

# On the isomorphism problem for generalized Baumslag-Solitar groups: invariants and flexible configurations

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## 1 Introduction

**Theorem A** (Corollary 6.27). *There is an algorithm that, given two GBS graph  $(\Gamma, \psi), (\Delta, \phi)$  with one qc-class and full-support gaps, decides whether the corresponding GBS groups are isomorphic or not.*

## 2 GBS graphs and quasi-conjugacy classes

In this section we set up the notation for the rest of the paper. We review the notions of conjugacy and quasi-conjugacy, and we show how they can be described by means of a finite amount of data. We provide explicit algorithms for dealing with them, and examples to illustrate these notions.

### 2.1 Graphs of groups

We consider graphs as combinatorial objects, following the notation of [Ser77]. A **graph** is a quadruple  $\Gamma = (V, E, \bar{\cdot}, \iota)$  consisting of a set  $V = V(\Gamma)$  of *vertices*, a set  $E = E(\Gamma)$  of *edges*, a map  $\bar{\cdot} : E \rightarrow E$  called *reverse* and a map  $\iota : E \rightarrow V$  called *initial endpoint*; we require that, for every edge  $e \in E$ , we have  $\bar{\bar{e}} \neq e$  and  $\bar{\bar{e}} = e$ . For an edge  $e \in E$ , we denote with  $\tau(e) = \iota(\bar{e})$  the *terminal endpoint* of  $e$ . A **path** in a graph  $\Gamma$ , with *initial endpoint*  $v \in V(\Gamma)$  and *terminal endpoint*  $v' \in V(\Gamma)$ , is a sequence  $\sigma = (e_1, \dots, e_\ell)$  of edges  $e_1, \dots, e_\ell \in E(\Gamma)$  for some integer  $\ell \geq 0$ , with the conditions  $\iota(e_1) = v$  and  $\tau(e_\ell) = v'$  and  $\tau(e_i) = \iota(e_{i+1})$  for  $i = 1, \dots, \ell - 1$ . A graph is **connected** if for every couple of vertices, there is a path going from one to the other. For a connected graph  $\Gamma$ , we define its **rank**  $\text{rk } \Gamma \in \mathbb{N} \cup \{+\infty\}$  as the rank of its fundamental group (which is a free group).

**Definition 2.1.** A **graph of groups** is a quadruple

$$\mathcal{G} = (\Gamma, \{G_v\}_{v \in V(\Gamma)}, \{G_e\}_{e \in E(\Gamma)}, \{\psi_e\}_{e \in E(\Gamma)})$$

consisting of a connected graph  $\Gamma$ , a group  $G_v$  for each vertex  $v \in V(\Gamma)$ , a group  $G_e$  for every edge  $e \in E(\Gamma)$  with the condition  $G_e = G_{\bar{e}}$ , and an injective homomorphism  $\psi_e : G_e \rightarrow G_{\tau(e)}$  for every edge  $e \in E(\Gamma)$ .

Let  $\mathcal{G} = (\Gamma, \{G_v\}_{v \in V(\Gamma)}, \{G_e\}_{e \in E(\Gamma)}, \{\psi_e\}_{e \in E(\Gamma)})$  be a graph of groups. Define the **universal group**  $\text{FG}(\mathcal{G})$  as the quotient of the free product  $(\ast_{v \in V(\Gamma)} G_v) \ast F(E(\Gamma))$  by the relations

$$\bar{e} = e^{-1} \quad \psi_{\bar{e}}(g) \cdot e = e \cdot \psi_e(g)$$

for  $e \in E(\Gamma)$  and  $g \in G_e$ .

Define the **fundamental group**  $\pi_1(\mathcal{G}, \tilde{v})$  of a graph of group  $\mathcal{G}$  with basepoint  $\tilde{v} \in V(\Gamma)$  to be the subgroup of  $\text{FG}(\mathcal{G})$  of the elements that can be represented by words such that, when going along the word, we read a path in the graph  $\Gamma$  from  $\tilde{v}$  to  $\tilde{v}$ . The fundamental group  $\pi_1(\mathcal{G}, \tilde{v})$  doesn't depend on the chosen basepoint  $\tilde{v}$ , up to isomorphism. Given an element  $g$  inside a vertex group  $G_v$ , we can take a path  $(e_1, \dots, e_\ell)$  from  $\tilde{v}$  to  $v$ : we define the conjugacy class  $[g] := [e_1 \dots e_\ell g \bar{e}_\ell \dots \bar{e}_1] \in \pi_1(\mathcal{G}, \tilde{v})$ , and notice that this doesn't depend on the chosen path.

### 2.2 Generalized Baumslag-Solitar groups

**Definition 2.2.** A **GBS graph of groups** is a finite graph of groups

$$\mathcal{G} = (\Gamma, \{G_v\}_{v \in V(\Gamma)}, \{G_e\}_{e \in E(\Gamma)}, \{\psi_e\}_{e \in E(\Gamma)})$$

such that each vertex group and each edge group is  $\mathbb{Z}$ .

A **Generalized Baumslag-Solitar group** is a group  $G$  isomorphic to the fundamental group of some GBS graph of groups.

**Definition 2.3.** A **GBS graph** is a couple  $(\Gamma, \psi)$  where  $\Gamma$  is a finite graph and  $\psi : E(\Gamma) \rightarrow \mathbb{Z} \setminus \{0\}$  is a function.

Given a GBS graph of groups  $\mathcal{G} = (\Gamma, \{G_v\}_{v \in V(\Gamma)}, \{G_e\}_{e \in E(\Gamma)}, \{\psi_e\}_{e \in E(\Gamma)})$ , the map  $\psi_e : G_e \rightarrow G_{\tau(e)}$  is an injective homomorphism  $\psi_e : \mathbb{Z} \rightarrow \mathbb{Z}$ , and thus coincides with multiplication by a unique non-zero integer  $\psi(e) \in \mathbb{Z} \setminus \{0\}$ . We define the GBS graph associated to  $\mathcal{G}$  as  $(\Gamma, \psi)$  associating to each edge  $e$  the factor  $\psi(e)$  characterizing the homomorphism  $\psi_e$ , see Figure 1. Giving a GBS graph of groups is equivalent to giving the corresponding GBS graph. In fact, the numbers on the edges are enough to reconstruct the injective homomorphisms and thus the graph of groups.

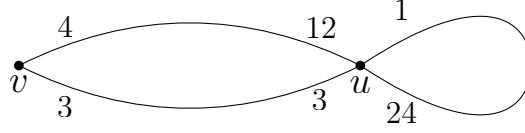


Figure 1: In the figure we can see a GBS graph  $(\Gamma, \psi)$  with two vertices  $v, u$  and three edges  $e_1, e_2, e_3$  (and their reverses). The edge  $e_1$  goes from  $v$  to  $u$  and has  $\psi(\bar{e}_1) = 4$  and  $\psi(e_1) = 12$ . The edge  $e_2$  goes from  $v$  to  $u$  and has  $\psi(\bar{e}_2) = \psi(e_2) = 3$ . The edge  $e_3$  goes from  $u$  to  $u$  and has  $\psi(\bar{e}_3) = 1$  and  $\psi(e_3) = 24$ .

Let  $\mathcal{G}$  be a GBS graph of groups and let  $(\Gamma, \psi)$  be the corresponding GBS graph. The universal group  $\text{FG}(\mathcal{G})$  has a presentation with generators  $V(\Gamma) \cup E(\Gamma)$ , the generator  $v \in V(\Gamma)$  representing the element 1 in  $\mathbb{Z} = G_v$ . The relations are given by  $\bar{e} = e^{-1}$  and  $u^{\psi(\bar{e})}e = ev^{\psi(e)}$  for every edge  $e \in E(\Gamma)$  with  $\iota(e) = u$  and  $\tau(e) = v$ .

### 2.3 Reduced affine representation of a GBS graph

**Definition 2.4.** For a GBS graph  $(\Gamma, \psi)$ , define its **set of primes**

$$\mathcal{P}(\Gamma, \psi) := \{r \in \mathbb{N} \text{ prime} : r \mid \psi(e) \text{ for some } e \in E(\Gamma)\}$$

Given a GBS graph  $(\Gamma, \psi)$ , consider the finitely generated abelian group

$$\mathbf{A} := \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{r \in \mathcal{P}(\Gamma, \psi)} \mathbb{Z}.$$

We denote with  $\mathbf{0} \in \mathbf{A}$  the neutral element. For an element  $\mathbf{a} = (a_0, a_r : r \in \mathcal{P}(\Gamma, \psi)) \in \mathbf{A}$  (with  $a_0 \in \mathbb{Z}/2\mathbb{Z}$  and  $a_r \in \mathbb{Z}$  for  $r \in \mathcal{P}(\Gamma, \psi)$ ), we denote  $\mathbf{a} \geq \mathbf{0}$  if  $a_r \geq 0$  for all  $r \in \mathcal{P}(\Gamma, \psi)$ ; notice that we aren't requiring any condition on  $a_0$ . We define the positive cone  $\mathbf{A}^+ := \{\mathbf{a} \in \mathbf{A} : \mathbf{a} \geq \mathbf{0}\}$ .

**Definition 2.5.** Let  $(\Gamma, \psi)$  be a GBS graph. Define its **(reduced) affine representation** to be the graph  $\Lambda = \Lambda(\Gamma, \psi)$  given by:

1.  $V(\Lambda) = V(\Gamma) \times \mathbf{A}^+$  is the disjoint union of copies of  $\mathbf{A}^+$ , one for each vertex of  $\Gamma$ .
2.  $E(\Lambda) = E(\Gamma)$  is the same set of edges as  $\Gamma$ , and with the same reverse map.
3. For an edge  $e \in E(\Lambda)$  we write the unique factorization  $\psi(e) = (-1)^{a_0} \prod_{r \in \mathcal{P}(\Gamma, \psi)} r^{a_r}$  and we define the terminal endpoint  $\tau_\Lambda(e) = (\tau_\Gamma(e), (a_0, a_r, \dots))$ , see Figure 2.

For a vertex  $v \in V(\Gamma)$  we denote  $\mathbf{A}_v^+ := \{v\} \times \mathbf{A}^+$  the corresponding copy of  $\mathbf{A}^+$ .

If  $\Lambda$  contains an edge going from  $p$  to  $q$ , then we denote  $p \longrightarrow q$ . If  $\Lambda$  contains edges from  $p_i$  to  $q_i$  for  $i = 1, \dots, m$ , then we denote

$$\begin{cases} p_1 \longrightarrow q_1 \\ \dots \\ p_m \longrightarrow q_m \end{cases}$$

This doesn't mean that  $p_1 \longrightarrow q_1, \dots, p_m \longrightarrow q_m$  are all the edges of  $\Lambda$ , but only that in a certain situation we are focusing on those edges. If we are focusing on a specific copy  $\mathbf{A}_v^+$  of  $\mathbf{A}^+$ , for some  $v \in V(\Gamma)$ , and we have edges  $(v, \mathbf{a}_i) \longrightarrow (v, \mathbf{b}_i)$  for  $i = 1, \dots, m$ , then we say that  $\mathbf{A}_v^+$  contains edges

$$\begin{cases} \mathbf{a}_1 \longrightarrow \mathbf{b}_1 \\ \dots \\ \mathbf{a}_m \longrightarrow \mathbf{b}_m \end{cases}$$

omitting the vertex  $v$ .

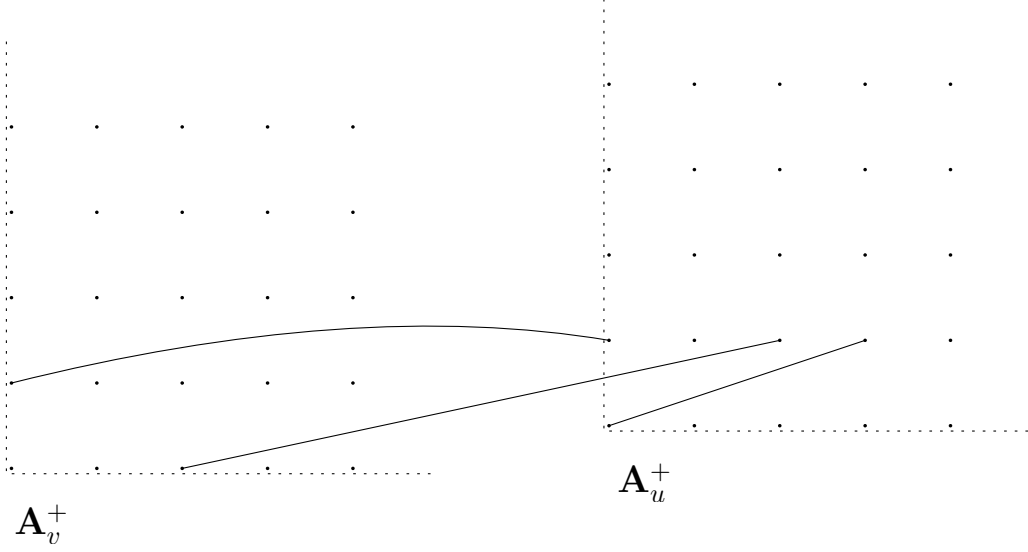


Figure 2: The affine representation  $\Lambda$  of the GBS graph  $(\Gamma, \psi)$  of Figure 1. The set of vertices consists of two copies  $\mathbf{A}_v^+$  and  $\mathbf{A}_u^+$  of the positive affine cone  $\mathbf{A}^+$ , associated to the two vertices  $v$  and  $u$  respectively. The edge  $e_1$  going from  $v$  to  $u$  was labeled with  $(\psi(\bar{e}_1), \psi(e_1)) = (4, 12) = (2^2 3^0, 2^2 3^1)$ , and thus now it goes from the point  $(2, 0)$  in  $\mathbf{A}_v^+$  to the point  $(2, 1)$  in  $\mathbf{A}_u^+$ . Similarly for  $e_2$  and  $e_3$ .

## 2.4 Support and control of vectors

The following notions for elements of  $\mathbf{A}$  will be widely used along the paper.

**Definition 2.6.** For  $\mathbf{x} \in \mathbf{A}$  define its **support** as the set

$$\text{supp}(\mathbf{x}) := \{r \in \mathcal{P}(\Gamma, \psi) : x_r \neq 0\}$$

*Remark 2.7.* Note that we omit the  $\mathbb{Z}/2\mathbb{Z}$  component from the definition of support.

**Definition 2.8.** Let  $\mathbf{a}, \mathbf{b}, \mathbf{w} \in \mathbf{A}^+$ . We say that  $\mathbf{a}, \mathbf{w}$  **controls**  $\mathbf{b}$  if any of the following equivalent conditions holds:

1. We have  $\mathbf{a} \leq \mathbf{b} \leq \mathbf{a} + k\mathbf{w}$  for some  $k \in \mathbb{N}$ .
2. We have  $\mathbf{b} - \mathbf{a} \geq \mathbf{0}$  and  $\text{supp}(\mathbf{b} - \mathbf{a}) \subseteq \text{supp}(\mathbf{w})$ .

## 2.5 Affine paths and conjugacy classes

Let  $(\Gamma, \psi)$  be a GBS graph and let  $\Lambda$  be its affine representation. Given a vertex  $p = (v, \mathbf{a}) \in V(\Lambda)$  and an element  $\mathbf{w} \in \mathbf{A}^+$ , we define the vertex  $p + \mathbf{w} := (v, \mathbf{a} + \mathbf{w}) \in V(\Lambda)$ . For two vertices  $p, p' \in V(\Lambda)$  we denote  $p' \geq p$  if  $p' = p + \mathbf{w}$  for some  $\mathbf{w} \in \mathbf{A}^+$ ; in particular this implies that both  $p, p'$  belong to the same  $\mathbf{A}_v^+$  for some  $v \in V(\Gamma)$ .

**Definition 2.9.** An **affine path** in  $\Lambda$ , with initial endpoint  $p \in V(\Lambda)$  and terminal endpoint  $p' \in V(\Lambda)$ , is a sequence  $(e_1, \dots, e_\ell)$  of edges  $e_1, \dots, e_\ell \in E(\Lambda)$  for some  $\ell \geq 0$ , such that there exist  $\mathbf{w}_1, \dots, \mathbf{w}_\ell \in \mathbf{A}^+$  satisfying the conditions  $\iota(e_1) + \mathbf{w}_1 = p$  and  $\tau(e_\ell) + \mathbf{w}_\ell = p'$  and  $\tau(e_i) + \mathbf{w}_i = \iota(e_{i+1}) + \mathbf{w}_{i+1}$  for  $i = 1, \dots, \ell - 1$ .

The elements  $\mathbf{w}_1, \dots, \mathbf{w}_\ell$  are called *translation coefficients* of the path; if they exist, then they are uniquely determined by the path and by the endpoints, and they can be computed algorithmically. They mean that an edge  $e \in E(\Lambda)$  connecting  $p$  to  $q$  allows us also to travel from  $p + \mathbf{w}$  to  $q + \mathbf{w}$  for every  $\mathbf{w} \in \mathbf{A}^+$ .

**Definition 2.10.** Let  $p, q \in V(\Lambda)$ .

1. We denote  $p \sim_c q$ , and we say that  $p, q$  are **conjugate**, if there is an affine path going from  $p$  to  $q$ .
2. We denote  $p \preceq_c q$  if  $p \leq q'$  for some  $q' \sim_c q$ .
3. We denote  $p \sim_{qc} q$ , and we say that  $p, q$  are **quasi-conjugate**, if  $p \preceq_c q$  and  $q \preceq_c p$ .

The relation  $\sim_c$  is an equivalence relations on the set  $V(\Lambda)$ . The relation  $\preceq_c$  is a pre-order on  $V(\Lambda)$ , and  $\sim_{qc}$  is the equivalence relation induced by the pre-order. Note that if  $p \sim_c p'$  then  $p + \mathbf{w} \sim_c p' + \mathbf{w}$  for all  $\mathbf{w} \in \mathbf{A}^+$ . Similarly, if  $p \sim_{qc} p'$  then  $p + \mathbf{w} \sim_{qc} p' + \mathbf{w}$  for all  $\mathbf{w} \in \mathbf{A}^+$ .

## 2.6 Minimal points, quasi-conjugacy support, linear algebra

We now show how conjugacy classes and quasi-conjugacy classes can be completely described by means of three pieces of data: the *minimal points*, the *quasi-conjugacy support* and the *linear algebra*.

**Definition 2.11.** Let  $(\Gamma, \psi)$  be a GBS graph and let  $p$  be a vertex of its affine representation  $\Lambda$ .

1. Define the **minimal points**

$$\min_{qc}(p) = \{q \in V(\Lambda) : q \sim_{qc} p \text{ and } q \preceq_c q' \text{ for all } q' \sim_{qc} p\}$$

2. Define the **quasi-conjugacy support**

$$\text{supp}_{qc}(p) = \bigcup \{\text{supp}(\mathbf{u}) : \mathbf{u} \in \mathbf{A}^+ \text{ and } p + \mathbf{u} \sim_{qc} p\} \subseteq \mathcal{P}(\Gamma, \psi).$$

3. Define the **linear algebra**

$$\text{lin}_{qc}(p) = \{\mathbf{r} \in \mathbf{A} : \mathbf{r} = \mathbf{w} - \mathbf{w}' \text{ for some } \mathbf{w}, \mathbf{w}' \in \mathbf{A}^+ \text{ such that } p + \mathbf{w} \sim_c p + \mathbf{w}' \sim_c p\}.$$

**Lemma 2.12.** The set  $\min_{qc}(p)$  is finite.

*Proof.* Fix a vertex  $u \in V(\Gamma)$ : since  $\Gamma$  has finitely many vertices, it's enough to prove that  $\min_{qc}(p) \cap \mathbf{A}_u^+$  is finite. Consider a set of variables  $\{x_i\}_{i \in \mathcal{P}(\Gamma, \psi)}$  and consider the ring of polynomials  $\mathbb{C}[x_i : i \in \mathcal{P}(\Gamma, \psi)]$ . For each  $(u, \mathbf{q}) \sim_{qc} p$  consider the monomial  $x^{\mathbf{q}} = \prod x_i^{q_i} \in \mathbb{C}[x_i, \dots]$ . Consider the ideal  $I = (x^{\mathbf{q}} : (u, \mathbf{q}) \sim_{qc} p) \subseteq \mathbb{C}[x_i, \dots]$  generated by these monomials. We have that  $I$  is a monomial ideal, i.e. it contains a polynomial  $P$  if and only if it contains all the monomial with non-zero coefficients of  $P$ . By Hilbert's basis theorem,  $I$  must be finitely generated, and thus we can find we find finitely many  $\mathbf{q}_1, \dots, \mathbf{q}_\ell \in \mathbf{A}^+$  such that  $I = (x^{\mathbf{q}_1}, \dots, x^{\mathbf{q}_\ell})$ . It follows that  $\min_{qc}(p) \cap \mathbf{A}_u^+ \subseteq \{(u, \mathbf{q}_1), \dots, (u, \mathbf{q}_\ell)\}$  must be finite, as desired.  $\square$

**Lemma 2.13.** If  $p \sim_{qc} q$  then  $\text{supp}_{qc}(p) = \text{supp}_{qc}(q)$ .

*Proof.* If  $p + \mathbf{u} \sim_{qc} p$  for some  $\mathbf{u} \in \mathbf{A}^+$ , then  $q + \mathbf{u} \sim_{qc} p + \mathbf{u} \sim_{qc} p \sim_{qc} q$ .  $\square$

**Lemma 2.14.** For all  $p \in V(\Lambda)$  there is  $\mathbf{w} \in \mathbf{A}^+$  such that  $p + \mathbf{w} \sim_c p$  and  $\text{supp}(\mathbf{w}) = \text{supp}_{qc}(p)$ .

*Proof.* Realize the quasi-conjugacy support as a finite union  $\text{supp}_{qc}(p) = \text{supp}(\mathbf{u}_1) \cup \dots \cup \text{supp}(\mathbf{u}_\ell)$  for  $\mathbf{u}_i \in \mathbf{A}^+$  with  $p + \mathbf{u}_i \sim_{qc} p$ . By definition of quasi-conjugacy, we can find  $\mathbf{v}_i \in \mathbf{A}^+$  such that  $p + \mathbf{u}_i + \mathbf{v}_i \sim_c p$ . Now set  $\mathbf{w} = (\mathbf{u}_1 + \mathbf{v}_1) + \dots + (\mathbf{u}_\ell + \mathbf{v}_\ell)$  and the conclusion follows.  $\square$

**Lemma 2.15.** We have that  $\text{lin}_{qc}(p)$  is a subgroup of  $\mathbf{A}$ .

*Proof.* Suppose that we are given  $\mathbf{r}, \mathbf{s} \in \text{lin}_{qc}(p)$ . Thus we have  $\mathbf{r} = \mathbf{w} - \mathbf{w}'$  and  $\mathbf{s} = \mathbf{u} - \mathbf{u}'$  for  $\mathbf{w}, \mathbf{w}', \mathbf{u}, \mathbf{u}' \in \mathbf{A}^+$  with  $p \sim_c p + \mathbf{w} \sim_c p + \mathbf{w}' \sim_c p + \mathbf{u} \sim_c p + \mathbf{u}'$ . In particular we have that  $p + \mathbf{w} + \mathbf{u} \sim_c p + \mathbf{w}' + \mathbf{u} \sim_c p + \mathbf{w}' + \mathbf{u}'$  and  $(\mathbf{w} + \mathbf{u}) - (\mathbf{w}' + \mathbf{u}') = \mathbf{r} + \mathbf{s}$ . Thus  $\mathbf{r} + \mathbf{s} \in \text{lin}_{qc}(p)$ . The conclusion follows.  $\square$

**Lemma 2.16** (Equivalent characterization of linear algebra). For  $p \in V(\Lambda)$  and  $\mathbf{r} \in \mathbf{A}$ , we have that  $\mathbf{r} \in \text{lin}_{qc}(p)$  if and only if it satisfies the following two conditions:

1.  $\Lambda$  contains vertices  $(u, \mathbf{a}) \sim_c (u, \mathbf{a} + \mathbf{r}) \sim_{qc} p$  for some  $u \in V(\Gamma)$  and  $\mathbf{a} \in \mathbf{A}$  with  $\mathbf{a}, \mathbf{a} + \mathbf{r} \geq \mathbf{0}$ .
2.  $\text{supp}(\mathbf{r}) \subseteq \text{supp}_{qc}(p)$

*Proof.* The definition of  $\text{lin}_{qc}(p)$  obviously implies the two conditions. Suppose now that  $\text{supp}(\mathbf{r}) \subseteq \text{supp}_{qc}(p)$  and there are vertices  $(u, \mathbf{a}) \sim_c (u, \mathbf{a} + \mathbf{r}) \sim_{qc} p$ . Since  $\text{supp}(\mathbf{r}) \subseteq \text{supp}_{qc}(p)$ , we can find  $\mathbf{w} \in \mathbf{A}^+$  such that  $p + \mathbf{w} \sim_c p$  and  $\mathbf{w} + \mathbf{r} \geq \mathbf{0}$ . Since  $p \sim_{qc} (u, \mathbf{a})$  we can find  $p' \sim_c p$  with  $(u, \mathbf{a}) \leq p'$ ; in particular we also have  $p + \mathbf{w} \sim_c p' + \mathbf{w}$  and  $p + (\mathbf{w} - \mathbf{r}) \sim_c p' + (\mathbf{w} - \mathbf{r})$ . But since  $(u, \mathbf{a}) \leq p' + \mathbf{w}$ , the condition  $(u, \mathbf{a}) \sim_c (u, \mathbf{a} + \mathbf{r})$  implies  $p' + \mathbf{w} \sim_c p + \mathbf{w} + \mathbf{r}$ . Therefore we have  $p + \mathbf{w} \sim_c p' + \mathbf{w} \sim_c p' + (\mathbf{w} - \mathbf{r}) \sim_c p + (\mathbf{w} - \mathbf{r})$ . The conclusion follows.  $\square$

*Remark 2.17.* Condition 2 of Lemma 2.16 can't be waived. In fact, one might try to define the linear algebra of a point  $p$  as the set of elements  $\mathbf{r} \in \mathbf{A}$  satisfying Condition 1 of Lemma 2.16. However, in general this definition gives a set which is different from the set  $\text{lin}_{qc}(p)$  of Definition 2.11, and which might not be a subgroup.

It follows that if  $p \sim_{qc} q$  then  $\text{min}_{qc}(p) = \text{min}_{qc}(q)$  and  $\text{supp}_{qc}(p) = \text{supp}_{qc}(q)$  and  $\text{lin}_{qc}(p) = \text{lin}_{qc}(q)$ ; thus, for a quasi-conjugacy class  $Q$ , the objects  $\text{min}_{qc}(Q), \text{supp}_{qc}(Q), \text{lin}_{qc}(Q)$  are well-defined. The following Proposition 2.18 explains how the minimal points and the quasi-conjugacy support give a complete description of a quasi-conjugacy class. The subsequent Proposition 2.19 explains how the linear algebra gives a complete description of the conjugacy classes contained in the quasi-conjugacy class.

**Proposition 2.18.** *Let  $(\Gamma, \psi)$  be a GBS graph and let  $p$  be a vertex of its affine representation  $\Lambda$ . Then the following are equivalent:*

1.  $q \sim_{qc} p$
2.  $q = m + \mathbf{w}$  for some  $m \in \text{min}_{qc}(p)$  and for some  $\mathbf{w} \in \mathbf{A}^+$  with  $\text{supp}(\mathbf{w}) \subseteq \text{supp}_{qc}(p)$ .

*Proof.* Immediate from the definitions.  $\square$

**Proposition 2.19.** *Let  $(\Gamma, \psi)$  be a GBS graph and let  $p$  be a vertex of its affine representation  $\Lambda$ . Then we have the following:*

1. For all  $\mathbf{w}, \mathbf{w}' \in \mathbf{A}^+$  with  $\text{supp}(\mathbf{w}), \text{supp}(\mathbf{w}') \subseteq \text{supp}_{qc}(p)$ , we have that

$$p + \mathbf{w} \sim_c p + \mathbf{w}' \Leftrightarrow \mathbf{w} - \mathbf{w}' \in \text{lin}_{qc}(p).$$

2. There is a natural inclusion of subgroups  $\text{lin}_{qc}(p) \leq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{\text{supp}_{qc}(p)} \leq \mathbf{A}$ .
3. There is a bijection between the following two sets:
  - (i) The set of conjugacy classes contained in the quasi-conjugacy class of  $p$ .
  - (ii) The quotient  $(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{\text{supp}_{qc}(p)})/\text{lin}_{qc}(p)$ .

*Remark 2.20.* The bijection in Item 3 of Proposition 2.19 is not canonic. It's canonic only up to translations in  $(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{\text{supp}_{qc}(p)})/\text{lin}_{qc}(p)$ .

*Proof.* Item 1 follows from Lemma 2.16 and Item 2 is trivial.

Take a conjugacy class  $Q$  contained in the quasi-conjugacy class of  $p$ . This means that we can find  $q \in Q$  such that  $q = p + \mathbf{w}$  for some  $\mathbf{w} \in \mathbf{A}^+$  with  $\text{supp}(\mathbf{w}) \subseteq \text{supp}_{qc}(p)$ . By Lemma 2.16, a different choice of  $q \in Q$  will change  $\mathbf{w}$  by adding an element of  $\text{lin}_{qc}(p)$ . Thus from  $Q$  we obtain a well-defined coset  $\mathbf{w} + \text{lin}_{qc}(p)$ . This defines a map from the set of conjugacy classes contained in the quasi-conjugacy class of  $p$  to  $(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{\text{supp}_{qc}(p)})/\text{lin}_{qc}(p)$ .

To prove surjectivity, take  $\mathbf{x} \in \mathbf{A}$  with  $\text{supp}(\mathbf{x}) \subseteq \text{supp}_{qc}(p)$  and consider the coset  $\mathbf{x} + \text{lin}_{qc}(p)$ . Since  $\text{supp}(\mathbf{x}) \subseteq \text{supp}_{qc}(p)$ , we can find  $\mathbf{w} \in \mathbf{A}^+$  such that  $p + \mathbf{w} \sim_c p$  and  $\mathbf{x} + \mathbf{w} \geq \mathbf{0}$ . In particular  $\mathbf{x} + \text{lin}_{qc}(p) = (\mathbf{x} + \mathbf{w}) + \text{lin}_{qc}(p)$  and this coset is realized by the conjugacy class of  $p + \mathbf{x} + \mathbf{w}$ .

To prove injectivity, suppose that we are given  $\mathbf{w}, \mathbf{w}' \in \mathbf{A}^+$  with  $\text{supp}(\mathbf{w}), \text{supp}(\mathbf{w}') \subseteq \text{supp}_{qc}(p)$ , and such that  $p + \mathbf{w}$  and  $p + \mathbf{w}'$  define the same coset  $\mathbf{w} + \text{lin}_{qc}(p) = \mathbf{w}' + \text{lin}_{qc}(p)$ . But then  $\mathbf{w} - \mathbf{w}' \in \text{lin}_{qc}(p) = \text{lin}_{qc}(p)$  and we find  $\mathbf{u}, \mathbf{u}' \in \mathbf{A}^+$  such that  $\mathbf{w} - \mathbf{w}' = \mathbf{u} - \mathbf{u}'$  and  $p + \mathbf{u} \sim_c p + \mathbf{u}' \sim_c p$ . This implies that  $p + \mathbf{w} \sim_c p + \mathbf{w} + \mathbf{u}' = p + \mathbf{w}' + \mathbf{u} \sim_c p + \mathbf{w}'$ , and injectivity follows.  $\square$

## 2.7 The graph of families and the modular map

Let  $(\Gamma, \psi)$  be a GBS graph and let  $\Lambda$  be its affine representation.

**Definition 2.21.** We say that  $p, q \in V(\Lambda)$  are *in the same family* if  $p \sim_{qc} q$  and they are of the form  $p = (v, \mathbf{a})$  and  $q = (v, \mathbf{b})$  with  $\text{supp}(\mathbf{b} - \mathbf{a}) \subseteq \text{supp}_{qc}(p)$ , for some  $v \in V(\Gamma)$  and  $\mathbf{a}, \mathbf{b} \in \mathbf{A}^+$ .

Being in the same family is an equivalence relation on  $V(\Lambda)$ ; an equivalence class is called **family** of vertices. Every family of vertices is contained in a quasi-conjugacy class (by definition), and it always contains at least one minimal point of the quasi-conjugacy class (by Proposition 2.19). In particular, each quasi-conjugacy class is partitioned into finitely many families.

**Definition 2.22.** Define the **graph of families**  $\mathcal{F} = \mathcal{F}(Q)$  of a quasi-conjugacy class  $Q$  as follows:

1.  $V(\mathcal{F})$  is the finite set of families contained in the quasi-conjugacy class  $Q$ .
2.  $E(\mathcal{F})$  is the set of triples  $(F, e, F') \in V(\mathcal{F}) \times E(\Lambda) \times V(\mathcal{F})$  such that there is  $\mathbf{w} \in \mathbf{A}^+$  with  $\iota(e) + \mathbf{w} \in F$  and  $\tau(e) + \mathbf{w} \in F'$ .
3. We set  $\overline{(F, e, F')} = (F', \bar{e}, F)$  and  $\iota((F, e, F')) = F$  and  $\tau((F, e, F')) = F'$ .

