 3 term-approximation:

$$\begin{aligned}\hat{\phi}_3(t) &= \hat{\phi}_2(t) - 0.788(\sqrt{2}\mathbf{1}_{\Delta_3}(t) - \sqrt{2}\mathbf{1}_{\Delta_4}(t)) \\ &= 2.1821\mathbf{1}_{\Delta_1}(t) - 0.0461\mathbf{1}_{\Delta_2}(t) + 2.1821\mathbf{1}_{\Delta_3}(t) - 0.0461\mathbf{1}_{\Delta_4}(t)\end{aligned}$$

4. 4-term approximation: $\hat{\phi}_4(t) = \hat{\phi}_3(t)$ since $\phi_2^H = 0$.

b.

The 4×4 Walsh-Hadamard matrix is given by:

$$\mathbf{W}_4 = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\\ 1 & -1 & 1 & -1\end{pmatrix}$$

and its columns are used to form a set of 4 orthonormal functions, $\{\chi_i^W(t)\}_{i=1}^4$, defined for $t \in [0,1]$.

(i)

We prove that $\mathbf{W}_4$ is unitary:

$$\mathbf{W}_4^*\mathbf{W}_4 = \frac{1}{4}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\\ 1 & -1 & 1 & -1\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\\ 1 & -1 & 1 & -1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = \mathbf{I}_4.$$

(ii)

The set of orthonormal Walsh-Hadamard functions $\{\chi_i^W(t)\}_{i=1}^4$, defined for $t \in [0,1]$:

$$\begin{aligned}\chi_1^W(t) &= \mathbf{1}_{\Delta_1}(t) + \mathbf{1}_{\Delta_2}(t) + \mathbf{1}_{\Delta_3}(t) + \mathbf{1}_{\Delta_4}(t) = \mathbf{1}_{[0,1)}(t)\\ \chi_2^W(t) &= \mathbf{1}_{\Delta_1}(t) + \mathbf{1}_{\Delta_2}(t) - \mathbf{1}_{\Delta_3}(t) - \mathbf{1}_{\Delta_4}(t) = \mathbf{1}_{[0,1/2)}(t) - \mathbf{1}_{[1/2,1)}(t)\\ \chi_3^W(t) &= \mathbf{1}_{\Delta_1}(t) - \mathbf{1}_{\Delta_2}(t) - \mathbf{1}_{\Delta_3}(t) + \mathbf{1}_{\Delta_4}(t) = \mathbf{1}_{\Delta_1}(t) - \mathbf{1}_{[\frac{1}{4},\frac{3}{4})}(t) + \mathbf{1}_{\Delta_4}(t)\\ \chi_4^W(t) &= \mathbf{1}_{\Delta_1}(t) - \mathbf{1}_{\Delta_2}(t) + \mathbf{1}_{\Delta_3}(t) - \mathbf{1}_{\Delta_4}(t)\end{aligned}$$

See Fig. 2.

(iii)

The best approximation is given by

$$\hat{\phi}(t) = \sum_{i=1}^{4}\langle\phi(t), \chi_i^W(t)\rangle\chi_i^W(t).$$

Calculating the coefficients:

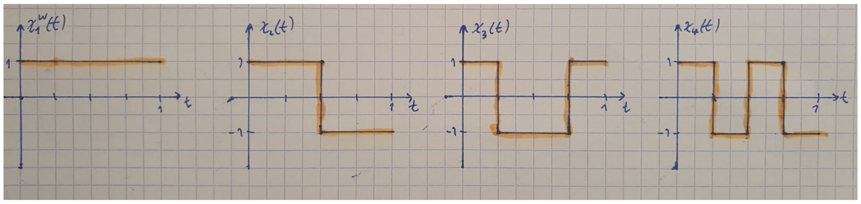

Figure 2: Set of Walsh-Hadamard functions

1.
$$\phi_1^W = \int_0^1 \phi(t)\,\mathrm{d}t = a + \frac{1}{2}c.$$

2.
$$\phi_2^W = \int_0^{1/2} \phi(t)\,\mathrm{d}t - \int_{1/2}^1 \phi(t)\,\mathrm{d}t = 2\left(\frac{1}{2}a + \frac{1}{4}c\right) - \left(a + \frac{1}{2}c\right) = 0.$$

