# 236200 – Signal, Image, and Data Processing
# Homework 4

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# Part I
# Theory

## 1 Inverting the Second Derivative Operator

### a.

By the definition of $\boldsymbol{\varphi}_{\text{data}} = \mathrm{H}\boldsymbol{\varphi}$ as

$$\varphi_{\text{data},j} = -\frac{1}{12}\varphi_{j-2[M]} + \frac{4}{3}\varphi_{j-1[M]} - \frac{5}{12}\varphi_{j[M]} + \frac{4}{3}\varphi_{j+1[M]} - \frac{1}{12}\varphi_{j+2[M]}$$

we have that

$$\begin{pmatrix} \varphi_{\text{data},1} \\ \varphi_{\text{data},2} \\ \varphi_{\text{data},3} \\ \vdots \\ \varphi_{\text{data},M} \end{pmatrix} = \begin{pmatrix} -\frac{5}{12} & \frac{4}{3} & -\frac{1}{12} & 0 & \dots & 0 & -\frac{1}{12} & \frac{4}{3} \\ \frac{4}{3} & -\frac{5}{12} & \frac{4}{3} & -\frac{1}{12} & 0 & \dots & 0 & -\frac{1}{12} \\ -\frac{1}{12} & \frac{4}{3} & -\frac{5}{12} & \frac{4}{3} & -\frac{1}{12} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{4}{3} & -\frac{1}{12} & 0 & \dots & 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{12} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_M \end{pmatrix},$$

and so $\mathbf{H}$ is a circulant matrix with first row $\begin{pmatrix} -\frac{5}{12} & \frac{4}{3} & -\frac{1}{12} & 0 & \dots & 0 & -\frac{1}{12} & \frac{4}{3} \end{pmatrix}$.

**b.**

We denote the inverse-filter matrix by $\mathbf{M} \in \mathbb{R}^{M\times M}$. In order to find it, we shall look for the eigenvalues of $\mathbf{H}$. Since $\mathbf{H}$ is circulant, its eigenvalues are given by

$$\begin{pmatrix} \lambda_1^{\mathrm{H}} \\ \lambda_2^{\mathrm{H}} \\ \lambda_3^{\mathrm{H}} \\ \vdots \\ \lambda_M^{\mathrm{H}} \end{pmatrix} = [\mathrm{DFT}]^* \begin{pmatrix} -\frac{5}{12} \\ \frac{4}{3} \\ -\frac{1}{12} \\ 0 \\ \cdots \\ 0 \\ -\frac{1}{12} \\ \frac{4}{3} \end{pmatrix}$$

(we multiply the $[\mathrm{DFT}]^*$ matrix by the first row of $\mathbf{H}$ written as column). Hence, for $1 \le k \le M$ we get the eigenvalue ($W = \exp\left(\frac{i2\pi}{M}\right)$):

$$\begin{aligned}
\lambda_k^{\mathrm{H}} &= -\frac{5}{12} + \frac{4}{3}W^{k-1} - \frac{1}{12}W^{2(k-1)} - \frac{1}{12}W^{(M-2)(k-1)} + \frac{4}{3}W^{(M-1)(k-1)} \\
&= -\frac{5}{12} + \frac{4}{3}(W^{k-1} + W^{(M-1)(k-1)}) - \frac{1}{12}(W^{2(k-1)} + W^{(M-2)(k-1)}) \\
&= -\frac{5}{12} + \frac{4}{3}\cdot 2\,\mathrm{Re}\left(W^{k-1}\right) - \frac{1}{12}\cdot 2\,\mathrm{Re}\left(W^{2(k-1)}\right) \\
&= -\frac{5}{12} + \frac{8}{3}\cos\left(\frac{2\pi(k-1)}{M}\right) - \frac{1}{6}\cos\left(\frac{4\pi(k-1)}{M}\right) \\
&= -\frac{5}{12} + \frac{8}{3}\cos\left(\frac{2\pi(k-1)}{M}\right) - \frac{1}{6}\left(2\cos^2\left(\frac{2\pi(k-1)}{M}\right) - 1\right) \\
&= -\frac{1}{3}\cos^2\left(\frac{2\pi(k-1)}{M}\right) + \frac{8}{3}\cos\left(\frac{2\pi(k-1)}{M}\right) - \frac{1}{4}.
\end{aligned}$$

(exactly the same eigenvalues as found in the implementation part). Now we ask ourselves, can this eigenvalue be zero? Denote $t = \cos\left(\frac{2\pi(k-1)}{M}\right)$. The polynomial $f(t) = -\frac{1}{3}t^2 + \frac{8}{3}t - \frac{1}{4}$ has two roots, $t = 4 - \frac{\sqrt{61}}{2} \approx 0.09$ and $t = 4 + \frac{\sqrt{61}}{2} \approx 7.09$, which is not possible $t \in [-1, 1]$). Now we get that if the polynomial has a root, then $k$ is given by

$$k-1 = \pm\frac{M}{2\pi}\arccos\left(4 - \frac{\sqrt{61}}{2}\right) + MK, \quad K \in \mathbb{Z} \Rightarrow k-1 = \arccos\left(4 - \frac{\sqrt{61}}{2}\right)M$$

as it is between 1 and $M$, and not possible as an integer (this is an index).

We conclude that none of $\mathbf{H}$'s eigenvalues is zero. Then $\mathbf{M}$ is a circulant matrix (and so is diagonalized by the $[\mathrm{DFT}]^*$ matrix), and its eigenvalues are

$$\lambda_k^{\mathrm{M}} = \frac{1}{\lambda_k^{\mathrm{H}}} = \frac{1}{-\frac{1}{3}\cos^2\left(\frac{2\pi(k-1)}{M}\right) + \frac{8}{3}\cos\left(\frac{2\pi(k-1)}{M}\right) - \frac{1}{4}}, \quad k = 1, 2, \ldots, M.$$

Then $\mathbf{M}$ is diagonalized by the $[\mathrm{DFT}]^*$ matrix (it is circulant), and its eigendecomposition is $\mathrm{M} = [\mathrm{DFT}]^* \mathbf{\Lambda_M} [\mathrm{DFT}]$ where $\mathbf{\Lambda_M} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

### c.

Yes, $\boldsymbol{\varphi}$ can be perfectly recovered using $\mathbf{M}$, since all of its eigenvalues are different then zero. Then

$$\begin{aligned}
\mathbf{M}\boldsymbol{\varphi}_{\mathbf{data}} &= \mathbf{MH}\boldsymbol{\varphi} \\
&= [\mathrm{DFT}]^* \mathbf{\Lambda_M} [\mathrm{DFT}][\mathrm{DFT}]^* \mathbf{\Lambda_H} [\mathrm{DFT}]\boldsymbol{\varphi} \\
&= [\mathrm{DFT}]^* \mathbf{\Lambda_M} \mathbf{\Lambda_H} [\mathrm{DFT}]\boldsymbol{\varphi} \\
&= [\mathrm{DFT}]^* \mathbf{I} [\mathrm{DFT}]\boldsymbol{\varphi} \\
&= \boldsymbol{\varphi}
\end{aligned}$$

since $\mathbf{\Lambda_M}\mathbf{\Lambda_H}$ is a diagonal matrix, with diagonal element $(k,k)$: $\lambda_k^{\mathbf{M}}\lambda_k^{\mathbf{H}} = \frac{1}{\lambda_k^{\mathbf{H}}}\lambda_k^{\mathbf{H}} = 1$, meaning it is $\mathbf{I}$.

## 2 Let's Randomize

### a.

Computing the mean of $\boldsymbol{\varphi}$, the $i$-th component is

$$\begin{aligned}
\mathrm{E}\{\varphi_i\} &= \mathrm{E}\{\varphi_i \mid K = i\} \Pr(K = i) + \mathrm{E}\{\varphi_i \mid K \neq i\} \Pr(K \neq i) \\
&= \mathrm{E}\{M + L\} \Pr(K = i) + \mathrm{E}\{M\} \Pr(K \neq i) \\
&= (\mathrm{E}\{M\} + \mathrm{E}\{L\}) \Pr(K = i) + \mathrm{E}\{M\} \Pr(K \neq i) \\
&= 0
\end{aligned}$$

since $\mathrm{E}\{M\} = 0$ and also $\mathrm{E}\{L\} = 0$:

$$\begin{aligned}
\mathrm{E}\{L\} &= \mathrm{E}\left\{L \,\middle|\, K \leq \frac{N}{2}\right\} \Pr\left(K \leq \frac{N}{2}\right) + \mathrm{E}\left\{L \,\middle|\, K > \frac{N}{2}\right\} \Pr\left(K > \frac{N}{2}\right) \\
&= \mathrm{E}\{L_1\} \Pr\left(K \leq \frac{N}{2}\right) + \mathrm{E}\{L_2\} \Pr\left(K > \frac{N}{2}\right) \\
&= 0
\end{aligned}$$

since $\mathrm{E}\{L_1\} = \mathrm{E}\{L_2\} = 0$. Note that in both calculations we used the fact that $M, L, K$ (and in the second $L_1, L_2$) are linearly independent random variables.

We conclude that $\mathrm{E}\{\boldsymbol{\varphi}\} = \mathbf{0}$.

**b.**

We compute $\mathbf{R}_{\boldsymbol{\varphi}}$, the autocorrelation matrix of $\boldsymbol{\varphi}$. The diagonal elements are ($i \in \{1, \dots, N\}$):

$$\begin{aligned}
r_{ii} &= \mathrm{E}\{\varphi_i^2\} \\
&= \mathrm{E}\{\varphi_i^2 \mid K = i\} \Pr(K = i) + \mathrm{E}\{\varphi_i^2 \mid K \neq i\} \Pr(K \neq i) \\
&= \mathrm{E}\{(M + L)^2\} \Pr(K = i) + \mathrm{E}\{M^2\} \Pr(K \neq i) \\
&= (\mathrm{E}\{M^2\} + \mathrm{E}\{L^2\}) \Pr(K = i) + \mathrm{E}\{M^2\} \Pr(K \neq i) \\
&= \left(c + \frac{N}{2}(a + b)\right) \Pr(K = i) + c \Pr(K \neq i) \\
&= c + \frac{N(a + b)}{2} \Pr(K = i) \\
&= c + \frac{N(a + b)}{2} \frac{1}{N} \\
&= c + \frac{a + b}{2}
\end{aligned}$$

since $K$ is distributed uniformly in $\{1, \dots, N\}$,

$$\begin{aligned}
\mathrm{E}\{(M + L)^2\} &= \mathrm{E}\{M^2\} + 2\,\mathrm{E}\{M\}\,\mathrm{E}\{L\} + \mathrm{E}\{L^2\} \\
&= \mathrm{E}\{M^2\} + 0 + \mathrm{E}\{L^2\} \\
&= \mathrm{E}\{M^2\} + \mathrm{E}\{L^2\},
\end{aligned}$$

and the r.v.'s are independent, and the following computation:

$$\begin{aligned}
\mathrm{E}\{L^2\} &= \mathrm{E}\left\{L^2 \,\middle|\, K \leq \frac{N}{2}\right\} \Pr\left(K \leq \frac{N}{2}\right) + \mathrm{E}\left\{L \,\middle|\, K > \frac{N}{2}\right\} \Pr\left(K > \frac{N}{2}\right) \\
&= \mathrm{E}\{L_1^2\} \Pr\left(K \leq \frac{N}{2}\right) + \mathrm{E}\{L_2^2\} \Pr\left(K > \frac{N}{2}\right) \\
&= Na \cdot \frac{1}{2} + Nb \cdot \frac{1}{2} \\
&= \frac{N(a + b)}{2}.
\end{aligned}$$

We have that $r_{ii} = c + \frac{a+b}{2}$.

Now we compute the other elements of the matrix. For some $i, j \in \{1, \dots, N\}$,

$i \neq j$, we compute:

$$
\begin{aligned}
r_{ij} &= \mathrm{E}\{\varphi_i \varphi_j\} \\
&= \mathrm{E}\{\varphi_i \varphi_j \mid K = i\} \Pr(K = i) + \mathrm{E}\{\varphi_i \varphi_j \mid K = j\} \Pr(K = j) \\
&+ \mathrm{E}\{\varphi_i \varphi_j \mid K \neq i, j\} \Pr(K \neq i, j) \\
&= \mathrm{E}\{(M + L)M\} \Pr(K = i) + \mathrm{E}\{M(M + L)\} \Pr(K = j) \\
&+ \mathrm{E}\{M^2\} \Pr(K \neq i, j) \\
&= (\mathrm{E}\{M^2\} + \mathrm{E}\{LM\}) \Pr(K = i) + (\mathrm{E}\{M^2\} + \mathrm{E}\{ML\}) \Pr(K = j) \\
&+ \mathrm{E}\{M^2\} \Pr(K \neq i, j) \\
&= c \Pr(K = i) + c \Pr(K = j) + c \Pr(K \neq i, j) \\
&= c
\end{aligned}
$$

since $\mathrm{E}\{LM\} = \mathrm{E}\{ML\} = 0$ (recall that $M, L$ are independent with $\mathrm{E}\{M\} = \mathrm{E}\{L\} = 0$; hence, for example, $\mathrm{E}\{ML\} = \mathrm{E}\{M\}\,\mathrm{E}\{L\} = 0$).

To sum up, the autocorrelation matrix of $\boldsymbol{\varphi}$ is

$$
\mathbf{R}_{\boldsymbol{\varphi}} = \begin{pmatrix}
c + \frac{a+b}{2} & c & c & \dots & c \\
c & c + \frac{a+b}{2} & c & \dots & c \\
c & c & c + \frac{a+b}{2} & \dots & c \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c & c & c & \dots & c + \frac{a+b}{2}
\end{pmatrix}.
$$

**c.**

We note that the autocorrelation matrix $\mathbf{R}_{\boldsymbol{\varphi}}$ is circulant; hence its PCA matrix is $[\mathrm{DFT}]^*$ for every $a, b$ and $c$.

# 3 Let's Randomize Again!

We compute $\mathbf{R}_{\boldsymbol{\varphi}}$, the autocorrelation matrix of $\boldsymbol{\varphi}$.

First, we compute the diagonal elements $r_{ii}$, $i \in \{1, \dots, N\}$. Assume that $i \leq \frac{N}{2}$, then

$$
\begin{aligned}
r_{ii} &= \mathrm{E}\{\varphi_i^2\} \\
&= \mathrm{E}\{\varphi_i^2 \mid K = i\} \Pr(K = i) + \mathrm{E}\{\varphi_i^2 \mid K \neq i\} \Pr(K \neq i) \\
&= \mathrm{E}\{(M + L)^2\} \Pr(K = i) + \mathrm{E}\{M^2\} \Pr(K \neq i) \\
&= (\mathrm{E}\{M^2\} + \mathrm{E}\{L^2\}) \Pr(K = i) + \mathrm{E}\{M^2\} \Pr(K \neq i) \\
&= \left(c + \frac{N}{2}(1 - c)\right) \Pr(K = i) + c \Pr(K \neq i) \\
&= c + \frac{N}{2}(1 - c) \Pr(K = i) \\
&= c + (1 - c) \\
&= 1
\end{aligned}
$$

because $\Pr(K=i)=\frac{1}{N/2}$ since $K$ is distributed uniformly in $\left\{1,\ldots,\frac{N}{2}\right\}$. In a very similar computation for $i>\frac{N}{2}$ (we divide to the cases where $K=i-\frac{N}{2}$ and $K\neq i-\frac{N}{2}$) we get the same result. Then $r_{ii}=1$ for all $i\in\{1,\ldots,N\}$.

Now we compute $r_{ij}=\mathrm{E}\{\varphi_i\varphi_j\}$ for $i\neq j$.

If $|j-i|\neq\frac{N}{2}$, then for instance if $i\leq\frac{K}{2},j>\frac{N}{2}$ we divide for cases where $K=i,K=j$ or $K\neq i,j$; in any case, the expectations are $\mathrm{E}\{M^2\}=c$ or $\mathrm{E}\{M(M+L)\}=c$, and the result is that $r_{ij}=c$.

The "interesting" case is where $|j-i|=\frac{N}{2}$. Assume that $i<j$. Then

$$\begin{aligned}
r_{ij}&=\mathrm{E}\{\varphi_i\varphi_j\}\\
&=\mathrm{E}\{\varphi_i\varphi_j\mid K=i\}\Pr(K=i)+\mathrm{E}\{\varphi_i\varphi_j\mid K\neq i\}\Pr(K\neq i)\\
&=\mathrm{E}\{(M+L)^2\}\Pr(K=i)+\mathrm{E}\{M^2\}\Pr(K\neq i)\\
&=(\mathrm{E}\{M^2\}+\mathrm{E}\{L^2\})\Pr(K=i)+\mathrm{E}\{M^2\}\Pr(K\neq i)\\
&=1
\end{aligned}$$

(same as done for $r_{ii}$).

To sum up, we have that

$$r_{ij}=\begin{cases}c & i\neq j \text{ and } |j-i|\neq\frac{N}{2}\\ 1 & i=j \text{ or } |j-i|=\frac{N}{2}\end{cases}$$

meaning that

$$\mathbf{R}_{\boldsymbol{\varphi}}=\begin{pmatrix}
1 & c & \ldots & c & 1 & c & \ldots & c\\
c & 1 & \ldots & c & c & 1 & \ldots & c\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
c & c & \ldots & 1 & c & c & \ldots & 1\\
1 & c & \ldots & c & 1 & c & \ldots & c\\
c & 1 & \ldots & c & c & 1 & \ldots & c\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
c & c & \ldots & 1 & c & c & \ldots & 1
\end{pmatrix}.$$

Note that this is a circulant matrix.

### b.

Since $\mathbf{R}_{\boldsymbol{\varphi}}$ his circulant, the vector containing its eigenvalues ca be obtained if we multiply the [DFT$^*$] matrix by the transpose of the first row of $\mathbf{R}_{\boldsymbol{\varphi}}$.

# 4 Wiener Filter

Since $\mathcal{H}$ is linear, there is a matrix $\mathbf{H}$ such that $\mathcal{H}\boldsymbol{\varphi}=\mathbf{H}\boldsymbol{\varphi}$.

The autocorrelation matrix of $\boldsymbol{\varphi}^*$ is given by

$$\begin{aligned}
\mathbf{R}_{\varphi^*} &= \mathrm{E}\{\boldsymbol{\varphi}^*(\boldsymbol{\varphi}^*)^T\} \\
&= \mathrm{E}\{(\mathbf{H}\boldsymbol{\varphi}+\mathbf{n})(\mathbf{H}\boldsymbol{\varphi}+\mathbf{n})^T\} \\
&= \mathrm{E}\{(\mathbf{H}\boldsymbol{\varphi}+\mathbf{n})(\boldsymbol{\varphi}^T\mathbf{H}^T+\mathbf{n}^T)\} \\
&= \mathrm{E}\{\mathbf{H}\boldsymbol{\varphi}\boldsymbol{\varphi}^T\mathbf{H}^T\} + \mathrm{E}\{\mathbf{H}\boldsymbol{\varphi}\mathbf{n}^T\} + \mathrm{E}\{\mathbf{n}\boldsymbol{\varphi}^T\mathbf{H}^T\} + \mathrm{E}\{\mathbf{n}\mathbf{n}^T\} \\
&= \mathbf{H}\,\mathrm{E}\{\boldsymbol{\varphi}\boldsymbol{\varphi}^T\}\mathbf{H}^T + \mathbf{H}\,\mathrm{E}\{\boldsymbol{\varphi}\}(\mathrm{E}\{\mathbf{n}\})^T + \mathrm{E}\{\mathbf{n}\}(\mathrm{E}\{\boldsymbol{\varphi}\})^T + \mathrm{E}\{\mathbf{n}\mathbf{n}^T\} \\
&= \mathbf{H}\mathbf{R}_{\varphi}\mathbf{H}^T + \mathbf{0} + \mathbf{0} + \mathbf{R}_{\mathrm{n}} \\
&= \mathbf{H}\mathbf{R}_{\varphi}\mathbf{H}^T + \mathbf{R}_{\mathbf{n}}
\end{aligned}$$

since $\mathrm{E}\{\mathbf{n}\} = \mathbf{0}$. Hence we have that

$$\mathbf{R}_{\varphi^*} = \mathbf{H}\mathbf{R}_{\varphi}\mathbf{H}^T + \mathbf{R}_{\mathbf{n}}.$$

## b.

We consider $\mathbf{R}_{\mathbf{n}} = \sigma_n^2\mathbf{I}$, where $\sigma_n > 0$. Then the Wiener Filter is

$$\mathbf{W} = \mathbf{R}_{\varphi}\mathbf{H}^T(\mathbf{H}\mathbf{R}_{\varphi}\mathbf{H}^T + \sigma_n\mathbf{I})^{-1} = \mathbf{R}_{\varphi}\mathbf{H}^T\mathbf{R}_{\varphi^*}^{-1}.$$

## c.

Assume that $\mathbf{A}$ is an $N \times N$ matrix diagonalized by the [DFT]$^*$ matrix. We show that $\mathbf{A}$ is circulant.

There is a diagonal matrix $\mathbf{\Lambda_A}$ such that

$$\begin{aligned}
\mathbf{A} &= [\mathrm{DFT}]^*\mathbf{\Lambda_A}[\mathrm{DFT}] \\
&= \sum_{k=1}^{N} \lambda_k \mathbf{d}_k \mathbf{d}_k^*
\end{aligned}$$

where $\lambda_k$ are the diagonal values of $\mathbf{\Lambda_A}$ ($\mathbf{A}$'s eigenvalues).

Now, we show that $\mathbf{d}_k\mathbf{d}_k^*$ is circulant for all $k = 1, \ldots, N$. Then also $\lambda_k\mathbf{d}_k\mathbf{d}_k^*$ is circulant, and $\mathbf{A}$ is circulant as a sum of circulant matrices, which concludes the proof.

Let some $k = 1, \ldots, N$. By definition of the [$\mathbf{DFT}$]$^*$ matrix,

$$\mathbf{d}_k = \begin{pmatrix} W^{0\cdot(k-1)} \\ W^{1\cdot(k-1)} \\ \vdots \\ W^{(N-1)\cdot(k-1)} \end{pmatrix} = \begin{pmatrix} \exp\left(\frac{i2\pi}{N} 0 \cdot (k-1)\right) \\ \exp\left(\frac{i2\pi}{N} 1 \cdot (k-1)\right) \\ \vdots \\ \exp\left(\frac{i2\pi}{N}(N-1)\cdot(k-1)\right) \end{pmatrix}$$

and so its $(s,t)$ component of $\mathbf{d}_k\mathbf{d}_k^*$ is

$$\begin{aligned}(\mathbf{d}_k\mathbf{d}_k^*)_{st} &= \exp\left(\frac{i2\pi}{N}(s-1)(k-1)\right)\cdot\exp\left(-\frac{i2\pi}{N}(k-1)(t-1)\right)\\ &= \exp\left(\frac{i2\pi}{N}(s-1-t+1)(k-1)\right)\\ &= \exp\left(\frac{i2\pi}{N}(s-t)(k-1)\right)\\ &= \exp\left(\frac{i2\pi}{N}((s-t)(\mathrm{mod}\, N))(k-1)\right)\end{aligned}$$

Since this entry depends only on the difference between $s,t$ modulo $N$, we conclude that $\mathbf{d}_k\mathbf{d}_k^*$ is circulant.

## d. and e.

No, in general the Wiener filter is not a shift-invariant system: recall that $\mathbf{W}$ is not necessarily circulant, and so not necessarily shift-invariant.

Now, the Wiener filter is

$$\mathbf{W} = \mathbf{R}_{\boldsymbol{\varphi}}\mathbf{H}^*(\mathbf{H}\mathbf{R}_{\boldsymbol{\varphi}}\mathbf{H}^* + \sigma_n^2\mathbf{I})^{-1}$$

and so we notice that if $\mathbf{R}_{\boldsymbol{\varphi}}$ and $\mathbf{H}$ are circulant, then $\mathbf{W}$ is circulant and so shift-invariant.

This is due to the fact that $\mathbf{H}\mathbf{R}_{\boldsymbol{\varphi}}\mathbf{H}^*$ is circulant as a product of circulant matrices; that $\mathbf{H}\mathbf{R}_{\boldsymbol{\varphi}}\mathbf{H}^* + \sigma_n^2\mathbf{I}$ is circulant as a sum of circulant matrices; that $(\mathbf{H}\mathbf{R}_{\boldsymbol{\varphi}}\mathbf{H}^* + \sigma_n^2\mathbf{I})^{-1}$ is circulant as an inverse of a circulant matrix; and finally that $\mathbf{W} = \mathbf{R}_{\boldsymbol{\varphi}}\mathbf{H}^*(\mathbf{H}\mathbf{R}_{\boldsymbol{\varphi}}\mathbf{H}^* + \sigma_n^2\mathbf{I})^{-1}$ as a product of circulant matrices.

Now we prove the above claims we used:

1. *The product of circulant matrices is circulant:* Let $\mathbf{A},\mathbf{B}$ two circulant matrices, then they are diagonalized by the $[\mathrm{DFT}]^*$ matrix:

   $$\mathbf{A} = [\mathrm{DFT}]^*\boldsymbol{\Lambda}_{\mathbf{A}}[\mathrm{DFT}], \quad \mathbf{B} = [\mathrm{DFT}]^*\boldsymbol{\Lambda}_{\mathbf{B}}[\mathrm{DFT}].$$

   Then

   $$\mathbf{AB} = [\mathrm{DFT}]^*\boldsymbol{\Lambda}_{\mathbf{A}}[\mathrm{DFT}][\mathrm{DFT}]^*\boldsymbol{\Lambda}_{\mathbf{B}}[\mathrm{DFT}] = [\mathrm{DFT}]^*\boldsymbol{\Lambda}_{\mathbf{A}}\boldsymbol{\Lambda}_{\mathbf{B}}[\mathrm{DFT}]$$

   and notice that $\boldsymbol{\Lambda}_{\mathbf{A}}\boldsymbol{\Lambda}_{\mathbf{B}}$ is diagonal as the product of two diagonal matrices. Then $\mathbf{AB}$ is diagonalized by the $[\mathrm{DFT}]^*$ matrix and hence circulant.

2. *The sum of circulant matrices is circulant:* Again, let $\mathbf{A},\mathbf{B}$ two circulant matrices, then they are diagonalized by the $[\mathrm{DFT}]^*$ matrix:

   $$\mathbf{A} = [\mathrm{DFT}]^*\boldsymbol{\Lambda}_{\mathbf{A}}[\mathrm{DFT}], \quad \mathbf{B} = [\mathrm{DFT}]^*\boldsymbol{\Lambda}_{\mathbf{B}}[\mathrm{DFT}].$$

Then

$$\mathbf{A}+\mathbf{B} = [\mathrm{DFT}]^*\mathbf{\Lambda_A}[\mathrm{DFT}] + [\mathrm{DFT}]^*\mathbf{\Lambda_B}[\mathrm{DFT}] = [\mathrm{DFT}]^*(\mathbf{\Lambda_A}+\mathbf{\Lambda_B})[\mathrm{DFT}]$$

and notice that $\mathbf{\Lambda_A}+\mathbf{\Lambda_B}$ is diagonal as the sum of two diagonal matrices. Then $\mathbf{A}+\mathbf{B}$ is diagonalized by the $[\mathrm{DFT}]^*$ matrix and hence circulant.

3. *The inverse of a circulant matrix is circulant:* Let $\mathbf{A}$ a circulant matrix, then it is diagonalized by the $[\mathrm{DFT}]^*$ matrix:

$$\mathbf{A} = [\mathrm{DFT}]^*\mathbf{\Lambda_A}[\mathrm{DFT}].$$

Then

$$\mathbf{A}^{-1} = [\mathrm{DFT}]^{-1}\mathbf{\Lambda_A}^{-1}([\mathrm{DFT}]^*)^{-1} = [\mathrm{DFT}]^*\mathbf{\Lambda_A}^{-1}[\mathrm{DFT}]$$

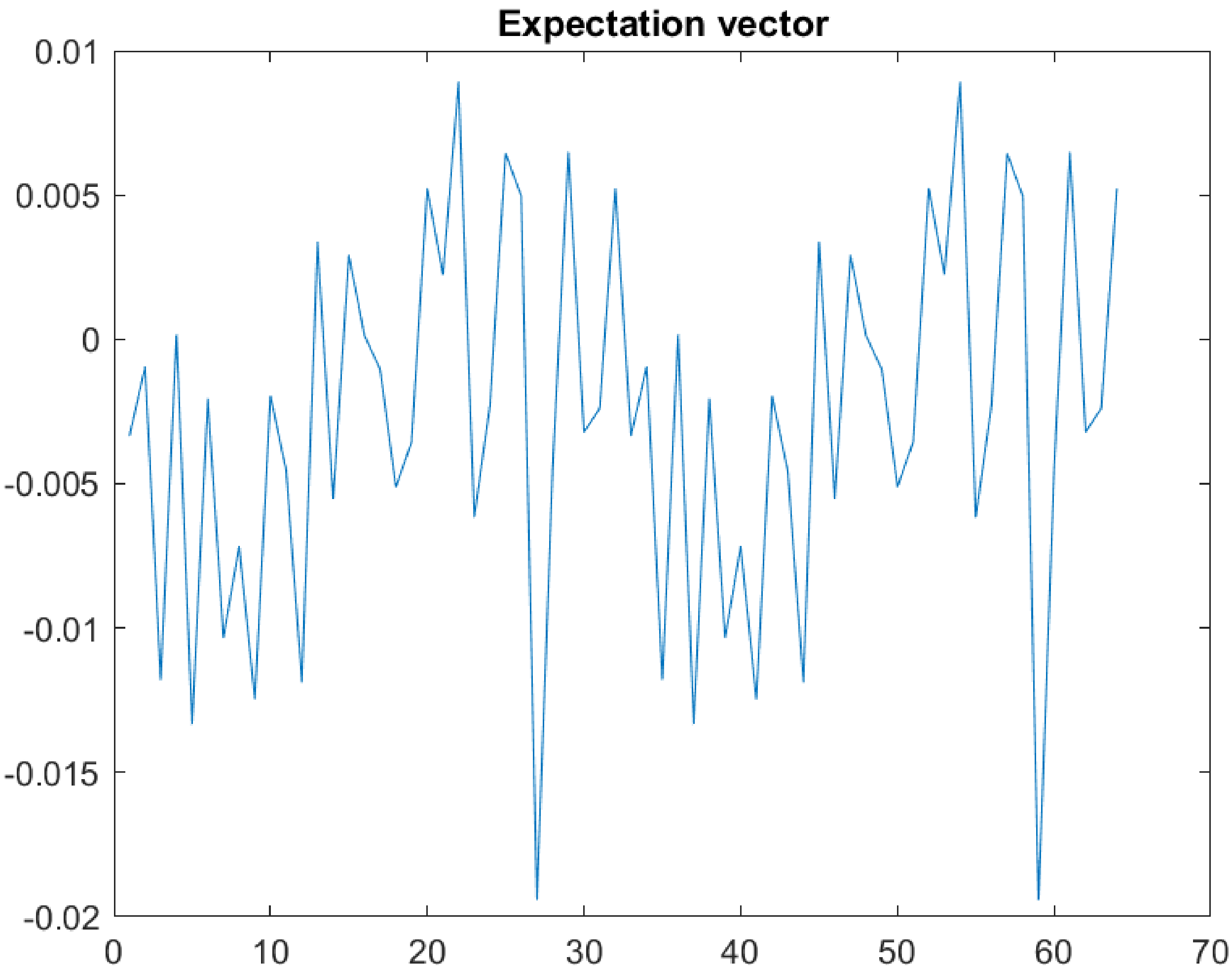
since $[\mathrm{DFT}]^{-1} = [\mathrm{DFT}]^*$, and notice that $\mathbf{\Lambda_A}^{-1}$ is diagonal as the inverse of a diagonal matrix. Then $\mathbf{A}^{-1}$ is diagonalized by the $[\mathrm{DFT}]^*$ matrix and hence circulant.