**Lemma 2.23.** Suppose that the endpoints of  $e \in E(\Lambda)$  lie in  $Q$ . Then there are unique families  $F, F' \in V(\mathcal{F})$  such that  $(F, e, F') \in E(\mathcal{F})$ , and they are the families containing  $\iota(e)$  and  $\tau(e)$  respectively.

*Proof.* If for some  $\mathbf{w} \in \mathbf{A}^+$  the vertex  $\iota(e) + \mathbf{w}$  lies in  $Q$ , then by definition of quasi-conjugacy support we have  $\text{supp}(\mathbf{w}) \subseteq \text{supp}_{qc}(Q)$ , and by definition of family we have that  $\iota(e) + \mathbf{w}$  lies in the same family as  $\iota(e)$ . Similarly for  $\tau(e)$ .  $\square$

If the endpoints of  $e \in E(\Lambda)$  don't lie in  $Q$ , then the graph of families can contain several edges of the form  $(F, e, F')$ , corresponding to different translates of  $e$ . However, it will always contain finitely many edges, as there are finitely many possible triples in  $V(\mathcal{F}) \times E(\Lambda) \times V(\mathcal{F})$ . Note also that the graph of families is connected: for every two families in the same quasi-conjugacy class, there must be an affine path connecting two points in the two families.

**Definition 2.24.** Define the **modular map**  $q : E(\Gamma) \rightarrow \mathbf{A}$  given by  $q(e) = \mathbf{b} - \mathbf{a}$  where  $\iota_\Lambda(e) = (v, \mathbf{a})$  and  $\tau_\Lambda(e) = (u, \mathbf{b})$  for some  $v, u \in V(\Gamma)$  and  $\mathbf{a}, \mathbf{b} \in \mathbf{A}^+$ .

Note that  $q(\bar{e}) = -q(e)$ . In particular, we obtain an homomorphism  $q : \pi_1(\Gamma) \rightarrow \mathbf{A}$ , which we call the **modular homomorphism**.

Let  $Q$  be a quasi-conjugacy class and let  $\mathcal{F} = \mathcal{F}(Q)$  be the corresponding graph of families. Then we have a modular map  $q_Q : E(\mathcal{F}) \rightarrow \mathbf{A}$  defined by  $q_Q((F, e, F')) = q(e)$ . Thus we can define a homomorphism  $q_Q : \pi_1(\mathcal{F}) \rightarrow \mathbf{A}$ , which we call **modular homomorphism of  $Q$** .

**Proposition 2.25.** Let  $Q$  be a quasi-conjugacy class and let  $\mathcal{F}(Q)$  be the corresponding graph of families. Then the image of  $q_Q : \pi_1(\mathcal{F}(Q)) \rightarrow \mathbf{A}$  is exactly  $\text{lin}_{qc}(Q)$ .

*Proof.* Fix  $p \in Q$  and take  $\mathbf{r} \in \text{lin}_{qc}(p)$ . Then there is an affine path  $(e_1, \dots, e_\ell)$ , with translation coefficients  $\mathbf{w}_1, \dots, \mathbf{w}_\ell$ , going from  $\iota(e_1) + \mathbf{w}_1 = p + \mathbf{w}$  to  $\tau(e_\ell) + \mathbf{w}_\ell = p + \mathbf{w}'$  with  $\mathbf{w}, \mathbf{w}' \in \mathbf{A}^+$  and  $\mathbf{w}' - \mathbf{w} = \mathbf{r}$ . For all  $i = 1, \dots, \ell - 1$  we have that  $\tau(e_i) + \mathbf{w}_i = \iota(e_{i+1}) + \mathbf{w}_{i+1}$  must lie in some family  $F_i \subseteq Q$ . Thus we obtain a path  $\sigma = ((F_1, e_1, F_2), \dots, (F_{\ell-1}, e_\ell, F_1))$  in  $\mathcal{F}(Q)$ . It's easy to check that  $q_Q(\sigma) = \mathbf{w}' - \mathbf{w} = \mathbf{r}$ .

Conversely, suppose that we are given a closed path  $\sigma = ((F_1, e_1, F_2), \dots, (F_{\ell-1}, e_\ell, F_1))$  in  $\mathcal{F}(Q)$ , for some families  $F_i \subseteq Q$  with  $i = 1, \dots, \ell$ . Then we can find  $\mathbf{w}_i \in \mathbf{A}^+$  such that  $\iota(e_i) + \mathbf{w}_i \in F_i$  and  $\tau(e_i) + \mathbf{w}_i \in F_{i+1}$ . It's easy to see that we can adjust  $\mathbf{w}_i$  in such a way that  $\tau(e_i) + \mathbf{w}_i = \iota(e_{i+1}) + \mathbf{w}_{i+1}$ , obtaining an affine path  $(e_1, \dots, e_\ell)$  from  $\iota(e_1) + \mathbf{w}_1$  to  $\tau(e_\ell) + \mathbf{w}_\ell$ . In particular, we can check that  $q_Q(\sigma) = \mathbf{w}_\ell - \mathbf{w}_1 \in \text{lin}_{qc}(p)$  as  $\iota(e_1) + \mathbf{w}_1$  and  $\tau(e_\ell) + \mathbf{w}_\ell$  lie in the same family  $F_1$ .

This proves that the image of the homomorphism  $q_Q : \pi_1(\mathcal{F}(Q)) \rightarrow \mathbf{A}$  is exactly  $\text{lin}_{qc}(p)$ , as desired.  $\square$

*Remark 2.26.* Applying the modular map  $q$  to a single edge isn't usually very meaningful. If the two endpoints of  $e$  are different vertices of  $\Gamma$ , then the difference  $\mathbf{b} - \mathbf{a}$  doesn't make much sense, as we are comparing powers of generators of different vertices. It only makes sense when applied to paths that begin and terminate at the same vertex.

However, if we are interested in studying a certain quasi-conjugacy class  $Q$ , then we need to refine the modular homomorphism even more. For example, if the two endpoints of  $e$  are the same vertex of  $\Gamma$ , but lie in different families of  $Q$ , then the difference  $\mathbf{b} - \mathbf{a}$  still needs to be ignored. This is because it will give some linear algebra which is in some sense "external" to the quasi-conjugacy class  $Q$ . This is why we consider the modular homomorphism  $q_Q : \pi_1(\mathcal{F}(Q)) \rightarrow \mathbf{A}$  instead.

For example, in [ACRK25b] we describe a procedure to make a GBS graph into a one-vertex GBS graph. The procedure will add a lot of loops to  $\pi_1(\Gamma)$ , adding noise to the modular homomorphism  $q : \pi_1(\Gamma) \rightarrow \mathbf{A}$ . However, the procedure will preserve the graphs of families and the modular homomorphisms associated with the quasi-conjugacy classes. Considering the graph of families makes the noise invisible to us.

## 2.8 Algorithmic computation of conjugacy classes

All the discussion of the previous Section 2.6 can be made algorithmic, as follows.

**Proposition 2.27.** *There is an algorithm that, given a GBS graph  $(\Gamma, \psi)$  and a vertex  $p$  of its affine representation  $\Lambda$ , computes the following:*

1.  $\min_{\text{qc}}(p)$ .
2.  $\text{supp}_{\text{qc}}(p)$ .
3. For every  $m \in \min_{\text{qc}}(p)$ , an affine path from  $m$  to a point  $\geq p$ .
4. An affine path from  $p$  to  $p + \mathbf{w}$  for some  $\mathbf{w} \in \mathbf{A}^+$  with  $\text{supp}(\mathbf{w}) = \text{supp}_{\text{qc}}(p)$ .

*Description of the algorithm.* In order to run the algorithm, it's not restrictive to ignore the component  $\mathbb{Z}/2\mathbb{Z}$ , just remember to add it to all minimal points at the end of the procedure. If the GBS graph  $(\Gamma, \psi)$  has negative labels, just change them with their absolute values. So here we work with  $\mathbf{A} = \mathbb{Z}^{\mathcal{P}(\Gamma, \psi)}$ . In each moment, the algorithm keeps in memory

1. A finite set of vertices  $M \subseteq V(\Lambda)$ .
2. A finite set of components  $S \subseteq \mathcal{P}(\Gamma, \psi)$ .
3. For every  $m \in M$ , an affine path  $\sigma_m$  from  $m$  to  $p + \mathbf{x}_m$  for some  $\mathbf{x}_m \in \mathbf{A}^+$  with  $\text{supp}(\mathbf{x}_m) \subseteq S$ .
4. An affine path  $\sigma$  from  $p$  to  $p + \mathbf{z}$  for some  $\mathbf{z} \in \mathbf{A}^+$  with  $\text{supp}(\mathbf{z}) = S$ .

**CLEAN UP ROUTINE.** For every  $m, m' \in M$  distinct we check whether  $m' \geq m$ . If so, we write  $m' = m + \mathbf{r}$  for some  $\mathbf{r} \in \mathbf{A}^+$  and we perform the following operations:

1. We change  $S$  into  $S \cup \text{supp}(\mathbf{r})$
2. We change the path  $\sigma$  into  $\sigma^d \bar{\sigma}_{m'} \sigma_m$  for some natural number  $d$  big enough.
3. We remove  $m'$  from  $M$ , and we forget the path  $\sigma_{m'}$ .

We reiterate until we find no couple  $m, m' \in M$  distinct with  $m' \geq m$ .

**MAIN ALGORITHM.** We initialize  $M = \{p\}$  with  $\sigma_p$  trivial. We initialize  $S = \emptyset$  with  $\sigma$  trivial. Then we repeat the following iteration until the two sets  $M, S$  don't remain unchanged.  
**Iteration:** We write a list of all the couples  $(m, e) \in M \times E(\Lambda)$ . For each couple  $(m, e)$ , we look for the minimum  $\mathbf{w} \in \mathbf{A}^+$  such that  $\iota(e) + \mathbf{w} = m + \mathbf{s}$  for some  $\mathbf{s} \in \mathbf{A}^+$  with  $\text{supp}(\mathbf{s}) \subseteq S$ . If it doesn't exist, then we just ignore that couple, and go on examining the other couples  $(m, e)$ . If it exists, then it's unique, and we proceed as follows:

1. Suppose that there is no  $m' \in M$  such that  $\tau(e) + \mathbf{w} \geq m'$ . In this case we change  $M$  into  $M \cup \{\tau(e) + \mathbf{w}\}$  and we set  $\sigma_{\tau(e) + \mathbf{w}} = \bar{\sigma}_m$ . Then we run the clean up routine, and we start a new iteration (i.e. we write again the list of all couples  $(m, e) \in M \times E(\Lambda)$  and so on).



2. Suppose that there is  $m' \in M$  such that  $\tau(e) + \mathbf{w} = m' + \mathbf{r}$  with  $\mathbf{r} \in \mathbf{A}^+$  and  $\text{supp}(\mathbf{r}) \not\subseteq S$ . Then we change  $S$  into  $S \cup \text{supp}(\mathbf{r})$ , and we change the path  $\sigma$  into  $\sigma^d \bar{\sigma}_m e \sigma_{m'}$  for some natural number  $d$  big enough. Then we run the clean up routine, and we start a new iteration (i.e. we write again the list of all couples  $(m, e) \in M \times E(\Lambda)$  and so on).
3. Suppose that there is  $m' \in M$  such that  $\tau(e) + \mathbf{w} \geq m'$ , and for each such  $m'$  we have that  $\tau(e) + \mathbf{w} = m' + \mathbf{r}$  with  $\text{supp}(\mathbf{r}) \subseteq S$ . Then we just ignore that couple, and go on examining the other couples  $(m, e)$ .

When we run a full iteration without changing the sets  $M, S$ , then we quit the cycle. The algorithm terminates and outputs  $\min_{\text{qc}}(p) = M$ ,  $\text{supp}_{\text{qc}}(p) = S$ , the affine paths  $\sigma_m$  for  $m \in M$ , the affine path  $\sigma$ .  $\square$

*Proof that the algorithm works.* Observe that we can add elements to the set  $S$  only finitely many times (since  $S$  is a subset of a finite set). As in the proof of Lemma 2.12, we can consider the ring of polynomials  $\mathbb{C}[x_i : i \in \mathcal{P}(\Gamma, \psi)]$  and the ideal  $(x^{\mathbf{m}} : (u, \mathbf{m}) \in M)$ . This ideal increases strictly every time that the algorithm adds a new element to  $M$ ; since rings of polynomial are Noetherian, this can happen only finitely many times. This proves that we can change the set  $M$  only finitely many times, and thus that the algorithm terminates.

It's easy to prove by induction that, in each moment while running the algorithm,  $M$  is contained in the quasi-conjugacy class of  $p$ ,  $S \subseteq \text{supp}_{\text{qc}}(p)$ , the paths  $\sigma_m$  for  $m \in M$  satisfy the property of Item 3, the path  $\sigma$  satisfies the property of Item 4. In particular, when during the algorithm we substitute  $\sigma$  with  $\sigma^d \bar{\sigma}_m e \sigma_{m'}$ , we are always able to (algorithmically) find an integer  $d$  such that the substitution gives another affine path starting at  $p$ .

When the algorithm terminates, consider the set of vertices  $M^{+S} = \{m + \mathbf{w} : m \in M \text{ and } \mathbf{w} \in \mathbf{A}^+ \text{ with } \text{supp}(\mathbf{w}) \subseteq S\}$  and observe that  $M^{+S}$  is contained in the quasi-conjugacy class of  $p$ . But since no couple  $(m, e)$  changes the sets  $M, S$ , we have that no translate of any edge  $e$  can be used to move from a point in  $M^{+S}$  to a point outside  $M^{+S}$ . This proves that  $M^{+S}$  is closed under quasi-conjugacy, and thus  $M^{+S}$  is the quasi-conjugacy class of  $p$ . Since every time we add a vertex to  $M$  we also run the clean up routine, this implies that  $M = \min_{\text{qc}}(p)$ . This also implies that  $S = \text{supp}_{\text{qc}}(p)$ . It follows that the output of the algorithm is correct.  $\square$

**Proposition 2.28.** *There is an algorithm that, given a GBS graph  $(\Gamma, \psi)$  and a vertex  $p$  of its affine representation  $\Lambda$ , computes the following:*

1. A finite set of generators for  $\text{lin}_{\text{qc}}(p)$ .
2. For each generator  $\mathbf{r}$ , an affine path  $\sigma_{\mathbf{r}}$  satisfying the condition of Item 1 of Lemma 2.16.

*Proof.* Let  $Q$  be the quasi-conjugacy class of  $p$ . By Proposition 2.27 we can algorithmically compute  $\text{supp}_{\text{qc}}(Q)$  and the finite set  $M = \min_{\text{qc}}(Q)$ . We can also algorithmically compute the graph of families  $\mathcal{F}(Q)$  (see Definition 2.22), as  $V(\mathcal{F}(Q))$  is a computable quotient of the finite set  $M$  and the set of edges is a computable subset of the finite set  $V(\mathcal{F}(Q)) \times E(\Gamma) \times V(\mathcal{F}(Q))$ . Moreover, we can give an algorithmic description of the modular homomorphism  $q_Q : \pi_1(\mathcal{F}(Q)) \rightarrow \mathbf{A}$ : given a loop  $\theta$  in  $\mathcal{F}(Q)$ , we can algorithmically compute  $q_Q(\theta) \in \mathbf{A}$ . In particular we can compute a set of generators for  $\pi_1(\mathcal{F}(Q))$  and their images through the homomorphism  $q_Q$ . According to Proposition 2.25, the images are the desired set of generators for  $\text{lin}_{\text{qc}}(Q)$  and the loops in  $\pi_1(\mathcal{F}(Q))$  are the corresponding affine paths (for some suitable translation coefficients), as desired.  $\square$

**Corollary 2.29.** *There is an algorithm that, given a GBS graph  $(\Gamma, \psi)$  and two vertices  $p, q$  of its affine representation  $\Lambda$ , does the following:*

- Decides whether  $p \sim_{\text{qc}} q$ , and in case they are, computes two affine paths, one from each of them to a vertex above the other.
- Decides whether  $p \sim_c q$ , and in case they are, computes an affine path from one to the other.

## 2.9 Examples

We show a couple of examples, to help the reader understand how conjugacy classes and quasi-conjugacy classes work. In the examples along the paper, for simplicity of notation, we will only deal with GBS graphs  $(\Gamma, \psi)$  with positive labels. For this reason we will use  $\mathbf{A} = \mathbb{Z}^{\mathcal{P}(\Gamma, \psi)}$  omitting the  $\mathbb{Z}/2\mathbb{Z}$  summand.

*Example 2.30.* Later in the paper (see Example 5.20), we will describe the isomorphism problem for this example. Consider the GBS graph  $(\Gamma, \psi)$  with a single vertex  $v$  and nine edges, with labels as in Figure 3. We call  $g$  the generator of  $G_v \cong \mathbb{Z}$ . Since the labels only use the prime factors 2 and 3, we have that  $\mathcal{P}(\Gamma, \psi) = \{2, 3\}$  and we use  $\mathbf{A} = \mathbb{Z}^2$ . The number  $2^a 3^b$  will be associated to the element  $(a, b) \in \mathbf{A}^+$  for  $a, b \in \mathbb{N}$ .

Consider the point  $p = (v, (1, 3))$  corresponding to the element  $g^{2^1 3^3} = g^{54}$ . The quasi-conjugacy class of  $p$  consists of only two points  $(v, (1, 3))$ ,  $(v, (3, 2))$ , corresponding to the elements  $g^{54}, g^{72}$  respectively, see Figure 4. In this case  $\text{supp}_{\text{qc}}(p) = \emptyset$  and  $\text{min}_{\text{qc}}(p)$  is the set of the two points of the quasi-conjugacy class, which is also a conjugacy class. This means that  $g^n$  for  $n \geq 1$  is conjugate to  $g^{54}$  if and only if  $n = 54, 72$ . In this case  $\text{lin}_{\text{qc}}(p) = 0$ .

Consider the point  $p = (v, (4, 1))$  corresponding to the element  $g^{2^4 3^1} = g^{48}$ . The quasi-conjugacy class of  $p$  consists of the points  $(v, (k, 1))$  for  $k \geq 4$ , corresponding to the elements  $g^{2^k \cdot 3}$  for  $k \geq 4$ . In this case  $\text{supp}_{\text{qc}}(p) = \{2\}$  and  $\text{min}_{\text{qc}}(p) = \{(v, (4, 1))\}$ . This quasi-conjugacy class is partitioned into two conjugacy classes, depending on the parity of  $k$ . This means that  $g^n$  for  $n \geq 1$  is conjugate to  $g^{48}$  if and only if  $n = 2^k \cdot 9$  for  $k \geq 4$  even. In this case  $\text{lin}_{\text{qc}}(p) = \langle (2, 0) \rangle$ .

Consider the point  $p = (v, (2, 3))$  corresponding to the element  $g^{108}$ . In this case  $\text{supp}_{\text{qc}}(p) = \{2, 3\}$  and  $\text{min}_{\text{qc}}(p) = \{(2, 3), (4, 2)\}$  and  $\text{lin}_{\text{qc}}(p) = \langle (2, 0), (0, 1) \rangle$ . The quasi-conjugacy class is partitioned into two conjugacy classes, as  $(\mathbb{Z}_{\text{qc}}^{\text{supp}}(p))/\text{lin}_{\text{qc}}(p) \cong \mathbb{Z}/2\mathbb{Z}$  which has cardinality 2.

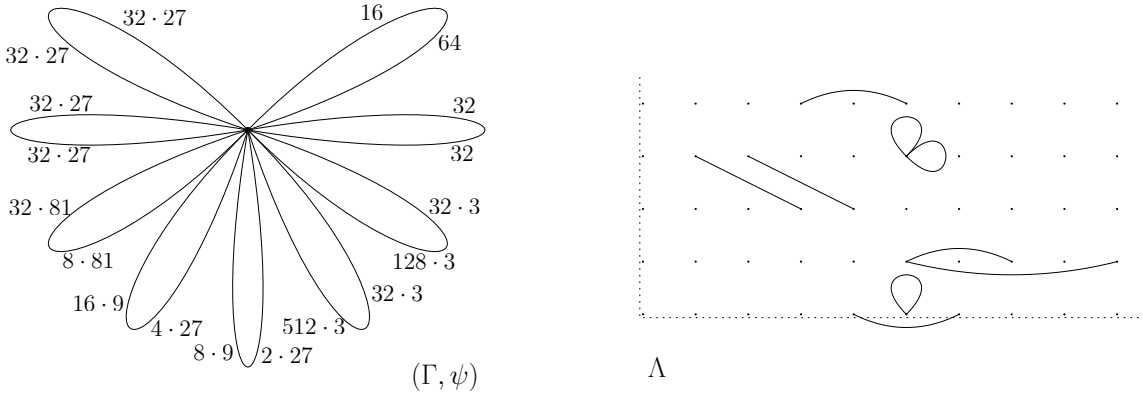


Figure 3: On the left a GBS graph  $(\Gamma, \psi)$  with one vertex and nine edges. On the right the corresponding affine representation  $\Lambda$ . On the horizontal axis the number of factors 2 and on the vertical axis the number of factors 3.

*Example 2.31.* This example is taken from [Wan23]. Later in the paper (see Example 4.21), we will describe the isomorphism problem for this example. Consider the GBS graph with a single vertex  $v$  and four edges, with labels as in Figure 5. The prime numbers of this GBS graph are  $\mathcal{P}(\Gamma, \psi) = \{2, 3, 5, 7\}$ . The number  $2^a 3^b 5^c 7^d$  will be associated to the element  $(a, b, c, d) \in \mathbf{A}^+$  for  $a, b, c, d \in \mathbb{N}$ . We call  $\Lambda$  the affine representation of  $(\Gamma, \psi)$ : this consists of a single copy  $\mathbf{A}_v^+$  of  $\mathbf{A}^+$  containing four edges

$$\begin{cases} (3, 0, 0, 0) \text{ --- } (1, 0, 1, 0) \\ (1, 1, 0, 0) \text{ --- } (0, 1, 1, 0) \\ (0, 1, 0, 1) \text{ --- } (1, 1, 1, 0) \\ (1, 0, 0, 1) \text{ --- } (1, 1, 1, 0) \end{cases}$$

Consider the point  $p = (v, (3, 0, 0, 0))$ . Its quasi-conjugacy class is finite, with  $\text{supp}_{\text{qc}}(p) = \emptyset$  and minimal points  $(3, 0, 0, 0)$ ,  $(1, 0, 1, 0)$  inside  $\mathbf{A}_v^+$ . It coincides with a conjugacy class and  $\text{lin}_{\text{qc}}(p) = 0$ .

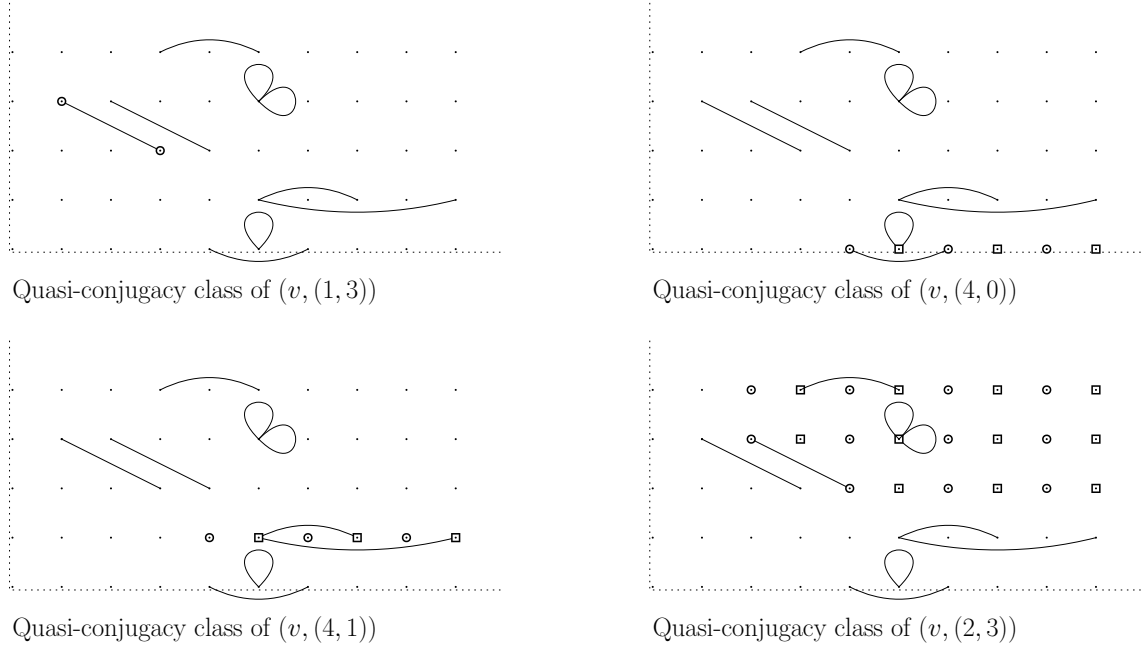


Figure 4: In the figure we see four copies of the same affine representation  $\Lambda$  associated with Example 2.30. In each copy, we put in evidence a different quasi-conjugacy class. We can see the quasi-conjugacy class of  $(1, 3)$  (top left), of  $(4, 0)$  (top right), of  $(4, 1)$  (bottom left), of  $(2, 3)$  (bottom right). To distinguish different conjugacy classes inside a common quasi-conjugacy class, we mark the vertices with different symbols (circles, squares).

Consider the point  $p = (v, (1, 1, 0, 0))$ . Its quasi-conjugacy class is finite, with  $\text{supp}_{\text{qc}}(p) = \emptyset$  and minimal points  $(1, 1, 0, 0)$ ,  $(0, 1, 1, 0)$  inside  $\mathbf{A}_v^+$ . It coincides with a conjugacy class and  $\text{lin}_{\text{qc}}(p) = 0$ .

Consider the point  $p = (v, (0, 1, 0, 1))$ . Its quasi-conjugacy class has  $\text{supp}_{\text{qc}}(p) = \{2, 3, 5, 7\}$  and minimal points  $(2, 1, 0, 0)$ ,  $(1, 1, 1, 0)$ ,  $(0, 1, 2, 0)$ ,  $(1, 0, 0, 1)$ ;  $(0, 1, 0, 1)$  inside  $\mathbf{A}_v^+$ . It coincides with a conjugacy class since  $\text{lin}_{\text{qc}}(p) = \mathbb{Z}^{\text{supp}_{\text{qc}}(p)}$ .

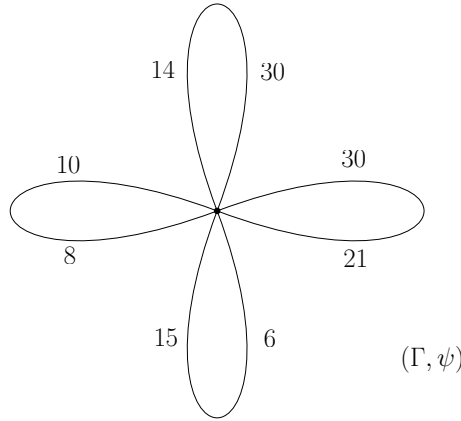


Figure 5: In the picture a GBS graph  $(\Gamma, \psi)$  with one vertex and four edges.

### 3 Moves on GBS graphs

In this section we define a family of moves that, given a GBS graph, produce another GBS graph with isomorphic fundamental group; these are called slide, induction, swap, connection (see [For06] [ACRK25b]). We recall some literature about the topic.

### 3.1 Sign-change move

Let  $(\Gamma, \psi)$  be a GBS graph. Let  $v \in V(\Gamma)$  be a vertex. Define the map  $\psi' : E(\Gamma) \rightarrow \mathbf{Z} \setminus \{0\}$  such that  $\psi'(e) = -\psi(e)$  if  $\tau(e) = v$  and  $\psi'(e) = \psi(e)$  otherwise. We say that the GBS graph  $(\Gamma, \psi')$  is obtained from  $(\Gamma, \psi)$  by means of a **vertex sign change**. If  $\mathcal{G}$  is the GBS graph of groups associated to  $(\Gamma, \psi)$ , then the vertex sign change move corresponds to changing the chosen generator for the vertex group  $G_v$ ; in particular it induces an isomorphism at the level of the universal group  $\text{FG}(\mathcal{G})$  and of the fundamental group  $\pi_1(\mathcal{G})$ .

Let  $(\Gamma, \psi)$  be a GBS graph. Let  $d \in E(\Gamma)$  be an edge. Define the map  $\psi' : E(\Gamma) \rightarrow \mathbf{Z} \setminus \{0\}$  such that  $\psi'(e) = -\psi(e)$  if  $e = d, \bar{d}$  and  $\psi'(e) = \psi(e)$  otherwise. We say that the GBS graph  $(\Gamma, \psi')$  is obtained from  $(\Gamma, \psi)$  by means of an **edge sign change**. If  $\mathcal{G}$  is the GBS graph of groups associated to  $(\Gamma, \psi)$ , then the vertex sign change move corresponds to changing the chosen generator for the edge group  $G_d$ ; in particular it induces an isomorphism at the level of the universal group  $\text{FG}(\mathcal{G})$  and of the fundamental group  $\pi_1(\mathcal{G})$ .

### 3.2 Slide move

Let  $(\Gamma, \psi)$  be a GBS graph. Let  $d, e$  be distinct edges with  $\tau(d) = \iota(e) = u$  and  $\tau(e) = v$ ; suppose that  $\psi(\bar{e}) = n$  and  $\psi(e) = m$  and  $\psi(d) = \ell n$  for some  $n, m, \ell \in \mathbf{Z} \setminus \{0\}$  (see Figure 6). Define the graph  $\Gamma'$  by replacing the edge  $d$  with an edge  $d'$ ; we set  $\iota(d') = \iota(d)$  and  $\tau(d') = v$ ; we set  $\psi(\bar{d}') = \psi(\bar{d})$  and  $\psi(d') = \ell m$ . We say that the GBS graph  $(\Gamma', \psi)$  is obtained from  $(\Gamma, \psi)$  by means of a **slide**.

At the level of the affine representation, we have an edge  $p \text{ --- } q$  and we have another edge with an endpoint at  $p + \mathbf{a}$  for some  $\mathbf{a} \in \mathbf{A}^+$ . The slide has the effect of moving the endpoint from  $p + \mathbf{a}$  to  $q + \mathbf{a}$  (see Figure 6).

$$\left\{ \begin{array}{l} p \text{ --- } q \\ r \text{ --- } p + \mathbf{a} \end{array} \right. \xrightarrow{\text{slide}} \left\{ \begin{array}{l} p \text{ --- } q \\ r \text{ --- } q + \mathbf{a} \end{array} \right.$$

### 3.3 Induction move

Let  $(\Gamma, \psi)$  be a GBS graph. Let  $e$  be an edge with  $\iota(e) = \tau(e) = v$ ; suppose that  $\psi(\bar{e}) = 1$  and  $\psi(e) = n$  for some  $n \in \mathbf{Z} \setminus \{0\}$ , and choose  $\ell \in \mathbf{Z} \setminus \{0\}$  and  $k \in \mathbf{N}$  such that  $\ell \mid n^k$ . Define the map  $\psi'$  equal to  $\psi$  except on the edges  $d \neq e, \bar{e}$  with  $\tau(e) = v$ , where we set  $\psi'(d) = \ell \cdot \psi(d)$ . We say that the GBS graph  $(\Gamma, \psi')$  is obtained from  $(\Gamma, \psi)$  by means of an **induction**.

At the level of the affine representation, we have an edge  $(v, \mathbf{0}) \text{ --- } (v, \mathbf{w})$ . We choose  $\mathbf{w}_1 \in \mathbf{A}^+$  such that  $\mathbf{w}_1 \leq k\mathbf{w}$  for some  $k \in \mathbf{N}$ ; we take all the endpoints of other edges lying in  $\mathbf{A}_v^+$ , and we translate them up by adding  $\mathbf{w}_1$  (see Figure 7).

### 3.4 Swap move

Let  $(\Gamma, \psi)$  be a GBS graph. Let  $e_1, e_2$  be distinct edges with  $\iota(e_1) = \tau(e_1) = \iota(e_2) = \tau(e_2) = v$ ; suppose that  $\psi(\bar{e}_1) = n$  and  $\psi(e_1) = \ell_1 n$  and  $\psi(\bar{e}_2) = m$  and  $\psi(e_2) = \ell_2 m$  and  $n \mid m$  and  $m \mid \ell_1^{k_1} n$  and  $m \mid \ell_2^{k_2} n$  for some  $n, m, \ell_1, \ell_2 \in \mathbf{Z} \setminus \{0\}$  and  $k_1, k_2 \in \mathbf{N}$  (see Figure 8). Define the graph  $\Gamma'$  by substituting the edges  $e_1, e_2$  with two edges  $e'_1, e'_2$ ; we set  $\iota(e'_1) = \tau(e'_1) = \iota(e'_2) = \tau(e'_2) = v$ ; we set  $\psi(\bar{e}'_1) = m$  and  $\psi(e'_1) = \ell_1 m$  and  $\psi(\bar{e}'_2) = n$  and  $\psi(e'_2) = \ell_2 n$ . We say that the GBS graph  $(\Gamma', \psi)$  is obtained from  $(\Gamma, \psi)$  by means of a **swap move**.

