DATA 605 Week 14 Homework

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Taylor Series

Taylor Series is defined as $f(x) = \sum\limits_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$.

Function 1: $f(x) = 1/(1-x)$

Find first several derivatives.

$f^0(c) = \frac{1}{(1-c)}$

$f'(c) = \frac{1}{(1-c)^2}$

$f''(c) = \frac{2}{(1-c)^3}$

$f'''(c) = \frac{6}{(1-c)^4}$

$f''''(c) = \frac{24}{(1-c)^5}$

Per definition,

$$
\begin{aligned}
f(x) &= \frac{1}{(1-c)0!}(x-c)^0 + \frac{1}{(1-c)^2 1!}(x-c)^1 + \frac{2}{(1-c)^3 2!}(x-c)^2 + \frac{6}{(1-c)^4 3!}(x-c)^3 + \frac{24}{(1-c)^5 4!}(x-c)^4 + \ldots \\
&= \frac{1}{(1-c)} + \frac{1}{(1-c)^2}(x-c) + \frac{2!}{(1-c)^3 2!}(x-c)^2 + \frac{3!}{(1-c)^4 3!}(x-c)^3 + \frac{4!}{(1-c)^5 4!}(x-c)^4 + \ldots \\
&= \frac{1}{(1-c)} + \frac{1}{(1-c)^2}(x-c) + \frac{1}{(1-c)^3}(x-c)^2 + \frac{1}{(1-c)^4}(x-c)^3 + \frac{1}{(1-c)^5}(x-c)^4 + \ldots \\
&= \sum_{n=0}^{\infty} \frac{1}{(1-c)^{n+1}}(x-c)^n
\end{aligned}
$$

The Maclaurin Series of $f(x)$, $c = 0$, $f(x) = \sum\limits_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots$.

It is easy to see that this series will only converge if $|x| < 1$, so the valid range is $(-1, 1)$.

Function 2: $f(x) = e^x$

Find first several derivatives.

$f^0(c) = e^c$

$f'(c) = e^c$

$f''(c) = e^c$

$f'''(c) = e^c$

$f''''(c) = e^c$

Per definition,

$$
\begin{aligned}
f(x) &= \frac{e^c}{0!}(x-c)^0 + \frac{e^c}{1!}(x-c)^1 + \frac{e^c}{2!}(x-c)^2 + \frac{e^c}{3!}(x-c)^3 + ... \\
&= e^c + e^c(x-c) + e^c\frac{(x-c)^2}{2!} + e^c\frac{(x-c)^3}{3!} + ... \\
&= e^c \sum_{n=0}^{\infty} \frac{(x-c)^n}{n!}
\end{aligned}
$$

The Maclaurin Series of $f(x)$, $c = 0$, $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ...$

Ratio Test:

$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n} = \frac{x \times x^n \times n!}{(n+1) \times n! \times x^n} = \frac{x}{n+1}$

$L = \lim_{n\to\infty} \frac{x}{n+1} = 0$ and $L < 1$.

This series will converge for any x, so the valid range is $(-\infty, \infty)$.

Function 3: $f(x) = ln(1+x)$

Find first several derivatives.

$f^0(c) = ln(1+c)$

$f'(c) = \frac{1}{c+1}$

$f''(c) = -\frac{1}{(c+1)^2}$

$f'''(c) = \frac{2}{(c+1)^3}$

$f''''(c) = -\frac{6}{(c+1)^4}$

Per definition,

$$
\begin{aligned}
f(x) &= \frac{ln(1+c)}{0!}(x-c)^0 + \frac{1}{(c+1)1!}(x-c)^1 - \frac{1}{(c+1)^2 2!}(x-c)^2 + \frac{2}{(c+1)^3 3!}(x-c)^3 - \frac{6}{(c+1)^4 4!}(x-c)^4 + ... \\
&= ln(1+c) + \frac{1}{(c+1)}(x-c) - \frac{1!}{(c+1)^2 2 \times 1!}(x-c)^2 + \frac{2!}{(c+1)^3 3 \times 2!}(x-c)^3 - \frac{3!}{(c+1)^4 4 \times 3!}(x-c)^4 + ... \\
&= ln(1+c) + \frac{1}{(c+1)}(x-c) - \frac{1}{2(c+1)^2}(x-c)^2 + \frac{1}{3(c+1)^3}(x-c)^3 - \frac{1}{4(c+1)^4}(x-c)^4 + ... \\
&= ln(1+c) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-c)^n}{n(c+1)^n}
\end{aligned}
$$

The Maclaurin Series of $f(x)$, $c = 0$, $f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$

Ratio Test:

$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1+1} x^{n+1}}{n+1} \times \frac{n}{(-1)^{n+1} x^n} = \frac{(-1)^{n+1} \times (-1) \times x \times x^n \times n}{(n+1)(-1)^{n+1} x^n} = \frac{-xn}{n+1}$

$L = \lim_{n\to\infty} |\frac{-xn}{n+1}| = |x|$

This series will only converge if $L < 1$ or $|x| < 1$, so then the valid range is $(-1, 1)$.