# Part2 – Matlab:

## a.

We choose a 10000 realizations of a random signal phi.

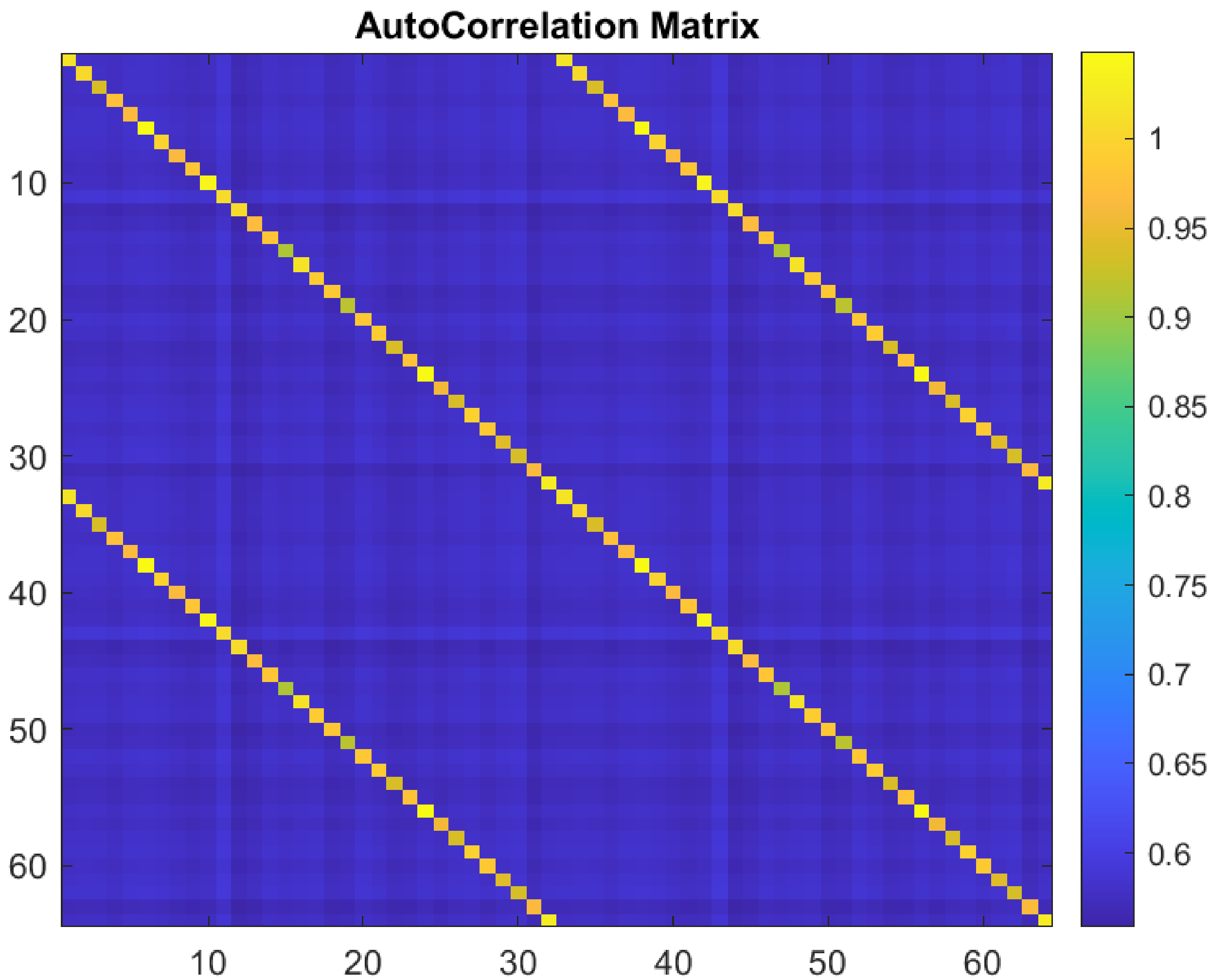
The expectation vector we've got is:




And:

```
Expectation vector MSE is:
   9.7048e-05
```

And the autocorrelation is:




And for 10000 realizations:

As we've got analytically.

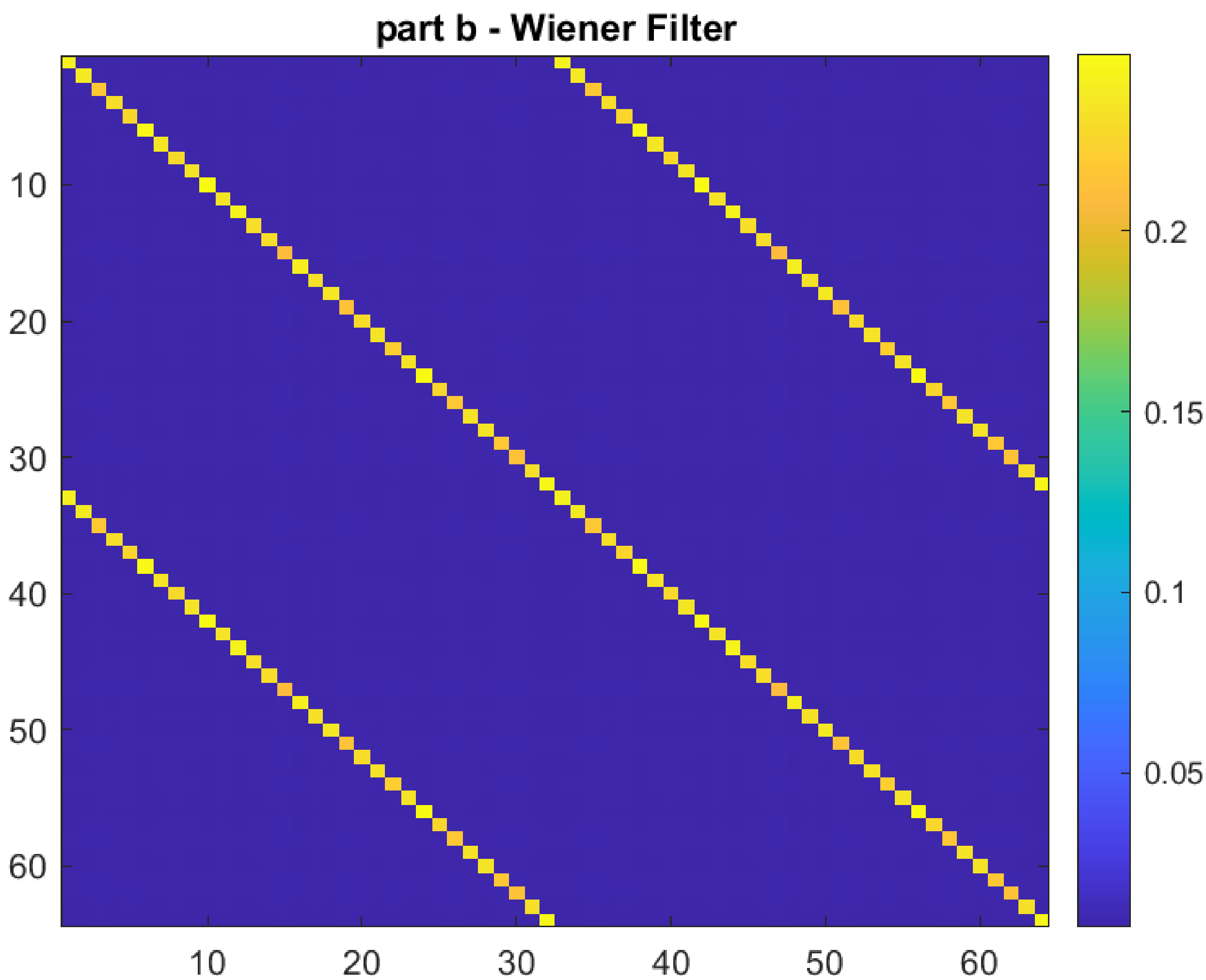
```
The MSE of the empirical AutoCorreation matrix and the one we found anayticly is:
   1.1484e-04
```

Although the in terms of MSE even with 10 realization we've got 0.10 MSE error, as we could observe, it is needed at least 1000 realizations to see this structures of diagonals that can be seen above, and with 10000 it also makes all the none diagonals even closer to c value, as is desirable. With a 1000 and higher number of realizations the MSE is dropped significantly.

b.

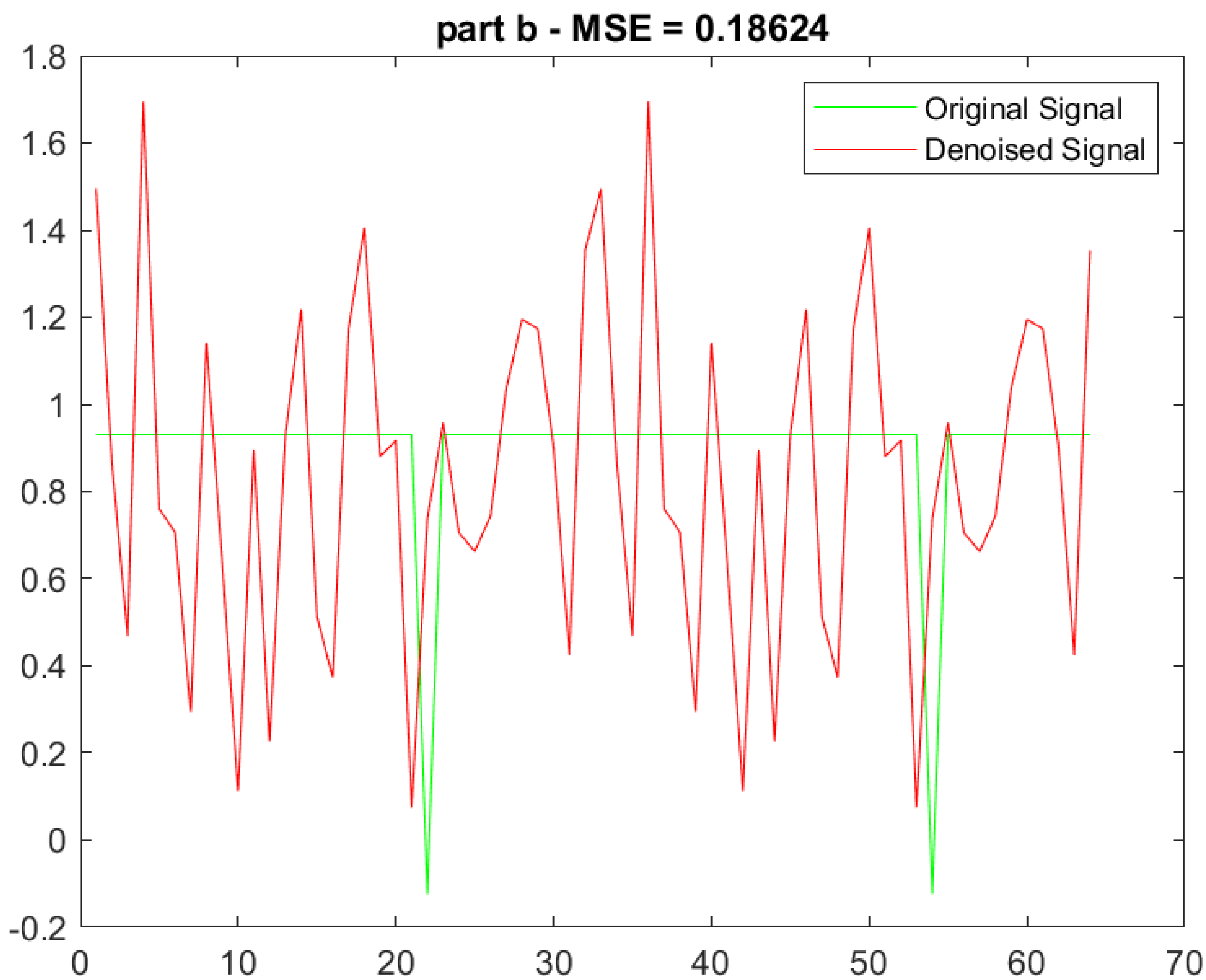
We’ve continued with the 10000 number of realizations, also for noise.

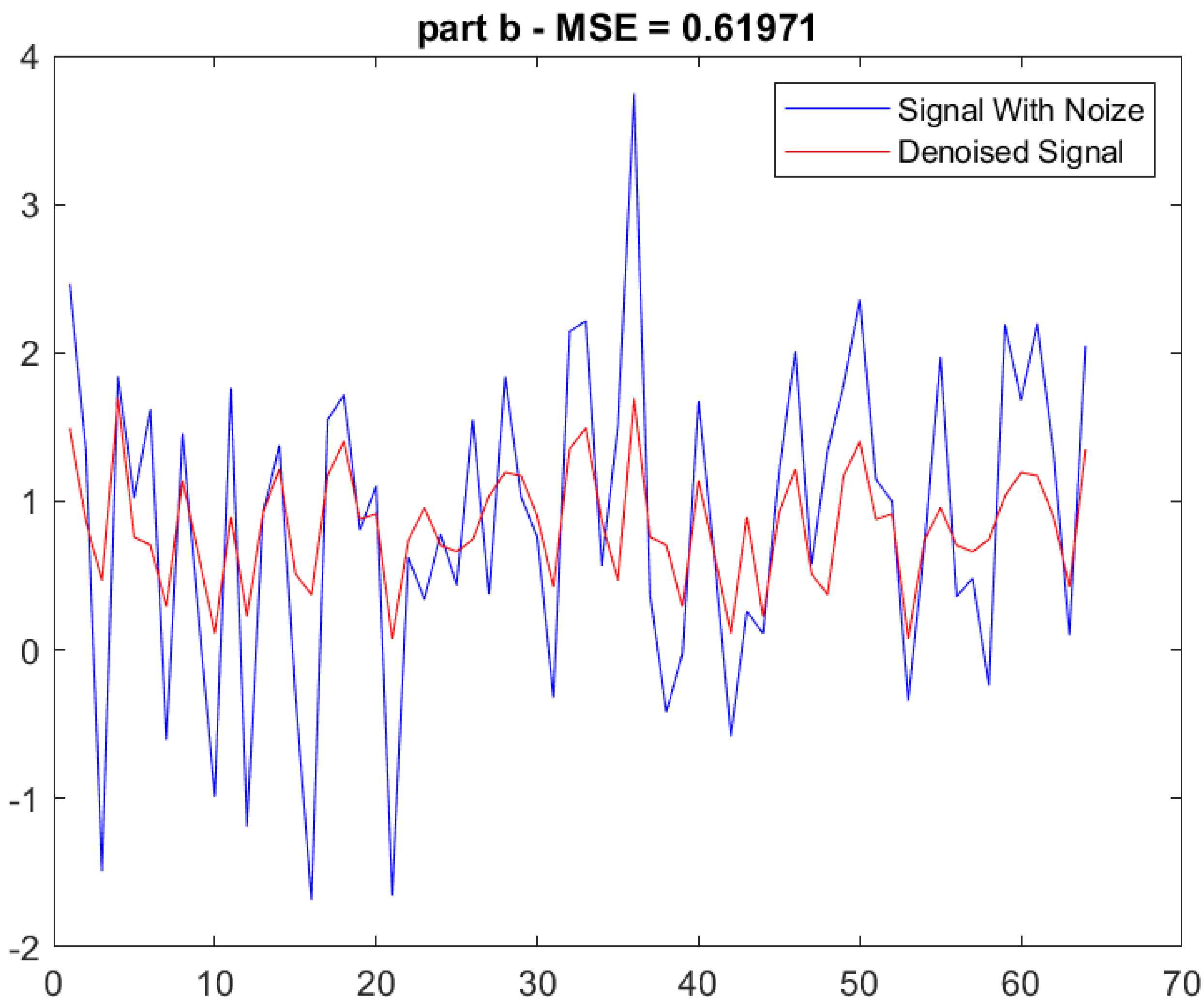
The Wiener filter that we’ve got using our Autocorrelation matrix is:



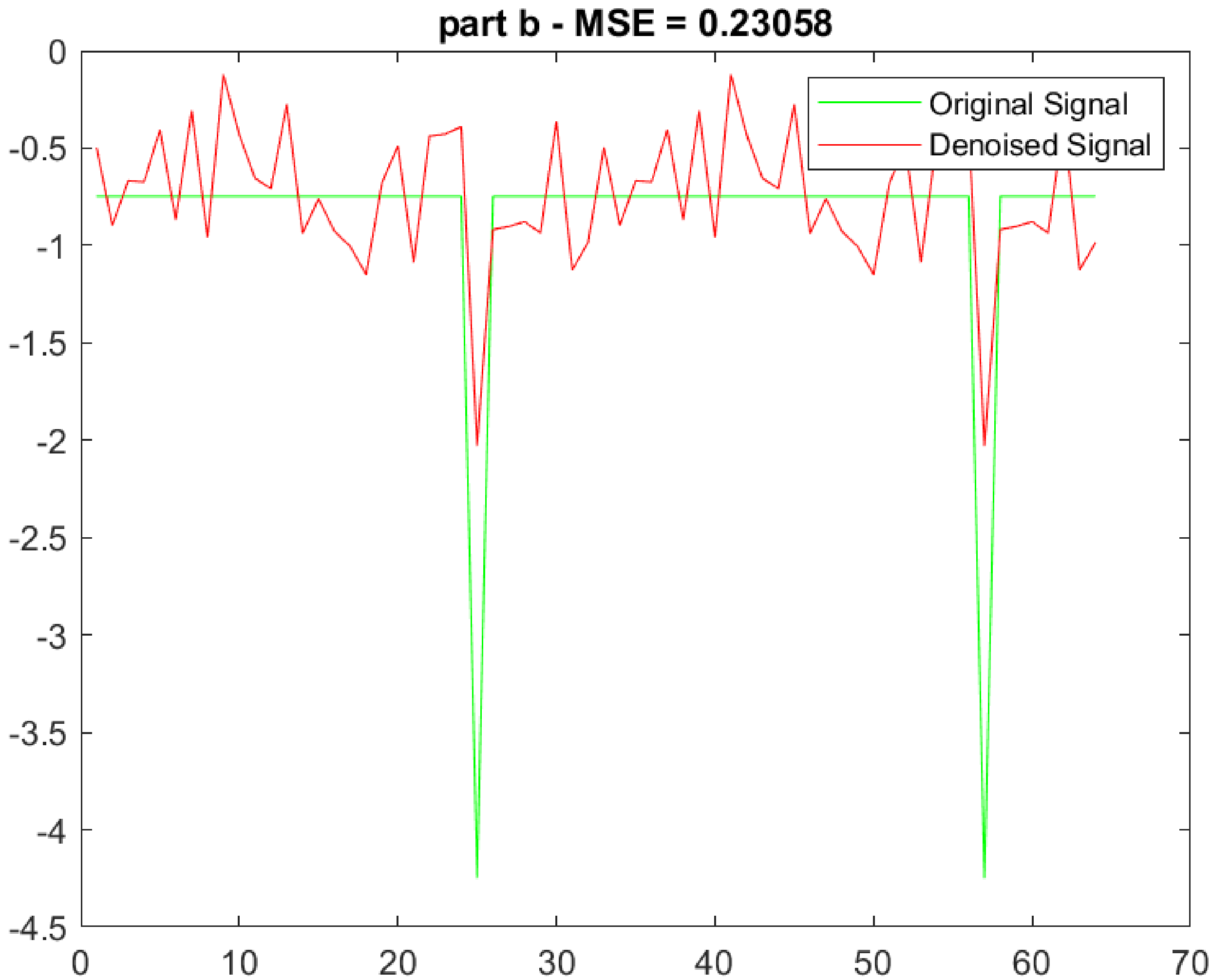

For 10000 realizations we’ve plotted each 2000 an example of a signal with the graph and the MSE:

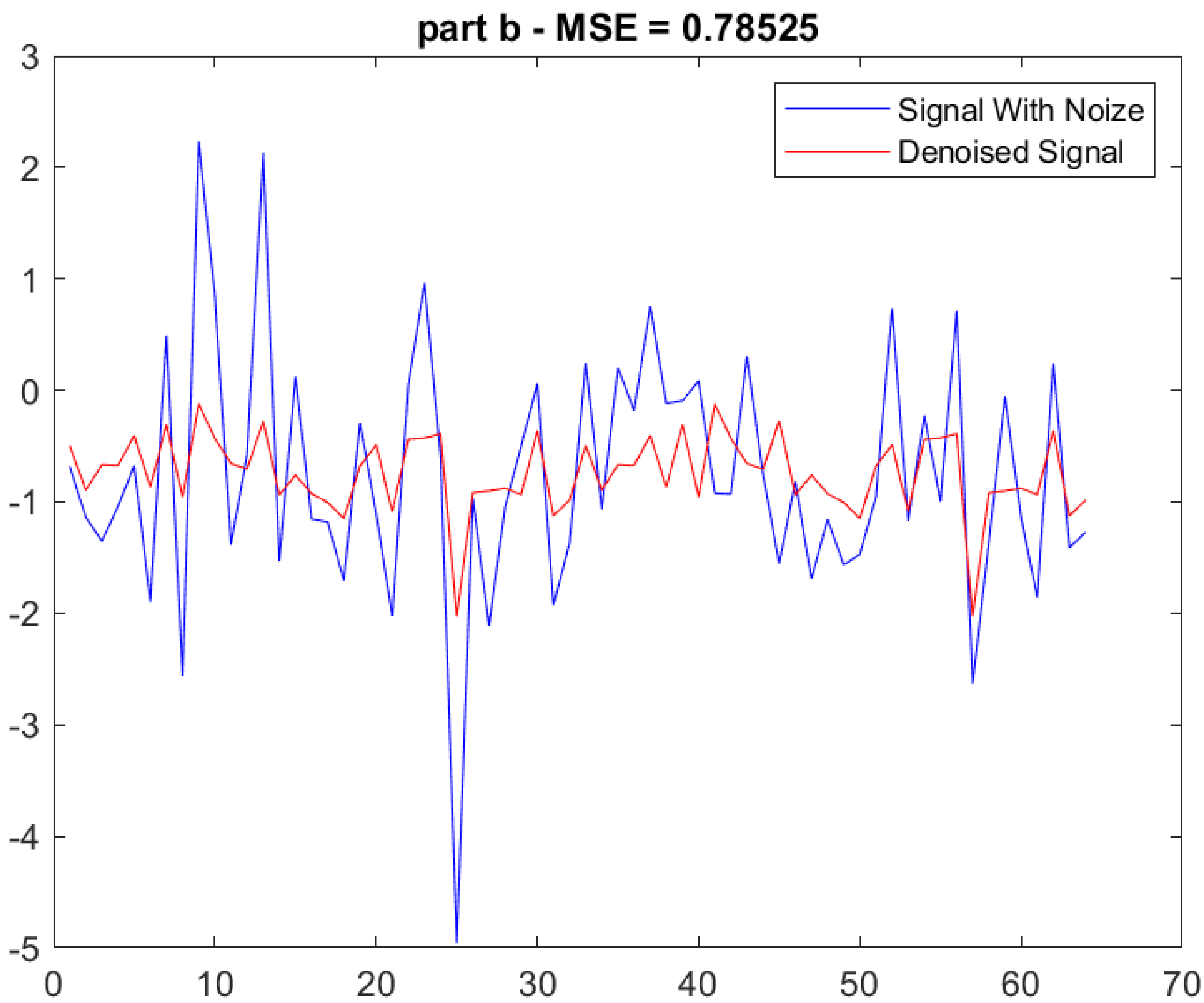
We’ve compared each clean signal to its restored signal (which is called ‘Denoised Signal’) and also the noised signal to the restored.