Let  $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{A}^+$  and  $p, q \in V(\Lambda)$  be such that  $p \leq q \leq p + k_1 \mathbf{w}_1$  and  $p \leq q \leq p + k_2 \mathbf{w}_2$  for some  $k_1, k_2 \in \mathbf{N}$ . At the level of the affine representation, we have an edge  $e_1$  going from  $p$  to  $p + \mathbf{w}_1$  and an edge  $e_2$  going from  $q$  to  $q + \mathbf{w}_2$ . The swap has the effect of substituting them with  $e'_1$  from  $q$  to  $q + \mathbf{w}_1$  and with  $e'_2$  from  $p$  to  $p + \mathbf{w}_2$  (see Figure 8).

$$\left\{ \begin{array}{l} p \text{ --- } p + \mathbf{w}_1 \\ q \text{ --- } q + \mathbf{w}_2 \end{array} \right. \xrightarrow{\text{swap}} \left\{ \begin{array}{l} q \text{ --- } q + \mathbf{w}_1 \\ p \text{ --- } p + \mathbf{w}_2 \end{array} \right.$$

### 3.5 Connection move

Let  $(\Gamma, \psi)$  be a GBS graph. Let  $d, e$  be distinct edges with  $\iota(d) = u$  and  $\tau(d) = \iota(e) = \tau(e) = v$ ; suppose that  $\psi(\bar{d}) = m$  and  $\psi(d) = \ell_1 n$  and  $\psi(\bar{e}) = n$  and  $\psi(e) = \ell n$  and  $\ell_1 \ell_2 = \ell^k$  for some  $m, n, \ell_1, \ell_2, \ell \in \mathbb{Z} \setminus \{0\}$  and  $k \in \mathbb{N}$  (see Figure 9). Define the graph  $\Gamma'$  by substituting the edges  $d, e$  with two edges  $d', e'$ ; we set  $\iota(d') = v$  and  $\tau(d') = \iota(e') = \tau(e') = u$ ; we set  $\psi(\bar{d}') = n$  and  $\psi(d') = \ell_2 m$  and  $\psi(\bar{e}') = m$  and  $\psi(e') = \ell m$ . We say that the GBS graph  $(\Gamma', \psi)$  is obtained from  $(\Gamma, \psi)$  by means of a **connection move**.

Let  $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2 \in \mathbf{A}^+$  and  $k \in \mathbb{N}$  be such that  $\mathbf{w}_1 + \mathbf{w}_2 = k \cdot \mathbf{w}$ . At the level of the affine representation, we have a two edges  $q \text{ --- } p + \mathbf{w}_1$  and  $p \text{ --- } p + \mathbf{w}$ . The connection move has the effect of replacing them with two edges  $p \text{ --- } p + \mathbf{w}_2$  and  $q \text{ --- } q + \mathbf{w}$  (see Figure 9).

$$\begin{cases} q \text{ --- } p + \mathbf{w}_1 \\ p \text{ --- } p + \mathbf{w} \end{cases} \xrightarrow{\text{connection}} \begin{cases} p \text{ --- } q + \mathbf{w}_2 \\ q \text{ --- } q + \mathbf{w} \end{cases}$$

*Remark 3.1.* In the definition of connection move, we also allow for the two vertices  $u, v$  to coincide.

### 3.6 Self-slide and reverse slide

The following Lemmas 3.2 and 3.3 introduce two additional moves that will be used in the next sections.

**Lemma 3.2** (Self-slide). *Let  $\mathbf{a}, \mathbf{b}, \mathbf{w} \in \mathbf{A}^+$  and  $\mathbf{x} \in \mathbf{A}$ . Suppose  $\mathbf{a}, \mathbf{w}$  controls  $\mathbf{b}$  and  $\mathbf{b} + 2\mathbf{x}$ . Then we can change*

$$\begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w} \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x} \end{cases} \quad \text{into} \quad \begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w} \\ \mathbf{b} + \mathbf{x} \text{ --- } \mathbf{b} + 2\mathbf{x} \end{cases}$$

*by means of a sequence of slides and swaps.*

*Proof.* See [ACRK25b]. □

**Lemma 3.3** (Reverse slide). *Let  $\mathbf{a}, \mathbf{b}, \mathbf{w} \in \mathbf{A}^+$  and  $\mathbf{x} \in \mathbf{A}$  with  $\mathbf{w} + \mathbf{x} \geq \mathbf{0}$ . Suppose  $\mathbf{a}, \mathbf{w}$  controls  $\mathbf{b}, \mathbf{b} + \mathbf{x}$  and suppose  $\mathbf{a}, \mathbf{w} + \mathbf{x}$  controls  $\mathbf{b}, \mathbf{b} + \mathbf{x}$ . If  $\mathbf{A}_v^+$  contains edges  $\mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w}$  and  $\mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x}$ , then we can change*

$$\begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w} \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x} \end{cases} \quad \text{into} \quad \begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w} + \mathbf{x} \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x} \end{cases}$$

*by means of a sequence of slides and swaps.*

*Proof.* See [ACRK25b]. □

### 3.7 Sequences of moves

In this section we recall how the isomorphism problem for generalized Baumslag-Solitar groups is reduced to checking whether two GBS graphs can be related by means of a sequence of moves. We need to introduce the notion of *totally reduced* GBS graph (see [ACRK25b]); we will only need the fact that every GBS graph  $(\Gamma, \psi)$  can be algorithmically changed to a totally reduced one.

**Definition 3.4.** *A GBS graph  $(\Gamma, \psi)$ , with affine representation  $\Lambda$ , is called **totally reduced** if the following conditions hold:*

1. *For every vertex  $v \in V(\Gamma)$ , if  $(v, \mathbf{0}) \sim_c (u, \mathbf{b})$  for  $u \in V(\Gamma)$  and  $\mathbf{b} \in \mathbf{A}^+$ , then  $u = v$ .*
2. *For every vertex  $v \in V(\Gamma)$ , in  $\Lambda$  there is an edge  $(v, \mathbf{0}) \text{ --- } (v, \mathbf{w})$  with  $\mathbf{w} \in \mathbf{A}^+$  such that, if  $(v, \mathbf{0}) \sim_c (v, \mathbf{b})$  for  $\mathbf{b} \in \mathbf{A}^+$ , then  $\mathbf{0}, \mathbf{w}$  controls  $\mathbf{b}$ .*

*Remark 3.5.* Condition 1 in the above Definition 3.4 coincides with the notion of *fully reduced* GBS graph introduced in [For06].

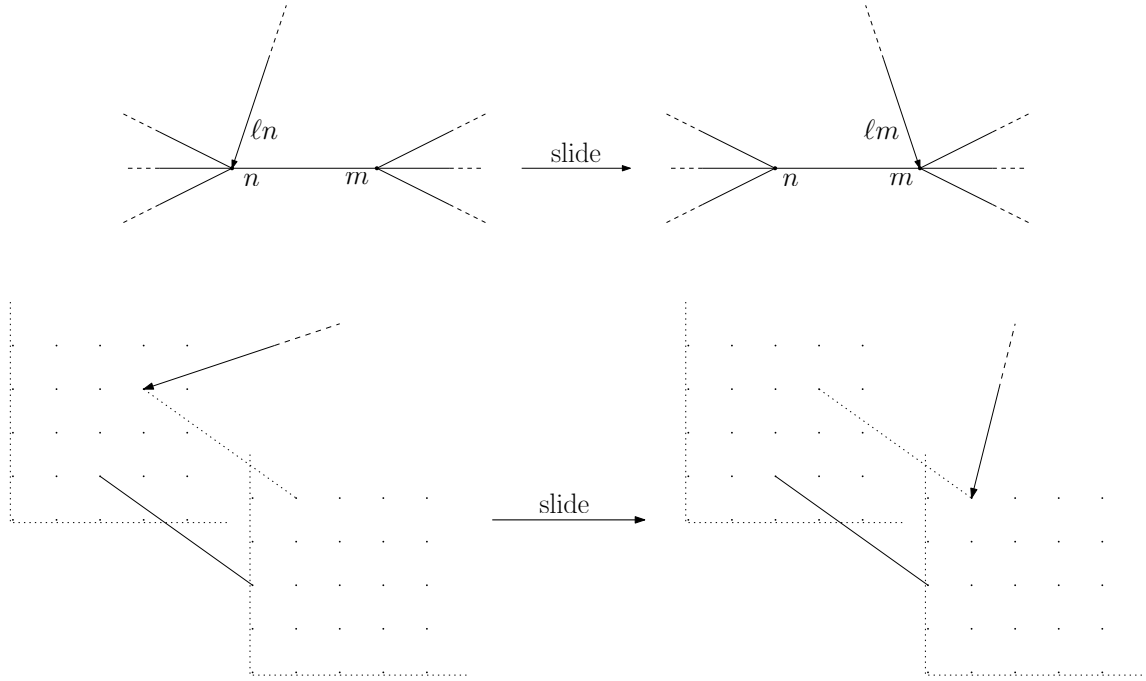


Figure 6: An example of a slide move. Above you can see the GBS graphs. Below you can see the corresponding affine representations.

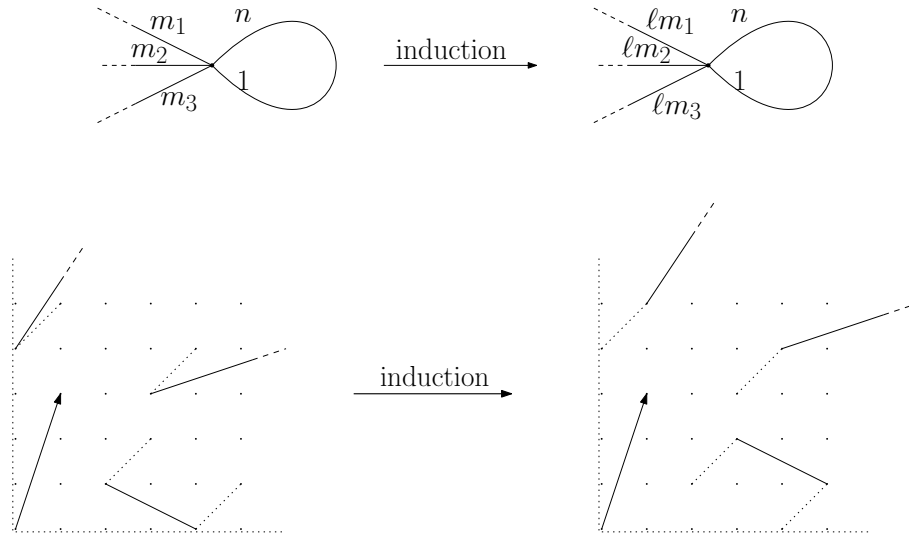


Figure 7: An example of an induction move. Above you can see the GBS graphs; here  $\ell \mid n^k$  for some integer  $k \geq 0$ . Below you can see the corresponding affine representations.

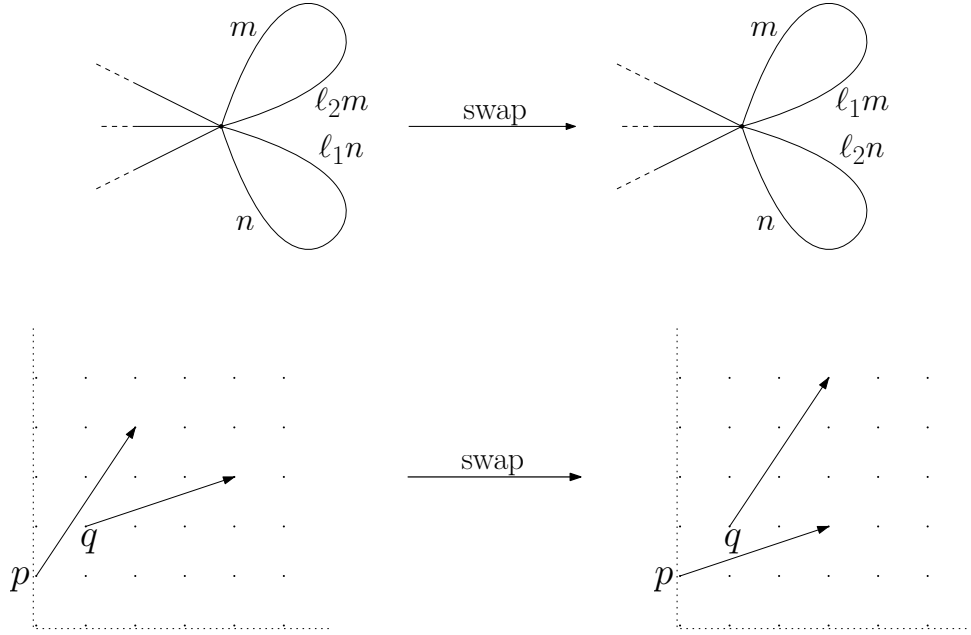


Figure 8: An example of a swap move. Above you can see the GBS graphs; here  $n \mid m \mid \ell_1^{k_1} n$  and  $n \mid m \mid \ell_2^{k_2} n$  for some integers  $k_1, k_2 \geq 0$ . Below you can see the corresponding affine representations.

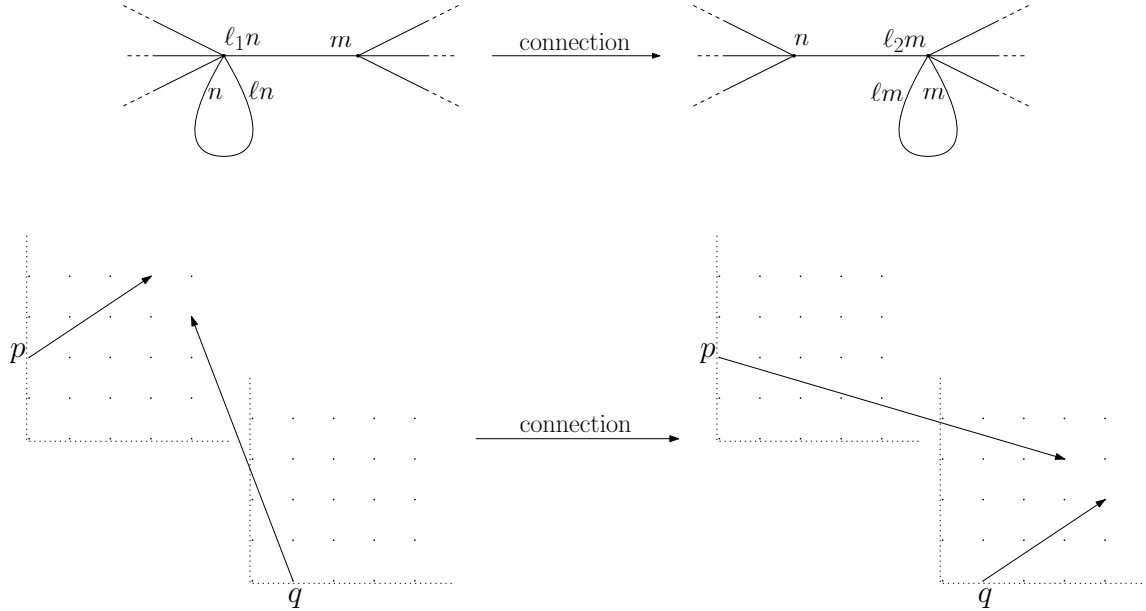


Figure 9: An example of a connection move. Above you can see the GBS graphs; here  $\ell_1 \ell_2 = \ell^k$  for some integer  $k \geq 0$ . Below you can see the corresponding affine representations.

**Proposition 3.6.** *There is an algorithm that, given a GBS graph  $(\Gamma, \psi)$ , computes a totally reduced GBS graph  $(\Gamma', \psi')$  such that the corresponding graphs of groups have isomorphic fundamental group.*

*Proof.* See [ACRK25b]. □

Here's one of the main results from [ACRK25b], which will be the starting point for all the results of this paper.

**Theorem 3.7.** *Let  $(\Gamma, \psi), (\Delta, \phi)$  be totally reduced GBS graphs and suppose that the corresponding graphs of groups have isomorphic fundamental group. Then  $|V(\Gamma)| = |V(\Delta)|$  and there is a sequence of slides, swaps, connections, sign-changes and inductions going from  $(\Delta, \phi)$  to  $(\Gamma, \psi)$ . Moreover, all the sign-changes and inductions can be performed at the beginning of the sequence.*

*Proof.* See [ACRK25b]. □

Thus it's enough to deal with the following problem: given two GBS graphs  $(\Gamma, \psi), (\Gamma', \psi')$ , determine whether there is a sequence of edge sign-changes, inductions, slides, swaps, connections going from one to the other. Note that a sign-change (resp. a slide, an induction, a swap, a connection) induces a natural bijection between the set of vertices of the graph before and after the move. Of course we can ignore the issue of guessing the bijection among the sets of vertices and the sign-changes at the beginning of the sequence, as these choices can be done only in finitely many ways. In what follows, we will also ignore the issue of guessing the inductions at the beginning of the sequence, as this hopefully represents a marginal issue - even though sometimes there are infinitely many possibilities, and thus this issue should be dealt with. In this paper, we will focus on the following question:

**Question 3.8.** *Given two totally reduced GBS graphs  $(\Gamma, \psi), (\Gamma', \psi')$  and a bijection  $b : V(\Gamma') \rightarrow V(\Gamma)$ , is there a sequence of (edge sign-changes,) slides, swaps, connections going from  $(\Gamma, \psi)$  to  $(\Gamma', \psi')$  and inducing the bijection  $b$  on the set of vertices?*

In [ACRK25b] we also show that the Question 3.8 can be reduced to the case of one-vertex graphs, and with all edge-labels positive and  $\neq 1$ . However, the tools we develop in this paper are general and don't make use of these additional assumptions.

### 3.8 Examples

Theorem 3.7 can already be used to explicitly describe, in some cases, the list of all the possible configurations which can be reached from a given one. We show through an example how this can help in solving the isomorphism problem for GBSs. As usual, in the examples we use  $\mathbf{A} = \mathbb{Z}^{\mathcal{P}(\Gamma, \psi)}$ , omitting the  $\mathbb{Z}/2\mathbb{Z}$  summand.

*Example 3.9.* Consider the GBS graph  $(\Gamma, \psi)$  with one vertex  $v$  and two edges, with labels 8, 48 and 27, 72. (see Figure 10). If we apply a connection move, we obtain another GBS graph with one vertex  $v$  and two edges, the new labels being 8, 108 and 27, 162. It's easy to see that, using slide moves, we can reach all the configurations of the following list  $L$ :

1. Two edges with labels 8, 48 and  $27, 72 \cdot 6^k$ , for some  $k \geq 0$  integer.
2. Two edges with labels 8,  $108 \cdot 6^k$  and 27, 162, for some  $k \geq 0$  integer.

But it's also easy to check that if we start at a configuration in  $L$ , then we can't apply a swap move, and if we apply a slide or a connection we fall again in a configuration listed in  $L$ . Thus  $L$  is a complete list of all the configurations which can be reached from  $(\Gamma, \psi)$  using only slides, swaps, connections. By Theorem 3.7 we can algorithmically determine whether a GBS graph  $(\Delta, \phi)$  encodes a GBS group isomorphic to the one of  $(\Gamma, \psi)$ . In order to do this, first we make  $(\Delta, \phi)$  into a totally reduced GBS graph (this can be done algorithmically). Then we apply sign-changes in all the (finitely many) possible ways, and for each of the resulting GBS graphs, we check whether it appears in the list  $L$ . Note that induction is not possible on  $(\Gamma, \psi)$ , so we don't need to worry about it in this example.



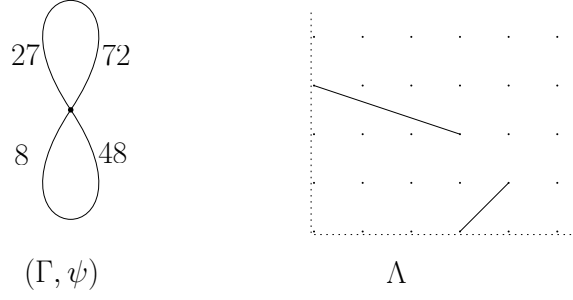


Figure 10: On the left a GBS graph  $(\Gamma, \psi)$  with one vertex and two edges. On the right the corresponding affine representation  $\Lambda$  (on the horizontal axis the number of factors 2, on the vertical axis the number of factors 3).

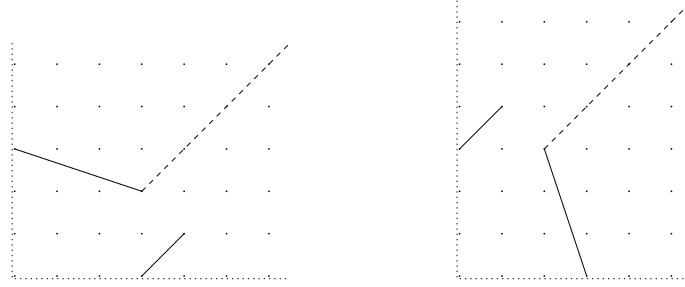


Figure 11: On the left, the configurations that can be reached from  $(\Gamma, \psi)$  by applying only slide moves - the endpoint of one of the edges is allowed to move along the dashed line, while the other three endpoints are required to stay fixed. On the right, the configurations that can be reached from  $(\Gamma, \psi)$  by applying a connection and then only slide moves - again, the endpoint of one of the edges is allowed to move along the dashed line, while the other endpoints stay fixed.

## 4 Independence of quasi-conjugacy classes

In this section we describe the basic invariants of the moves (which are also essentially isomorphism invariants). These are listed in Definition 4.3, and are the set of primes, the conjugacy and quasi-conjugacy classes, the number of edges in each conjugacy class.

We divide slide moves into two types: *internal* (when they involve two edges in the same quasi-conjugacy class) and *external* (otherwise). We introduce a notion of *external equivalence*, we show that it's an invariant of the moves, and that it can be used to describe external slide moves (without any need to know the exact position of the edges). We show that edges in different quasi-conjugacy classes can only interact with each other using the external equivalence. For the isomorphism problem, this means that distinct quasi-conjugacy classes can be dealt with separately and independently of each other, see Theorem 4.9.

Finally, we introduce the notions of *minimal region*. Every endpoint of every edge must lie above at least one minimal region, making them into excellent "starting points" for slide moves. We show that each of these regions is forced to contain at least one endpoint of some edge, see Lemma 4.17. This decreases a lot the amount of possible configurations that one has to consider.

We show with concrete examples how helpful these notions are when trying to describe the isomorphism problem.

### 4.1 Basic invariants of the moves

Let  $(\Gamma, \psi)$  be a GBS graph and let  $\Lambda$  be its affine representation. In Section 2.5 we had defined a partition of the set of vertices  $V(\Lambda)$ , given by the equivalence relation of conjugacy. We had also defined a pre-order  $\preceq_c$  on  $V(\Lambda)$ , by saying that  $p \preceq_c q$  if  $p \leq q' \sim_c q$  for some other vertex  $q'$ . This pre-order  $\preceq_c$  induces another partition of  $V(\Lambda)$ , given by the equivalence relation of quasi-

conjugacy. At the same time  $\preceq_c$  also defines an order relation on the set of quasi conjugacy classes. These relations can be extended to edges, as follows.

**Definition 4.1.** For  $e, f \in E(\Lambda)$  we denote:

1.  $e \sim_c f$  if  $\iota(e) \sim_c \iota(f)$ , and we say that  $e, f$  are **conjugate**.
2.  $e \preceq_c f$  if  $\iota(e) \preceq_c \iota(f)$
3.  $e \sim_{qc} f$  if  $\iota(e) \sim_{qc} \iota(f)$ , and we say that  $e, f$  are **quasi-conjugate**.

We observe that for  $e \in E(\Lambda)$  we have  $\iota(e) \sim_c \tau(e)$ . In particular, in the above Definition 4.1 we can use  $\tau(e), \tau(f)$  instead of  $\iota(e), \iota(f)$ , as we prefer. As for vertices, conjugacy and quasi-conjugacy are equivalence relations on the sets of edges, and  $\preceq_c$  induces a partial order on the set of quasi-conjugacy classes of edges. In particular, we obtain a finite poset of quasi-conjugacy classes of edges with the order relation  $\preceq_c$ .

**Definition 4.2.** For  $e \in E(\Gamma)$  we say that:

1.  $e$  belongs to the conjugacy class  $C$  if  $\iota(e) \in C$ . We define the set

$$E_c(\Lambda, C) := \{e \in E(\Lambda) : \iota(e) \in C\}$$

of the edges in the conjugacy class  $C$ .

2.  $e$  belongs to the quasi-conjugacy class  $Q$  if  $\iota(e) \in Q$ . We define the set

$$E_{qc}(\Lambda, Q) := \{e \in E(\Lambda) : \iota(e) \in Q\}$$

of the edges quasi-conjugate to  $Q$ .

We are now ready to define the basic invariants of a GBS graph.

**Definition 4.3** (Basic invariants). Let  $(\Gamma, \psi)$  be a GBS graph with affine representation  $\Lambda$ . We define the **basic invariants** of  $(\Gamma, \psi)$  as the following list of data:

1. The set of primes  $\mathcal{P}(\Gamma, \psi)$ .
2. The finite list of quasi-conjugacy classes  $Q$  containing at least one edge. For each such quasi-conjugacy class we list the finite data  $\min_{qc}(Q), \text{supp}_{qc}(Q), \text{lin}_{qc}(Q)$ . For each such quasi-conjugacy class, we write the number  $|E_{qc}(\Lambda, Q)|$ .
3. The finite list of conjugacy classes  $C \subseteq Q$  containing at least one edge, for each of them the number  $|E_c(\Lambda, C)|$ .

By direct check with the definitions, we have that the basic invariants are, indeed, invariant by slides, swaps, connections. Sign-changes and induction will preserve the invariants of Items 1 and 2. An edge sign-change will change the numbers  $|E_c(\Lambda, C)|$ , as it will move an edge from a conjugacy class  $C$  to the conjugacy class  $C + \mathbf{e}$  where  $\mathbf{e} \in \mathbf{A}$  is the 2-torsion element. A vertex sign-change or an induction will change the structure of the conjugacy classes  $C$  and the numbers  $|E_c(\Lambda, C)|$ ; however, we point out that these changes occur in quite a controlled way, as vertex sign-changes and inductions are just moving the points inside a copy  $\mathbf{A}_v^+ \subseteq V(\Lambda)$  by translation. In particular, we have the following:

**Corollary 4.4.** Let  $(\Gamma, \psi), (\Gamma', \psi')$  be two GBS graph and suppose that there is a sequence of slides, swaps, connections going from one to the other and inducing a bijection  $V(\Gamma) = V(\Gamma')$ . Then this induces a bijection  $V(\Lambda) = V(\Lambda')$ , and  $(\Gamma, \psi), (\Gamma', \psi')$  have the same basic invariants (Definition 4.3).

## 4.2 External equivalence

Let  $(\Gamma, \psi)$  be a GBS graph and let  $\Lambda$  be its affine representation.

**Definition 4.5.** Let  $p, q \in V(\Lambda)$  with  $p \sim_{qc} q$ .

1. We say that  $p, q$  are **externally equivalent**, denoted  $p \sim_{ee} q$ , if there is an affine path, going from  $p$  to  $q$ , which doesn't use edges quasi-conjugate to  $p$  and  $q$ .
2. We denote  $p \preceq_{ee} q$  if  $p \leq q'$  for some  $q' \sim_{ee} q$ .
3. We say that  $p, q$  are **quasi-externally equivalent**, denoted  $p \sim_{qee} q$ , if  $p \preceq_{ee} q$  and  $q \preceq_{ee} p$ .

The (quasi-)external equivalence class of  $p$  is its (quasi-)conjugacy class in the GBS graph where we remove all edges quasi-conjugate to  $p$ . In particular, they can be described in terms of the same data as for (quasi-)conjugacy classes.

**Definition 4.6.** Let  $(\Gamma, \psi)$  be a GBS graph and let  $p$  be a vertex of its affine representation  $\Lambda$ .

1. Define the **external minimal points**

$$\min_{qee}(p) = \{q \in V(\Lambda) : q \sim_{qee} p \text{ and } q \preceq_{ee} q' \text{ for all } q' \sim_{qee} p\}$$

2. Define the **external support**

$$\text{supp}_{qee}(p) = \bigcup \{\text{supp}(\mathbf{u}) : \mathbf{u} \in \mathbf{A}^+ \text{ and } p + \mathbf{u} \sim_{qee} p\} \subseteq \mathcal{P}(\Gamma, \psi).$$

3. Define the **external linear algebra**

$$\text{lin}_{qee}(p) = \{\mathbf{r} \in \mathbf{A} : \mathbf{r} = \mathbf{w} - \mathbf{w}' \text{ for some } \mathbf{w}, \mathbf{w}' \in \mathbf{A}^+ \text{ such that } p + \mathbf{w} \sim_{ee} p + \mathbf{w}' \sim_{ee} p\}.$$

Lemmas 2.12 2.14 2.15 2.16 hold for external equivalence in the exact same way as they do for conjugacy. Similarly, it's easy to see that  $\min_{qee}(p), \text{supp}_{qee}(p), \text{lin}_{qee}(p)$  are invariant if we change  $p$  by quasi-external equivalence. Propositions 2.18 and 2.19 also hold for external equivalence, meaning that  $\min_{qee}(p), \text{supp}_{qee}(p), \text{lin}_{qee}(p)$  can be used to describe external equivalence class using a finite set of data. Finally, the algorithms of Propositions 2.27 2.28 and Corollary 2.29 can be used for external equivalence as well, making everything algorithmically computable.

The following Lemma 4.7 shows that the external-equivalence classes are uniquely determined by the basic invariants.

**Lemma 4.7.** Let  $(\Gamma, \psi), (\Gamma', \psi')$  be GBS graphs with a bijection  $V(\Gamma) = V(\Gamma')$ . Suppose that they have the same set of primes and the same quasi-conjugacy classes containing at least one edge. Then  $(\Gamma, \psi), (\Gamma', \psi')$  have the same external equivalence and quasi-external equivalence relations.

*Proof.* Let  $p \sim_{qc} q$  be two vertices in a quasi-conjugacy class  $Q$  in the affine representation  $\Lambda$  of  $(\Gamma, \psi)$ . Then  $p \sim_{ee} q$  if and only if there is a sequence  $p = p_1, p_2, \dots, p_{\ell-1}, p_{\ell} = q$  with the following properties:

1. For all  $i = 1, \dots, \ell - 1$  there is a quasi-conjugacy class  $Q_i \neq Q$  containing at least one edge, and points  $r_i \sim_c s_i \in Q_i$ .
2. For all  $i = 1, \dots, \ell - 1$  we have that  $p_i = r_i + \mathbf{w}_i$  and  $p_{i+1} = s_i + \mathbf{w}_i$  for some  $\mathbf{w}_i \in \mathbf{A}^+$ .

If  $p \sim_{ee} q$  then the existence of such a sequence follows from the definition. Conversely, if there is such a sequence, then we can easily construct an affine path from  $p$  to  $q$  which doesn't use any edge in  $Q$ . But the existence of such a sequence only depends on the quasi-conjugacy classes containing at least one edge. The conclusion follows.  $\square$

### 4.3 External slide moves

The following Lemma 4.8 introduces a new move, based on external equivalence classes. As we will see, this is the key to separate the main Question 3.8 into several independent problems, each of them internal to a single quasi-conjugacy class.

**Lemma 4.8** (External slides). *Let  $(\Gamma, \psi)$  be a GBS graph and let  $\Lambda$  be its affine representation. Suppose that  $\Lambda$  contains an edge  $p \text{ --- } q$  and that  $q \sim_{ee} q'$  for some  $p, q, q' \in V(\Lambda)$ . Then we can change  $p \text{ --- } q$  into  $p \text{ --- } q'$  by means of a sequence of slide moves.*

*Proof.* Since  $q \sim_{ee} q'$  there must be an affine path from  $q$  to  $q'$  that doesn't use any edge quasi-conjugate to  $q$ . In particular, the affine path doesn't use the edge  $p \text{ --- } q$ . Thus we can slide  $p \text{ --- } q$  along the affine path in order to change it into  $p \text{ --- } q'$ .  $\square$

Let  $(\Gamma, \psi)$  be a GBS graph and let  $\Lambda$  be its affine representation. Observe that, if we perform a swap or a connection move involving two edges  $e, f$ , then  $f \sim_{qc} e$ . Instead, when changing an edge  $e$  by means of a slide over an edge  $f$ , there are two possibilities:

1.  $f \sim_{qc} e$ , in which case we say that the slide is **internal**.
2.  $f \not\sim_{qc} e$ , in which case we must have  $f \preceq_c e$ , and the slide move is a particular case of the external slide move of Lemma 4.8.

