DATA 605 Week 8 Homework

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Chapter 7.2 Exercise 11

A company buys 100 lightbulbs, each of which has an exponential lifetime of 1000 hours. What is the expected time for the first of these bulbs to burn out?

SOLUTION

Per problem 10, the density of minimum value among n independent random variables with an exponential density has mean μ/n, where μ is mean of exponential density of individual variable. Per problem, $n = 100$ and $\mu = 1000$, so the expected time for the first of the bulbs to burn out is $E(M) = \mu/n = 1000/100 = 10$.

Problem 10 is confirmed/solved by Jaan Bernberg in our Week 8 Discussion Board (http://rpubs.com/jbrnbrg/DiscussWk8_605).

Additionally, in researching this problem, I came across a note that lifetime of lightbulbs is not necessarily an independent event as there are factors that will influence multiple lightbulbs (such as spikes in the electric grid).

Chapter 7.2 Exercise 14

Assume that X_1 and X_2 are independent random variables, each having an exponential density with parameter λ. Show that $Z = X_1 - X_2$ has density $f_Z(z) = (1/2)e^{-\lambda|z|}$.

SOLUTION

$f_Z(z) = (1/2)e^{-\lambda|z|}$ can be re-written as $f_Z(z) = \begin{cases} (1/2)e^{-\lambda z}, & \text{if } z \geq 0, \\ (1/2)e^{\lambda z}, & \text{if } z < 0. \end{cases}$

Since X_1 and X_2 have exponential density, their PDF is

$f_{X_1}(x) = f_{X_2}(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$

$$
\begin{aligned}
f_Z(z) &= f_{X_1+(-X_2)}(z) \\
&= \int_{-\infty}^{\infty} f_{-X_2}(z-x_1) f_{X_1}(x_1) dx_1 \\
&= \int_{-\infty}^{\infty} f_{X_2}(x_1-z) f_{X_1}(x_1) dx_1 \\
&= \int_{-\infty}^{\infty} \lambda e^{-\lambda(x_1-z)} \lambda e^{-\lambda x_1} dx_1 \\
&= \int_{-\infty}^{\infty} \lambda^2 e^{-\lambda x_1+\lambda z} e^{-\lambda x_1} dx_1 \\
&= \int_{-\infty}^{\infty} \lambda^2 e^{\lambda z-\lambda x_1-\lambda x_1} dx_1 \\
&= \int_{-\infty}^{\infty} \lambda^2 e^{\lambda(z-2x_1)} dx_1
\end{aligned}
$$

Consider $z = x_1 - x_2$, then $x_2 = x_1 - z$.

If $z \geq 0$, then $x_2 = (x_1 - z) \geq 0$, and $x_1 \geq z$, and, using WolframAlpha, $f_Z(z) = \int_z^{\infty} \lambda^2 e^{\lambda(z-2x_1)} dx_1 = \frac{1}{2}\lambda e^{-\lambda z}$.

If $z < 0$, then $x_2 = (x_1 - z) \geq 0$, and $x_1 \geq 0$, and $f_Z(z) = \int_0^{\infty} \lambda^2 e^{\lambda(z-2x_1)} dx_1 = \frac{1}{2}\lambda e^{\lambda z}$.

Combining two sides we get $f_Z(z) = \begin{cases} (1/2)e^{-\lambda z}, & \text{if } z \geq 0, \\ (1/2)e^{\lambda z}, & \text{if } z < 0. \end{cases}$

Chapter 8.2 Exercise 1

Let X be a continuous random variable with mean $\mu = 10$ and variance $\sigma^2 = 100/3$. Using Chebyshev's Inequality, find an upper bound for the following probabilities.

a. $P(|X - 10| \geq 2)$
b. $P(|X - 10| \geq 5)$
c. $P(|X - 10| \geq 9)$
d. $P(|X - 10| \geq 20)$

SOLUTION

Chebyshev Inequality: $P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$ or, per example 8.4, $P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$.

Per problem, $\mu = 10$ and $\sigma = \sqrt{\frac{100}{3}} = \frac{10}{\sqrt{3}}$.

If $\epsilon = k\sigma$, then $k = \frac{\epsilon}{\sigma} = \frac{\epsilon\sqrt{3}}{10}$.

Let u be upper bound in Chebyshev's Inequality, then $u = \frac{1}{k^2} = \frac{1}{(\epsilon\sqrt{3}/10)^2} = \frac{100}{3\epsilon^2}$.

a. $\epsilon = 2$, the upper bound is $u = \frac{100}{3\times2^2} = \frac{25}{3} \approx 8.3333$. Since probability cannot be greater than 1, the upper bound is 1.
b. $\epsilon = 5$, the upper bound is $u = \frac{100}{3\times5^2} = \frac{4}{3} \approx 1.3333$. Since probability cannot be greater than 1, the upper bound is 1.
c. $\epsilon = 9$, the upper bound is $u = \frac{100}{3\times9^2} = \frac{100}{243} \approx 0.4115$.
d. $\epsilon = 20$, the upper bound is $u = \frac{100}{3\times20^2} = \frac{1}{12} \approx 0.0833$.