3.
$$\begin{aligned}
\phi_3^W &= \int_0^{1/4} \phi(t)\,\mathrm{d}t - \int_{1/4}^{3/4} \phi(t)\,\mathrm{d}t + \int_{3/4}^1 \phi(t)\,\mathrm{d}t \\
&= 2\left(\frac{a}{4} + \frac{1}{2\pi}b + \left(\frac{1}{8} + \frac{1}{4\pi}\right)c\right) - 2\left(\frac{3a}{4} - \frac{1}{2\pi}b + \left(\frac{3}{8} - \frac{1}{4\pi}\right)c\right) \\
&+ a + \frac{1}{2}c \\
&= \frac{2}{\pi}b + \frac{1}{\pi}c \approx 0.637b + 0.318c.
\end{aligned}$$

4.
$$\begin{aligned}
\phi_4^W &= \int_0^{1/4} \phi(t)\,\mathrm{d}t - \int_{1/4}^{1/2} \phi(t)\,\mathrm{d}t + \int_{1/2}^{3/4} \phi(t)\,\mathrm{d}t - \int_{3/4}^1 \phi(t)\,\mathrm{d}t \\
&= 2\left(\frac{a}{4} + \frac{1}{2\pi}b + \left(\frac{1}{8} + \frac{1}{4\pi}\right)c\right) - 2\left(\frac{1}{2}a + \frac{1}{4}c\right) \\
&+ 2\left(\frac{3a}{4} - \frac{1}{2\pi}b + \left(\frac{3}{8} - \frac{1}{4\pi}\right)c\right) - \left(a + \frac{1}{2}c\right) \\
&= 0
\end{aligned}$$

Then

$$\begin{aligned}
\hat{\phi}(t) &= \left(a + \frac{1}{2}c\right)\mathbf{1}_{[0,1]}(t) - \left(\frac{2}{\pi}b + \frac{1}{\pi}c\right)\left(\mathbf{1}_{\Delta_1}(t) - \mathbf{1}_{\left[\frac{1}{4},\frac{3}{4}\right)}(t) + \mathbf{1}_{\Delta_4}(t)\right) \\
&= \left(a - \frac{2}{\pi}b + \left(\frac{1}{2} - \frac{1}{\pi}\right)c\right)\mathbf{1}_{\Delta_1}(t) + \left(a + \frac{2}{\pi}b + \left(\frac{1}{2} + \frac{1}{\pi}\right)c\right)\mathbf{1}_{\left[\frac{1}{4},\frac{3}{4}\right]} \\
&+ \left(a - \frac{2}{\pi}b + \left(\frac{1}{2} - \frac{1}{\pi}\right)c\right)\mathbf{1}_{\Delta_4}(t).
\end{aligned}$$

Now for the MSE.

$$\sum_{i=1}^{4}(\phi_i^W)^2 = \left(a + \frac{1}{2}c\right)^2 + \frac{1}{\pi^2}(2b+c)^2$$
$$= a^2 + 0.405b^2 + 0.351c^2 + ac + 0.405bc.$$

Hence
$$\mathrm{MSE}(4) = 0.095b^2 + 0.024c^2 - 0.5ac - 0.155bc.$$

(iv)

Assume $a \geq b \geq 0$ and $c \geq 0$. Then

$$\chi_1^W > \chi_3^W > \chi_2^W = \chi_4^W = 0.$$

Then:

- 1-term approximation:
$$\hat{\phi}_1(t) = \phi_1^W \chi_1(t) = \left(a + \frac{1}{2}\right) \mathbf{1}_{[0,1]}(t).$$