In particular, this means that we can deal with distinct quasi-conjugacy classes separately and independently from each other.

**Theorem 4.9** (Independence of quasi-conjugacy classes). *Let  $(\Gamma, \psi), (\Gamma', \psi')$  be two GBS graphs with a bijection  $V(\Gamma) = V(\Gamma')$ . Then the following are equivalent:*

1. *There is a sequence of slides, swaps, connections going from  $(\Gamma, \psi)$  to  $(\Gamma', \psi')$  inducing the given bijection.*
2. *For every quasi-conjugacy class  $Q$ , there is a sequence of swaps, connections, internal slides, external slides (Lemma 4.8) - each of them involving only edges in  $Q$  - going from the configuration in  $(\Gamma, \psi)$  to the configuration in  $(\Gamma', \psi')$ .*

*Proof.* Every slide move which involves two edges from different quasi-conjugacy classes can be replaced by an external slide as in Lemma 4.8. But external slides don't depend on the position of the edges in the other quasi-conjugacy classes (but only on the external equivalence relation). Thus, different moves, involving edges in different quasi-conjugacy classes, now commute. Thus we can deal with each quasi-conjugacy class separately and independently.  $\square$

In some particular cases, the possible configurations of certain quasi-conjugacy classes are extremely easy to describe.

**Corollary 4.10.** *Let  $(\Gamma, \psi)$  be a GBS graph with affine representation  $\Lambda$ , and let  $Q$  be a quasi-conjugacy class. Suppose that  $Q$  contains only one edge  $e$ . Then, all the configurations inside  $Q$  obtained by sequences of external slides, internal slides, swaps, connections, can also be obtained by performing two external-slides on the two endpoints of  $e$ .*

*Proof.* With only one edge in the quasi-conjugacy class, it's not possible to perform internal slides, swaps, connections.  $\square$

**Corollary 4.11.** *Let  $(\Gamma, \psi)$  be a GBS graph with affine representation  $\Lambda$ , and let  $Q$  be a quasi-conjugacy class. Suppose that, for all  $p, p' \in Q$ , we have that  $p \sim_c p'$  if and only if  $p \sim_{ee} p'$ . Then, all the configurations inside  $Q$  obtained by sequences of external slides, internal slides, swaps, connections, can also be obtained by performing external-slides only, and once on every endpoint of every edge.*

*Proof.* It's easy to manually check that a single move between internal slide, swap, connection, can be substituted with external slides. The conclusion follows.  $\square$

**Corollary 4.12.** *Let  $(\Gamma, \psi)$  be a GBS graph with affine representation  $\Lambda$ , and let  $Q$  be a quasi-conjugacy class. Suppose that  $\text{supp}_{\text{qc}}(Q) = \emptyset$ . Then only finitely many configurations can be obtained in  $Q$  by means of sequences of external slides, internal slides, swaps, connections.*

*Proof.* If  $\text{supp}_{\text{qc}}(Q) = \emptyset$  then  $Q$  is finite, and thus finitely many configurations are possible.  $\square$

#### 4.4 Mobile edges

Clay and Forester introduced a notion of *mobile edge* (see Definition 3.12 in [CF08]), which played an important role in the subsequent literature on GBSs (e.g. [Dud17] [Wan23]). An edge  $e \in E(\Lambda)$  is mobile if and only if it belongs to a quasi-conjugacy class  $Q$  with  $\text{supp}_{\text{qc}}(Q) \neq \emptyset$ .

As a corollary of the above discussion, we recover the following well-known results:

**Corollary 4.13** ([For06, Theorem 8.2]). *Let  $(\Gamma, \psi)$  be a GBS graph and suppose that  $q(\pi_1(\Gamma)) \cap \mathbf{A}^+ = \{\mathbf{0}\}$ . Then only finitely many configurations can be obtained from  $(\Gamma, \psi)$  by means of sequences of sign-changes, inductions, slides, swaps, connections.*

*Proof.* If  $q(\pi_1(\Gamma)) \cap \mathbf{A}^+ = \{\mathbf{0}\}$  then for every quasi-conjugacy class  $Q$  we must have  $\text{supp}_{\text{qc}}(Q) = \emptyset$ , and the result follows from Corollary 4.12.  $\square$

**Corollary 4.14** ([Dud17, Theorem 2]). *There is an algorithm that, given GBS graphs  $(\Gamma, \psi), (\Delta, \phi)$  such that  $(\Gamma, \psi)$  has at most one mobile edge, decides whether the corresponding GBS groups are isomorphic or not.*

*Proof.* Let  $Q$  be a quasi-conjugacy class in the affine representation  $\Lambda$  of  $(\Gamma, \psi)$ . If  $Q$  contains at least one edge, then either  $\text{supp}_{\text{qc}}(Q) = \emptyset$  (and we can describe the isomorphism problem using Corollary 4.12) or  $Q$  contains only one edge (and we can decide the isomorphism problem using Corollary 4.10).  $\square$

#### 4.5 Minimal regions of a quasi-conjugacy class

Let  $(\Gamma, \psi)$  be a GBS graph and let  $\Lambda$  be its affine representation. Let  $Q \subseteq V(\Lambda)$  be a quasi-conjugacy class.

**Definition 4.15.** *A quasi-external equivalence class  $R \subseteq Q$  is called **region** of  $Q$ .*

The set of regions of  $Q$  is a partition of  $Q$ . Moreover, the relation  $\preceq_{\text{ee}}$  makes the set of regions of  $Q$  into a partially ordered set. A **minimal region** of  $Q$  is a region  $M \subseteq Q$  which is minimal by  $\preceq_{\text{ee}}$ . Every minimal region must contain at least one point from  $\min_{\text{qc}}(Q)$  - possibly more than one. Each quasi-conjugacy class contains finitely many minimal regions, and always at least one.

*Remark 4.16.* Not every point in  $\min_{\text{qc}}(Q)$  needs to belong to a minimal region.

The reason for which we are interested in the minimal regions of  $Q$  is that they often give us important information about the position of the edges in the quasi-conjugacy class (especially when the quasi-conjugacy class contains few edges compared to the number of minimal regions, or when the configuration is particularly "rigid").

**Lemma 4.17.** *Let  $Q$  be a quasi-conjugacy class containing at least one edge. Then every minimal region of  $Q$  must contain at least one endpoint of one edge. In particular, if  $Q$  contains  $d \geq 1$  minimal regions, then  $Q$  contains at least  $\lceil \frac{d}{2} \rceil$  edges.*

*Proof.* Edges from other quasi-conjugacy classes don't allow us to change region. Therefore, if we want to move away from a minimal region  $M$ , we need to do that using an edge belonging to our quasi-conjugacy class  $Q$ . By minimality, this means that there must be an edge with an endpoint in  $M$ .  $\square$

*Remark 4.18.* The lower bound  $\lceil \frac{d}{2} \rceil$  in the above Lemma 4.17 is optimal.

**Definition 4.19.** *An edge  $e$  is called **floating** if none of  $\iota(e), \tau(e)$  is in a minimal region.*

Note that the number of floating edges isn't invariant when performing moves.

## 4.6 Examples

As usual, in the examples we use  $\mathbf{A} = \mathbb{Z}^{\mathcal{P}(\Gamma, \psi)}$ , omitting the  $\mathbb{Z}/2\mathbb{Z}$  summand.

*Example 4.20.* Later in the paper (see Example 5.21) we will describe the isomorphism problem for this example. Consider the GBS graph  $(\Gamma, \psi)$  with one vertex  $v$  and four edges, as in Figure 12, and call  $\Lambda$  its affine representation. The edges are partitioned into two different quasi-conjugacy classes  $B \preceq_c C$ , with  $B$  containing one edge and  $C$  containing three edges (see Figure 13). The quasi-conjugacy class  $B$  has exactly one minimal region  $N = \{(1, 0)\}$ . The quasi-conjugacy class  $C$  has exactly two minimal regions, namely  $M_1 = \{(k, 1) : k \geq 1\}$  and  $M_2 = \{(0, 3)\}$ .

Suppose that  $(\Delta, \phi)$  is a GBS graph which is obtained from  $(\Gamma, \psi)$  by means of a sequence of slides, swaps, connections. By Corollary 4.4  $(\Delta, \phi)$  must have the same quasi-conjugacy classes and the same regions as  $(\Gamma, \psi)$ , and that it must have exactly one edge in  $B$  and three in  $C$ . By Theorem 4.9 we deduce that  $(\Delta, \phi)$  must contain an edge  $(1, 0) \text{ --- } (4, 0)$  since that edge is alone in the quasi-conjugacy class  $B$ , and thus by Corollary 4.10 it can only change by external slides, which are trivial in this case. Lemma 4.17 tells us that among the three edges contained in  $C$ , at least one must have an endpoint in  $M_1$  and at least one must have an endpoint in  $M_2$ .

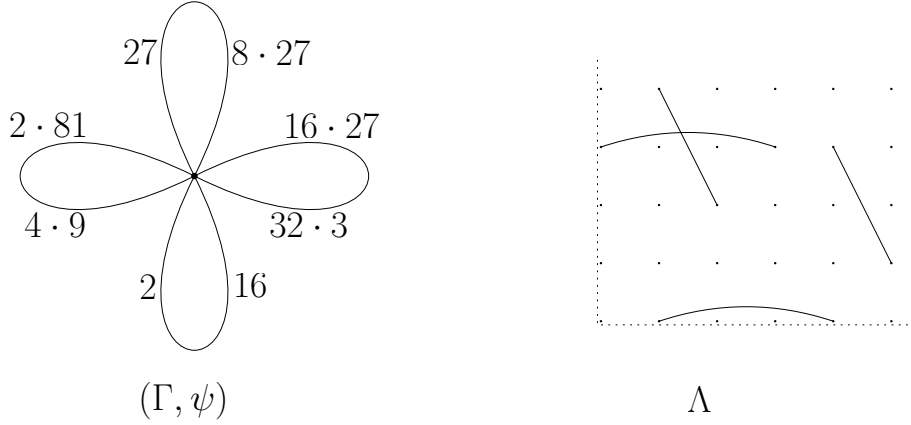


Figure 12: The GBS graph  $(\Gamma, \psi)$  and the affine representation  $\Lambda$  of Example 4.20.

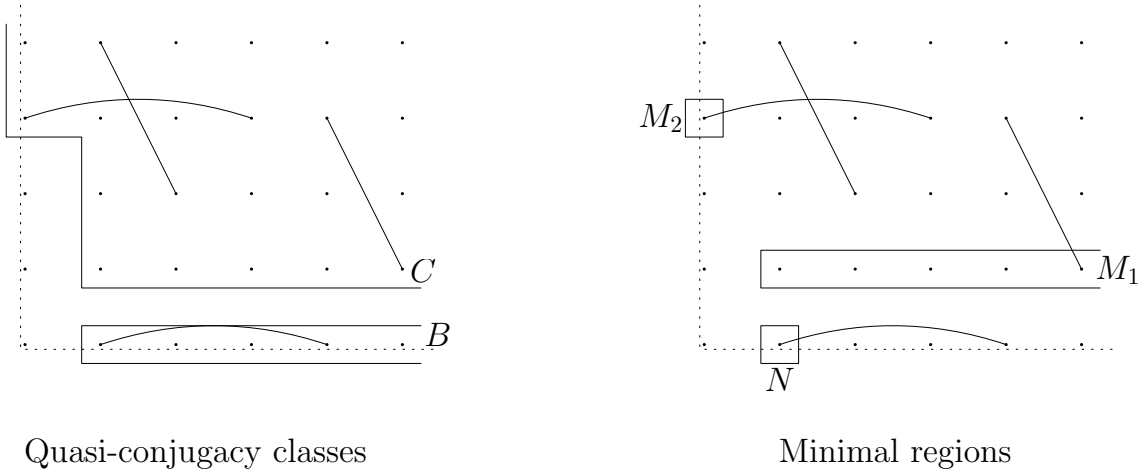


Figure 13: The quasi-conjugacy classes and the minimal regions for the GBS graph of Example 4.20.

*Example 4.21.* We can now describe the isomorphism problem for the GBS graph of Example 2.31. Using the same notation as in Example 2.31, call  $(\Gamma, \psi)$  the GBS graph, with a single vertex  $v$ , and  $\Lambda$  its affine representation.

The edges of  $\Lambda$  are partitioned into three conjugacy classes: one in the conjugacy class  $A$  of the point  $(3, 0, 0, 0)$  in  $\mathbf{A}_v^+$ , one in the conjugacy class  $B$  of the point  $(1, 1, 0, 0)$  in  $\mathbf{A}_v^+$ , and two in the conjugacy class  $C$  of the point  $(0, 1, 0, 1)$  in  $\mathbf{A}_v^+$ . For a description of the quasi-conjugacy classes  $A, B, C$  see Example 2.31. These three quasi-conjugacy classes form a poset given by  $A \preceq_c C$  and  $B \preceq_c C$ . By Theorem 4.9 we can examine different quasi-conjugacy classes independently.

The quasi-conjugacy class  $A$  only contains the edge  $(3, 0, 0, 0) \text{ --- } (1, 0, 1, 0)$ , so by Corollary 4.10 that edge can't be changed by any move. A similar analysis on  $B$  tells us that the edge  $(1, 1, 0, 0) \text{ --- } (0, 1, 1, 0)$  will not change either.

The quasi-conjugacy class  $C$  has three minimal regions  $M_1 = \{(1, 0, 0, 1)\}$  and  $M_2 = \{(0, 1, 0, 1)\}$  and  $M_3 = \{(a, 1, c, 0) : a + c \geq 2\}$ . By Lemma 4.17, along any sequence of slides, swaps, connections, we will always have (at least) one endpoint in  $M_1$ , one endpoint in  $M_2$ , one endpoint in  $M_3$ . We only need to understand where the fourth endpoint might be. Consider the following list  $L$  of configurations:

1. For  $a + c, a' + c' \geq 2$  the configurations

$$\begin{cases} (1, 0, 0, 1) \text{ --- } (a, 1, c, 0) \\ (0, 1, 0, 1) \text{ --- } (a', 1, c', 0) \end{cases}$$

2. For  $a + c \geq 2$  and  $b, d \geq 0$  the configurations

$$\begin{cases} (1, 0, 0, 1) \text{ --- } (a, 1, c, 0) \\ (0, 1, 0, 1) \text{ --- } (1 + b, 0, d, 1) \end{cases}$$

3. For  $a + c \geq 2$  and  $b, d \geq 0$  the configurations

$$\begin{cases} (1, 0, 0, 1) \text{ --- } (b, 1, d, 1) \\ (0, 1, 0, 1) \text{ --- } (a, 1, c, 0) \end{cases}$$

It's easy to see that all the configurations in the list  $L$  can be reached from  $(\Gamma, \psi)$  by means of internal slide moves and external slide moves. Moreover, if we are at any configuration in  $L$ , then no swap or connection is possible, and by performing an (internal or external) slide we end up at another configuration in  $L$ . It follows that  $L$  is a complete list of all the configuration that can be obtained from  $(\Gamma, \psi)$  by means of slides, swaps, connections. If we are given a GBS graph  $(\Delta, \phi)$ , and we want to know whether it encodes the same group as  $(\Gamma, \psi)$ , then we just change  $(\Delta, \phi)$  into a totally reduced GBS graph (this can be done algorithmically), we perform sign-changes in all the possible (finitely many) ways, and we check whether one of the resulting graphs appears in the list  $L$ . Note that, since induction can't be performed on  $(\Gamma, \psi)$ , we don't have to worry about it in this case.

## 5 Configurations based on linear algebra

### 5.1 Relative linear algebra of a point

Let  $(\Gamma, \psi)$  be a GBS graph with affine representation  $\Lambda$ .

**Definition 5.1.** For  $p \in V(\Lambda)$  and  $e_1, \dots, e_k \in E(\Lambda)$ , we define the **relative linear algebra** as

$$\text{lin}(p \text{ rel } e_1, \dots, e_k) = \{\mathbf{r} \in \mathbf{A} : \mathbf{r} = \mathbf{w} - \mathbf{w}' \text{ for some } \mathbf{w}, \mathbf{w}' \in \mathbf{A}^+ \text{ such that } p + \mathbf{w} \sim_c p + \mathbf{w}' \sim_c p \text{ and there is an affine path from } p + \mathbf{w} \text{ to } p + \mathbf{w}' \text{ which doesn't use } e_1, \bar{e}_1, \dots, e_k, \bar{e}_k\}.$$

**Lemma 5.2.** We have that  $\text{lin}(p \text{ rel } e_1, \dots, e_k)$  is a subgroup of  $\mathbf{A}$ .

*Proof.* Suppose that we are given  $\mathbf{r}, \mathbf{s} \in \text{lin}(p \text{ rel } e_1, \dots, e_k)$ . Thus we have  $\mathbf{r} = \mathbf{w} - \mathbf{w}'$  and  $\mathbf{s} = \mathbf{u} - \mathbf{u}'$ , where  $\mathbf{w}, \mathbf{w}', \mathbf{u}, \mathbf{u}' \in \mathbf{A}^+$  are such that  $p + \mathbf{w}, p + \mathbf{w}', p + \mathbf{u}, p + \mathbf{u}' \sim_c p$  and there are affine paths from  $p + \mathbf{w}$  to  $p + \mathbf{w}'$ , and from  $p + \mathbf{u}$  to  $p + \mathbf{u}'$ , which don't use  $e_1, \dots, e_k$ . It follows that there is an affine path from  $p + \mathbf{w} + \mathbf{u}$  to  $p + \mathbf{w}' + \mathbf{u}'$  which doesn't use  $e_1, \dots, e_k$ , and  $(\mathbf{w} + \mathbf{u}) - (\mathbf{w}' + \mathbf{u}') = \mathbf{r} + \mathbf{s}$ . Thus  $\mathbf{r} + \mathbf{s} \in \text{lin}(p \text{ rel } e_1, \dots, e_k)$ . The conclusion follows.  $\square$

**Definition 5.3.** We say that an edge  $e \in E(\Lambda)$  has **internal support** if  $e$  is given by  $(u, \mathbf{a}) \longrightarrow (u, \mathbf{b})$  for some  $u \in V(\Gamma)$  and  $\mathbf{a}, \mathbf{b} \in \mathbf{A}^+$  with  $\text{supp}(\mathbf{b} - \mathbf{a}) \subseteq \text{supp}_{\text{qc}}(\iota(e))$ .

In what follows, we will always consider the linear algebra of  $p$  relative to edges belonging to its quasi-conjugacy class and with internal support.

**Lemma 5.4** (Equivalent characterization of relative linear algebra). *Let  $p \in V(\Lambda)$  and  $e_1, \dots, e_k \in E_c(\Lambda, p)$  be with internal support. Then, for  $\mathbf{r} \in \mathbf{A}$ , we have that  $\mathbf{r} \in \text{lin}(p \text{ rel } e_1, \dots, e_k)$  if and only if the following two conditions hold:*

1.  $\Lambda$  contains vertices  $(u, \mathbf{a}), (u, \mathbf{a} + \mathbf{r}) \sim_{\text{qc}} p$ , for some  $u \in V(\Gamma)$  and  $\mathbf{a} \in \mathbf{A}^+$ , such that there is an affine path from  $(u, \mathbf{a})$  to  $(u, \mathbf{a} + \mathbf{r})$  which doesn't use the edges  $e_1, \bar{e}_1, \dots, e_k, \bar{e}_k$ .
2.  $\text{supp}(\mathbf{r}) \subseteq \text{supp}_{\text{qc}}(p)$ .

*Proof.* Without loss of generality, we can assume that  $e_1, \dots, e_k$  belong to quasi-conjugacy classes  $\preceq_c p$ ; in particular  $\text{supp}(e_1), \dots, \text{supp}(e_k) \subseteq \text{supp}_{\text{qc}}(p)$ . If  $\mathbf{r} \in \text{lin}(p \text{ rel } e_1, \dots, e_k)$  then clearly the two conditions hold. Conversely, suppose that  $\text{supp}(\mathbf{r}) \subseteq \text{supp}_{\text{qc}}(p)$ , and that there are vertices  $(u, \mathbf{a}), (u, \mathbf{a} + \mathbf{r}) \sim_{\text{qc}} p$  and an affine path from  $(u, \mathbf{a})$  to  $(u, \mathbf{a} + \mathbf{r})$  which doesn't use  $e_1, \dots, e_k$ .

Since  $e_i$  has internal support, we have that  $e_i$  goes from  $(u_i, \mathbf{a}_i)$  to  $(u_i, \mathbf{b}_i)$  for some  $u_i \in V(\Gamma)$  and  $\mathbf{a}_i, \mathbf{b}_i \in \mathbf{A}^+$  with  $\text{supp}(\mathbf{b}_i - \mathbf{a}_i) \subseteq \text{supp}_{\text{qc}}(p)$ . Let's take  $\mathbf{w} \in \mathbf{A}^+$  with  $p + \mathbf{w} \sim_c p$  and such that  $\mathbf{w} + \mathbf{b}_i - \mathbf{a}_i, \mathbf{w} + \mathbf{a}_i - \mathbf{b}_i \geq \mathbf{0}$  for  $i = 1, \dots, r$  and  $\mathbf{w} + \mathbf{r}, \mathbf{w} - \mathbf{r} \geq \mathbf{0}$ .

Since  $p \sim_{\text{qc}} (u, \mathbf{a})$  we can find  $p' \sim_c p$  with  $p' \geq (u, \mathbf{a})$ . Let's say that the affine path from  $p$  to  $p'$  uses the edges  $e_1, \bar{e}_1, \dots, e_k, \bar{e}_k$  exactly  $\lambda \geq 0$  times in total: then we can use the same path, but removing the occurrences of  $e_1, \bar{e}_1, \dots, e_k, \bar{e}_k$ , in order to go from  $p + \lambda \mathbf{w}$  to some point  $p''$  with  $p'' \geq p'$ .

Now we have an affine path from  $p + \lambda \mathbf{w}$  to  $p''$  which doesn't use the edges  $e_1, \bar{e}_1, \dots, e_k, \bar{e}_k$ ; note that  $(u, \mathbf{a}) \leq p''$ . This induces affine paths from  $p + \lambda \mathbf{w} + \mathbf{w}$  to  $p'' + \mathbf{w}$  and from  $p + \lambda \mathbf{w} + (\mathbf{w} + \mathbf{r})$  to  $p'' + \mathbf{w} + \mathbf{r}$ . By hypothesis there is an affine path from  $(u, \mathbf{a})$  to  $(u, \mathbf{a} + \mathbf{r})$  which doesn't use  $e_1, \bar{e}_1, \dots, e_k, \bar{e}_k$ , and this induces a path from  $p'' + \mathbf{w}$  to  $p'' + \mathbf{w} + \mathbf{r}$  with the same property. Concatenating these three paths, we obtain an affine path from  $p + \lambda \mathbf{w} + \mathbf{w}$  to  $p + \lambda \mathbf{w} + (\mathbf{w} + \mathbf{r})$  which doesn't use  $e_1, \bar{e}_1, \dots, e_k, \bar{e}_k$ . The conclusion follows.  $\square$

If  $p \sim_{\text{qc}} p'$  and  $e_1, \dots, e_k$  have internal support, then  $\text{lin}(p \text{ rel } e_1, \dots, e_k) = \text{lin}(p' \text{ rel } e_1, \dots, e_k)$ .

**Lemma 5.5.** Let  $p \in V(\Lambda)$  and  $e_1, \dots, e_k \in E_c(\Lambda, p)$  be with internal support. Let's say that  $e_i$  is given by  $(v_i, \mathbf{a}_i) \longrightarrow (v_i, \mathbf{a}_i + \mathbf{x}_i)$  with  $\mathbf{a}_i \in \mathbf{A}^+$  and  $\mathbf{x}_i \in \mathbf{A}$ , for  $i = 1, \dots, k$ . Then

$$\langle \mathbf{x}_1, \dots, \mathbf{x}_k, \text{lin}(p \text{ rel } e_1, \dots, e_k) \rangle = \text{lin}_{\text{qc}}(p)$$

*Proof.* TO DO  $\square$



**Lemma 5.6.** *There is an algorithm that, given a GBS graph  $(\Gamma, \psi)$ , a vertex  $p$  in its affine representation  $\Lambda$  and edges  $e_1, \dots, e_k \in E(\Lambda)$  with internal support, computes a finite set of generators for  $\text{lin}(p \text{ rel } e_1, \dots, e_k)$ .*

*Proof.* TO DO □

## 5.2 Relative slide

The following fundamental Proposition 5.7 is the reason why we introduced the relative linear algebra. It tells us that, whenever two edges are in a position that allows for enough interaction, we can change them, using the linear algebra relative to the edge to be changed, as we wish. Note that the hypothesis required by Proposition 5.7 are reasonably weak. The subsequent Proposition 5.9 tells us that, if an edge  $f$  has both its endpoints controlled by another edge, then  $f$  can be translated at will along its conjugacy class. Let  $(\Gamma, \psi)$  be a GBS graph with affine representation  $\Lambda$ .

**Proposition 5.7** (Relative slide). *Let  $p = (v, \mathbf{a}) \in V(\Lambda)$  and  $e, f \in E(\Lambda)$  be such that:*

1.  $e$  is given by  $\mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w}$  inside  $\mathbf{A}_v^+$ , for some  $\mathbf{w} \in \mathbf{A}^+$ .
2.  $f$  is given by  $\mathbf{b} \text{ --- } \mathbf{c}$  inside  $\mathbf{A}_v^+$ , for some  $\mathbf{b}, \mathbf{c} \in \mathbf{A}^+$ .
3.  $\mathbf{c} \geq \mathbf{a}$  and  $\text{supp}(\mathbf{c} - \mathbf{b}) \subseteq \text{supp}(\mathbf{w})$ .

*Then, for all  $\mathbf{r} \in \text{lin}(p \text{ rel } f)$  with  $\mathbf{c} + \mathbf{r} \geq \mathbf{a}$ , there is a sequence of slides and swaps changing*

$$\begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w} \\ \mathbf{b} \text{ --- } \mathbf{c} \end{cases} \quad \text{into} \quad \begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w} \\ \mathbf{b} \text{ --- } \mathbf{c} + \mathbf{r} \end{cases}$$

*Proof.* Since  $\mathbf{r} \in \text{lin}(p \text{ rel } f)$ , we can find  $\mathbf{u} \geq -\mathbf{r}, \mathbf{0}$  such that  $(v, \mathbf{a} + \mathbf{u}) \sim_c (v, \mathbf{a})$  and there is an affine path going from  $(v, \mathbf{a} + \mathbf{u})$  to  $(v, \mathbf{a} + \mathbf{u} + \mathbf{r})$  which doesn't use the edge  $f$ .

Since  $\text{supp}(\mathbf{c} - \mathbf{b}) \subseteq \text{supp}(\mathbf{w})$ , we can find  $k \geq 0$  such that  $k\mathbf{w} + (\mathbf{b} - \mathbf{c}), k\mathbf{w} + (\mathbf{c} - \mathbf{b}) \geq \mathbf{0}$ . Consider an affine path going from  $(v, \mathbf{a})$  to  $(v, \mathbf{a} + \mathbf{u})$ ; for every occurrence of  $f$  or  $\bar{f}$  in such path, substitute such occurrence with  $k$  occurrences of the edge  $e$  instead. We obtain an affine path going from  $(v, \mathbf{a})$  to  $(v, \mathbf{a} + \mathbf{u}')$  for some  $\mathbf{u}' \in \mathbf{A}^+$  with  $\mathbf{u}' \geq \mathbf{u}$ ; notice that such path doesn't use the edge  $f$ . Since  $\mathbf{c} \geq \mathbf{a}$ , we also have a path from  $(v, \mathbf{c})$  to  $(v, \mathbf{c} + \mathbf{u}')$  which doesn't use the edge  $f$ . Since  $\mathbf{c} + \mathbf{r} \geq \mathbf{a}$ , we also have a path from  $(v, \mathbf{c} + \mathbf{r})$  to  $(v, \mathbf{c} + \mathbf{r} + \mathbf{u}')$  which doesn't use  $f$ .

The path from  $(v, \mathbf{a} + \mathbf{u})$  to  $(v, \mathbf{a} + \mathbf{u} + \mathbf{r})$  which doesn't use  $f$  also gives us a path from  $(v, \mathbf{c} + \mathbf{u}')$  to  $(v, \mathbf{c} + \mathbf{r} + \mathbf{u}')$  which doesn't use  $f$  (since  $\mathbf{c} + \mathbf{u}' \geq \mathbf{a} + \mathbf{u}$ ). Therefore, by means of a sequence of slide moves, we can change  $\mathbf{b} \text{ --- } \mathbf{c}$  into  $\mathbf{b} \text{ --- } \mathbf{c} + \mathbf{u}'$  and then into  $\mathbf{b} \text{ --- } \mathbf{c} + \mathbf{r} + \mathbf{u}'$  and finally into  $\mathbf{b} \text{ --- } \mathbf{c} + \mathbf{r}$ . The conclusion follows. □

## 5.3 Floating edges

**Definition 5.8.** *An edge  $f$  is called **floating** if it has no endpoint in any minimal region.*

Note that the property of being floating, as well as the number of floating edges, is not invariant by sequences of moves (nor under isomorphism). The next Proposition 5.9 shows that a floating edge can be moved wherever is convenient inside its quasi-conjugacy class.

**Proposition 5.9** (Translation of controlled floating edges). *Let  $p = (v, \mathbf{a}) \in V(\Lambda)$  and  $e, f \in E(\Lambda)$  be distinct edges such that:*

1.  $e$  is given by  $(v, \mathbf{a}) \text{ --- } (v, \mathbf{a} + \mathbf{w})$  for some  $\mathbf{w} \in \mathbf{A}^+$ .
2.  $f$  is given by  $(v, \mathbf{b} + \mathbf{u}) \text{ --- } (v, \mathbf{b} + \mathbf{u}')$  for some  $\mathbf{b}, \mathbf{u}, \mathbf{u}' \in \mathbf{A}^+$ .
3.  $\mathbf{a}, \mathbf{w}$  controls  $\mathbf{b}, \mathbf{b} + \mathbf{u}, \mathbf{b} + \mathbf{u}'$ .

*Then, for all  $(u, \mathbf{c}) \sim_c (v, \mathbf{b})$ , there is a sequence of slides and swaps changing  $f$  from*

$$(v, \mathbf{b} + \mathbf{u}) \text{ --- } (v, \mathbf{b} + \mathbf{u}') \quad \text{into} \quad (u, \mathbf{c} + \mathbf{u}) \text{ --- } (u, \mathbf{c} + \mathbf{u}')$$

*Proof.* We can find an affine path  $\sigma$ , which doesn't use  $f$ , from  $(u, \mathbf{c})$  to  $(v, \mathbf{d})$  for some  $\mathbf{d} \geq \mathbf{b}$ . We can find an affine path  $\tau$  from  $(v, \mathbf{b})$  to  $(v, \mathbf{d})$ . We choose  $k \in \mathbb{N}$  very big, and we consider the affine path  $e^k \tau e^{-k}$ , from  $(v, \mathbf{b})$  to  $(v, \mathbf{d})$ . Suppose that  $\tau$  contains in total  $h \in \mathbb{Z}$  occurrences of  $f$  (counted with sign), and let  $\tau'$  be the sequence obtained by removing those occurrences. Then  $e^k f^h \tau' e^{-k}$  is an affine path going from  $(v, \mathbf{b})$  to  $(v, \mathbf{d})$  (since  $k$  was chosen very big).

We now change  $(v, \mathbf{b} + \mathbf{u}) \longrightarrow (v, \mathbf{b} + \mathbf{u}')$  by slide along  $e^k$  into  $(v, \mathbf{b} + \mathbf{u} + k\mathbf{w}) \longrightarrow (v, \mathbf{b} + \mathbf{u} + k\mathbf{w})$ , then by self-slides (Lemma 3.2) into  $(v, \mathbf{b} + \mathbf{u} + k\mathbf{w} + h(\mathbf{u}' - \mathbf{u})) \longrightarrow (v, \mathbf{b} + \mathbf{u}' + k\mathbf{w} + h(\mathbf{u}' - \mathbf{u}))$ , then by slide along  $\tau'$  into  $(v, \mathbf{d} + \mathbf{u} + k\mathbf{w}) \longrightarrow (v, \mathbf{d} + \mathbf{u}' + k\mathbf{w})$ , then by slide along  $e^{-k}$  into  $(v, \mathbf{d} + \mathbf{u}) \longrightarrow (v, \mathbf{d} + \mathbf{u}')$ , and finally by slide along  $\sigma$  into  $(u, \mathbf{c} + \mathbf{u}) \longrightarrow (u, \mathbf{c} + \mathbf{u}')$ , as desired.  $\square$

## 5.4 Cliques

Let  $(\Gamma, \psi)$  be a GBS graph and let  $\Lambda$  be its affine representation.