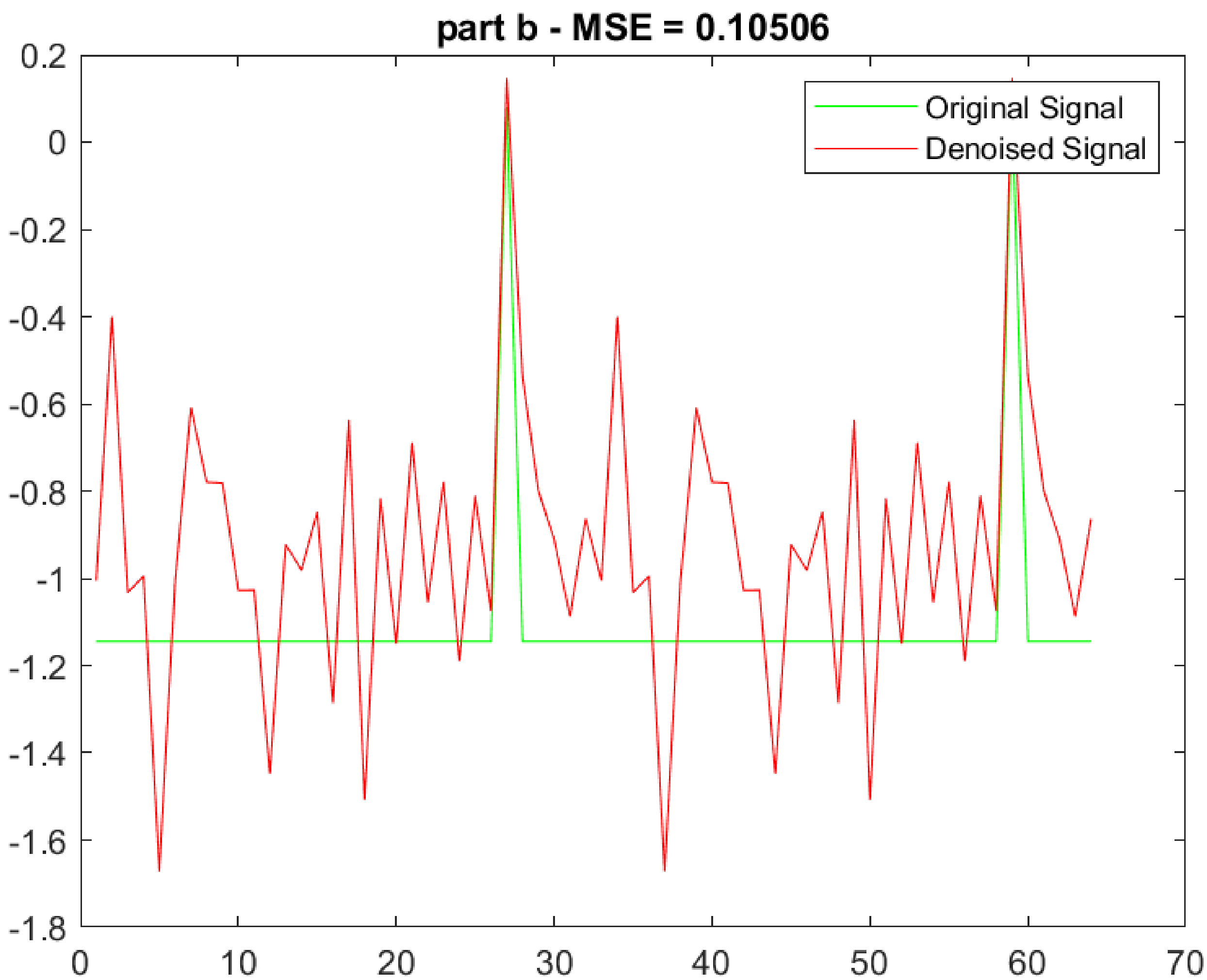


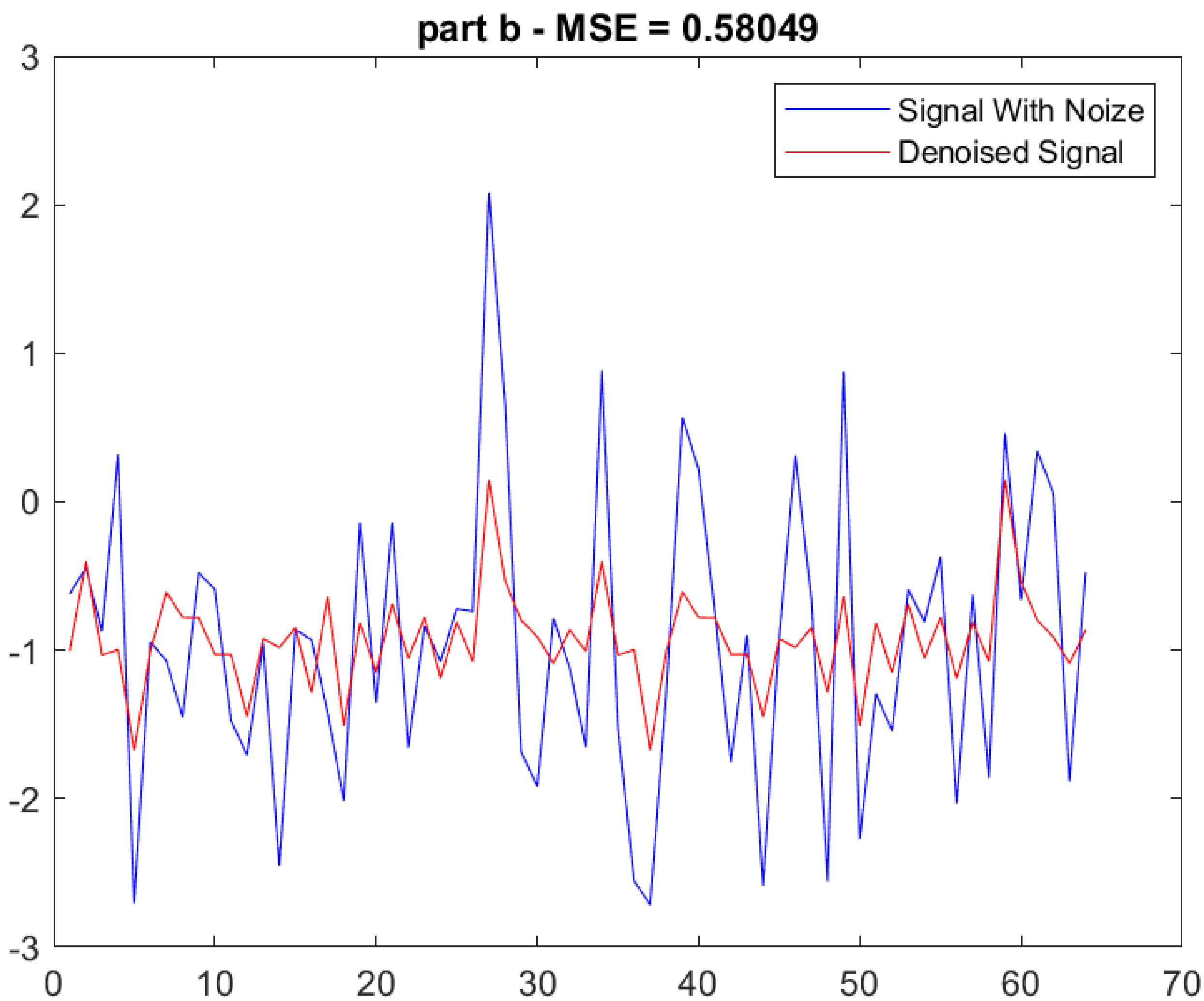


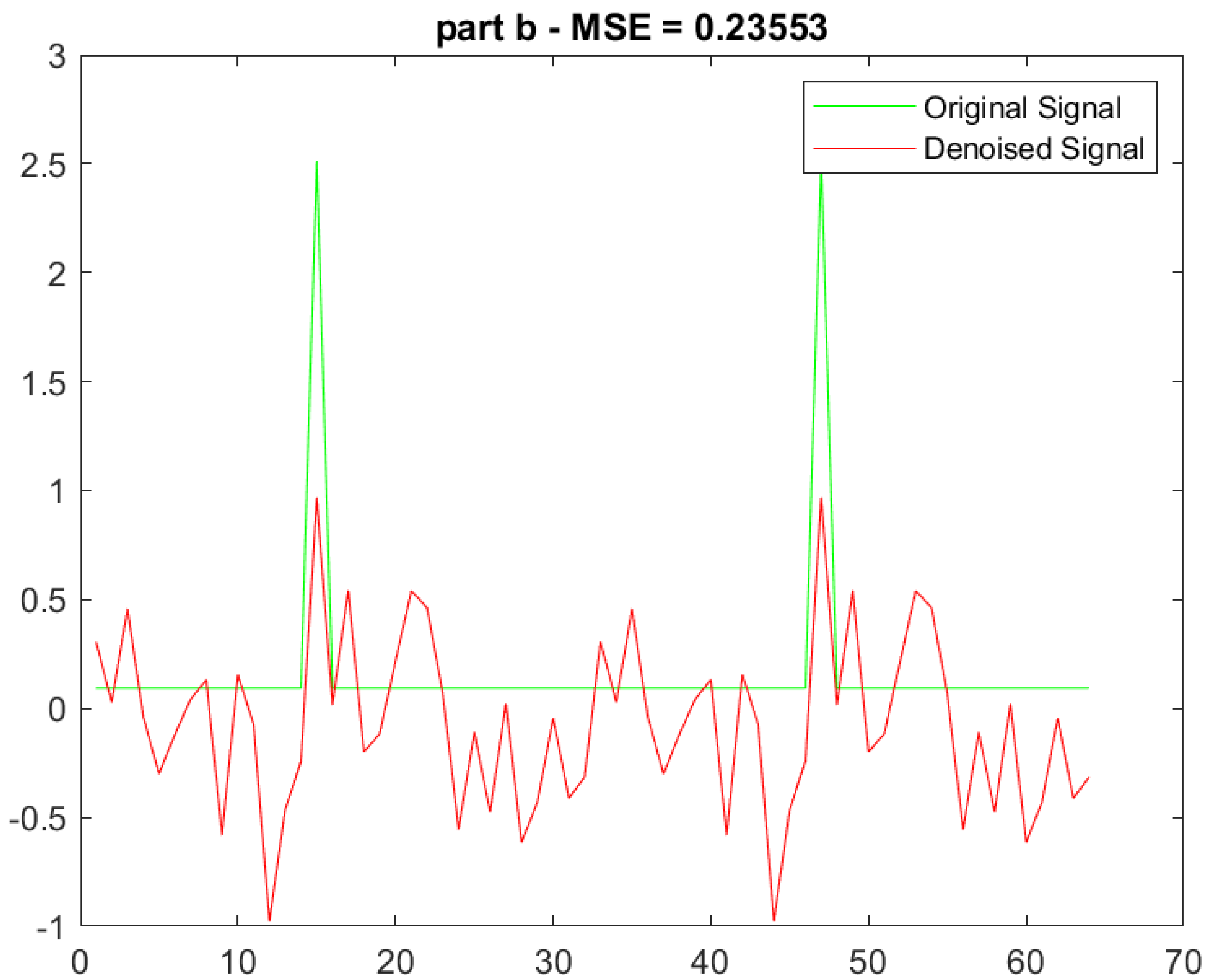
part b - MSE = 0.61971
Signal With Noize
Denoised Signal
4
3
2
1
0
-1
-2
0
10
20
30
40
50
60
70


part b - MSE = 0.23058
Original Signal
Denoised Signal
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
0
10
20
30
40
50
60
70

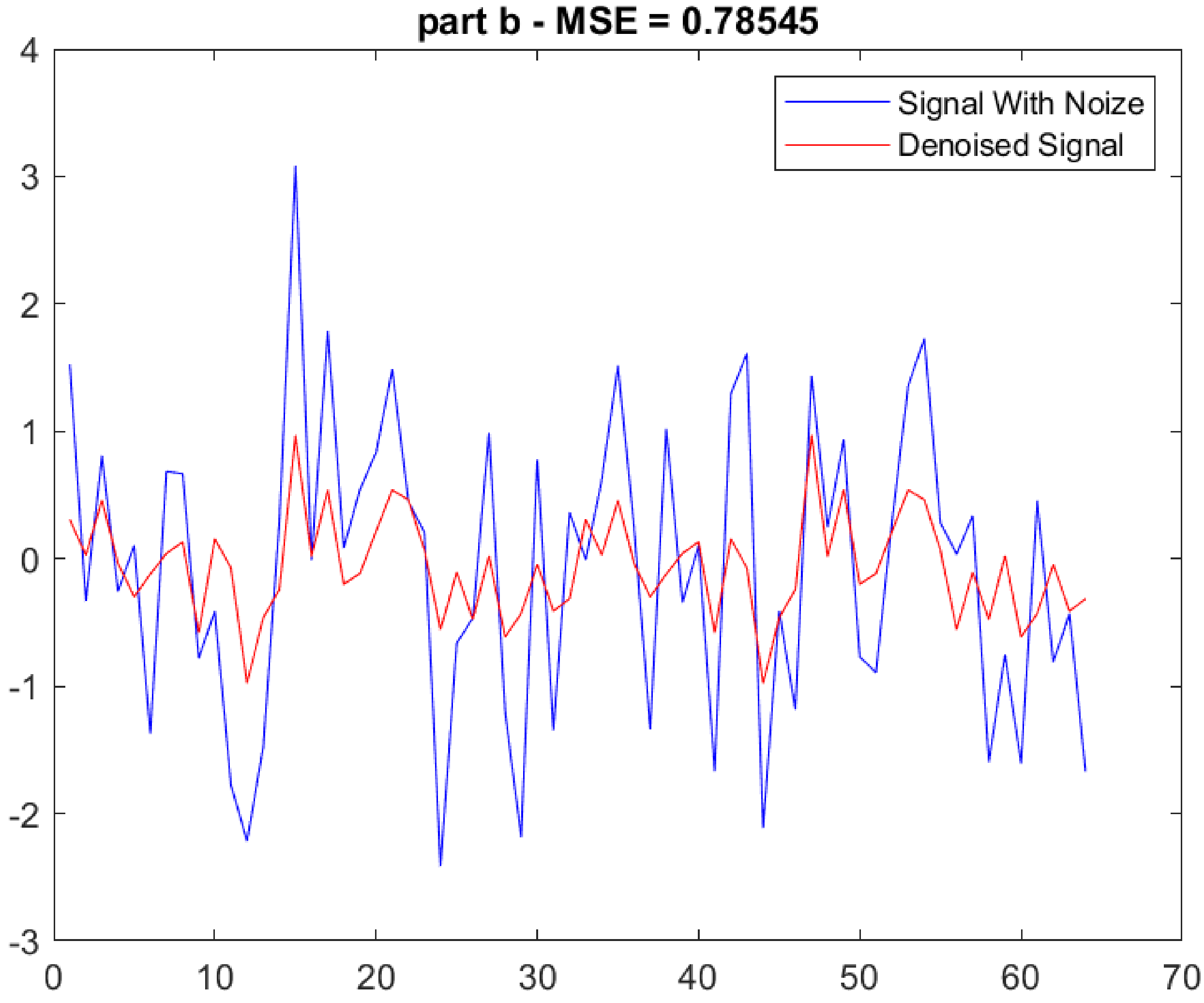

part b - MSE = 0.78525
Signal With Noize
Denoised Signal
3
2
1
0
-1
-2
-3
-4
-5
0
10
20
30
40
50
60
70

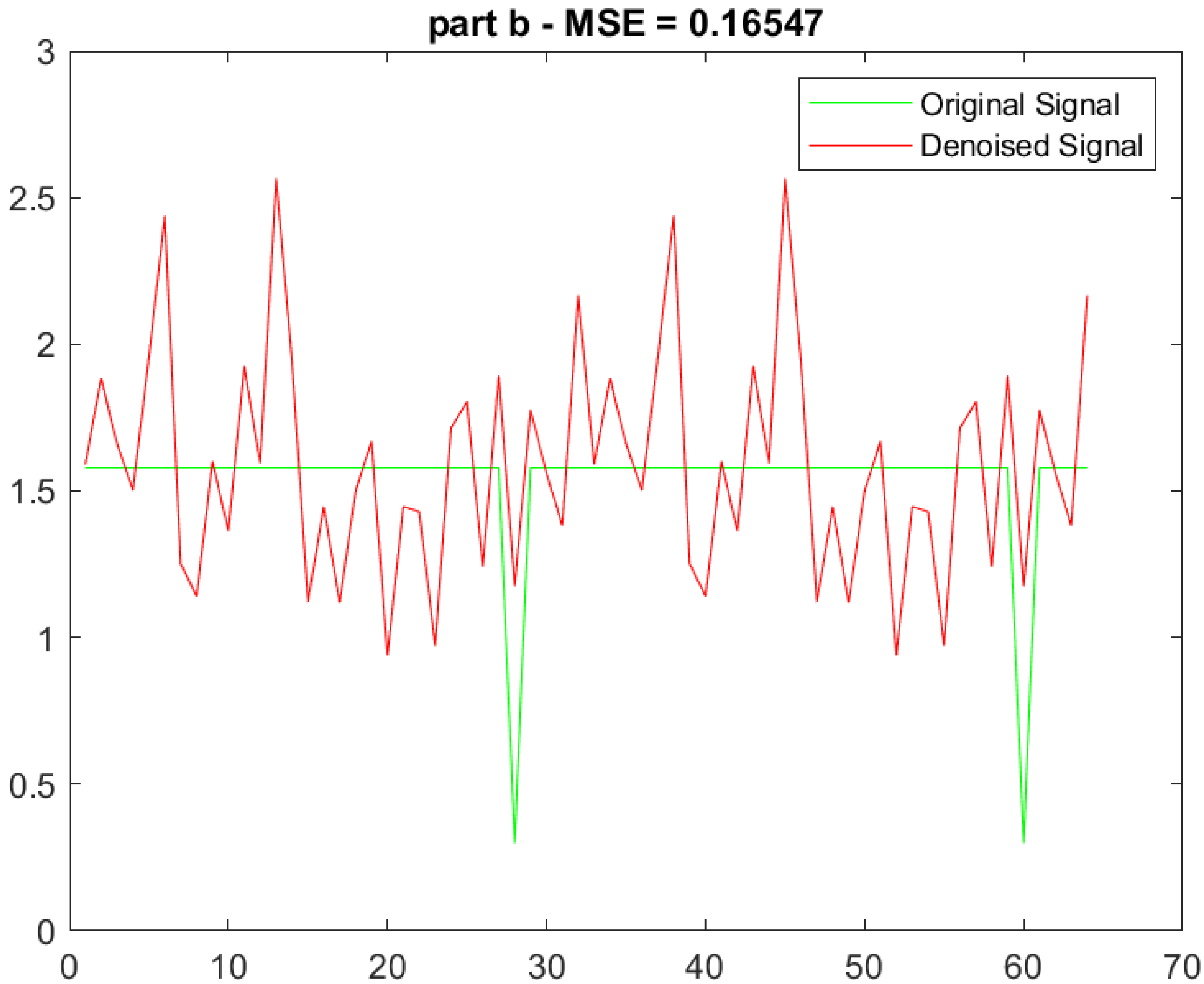

part b - MSE = 0.10506
Original Signal
Denoised Signal
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
-1.8
0
10
20
30
40
50
60
70


part b - MSE = 0.58049
Signal With Noize
Denoised Signal
3
2
1
0
-1
-2
-3
0
10
20
30
40
50
60
70

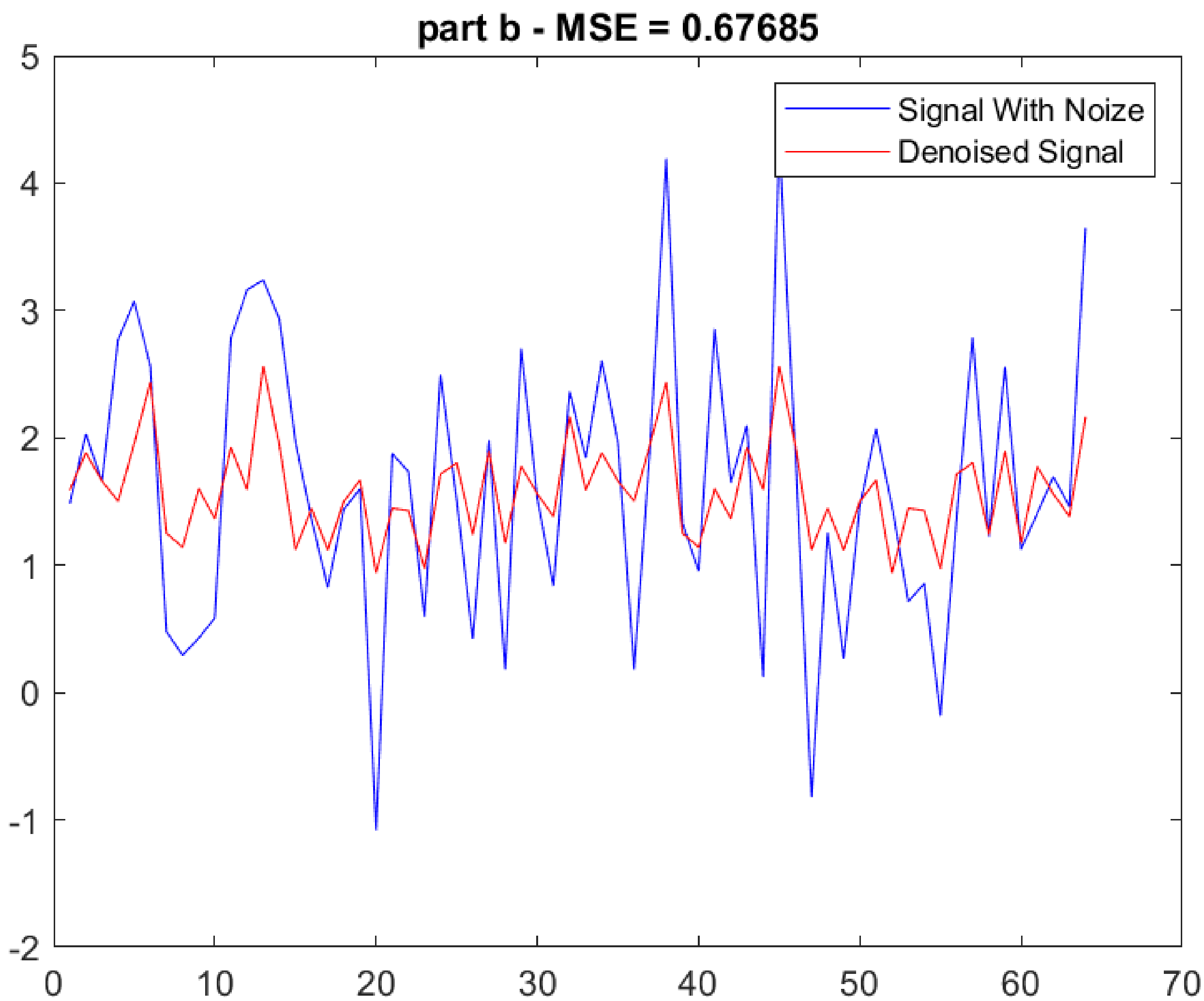


part b - MSE = 0.23553
Original Signal
Denoised Signal
3
2.5
2
1.5
1
0.5
0
-0.5
-1
0
10
20
30
40
50
60
70


part b - MSE = 0.78545
Signal With Noize
Denoised Signal
4
3
2
1
0
-1
-2
-3
0
10
20
30
40
50
60
70



part b - MSE = 0.16547
Original Signal
Denoised Signal
3
2.5
2
1.5
1
0.5
0
0
10
20
30
40
50
60
70



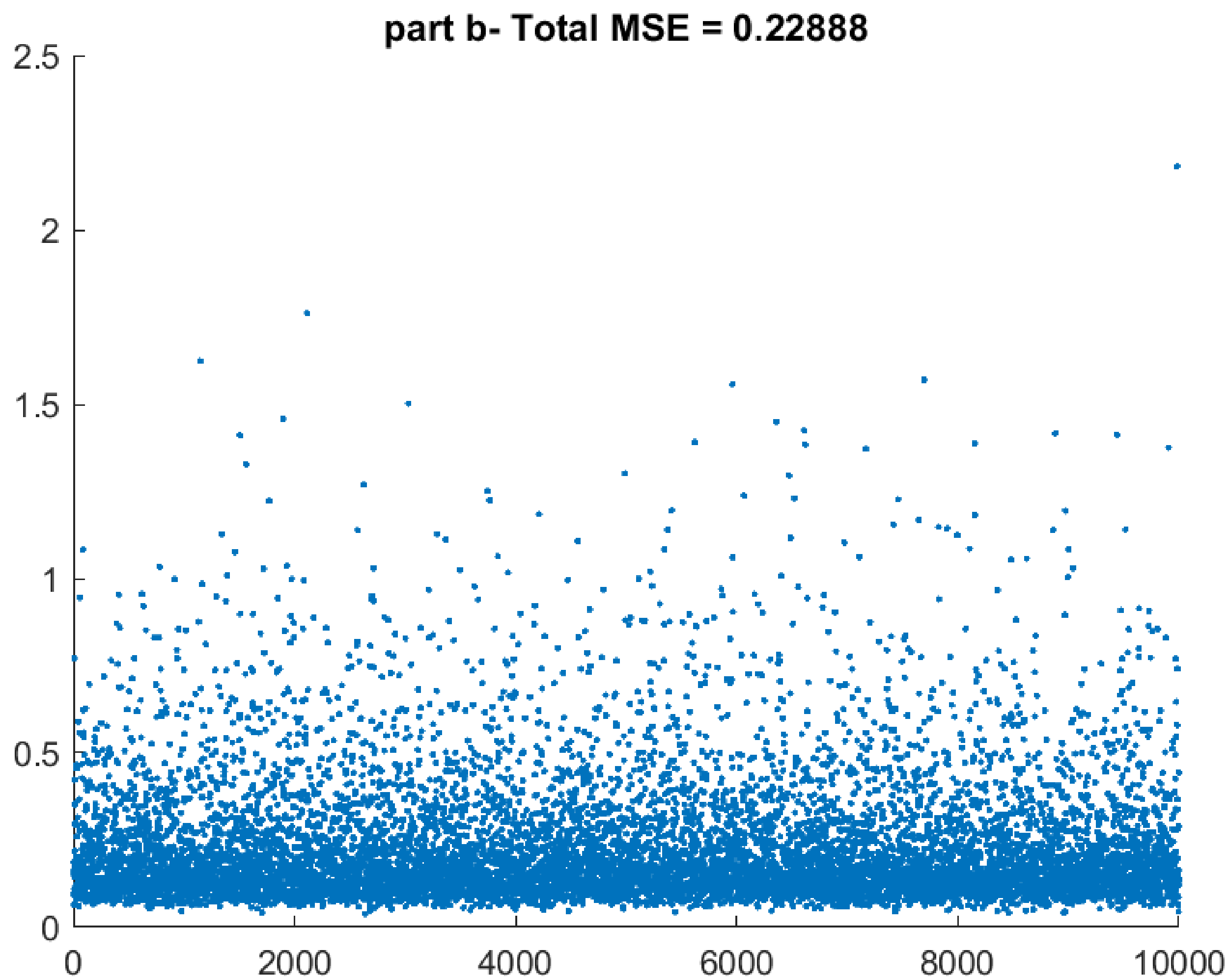

As can be observed from the graphs above:

In the green-red(clean-denoised) graphs in some cases we can see clearly correspondence of the spikes in the clean signal to the spikes in the resotred signal.

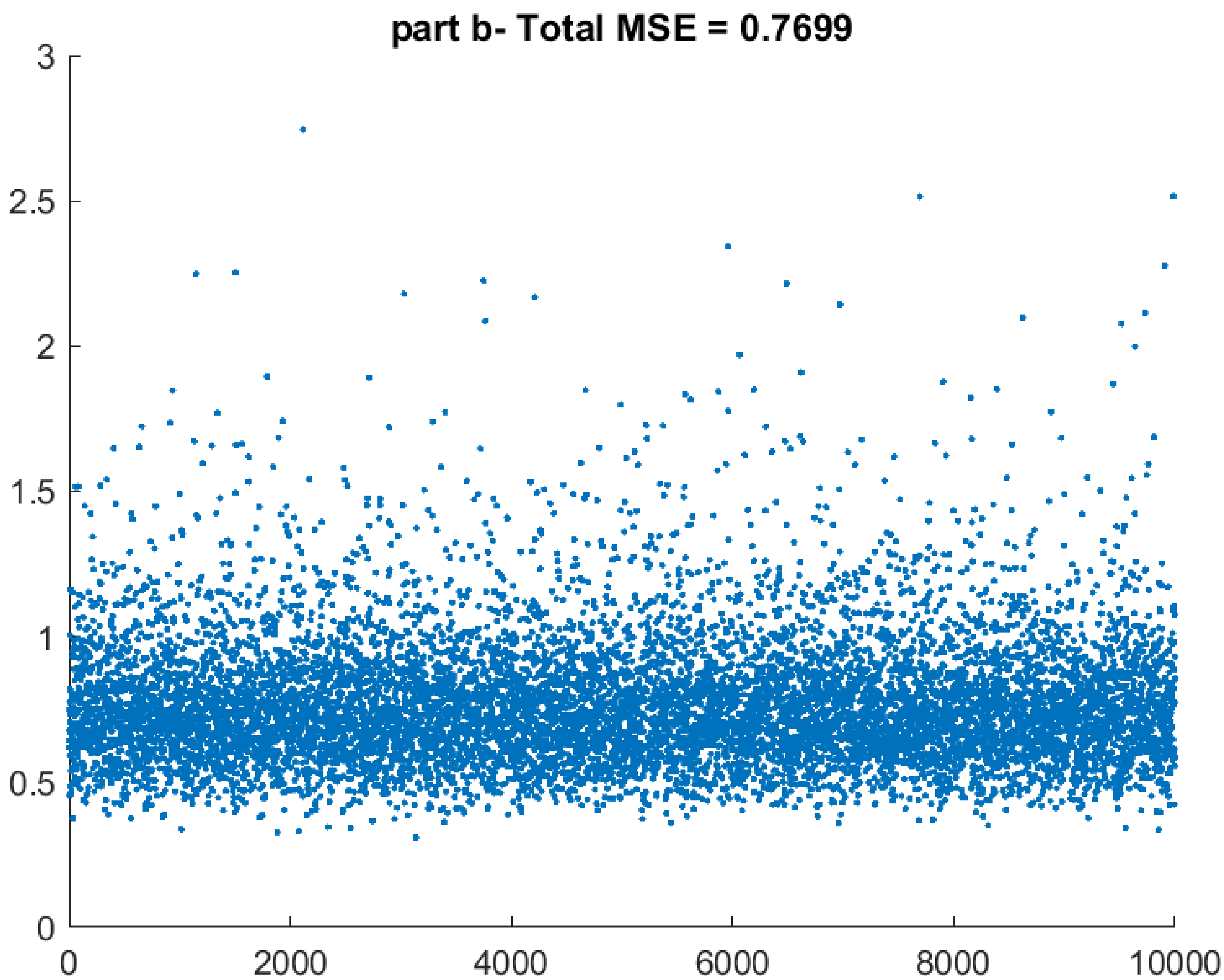
The blue-red graphs teach us about another aspect, which is the big difference in variance between the clean and restored signals of the resotred signal, that caused by the non-ideal restoration, which in some way similar to the noised correspondng signals. That's why in addition to the corresponding spikes there is a lot of disturbance, which is cause becaused of the non-ideal restoration.

It also expectable that the MSE of the restored signal and the clean signal is better then the MSE of the restored and the noised, as this is exactly what Wiener filter does.

The total MSE is of the restored signal and the clean signal is:

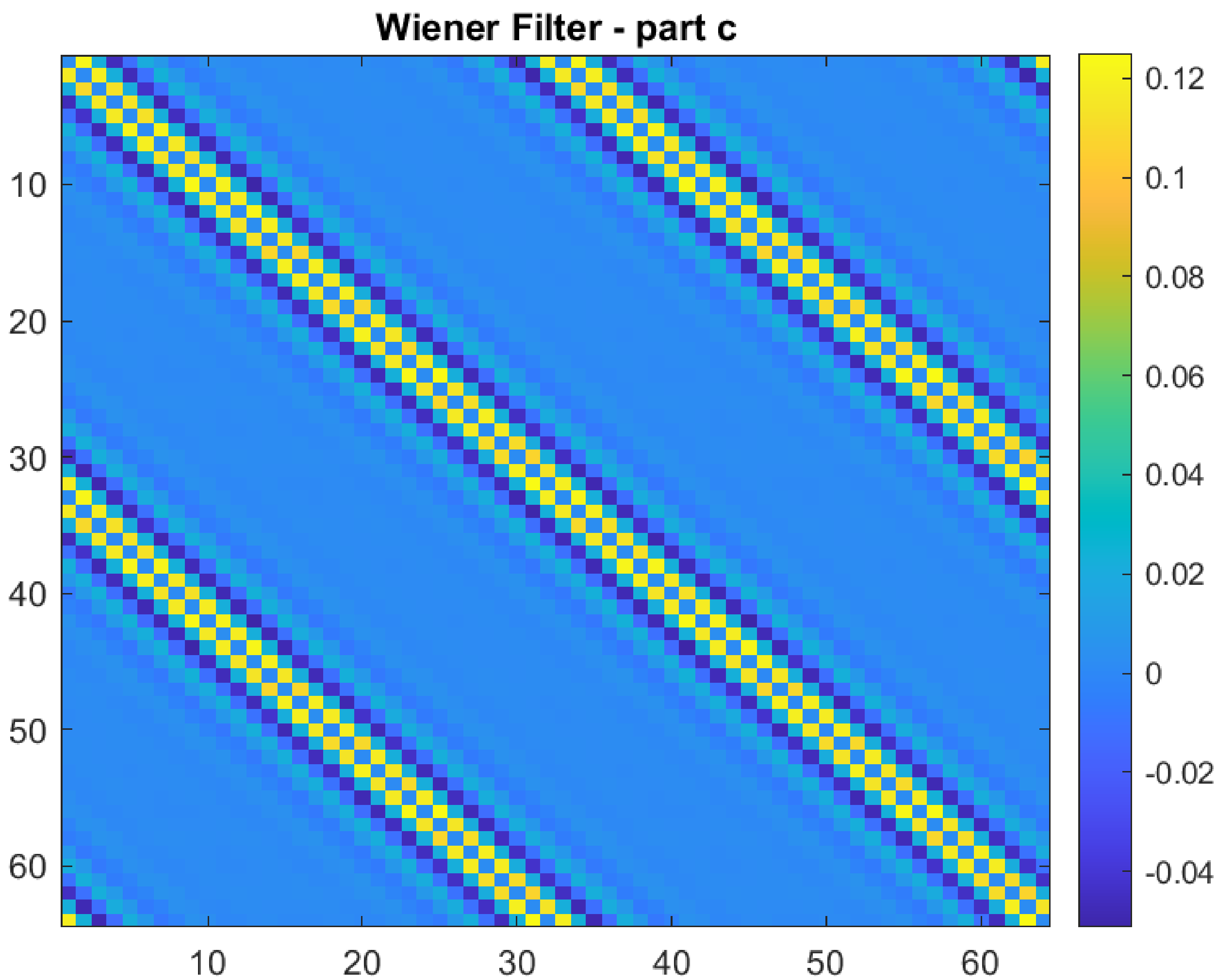



And the total MSE of the resotred signal and the noisy signal is:

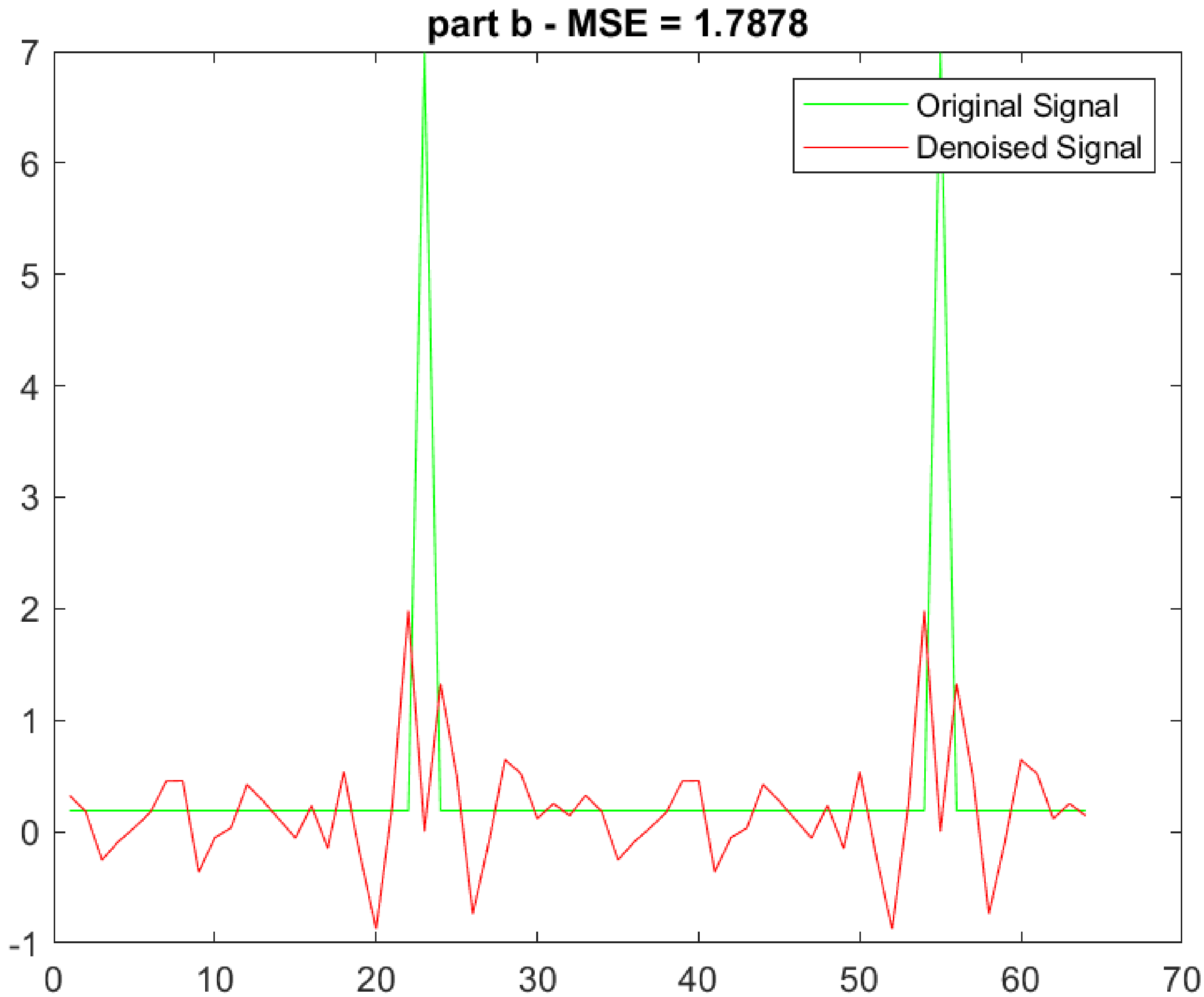



c.