- 2-term approximation:
$$\begin{aligned}\hat{\phi}_2(t) &= \hat{\phi}_1(t) + \phi_3^W \chi_3(t)\\ &= \left(a + \frac{1}{2}c\right) \mathbf{1}_{[0,1]}(t) - \left(\frac{2}{\pi}b + \frac{1}{\pi}c\right)\left(\mathbf{1}_{\Delta_1}(t) - \mathbf{1}_{\left[\frac{1}{4},\frac{3}{4}\right)}(t) + \mathbf{1}_{\Delta_4}(t)\right)\\ &= \left(a - \frac{2}{\pi}b + \left(\frac{1}{2} - \frac{1}{\pi}\right)c\right)\mathbf{1}_{\Delta_1}(t) + \left(a + \frac{2}{\pi}b + \left(\frac{1}{2} + \frac{1}{\pi}\right)c\right)\mathbf{1}_{\left[\frac{1}{4},\frac{3}{4}\right]}(t)\\ &\quad + \left(a - \frac{2}{\pi}b + \left(\frac{1}{2} - \frac{1}{\pi}\right)c\right)\mathbf{1}_{\Delta_4}(t)\end{aligned}$$

- 3-term & 4-term approximations: $\hat{\phi}_4(t) = \hat{\phi}_3(t) = \hat{\phi}_2(t)$ since $\phi_4^W = \phi_3^W = 0$.

(v)

Assume $a = \frac{1}{\pi}, b = 1$ and $c = \frac{3}{2}$. Then

$$\phi_1^W = \frac{1}{\pi} + \frac{3}{4} \approx 1.068, \quad \phi_2^W = 0, \quad \phi_3^W = \frac{2}{\pi} + \frac{3}{2\pi} \approx 1.114, \quad \phi_4^W = 0.$$

Then:
$$\phi_3^W > \phi_1^W > \phi_2^W = \phi_4^W = 0.$$

and so we get:

- 1-term approximation:

$$\hat{\phi}_1(t) = \phi_3^W \chi_3^W = 1.114(\mathbf{1}_{\Delta_1}(t) - \mathbf{1}_{[\frac{1}{4},\frac{3}{4})}(t) + \mathbf{1}_{\Delta_4}(t))$$

- 2-term approximation:

$$\begin{aligned}\hat{\phi}_2(t) = \hat{\phi}_1(t) + \phi_1^W \chi_1^W \\ &= 1.114(\mathbf{1}_{\Delta_1}(t) - \mathbf{1}_{[\frac{1}{4},\frac{3}{4})}(t) + \mathbf{1}_{\Delta_4}(t)) + 1.068\mathbf{1}_{[0,1]} \\ &= 2.182\mathbf{1}_{\Delta_1}(t) - 0.046\mathbf{1}_{[\frac{1}{4},\frac{3}{4})} + 2.182\mathbf{1}_{\Delta_4}.\end{aligned}$$

- 3-term & 4-term approximations: $\hat{\phi}_4(t) = \hat{\phi}_3(t) = \hat{\phi}_2(t)$ since $\phi_4^W = \phi_3^W = 0$.

3 Bit Allocation of a Two-Dimensional Signal

We consider a function ϕ: $[0,1] \times [0,1] \to \mathbb{R}$ by

$$\phi(x,y) = x^a y$$

where $a \geq 1$.

a.

In this problem, $\phi_L = 0$ (at $(0,0)$) and $\phi_H = 1$ (at $(1,1)$; this is due to the fact that ϕ is increasing in both x and y axes).

Now, we shall find the energies of ϕ'_x and ϕ'_y, which are:

$$\phi'_x = ax^{a-1}y, \quad \phi'_y = x^a.$$

$$\begin{aligned}\text{Energy}(\phi'_x) &= \int_0^1 \int_0^1 (\phi'_x)^2 \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^1 \int_0^1 a^2 x^{2(a-1)} y^2 \, \mathrm{d}x \, \mathrm{d}y \\ &= a^2 \int_0^1 y^2 \left. \frac{x^{2a-1}}{2a-1} \right|_{x=0}^1 \\ &= \frac{a^2}{2a-1} \left. \frac{y^3}{3} \right|_{y=0}^1 \\ &= \frac{a^2}{3(2a-1)} = \frac{a^2}{6a-3}\end{aligned}$$

and

$$
\begin{aligned}
\text{Energy}(\phi'_y) &= \int_0^1 \int_0^1 (\phi'_y)^2 \,\mathrm{d}x \,\mathrm{d}y \\
&= \int_0^1 \int_0^1 x^{2a} \,\mathrm{d}x \,\mathrm{d}y \\
&= \int_0^1 \left. \frac{x^{2a+1}}{2a+1} \right|_{x=0}^{1} \mathrm{d}y \\
&= \frac{1}{2a+1} \int_0^1 \mathrm{d}y \\
&= \frac{1}{2a+1}.
\end{aligned}
$$

b.