**Definition 5.10** (Cliques). *Let  $v \in V(\Gamma)$  be a vertex and  $k \geq 2$ . A  $k$ -clique at  $v$  is a family of edges  $e_1, \dots, e_k \in E(\Lambda)$  satisfying the following conditions:*

1. *For  $i = 1, \dots, k$ , we have that  $e_i$  is given by  $\mathbf{a}_i \longrightarrow \mathbf{a}_i + \mathbf{w}_i$  inside  $\mathbf{A}_v^+$ , for some  $\mathbf{a}_i, \mathbf{w}_i \in \mathbf{A}^+$ .*
2. *For all  $i, j = 1, \dots, k$ , we have that  $\mathbf{a}_i + \mathbf{w}_i \geq \mathbf{a}_j$ .*

*In addition, the clique is called **full-support** if  $\text{supp}(\mathbf{w}_1) = \dots = \text{supp}(\mathbf{w}_k) = \text{supp}_{\text{qc}}((v, \mathbf{a}_1))$ .*

The edges in a clique all belong to a common quasi-conjugacy class. As we will see, there are substantial differences in the behaviour of 2-cliques compared to  $k$ -cliques for  $k \geq 3$ . Thus we also provide immediately the following definition.

**Definition 5.11** (Controlled and weak 2-cliques). *Let  $v \in V(\Gamma)$  be a vertex and let  $\mathbf{a}_1 \longrightarrow \mathbf{a}_1 + \mathbf{w}_1$  and  $\mathbf{a}_2 \longrightarrow \mathbf{a}_2 + \mathbf{w}_2$  be a 2-clique at  $v$ , for some  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{w}_1, \mathbf{w}_2 \in \mathbf{A}^+$ . We say that the clique is*

1. ***controlled** if  $\mathbf{a}_2 \geq \mathbf{a}_1$  or  $\mathbf{a}_1 \geq \mathbf{a}_2$ .*
2. ***weak** if  $\mathbf{a}_2 \not\geq \mathbf{a}_1$  and  $\mathbf{a}_1 \not\geq \mathbf{a}_2$ .*

The following Lemma 5.12 shows that every clique can be made into a full-support clique. The following Lemma 5.13 shows that, if an edge controls the two endpoints of another edge, then we can make them into a full-support controlled 2-clique.

**Lemma 5.12** (Getting full support). *Let  $p = (v, \mathbf{a}) \in V(\Lambda)$  and  $e, f \in E(\Lambda)$  be such that:*

1.  *$e$  is given by  $\mathbf{a} \longrightarrow \mathbf{a} + \mathbf{w}$  inside  $\mathbf{A}_v^+$ , for some  $\mathbf{w} \in \mathbf{A}^+$ .*
2.  *$f$  is given by  $\mathbf{b} \longrightarrow \mathbf{b} + \mathbf{u}$  inside  $\mathbf{A}_v^+$ , for some  $\mathbf{b}, \mathbf{u} \in \mathbf{A}^+$ .*
3.  *$\mathbf{a} + \mathbf{w}, \mathbf{b} + \mathbf{u} \geq \mathbf{a}, \mathbf{b}$*

*Then there is a sequence of slides and swaps changing*

$$\begin{cases} \mathbf{a} \longrightarrow \mathbf{a} + \mathbf{w} \\ \mathbf{b} \longrightarrow \mathbf{b} + \mathbf{u} \end{cases} \quad \text{into} \quad \begin{cases} \mathbf{a} \longrightarrow \mathbf{a} + \mathbf{w}' \\ \mathbf{b} \longrightarrow \mathbf{b} + \mathbf{u}' \end{cases}$$

*for some  $\mathbf{w}', \mathbf{u}' \in \mathbf{A}^+$  such that  $\mathbf{a} + \mathbf{w}', \mathbf{b} + \mathbf{u}' \geq \mathbf{a}, \mathbf{b}$  and  $\text{supp}(\mathbf{w}') = \text{supp}(\mathbf{u}') = \text{supp}_{\text{qc}}(p)$ .*

*Proof.* We take  $\mathbf{r} \in \text{lin}(p \text{ rel } e, f)$  and  $k \geq 0$  such that  $k(\mathbf{u} + \mathbf{w}) + \mathbf{r} \geq \mathbf{0}$  and  $\text{supp}(k(\mathbf{u} + \mathbf{w}) + \mathbf{r}) = \text{supp}_{\text{qc}}(p)$ . We now slide to change  $e$  from  $\mathbf{a} \longrightarrow \mathbf{a} + \mathbf{w}$  into  $\mathbf{a} \longrightarrow \mathbf{a} + \mathbf{w} + \mathbf{u}$ . Now, since  $\text{supp}(\mathbf{u}) \subseteq \text{supp}(\mathbf{w} + \mathbf{u})$ , we can perform a relative slide move (Proposition 5.7) to change  $f$  from  $\mathbf{b} \longrightarrow \mathbf{b} + \mathbf{u}$  into  $\mathbf{b} \longrightarrow \mathbf{b} + \mathbf{u} + k(\mathbf{u} + \mathbf{w}) + \mathbf{r}$ . The conclusion follows performing one last slide move.  $\square$

**Lemma 5.13** (Creating controlled 2-cliques). *Let  $p = (v, \mathbf{a}) \in V(\Lambda)$  and  $e, f \in E(\Lambda)$  be such that:*

1.  *$e$  is given by  $\mathbf{a} \longrightarrow \mathbf{a} + \mathbf{w}$  inside  $\mathbf{A}_v^+$ , for some  $\mathbf{w} \in \mathbf{A}^+$ .*

2.  $f$  is given by  $\mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x}$  inside  $\mathbf{A}_v^+$ , for some  $\mathbf{b} \in \mathbf{A}^+$  and  $\mathbf{x} \in \mathbf{A}$ .
3.  $\mathbf{a}, \mathbf{w}$  controls  $\mathbf{b}, \mathbf{b} + \mathbf{x}$ .

Then there is a sequence of slides and swaps changing

$$\begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w} \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x} \end{cases} \quad \text{into} \quad \begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w}' \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x}' \end{cases}$$

for some  $\mathbf{w}', \mathbf{x}' \in \mathbf{A}^+$  such that  $\mathbf{a} + \mathbf{w}', \mathbf{b} + \mathbf{x}' \geq \mathbf{a}, \mathbf{b}$  and  $\text{supp}(\mathbf{w}') = \text{supp}(\mathbf{x}') = \text{supp}_{\text{qc}}(p)$ .

*Proof.* Call  $p = \iota(e)$ . Since  $\text{supp}(\mathbf{x}) \subseteq \text{supp}(\mathbf{w}) \subseteq \text{supp}_{\text{qc}}(p)$ , we can find  $\mathbf{r} \in \text{lin}(p \text{ rel } f)$  such that  $\mathbf{r}, \mathbf{x} + \mathbf{r} \geq \mathbf{0}, \mathbf{b} - \mathbf{a}$  and  $\text{supp}(\mathbf{x} + \mathbf{r}) = \text{supp}_{\text{qc}}(p)$ . By Proposition 5.7 and by Lemma 3.3 we can change

$$\begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w} \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x} \end{cases} \xrightarrow{\text{slides}} \begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w} \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x} + \mathbf{r} \end{cases} \xrightarrow{\text{reverse slide}} \begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w} + \mathbf{x} + \mathbf{r} \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x} + \mathbf{r} \end{cases}$$

and the conclusion follows.  $\square$

## 5.5 Flexible configurations

The following Theorem 5.14 shows that we can change as we want with the edges of a  $k$ -clique, provided that we preserve the fact that it is a clique, plus some reasonable linear algebra condition. As we will see in the next TO DO Section, the hypothesis  $k \geq 3$  is fundamental to make the statement work. The notion of Nielsen equivalent ordered  $k$ -tuple of elements of an abelian group is defined in Appendix A. Let  $(\Gamma, \psi)$  be a GBS graph with affine representation  $\Lambda$ .

**Theorem 5.14** (Single clique with  $k \geq 3$ ). *Let  $k \geq 3$ . Let  $v \in V(\Gamma)$  and let  $\mathbf{a}_i, \mathbf{w}_i, \mathbf{w}'_i \in \mathbf{A}^+$  for  $i = 1, \dots, k$ . Call  $p = (v, \mathbf{a}_1) \in V(\Lambda)$ . Suppose that:*

1. For  $i = 1, \dots, k$ , there is  $e_i \in E(\Lambda)$  given by  $\mathbf{a}_i \text{ --- } \mathbf{a}_i + \mathbf{w}_i$  inside  $\mathbf{A}_v^+$ .
2. For  $i, j = 1, \dots, k$ , we have that  $\mathbf{a}_i + \mathbf{w}_i, \mathbf{a}_i + \mathbf{w}'_i \geq \mathbf{a}_j$ .
3.  $\langle \mathbf{w}_1, \dots, \mathbf{w}_k, \text{lin}(p \text{ rel } e_1, \dots, e_k) \rangle = \langle \mathbf{w}'_1, \dots, \mathbf{w}'_k, \text{lin}(p \text{ rel } e_1, \dots, e_k) \rangle \leq \mathbf{A}$ .
4. In the abelian group

$$\langle \mathbf{w}_1, \dots, \mathbf{w}_k, \text{lin}(p \text{ rel } e_1, \dots, e_k) \rangle / \text{lin}(p \text{ rel } e_1, \dots, e_k) = \text{lin}_{\text{qc}}(p) / \text{lin}(p \text{ rel } e_1, \dots, e_k)$$

the  $k$ -tuple  $[\mathbf{w}'_1], \dots, [\mathbf{w}'_k]$  is Nielsen equivalent to  $[\mathbf{w}_1], \dots, [\mathbf{w}_k]$ .

Then there is a sequence of slides changing

$$\begin{cases} \mathbf{a}_1 \text{ --- } \mathbf{a}_1 + \mathbf{w}_1 \\ \dots \\ \mathbf{a}_k \text{ --- } \mathbf{a}_k + \mathbf{w}_k \end{cases} \quad \text{into} \quad \begin{cases} \mathbf{a}_1 \text{ --- } \mathbf{a}_1 + \mathbf{w}'_1 \\ \dots \\ \mathbf{a}_k \text{ --- } \mathbf{a}_k + \mathbf{w}'_k \end{cases}$$

*Observation 5.15.* In the setting of Theorem 5.14, if there are  $\mathbf{a}_j \geq \mathbf{a}_i$  for some  $i \geq j$ , then it's also possible to perform a swap move: this has the effect of permuting the  $k$ -tuple  $[\mathbf{w}_1], \dots, [\mathbf{w}_k]$ . This can be relevant when computing the Nielsen equivalence class of the  $k$ -tuples  $[\mathbf{w}_1], \dots, [\mathbf{w}_k]$ : if the  $k$ -tuple is a minimal set of generators for the abelian group  $\langle \mathbf{w}_1, \dots, \mathbf{w}_k, \text{lin}(p \text{ rel } e_1, \dots, e_k) \rangle / \text{lin}(p \text{ rel } e_1, \dots, e_k)$ , then the swap will change the sign of the determinant of the matrix. On the contrary, performing a connection move between two edges of a clique will always preserve the Nielsen equivalence class of the ordered  $k$ -tuple  $[\mathbf{w}_1], \dots, [\mathbf{w}_k]$ .

*Proof.* The proof is based on the results of Appendix B. By Lemma 5.12 we can assume that  $\text{supp}(\mathbf{w}_i) = \text{supp}(\mathbf{w}'_i) = \text{supp}_{\text{qc}}(p)$  for  $i = 1, \dots, k$ . Let  $I = \text{supp}_{\text{qc}}(p) \cup \{0\}$ . Let  $M = \max\{|a_{ir} - a_{jr}| : i, j = 1, \dots, k \text{ and } r \in I\}$ .

Consider the abelian group  $\mathbb{Z}^I$  and consider the natural map  $\pi : \mathbb{Z}^I \rightarrow \mathbf{A}$ , and let  $v = (2, 0, 0, \dots, 0) \in \mathbb{Z}^I$  be the generator of  $\ker \pi$ . We choose liftings  $w_i, w'_i \in \mathbb{Z}^I$  such that  $\pi(w_i) = \mathbf{w}_i, \pi(w'_i) = \mathbf{w}'_i$  for  $i = 1, \dots, k$ ; since  $\text{supp}(\mathbf{w}) = \text{supp}(\mathbf{w}') = \text{supp}(\mathbf{x}_i) = \text{supp}(\mathbf{x}'_i) = I \setminus \{0\}$ , we can choose  $w_i, w'_i$  in such a way that they are all  $M$ -big (according to Appendix B). We also choose a basis  $\mathbf{v}_1, \dots, \mathbf{v}_s$  for  $\text{lin}((v, \mathbf{a}_1) \text{ rel } e_1, \dots, e_k)$  and we choose liftings  $v_1, \dots, v_s \in \mathbb{Z}^I$ . By Theorem B.1, there is a sequence of Nielsen moves and Nielsen moves relative to  $v, v_1, \dots, v_s$  going from  $w_1, \dots, w_k$  to  $w'_1, \dots, w'_k$  such that every  $k$ -tuple along the sequence is  $M$ -big.

In particular, this means that we can change

$$\begin{cases} \mathbf{a}_1 & \text{---} & \mathbf{a}_1 + \pi(w_1) \\ \dots & & \dots \\ \mathbf{a}_k & \text{---} & \mathbf{a}_k + \pi(w_k) \end{cases} \quad \text{into} \quad \begin{cases} \mathbf{a}_1 & \text{---} & \mathbf{a}_1 + \pi(w'_1) \\ \dots & & \dots \\ \mathbf{a}_k & \text{---} & \mathbf{a}_k + \pi(w'_k) \end{cases}$$

by means of slide moves and reverse slide moves (Lemma 3.3) and relative slide moves (Proposition 5.7). The conclusion follows.  $\square$

The next Theorem 5.16 shows that a similar conclusion can be obtained if we consider several cliques inside the same quasi-conjugacy class. The interesting point here is that, even if the cliques are completely disjoint, they all interact together, as if they were a single larger clique. Note that we require that there are  $r \geq 2$  cliques, but we now allow for each clique to contain 2 edges only.

**Theorem 5.16** (Multiple cliques). *Let  $r \geq 2$  and let  $k_1, \dots, k_r \geq 2$ . Let  $v_s \in V(\Gamma)$  and let  $\mathbf{a}_{si}, \mathbf{w}_{si}, \mathbf{w}'_{si} \in \mathbf{A}^+$  for  $s = 1, \dots, r$  and  $i = 1, \dots, k_s$ . Call  $p_s = (v_s, \mathbf{a}_{s1}) \in V(\Lambda)$  for  $s = 1, \dots, r$ . Suppose that:*

1.  $p_1 \sim_{\text{qc}} p_2 \sim_{\text{qc}} \dots \sim_{\text{qc}} p_r$ .
2. For  $s = 1, \dots, r$  and  $i = 1, \dots, k_s$ , there is  $e_{si} \in E(\Lambda)$  given by  $\mathbf{a}_{si} \text{---} \mathbf{a}_{si} + \mathbf{w}_{si}$  inside  $\mathbf{A}_{v_s}^+$ .
3. For  $s = 1, \dots, r$  and  $i, j = 1, \dots, k_s$  we have that  $\mathbf{a}_{si} + \mathbf{w}_{si}, \mathbf{a}_{si} + \mathbf{w}'_{si} \geq \mathbf{a}_{sj}$ .
4.  $\langle \mathbf{w}_{11}, \dots, \mathbf{w}_{21}, \dots, \text{lin}(p_1 \text{ rel } e_{11}, \dots, e_{21}, \dots) \rangle = \langle \mathbf{w}'_{11}, \dots, \mathbf{w}'_{21}, \dots, \text{lin}(p_1 \text{ rel } e_{11}, \dots, e_{21}, \dots) \rangle$ .
5. In the abelian group

$$\frac{\langle \mathbf{w}_{11}, \dots, \mathbf{w}_{21}, \dots, \text{lin}(p_1 \text{ rel } e_{11}, \dots, e_{21}, \dots) \rangle}{\text{lin}(p_1 \text{ rel } e_{11}, \dots, e_{21}, \dots)} = \frac{\text{lin}_{\text{qc}}(p_1)}{\text{lin}(p_1 \text{ rel } e_{11}, \dots, e_{21}, \dots)}$$

the  $(k_1 + \dots + k_s)$ -tuple  $[\mathbf{w}'_{11}], \dots, [\mathbf{w}'_{21}], \dots$  is Nielsen equivalent to  $[\mathbf{w}_{11}], \dots, [\mathbf{w}_{21}], \dots$

Then there is a sequence of slides changing

$$\begin{cases} (v_1, \mathbf{a}_{11}) & \text{---} & (v_1, \mathbf{a}_{11} + \mathbf{w}_{11}) \\ \dots & & \dots \\ (v_2, \mathbf{a}_{21}) & \text{---} & (v_2, \mathbf{a}_{21} + \mathbf{w}_{21}) \\ \dots & & \dots \end{cases} \quad \text{into} \quad \begin{cases} (v_1, \mathbf{a}_{11}) & \text{---} & (v_1, \mathbf{a}_{11} + \mathbf{w}'_{11}) \\ \dots & & \dots \\ (v_2, \mathbf{a}_{21}) & \text{---} & (v_2, \mathbf{a}_{21} + \mathbf{w}'_{21}) \\ \dots & & \dots \end{cases}$$

*Proof.* TO DO  $\square$

## 5.6 Flexibility of 2-cliques with additional assumptions

As we will see in the next TO DO Section, Theorem 5.14 becomes false, in general, when one considers 2-cliques (especially for weak 2-cliques, but also for controlled ones, if the relative linear algebra is bad enough). It remains true for controlled 2-cliques if one makes some additional assumption. Let  $(\Gamma, \psi)$  be a GBS graph with affine representation  $\Lambda$ .

**Proposition 5.17** (Controlled 2-cliques without relative). *Let  $p = (v, \mathbf{a}) \in V(\Lambda)$  be a vertex and let  $\mathbf{b}, \mathbf{w}, \mathbf{w}' \in \mathbf{A}^+$  and  $\mathbf{x}, \mathbf{x}' \in \mathbf{A}$  be such that:*

1. *There is  $e \in E(\Lambda)$  given by  $\mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w}$  inside  $\mathbf{A}_v^+$ .*
2. *There is  $f \in E(\Lambda)$  is given by  $\mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x}$  inside  $\mathbf{A}_v^+$ .*
3.  *$\mathbf{a}, \mathbf{w}$  controls  $\mathbf{b}, \mathbf{b} + \mathbf{x}$ .*
4.  *$\mathbf{a}, \mathbf{w}'$  controls  $\mathbf{b}, \mathbf{b} + \mathbf{x}'$ .*
5.  *$\langle \mathbf{w}, \mathbf{x} \rangle = \langle \mathbf{w}', \mathbf{x}' \rangle \leq \mathbf{A}$ .*

*Then there is a sequence of slides and swaps changing*

$$\begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w} \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x} \end{cases} \quad \text{into} \quad \begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w}' \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x}' \end{cases}$$

*Proof.* TO DO cite □

**Proposition 5.18** (Controlled 2-cliques with full-support relative). *Let  $p = (v, \mathbf{a}) \in V(\Lambda)$  be a vertex and let  $\mathbf{b}, \mathbf{w}, \mathbf{w}' \in \mathbf{A}^+$  and  $\mathbf{x}, \mathbf{x}' \in \mathbf{A}$  be such that:*

1. *There is  $e \in E(\Lambda)$  given by  $\mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w}$  inside  $\mathbf{A}_v^+$ .*
2. *There is  $f \in E(\Lambda)$  is given by  $\mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x}$  inside  $\mathbf{A}_v^+$ .*
3.  *$\mathbf{a}, \mathbf{w}$  controls  $\mathbf{b}, \mathbf{b} + \mathbf{x}$ .*
4.  *$\mathbf{a}, \mathbf{w}'$  controls  $\mathbf{b}, \mathbf{b} + \mathbf{x}'$ .*
5.  *$\langle \mathbf{w}, \mathbf{x}, \text{lin}(p \text{ rel } e, f) \rangle = \langle \mathbf{w}', \mathbf{x}', \text{lin}(p \text{ rel } e, f) \rangle \leq \mathbf{A}$ .*
6. *In the abelian group*

$$\langle \mathbf{w}, \mathbf{x}, \text{lin}(p \text{ rel } e, f) \rangle / \text{lin}(p \text{ rel } e, f) = \text{lin}_{\text{qc}}(p) / \text{lin}(p \text{ rel } e, f)$$

*the couple  $[\mathbf{w}'], [\mathbf{x}']$  is Nielsen equivalent to a permutation of  $[\mathbf{w}], [\mathbf{x}]$ .*

7. *There is  $\mathbf{r} \in \text{lin}(p \text{ rel } e, f)$  with  $\mathbf{r} \geq \mathbf{0}$  and  $\text{supp}(\mathbf{r}) = \text{supp}_{\text{qc}}(p)$ .*

*Then there is a sequence of slides and swaps changing*

$$\begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w} \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x} \end{cases} \quad \text{into} \quad \begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w}' \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x}' \end{cases}$$

*Proof.* By Lemma 5.13 we can assume that  $\text{supp}(\mathbf{w}) = \text{supp}(\mathbf{w}') = \text{supp}_{\text{qc}}(p)$ . Using slide moves, we can assume that  $\mathbf{x} \geq \mathbf{w} \geq \mathbf{0}$  and thus  $\text{supp}(\mathbf{x}) = \text{supp}_{\text{qc}}((v, \mathbf{a}))$ . Similarly we can assume that  $\mathbf{x}' \geq \mathbf{w}' \geq \mathbf{0}$  and  $\text{supp}(\mathbf{x}') = \text{supp}_{\text{qc}}((v, \mathbf{a}))$ . Up to performing a swap move, we can assume that  $[\mathbf{w}'], [\mathbf{x}']$  is Nielsen equivalent to  $[\mathbf{w}], [\mathbf{x}]$  in  $\langle \mathbf{w}, \mathbf{x}, \text{lin}(p \text{ rel } e, f) \rangle / \text{lin}(p \text{ rel } e, f)$ .

Now using Proposition B.4 of Appendix B we are able to change

$$\begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w} \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x} \end{cases} \quad \text{into} \quad \begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w}' \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x}' \end{cases}$$

□

**Proposition 5.19** (Controlled 2-cliques with finite abelian group). *Let  $p = (v, \mathbf{a}) \in V(\Lambda)$  and let  $\mathbf{w}, \mathbf{w}', \mathbf{b} \in \mathbf{A}^+$  and  $\mathbf{x}, \mathbf{x}' \in \mathbf{A}$  be such that:*

1. *There is  $e \in E(\Lambda)$  given by  $\mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w}$  inside  $\mathbf{A}_v^+$ .*
2. *There is  $f \in E(\Lambda)$  given by  $\mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x}$  inside  $\mathbf{A}_v^+$ .*

3.  $\mathbf{a}, \mathbf{w}$  controls  $\mathbf{b}, \mathbf{b} + \mathbf{x}$ .
4.  $\mathbf{a}, \mathbf{w}'$  controls  $\mathbf{b}', \mathbf{b}' + \mathbf{x}'$ .
5.  $\langle \mathbf{w}, \mathbf{x}, \text{lin}(p \text{ rel } e, f) \rangle = \langle \mathbf{w}', \mathbf{x}', \text{lin}(p \text{ rel } e, f) \rangle \leq \mathbf{A}$ .
6. In the abelian group

$$\langle \mathbf{w}, \mathbf{x}, \text{lin}(p \text{ rel } e, f) \rangle / \text{lin}(p \text{ rel } e, f) = \text{lin}_{\text{qc}}(p) / \text{lin}(p \text{ rel } e, f)$$

the couple  $[\mathbf{w}'], [\mathbf{x}']$  is Nielsen equivalent to a permutation of  $[\mathbf{w}], [\mathbf{x}]$ .

7. The abelian group  $\langle \mathbf{w}, \mathbf{x}, \text{lin}(p \text{ rel } e, f) \rangle / \text{lin}(p \text{ rel } e, f)$  is finite.

Then there is a sequence of slides and swaps changing

$$\begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w} \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x} \end{cases} \quad \text{into} \quad \begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w}' \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x}' \end{cases}$$

*Proof.* By Lemma 5.13 we can assume that  $\text{supp}(\mathbf{w}) = \text{supp}_{\text{qc}}(p)$ . Up to sliding, we can also assume that  $\mathbf{x} \geq \mathbf{w} \geq \mathbf{0}$ , in particular implying that  $\text{supp}(\mathbf{x}) = \text{supp}_{\text{qc}}(p)$ . Similarly, we can assume that  $\mathbf{x}' \geq \mathbf{w}' \geq \mathbf{0}$  and  $\text{supp}(\mathbf{x}') = \text{supp}(\mathbf{w}') = \text{supp}_{\text{qc}}(p)$ . The group

$$\langle \mathbf{w}, \mathbf{x}, \text{lin}(p \text{ rel } e, f) \rangle / \text{lin}(p \text{ rel } e, f) = \langle \mathbf{w}', \mathbf{x}', \text{lin}(p \text{ rel } e, f) \rangle / \text{lin}(p \text{ rel } e, f)$$

is an abelian group generated by (at most) two elements, and thus is isomorphic to one of the following: a finite abelian group,  $\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}, \mathbb{Z}^2$  for some  $n \geq 2$ . By hypothesis it's not virtually cyclic; we now deal with the remaining two cases.

By hypothesis,  $\langle \mathbf{w}, \mathbf{x}, \text{lin}(p \text{ rel } e, f) \rangle / \text{lin}(p \text{ rel } e, f)$  is isomorphic to a finite abelian group, let's say of order  $n \geq 1$ . By hypothesis there is a finite sequence of Nielsen moves changing  $[\mathbf{w}], [\mathbf{x}]$  into either  $[\mathbf{w}'], [\mathbf{x}']$  or  $[\mathbf{x}'], [\mathbf{w}']$ ; since we are working within a finite abelian group, all of these Nielsen moves can be taken to be additions (since, for example, substituting  $[\mathbf{w}]$  with  $[\mathbf{w}] - [\mathbf{x}]$  is the same as substituting  $[\mathbf{w}]$  with  $[\mathbf{w}] + (n-1)[\mathbf{x}]$ ). For every Nielsen move on  $[\mathbf{w}], [\mathbf{x}]$  we can perform the corresponding slide move or reverse slide move (Lemma 3.3) to change  $\mathbf{w}, \mathbf{x}$  accordingly. We obtain  $[\mathbf{w}] = [\mathbf{w}']$  and  $[\mathbf{x}] = [\mathbf{x}']$  (or exchanged, in which case we just perform a swap move). By means of relative slides (Proposition 5.7) and swaps we finally obtain  $\mathbf{w} = \mathbf{w}'$  and  $\mathbf{x} = \mathbf{x}'$ , as desired.  $\square$

## 5.7 Examples

As usual, in the examples we use  $\mathbf{A} = \mathbb{Z}^{\mathcal{P}(\Gamma, \psi)}$ , omitting the  $\mathbb{Z}/2\mathbb{Z}$  summand.

*Example 5.20.* We can now describe the isomorphism problem for the GBS graph of Example 2.30. Using the same notation as in Example 2.30, call  $(\Gamma, \psi)$  the GBS graph, with a single vertex  $v$ , and  $\Lambda$  its affine representation. The edges of  $\Lambda$  are partitioned into four quasi-conjugacy classes  $C_1, C_2, C_3, C_4$ , see Figure 14, which had been described in Example 2.30. Suppose that we try and perform a sequence of slides, swaps, connections starting from this GBS graph.

The quasi-conjugacy class  $C_1$  contains only the edge  $(1, 3) \text{ --- } (3, 2)$ , and thus by Corollary 4.10 that edge can't be changed. In the quasi-conjugacy classes  $C_3, C_4$ , the two notions of conjugacy and external equivalence coincide, and thus by Corollary 4.11 we can change the endpoints of the edges at will using external slides, and that's a complete list of all configurations that can be reached with slides, swaps, connections.

The quasi-conjugacy class  $C_2$  contains two edges, belonging to two distinct conjugacy classes, and one minimal region  $M_2 = \{(4, 0)\}$ ; we have  $\text{lin}_{\text{qc}}(C_2) = \langle (2, 0) \rangle$ . For simplicity of notation, call  $\mathbf{a} = (4, 0)$  and  $\mathbf{u} = (2, 0)$ . Let's call  $e$  the edge belonging to the conjugacy class of  $(4, 0)$ : by Lemma 4.17 the edge  $e$  must always be given by  $(4, 0) \text{ --- } (4 + 2a, 0)$  for some integer  $a \geq 0$ . If  $a = 0$  then  $M_2$  would be a quasi-conjugacy class by itself, contradiction, so we also always have  $a \neq 0$ . Let's call  $f$  the edge belonging to the other conjugacy class: the edge  $f$  must always be given by  $(5 + 2b, 0) \text{ --- } (5 + 2c, 0)$  for some integers  $b, c \geq 0$ . Finally, we must always have that

$\langle (2a, 0), (2b - 2c, 0) \rangle = \langle (2, 0) \rangle$ , otherwise the linear algebra of  $C_2$  wouldn't be the correct one. On the other hand, by Proposition 5.17, every configuration satisfying the above conditions can actually be reached. To summarize, the configurations for  $C_2$ , that we can reach with a sequence of slides, swaps, connections, are exactly the ones such that:

1.  $e$  is given by  $(4, 0) \text{ --- } (4 + 2a, 0)$  for some integer  $a \geq 1$ .
2.  $f$  is given by  $(5 + 2b, 0) \text{ --- } (5 + 2c, 0)$  for some integers  $b, c \geq 0$ .
3.  $(a, b - c) = 1$ .

Finally, we take into account the (finitely many) sign-changes, and we observe that induction moves don't apply to this example. This gives algorithmic solution to the isomorphism problem for an arbitrary GBS graph against this specific one.

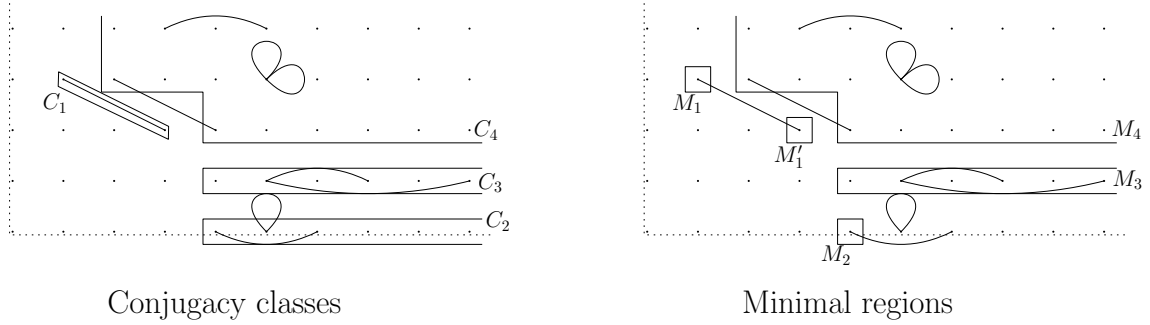


Figure 14: TO DO

*Example 5.21.* We can now describe the isomorphism problem for the GBS graph of Example 4.20. Using the same notation as in Example 4.20, call  $(\Gamma, \psi)$  the GBS graph, with a single vertex  $v$ , and  $\Lambda$  its affine representation. The quasi-conjugacy classes  $B, C$  (which are the only ones containing edges) and the minimal regions have already been described in Example 4.20, see Figure 13. Suppose we try and change the GBS graph by means of slides, swaps, connections. The quasi-conjugacy class  $B$  contains only one edge  $(1, 0) \text{ --- } (4, 0)$ , and thus by Corollary 4.10 that edge will always stay there. The quasi-conjugacy class  $C$  is characterized by  $\min_{qc}(C) = \{(1, 1), (0, 3)\}$  and  $\text{supp}_{qc}(C) = \{2, 3\}$  and  $\text{lin}_{qc}(C) = \langle (3, 0), (2, 2) \rangle$ . In particular  $C$  is partitioned into six conjugacy classes; the three edges belong to three different conjugacy classes.

Suppose that we are given another GBS graph  $(\Gamma', \psi')$  with the same conjugacy classes, and the same number of edges in each conjugacy class. Of course we must have an edge  $(1, 0) \text{ --- } (4, 0)$  in  $B$ . Let's call  $e_1, e_2, e_3$  the other three edges, with  $e_1 \sim_c (2, 1)$  and  $e_2 \sim_c (3, 0)$  and  $e_3 \sim_c (2, 2)$ . We observe that, by Lemma 4.17, there must always be at least one edge with an endpoint in  $M_2$ , and by looking at the conjugacy classes, this edge must be  $e_2$ . We deal with cases.

CASE 1: The other endpoint of  $e_2$  falls inside  $M_1$ . In this case  $e_2$  must be given by  $(0, 3) \text{ --- } (a, 1)$  for some integer  $a \geq 1$ , and up to external slide we can assume that  $e_2$  is given by  $(0, 3) \text{ --- } (1, 1)$ . If we now keep  $e_2$  fixed, we can move the endpoints of  $e_1, e_3$  at will inside their conjugacy class. In particular, we can arrive at the configuration with  $e_1$  given by  $(1, 3) \text{ --- } (2, 1)$  and  $e_3$  given by  $(1, 4) \text{ --- } (2, 2)$ , from which we can reach  $(\Gamma, \psi)$ .

