DATA 605 Week 9 Homework

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Problem 1: Chapter 9.3 Exercise 11

The price of one share of stock in the Pilsdorff Beer Company is given by Y_n on the nth day of the year. Finn observes that the differences $X_n = Y_{n+1} - Y_n$ appear to be independent random variables with a common distribution having mean $\mu = 0$ and variance $\sigma^2 = 1/4$. If $Y_1 = 100$, estimate the probability that Y_{365} is

a. ≥ 100
b. ≥ 110
c. ≥ 120

SOLUTION

Since X_n is an independent random variable, then its sum S_n is normally distributed.

$$\begin{aligned}
S_n &= X_1 + X_2 + X_3 ... + X_n \\
&= (Y_2 - Y_1) + (Y_3 - Y_2) + (Y_4 - Y_3) + ... + (Y_{n+1} - Y_n) \\
&= Y_2 - Y_1 + Y_3 - Y_2 + Y_4 - Y_3 + ... + Y_{n+1} - Y_n \\
&= Y_{n+1} - Y_1 \\
&= Y_{n+1} - 100
\end{aligned}$$

Mean of S_n: $\mu_{S_n} = \mu_{X_1} + \mu_{X_2} + \mu_{X_3} + ... + \mu_{X_n} = n\mu_X = 0$

Variance of S_n: $\sigma^2_{S_n} = \sigma^2_{X_1} + \sigma^2_{X_2} + \sigma^2_{X_3} + ... + \sigma^2_{X_n} = n\sigma^2_X = n \times 1/4 = n/4$

Standard deviation of S_n: $\sigma_{S_n} = \sqrt{n}/2$

Consider $n = 364$, then $S_{364} = Y_{365} - 100$ and $Y_{365} = S_{364} + 100$.

$\sigma^2_{S_{364}} = 364/4 = 91$ and $\sigma_{S_{364}} = \sqrt{91}$

Part (a)

$$\begin{aligned}
P(Y_{365} \geq 100) &= P(S_{364} + 100 \geq 100) \\
&= P(S_{364} \geq 0)
\end{aligned}$$

Since S_n is normally distributed with mean 0 and normal distribution is symmetric, exactly half of values will be greater than mean.

ANSWER: $P(S_{364} \geq 0) = 0.5$

Part (b)

$$\begin{aligned}
P(Y_{365} \geq 110) &= P(S_{364} + 100 \geq 110) \\
&= P(S_{364} \geq 10) \\
&= P(S^*_{364} \geq 10/\sqrt{91})
\end{aligned}$$

```r
z <- 10/sqrt(91)
pnorm(z, lower.tail=FALSE)
```

```
## [1] 0.1472537
```

ANSWER: $P(Y365 \geq 110) \approx 0.14725$

Part (c)

$$\begin{aligned}
P(Y_{365} \geq 120) &= P(S_{364} + 100 \geq 120) \\
&= P(S_{364} \geq 20) \\
&= P(S^*_{364} \geq 20/\sqrt{91})
\end{aligned}$$

```r
z <- 20/sqrt(91)
pnorm(z, lower.tail=FALSE)
```

```
## [1] 0.01801584
```

ANSWER: $P(Y365 \geq 120) \approx 0.01802$

Problem 2

Calculate the expected value and variance of the binomial distribution using the moment generating function.

SOLUTION

For binomial distribution, $P(X = k) = \binom{n}{k} p^k q^{n-k}$, where $q = 1 - p$.

I have used the following chapter found at the site of The University of Arizona: http://math.arizona.edu/~tgk/464_10/chap4_9_29.pdf.

The moment generating function is $M_X(t) = (q + pe^t)^n$.

The first moment is $M'_X(t) = n(q + pe^t)^{n-1}pe^t$.

The expected value is the first moment evaluated at $t = 0$:

$$\begin{aligned}
E(X) = M'_X(0) &= n(q + pe^0)^{n-1}pe^0 \\
&= n(q + p)^{n-1}p \\
&= np(1 - p + p)^{n-1} \\
&= np1^{n-1} \\
&= np
\end{aligned}$$

The second moment is $M''_X(t) = n(n-1)(q + pe^t)^{n-2}p^2e^{2t} + n(q + pe^t)^{n-1}pe^t$.

Evaluate the second moment at $t = 0$:

$$
\begin{aligned}
E(X^2) = M_X''(0) &= n(n-1)(q + pe^0)^{n-2}p^2e^0 + n(q + pe^0)^{n-1}pe^0 \\
&= n(n-1)(1 - p + p)^{n-2}p^2 + n(1 - p + p)^{n-1}p \\
&= n(n-1)p^2 + np
\end{aligned}
$$

The variance is $V(X) = E(X^2) - E(X)^2$:

$$
\begin{aligned}
V(X) &= n(n-1)p^2 + np - n^2p^2 \\
&= np((n-1)p + 1 - np) \\
&= np(np - p + 1 - np) \\
&= np(1 - p) \\
&= npq
\end{aligned}
$$

We arrived at the known definitions for binomial distribution - $E(X) = np$ and $V(X) = npq$.

Problem 3

Calculate the expected value and variance of the exponential distribution using the moment generating function.

SOLUTION

For exponential distribution, $f(x) = \lambda e^{-\lambda x}$.

The moment generating function is $M_X(t) = \frac{\lambda}{\lambda - t}, t < \lambda$.

Using WolframAlpha, we get $M_X'(t) = \frac{\lambda}{(\lambda-t)^2}$ and $M_X''(t) = \frac{2\lambda}{(\lambda-t)^3}$.

Expected value:

$$
\begin{aligned}
E(X) = M_X'(0) &= \frac{\lambda}{(\lambda - 0)^2} \\
&= \frac{\lambda}{\lambda^2} \\
&= \frac{1}{\lambda}
\end{aligned}
$$

Variance:

$$
\begin{aligned}
V(X) = E(X^2) - E(X)^2 &= M_X''(0) - M_X'(0)^2 \\
&= \frac{2\lambda}{(\lambda - 0)^3} - \frac{1}{\lambda^2} \\
&= \frac{2\lambda}{\lambda^3} - \frac{1}{\lambda^2} \\
&= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\
&= \frac{1}{\lambda^2}
\end{aligned}
$$

We arrived at the known definitions for binomial distribution - $E(X) = 1/\lambda$ and $V(X) = 1/\lambda^2$.