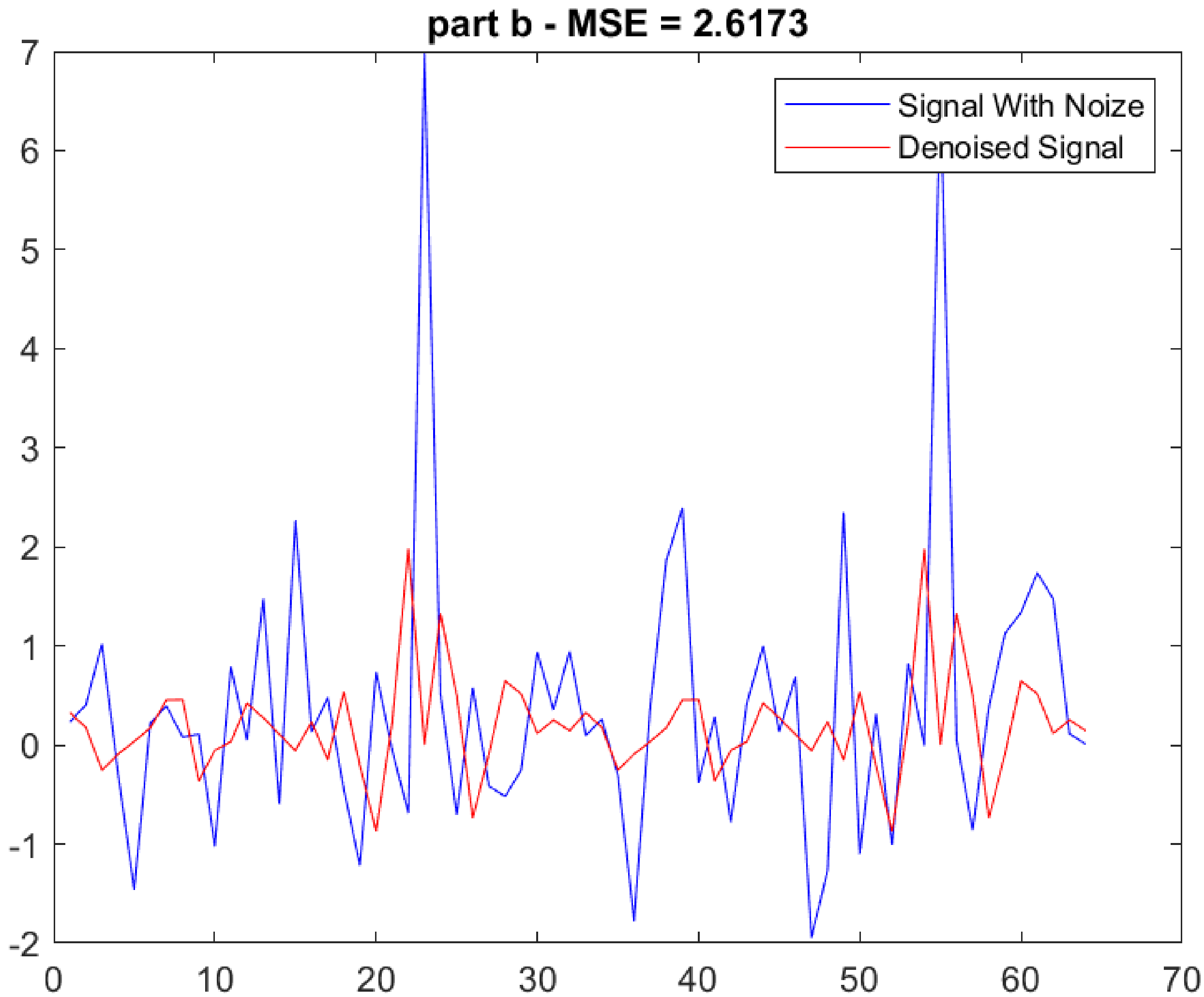
The Wiener filter:

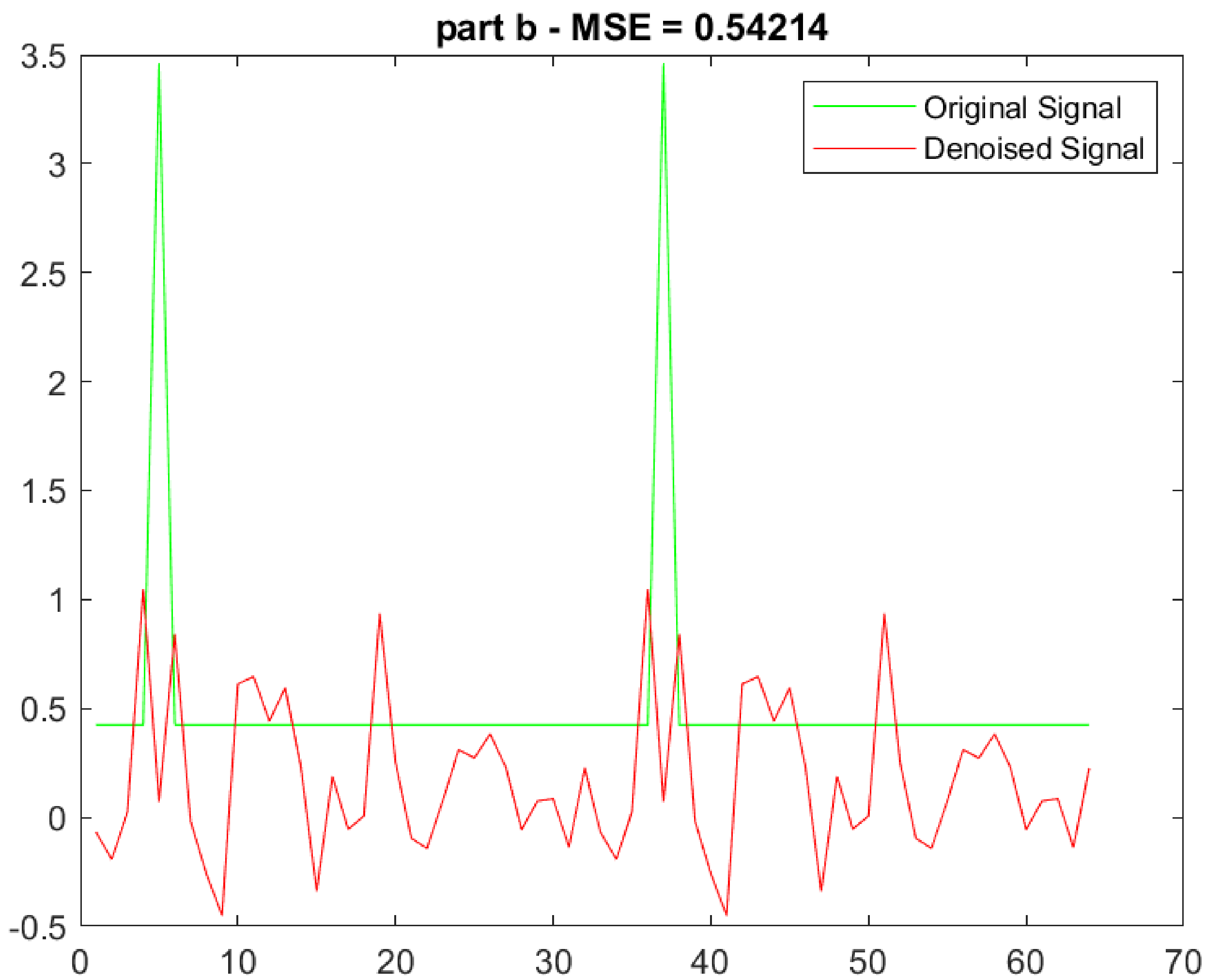



And the graphs as before:

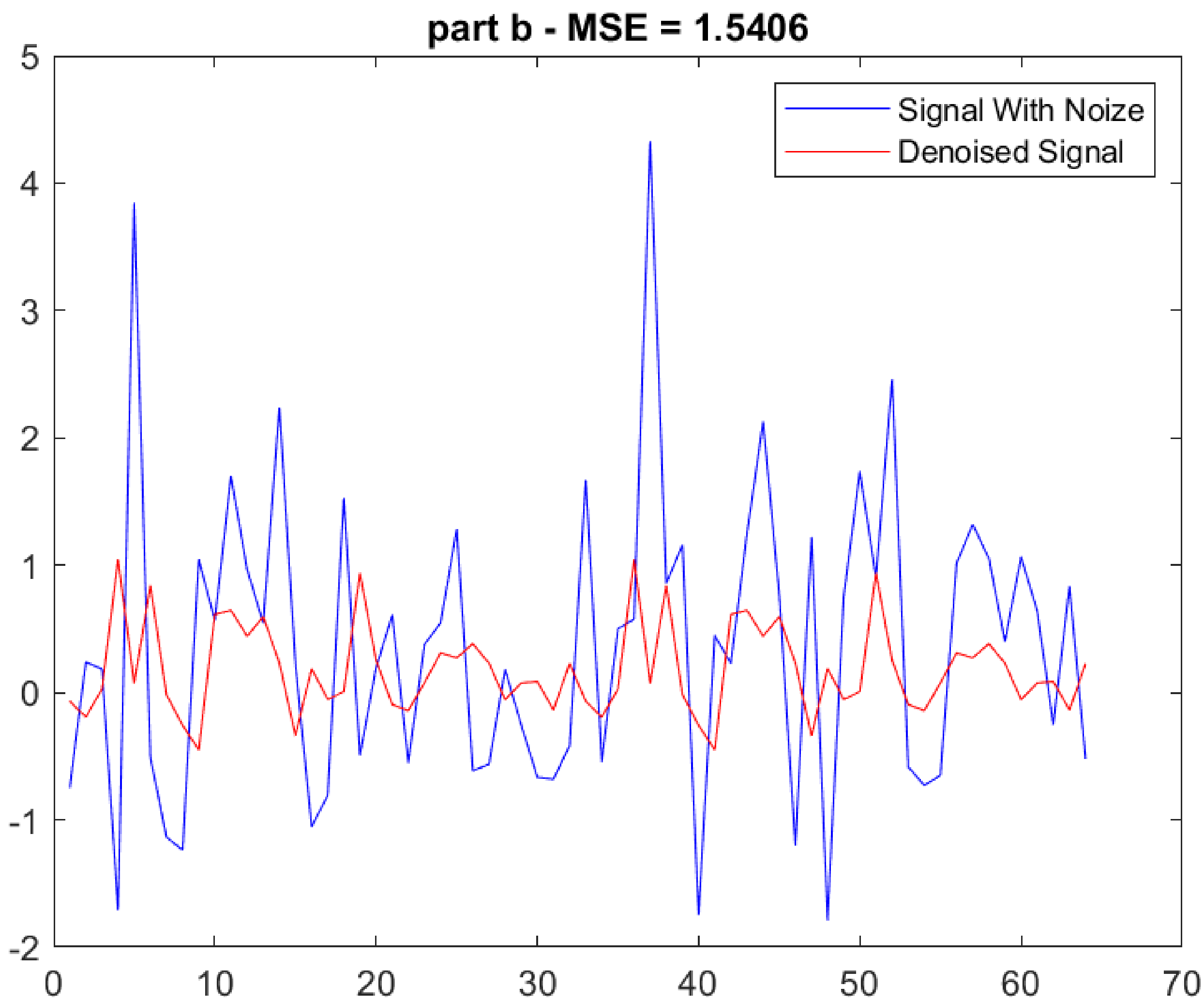


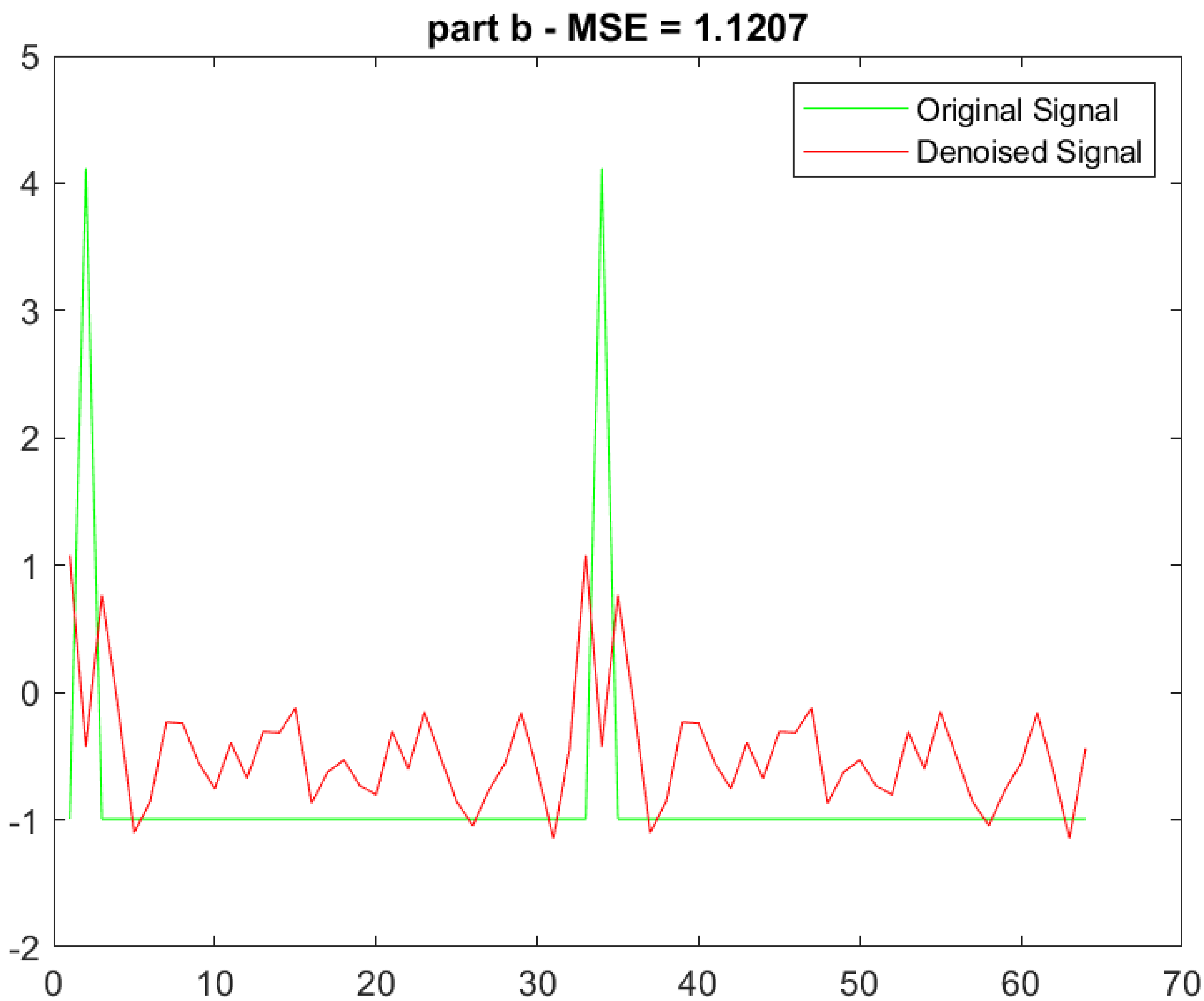
part b - MSE = 1.7878
Original Signal
Denoised Signal
7
6
5
4
3
2
1
0
-1
0
10
20
30
40
50
60
70

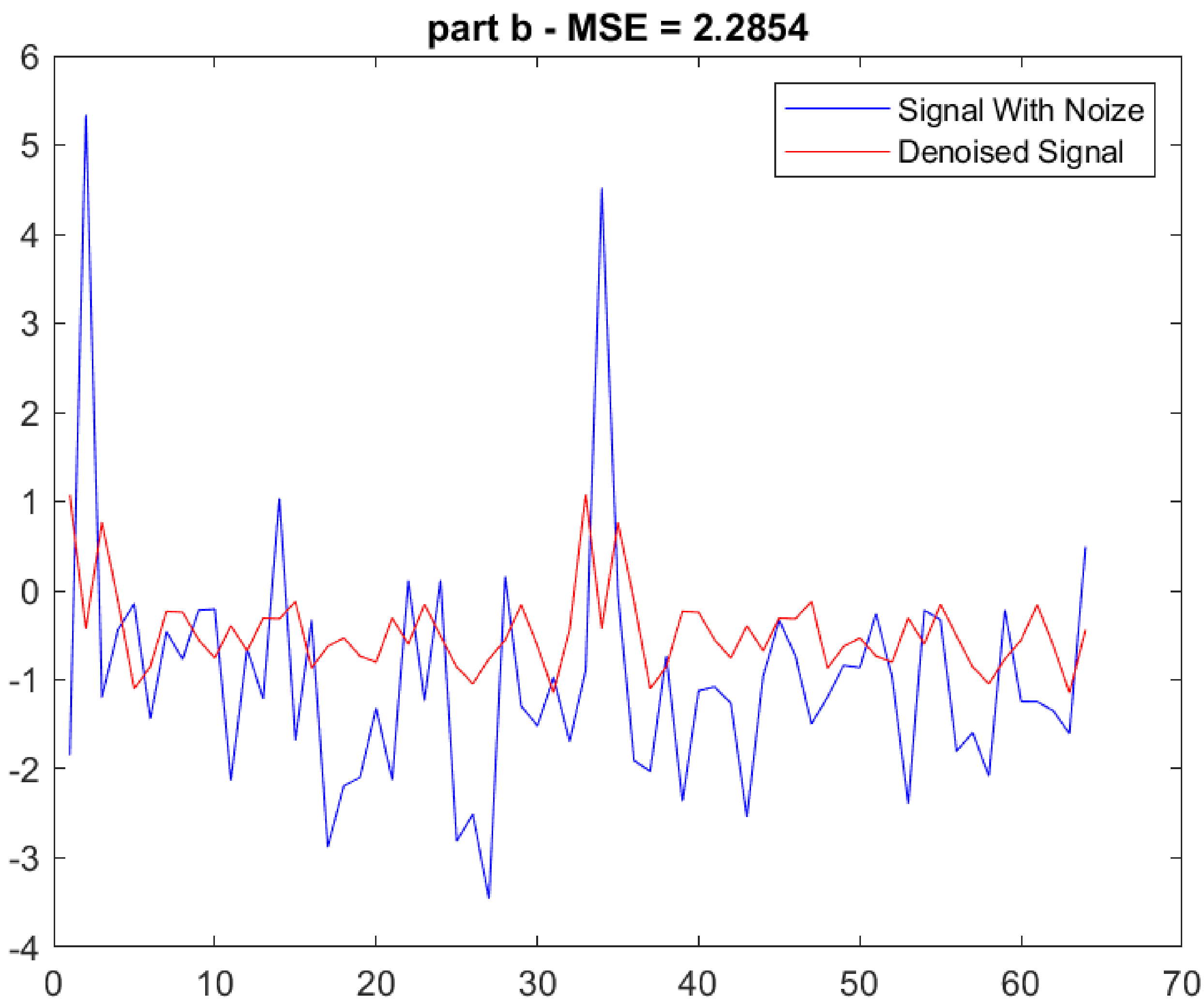

part b - MSE = 2.6173
Signal With Noize
Denoised Signal
7
6
5
4
3
2
1
0
-1
-2
0
10
20
30
40
50
60
70



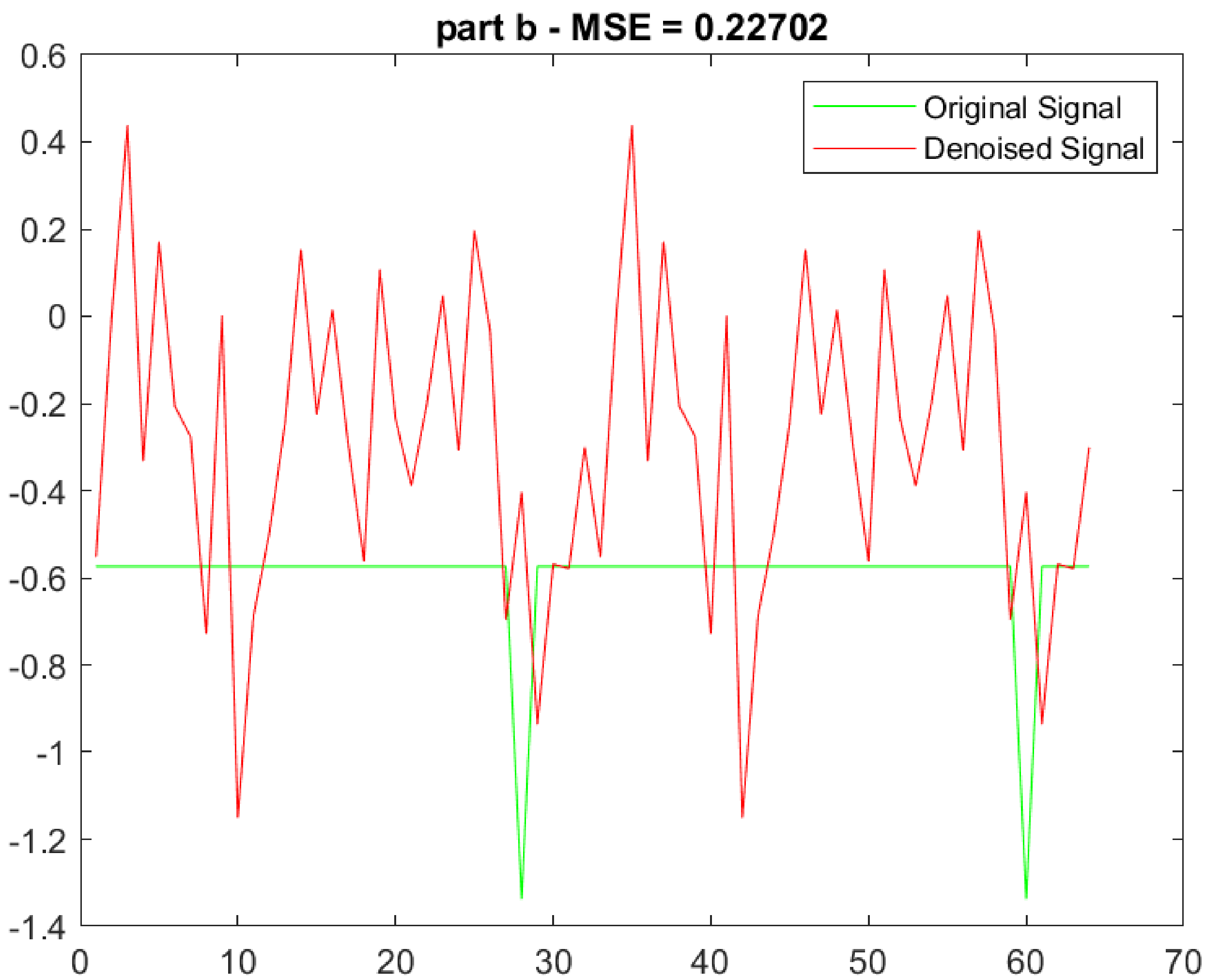
part b - MSE = 0.54214
Original Signal
Denoised Signal
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
0
10
20
30
40
50
60
70

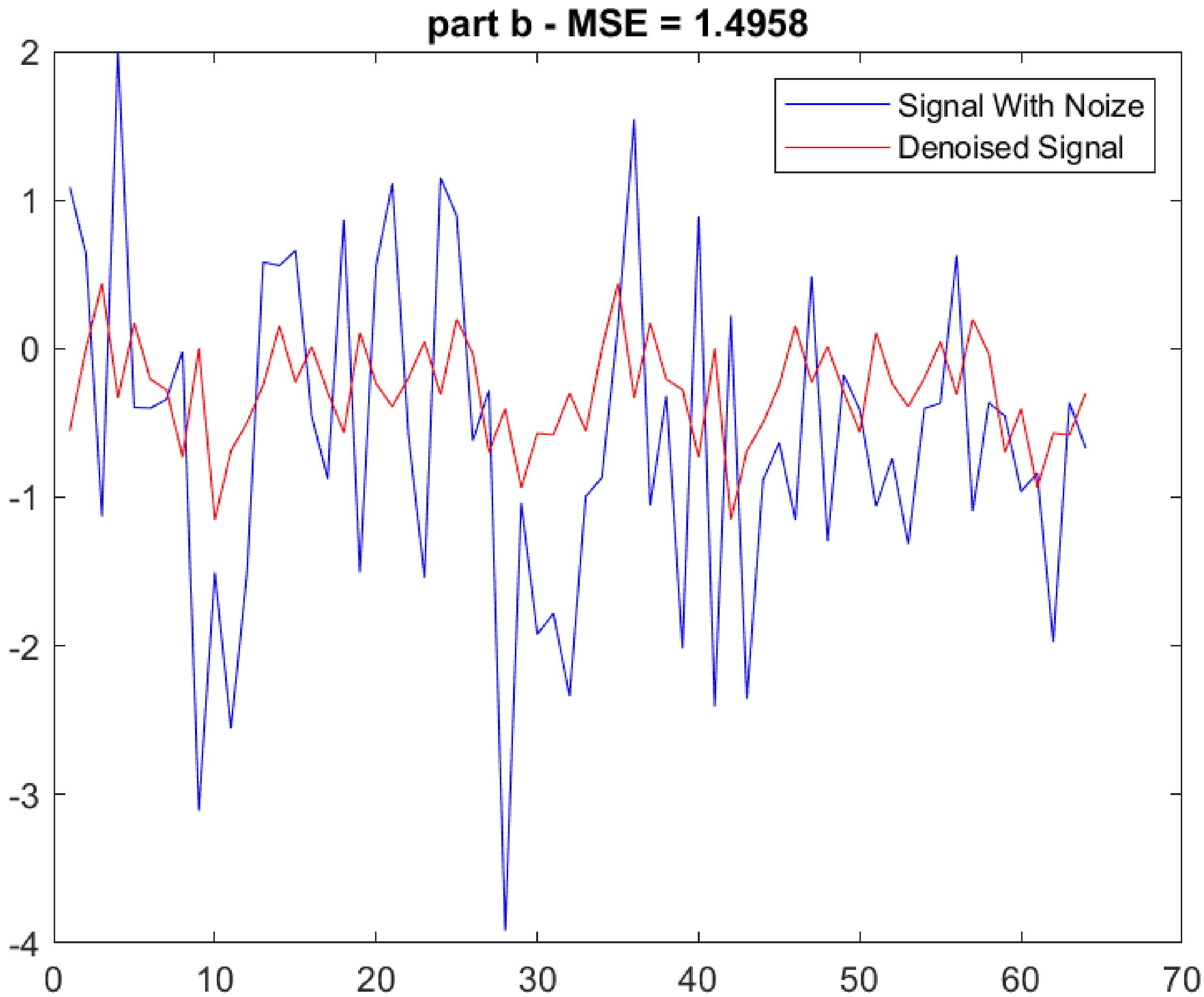

part b - MSE = 1.5406
Signal With Noize
Denoised Signal
5
4
3
2
1
0
-1
-2
0
10
20
30
40
50
60
70


part b - MSE = 1.1207
Original Signal
Denoised Signal
5
4
3
2
1
0
-1
-2
0
10
20
30
40
50
60
70