CASE 2:  $e_2$  is given by  $(0, 3) \text{ --- } (a, b)$  for some integers  $(a, b) \geq (1, 3)$ . By Lemma 4.17, we obtain that  $e_1$  must have one endpoint inside  $M_1$ . It can't have both endpoints inside  $M_1$ , otherwise  $M_1$  would be a quasi-conjugacy class by itself, contradiction; up to external slide, we can assume that  $e_1$  is of the form  $(2, 1) \text{ --- } (c, d)$  for some integers  $(c, d) \geq (0, 3)$ . Up to external slide, we can obtain  $(a, b) \geq (0, 3), (2, 1)$ , and up to sliding  $e_1$  along  $e_2$ , we can obtain  $(c, d) \geq (0, 3), (2, 1)$ ; thus by Lemma 5.12 we can make them into a full-support 2-clique. Now it's easy to manipulate  $e_1, e_2, e_3$  in order to make them into a full-support 3-clique. Thus by Theorem 5.14 we can pass from one configuration to the other - here we are using that the quotient  $\text{lin}_{qc}(C)/\text{lin}(C \text{ rel } e_1, e_2, e_3)$  is 2-generated, and thus every triple of generators is Nielsen equivalent to any other. Thus, from all of these configurations we can reach  $(\Gamma, \psi)$ .

CASE 3:  $e_2$  is given by  $(0, 3) \text{ --- } (0, a)$  for some integer  $a \geq 3$ . We can't have  $a = 3$ , otherwise  $M_2$  would be a quasi-conjugacy class by itself, contradiction; thus  $a \geq 4$ . One of  $e_1, e_3$  must be of the form  $(0, b) \text{ --- } (c, d)$  for integers  $b \geq 3$  and  $(c, d) \geq (1, 1)$ , otherwise the set  $\{(0, k) : k \geq 3\}$  would be a (union of) quasi-conjugacy class(es) by itself, contradiction. But then we can perform a connection move and reduce ourselves to case 2.

To summarize, a GBS graph can be obtained from  $(\Gamma, \psi)$  by means of a sequence of slides, swaps, connections, if and only if it has the same conjugacy classes and the same number of edges in each conjugacy class as  $(\Gamma, \psi)$ . This characterizes the isomorphism problem for  $(\Gamma, \psi)$ . As usual, it remains to check the (finitely many) sign-changes. Note that induction doesn't apply in this example.

*Example 5.22.* Consider the three GBS graphs  $(\Gamma_1, \psi_1), (\Gamma_2, \psi_2), (\Gamma_3, \psi_3)$  as in Figure 15, with affine representations  $\Lambda_1, \Lambda_2, \Lambda_3$  respectively. It's easy to see that the three GBS graphs have the same set of primes  $\mathcal{P} = \{2, 3, 5\}$ , the same conjugacy classes and external equivalence classes, and the same number of edges belonging to each conjugacy class.

To be precise, there is a conjugacy class  $B$  satisfying  $\min_{\text{qc}}(B) = \{(3, 0, 0)\}$  and  $\text{supp}_{\text{qc}}(B) = \{2\}$  and  $\text{lin}_{\text{qc}}(B) = \langle (7, 0, 0) \rangle$ ; this conjugacy class contains one edge  $(3, 0, 0) \text{ --- } (10, 0, 0)$  that can't be changed along any sequence of slides, swaps, connections.

There is a conjugacy class  $C$  satisfying  $\min_{\text{qc}}(C) = \{(2, 1, 0), (1, 2, 0), (0, 3, 0)\}$  and  $\text{supp}_{\text{qc}}(C) = \{2, 3, 5\}$  and  $\text{lin}_{\text{qc}}(C) = \langle (1, 0, 0), (0, 1, 0), (0, 0, 1) \rangle$ . The three minimal points are also the three minimal regions  $M_1 = \{(2, 1, 0)\}$  and  $M_2 = \{(1, 2, 0)\}$  and  $M_3 = \{(0, 3, 0)\}$ . This quasi-conjugacy class also coincides with a conjugacy class, and contains three edges,  $e_1$  given by  $(2, 1, 0) \text{ --- } (2, 1, 0) + \mathbf{x}_1$ , and  $e_2$  given by  $(1, 2, 0) \text{ --- } (1, 2, 0) + \mathbf{x}_2$ , and  $e_3$  given by  $(0, 3, 0) \text{ --- } (0, 3, 0) + \mathbf{x}_3$ , for some  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbf{A}$  which are different in the three examples.

The question is: which ones of these GBS graphs represents groups isomorphic to each other? The edges  $e_1, e_2, e_3$  form a clique, and thus we can apply Theorem 5.14 to try and go from one configuration to another. However, we need to check the Nielsen equivalence type of the (ordered) triple  $[\mathbf{x}_1], [\mathbf{x}_2], [\mathbf{x}_3]$  in the quotient

$$\frac{\text{lin}_{\text{qc}}(C)}{\text{lin}(C \text{ rel } e_1, e_2, e_3)} = \frac{\langle (1, 0, 0), (0, 1, 0), (0, 0, 1) \rangle}{\langle (7, 0, 0) \rangle} \cong \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}^2.$$

By Proposition A.5 we need to look at the determinant of the corresponding matrix modulo 7. The three matrices for  $(\Gamma_1, \psi_1), (\Gamma_2, \psi_2), (\Gamma_3, \psi_3)$  are given by

$$\begin{pmatrix} 3 & 2 & 2 \\ 7 & 5 & 4 \\ 6 & 4 & 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 6 & 4 & 4 \\ 7 & 5 & 4 \\ 6 & 4 & 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 9 & 6 & 6 \\ 7 & 5 & 4 \\ 6 & 4 & 5 \end{pmatrix}$$

respectively, and have determinants congruent to 1, 2, 3 modulo 7 respectively. So Theorem 5.14 doesn't allow us to jump from one to the other.

And in fact the three groups are pairwise non-isomorphic. Suppose that we start from one of  $(\Gamma_1, \psi_1), (\Gamma_2, \psi_2), (\Gamma_3, \psi_3)$  and we perform a sequence of slides, swaps, connections. By Lemma 4.17, in each moment there must be at least one endpoint of some edge at  $(2, 1, 0)$ , one at  $(1, 2, 0)$ , and one at  $(0, 3, 0)$ . We would like to deduce that the configuration can be written uniquely as

$$\begin{cases} (2, 1, 0) \text{ --- } (2, 1, 0) + \mathbf{x}_1 \\ (1, 2, 0) \text{ --- } (1, 2, 0) + \mathbf{x}_2 \\ (0, 3, 0) \text{ --- } (0, 3, 0) + \mathbf{x}_3 \end{cases}$$

for some  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbf{A}$ . This is true, even though one has to be careful in to the degenerate cases where some edge connects two minimal regions. To be precise, there must be some edge with one endpoint in  $\{(2, 1, 0), (1, 2, 0), (0, 3, 0)\}$  and the other endpoint outside; without loss of generality, assume it has an endpoint in  $(0, 3, 0)$ . Then we call such edge  $(0, 3, 0) \text{ --- } (0, 3, 0) + \mathbf{x}_3$ , uniquely determining  $\mathbf{x}_3$ . Now there must be some edge with one endpoint in  $\{(2, 1, 0), (1, 2, 0)\}$  and the other endpoint outside; without loss of generality, assume it has an endpoint in  $(1, 2, 0)$ . Then we call such edge  $(1, 2, 0) \text{ --- } (1, 2, 0) + \mathbf{x}_2$ , uniquely determining  $\mathbf{x}_2$ . Finally, with a similar reasoning we uniquely determine  $\mathbf{x}_1$ .



Now we write the three vectors  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$  as columns of a  $3 \times 3$  matrix, and we compute the determinant modulo 7. This is invariant by external slides (since the determinant is taken modulo 7), by internal slides (which correspond to column operations on the matrix), by connections (which correspond to exchanging two columns, changing the sign of one column, and then performing a few column operation), and swaps can never be applied. It follows that the determinant modulo 7 must be the same along any sequence of slides, swaps, connections. Since in the three graphs  $(\Gamma_1, \psi_1), (\Gamma_2, \psi_2), (\Gamma_3, \psi_3)$  the determinant has different values, there is no sequence of slides, swaps, connections going from one of them to another. Finally, one easily check that sign-changes can't help finding an isomorphism in this case. Induction doesn't apply to these examples.

So these are three GBS graphs which exhibit a very similar behaviour - same conjugacy classes, same number of edges in each conjugacy class - but are not isomorphic; and the isomorphism problem in this case is decided by a finer invariant, which is this determinant modulo 7. In this specific case, if the determinant modulo 7 would have been the same, then we would have immediately been able to provide an isomorphism using Theorem 5.14. As we will see in the next sections TO DO, this is not always the case.

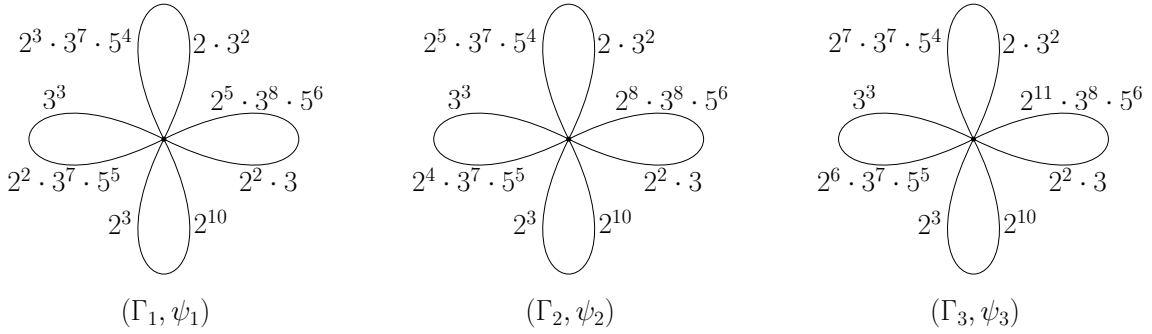


Figure 15: TO DO

*Example 5.23.* Consider the GBS graph  $(\Gamma, \psi)$  as in Figure 16, with affine representation  $\Lambda$ . The graph  $\Gamma$  has two vertices  $v, u$  and five edges  $e_1, e_2, e_3, e_4, e_5$ , which in the affine representation are given by

$$\begin{cases} (v, (3, 0)) \text{ --- } (v, (4, 3)) \\ (v, (1, 2)) \text{ --- } (v, (5, 2)) \\ (v, (3, 3)) \text{ --- } (u, (1, 1)) \\ (u, (4, 0)) \text{ --- } (u, (5, 3)) \\ (u, (0, 2)) \text{ --- } (u, (4, 4)) \end{cases}$$

respectively. There is a conjugacy class  $C$ , which has minimal points  $(v, (3, 0)), (v, (1, 2)), (u, (4, 0)), (u, (1, 1)), (u, (0, 2))$  and  $\text{supp}_{\text{qc}}(C) = \{2, 3\}$  and  $\text{lin}_{\text{qc}}(C) = \langle (2, 0), (1, 1) \rangle$ , and which contains all five the edges.

The two edges  $e_1, e_2$  form a 2-clique inside  $\mathbf{A}_v^+$ , and the two edges  $e_4, e_5$  form a 2-clique inside  $\mathbf{A}_u^+$ . These two 2-cliques can be written as

$$\begin{cases} (v, (3, 0)) \text{ --- } (v, (4, 3)) + \mathbf{x}_1 \\ (v, (1, 2)) \text{ --- } (v, (5, 2)) + \mathbf{x}_2 \end{cases} \quad \text{and} \quad \begin{cases} (u, (4, 0)) \text{ --- } (u, (5, 3)) + \mathbf{x}_4 \\ (u, (0, 2)) \text{ --- } (u, (4, 4)) + \mathbf{x}_5 \end{cases}$$

where  $\mathbf{x}_1 = (1, 3)$  and  $\mathbf{x}_2 = (4, 0)$  and  $\mathbf{x}_4 = (1, 3)$  and  $\mathbf{x}_5 = (4, 2)$ . The two cliques belong to the same quasi-conjugacy class, and thus they can play together (according to Theorem 5.16) as if they were a single 4-clique. This means that we can change  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4, \mathbf{x}_5$  all together at will using Nielsen transformations; provided that we respect some inequality constraints - but the constraints aren't very relevant, according to Theorem B.1.

For example,  $e_2$  is given by  $(v, (1, 2)) \text{ --- } (v, (5, 2))$ , and suppose that we want to change it into  $(v, (1, 2)) \text{ --- } (v, (5, 2)) + (4, 2)$ . We observe that  $(4, 2)$  is exactly the difference between the two endpoints of  $e_5$ . Thus we slide the endpoint of  $e_2$  along the affine path  $e_1 e_3 e_5 \bar{e}_3 \bar{e}_1$  and this changes

the endpoint of  $e_2$  from  $(v, (5, 2))$  to  $(v, (6, 5))$  to  $(u, (4, 3))$  to  $(u, (4, 3)) + (4, 2)$  to  $(v, (6, 5)) + (4, 2)$  to  $(v, (5, 2)) + (4, 2)$  as desired. This procedure can be performed in general, and is described by the relative slide move of Proposition 5.7.

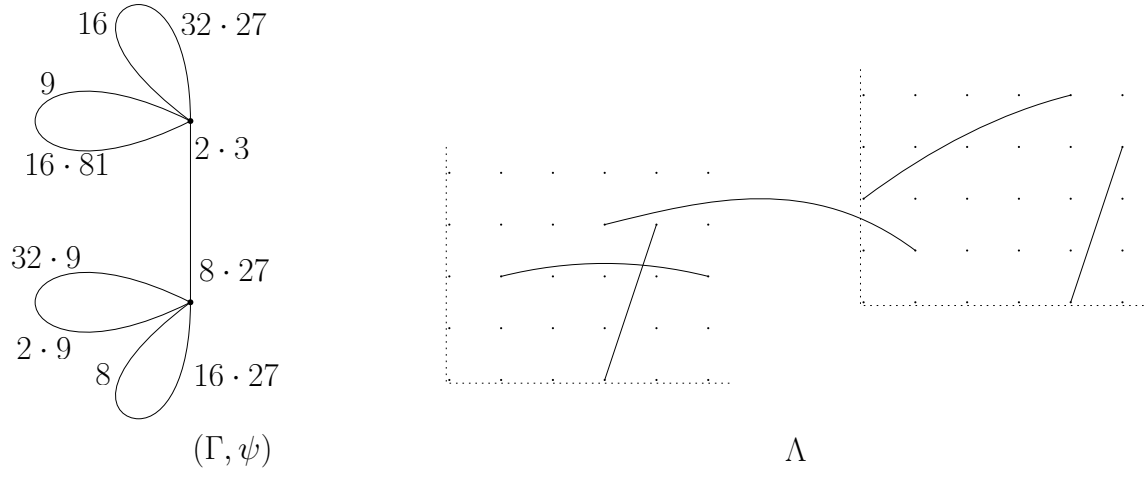


Figure 16: TO DO

## 6 GBSs with one qc-class and full-support gaps

The aim of this section is to classify GBS graphs with one qc-class and full-support gaps.

### 6.1 One qc-class

**Definition 6.1.** Let  $(\Gamma, \psi)$  be a GBS graph with affine representation  $\Lambda$ . We say that  $(\Gamma, \psi)$  has **one qc-class** if all of its edges belong to the same quasi-conjugacy class.

Note that the property of having one qc-class is an isomorphism invariant. In this case the external equivalence relation is trivial in that quasi-conjugacy class, and in particular the minimal regions are essentially points (to be formal, couples of points, with only the  $\mathbb{Z}/2\mathbb{Z}$  component changing). Along this section, we will abuse notation and talk about minimal points instead of minimal regions.

In what follows, we will also assume that the quasi-conjugacy class has non-trivial quasi-conjugacy support: for a quasi-conjugacy class  $Q$  with  $\text{supp}_{\text{qc}}(Q) = \emptyset$ , the isomorphism problem can be solved using Corollary 4.12.

### 6.2 Graph of families

Let  $(\Gamma, \psi)$  be a totally reduced GBS graph with affine representation  $\Lambda$ . Suppose that  $(\Gamma, \psi)$  has one qc-class  $Q$ . Then we will consider the graph of families  $\mathcal{F}(Q)$  as defined in Section 2.7. In particular we have the following properties:

1. Each minimal point of  $Q$  belongs to exactly one family. Each family contains at least one minimal point. Therefore, the finite set of minimal points of  $Q$  is partitioned into  $|V(\mathcal{F}(Q))|$  non-empty subsets.
2. According to Lemma 2.23, we have exactly one edge in  $\mathcal{F}(Q)$  for every edge in  $\Gamma$ . In other words, we have that  $E(\mathcal{F}(Q)) = E(\Gamma) = E(\Lambda)$ .
3. We have the modular homomorphism  $q_Q : \pi_1(\mathcal{F}(Q)) \rightarrow \mathbf{A}$  as defined in Section 2.7.

**Definition 6.2.** A family  $F \in V(\mathcal{F}(Q))$  is called **individual** if it contains exactly one minimal point, and **non-individual** otherwise.

*Remark 6.3.* In the particular case when  $\text{supp}_{\text{qc}}(Q) = \mathcal{P}(\Gamma, \psi)$ , the graph of families is exactly  $\mathcal{F}(Q) = \Gamma$ . For simplicity, the reader can go through this whole section with this extra hypothesis in mind (this extra assumption won't make any of the arguments easier).

### 6.3 One endpoint at each minimal point

Let  $(\Gamma, \psi)$  be a totally reduced GBS graph with affine representation  $\Lambda$ . Suppose that  $(\Gamma, \psi)$  has one qc-class  $Q$  with  $\text{supp}_{\text{qc}}(Q) \neq \emptyset$ .

**Definition 6.4.** We say that  $(\Gamma, \psi)$  is **clean** if every minimal point contains exactly one endpoint of exactly one edge, and no edge has both endpoints at minimal points.

If  $\text{supp}_{\text{qc}}(Q) = \emptyset$ , then every endpoint of every edge must lie at a minimal point, and thus the isomorphism problem is described through finitely many GBS graphs. If instead  $\text{supp}_{\text{qc}}(Q) \neq \emptyset$ , then we can arrange for the GBS graph to be clean, as we now explain.

Suppose that  $\text{supp}_{\text{qc}}(Q) \neq \emptyset$ . Define the graph  $\mathcal{L}$  as follows:

- $V(\mathcal{L})$  is the set of minimal points of  $(\Gamma, \psi)$ , together with an extra vertex  $v_{\text{floating}}$ .
- $E(\mathcal{L}) = E(\Lambda)$ , with the same reverse map. For  $e \in E(\Lambda)$ , if  $\tau_\Lambda(e)$  is a minimal point then we set  $\tau_{\mathcal{L}}(e) = \tau_\Lambda(e)$ , otherwise we set  $\tau_{\mathcal{L}}(e) = v_{\text{floating}}$ .

Note that loops in  $\mathcal{L}$  at  $v_{\text{floating}}$  correspond to floating edges in  $(\Gamma, \psi)$ . Since  $(\Gamma, \psi)$  has one qc-class  $Q$ , we must have that  $\mathcal{L}$  is connected. Note that  $(\Gamma, \psi)$  is clean if and only if in  $\mathcal{L}$  every minimal point  $m$  has valence one, and the only edge at  $m$  connects  $m$  to  $v_{\text{floating}}$ .

We now define a *clean projection* from  $(\Gamma, \psi)$  to a clean GBS graph. Choose a maximal tree of  $\mathcal{L}$ . Define the GBS graph  $(\Gamma', \psi')$  as follows. If  $e \in E(\mathcal{L})$  is an edge which doesn't lie in the maximal tree, then we slide  $\tau(e)$  along the maximal tree until it ends up at  $v_{\text{floating}}$ . We do this for all endpoints of all edges that don't lie in the chosen maximal tree. Now, if  $e \in E(\mathcal{L})$  lies in the maximal tree, we consider its two endpoints  $\iota(e), \tau(e)$ , we take the one that lies nearer to  $v_{\text{floating}}$ , and we slide it along the maximal tree until it ends up at  $v_{\text{floating}}$ . We do this for all edges that lie in the chosen maximal tree. The result of this procedure is a clean GBS graph  $(\Gamma', \psi')$ .

**Lemma 6.5.** *The above clean projection has the following properties:*

1. *For a fixed maximal tree, performing the slide moves in a different order will produce the same clean GBS graph  $(\Gamma', \psi')$ .*
2. *For different choices of maximal trees, producing clean GBS graphs  $(\Gamma', \psi'), (\Gamma'', \psi'')$ , there is a sequence of edge sign-changes and slides going from  $(\Gamma', \psi')$  to  $(\Gamma'', \psi'')$  and such that every GBS graph along the sequence is clean.*

*Proof.* This is an easy check with the definitions. □

**Lemma 6.6.** *Let  $(\Gamma, \psi), (\Delta, \phi)$  be related by a edge sign-change, slide, swap or connection; let  $(\Gamma', \psi'), (\Delta', \phi')$  be their respective clean projections. Then there is a sequence of edge sign-changes, slides, swaps, connections passing from  $(\Gamma', \psi')$  to  $(\Delta', \phi')$  and going only through clean GBS graphs.*

*Proof.* This is an easy check with the definitions. □

Thanks to the above Lemma 6.5 and Lemma 6.6, for the rest of this section we will be able to assume that our GBS graphs are clean, and that all the sequences of moves go only through clean GBS graphs.

In particular, if  $(\Gamma, \psi)$  is a GBS graph with one qc-class  $Q$ , and if  $\text{supp}_{\text{qc}}(Q) \neq \emptyset$ , then a strong version of Lemma 4.17 holds. To be precise, up to taking the clean projection of  $(\Gamma, \psi)$ , for every minimal point  $m$  we have exactly one edge with exactly one endpoint at  $m$ , and all the other edges are floating edges. In particular, a clean GBS graph  $(\Gamma, \psi)$  has at least as many edges as minimal points; moreover, the inequality is strict if and only if  $(\Gamma, \psi)$  has floating edges. This motivates the following definition.

**Definition 6.7.** *Let  $(\Gamma, \psi)$  be a GBS graph with one qc-class  $Q$ , and with  $\text{supp}_{\text{qc}}(Q) \neq \emptyset$ . We say that  $(\Gamma, \psi)$  has **floating pieces** if it has strictly more edges than minimal points.*

TO DO maybe it's more convenient to take some time here and explain the structure of the clean projection. Basically, all the rank between the minimal points is converted into loops based at the other endpoint of the "escaping edges". If the maximal tree contains several escaping edges for the same connected component of minimal points, then there will also be an edge connecting the two "escaping points". Maybe a figure might help.

## 6.4 One endpoint in each individual family

Let  $(\Gamma, \psi)$  be a totally reduced GBS graph with affine representation  $\Lambda$ . Suppose that  $(\Gamma, \psi)$  has one qc-class  $Q$  with  $\text{supp}_{\text{qc}}(Q) \neq \emptyset$ .

**Definition 6.8.** *If  $F \in V(\mathcal{F}(Q))$  is an individual family, then we say that  $(\Gamma, \psi)$  is  **$F$ -polished** if  $F$  contains exactly one endpoint of one edge.*

Let  $F \in V(\mathcal{F}(Q))$  be an individual family, and let  $m$  be the unique minimal point in  $F$ . A **set of  $F$ -polishing data** is given by pairwise distinct edges  $e_1, \dots, e_n, e$  satisfying the following properties:

1. For  $j = 1, \dots, n$  we have that  $e_j$  goes from  $m + \mathbf{a}_j$  to  $m + \mathbf{b}_j$ .

2. For  $j = 1, \dots, n$  we have  $\text{supp}(\mathbf{a}_j) \subseteq (\text{supp}(\mathbf{b}_{j-1}) \cup \dots \cup \text{supp}(\mathbf{b}_1))$ .
3. The edge  $e$  goes from  $m + \mathbf{c}$  to  $p$  for some  $p \not\geq m$ .
4. We have  $\text{supp}(\mathbf{c}) \subseteq (\text{supp}(\mathbf{b}_n) \cup \dots \cup \text{supp}(\mathbf{b}_1))$ .

If  $p$  belongs to a family  $F' \in V(\mathcal{F}(Q))$ , then we say that the set of polishing data is **pointing to**  $F'$ . As in [ACRK25b] we can choose natural numbers  $k_1, \dots, k_n$  big enough, and define

$$\mathbf{w}_j = (\mathbf{b}_j - \mathbf{a}_j) + k_{j-1}(\mathbf{b}_{j-1} - \mathbf{a}_{j-1}) + k_{j-1}k_{j-2}(\mathbf{b}_{j-2} - \mathbf{a}_{j-2}) + \dots + k_{j-1}k_{j-2} \dots k_1(\mathbf{b}_1 - \mathbf{a}_1)$$

for  $j = 1, \dots, n$ , and  $\mathbf{w} = k_n \mathbf{w}_n - \mathbf{c}$ . This allows us to define an  $F$ -**polishing procedure** that changes the edges

$$\left\{ \begin{array}{l} m + \mathbf{a}_1 \text{ --- } m + \mathbf{b}_1 \\ m + \mathbf{a}_2 \text{ --- } m + \mathbf{b}_2 \\ m + \mathbf{a}_3 \text{ --- } m + \mathbf{b}_3 \\ \dots \\ m + \mathbf{a}_n \text{ --- } m + \mathbf{b}_n \\ m + \mathbf{c} \text{ --- } p \end{array} \right. \quad \text{into} \quad \left\{ \begin{array}{l} m + \mathbf{a}_1 \text{ --- } p + \mathbf{w} + \mathbf{a}_1 \\ p + \mathbf{w} + \mathbf{a}_2 \text{ --- } p + \mathbf{w} + \mathbf{a}_2 + \mathbf{w}_1 \\ p + \mathbf{w} + \mathbf{a}_3 \text{ --- } p + \mathbf{w} + \mathbf{a}_3 + \mathbf{w}_2 \\ \dots \\ p + \mathbf{w} + \mathbf{a}_n \text{ --- } p + \mathbf{w} + \mathbf{a}_n + \mathbf{w}_{n-1} \\ p \text{ --- } p + \mathbf{w}_n \end{array} \right.$$

and each other endpoints of the form  $m + \mathbf{d}$  into  $p + \mathbf{w} + \mathbf{d}$ .

In the same way as in [ACRK25b], we can see that for  $k_1, \dots, k_n$  big enough the  $F$ -polishing procedure can be obtained through a sequence of moves, giving an  $F$ -polished GBS graph as a result.

**Lemma 6.9.** *For different choices of the set of  $F$ -polishing data and of big enough constants, producing  $F$ -polished GBS graphs  $(\Gamma', \psi'), (\Gamma'', \psi'')$ , there is a sequence of edge sign-changes, slides, swaps, connections going from  $(\Gamma', \psi')$  to  $(\Gamma'', \psi'')$  and such that every GBS graph along the sequence is  $F$ -polished.*

*Proof.* Analogous to [ACRK25b]. □

**Lemma 6.10.** *Let  $(\Gamma, \psi), (\Delta, \psi)$  be related by an edge sign-change, slide, swap or connection; let  $(\Gamma', \psi'), (\Delta', \psi')$  be corresponding  $F$ -polished GBS graphs. Then there is a sequence of edge sign-changes, slides, swaps, connections passing from  $(\Gamma', \psi')$  to  $(\Delta', \psi')$  and going only through  $F$ -polished GBS graphs.*

*Proof.* Analogous to [ACRK25b]. □

Note that, if  $F, G \in V(\mathcal{F}(Q))$  are two individual families, and if  $(\Gamma, \psi)$  is  $G$ -polished, then there must be a set of  $F$ -polishing data pointing to some  $F' \neq G$  (unless  $F, G$  are the only families). If we use such a set of data to perform the above procedure, the result will be  $G$ -polished. Moreover, if we restrict only to set of data pointing to families  $F' \neq G$ , then Lemma 6.9 and Lemma 6.10 still hold, producing sequences of moves going only through  $G$ -polished GBS graphs. In particular, in what follows we can assume that all our GBS graphs are  $F$ -polished with respect to all individual families  $F \in V(\mathcal{F}(Q))$ , and we can consider only sequences of moves going through  $F$ -polished GBS graphs.

*Remark 6.11.* The  $F$ -polishing procedure is also compatible with the notion of clean. All of the above discussion, Lemma 6.9 and Lemma 6.10 still hold if we add the additional requirement that all GBS graphs appearing are clean. In particular, we can restrict our attention to GBS graphs which are clean and  $F$ -polished for all individual families  $F \in V(\mathcal{F}(Q))$ , and to sequences of moves going only through GBS graphs of this kind.

## 6.5 Full-support gaps

Let  $(\Gamma, \psi)$  be a GBS graph with affine representation  $\Lambda$ . Suppose that we are given two vertices  $p, q \in V(\Lambda)$  with  $q \geq p$ : then we can write  $q = p + \mathbf{w}$  for a unique  $\mathbf{w} \in \mathbf{A}^+$ , which we call the *gap* from  $p$  to  $q$ . We say that the gap from  $p$  to  $q$  is *full-support* if  $\text{supp}(\mathbf{w}) = \text{supp}_{\text{qc}}(p)$  (which in particular implies that  $q \sim_{\text{qc}} p$ ).

**Definition 6.12.** Let  $(\Gamma, \psi)$  be a GBS graph and let  $Q$  be a quasi-conjugacy class. We say that  $Q$  has **full-support gaps** if for every edge  $e$  in  $Q$  we have that either  $\tau(e)$  lies in a minimal region, or  $\tau(e)$  lies above a minimal point of  $Q$  with full-support gap.

This means that, whenever an end-point of an edge that falls strictly above a minimal region, the gap can be taken to have the full quasi-conjugacy support. Note that the property of being full-support gap isn't an isomorphism invariant, nor invariant under performing moves.

## 6.6 GBSs graphs with floating pieces

Let  $(\Gamma, \psi)$  be a totally reduced GBS graph with affine representation  $\Lambda$ . Suppose that  $(\Gamma, \psi)$  has one qc-class  $Q$  with  $\text{supp}_{\text{qc}}(Q) \neq \emptyset$ , and that  $(\Gamma, \psi)$  is clean. Suppose that  $(\Gamma, \psi)$  has floating pieces.

We choose a maximal tree for the graph of families  $\mathcal{F}(Q)$  and we call  $e_1, \dots, e_r$  the edges outside the maximal tree (with the convention that, if we consider an edge, we don't consider its reverse). For  $i = 1, \dots, r$ , the edge  $e_i$  determines a unique reduced loop  $\sigma_i$  in  $\mathcal{F}(Q)$ , that crosses  $e_i$  once and then goes back through the maximal tree. We define  $\mathbf{x}_i = q_Q(\sigma_i)$  for  $i = 1, \dots, r$ .

**Definition 6.13.** We define the **linear invariant** of  $Q$  as the  $r$ -tuple  $\mathbf{x}_1, \dots, \mathbf{x}_r \in \text{lin}_{\text{qc}}(Q)$  considered up to Nielsen equivalence and permutation.

Taking  $\bar{e}_i$  instead of  $e_i$  or taking the  $e_i$  in a different order will give the same linear invariant. Similarly, a different choice of maximal tree for  $\mathcal{F}(Q)$  will give the same linear invariant. Edge sign-changes, slides, swaps, connections don't change the linear invariant either. This motivates the terminology *invariant*. Note that  $\mathbf{x}_1, \dots, \mathbf{x}_r$  are generators for  $\text{lin}_{\text{qc}}(Q)$ , which is isomorphic to either  $\mathbb{Z}^k$  or  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^k$  for some  $k \in \mathbb{N}$ . Thus in this case all  $k$ -tuples of generators for  $\text{lin}_{\text{qc}}(Q)$  are Nielsen equivalent to each other up to permutation.

We now define a normal form for GBS graphs with floating pieces. Fix a family  $F_0 \in V(\mathcal{F}(Q))$ . For every family  $F \neq F_0$  we fix a minimal point  $m_F$  in that family.

**Definition 6.14** (Normal form for GBSs with floating pieces). We say that  $(\Gamma, \psi)$  is in **normal form** (with respect to the family  $F_0$  and the minimal points  $\{m_F\}_{F \neq F_0}$ ) if the following conditions hold:

1. For every family  $F \neq F_0$  and for the minimal point  $m_F$ , take the unique edge  $e$  with  $\iota(e) = m_F$ . We require that  $\tau(e)$  is in  $F_0$ , and lies above all minimal points of  $F_0$  with full-support gap.
2. For every family  $F$  and for every minimal point  $m$  (except  $m_F$ ), take the unique edge  $e$  with  $\iota(e) = m$ . We require that  $\tau(e)$  is in  $F$ , and lies above all minimal points of  $F$  with full-support gap.
3. For every floating edge  $e$ , we require that  $\iota(e), \tau(e)$  are in  $F_0$ , and lie above all minimal points of  $F_0$  with full-support gap.