The bit allocation optimization problem is:

$$
\underset{N_x, N_y, b}{\text{minimize}} \ \text{MSE}^{\text{total}}(N_x, N_y, b)
$$

subject to $N_x N_y b \leq B$, where

$$
\text{MSE}^{\text{total}}(N_x, N_y, b) = \frac{1}{12N_x^2} \text{Energy}(\phi'_x) + \frac{1}{12N_y^2} \text{Energy}(\phi'_y) + \frac{1}{12} \frac{(\phi_H - \phi_L)^2}{2^{2b}}
$$

(as seen in the tutorial).

c.

Now we consider a function $\psi \colon [0,1] \times [0,1] \to \mathbb{R}$ by

$$
\psi(x, y) = axy
$$

where $a \geq 1$, and we assuming that B is big enough to forget about bit constraint.

First, we notice that if $a = 1$, ϕ and ψ are exactly the same functions, and therefore $N'_x = N_x, N_y = N'_y, b' = b$. Now we assume that $a > 1$.

Notice that $\phi_L = 0$ (at $(0,0)$) and $\phi_H = a$ (at $(1,1)$).

The energies of $\psi'_x = ay$ and $\psi'_y = ax$ are

$$\begin{aligned}
\mathrm{Energy}(\psi'_x) &= \int_0^1 \int_0^1 (\psi'_x)^2 \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_0^1 \int_0^1 a^2 y^2 \, \mathrm{d}x \, \mathrm{d}y \\
&= a^2 \int_0^1 y^2 \int_0^1 \mathrm{d}x \, \mathrm{d}y \\
&= a^2 \int_0^1 y^2 \, \mathrm{d}y \\
&= a^2 \left. \frac{y^3}{3} \right|_0^1 \\
&= \frac{a^2}{3}
\end{aligned}$$

Similarly, we get

$$\mathrm{Energy}(\psi'_y) = \frac{a^2}{3}.$$

The total MSE of $\psi(x, y)$ is ($N'_x N'_y b' = B$):

$$\begin{aligned}
\mathrm{MSE}^{\psi}(N'_x, N'_y, b') &= \frac{1}{12(N'_x)^2} \frac{a^2}{3} + \frac{1}{12(N'_y)^2} \frac{a^2}{3} + \frac{1}{12} \frac{a^2}{2^{2b'}} \\
&= \frac{a^2}{12} \left(\frac{1}{3(N'_x)^2} + \frac{1}{3(N'_y)^2} + \frac{1}{2^{2b'}} \right)
\end{aligned}$$

The total MSE of $\phi(x, y)$ is ($N_x N_y b = B$):

$$\begin{aligned}
\mathrm{MSE}^{\phi}(N_x, N_y, b) &= \frac{1}{12N_x^2} \frac{a^2}{6a - 3} + \frac{1}{12N_y^2} \frac{1}{2a + 1} + \frac{1}{12} \frac{1}{2^{2b}} \\
&= \frac{1}{12} \left(\frac{1}{3 \left(\frac{\sqrt{2a-1}}{a} N_x \right)^2} + \frac{1}{3 \left(\sqrt{\frac{2a+1}{3}} N_y \right)^2} + \frac{1}{2^{2b}} \right)
\end{aligned}$$

We look at the following MSE problems as the same one, and hence demand:

$$\begin{aligned}
N'_x &= \frac{\sqrt{2a-1}}{a} N_x, \\
N'_y &= \sqrt{\frac{2a+1}{3}} N_y, \\
b' &= b.
\end{aligned}$$

4 On Hadamard matrices

Let $n \in \mathbb{N}^*$ a positive integer, $N = 2^n$, and consider the Hadamard matrix: $H_{2^n} = H_N$.

a.

We prove that H_N is a symmetric, real and unitary matrix.