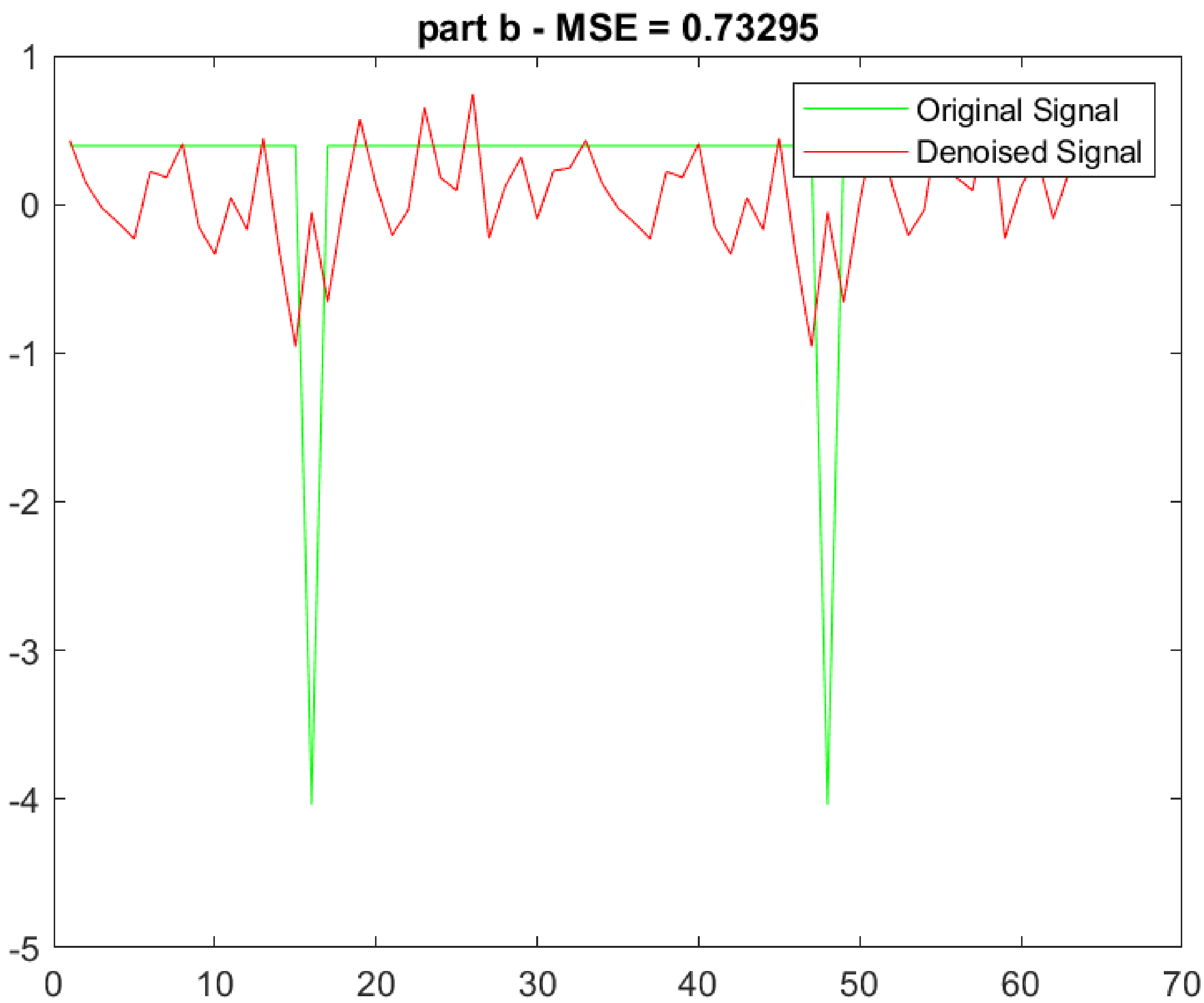


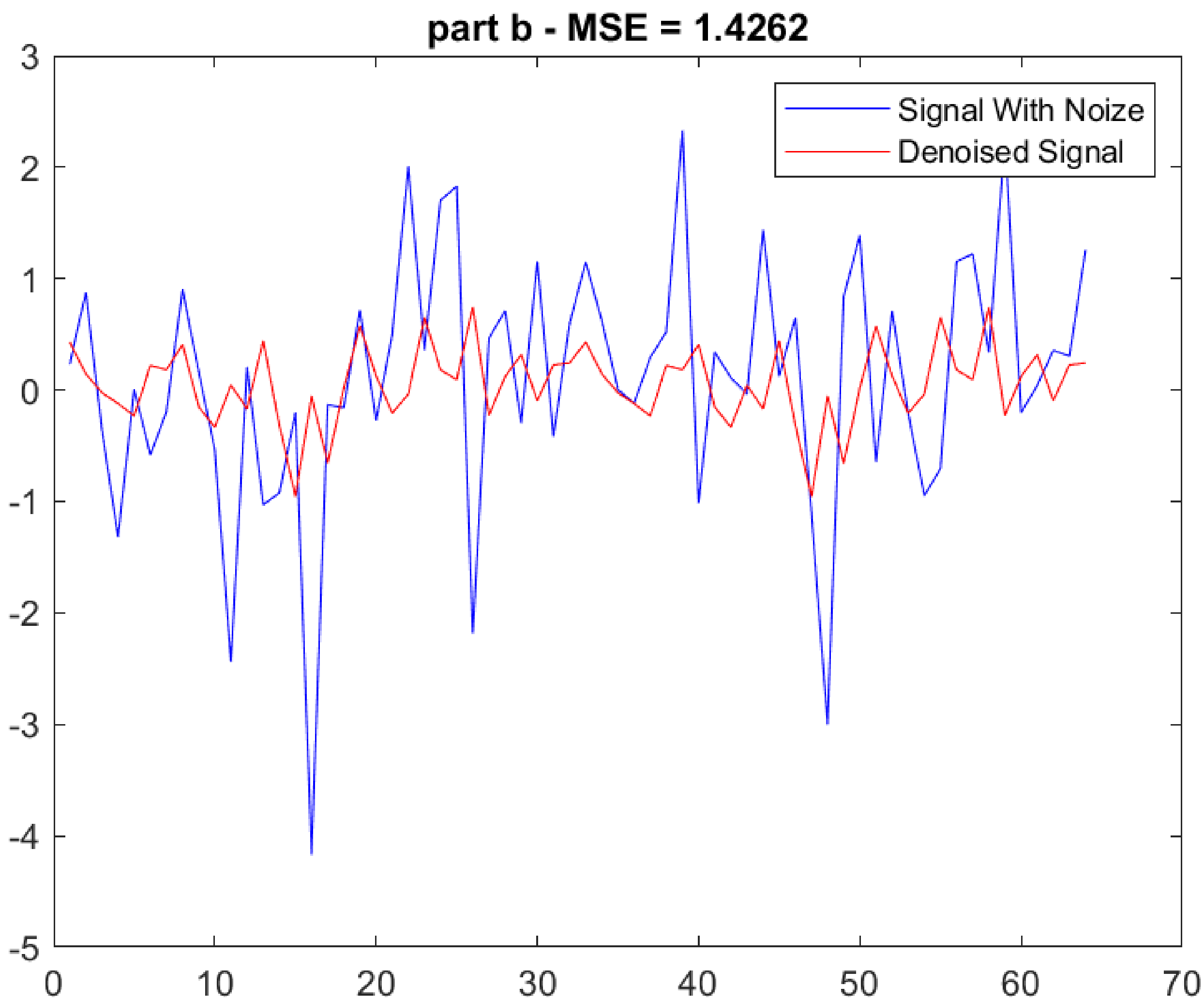
part b - MSE = 2.2854
Signal With Noize
Denoised Signal
6
5
4
3
2
1
0
-1
-2
-3
-4
0
10
20
30
40
50
60
70


part b - MSE = 0.22702
Original Signal
Denoised Signal
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
0
10
20
30
40
50
60
70



part b - MSE = 1.4958
Signal With Noize
Denoised Signal
2
1
0
-1
-2
-3
-4
0
10
20
30
40
50
60
70

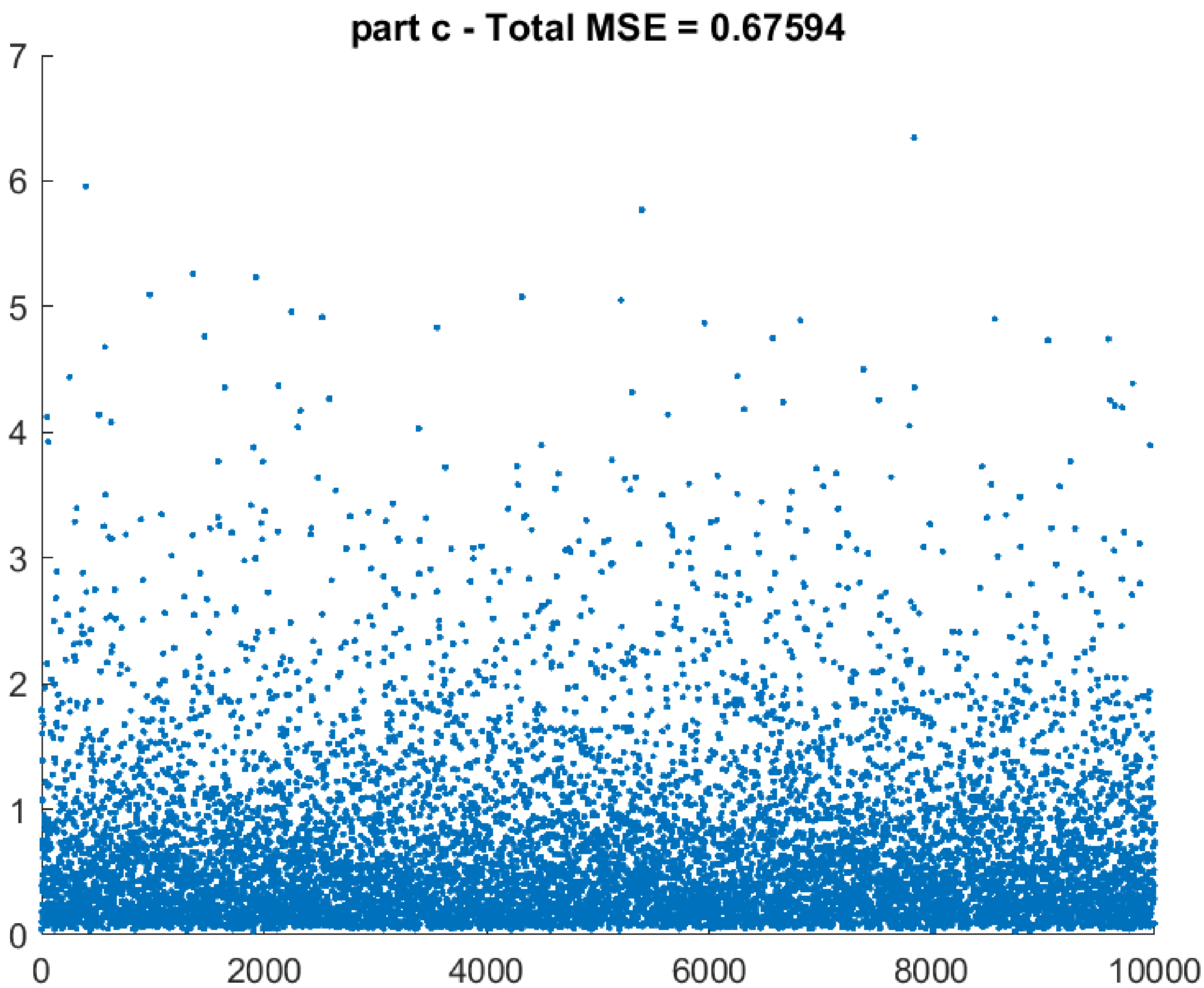

part b - MSE = 0.73295
Original Signal
Denoised Signal
1
0
-1
-2
-3
-4
-5
0
10
20
30
40
50
60
70

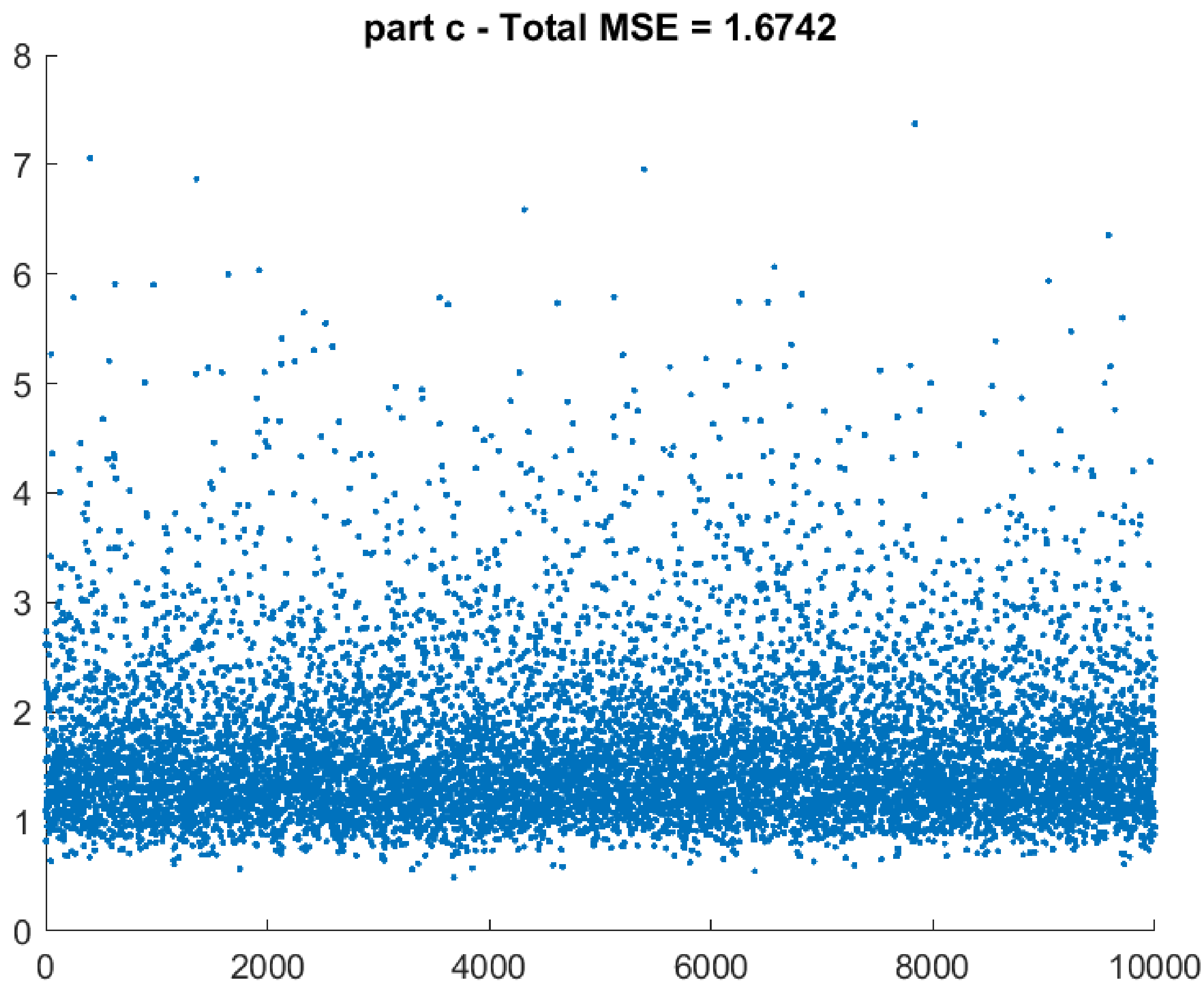



In this section, we can generally notice that the MSE in both graphs is increased, what caused by the additional difficulty of the Wiener filter to restore the distortion of H which was added.

It can be observed as previously that the area of the spikes is mostly restored with some correspondence to the clean signal and other distortion in the restored signal is close to the noisy signal.

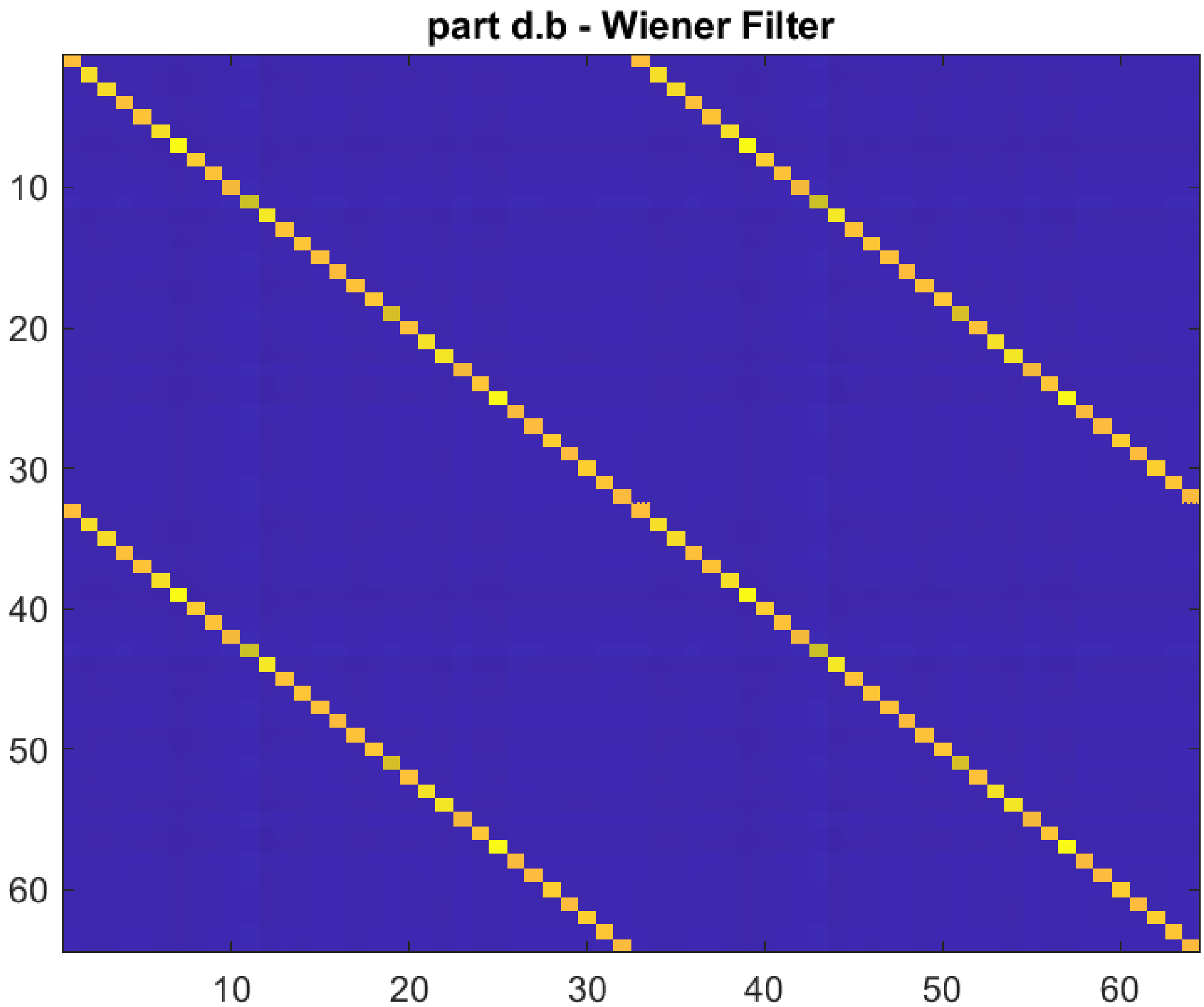
The average MSEs are the following (first the clean vs restored and then the noisy vs restored):


part c - Total MSE = 0.67594
7
6
5
4
3
2
1
0
0
2000
4000
6000
8000
10000

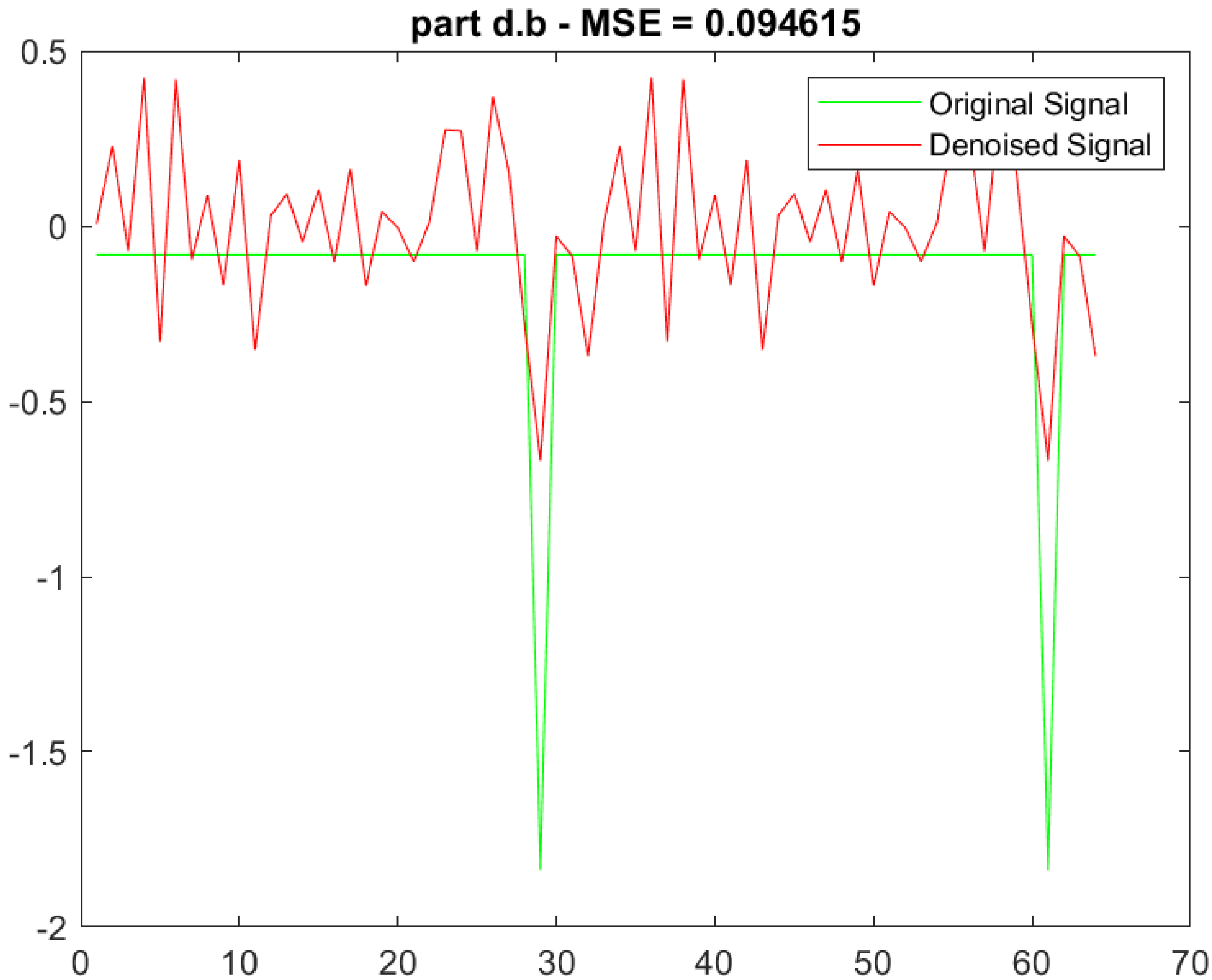

part c - Total MSE = 1.6742
8
7
6
5
4
3
2
1
0
0
2000
4000
6000
8000
10000

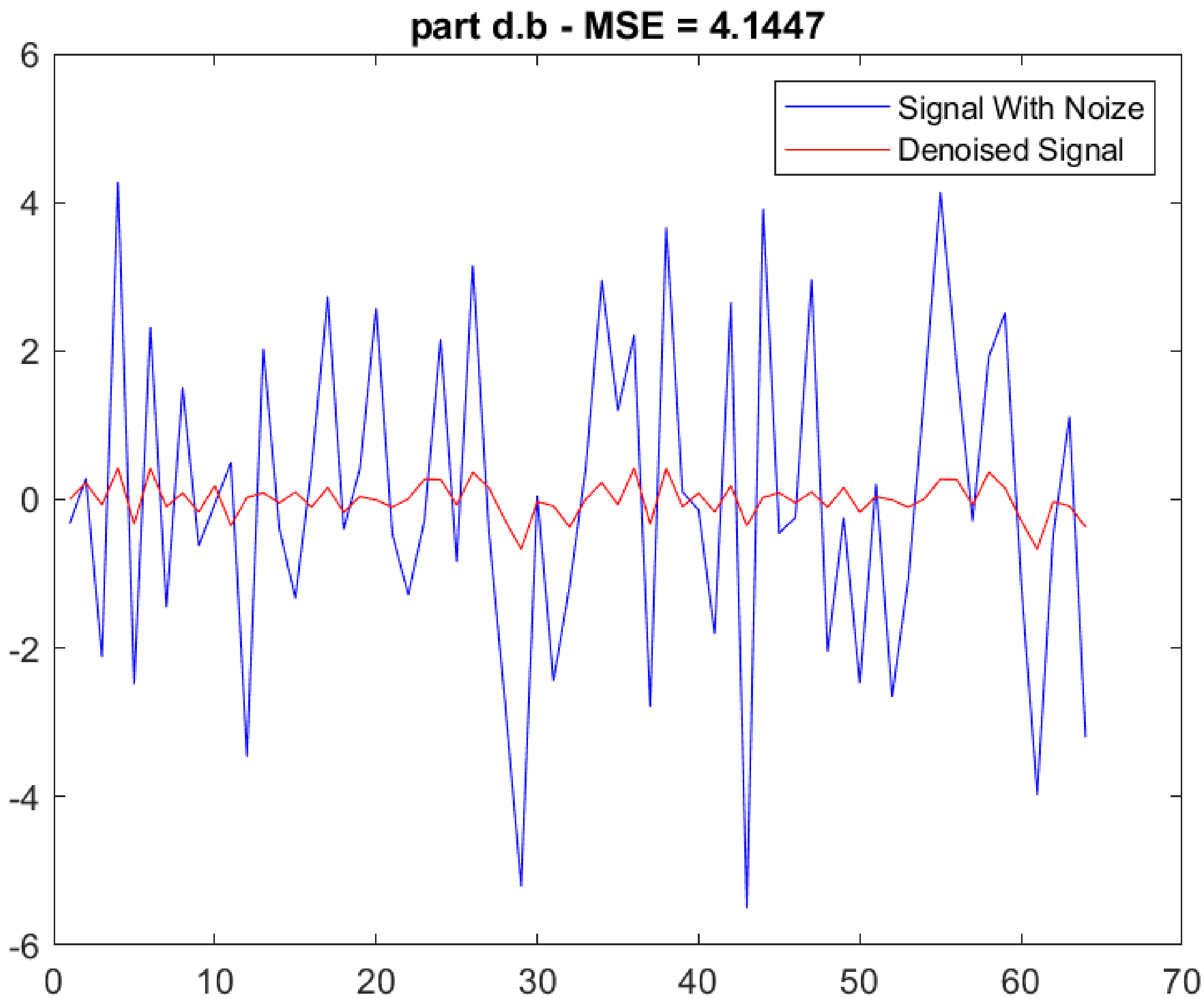
d.