**Proposition 6.15.** Suppose that  $(\Gamma, \psi)$  has full-support gaps. Then, for every family  $F_0$  and minimal points  $\{m_F\}_{F \neq F_0}$ , we can bring  $(\Gamma, \psi)$  to normal form by means of a sequence of slides, swaps, connections.

*Proof.* For every minimal point  $m$ , let  $e_m$  be the unique edge such that  $\iota(e_m) = m$ . We say that  $e_m$  is *positive* if  $\tau(e_m)$  is above  $m$  with full-support gap.

In steps 1 and 2 we maximize the number of positive edges, and we observe that the remaining edges at the minimal points for a tree in the graph of families  $\mathcal{F}(Q)$ . In step 3 we change the

tree, and we assume that all non-positive edges at the minimal points point at  $F_0$ . In step 4, we make all positive edges very long (taking advantage of floating edges). In step 5, we arrange for the non-positive edges to be at the prescribed minimal points  $m_F$ .

STEP 1: If  $e_m$  isn't positive, then we choose a minimal point  $m' \neq m$  such that  $\tau(e_m)$  is above  $m'$  with full-support gap. If  $e_{m'}$  isn't positive, then we choose  $m''$  such that  $\tau(e_{m'})$  is above  $m''$  with full-support gap; and so on. If the sequence doesn't terminate with a positive edge, then we find a cycle, and by sliding one edge of the cycle along the others, we increase the number of positive edges. Note that the hypothesis of having full-support gaps is preserved.

We reiterate this argument until, for every sequence  $m, m', m'', \dots$ , the sequence eventually ends at some  $m^{(k)}$  with  $e_{m^{(k)}}$  positive. At this point, we can slide  $\tau(e_m)$  along  $e_{m'}, e_{m''}, \dots$  in order to get that  $\tau(e_m)$  is above  $m^{(k)}$  with full-support gap. Similarly for all the other non-positive edges

Thus we can assume that, for all minimal points  $m$ , either the edge  $e_m$  is positive, or  $\tau(e_m)$  lies above some other minimal point  $m'$  with full-support gap and with  $e_{m'}$  positive.

STEP 2: Now consider the graph of families  $\mathcal{F}(Q)$ , and remove the floating edges to get a graph  $\mathcal{F}'$ .

Suppose that  $\mathcal{F}'$  contains an embedded (unoriented) closed cycle  $\gamma$  containing a non-positive edge  $e_m$ . Thus  $\tau(e_m)$  lies above some minimal point  $m'$  with full-support gap and with  $e_{m'}$  positive. By sliding  $\tau(e_m)$  along  $e_{m'}$  many times, we can assume that all components of  $\tau(e_m)$  in  $\text{supp}_{\text{qc}}(Q)$  are very big. We can then slide  $\tau(e_m)$  along the other edges of  $\gamma$  until it becomes positive. This increases the number of positive edges.

By reiterating this procedure, we can assume that every embedded closed cycle in  $\mathcal{F}'$  is a single loop consisting of a positive edge. This means that  $\mathcal{F}'$  is a tree plus some loops at the vertices consisting of positive edges.

STEP 3: We now want all non-positive edges of  $\mathcal{F}'$  to point towards a prescribed family  $F_0$ . As we said before, non-positive edges of  $\mathcal{F}'$  form a tree. Suppose that  $e_m$  is a non-positive edge, and  $\tau(e_m)$  is more far from  $F_0$  than  $\iota(e_m)$  (in the tree). Then we have that  $\tau(e_m)$  lies above  $m'$  such that  $e_{m'}$  is positive, and thus we can perform a connection move. This substitutes  $e_m, e_{m'}$  with  $f_m, f_{m'}$  respectively, where  $f_m$  is positive and  $f_{m'}$  is non-positive; what we gain is that now  $\tau(f_{m'})$  is more near to  $F_0$  than  $\iota(f_{m'})$ . If some other non-positive edge  $e_{m''}$  has  $\tau(e_{m''})$  above  $m'$ , then we now slide  $\tau(e_{m''})$  along  $f_{m'}$ , in such a way that  $\tau(e_{m''})$  ends up above  $m$  with positive edge  $f_m$ .

Thus we can assume that all non-positive edges point towards  $F_0$ . For every non-positive edge  $e_m$ , we have that  $\tau(e_m)$  is above a positive edge, and thus up to slide we can make all components of  $\tau(e_m)$  in  $\text{supp}_{\text{qc}}(Q)$  very big, and then slide  $\tau(e_m)$  so that it falls in  $F_0$ , and above all minimal points of  $F_0$  with full-support gap.

Thus now  $\mathcal{F}'$  is made of non-positive edges (one for each family  $F \neq F_0$ , and all pointing to  $F_0$ , and with terminal endpoint with all components in  $\text{supp}_{\text{qc}}(Q)$  very big), and of positive edges at every other minimal point.

STEP 4: Up to slides, we can assume that every floating edge has both endpoints in  $F_0$ .

Let  $e_m$  be a positive edge given by  $m \xrightarrow{\mathbf{u}} m + \mathbf{u}$  for  $\mathbf{u} \in \mathbf{A}^+$  with  $\text{supp}(\mathbf{u}) = \text{supp}_{\text{qc}}(Q)$ . Take a floating edge  $e$  and, up to slides, assume that  $\iota(e)$  lies above  $m$  and  $\tau(e) = \iota(e) + \mathbf{w}$  for some  $\mathbf{w} \in \mathbf{A}^+$  with  $\text{supp}(\mathbf{w}) = \text{supp}_{\text{qc}}(Q)$  and with all non-zero components of  $\mathbf{w}$  very big. Then we perform a swap move, and we get an edge  $m \xrightarrow{\mathbf{w}} m + \mathbf{w}$  and a floating edge. In other words, we can assume that all components in  $\text{supp}_{\text{qc}}(Q)$  of all positive edges are very big.

STEP 5: For a family  $F \neq F_0$ , take a prescribed minimal point  $m_F$ . If  $e_{m_F}$  is positive, then there is another minimal point  $m'$  with  $e_{m'}$  non-positive. We have that  $\tau(e_{m'})$  lies above some minimal point  $m''$  in  $F_0$ , with  $e_{m''}$  positive. We now perform a connection to change  $e_{m'}, e_{m''}$  into  $f_{m'}, f_{m''}$  with  $f_{m'}$  positive. Then we perform a connection to change  $e_{m_F}, f_{m''}$  into  $g_{m_F}, g_{m''}$  with  $g_{m''}$  positive. In this way, the edges at  $m', m''$  are now positive, and the edge at  $m_F$  is not.

Thus the resulting GBS graph is in normal form, as desired.  $\square$

**Proposition 6.16.** *Suppose that  $(\Gamma, \psi), (\Delta, \phi)$  are in normal form. Suppose that they have the same  $\text{min}_{\text{qc}}(Q), \text{supp}_{\text{qc}}(Q), \text{lin}_{\text{qc}}(Q)$  and the same number of edges in each conjugacy class (up to edge sign-changes). Suppose that they have the same linear invariant. Then there is a sequence of edge sign-changes, slides, swaps, connections going from one to the other.*

*Proof.* Let  $\Lambda$  be the affine representation of  $(\Gamma, \psi)$ . We have the edges  $m_F \text{ --- } c_F$  for  $F \neq F_0$  and  $c_F \in F_0$ . We can easily assume that all the other edges are given by  $p_i \text{ --- } p_i + \mathbf{x}_i$  with  $\mathbf{x}_i \in \mathbf{A}^+$  and  $\text{supp}(\mathbf{x}_i) = \text{supp}_{\text{qc}}(Q)$ , for  $i = 1, \dots, r$ .

It's fairly easy to see that, by means of swap moves, and up to translating the floating edges with slide moves, we can permute the vectors  $\mathbf{x}_1, \dots, \mathbf{x}_r$  as we want.

Once we are able to permute them as we want, we can also perform Nielsen moves among them (since we are able, for example, to change a floating edge by slide over another edge), provided that we keep all components in  $\text{supp}_{\text{qc}}(Q)$  of all  $\mathbf{x}_i$  big enough, so that the graph remains in normal form.

The linear invariant is equal to the  $r$ -tuple  $\mathbf{x}_1, \dots, \mathbf{x}_r$ , up to Nielsen equivalence and permutation, which is the same in  $(\Gamma, \psi)$  and  $(\Delta, \phi)$  by assumption. In particular, using the results of Appendix B, we can perform moves to make  $\mathbf{x}_1, \dots, \mathbf{x}_r$  to be the same in  $(\Gamma, \psi)$  and  $(\Delta, \phi)$ .

Now, the  $p_i$  which are minimal points are already the same in  $(\Gamma, \psi)$  and  $(\Delta, \phi)$ , by assumption (up to performing some edge sign-change). The  $p_i$  which are non-minimal can be made to be the same by slides and self-slides (Lemma 3.2) and permutations of  $\mathbf{x}_1, \dots, \mathbf{x}_r$ , since we are assuming that there are the same number of edges in each conjugacy class in  $(\Gamma, \psi)$  and  $(\Delta, \phi)$ .

Finally, we can change the endpoints  $c_F$  by any combination of  $\mathbf{x}_1, \dots, \mathbf{x}_r$  (since we can slide  $c_F$  over at least one  $\mathbf{x}_i$ , and we can permute  $\mathbf{x}_1, \dots, \mathbf{x}_r$  as we want in any moment). Since  $(\Gamma, \psi)$  and  $(\Delta, \phi)$  have the same conjugacy classes, we can arrange for the  $c_F$  to be the same too.

The conclusion follows.  $\square$

## 6.7 Rigid vectors

Let  $(\Gamma, \psi)$  be a totally reduced GBS graph with affine representation  $\Lambda$ . Suppose that  $(\Gamma, \psi)$  has one qc-class  $Q$  with  $\text{supp}_{\text{qc}}(Q) \neq \emptyset$ , and that  $(\Gamma, \psi)$  is clean. Suppose that  $(\Gamma, \psi)$  doesn't have floating pieces.

**Definition 6.17.** A **rigid cycle** is a sequence of  $\ell \geq 1$  edges  $m_1 \text{ --- } m_2 + \mathbf{w}_1$ ,  $m_2 \text{ --- } m_3 + \mathbf{w}_2$ ,  $\dots$ ,  $m_\ell \text{ --- } m_1 + \mathbf{w}_\ell$  with the following properties:

1.  $m_1, \dots, m_\ell$  are pairwise distinct minimal points.
2.  $\mathbf{w}_1, \dots, \mathbf{w}_\ell \in \mathbf{A}^+$ .
3. There are no distinct minimal points  $m, m'$  such that  $m + \mathbf{w}_1 + \dots + \mathbf{w}_\ell \geq m'$ .

For a rigid cycle we define the associated **rigid vector** as  $\mathbf{w} = \mathbf{w}_1 + \dots + \mathbf{w}_\ell \in \mathbf{A}^+$ .

In the above hypothesis, we must have that  $\text{supp}(\mathbf{w}) \neq \emptyset$  since  $(\Gamma, \psi)$  is clean. In a rigid cycle, every endpoint of every edge lies above a unique minimal point. In fact, if  $m_{i+1} + \mathbf{w}_i \geq m'$  for some minimal point  $m'$ , then in particular  $m_{i+1} + \mathbf{w} \geq m'$ , and by Item 3 of Definition 6.17 this implies that  $m' = m_{i+1}$ . In particular, rigid cycles are pairwise disjoint. Note also that  $\mathbf{w} \in \text{lin}_{\text{qc}}(Q)$ , since we can, for example, slide one edge of the rigid cycle along the others to obtain an edge  $m \text{ --- } m + \mathbf{w}$ .

**Definition 6.18.** Define the set of **rigid vectors** of  $(\Gamma, \psi)$  as the multi-set of the vectors  $\mathbf{w}$  associated with the rigid cycles of  $(\Gamma, \psi)$ .

**Proposition 6.19.** The multi-set of rigid vectors of  $(\Gamma, \psi)$  is invariant under edge sign-changes, slides, swaps, connections.

*Proof.* Immediate check with the definitions.  $\square$

## 6.8 GBS graphs with no floating pieces

Let  $(\Gamma, \psi)$  be a totally reduced GBS graph with affine representation  $\Lambda$ . Suppose that  $(\Gamma, \psi)$  has one qc-class  $Q$  with  $\text{supp}_{\text{qc}}(Q) \neq \emptyset$ , and that  $(\Gamma, \psi)$  is clean. Suppose that  $(\Gamma, \psi)$  doesn't have floating pieces.

We denote with  $\mathcal{M}$  the finite set of minimal points of  $Q$ . This is partitioned into the non-empty sets  $\mathcal{M}_F$  for  $F \in V(\mathcal{F}(Q))$ , according to which family a given minimal point belongs. We give



an orientation to each edge of  $\Lambda$ : we say that  $e$  is the *oriented edge* of a pair  $e, \bar{e} \in E(\Lambda)$  if  $\iota(e)$  is a minimal point. Note that exactly one of  $e, \bar{e}$  is the oriented edge. Since  $(\Gamma, \psi)$  is clean and doesn't have floating pieces, we have a bijection between the set of oriented edges and  $\mathcal{M}$ : the oriented edge  $e$  corresponds to the minimal point  $\iota(e) \in \mathcal{M}$ . Recall that  $E(\mathcal{F}(Q)) = E(\Lambda)$ , and in particular  $\mathcal{F}(Q)$  is an oriented graph. We fix a family  $F_0 \in V(\mathcal{F}(Q))$ , which will play the role of basepoint.

Let  $\mathbf{R} = \{\mathbf{r}_1, \dots, \mathbf{r}_k\}$  be the multi-set of rigid vectors as in Definition 6.18. For  $i = 1, \dots, k$ , choose an oriented edge  $e_i$  in the rigid cycle corresponding to  $\mathbf{r}_i$ . Removing  $e_1, \dots, e_k$  leaves the graph  $\mathcal{F}(Q)$  connected (as they are part of embedded cycles with disjoint sets of edges). Thus we can take a maximal tree  $T$  for  $\mathcal{F}(Q)$  which doesn't contain  $e_1, \dots, e_k$ . We call  $t_1, \dots, t_h$  the oriented edges in the maximal tree  $T$ , and we call  $f_1, \dots, f_\ell$  the remaining oriented edges. This means that  $e_1, \dots, e_k, f_1, \dots, f_\ell, t_1, \dots, t_h$  is a list of all the oriented edges of  $\mathcal{F}(Q)$ , each appearing exactly once. In particular  $\mathcal{M} = \{\iota(e_1), \dots, \iota(e_k), \iota(f_1), \dots, \iota(f_\ell), \iota(t_1), \dots, \iota(t_h)\}$ .

For  $i = 1, \dots, \ell$  we consider the path  $\sigma_i$  which goes through  $f_i$  and then back to the starting point through the maximal tree  $T$ . For  $i = 1, \dots, \ell$  we define  $\mathbf{x}_i = q_Q(\sigma_i) + \langle \mathbf{R} \rangle \in \frac{\text{lin}_{\text{qc}}(Q)}{\langle \mathbf{R} \rangle}$ . We consider the tree  $T$  with base-point  $F_0$ : every edge has an orientation, which is either pointing towards  $F_0$ , or away from  $F_0$ . We call  $\epsilon \in \mathbb{N}$  the number of edges pointing away from  $F_0$ .

We define the map  $\Theta : \mathcal{M} \rightarrow \mathbf{R} \sqcup \frac{\text{lin}_{\text{qc}}(Q)}{\langle \mathbf{R} \rangle} \sqcup (V(\mathcal{F}(Q)) \setminus \{F_0\})$  as follows:

1.  $\Theta(\iota(e_i)) = \mathbf{r}_i \in \mathbf{R}$  for  $i = 1, \dots, k$ .
2.  $\Theta(\iota(f_i)) = \mathbf{x}_i \in \frac{\text{lin}_{\text{qc}}(Q)}{\langle \mathbf{R} \rangle}$  for  $j = 1, \dots, \ell$ .
3.  $\Theta(\iota(t_i)) \in V(\mathcal{F}(Q)) \setminus \{F_0\}$  is defined as follows. If  $t_i$  is pointing towards  $F_0$ , we set  $\Theta(\iota(t_i))$  as the family of  $\iota(t_i)$ . If  $t_i$  is pointing away from  $F_0$ , we set  $\Theta(\iota(t_i))$  as the family of  $\tau(t_i)$ .

We define the map  $(-1)^\epsilon \Theta$  by pre-composing  $\Theta$  with any permutation of  $\mathcal{M}$  of sign  $(-1)^\epsilon$ .

**Definition 6.20.** We define the **linear invariant** of  $(\Gamma, \psi)$  as the map  $(-1)^\epsilon \Theta$ , considered up to

1. Nielsen equivalence of  $\mathbf{x}_1, \dots, \mathbf{x}_\ell$  in the group  $\frac{\text{lin}_{\text{qc}}(Q)}{\langle \mathbf{R} \rangle}$ .
2. Pre-composing  $\Theta$  with an even permutation of  $\mathcal{M}$ .

TO DO explanation of the meaning. Say that it makes sense if one thinks about the normal form.  $\Theta(m) = \mathbf{r}$  means that we have a rigid edge  $m \rightarrow m + \mathbf{r}$ .  $\Theta(m) = \mathbf{x}$  means that we have a (non-rigid) edge  $m \rightarrow m + \mathbf{x}$ . For each family  $F \neq F_0$  there is a unique edge going from a minimal point of  $F$  to a non-minimal point inside  $F_0$ :  $\Theta(m) = F$  morally means that this edge starts at the minimal point  $m$  (ignore the fact that  $m$  might not belong to  $F$ , and everything works).

**Lemma 6.21.** A different choice of the edges in the rigid cycles and of the maximal tree  $T$  produces the same linear invariant.

*Proof.* Take an embedded cycle  $\gamma$  in  $\mathcal{F}(Q)$ , and call  $\mathbf{x} = q_Q(\gamma)$ . Suppose that  $\gamma = \dots f_1^{\eta_1} \dots f_2^{\eta_2} \dots$  for some oriented edges  $f_1, f_2$  and for some  $\eta_1, \eta_2 = \pm 1$ . Suppose that some maximal tree  $T_1$  contains  $f_1$  but not  $f_2$ , and let  $T_2$  be the maximal tree obtained from  $T_1$  by removing  $f_1$  and adding  $f_2$ . Suppose also that both  $T_1, T_2$  are missing exactly one edge for each rigid cycle. Let  $\Theta_1, \Theta_2$  be the maps associated with  $T_1, T_2$  respectively.

For an edge  $e$  not appearing in  $\gamma$  we have the following. If  $e \in T_1 \cap T_2$  then  $\Theta_1(e) = \Theta_2(e)$  and the contribution to  $\epsilon$  doesn't change. For  $e \notin T_1 \cup T_2$  we have that  $\Theta_1(e) - \Theta_2(e)$  is  $\mathbf{0}$  or  $\pm \mathbf{x}$ .

For the edges  $e$  appearing in  $\gamma$ , the values of  $\Theta_1, \Theta_2$  will be  $\pm \mathbf{x}$  and  $F_1, F_2, \dots, F_r$  (the vertices appearing in  $\gamma$ , except the one nearest to  $F_0$ ), in some order. When changing from  $T_1$  from  $T_2$ , these values get permuted (the permutation consisting of a single cycle), the contributions to  $\epsilon$  might change, and  $\pm \mathbf{x}$  might change sign. It's a routine check with the cases that these changes compensate each other, i.e.  $\pm \mathbf{x}$  changes sign if and only if the sign of the permutation of the values, plus the sum of the contributions to  $\epsilon$ , is negative.

Thus we can change from  $T_1$  to  $T_2$  without affecting the linear invariant. But with changes of this kind, we can go from any tree (containing all edges in each rigid cycle except one) to any other tree (containing all edges in each rigid cycle except one). The conclusion follows.  $\square$

**Proposition 6.22.** *The linear invariant of  $(\Gamma, \psi)$  is invariant under edge sign-changes, slides, swaps, connections.*

*Proof.* Immediate check with the definitions (using Lemma 6.21).  $\square$

We now define a normal form for GBS graphs with no floating pieces. Fix a non-individual family  $F_0 \in V(\mathcal{F}(Q))$ . For every family  $F \neq F_0$  we fix a minimal point  $m_F$  in that family.

**Definition 6.23** (Normal form for GBSs with no floating pieces). *Suppose that  $(\Gamma, \psi)$  has full-support gaps. We say that  $(\Gamma, \psi)$  is in **normal form** (with respect to the non-individual family  $F_0$  and the minimal points  $\{m_F\}_{F \neq F_0}$ ) if the following conditions hold:*

1. *For every family  $F \neq F_0$  and for the minimal point  $m_F$ , take the unique edge  $e$  with  $\iota(e) = m_F$ . We require that  $\tau(e)$  is in  $F_0$ , and lies above all minimal points of  $F_0$  with full-support gap.*
2. *For every family  $F$  and for every minimal point  $m$  in  $F$  (except possibly  $m_F$ ), take the unique edge  $e$  with  $\iota(e) = m$ . We require that  $\tau(e)$  is in  $F$ . If  $e$  isn't a rigid cycle, we also require that  $\tau(e)$  lies above all minimal points of  $F$  with full-support gap.*

**Proposition 6.24.** *Suppose that  $(\Gamma, \psi)$  has full-support gaps. Then, for every non-individual family  $F_0$  and minimal points  $\{m_F\}_{F \neq F_0}$ , we can bring  $(\Gamma, \psi)$  to normal form by means of a sequence of slides, swaps, connections.*

*Proof.* For every minimal point  $m$ , let  $e_m$  be the unique edge such that  $\iota(e_m) = m$ . We say that  $e_m$  is *positive* if  $\tau(e_m)$  is above  $m$  with full-support gap.

The first three steps in the proof of Proposition 6.15 apply here. Therefore, we can assume that every edge is positive, except for the edges  $e_{m_F}$  for  $F \neq F_0$ , for which  $\tau(e_{m_F})$  lies in  $F_0$  and above all minimal points of  $F_0$  with full-support gap.

Let  $F \neq F_0$  be a family containing at least two minimal points  $m, m_F$ , and let  $m_1, m_2$  be two minimal points in  $F_0$ . Then we have edges  $m_1 \text{ --- } m_1 + \mathbf{x}$  and  $m_2 \text{ --- } m_2 + \mathbf{y}$  and  $m \text{ --- } m + \mathbf{z}$  and  $m_F \text{ --- } p$  with  $p$  above both  $m_1, m_2$ . Then we can perform connections to change

$$\begin{aligned} \begin{cases} m_1 \text{ --- } m_1 + \mathbf{x} \\ m_2 \text{ --- } m_2 + \mathbf{y} \\ m \text{ --- } m + \mathbf{z} \\ m_F \text{ --- } p \end{cases} &\text{ into } \begin{cases} m_1 \text{ --- } q \\ m_2 \text{ --- } m_2 + \mathbf{y} \\ m \text{ --- } m + \mathbf{z} \\ m_F \text{ --- } m_F + \mathbf{x} \end{cases} &\text{ into } \begin{cases} m_1 \text{ --- } m_1 + \mathbf{z} \\ m_2 \text{ --- } m_2 + \mathbf{y} \\ m \text{ --- } p' \\ m_F \text{ --- } m_F + \mathbf{x} \end{cases} \\ & &\text{ into } \begin{cases} m_1 \text{ --- } m_1 + \mathbf{z} \\ m_2 \text{ --- } q' \\ m \text{ --- } m + \mathbf{y} \\ m_F \text{ --- } m_F + \mathbf{x} \end{cases} &\text{ into } \begin{cases} m_1 \text{ --- } m_1 + \mathbf{z} \\ m_2 \text{ --- } m_2 + \mathbf{x} \\ m \text{ --- } m + \mathbf{y} \\ m_F \text{ --- } p'' \end{cases} \end{aligned}$$

ultimately getting a cyclic permutation of  $\mathbf{x}, \mathbf{y}, \mathbf{z}$  (here  $p, p', p''$  lie in  $F_0$  above  $m_1, m_2$ , while  $q, q'$  lie in  $F$  above  $m, m_F$ ).

Now suppose that some positive edge  $e_m$  given by  $m \text{ --- } m + \mathbf{w}$  isn't a rigid edge, but  $m + \mathbf{w}$  doesn't lie above all minimal points in its family. Then we can find two minimal points  $m', m''$  such that  $m' + \mathbf{w} \geq m''$ . By means of cyclic permutations as explained above, we can bring  $\mathbf{w}$  to  $F_0$ , and then to  $m'$ , so that we end up with an edge  $m' \text{ --- } m' + \mathbf{w}$ . Using that  $m' + \mathbf{w} \geq m''$ , we can perform slides and make all components of  $\mathbf{w}$  in  $\text{supp}_{\text{qc}}(Q)$  very big. Finally, by means of other cyclic permutations, we bring  $\mathbf{w}$  back to  $m$ .

By reiterating this procedure, we eventually obtain a GBS graph in normal form, as desired.  $\square$

**Proposition 6.25.** *Suppose that  $(\Gamma, \psi), (\Delta, \phi)$  are in normal form. Suppose that they have the same  $\min_{\text{qc}}(Q), \text{supp}_{\text{qc}}(Q), \text{lin}_{\text{qc}}(Q)$  and the same number of edges in each conjugacy class (up to edge sign-changes). Suppose that they have the same multi-set of rigid edges and the same linear invariant. Suppose that either  $F_0$  contains at least three minimal points, or there is at least another non-individual family besides  $F_0$ . Then there is a sequence of edge sign-changes, slides, swaps, connections going from one to the other.*

*Proof.* Up to edge sign-changes, we can assume that the minimal points are exactly the same.

As in the proof of Proposition 6.24, we can perform cyclic permutations on the vectors at the edges, by means of connections. In particular, we can assume that all rigid edges are at the same minimal points in  $(\Delta, \phi)$  as in  $(\Gamma, \psi)$ .

If we have a (possibly rigid) edge  $m \text{ --- } m + \mathbf{x}$  and a non-rigid edge  $m' \text{ --- } m' + \mathbf{y}$ , then with a sequence of slides we can change  $\mathbf{y}$  into  $\mathbf{y} \pm \mathbf{x}$ , provided that all components in  $\text{supp}_{\text{qc}}(Q)$  of  $\mathbf{y}$  remain big enough. Now we deal with two cases: if there is at least one rigid edge, then we want to use Proposition B.4; if there is no rigid edge, then by hypothesis there are at least 3 positive vectors, and we can use Theorem B.1. In both cases, since  $(\Gamma, \psi), (\Delta, \phi)$  have the same linear invariant, we can obtain that the values of  $\mathbf{x}$  are the same in  $(\Delta, \phi)$  as in  $(\Gamma, \psi)$ .

Finally, for  $F \neq F_0$  and for the edge  $m_F \text{ --- } c_F$ , we can change  $c_F$  to any other point in the same conjugacy class, by means of slide moves (and connections, if we would need to add/subtract some vector which lies in  $F$ ). In particular, we can set also these to be the same in  $(\Delta, \phi)$  as in  $(\Gamma, \psi)$ . The conclusion follows.  $\square$

## 6.9 The algorithm

**Theorem 6.26.** *There is an algorithm that, given two GBS graph  $(\Gamma, \psi), (\Delta, \phi)$  with one qc-class and full-support gaps, decides whether there is a sequence of edge sign-changes, slides, swaps, connections going from one to the other. In case such a sequence exists, the algorithm also computes one such sequence.*

*Proof.* First of all, we check that the conjugacy class  $Q$  has the same  $\min_{\text{qc}}(Q), \text{supp}_{\text{qc}}(Q), \text{lin}_{\text{qc}}(Q)$  and the same number of edges in each conjugacy class (up to edge sign-changes), otherwise by Corollary 4.4 we can't change  $(\Gamma, \psi)$  into  $(\Delta, \phi)$ .

If  $(\Gamma, \psi)$  has floating pieces, then  $(\Delta, \phi)$  must have floating pieces too. We make both of them clean and we compute the linear invariant: this must be the same, otherwise there is no sequence of moves going from one to the other. If they have the same linear invariant, then we bring both of them in normal form as in Proposition 6.15, and then we can go from one to the other by Proposition 6.16.

If  $(\Gamma, \psi)$  has no floating pieces, but it has a family with at least three minimal points, or at least two families with at least two minimal points each. Then  $(\Delta, \phi)$  must have the same property (as it has the same families and minimal points). We make both of them clean and we compute the multi-set of rigid edges and the linear invariant: these must be the same, otherwise there is no sequence of moves going from one to the other. If they have the same multi-set of rigid edges and the same linear invariant, then we bring both of them in normal form as in Proposition 6.24, and then we can go from one to the other by Proposition 6.25.

Finally, suppose that  $(\Gamma, \psi)$  has no floating pieces, and all families of  $(\Gamma, \psi)$  are individual, except at most one with at most two minimal points; we call  $F_0$  such family. Note that  $(\Delta, \phi)$  must have the same property (as it has the same families and minimal points). In this case, we make both  $(\Gamma, \psi)$  and  $(\Delta, \phi)$  clean and  $F$ -polished for each family  $F \neq F_0$ , as in Section 6.4. From now on, we only deal with sequences of moves involving GBS graphs which are  $F$ -polished for each  $F \neq F_0$ .

For every family  $F \neq F_0$ , we take the unique minimal point  $m_F$  in that family, and the unique edge  $m_F \text{ --- } c_F$  with  $c_F \in F_0$ , and we observe that  $c_F$  can be changed to any other point in the same conjugacy class, by means of slide moves only; in particular, in any moment, we can arrange them to be the same in  $(\Gamma, \psi)$  as in  $(\Delta, \phi)$ . Let  $(\Gamma', \psi'), (\Delta', \phi')$  be the GBS graphs obtained from  $(\Gamma, \psi), (\Delta, \phi)$  respectively, by removing all edges with endpoints outside  $F_0$ . Then there is a sequence of moves going from  $(\Gamma, \psi)$  to  $(\Delta, \phi)$  if and only if there is a sequence of moves going from  $(\Gamma', \psi')$  to  $(\Delta', \phi')$ .

But  $(\Gamma', \psi'), (\Delta', \phi')$  are GBS graphs with one vertex and at most two edges each. The conclusion thus follows from [?] TO DO fix reference.  $\square$

**Corollary 6.27.** *There is an algorithm that, given two GBS graph  $(\Gamma, \psi), (\Delta, \phi)$  with one qc-class and full-support gaps, decides whether the corresponding GBS groups are isomorphic or not. In case they are, the algorithm also computes a sequence of sign-changes, inductions, slides, swaps, connections going from  $(\Gamma, \psi)$  to  $(\Delta, \phi)$ .*

*Proof.* By Theorem 6.26, and since vertex sign-changes can be guessed in finitely many ways, we only have to deal with induction.

But if  $(\Gamma, \psi)$  allows for induction, is totally reduced, and has one qc-class, then it must have just one vertex, and by means of a sequence of moves we can make it into a controlled GBS graph. In this case, the thesis follows from [ACRK25b].  $\square$

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## A Nielsen equivalence in finitely generated abelian groups

We consider elements of  $\mathbb{Z}^n$  as column vectors. We denote with  $\mathcal{M}_{n \times k}(\mathbb{Z})$  the set of matrices with  $n$  rows and  $k$  columns and integer coefficients. For  $M \in \mathcal{M}_{n \times k}(\mathbb{Z})$  we denote with  $M_i^j$  the integer number in the  $i$ -th row and  $j$ -th column of  $M$ , for  $i = 1, \dots, n$  and  $j = 1, \dots, k$ ; we denote with  $\text{span}(M)$  the subgroup of  $\mathbb{Z}^n$  generated by the columns of  $M$ . A **column operation** on a matrix consists of choosing a column and adding or subtracting it from another column. A **(column) reordering operation** consists of interchanging two columns of the matrix. Notice that column and column reordering operations don't change  $\text{span}(M)$ . A **row operation** consists of choosing a row and adding or subtracting it from another row. Row operations correspond to changing basis for the free abelian group  $\mathbb{Z}^n$ .

### Smith normal form

**Definition A.1.** A matrix  $S \in \mathcal{M}_{n \times k}(\mathbb{Z})$  is in **Smith normal form** if it satisfies the following conditions:

1.  $S_i^j = 0$  whenever  $i \neq j$ .
2.  $S_i^i \geq 0$  for  $i = 1, \dots, m$  where  $m = \min\{n, k\}$ .
3. We have  $S_{i+1}^{i+1} \mid S_i^i$  for all  $i = 1, \dots, m-1$ .

Every matrix is equivalent, up to row and column operations and reordering operations, to a unique matrix in Smith normal form. An interesting feature is that the integers  $S_i^i$  can be computed directly from the initial matrix, as follows. For  $M \in \mathcal{M}_{n \times k}(\mathbb{Z})$ , define  $D_\ell(M) \geq 0$  as the greatest common divisor of all the determinants of the  $\ell \times \ell$  minors of  $M$ , for  $1 \leq \ell \leq m$  where  $m = \min\{n, k\}$ . Notice that, if  $M$  has rank  $r$ , then  $D_\ell(M) = 0$  if and only if  $\ell \geq r+1$ .