- H_N is symmetric: by induction on N. For $N = 1$, $H_1 = (1)$ which is symmetric. For $N = 2$,

$$H_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

  which is also symmetric. Now assume that $H_{\frac{1}{2}N}$ is symmetric; we show that H_N is also symmetric. Indeed,

$$H_N = \frac{1}{\sqrt{2}} \begin{pmatrix} H_{\frac{1}{2}N} & H_{\frac{1}{2}N} \\ H_{\frac{1}{2}N} & -H_{\frac{1}{2}N} \end{pmatrix}$$

  and so

$$H_N^{\mathrm{T}} = \frac{1}{\sqrt{2}} \begin{pmatrix} H_{\frac{1}{2}N}^{\mathrm{T}} & H_{\frac{1}{2}N}^{\mathrm{T}} \\ H_{\frac{1}{2}N}^{\mathrm{T}} & -H_{\frac{1}{2}N}^{\mathrm{T}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} H_{\frac{1}{2}N} & H_{\frac{1}{2}N} \\ H_{\frac{1}{2}N} & -H_{\frac{1}{2}N} \end{pmatrix} = H_N$$

  because by induction hypothesis $H_{\frac{1}{2}N}^{\mathrm{T}} = H_{\frac{1}{2}N}$.

- H_N is real: this is pretty clear; doing it formally by induction, H_1, H_2 are real, and if we assume that $H_{\frac{1}{2}N}$ is real then also H_N by its construction (see above).

- H_N is unitary: We show by induction over n. For $N = 1$, $H_1^* H_1 = (1)(1) = (1) = I_1$. For $N = 2$:

$$H_2^* H_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

  Now assume that $H_{\frac{1}{2}N}$ is unitary. So H_N is unitary too:

$$\begin{aligned} H_N^* H_N &= \frac{1}{2} \begin{pmatrix} H_{\frac{1}{2}N}^* & H_{\frac{1}{2}N}^* \\ H_{\frac{1}{2}N}^* & -H_{\frac{1}{2}N}^* \end{pmatrix} \begin{pmatrix} H_{\frac{1}{2}N} & H_{\frac{1}{2}N} \\ H_{\frac{1}{2}N} & -H_{\frac{1}{2}N} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 2H_{\frac{1}{2}N}^* H_{\frac{1}{2}N} & 0 \\ 0 & 2H_{\frac{1}{2}N}^* H_{\frac{1}{2}N} \end{pmatrix} \\ &= \begin{pmatrix} I_{\frac{1}{2}N} & 0 \\ 0 & I_{\frac{1}{2}N} \end{pmatrix} \\ &= I_N \end{aligned}$$

  since $H_{\frac{1}{2}N}^* H_{\frac{1}{2}N} = I_{\frac{1}{2}N}$ by induction hypothesis.

Now, we show that H_N can be written as $H_N = \lambda_N A_N$ where $\lambda_N = \frac{1}{\sqrt{N}} \in \mathbb{R}$ and A_N is a matrix with only ± 1 entries. We do it by induction over N. For $N = 1$, $H_1 = \frac{1}{\sqrt{1}}(1)$. For $N = 2$,

$$H_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

so indeed $\lambda_1 = \frac{1}{\sqrt{2}}$ and $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ a matrix with only ± 1 entries. Now, we assume $H_{\frac{1}{2}N}$ can be written as $H_{\frac{1}{2}N} = \lambda_{\frac{1}{2}N} A_{\frac{1}{2}N}$. Then:

$$\begin{aligned} H_N &= \frac{1}{\sqrt{2}} \begin{pmatrix} H_{\frac{1}{2}N} & H_{\frac{1}{2}N} \\ H_{\frac{1}{2}N} & -H_{\frac{1}{2}N} \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} \lambda_{\frac{1}{2}N} A_{\frac{1}{2}N} & \lambda_{\frac{1}{2}N} A_{\frac{1}{2}N} \\ \lambda_{\frac{1}{2}N} A_{\frac{1}{2}N} & -\lambda_{\frac{1}{2}N} A_{\frac{1}{2}N} \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \lambda_{\frac{1}{2}N} \begin{pmatrix} A_{\frac{1}{2}N} & A_{\frac{1}{2}N} \\ A_{\frac{1}{2}N} & -A_{\frac{1}{2}N} \end{pmatrix} \\ &= \frac{1}{\sqrt{N}} A_N \end{aligned}$$

(since $\lambda_{\frac{1}{2}N} = \frac{1}{\sqrt{\frac{1}{2}N}}$) where

$$A_N = \begin{pmatrix} A_{\frac{1}{2}N} & A_{\frac{1}{2}N} \\ A_{\frac{1}{2}N} & -A_{\frac{1}{2}N} \end{pmatrix}$$

a matrix whose entries are ± 1 since $A_{\frac{1}{2}N}$ is so (by induction hypothesis).

b.