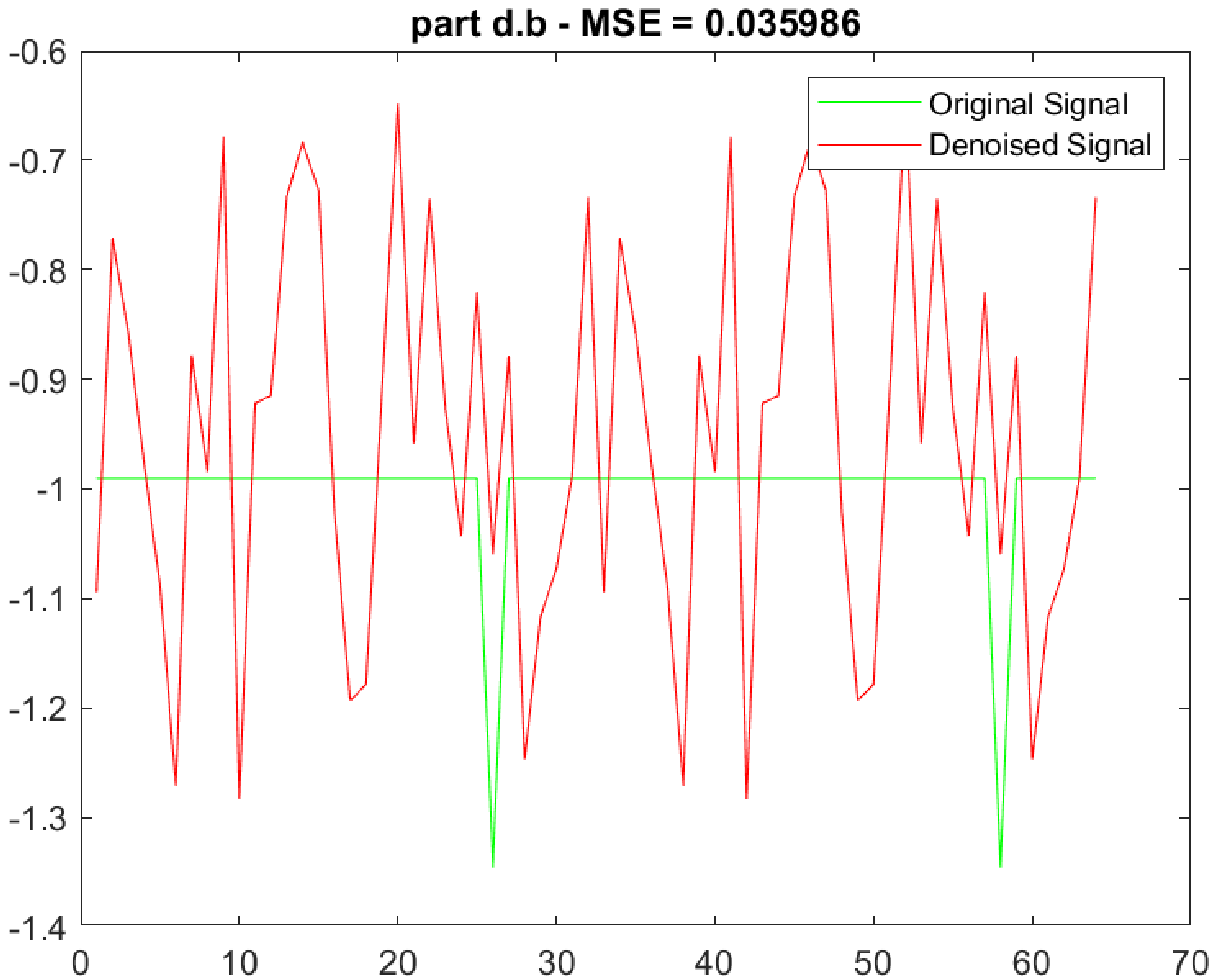
The Wiener filter with the new sigma without the distortion matrix is:

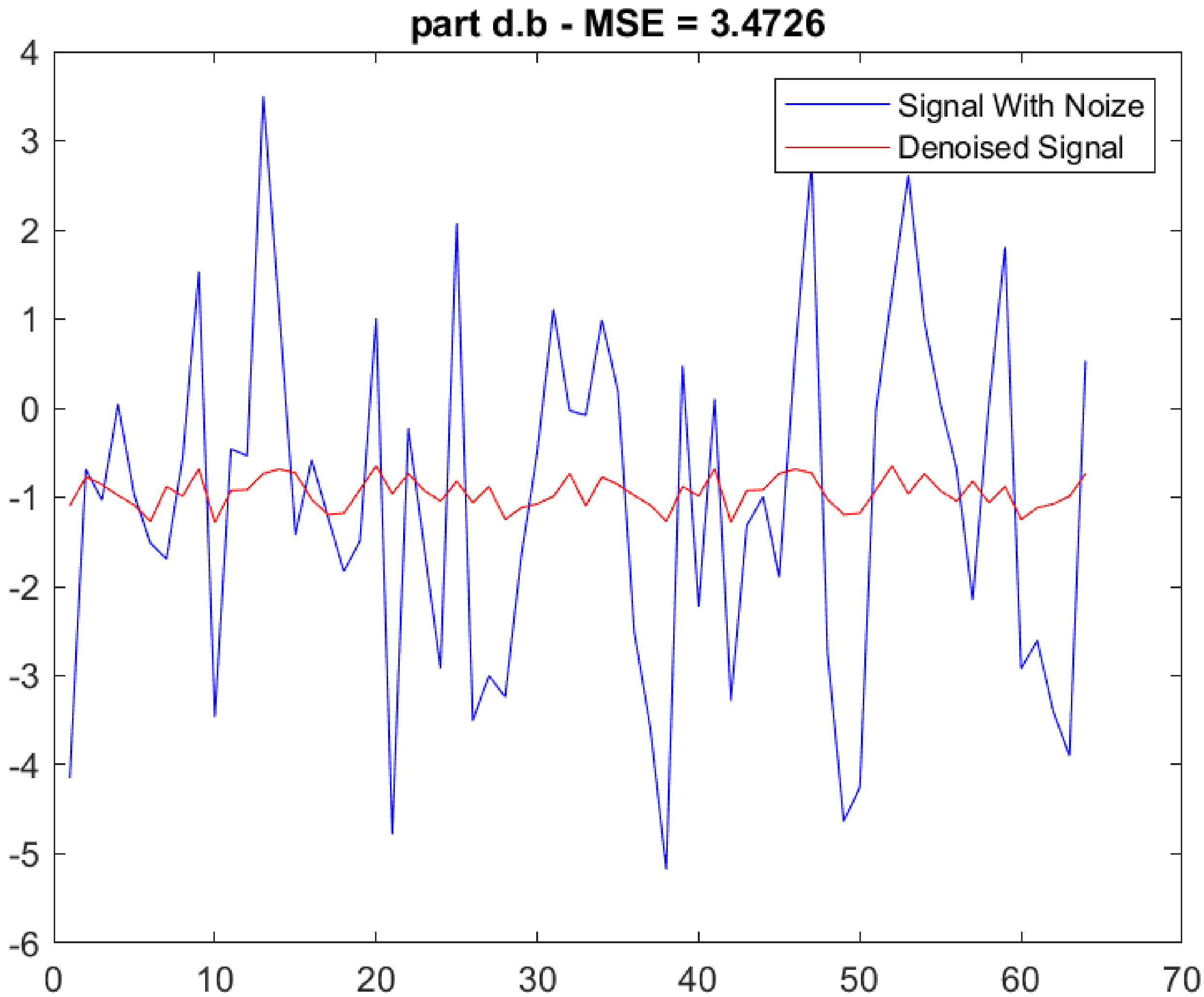



The graphs of the sampled signal to present (only with the noise and without the distortion):

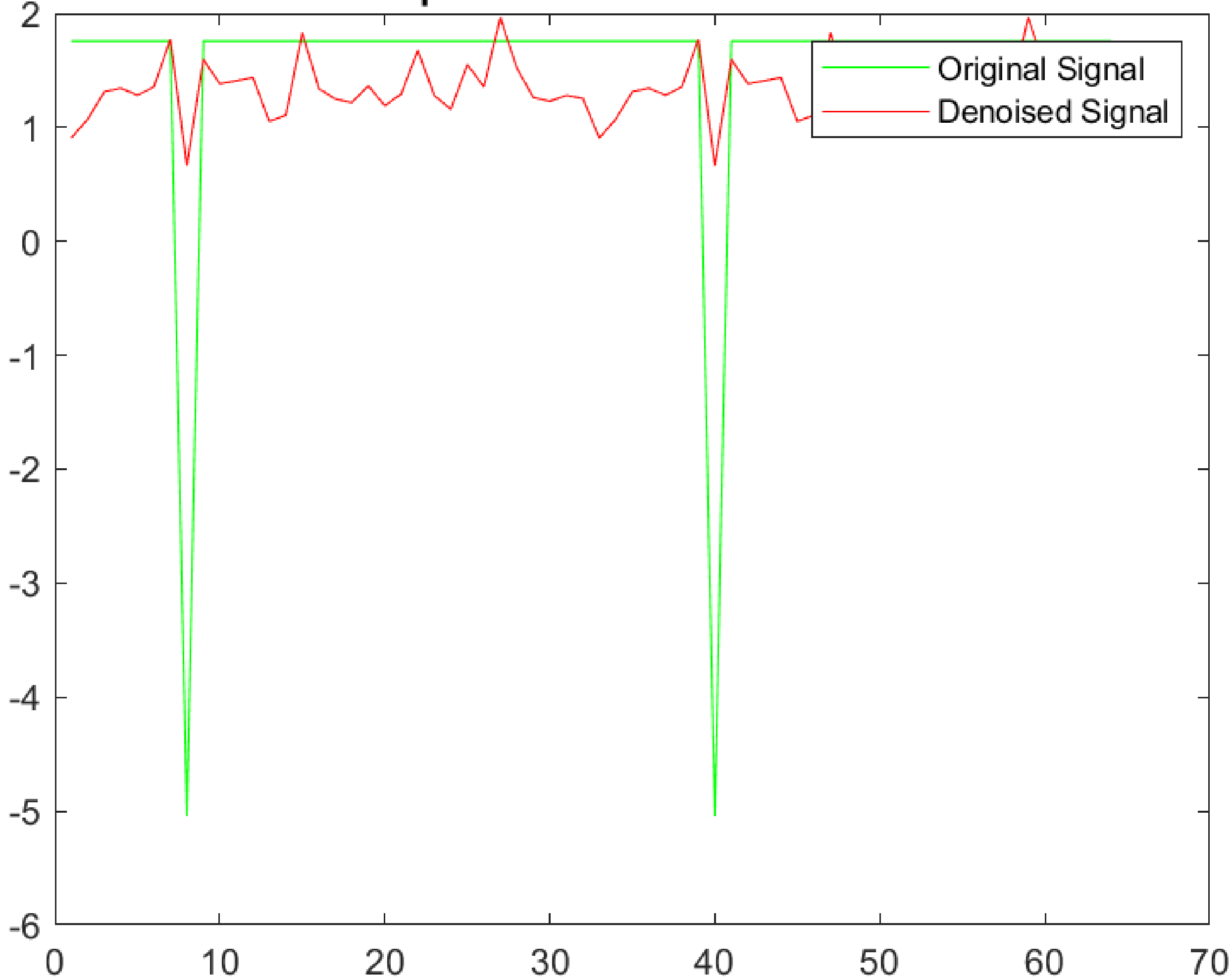

part d.b - MSE = 0.094615
Original Signal
Denoised Signal
0.5
0
-0.5
-1
-1.5
-2
0
10
20
30
40
50
60
70

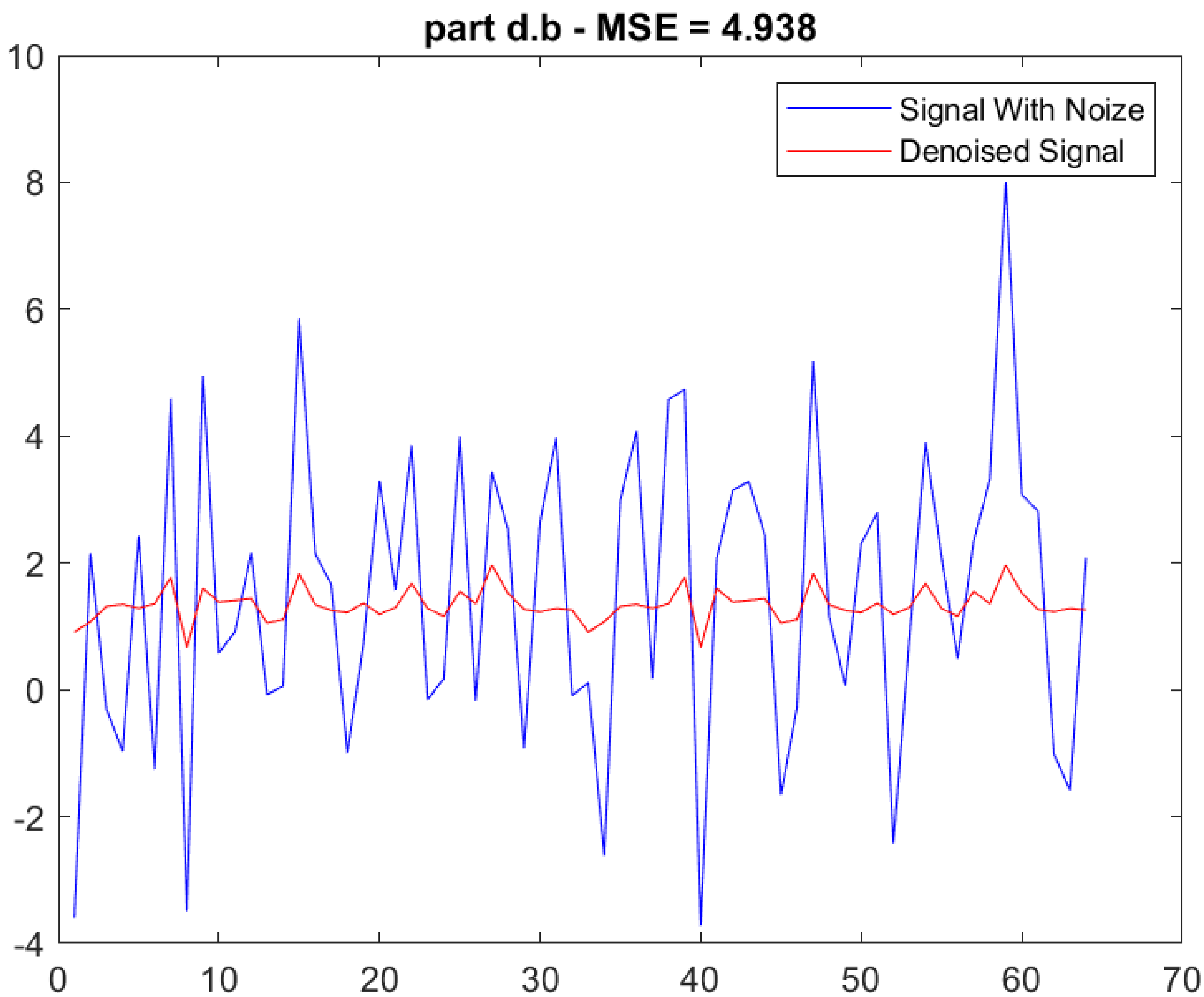

part d.b - MSE = 4.1447
Signal With Noize
Denoised Signal
6
4
2
0
-2
-4
-6
0
10
20
30
40
50
60
70


part d.b - MSE = 0.035986
Original Signal
Denoised Signal
-0.6
-0.7
-0.8
-0.9
-1
-1.1
-1.2
-1.3
-1.4
0
10
20
30
40
50
60
70

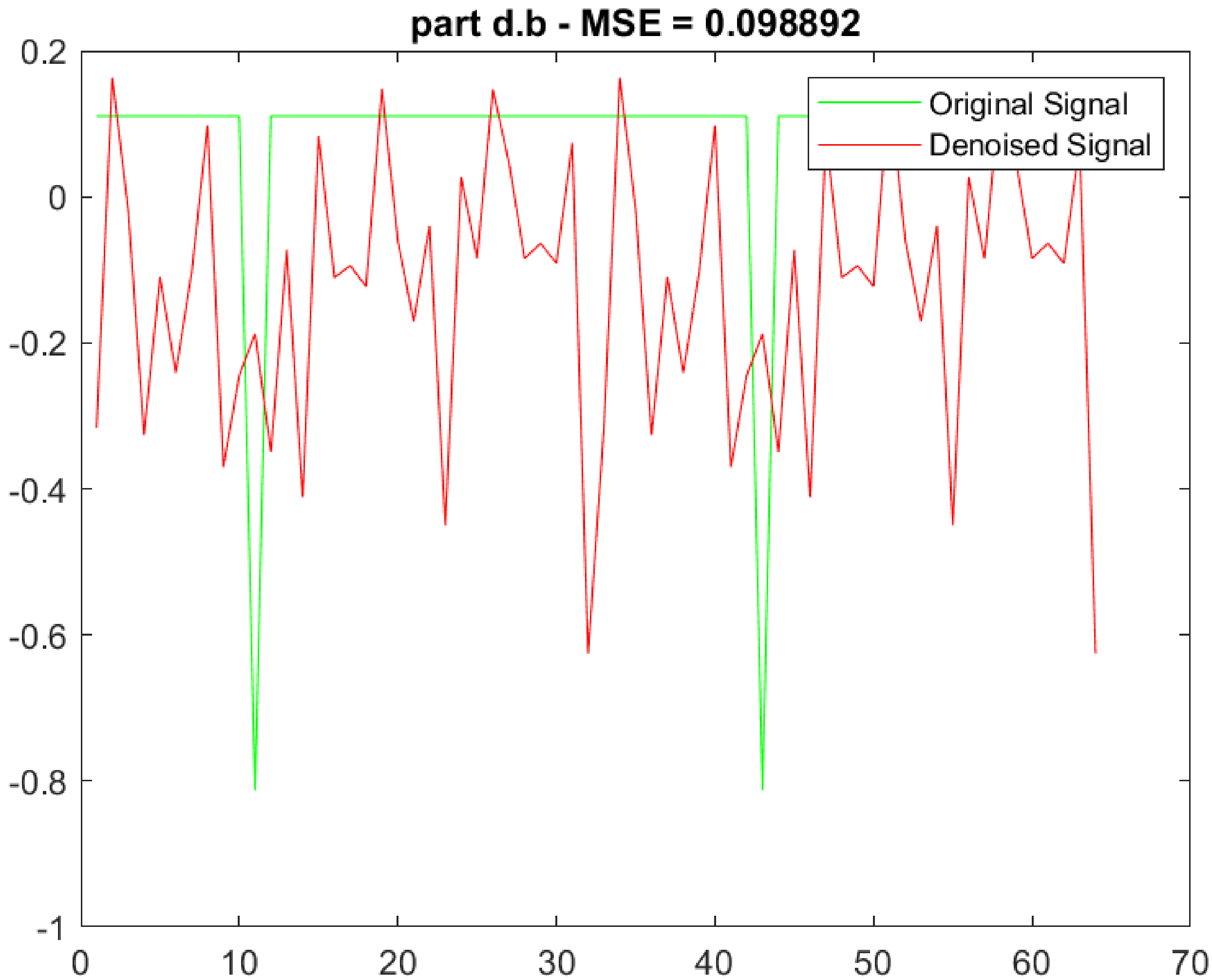

part d.b - MSE = 3.4726
Signal With Noize
Denoised Signal
4
3
2
1
0
-1
-2
-3
-4
-5
-6
0
10
20
30
40
50
60
70

**part d.b - MSE = 1.2239**

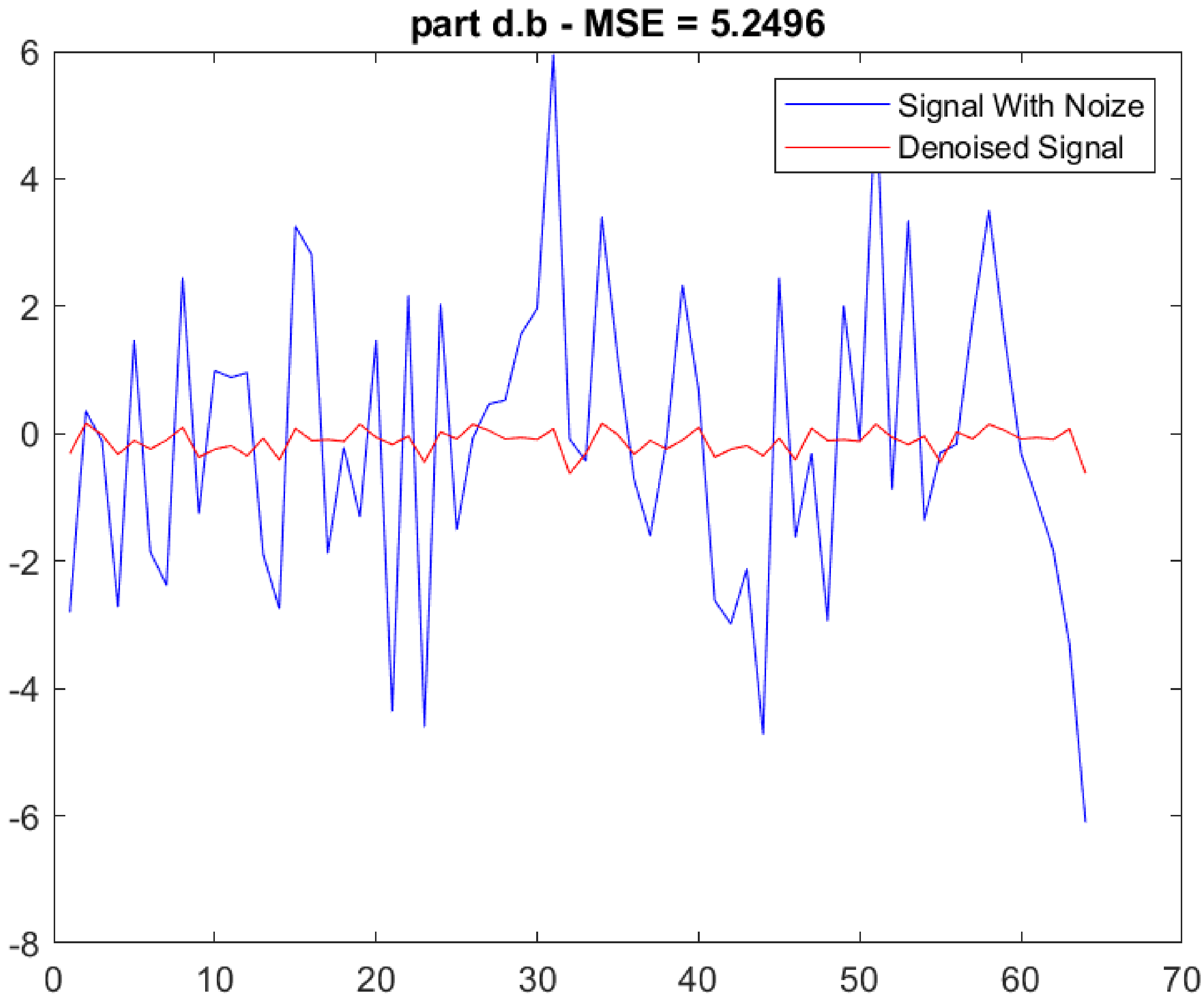


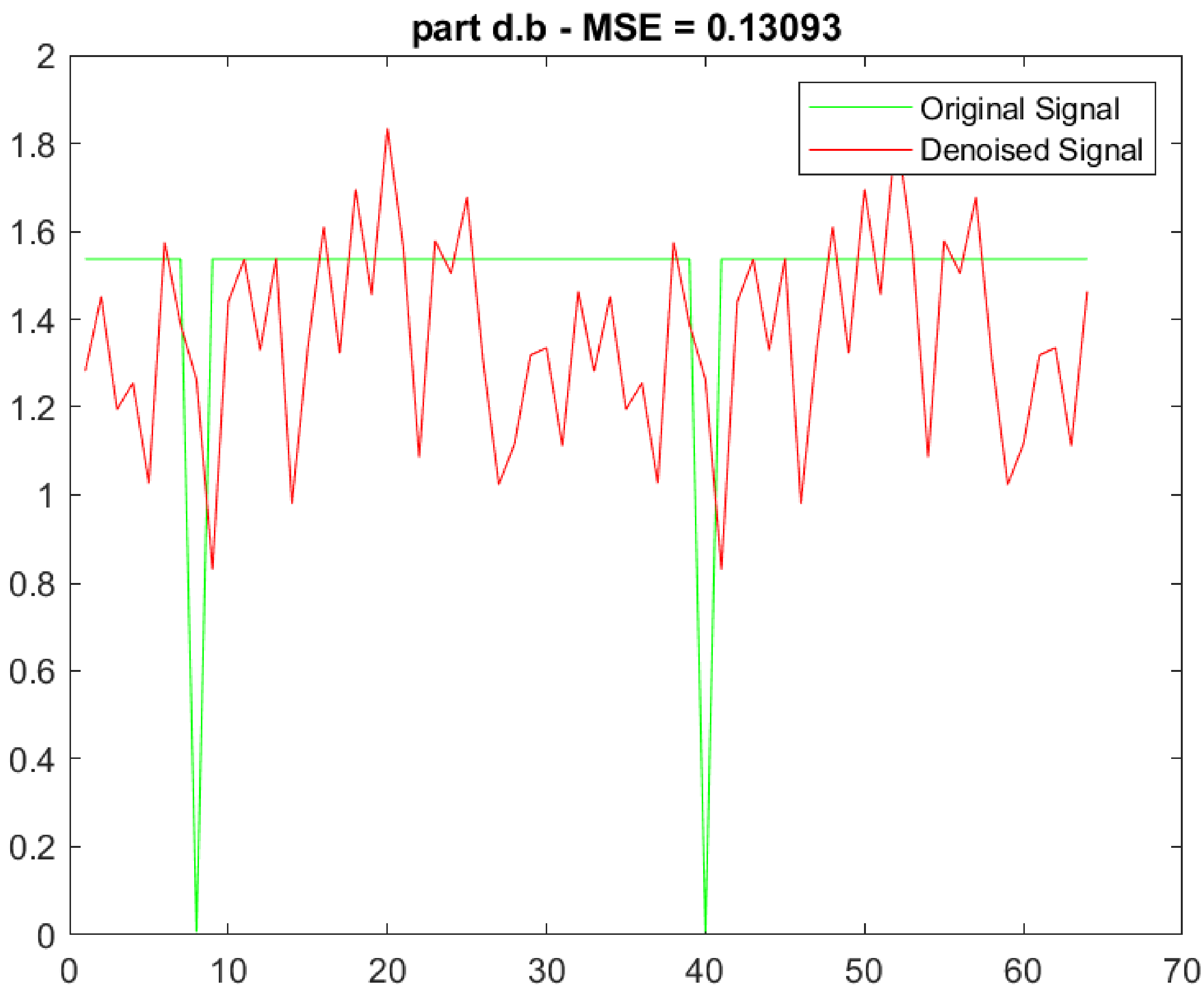


part d.b - MSE = 4.938
Signal With Noize
Denoised Signal
10
8
6
4
2
0
-2
-4
0
10
20
30
40
50
60
70

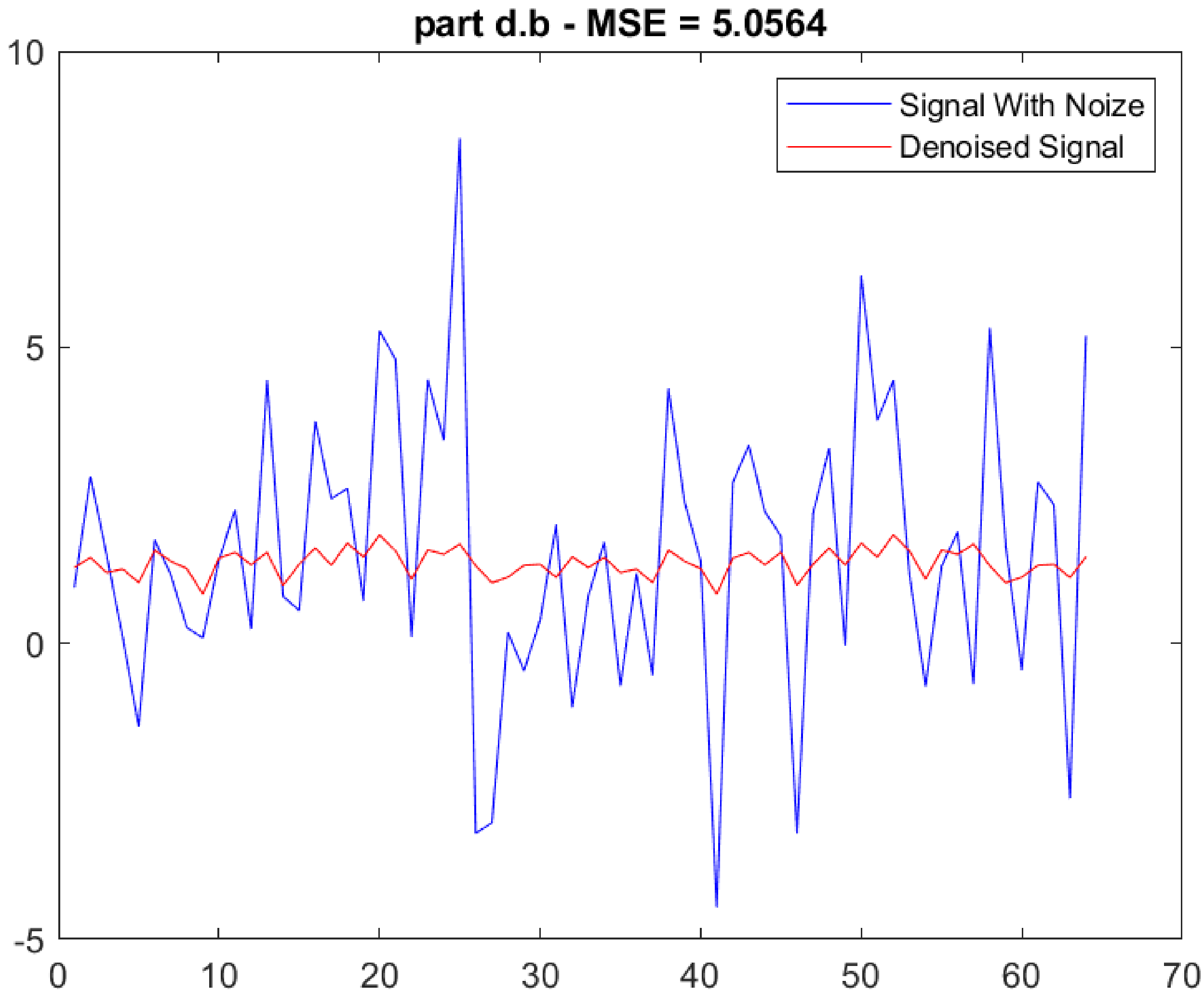


part d.b - MSE = 0.098892
Original Signal
Denoised Signal
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
10
20
30
40
50
60
70

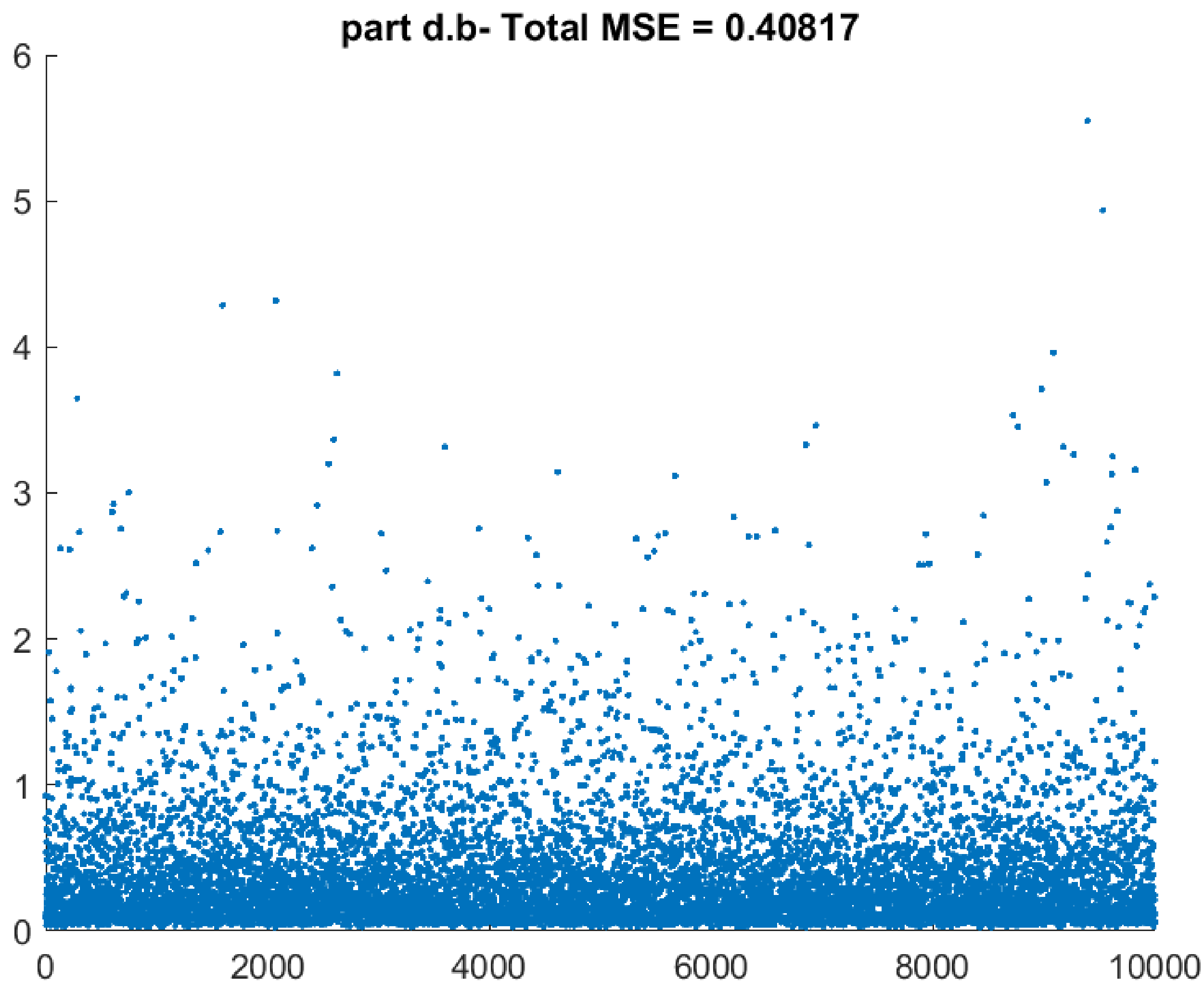

part d.b - MSE = 5.2496
Signal With Noize
Denoised Signal
6
4
2
0
-2
-4
-6
-8
0
10
20
30
40
50
60
70

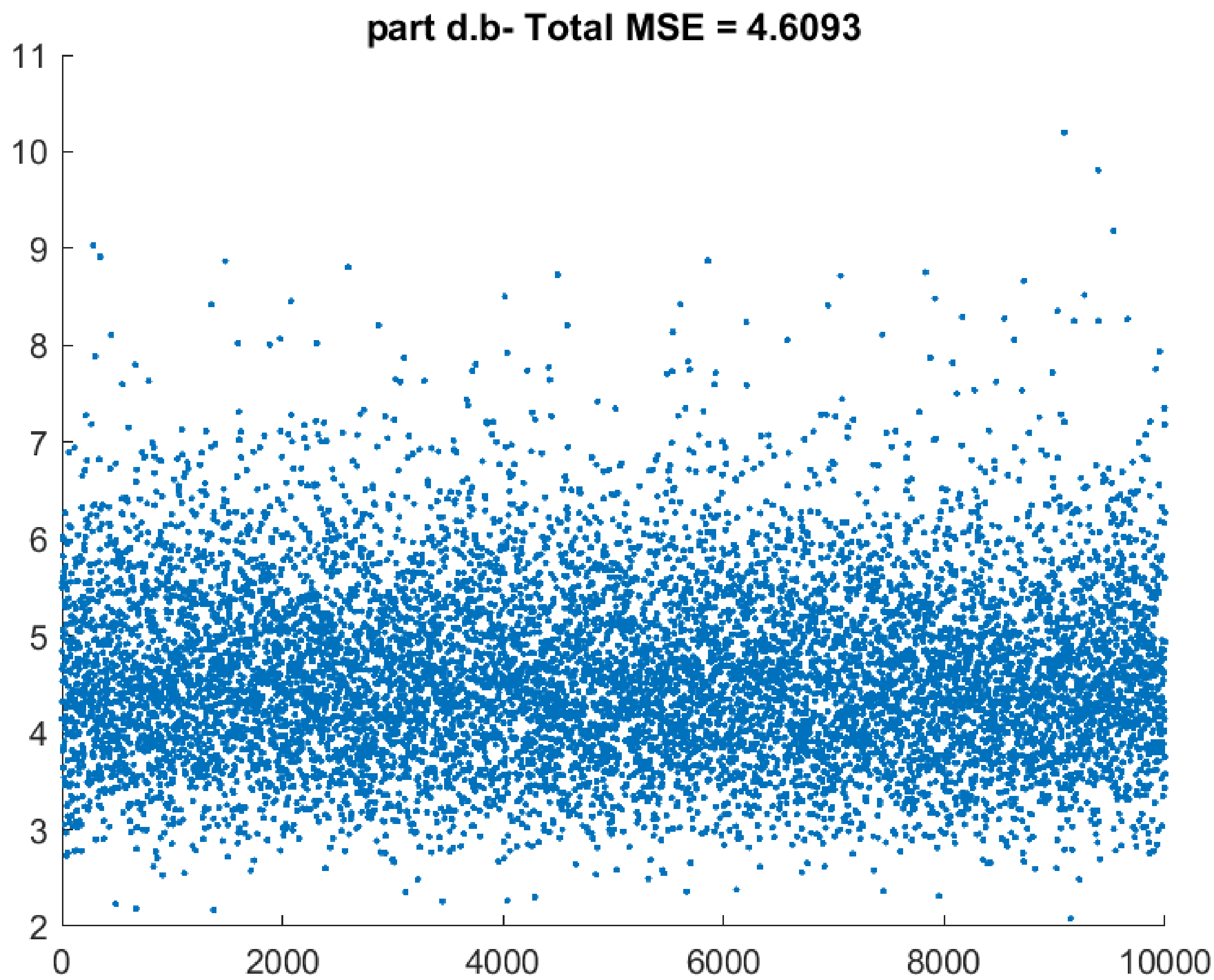


part d.b - MSE = 0.13093
Original Signal
Denoised Signal
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
50
60
70




And in total (restored vs clean and restored vs noisy):


part d.b- Total MSE = 0.40817
6
5
4
3
2
1
0
0
2000
4000
6000
8000
10000

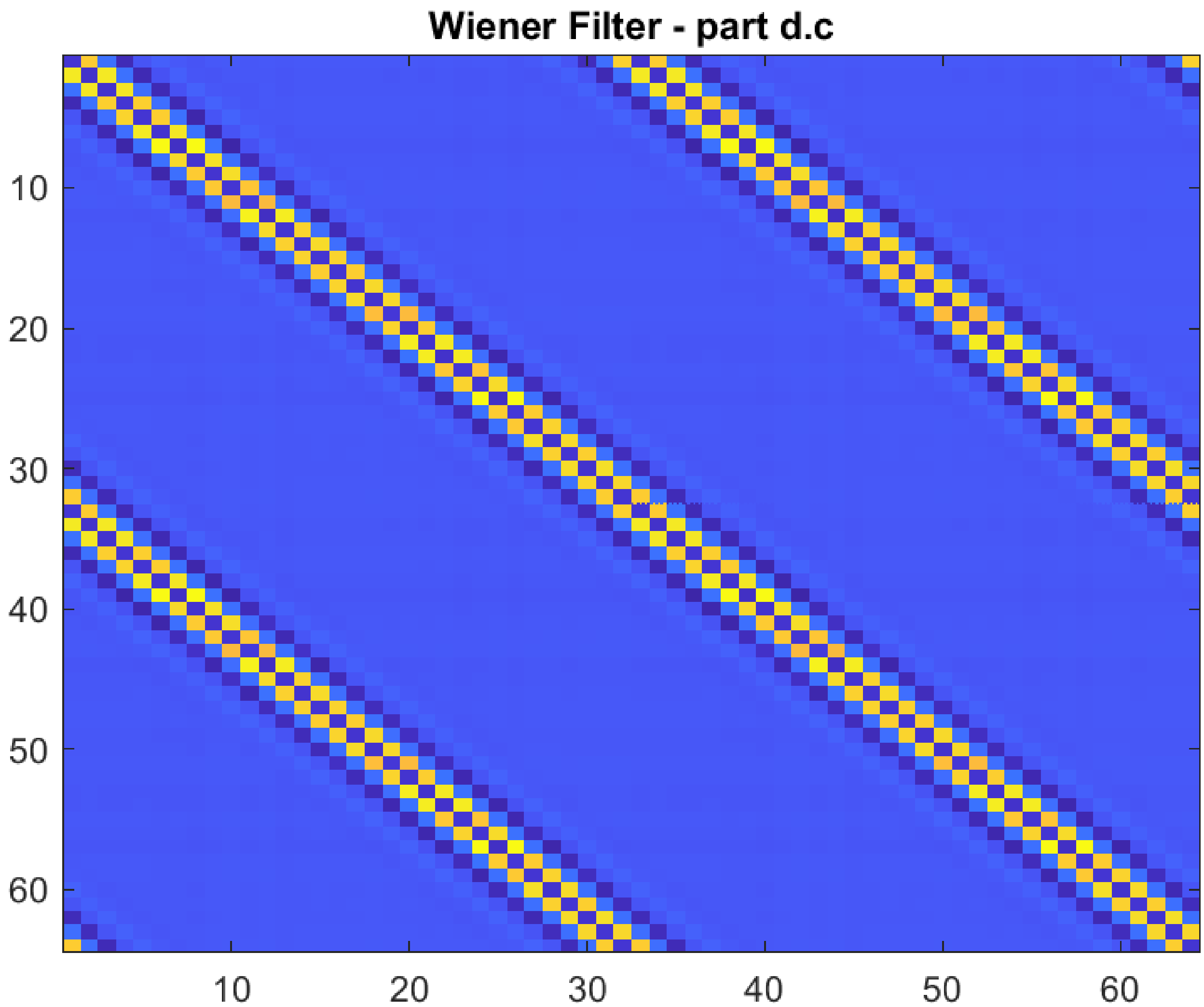



Comparing to the result we've got previously with smaller sigma, here we can observe that the MSE in total is bigger by factor 2 in average, which is an expectable effect due to enraged noise.

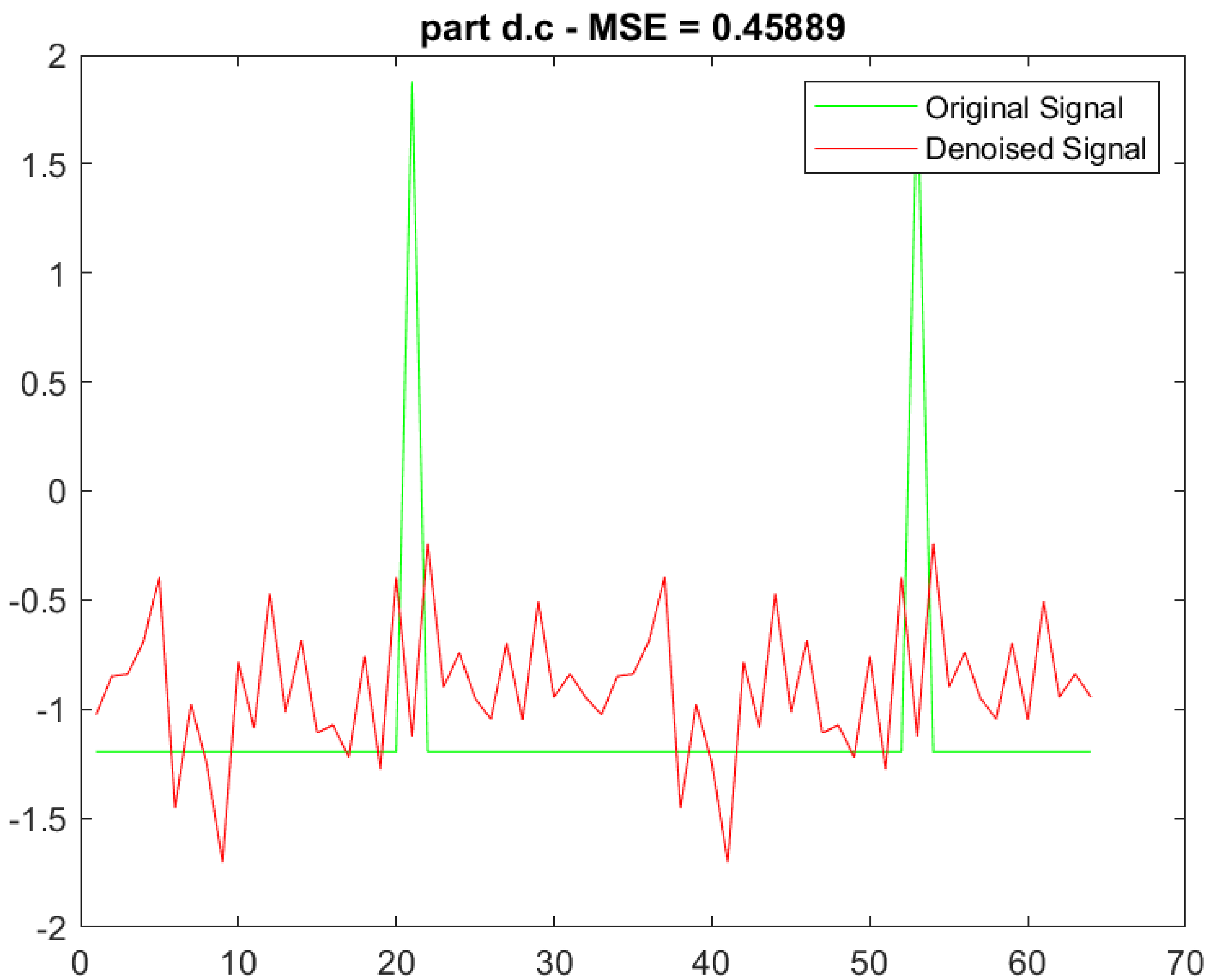
A second predictable outcome is significantly enlarged MSE error between the restored signal and the noisy signal caused by the same enlargement in noise (and as expected the larger difference between the curves behaviors).

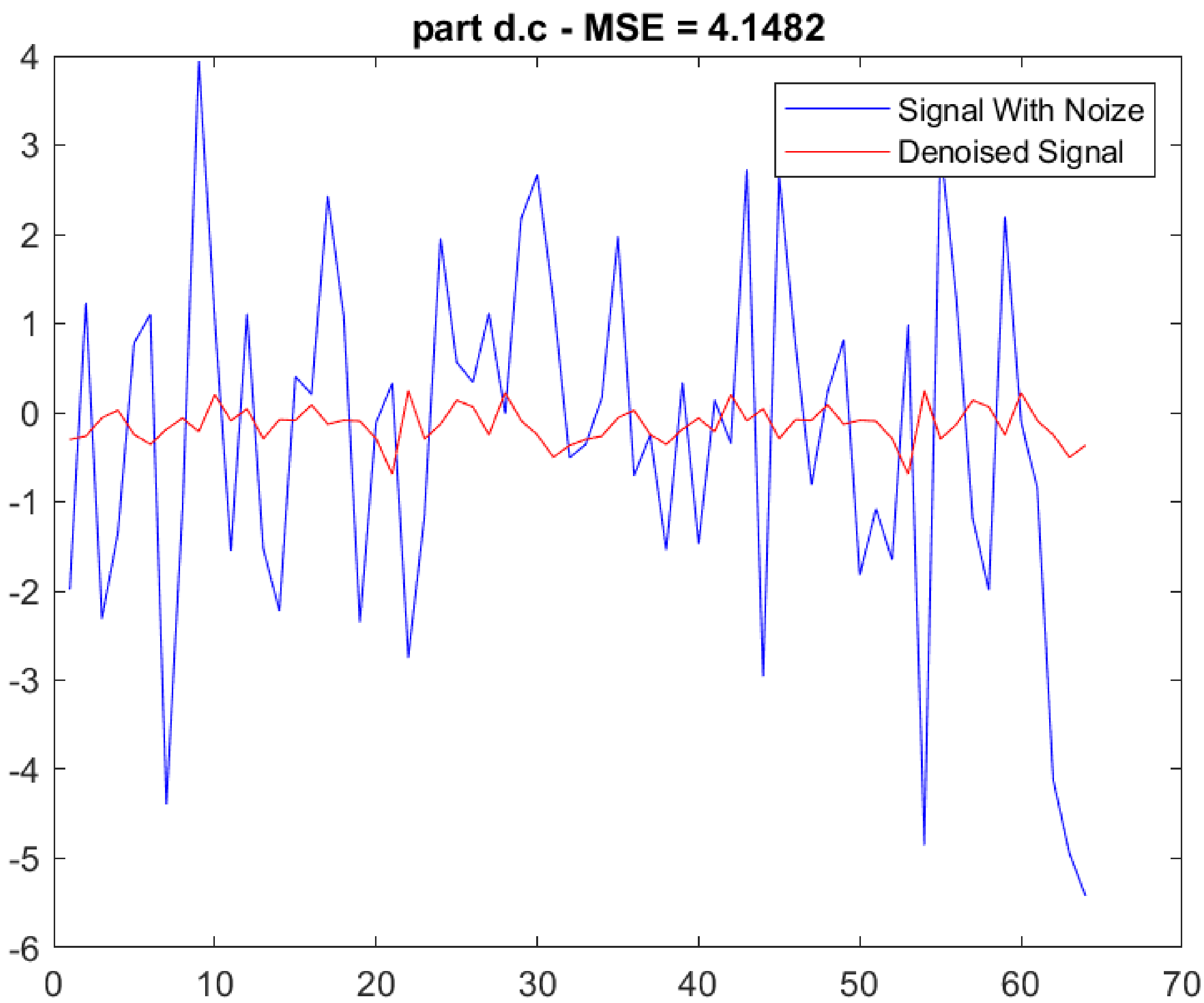
Another interesting outcome is that, although in some cases the spikes between the restored and the clean signals are corresponding to each other, in some cases the signals are similar only in the average sense, thus the restored signal have a big noisy behavior but in average behave like the straight green line.

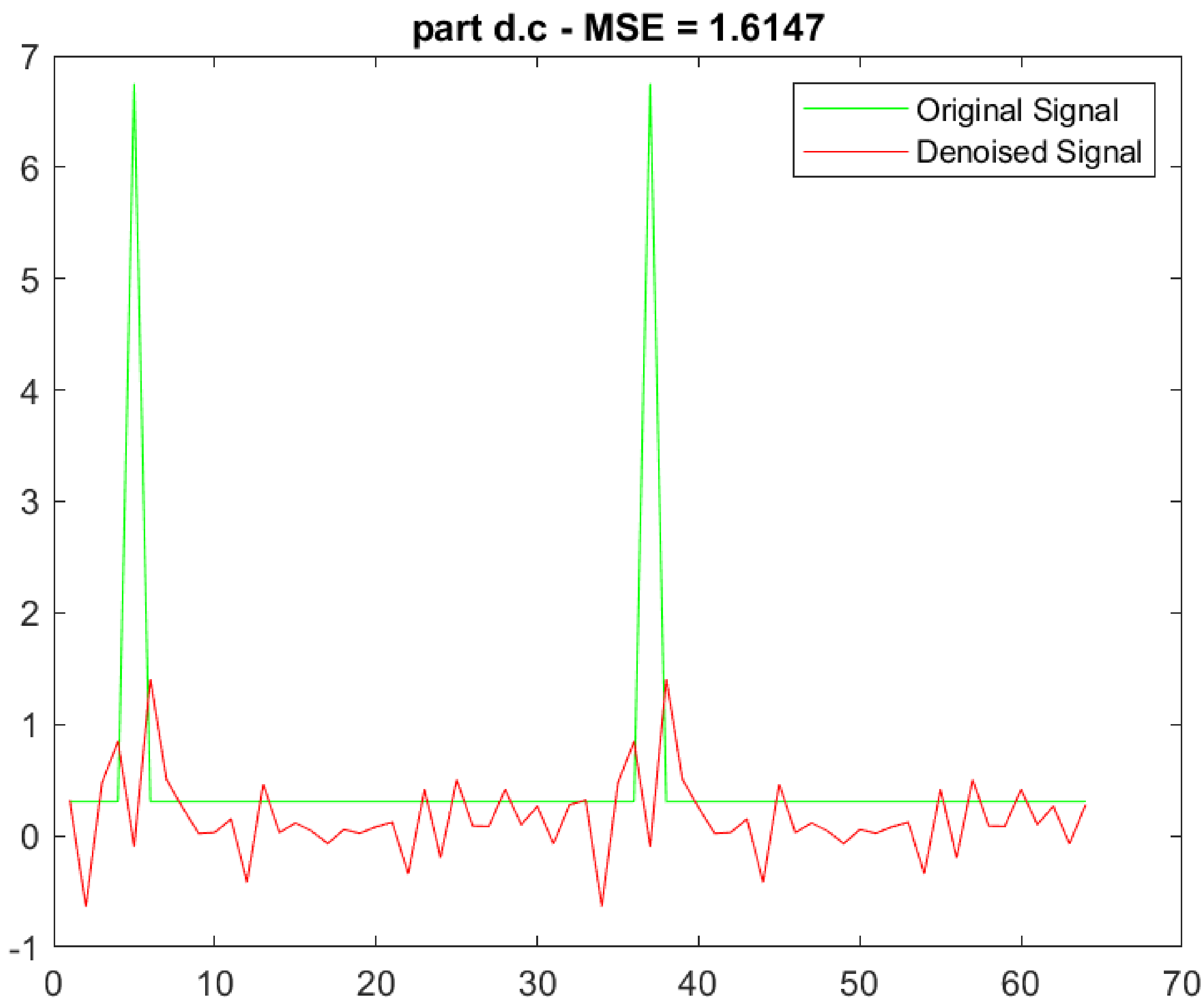
The Wiener Filter with the distortion and the new noise of sigma=5:

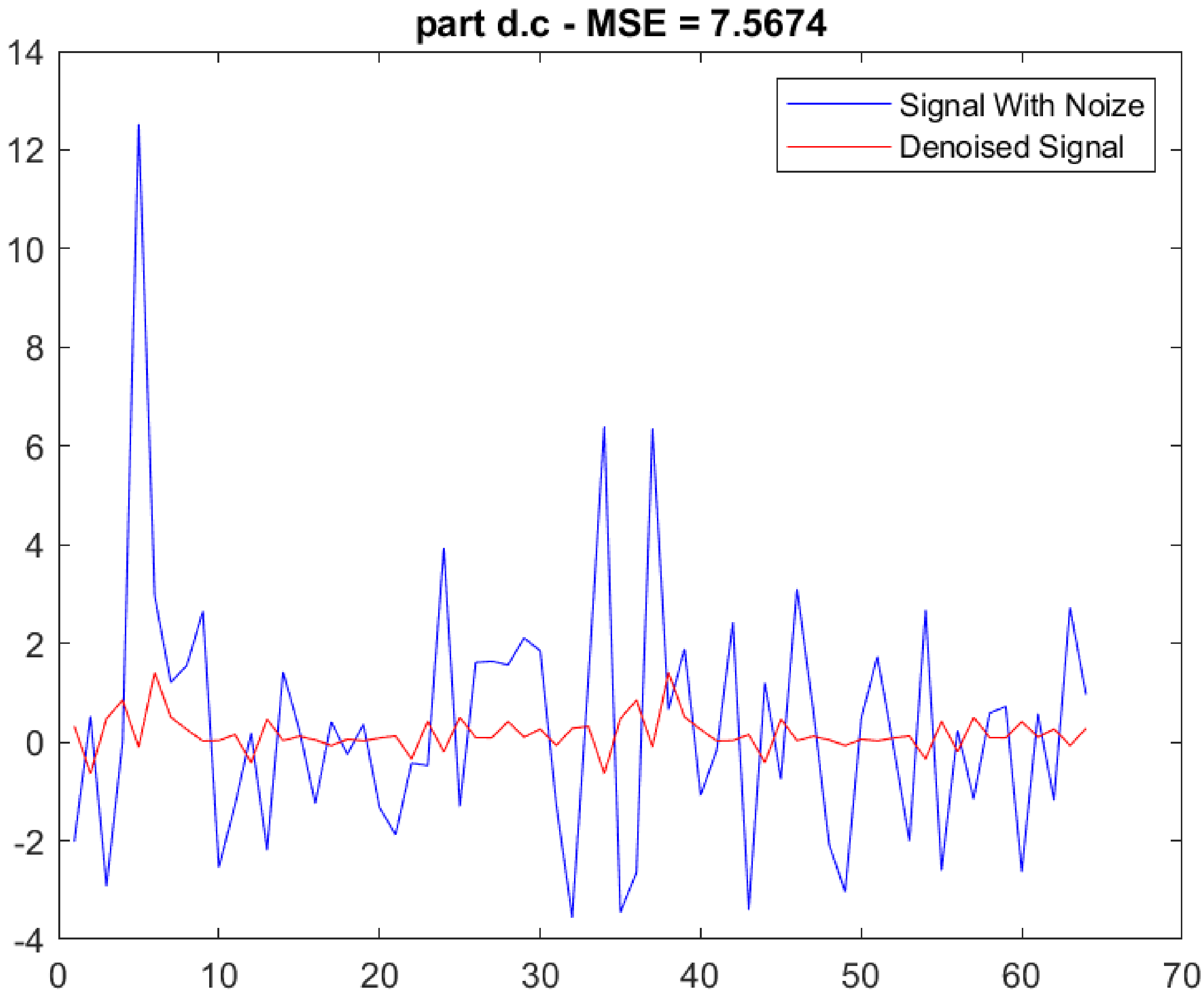



The graphs in same structure that they were presented similarly above:

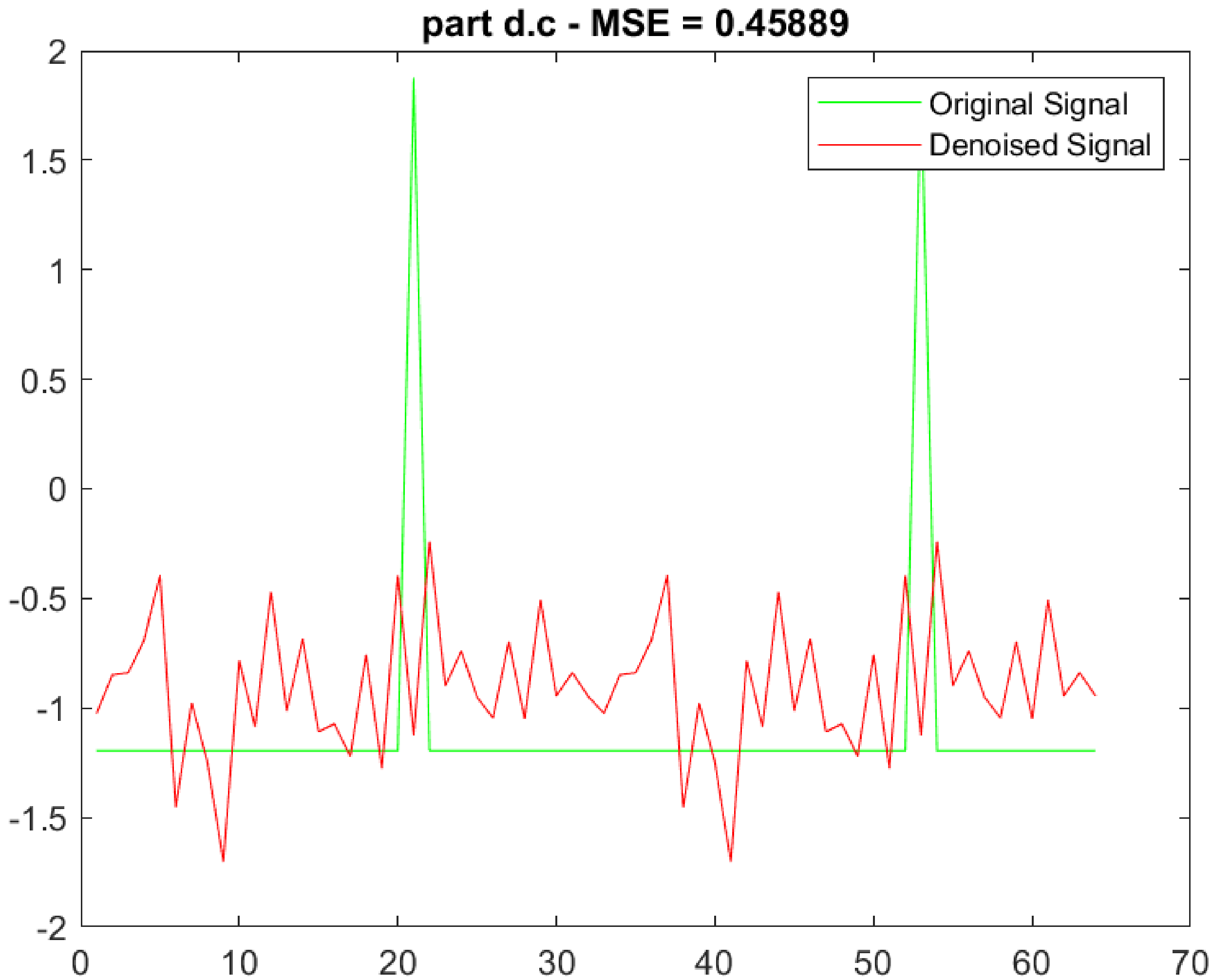

part d.c - MSE = 0.45889
Original Signal
Denoised Signal
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
10
20
30
40
50
60
70

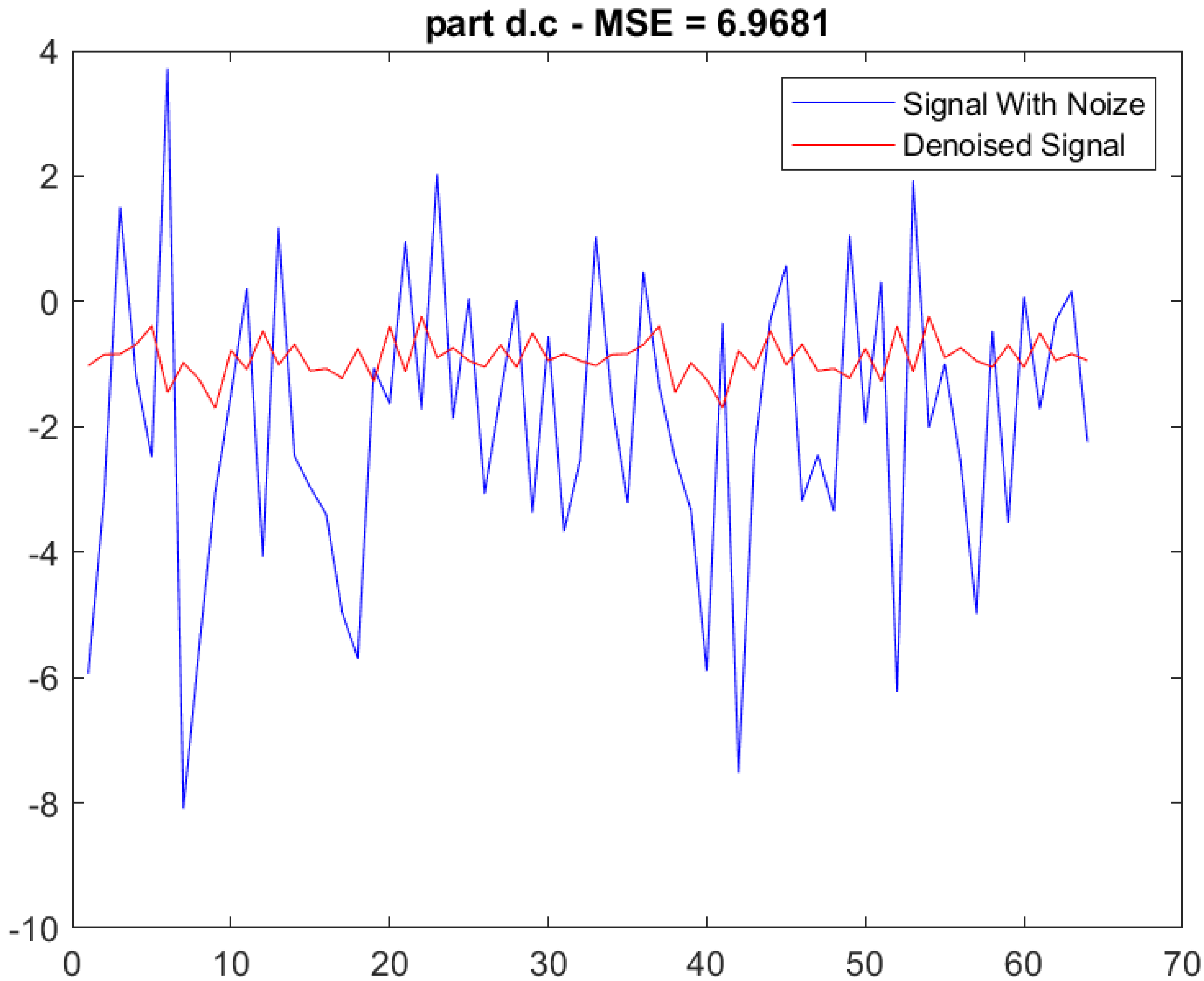

part d.c - MSE = 4.1482
Signal With Noize
Denoised Signal
4
3
2
1
0
-1
-2
-3
-4
-5
-6
0
10
20
30
40
50
60
70