(i)

Let s_1, s_2 two sequences of numbers of same length. Denote by s_1^{last} the last digit of s_1, and by s_2^{first} the first digit of s_2. Then

$$S(s_1 s_2) = \begin{cases} S(s_1) + S(s_2) & \text{if } s_1^{\text{last}} = s_2^{\text{first}} \\ S(s_1) + S(s_2) + 1 & \text{if } s_1^{\text{last}} \neq s_2^{\text{first}} \end{cases}.$$

Indeed, if $s_1^{\text{last}} = s_2^{\text{first}}$ the total number of sign changes is the number of sign changes in s_1 plus the number of sign changes in s_2: no new sign change is created, as in the case where $s_1^{\text{last}} \neq s_2^{\text{first}}$, so we have to add 1 to the number of sign changes in $s_1 s_2$.

(ii)

We denote by r_i the i-th row of H_N. We prove the ensemble equality:

$$\{S(r_1), \ldots, S(r_N)\} = \{0, \ldots, N-1\},$$

i.e. that the number of changes of sign in the rows of H_N are the first N integers starting at 0.

We show the equality by induction over N. For $N = 2$, $S(r_1) = 0, S(r_2) = 1$ and so $\{S(r_1), S(r_2)\} = \{0, 1\}$. Now assume correctness for $\frac{1}{2}N$ and prove for N:

$$H_N = \frac{1}{\sqrt{N}} \begin{pmatrix} A_{\frac{1}{2}N} & A_{\frac{1}{2}N} \\ A_{\frac{1}{2}N} & -A_{\frac{1}{2}N} \end{pmatrix}$$

Now, by the last section we obtain

$$S(r_i^N) = \begin{cases} 2S(r_i^{\frac{1}{2}N}) & \text{if } r_i^{\frac{1}{2}N,\text{ last}} = r_i^{\frac{1}{2}N,\text{ first}} \\ 2S(r_i^{\frac{1}{2}N}) + 1 & \text{if } r_i^{\frac{1}{2}N,\text{ last}} \neq r_i^{\frac{1}{2}N,\text{ first}} \end{cases}.$$

(r_i^N is the i-th row of the Hadamard matrix H_N; $r_i^{\frac{1}{2}N}$ is the i-th row of the Hadamard matrix $H_{\frac{1}{2}N}$).

By induction hypothesis,

$$\{S(r_i^{\frac{1}{2}N})\}_{i=1}^{\frac{1}{2}N} = \left\{0, \ldots, \frac{1}{2}N - 1\right\}.$$

Notice that every row $r_i^{\frac{1}{2}N}$ appears twice: in the upper part, where it gives a row $r_i^N = (r_i^{\frac{1}{2}N}\ \ r_i^{\frac{1}{2}N})$ to A_N, and in the lower part, where it gives a row $r_{i+N/2}^N = (r_i^{\frac{1}{2}N}\ \ r_i^{\frac{1}{2}N})$ to A_N. In one row there is a sign change in the concatenation point, in the other there is no sign change. Therefore, $\{S(r_i^N), S(r_{i+N/2}^N)\} = \{2S(r_i^{N/2}), 2S(r_i^{N/2}) + 1\}$, and therefore

$$\begin{aligned} \{S(r_1^N), \ldots, S(r_N^N)\} &= \{2S(r_i^{N/2}) : i = 1, \ldots, N/2\} \cup \{2S(r_i^{i/2}) + 1 : i = 1, \ldots, N/2\} \\ &= \{0, \ldots, N - 1\} \end{aligned}$$

since the first set contains all the even elements up to N and the second set contains all the odd elements up to N. $\square$