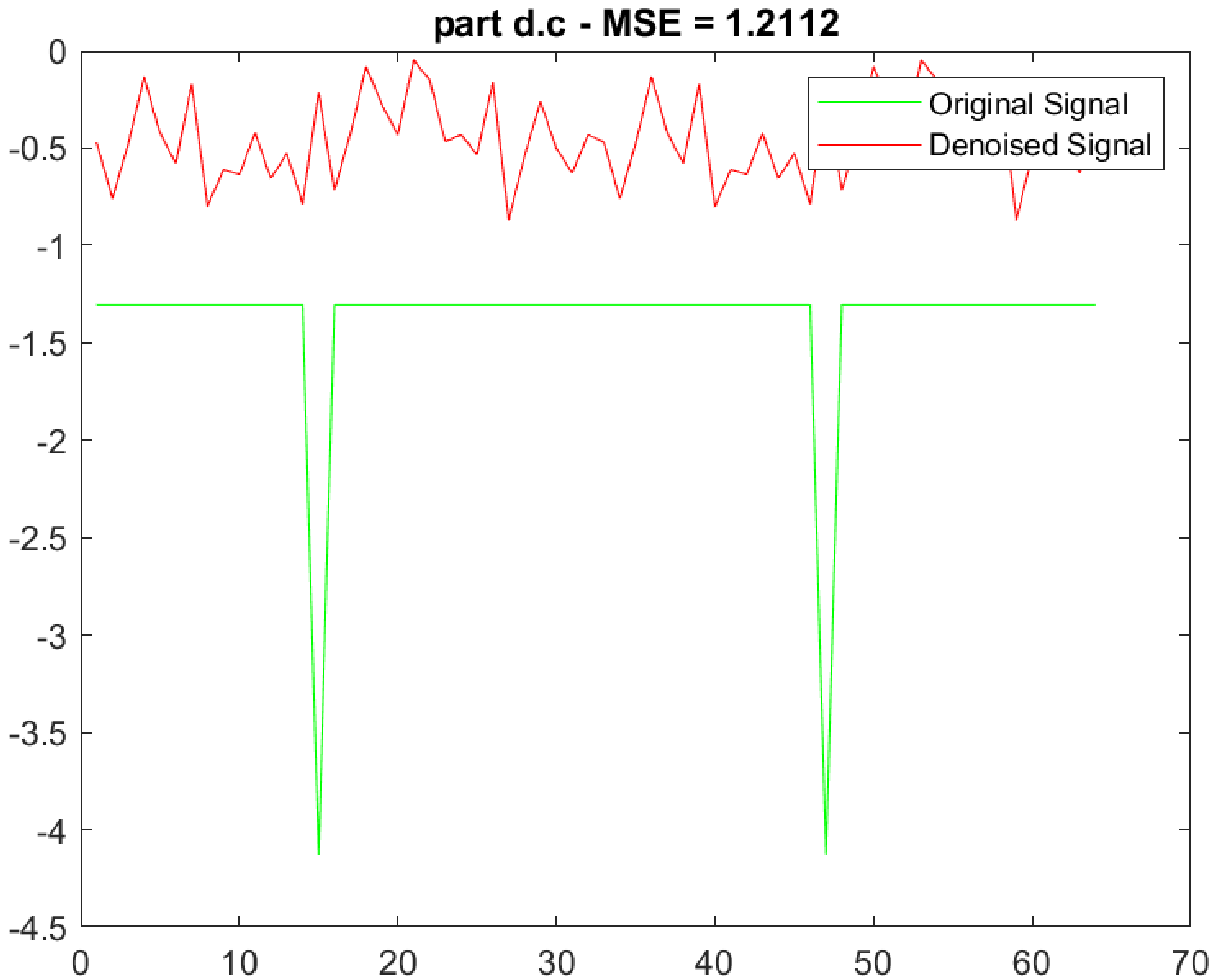

part d.c - MSE = 1.6147
Original Signal
Denoised Signal
7
6
5
4
3
2
1
0
-1
0
10
20
30
40
50
60
70



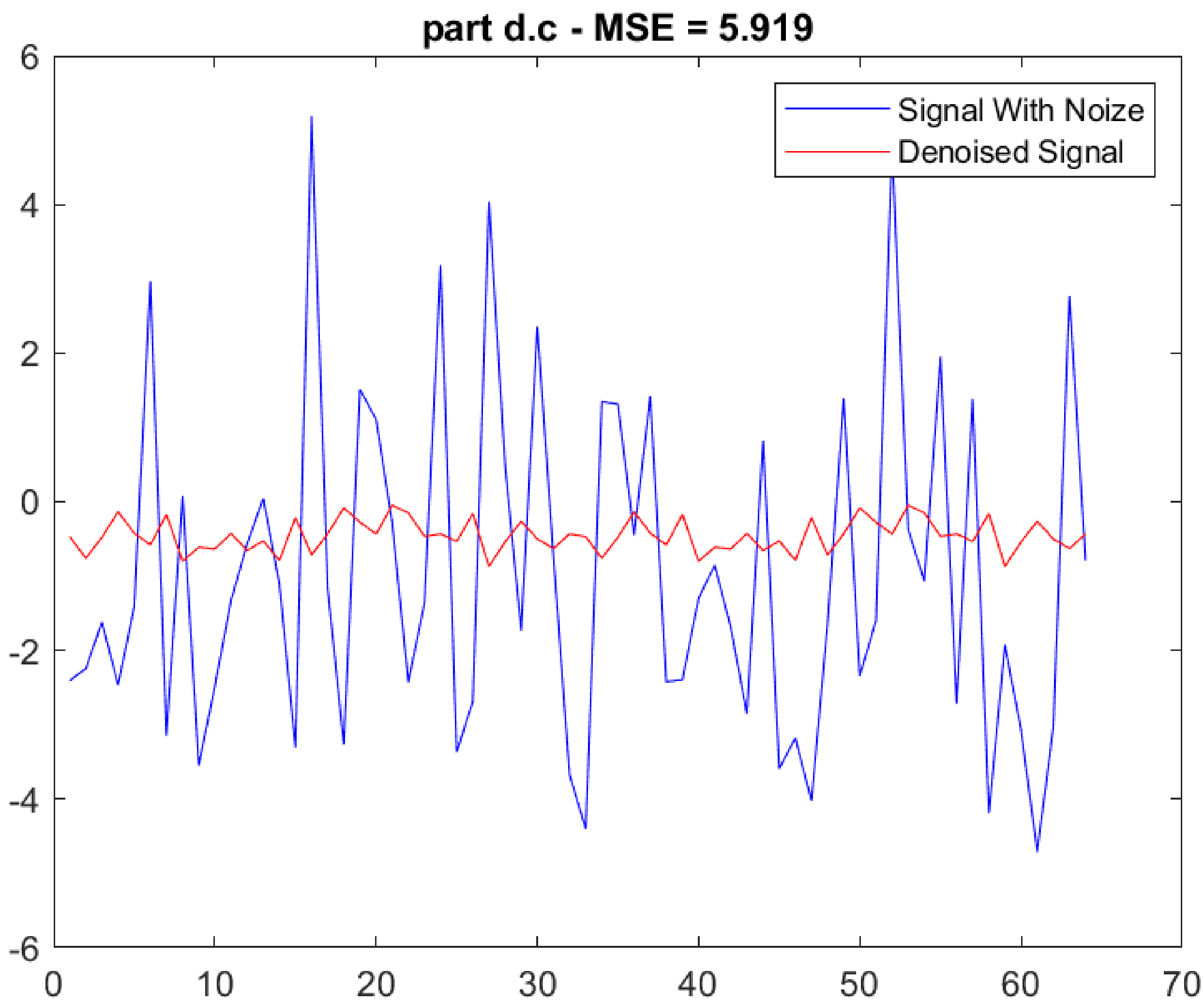
part d.c - MSE = 7.5674
Signal With Noize
Denoised Signal
14
12
10
8
6
4
2
0
-2
-4
0
10
20
30
40
50
60
70

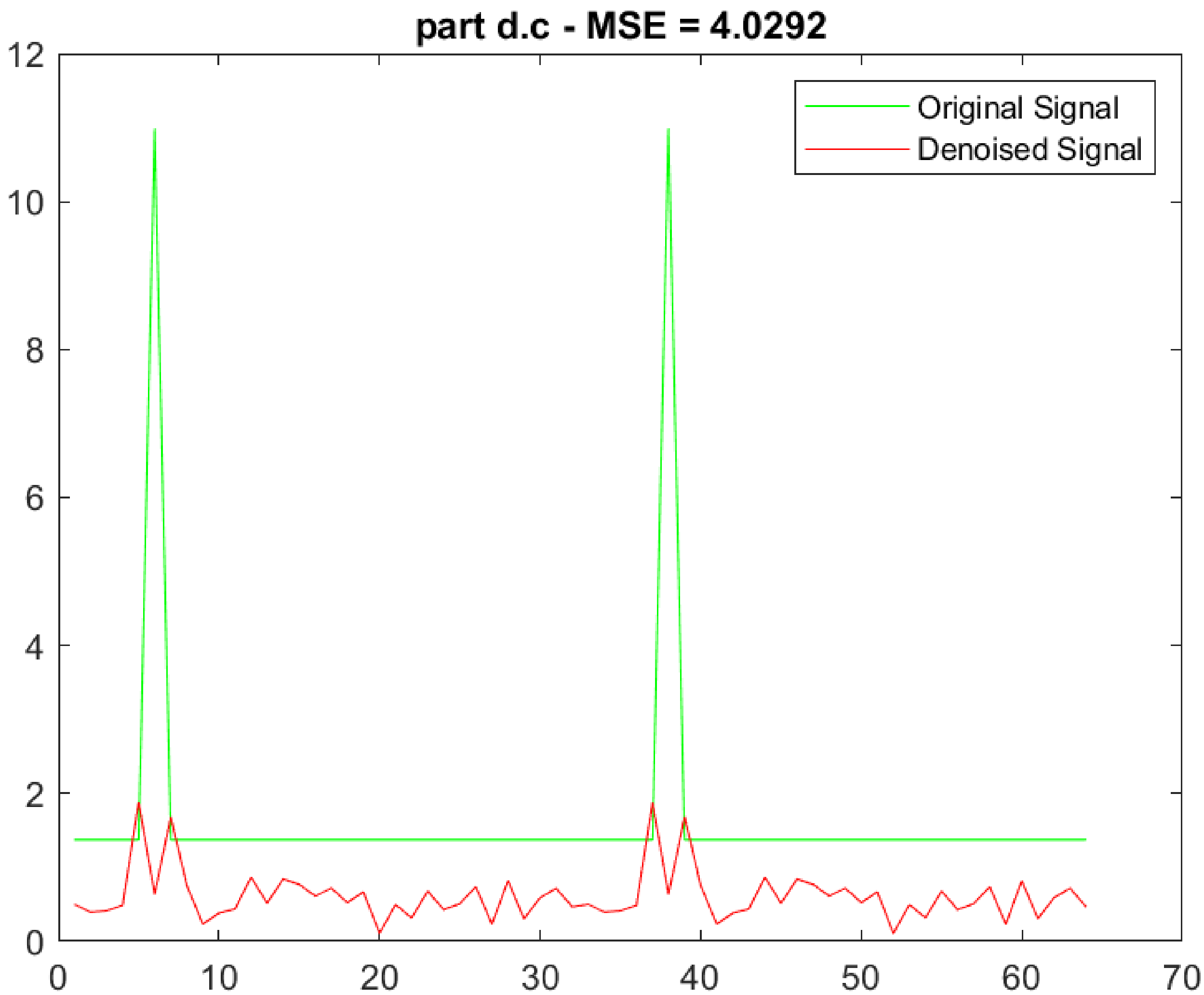

part d.c - MSE = 0.45889
Original Signal
Denoised Signal
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
10
20
30
40
50
60
70


part d.c - MSE = 6.9681
Signal With Noize
Denoised Signal
4
2
0
-2
-4
-6
-8
-10
0
10
20
30
40
50
60
70

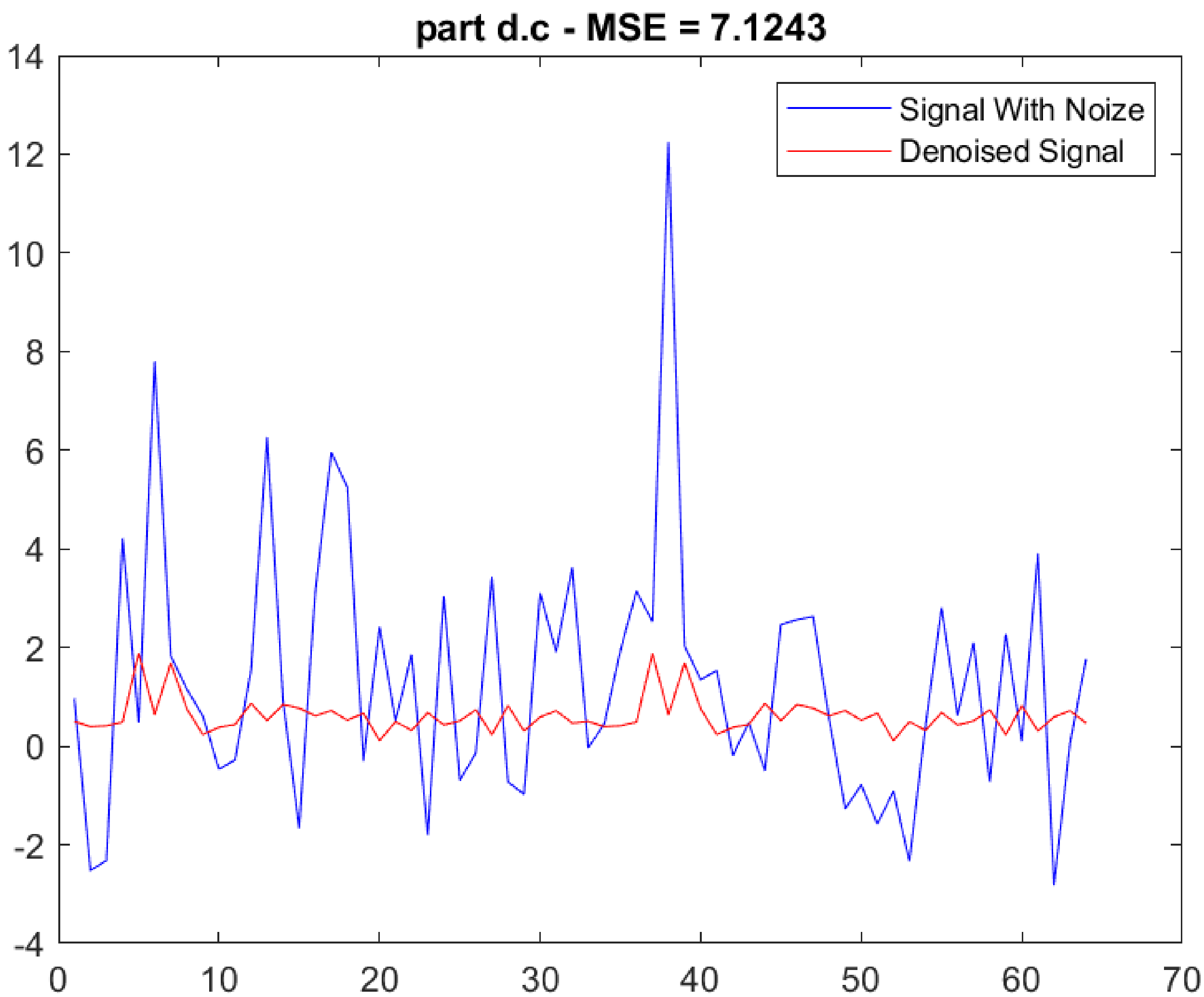


part d.c - MSE = 1.2112
Original Signal
Denoised Signal
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
0
10
20
30
40
50
60
70


part d.c - MSE = 5.919
Signal With Noize
Denoised Signal
6
4
2
0
-2
-4
-6
0
10
20
30
40
50
60
70

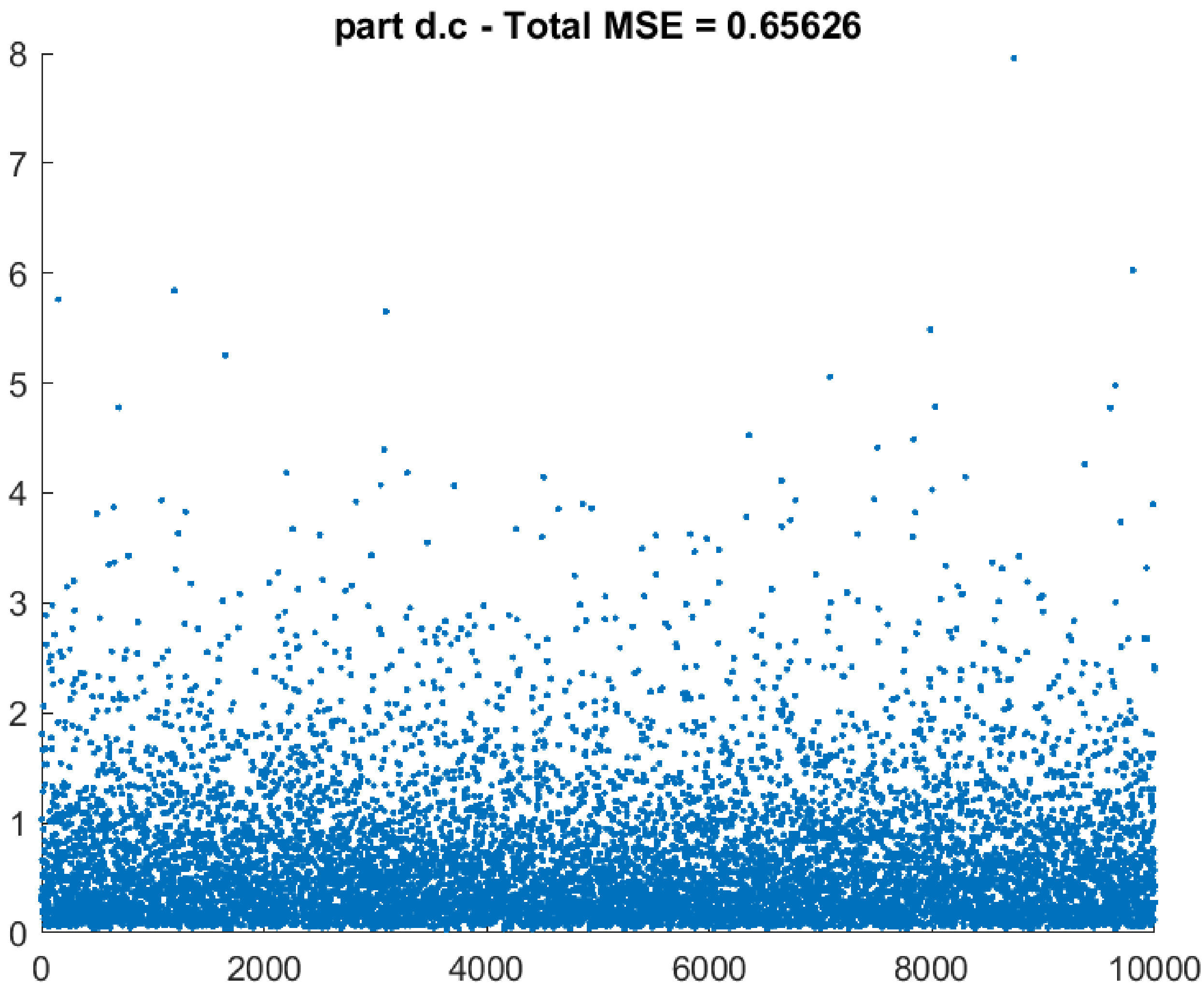


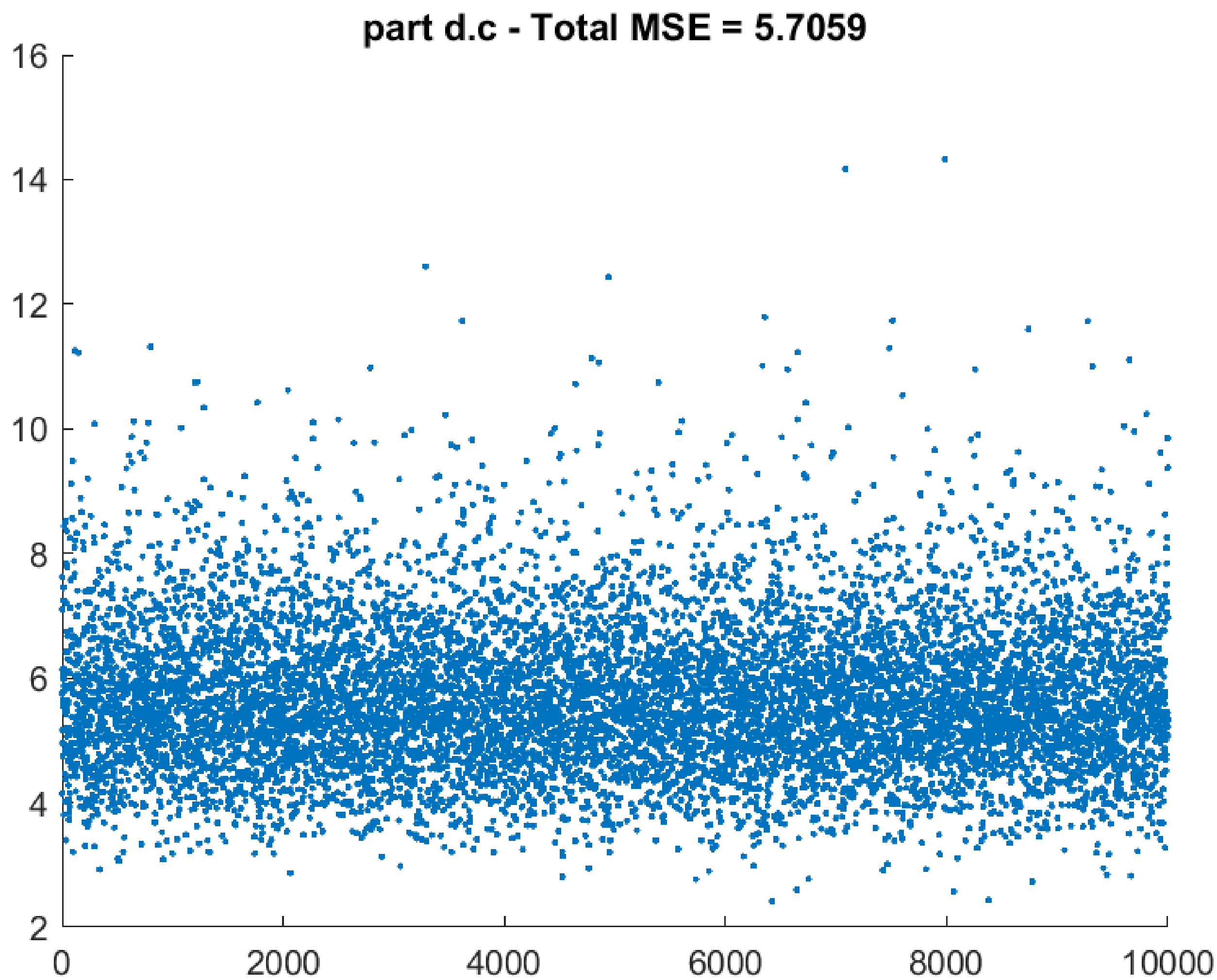
part d.c - MSE = 4.0292
Original Signal
Denoised Signal
0
2
4
6
8
10
12
0
10
20
30
40
50
60
70



part d.c - MSE = 7.1243
Signal With Noize
Denoised Signal
14
12
10
8
6
4
2
0
-2
-4
0
10
20
30
40
50
60
70

And the Total (viewed same as previously):






We can see that in some cases the MSE could be bigger then in d.b and c sections, which is visually cann be observed as in average the clean signal can be significantly above or under the restored signal, but in average it the MSE is bigger as expected then the previous undistorted version and similar to the c section total MSE, which can be a result of some randomness, because it is more expectable, that it will be greater then in c section as the noise is bigger.

It also can be noticed that the outliers are more distant from the 0 MSE comparing to section c and d.b, as expected with bigger variance. In some cases the spikes are also not corresponding comparing th restored signal to the clean one.

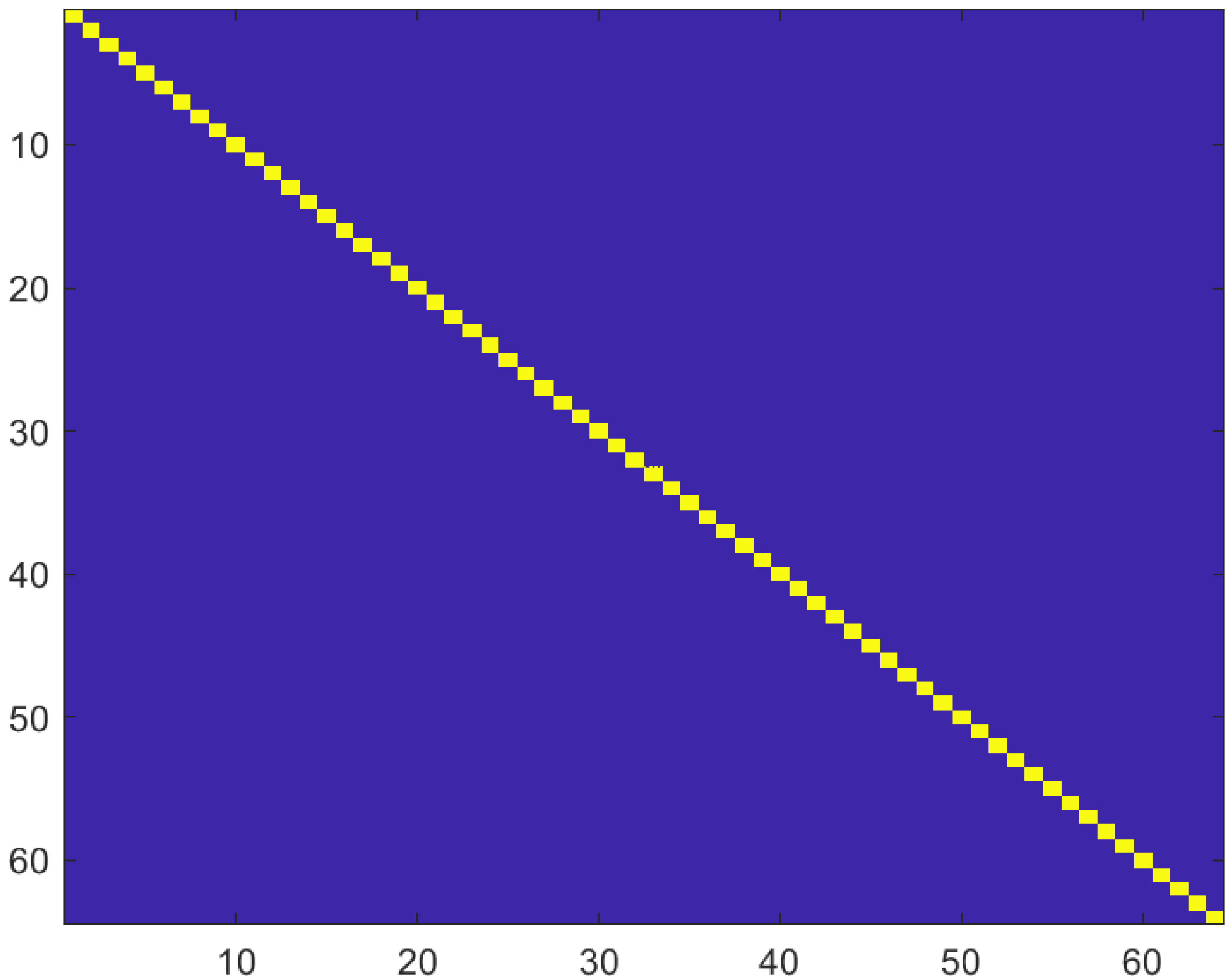
It is seen that Wiener filter doing less good job then previously.

The second type of graphs (noisy vs clean):

The MSE is as expected enlarged, as can be observed visually.

e.

H_H_pseudo_inv = H_pseudo_inv_H are:




We've observed that our H matrix is of full rank as we've seen from computing its eigenvalues, thus the pseudo-inverse is strictly the same as the inverse, and as a result of this, both H * H_pseudo_inv and

H_pseudo_inv * H are equal to Identity matrix.

That means that there aren't such phi_1 and phi_2 (which are different since the norm of their difference is posiive) for which: H_pseudo_inv * phi_1 = H_pseudo_inv * phi_2.

Because if we'd assumed that there are such, then:

H* H_pseudo_inv * phi_1 = H*H_pseudo_inv * phi_2

Then:

H * H^{-1} * phi_1 = H * H^{-1} * phi_2

Thus we would get:

$Phi_1 = Phi_2$ in contradiction to the fact that they are different.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