**Lemma A.2.** Let  $M \in \mathcal{M}_{n \times k}(\mathbb{Z})$  be a matrix of rank  $r$ . Let  $S$  be the unique matrix in Smith normal form obtained from  $M$  by means of row and column operations. Then we have the following:

1.  $D_1(M) \mid D_2(M) \mid \dots \mid D_m(M)$  where  $m = \min\{n, k\}$ .
2.  $S_i^i = D_{m+1-i}(M) = 0$  for  $i = 1, \dots, m-r$ .
3.  $S_i^i = D_{m+1-i}(M)/D_{m-i}(M)$  for  $i = m-r+1, \dots, m-1$ .
4.  $S_m^m = D_1(M)$ .

*Proof.* It's immediate to notice that  $D_\ell(M)$  is invariant under row and column operations, as well as column swap operation. But for a matrix in Smith normal the thesis holds, so it holds for all matrices.  $\square$

### Finitely generated abelian groups

Every finitely generated abelian group is isomorphic to

$$\mathbb{Z}/d_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/d_n\mathbb{Z}$$

for unique integers numbers  $n, d_1, \dots, d_n \geq 0$  satisfying  $1 \neq d_n \mid d_{n-1} \mid \dots \mid d_1$ . Notice that some  $d_i$  can be equal to 0, producing  $\mathbb{Z}$  summands.

**Lemma A.3.** Let  $M \in \mathcal{M}_{n \times k}(\mathbb{Z})$ . Let  $S \in \mathcal{M}_{n \times k}(\mathbb{Z})$  be the Smith normal form of  $M$ . Then  $\mathbb{Z}^n/\text{span}(M)$  is isomorphic to  $\mathbb{Z}/d_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/d_n'\mathbb{Z}$  where

1.  $n' \leq \min\{n, k\}$  is the maximum integer such that  $S_{n'}^{n'} \neq 1$ .
2.  $d_i = S_i^i$  for  $i = 1, \dots, n'$ .

*Proof.* Column and reordering operations don't change  $\text{span}(M)$ , and in particular they don't change the quotient  $\mathbb{Z}^n/\text{span}(M)$ . Row operations correspond to changing basis for  $\mathbb{Z}^n$ , so they don't change the isomorphism type of  $\mathbb{Z}^n/\text{span}(M)$ . Thus we have that  $\mathbb{Z}^n/\text{span}(M)$  is isomorphic to  $\mathbb{Z}^n/\text{span}(S)$ . The conclusion follows.  $\square$



## Nielsen equivalence in finitely generated abelian groups

Let  $A$  be a finitely generated abelian group. Let  $(w_1, \dots, w_k)$  be an ordered  $k$ -tuple of elements of  $A$ : a **Nielsen move** on  $(w_1, \dots, w_k)$  is any operation that consists of the substitution  $w_i$  with  $w_i + w_j$  or with  $w_i - w_j$ , for some  $j \neq i$ . Two ordered  $k$ -tuples are **Nielsen equivalent** if there is a sequence of Nielsen moves going from one to the other. Our aim in this section is to classify the  $k$ -tuples of generators for  $A$  up to Nielsen equivalence (we ignore  $k$ -tuples that don't generate the whole group  $A$ , as they would require us to change the ambient group we are working in).

Fix an isomorphism

$$A = \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/d_n\mathbb{Z}$$

for integers  $n, d_1, \dots, d_n \geq 0$  with  $1 \neq d_n \mid d_{n-1} \mid \dots \mid d_1$ . Then we can represent elements of  $A$  as column vectors with  $n$  entries, where the  $i$ -th entry is an element of  $\mathbb{Z}/d_i\mathbb{Z}$  for  $i = 1, \dots, n$ . We represent an ordered  $k$ -tuple  $(w_1, \dots, w_k)$  as a matrix  $M = M(w_1, \dots, w_k)$  with  $n$  rows and  $k$  columns. Nielsen moves correspond to column operations on the matrix  $M$ .

**Lemma A.4.** *The group  $A = \mathbb{Z}/d_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/d_n\mathbb{Z}$  can't be generated by less than  $n$  elements.*

*Proof.* Take a prime  $p \mid d_n$ . Consider the surjective projection map from  $\mathbb{Z}/d_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/d_n\mathbb{Z}$  to  $(\mathbb{Z}/p\mathbb{Z})^n$ . But  $(\mathbb{Z}/p\mathbb{Z})^n$  can't be generated by less than  $n$  elements.  $\square$

**Proposition A.5.** *For every  $k \geq n+1$ , every two ordered  $k$ -tuples of generators for  $A$  are Nielsen equivalent. Two ordered  $n$ -tuples of generators for  $A$  are Nielsen equivalent if and only if the corresponding matrices have the same determinant modulo  $d_n$ .*

*Proof.* Let  $(w_1, \dots, w_k)$  and  $(u_1, \dots, u_k)$  be  $k$ -tuples of generators for  $A$  and let  $M, N \in \mathcal{M}_{n \times k}(\mathbb{Z})$  be the matrices whose columns represent  $w_1, \dots, w_k$  and  $u_1, \dots, u_k$  respectively. If  $k = n$  and  $w_1, \dots, w_k$  and  $u_1, \dots, u_k$  are Nielsen equivalent, then there must be a sequence of column operations going from  $M$  to  $N$ , and in particular the two matrices must have the same determinant.

We look at the first row of  $M$ , and using column operations we perform the Euclidean algorithm; we obtain a new matrix  $M_1^1 \neq 0$  and  $M_1^j = 0$  for  $j = 2, \dots, k$ . But since  $w_1, \dots, w_k$  generate the whole group, we must have that  $M_1^1$  is coprime with  $d_1$ . We add  $d_1$  to  $M_1^1$  (we can, since elements of the first row are elements of  $\mathbb{Z}/d_1\mathbb{Z}$ ), and then we perform the Euclidean algorithm again, obtaining a new matrix with  $M_1^1 = 1$  and  $M_1^j = 0$  for  $j = 2, \dots, k$ .

We reiterate the same reasoning by induction. If  $k \geq n+1$  we obtain a matrix with  $M_i^i = 1$  for  $i = 1, \dots, n$  and  $M_i^j = 0$  for  $j \neq i$ . Performing the same procedure on  $N$  we obtain the same matrix, and in particular there is a sequence of column operations going from  $M$  to  $N$ , and thus  $w_1, \dots, w_k$  is Nielsen equivalent to  $u_1, \dots, u_k$ . If  $k = n$ , we obtain a matrix with  $M_i^i = 1$  for  $i = 1, \dots, n-1$  and  $M_n^n$  invertible modulo  $d_n$  and  $M_i^j = 0$  for  $j \neq n$ . Notice that the determinant of  $M$  modulo  $d_n$  is an invariant under column operations, and for the matrix reduced in this form it's equal to  $M_n^n$ . If the determinant of  $M$  was equal to the one of  $N$ , then performing the same procedure on  $N$  we obtain the same matrix, and in particular there is a sequence of column operations going from  $M$  to  $N$ , and thus  $(w_1, \dots, w_k)$  is Nielsen equivalent to  $(u_1, \dots, u_k)$ .  $\square$

We will also need the following two lemmas.

**Lemma A.6.** *Let  $a \in A$  be an element represented by a vector  $(a_1, \dots, a_n)$ , and suppose that  $\gcd(a_1, \dots, a_{n-1}, a_n) = 1$ . Let  $(a, w_2, \dots, w_k)$ ,  $(a, u_2, \dots, u_k)$  be Nielsen equivalent  $k$ -tuples of generators for  $A$  with  $k \geq n+1$ . Then  $([w_2], \dots, [w_k])$ ,  $([u_2], \dots, [u_k])$  are Nielsen equivalent  $(k-1)$ -tuples of generators for  $A/\langle a \rangle$ .*

*Proof.* Let  $R \in \mathcal{M}_{n \times n}(\mathbb{Z})$  be the matrix given by  $R_i^i = d_i$  for  $i = 1, \dots, n$  and  $R_i^j = 0$  for  $j \neq i$ . We have that  $A = \mathbb{Z}^n / \text{span}(R)$ . Let  $R'$  be the matrix obtained from  $R$  by adding an extra column, equal to  $(a_1, \dots, a_n)$ . Then  $A/\langle a \rangle = \mathbb{Z}^n / \text{span}(R')$ . Since  $\gcd(a_1, \dots, a_{n-1}) = 1$ , we see that  $D_1(R') = 1$  and  $D_2(R') = d_n \neq 1$ . In particular, by Lemmas A.2, A.3 and A.4 we have that the minimum number of generators for  $A/\langle a \rangle$  is  $n-1$ . But then by Proposition A.5 the  $(k-1)$ -tuples of generators  $([w_2], \dots, [w_k])$ ,  $([u_2], \dots, [u_k])$  for  $A/\langle a \rangle$  are Nielsen equivalent, since  $k-1 \geq n$ .  $\square$

**Lemma A.7.** *Let  $a \in A$  be an element represented by a vector  $(a_1, \dots, a_n)$ , and suppose that  $\gcd(a_1, \dots, a_{n-1}) = 1$ . Let  $(a, w_2, \dots, w_n)$ ,  $(a, u_2, \dots, u_n)$  be Nielsen equivalent  $n$ -tuples of generators for  $A$ . Then  $([w_2], \dots, [w_n])$ ,  $([u_2], \dots, [u_n])$  are Nielsen equivalent  $(n-1)$ -tuples of generators for  $A/\langle a \rangle$ .*

*Proof.* Let  $R \in \mathcal{M}_{n \times n}(\mathbb{Z})$  be the matrix given by  $R_i^i = d_i$  for  $i = 1, \dots, n$  and  $R_i^j = 0$  for  $j \neq i$ . We have that  $A = \mathbb{Z}^n / \text{span}(R)$ . Let also  $M, N \in \mathcal{M}_{n \times k}(\mathbb{Z})$  be matrices whose columns represent  $a, w_2, \dots, w_n$  and  $a, u_2, \dots, u_n$  respectively. By Proposition A.5  $M$  and  $N$  must have the same determinant modulo  $d_n$ .

Let  $R'$  be the matrix obtained from  $R$  by adding an extra column, equal to the first column of  $M$  and of  $N$ . We have that  $A/\langle a \rangle = \mathbb{Z}^n / \text{span}(R')$  and  $D_1(R') = 1$  and  $D_2(R') = d_n$  (since  $\gcd(a_1, \dots, a_{n-1}) = 1$ ). Thus, by Proposition A.5, we only need to check that the matrices representing  $([w_2], \dots, [w_n])$  and  $([u_2], \dots, [u_n])$  have the same determinant modulo  $d_n$ .

We perform row operations at the same time on  $R, M, N$ , running the Euclidean algorithm on the first column of  $M$  (which is also the first column on  $N$ ). We obtain matrices  $\bar{R}, \bar{M}, \bar{N}$  such that the first column of  $\bar{M}$  (which is also the first column of  $\bar{N}$ ) is equal to  $(1, 0, \dots, 0, 0)$ . Notice that the determinant of  $\bar{M}$  is the same as the one of  $\bar{N}$  modulo  $d_n$ , since row operations don't change the determinant. Since row operations correspond to changing basis of  $\mathbb{Z}^n$ , we have an isomorphism  $A \cong \mathbb{Z}^n / \text{span}(\bar{R})$ , with the two matrices  $\bar{M}$  and  $\bar{N}$  representing the two  $k$ -tuples  $a, w_2, \dots, w_n$  and  $a, u_2, \dots, u_n$ . Let  $\bar{M}''$  be the minor of  $\bar{M}$  obtained by cancelling the first row and the first column of  $\bar{M}$ , and define  $\bar{N}''$  analogously; let also  $\bar{R}''$  be the matrix obtained from  $\bar{R}$  by removing the first row. The isomorphism  $A \cong \mathbb{Z}^n / \text{span}(\bar{R})$  induces an isomorphism  $A/\langle a \rangle \cong \mathbb{Z}^{n-1} / \text{span}(\bar{R}'')$  and the matrices  $\bar{M}''$  and  $\bar{N}''$  represent the  $(n-1)$ -tuples  $([w_2], \dots, [w_n])$  and  $([u_2], \dots, [u_n])$ . But since the first column of  $\bar{M}$  and of  $\bar{N}$  is  $(1, 0, \dots, 0)$  and  $\bar{M}, \bar{N}$  have the same determinant modulo  $d_n$ , it follows that  $\bar{M}''$  and  $\bar{N}''$  have the same determinant modulo  $d_n$ . The conclusion follows.  $\square$

## B Nielsen equivalence for big vectors

Fix a free abelian group  $\mathbb{Z}^n$ . Let  $(w_1, \dots, w_k)$  be an ordered  $k$ -tuple of vectors of  $\mathbb{Z}^n$ . For an integer  $M \geq 1$  we say that an ordered  $k$ -tuple  $(w_1, \dots, w_k)$  is  **$M$ -big** if each component of each of  $w_1, \dots, w_k$  is  $\geq M$ . Our aim is to classify the  $M$ -big ordered  $k$ -tuples of vectors up to Nielsen moves. This means that, given two  $M$ -big ordered  $k$ -tuples, we want to know whether there is a sequence of Nielsen moves that goes from one to the other, and such that all the  $k$ -tuples that we write along the sequence are  $M$ -big.

We are going to need also a relative version of the same problem, that we now introduce. Let  $v_1, \dots, v_h$  be vectors of  $\mathbb{Z}^n$  and let  $(w_1, \dots, w_k)$  be an ordered  $k$ -tuple of vectors of  $\mathbb{Z}^n$ . A **Nielsen move relative to  $v_1, \dots, v_h$** , also called **relative Nielsen move** when there is no ambiguity, is any operation that consists of substituting  $w_i$  with  $w_i + v_j$  or  $w_i - v_j$  for some  $i \in \{1, \dots, k\}$  and  $j \in \{1, \dots, h\}$ . Fixed vectors  $v_1, \dots, v_h$ , we want to classify the  $M$ -big ordered  $k$ -tuples up to Nielsen moves and relative Nielsen moves. This means that, given two  $M$ -big ordered  $k$ -tuples, we want to know whether there is a sequence of Nielsen moves and relative Nielsen moves that goes from one to the other, and such that all the  $k$ -tuples that we write along the sequence are  $M$ -big.

Given vectors  $v_1, \dots, v_h \in \mathbb{Z}^n$  and an ordered  $k$ -tuple  $(w_1, \dots, w_k)$ , we define the finitely generated abelian group

$$A(w_1, \dots, w_k; v_1, \dots, v_h) = \langle v_1, \dots, v_h, w_1, \dots, w_k \rangle / \langle v_1, \dots, v_h \rangle$$

It is immediate to see that  $A$  is invariant if we change the  $k$ -tuple by Nielsen moves and relative Nielsen moves. Moreover, we can look at the ordered  $k$ -tuple of generators  $([w_1], \dots, [w_k])$  for the abelian group  $A$ , and we notice that Nielsen moves on  $(w_1, \dots, w_k)$  correspond to Nielsen moves on  $([w_1], \dots, [w_k])$ , while relative Nielsen moves on  $(w_1, \dots, w_k)$  correspond to the identity on  $([w_1], \dots, [w_k])$ .

The problem of classification of  $M$ -big ordered  $k$ -tuples of vectors up to Nielsen moves and relative Nielsen moves, and the problem of classification of ordered  $k$ -tuples of generators for  $A$  up to Nielsen equivalence, are strictly related. In fact, we will prove that they are equivalent for  $k \geq 3$ , regardless of  $M$ . The main result of Appendix B is the following theorem:

**Theorem B.1.** Let  $v_1, \dots, v_h \in \mathbb{Z}^n$  and let  $M \geq 1$ . Let  $k \geq 3$  and let  $(w_1, \dots, w_k), (w'_1, \dots, w'_k)$  be  $M$ -big ordered  $k$ -tuples. Then the following are equivalent:

1. There is a sequence of Nielsen moves and Nielsen moves relative to  $v_1, \dots, v_h$  going from  $(w_1, \dots, w_k)$  to  $(w'_1, \dots, w'_k)$  such that all the  $k$ -tuples that we write along the sequence are  $M$ -big.
2. We have that  $A(w_1, \dots, w_k; v_1, \dots, v_h) = A(w'_1, \dots, w'_k; v_1, \dots, v_h) = A$  and the two ordered  $k$ -tuples of generators  $([w_1], \dots, [w_k]), ([w'_1], \dots, [w'_k])$  are Nielsen equivalent in  $A$ .

## Preliminary results

We provide a couple of preliminary lemmas, that will be useful in what follows. Fix  $n \geq 1$  and vectors  $v_1, \dots, v_h$  in  $\mathbb{Z}^n$ ; fix also an integer  $M \geq 1$ .

**Lemma B.2.** Let  $k \geq 1$  and let  $(w_1, \dots, w_k)$  be an  $M$ -big ordered  $k$ -tuple. Then for every even permutation  $\sigma$  of  $\{1, \dots, k\}$  there is a sequence of Nielsen moves going from  $(w_1, \dots, w_k)$  to  $(w_{\sigma(1)}, \dots, w_{\sigma(k)})$  such that all the  $k$ -tuples that we write along the sequence are  $M$ -big.

*Proof.* Consider the sequence of Nielsen moves

$$(w_1, w_2, w_3) \rightarrow (w_1, w_2 + w_3, w_3) \rightarrow (w_1, w_2 + w_3, w_3 + w_1) \rightarrow (w_1 + w_2 + w_3, w_2 + w_3, w_3 + w_1) \rightarrow \\ \rightarrow (w_2, w_2 + w_3, w_3 + w_1) \rightarrow (w_2, w_3, w_3 + w_1) \rightarrow (w_2, w_3, w_1)$$

and notice that in the same way we can obtain any permutation which is a 3-cycle. But 3-cycles generate all even permutations; the conclusion follows.  $\square$

**Lemma B.3.** Let  $k \geq 2$  and let  $(w_1, w_2, \dots, w_k), (w'_1, w_2, \dots, w_k)$  be  $M$ -big ordered couples. Suppose that  $w'_1 = w_1 + \lambda_2 w_2 + \dots + \lambda_k w_k + \mu_1 v_1 + \dots + \mu_h v_h$  for some  $\lambda_2, \dots, \lambda_k, \mu_1, \dots, \mu_h \in \mathbb{Z}$ . Then there is a sequence of Nielsen moves and relative Nielsen moves going from  $(w_1, w_2, \dots, w_k)$  to  $(w'_1, w_2, \dots, w_k)$  such that all the  $k$ -tuples that we write along the sequence are  $M$ -big.

*Proof.* We choose an arbitrary positive integer  $C$  which is much bigger than all the components of  $w_2, \dots, w_k, v_1, \dots, v_h$  and than all of  $\lambda_2, \dots, \lambda_k, \mu_1, \dots, \mu_h$ . We proceed as follows

$$(w_1, w_2, \dots, w_k) \xrightarrow{*} (w_1 + Cw_2, w_2, \dots, w_k) \xrightarrow{*} (w'_1 + Cw_2, w_2, \dots, w_k) \xrightarrow{*} (w'_1, w_2, \dots, w_k)$$

where  $\xrightarrow{*}$  denotes a finite sequence of Nielsen moves and relative Nielsen moves. It's evident that the first and the third  $\xrightarrow{*}$  can be performed in such a way that every  $k$ -tuple that we write along the sequence is  $M$ -big. All the  $k$ -tuples that we write during the second  $\xrightarrow{*}$  are  $M$ -big, provided that  $C$  had been chosen big enough.  $\square$

**Proposition B.4.** Let  $k \geq 1$  and let  $(w_1, \dots, w_k), (w'_1, \dots, w'_k)$  be  $M$ -big ordered  $k$ -tuples such that  $A(w_1, \dots, w_k; v_1, \dots, v_h) = A(w'_1, \dots, w'_k; v_1, \dots, v_h) = A$ . Suppose there is a vector in  $\langle v_1, \dots, v_h \rangle$  such that all of its components are strictly positive. Then the following are equivalent:

1. There is a sequence of Nielsen moves and relative Nielsen moves going from  $(w_1, \dots, w_k)$  to  $(w'_1, \dots, w'_k)$  such that all the  $k$ -tuples that we write along the sequence are  $M$ -big.
2. The ordered  $k$ -uples of generators  $([w_1], \dots, [w_k]), ([w'_1], \dots, [w'_k])$  are Nielsen equivalent in  $A$ .

*Proof.*  $1 \Rightarrow 2$ . Trivial.

$2 \Rightarrow 1$ . Let  $v$  be a vector in  $\langle v_1, \dots, v_h \rangle$  with all components strictly positive. For every element  $[w] \in A$  we can construct a lifting  $w \in \langle w_1, \dots, w_k, v_1, \dots, v_h \rangle$  such that all the components of  $w$  are  $\geq M$ ; in fact, we can just pick any lifting and add  $v$  repeatedly until the components become all  $\geq M$ .

Suppose there is a sequence of Nielsen moves going from  $([w_1], \dots, [w_k])$  to  $([w'_1], \dots, [w'_k])$  in  $A$ . Let  $([w_1^r], \dots, [w_k^r])$  be such a sequence, for  $r = 1, \dots, R$ , so that  $[w_1^1] = [w_1]$  and  $[w_k^R] = [w'_k]$  for  $i = 1, \dots, k$ . Let  $w_i^r$  be an  $M$ -big lifting of  $[w_i^r]$  for  $i = 1, \dots, k$  and  $r = 1, \dots, R$ , and we can take  $w_i^1 = w_i$  and  $w_i^R = w'_i$  for  $i = 1, \dots, k$ . By Lemma B.3, it's possible to pass from  $(w_1^r, \dots, w_k^r)$  to  $(w_1^{r+1}, \dots, w_k^{r+1})$  with a sequence of Nielsen moves and relative Nielsen moves. The conclusion follows.  $\square$

## Nielsen equivalence for three or more big vectors

Fix  $n \geq 1$  and vectors  $v_1, \dots, v_h$  in  $\mathbb{Z}^n$ ; fix also an integer  $M \geq 1$ . This section is dedicated to the proof of Theorem B.1. The idea is to try and apply Proposition B.4. When we don't have a positive vector in  $\langle v_1, \dots, v_h \rangle$ , we essentially use one of the  $w_i$ s instead. This means that, given two  $M$ -big ordered  $k$ -tuples  $(w_1, \dots, w_k), (w'_1, \dots, w'_k)$ , we want to manipulate them until they end up with a common vector. The following Proposition B.5 explains how to manipulate one of the two  $k$ -tuples in order to get a vector "almost" in common with the other, up to an integer multiple; the next Propositions B.7 and B.8 are aimed at removing the integer multiple, and to make the vector "really" in common to the  $k$ -tuples.

**Proposition B.5.** *Let  $k \geq 3$  and let  $(w_1, \dots, w_k), (w'_1, \dots, w'_k)$  be  $M$ -big ordered  $k$ -tuples such that  $A(w_1, \dots, w_k; v_1, \dots, v_h) = A(w'_1, \dots, w'_k; v_1, \dots, v_h) = A$ . Then there is an  $M$ -big ordered  $k$ -tuple  $(w''_1, \dots, w''_k)$  such that:*

1. *There is a sequence of Nielsen moves and relative Nielsen moves going from  $(w_1, \dots, w_k)$  to  $(w''_1, \dots, w''_k)$  such that all the  $k$ -tuples that we write along the sequence are  $M$ -big.*
2. *We have  $w'_1 = d(w''_1 - w''_2) + \mu_1 v_1 + \dots + \mu_h v_h$  for some  $d, \mu_1, \dots, \mu_h \in \mathbb{Z}$ .*

*Proof.* Let  $w'_1 = \lambda_1 w_1 + \dots + \lambda_k w_k + \mu_1 v_1 + \dots + \mu_h v_h$  for  $\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_h \in \mathbb{Z}$ . We notice that for  $i \neq j$  we have the identity

$$\lambda_i w_i + \lambda_j w_j = (\lambda_i - \lambda_j) w_i + \lambda_j (w_j + w_i)$$

For an ordered  $k$ -tuple of integers  $(\lambda_1, \dots, \lambda_k)$  we define a *subtraction* to be any operation that consists of substituting  $\lambda_i$  with  $\lambda_i - \lambda_j$  for some  $i \neq j$ . We can perform any sequence of subtractions to the  $k$ -tuple of integers  $(\lambda_1, \dots, \lambda_k)$ , as these are achieved by means of Nielsen moves on the  $k$ -tuple of vectors  $(w_1, \dots, w_k)$  that keep it  $M$ -big.

We now work on the ordered  $k$ -tuple of integers  $(\lambda_1, \dots, \lambda_k)$ . As  $k \geq 3$ , it's easy to see that by means of subtractions we can assume that  $\lambda_k < 0$ . We now subtract  $\lambda_k$  from each of  $\lambda_1, \dots, \lambda_{k-1}$  enough times, in order to get  $\lambda_1, \dots, \lambda_{k-1} > 0$ . By means of Lemma B.6, we can subtract  $\lambda_k$  from  $\lambda_1$  the right amount of times, in order to get  $\gcd(\lambda_1, \dots, \lambda_{k-1}) = \gcd(\lambda_1, \dots, \lambda_k)$ . We now use subtractions on the non-negative integers  $\lambda_1, \dots, \lambda_{k-1}$  in order to perform the Euclidean algorithm (and here we need  $k \geq 3$ ): we reduce to a situation of the type  $d, 0, \dots, 0, -cd$  for some integers  $c, d \geq 1$ . From this we obtain  $d, -d, 0, \dots, 0, -cd$  and finally  $d, -d, 0, \dots, 0$ . The conclusion follows.  $\square$

**Lemma B.6.** *Let  $a, b, c \in \mathbb{Z}$  with  $b \neq 0$ . Then there is  $\lambda \in \mathbb{Z}$  such that*

$$\gcd(a, b, c) = \gcd(a + \lambda c, b)$$

*Moreover, the same holds with  $\lambda'$  instead of  $\lambda$ , where  $\lambda'$  is any element of the coset  $\lambda + b\mathbb{Z}$ .*

*Proof.* Let  $p \mid b$  be a prime. If  $v_p(c) > v_p(a)$  then we have  $v_p(a + \lambda c) = v_p(a) = \min\{v_p(a), v_p(c)\}$ . If  $v_p(c) < v_p(a)$  then we impose the condition  $\lambda \not\equiv 0 \pmod{p}$ ; this implies that  $v_p(a + \lambda c) = v_p(c) = \min\{v_p(a), v_p(c)\}$ . If  $v_p(c) = v_p(a)$  then we write  $a = p^{v_p(a)} a'$  and  $c = p^{v_p(c)} c'$  with  $a', c'$  coprime with  $p$ , and we impose the condition  $\lambda \not\equiv -a'/c' \pmod{p}$ ; this implies that  $v_p(a + \lambda c) = v_p(a) = v_p(c)$ .

By the Chinese remainder theorem, we can find  $\lambda$  satisfying the above conditions for all primes  $p \mid b$ . Now for every prime  $p$  that divides  $b$  we have  $v_p(\gcd(a + \lambda c, b)) = \min\{v_p(a + \lambda c), v_p(b)\} = \min\{v_p(a), v_p(b), v_p(c)\} = v_p(\gcd(a, b, c))$ , and for every prime  $p$  that doesn't divide  $b$  we have  $v_p(\gcd(a + \lambda c, b)) = 0 = v_p(\gcd(a, b, c))$ . It follows that  $\gcd(a + \lambda c, b) = \gcd(a, b, c)$ , as desired.  $\square$

**Proposition B.7.** *Let  $k \geq 2$  and let  $(w_1, \dots, w_k)$  be an  $M$ -big ordered  $k$ -tuple. Suppose  $A = A(w_1, \dots, w_k; v_1, \dots, v_h) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus A'$  for some finitely generated abelian group  $A'$ . Then there is an  $M$ -big ordered  $k$ -tuple  $(w'_1, \dots, w'_k)$  such that:*

1. *There is a sequence of Nielsen moves and relative Nielsen moves going from  $(w_1, \dots, w_k)$  to  $(w'_1, \dots, w'_k)$  such that each  $k$ -tuple that we write along the sequence is  $M$ -big.*
2. *The first two components of  $[w'_1] \in A \cong \mathbb{Z} \oplus \mathbb{Z} \oplus A'$  are coprime, and  $[w'_1]$  is not a proper power.*

*Proof.* Let  $[w_i]$  be equal to the vector  $(\alpha_i, \beta_i, \gamma_i)$  through the isomorphism  $A \cong \mathbb{Z} \oplus \mathbb{Z} \oplus A'$ , for  $i = 1, \dots, k$  and  $\alpha_i, \beta_i \in \mathbb{Z}$  and  $\gamma_i \in A'$ . Since  $[w_1], \dots, [w_k]$  generate  $A$ , we must have  $\alpha_i \beta_j - \alpha_j \beta_i \neq 0$  for some  $i, j \in \{1, \dots, k\}$ . By Lemma B.2 we can reorder the vectors with an arbitrary even permutation. Thus we can assume

$$\alpha_1 \beta_2 - \alpha_2 \beta_1 \neq 0.$$

If  $\alpha_2 = 0$  then  $\alpha_1 \neq 0$  (since  $\alpha_1 \beta_2 - \alpha_2 \beta_1 \neq 0$ ) and we substitute  $w_2$  with  $w_2 + w_1$ . Thus we can assume

$$\alpha_2 \neq 0.$$

By Lemma B.6 we find  $\lambda \geq 0$  such that  $\gcd(\alpha_1 + \lambda \alpha_3, \alpha_2) = \gcd(\alpha_1, \alpha_2, \alpha_3)$ . We substitute  $w_1$  with  $w_1 + \lambda w_3$  and thus we can assume  $\gcd(\alpha_1, \alpha_2) = \gcd(\alpha_1, \alpha_2, \alpha_3)$ . Since  $\lambda$  can be chosen modulo  $\alpha_2$ , we can also take  $\lambda$  in such a way that  $\alpha_1 \beta_2 - \alpha_2 \beta_1$  remains  $\neq 0$ . Reiterating the same reasoning, we obtain  $\gcd(\alpha_1, \alpha_2) = \gcd(\alpha_1, \dots, \alpha_k)$ , which is 1 since  $[w_1], \dots, [w_k]$  generate  $A$ . Thus we can assume

$$\gcd(\alpha_1, \alpha_2) = 1.$$

By Lemma B.6 we find  $C \geq 0$  such that  $\gcd(\alpha_1 + C\alpha_2, \alpha_1 \beta_2 - \alpha_2 \beta_1) = \gcd(\alpha_1, \alpha_2, \alpha_1 \beta_2 - \alpha_2 \beta_1) = 1$ . We now have that

$$\gcd(\alpha_1 + C\alpha_2, \alpha_1 \beta_2 - \alpha_2 \beta_1 - \beta_2(\alpha_1 + C\alpha_2)) = 1$$

$$\gcd(\alpha_1 + C\alpha_2, -\alpha_2(\beta_1 + C\beta_2)) = 1$$

$$\gcd(\alpha_1 + C\alpha_2, \beta_1 + C\beta_2) = 1$$

We substitute  $w_1$  with  $w_1 + Cw_2$  and thus obtain that  $(\alpha_1, \beta_1) = 1$ . This also implies that the element  $[w_1]$  can't be a proper power in  $A$ .  $\square$

**Proposition B.8.** *Let  $k \geq 3$  and let  $(w_1, \dots, w_k)$  be an  $M$ -big ordered  $k$ -tuple. Suppose that  $A = A(w_1, \dots, w_k; v_1, \dots, v_h) \cong \mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus A'$  for some  $d_2 \geq 2$  and  $A'$  finitely generated abelian group. Then there is an  $M$ -big ordered  $k$ -tuple  $(w'_1, \dots, w'_k)$  such that:*

1. *There is a sequence of Nielsen moves and relative Nielsen moves going from  $(w_1, \dots, w_k)$  to  $(w'_1, \dots, w'_k)$  such that each  $k$ -tuple that we write along the sequence is  $M$ -big.*
2. *The first two components of  $[w'_1] \in A \cong \mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus A'$  are coprime, and  $[w'_1]$  is not a proper power.*

*Proof.* Let  $[w_i]$  be equal to the vector  $(\alpha_i, \beta_i, \gamma_i)$  through the isomorphism  $A \cong \mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus A'$ , for  $i = 1, \dots, k$  and  $\alpha_i \in \mathbb{Z}$  and  $\beta_i \in \mathbb{Z}/d_2\mathbb{Z}$  and  $\gamma_i \in A'$ . Let also  $p$  be a prime that divides  $d_2$ .

Since  $[w_1], \dots, [w_k]$  generate  $A$ , we must have  $\alpha_i \beta_j - \alpha_j \beta_i \not\equiv 0 \pmod p$  for some  $i, j \in \{1, \dots, k\}$ . By Lemma B.2 we can reorder the vectors with an arbitrary even permutation. Thus we can assume

$$\alpha_1 \beta_2 - \alpha_2 \beta_1 \not\equiv 0 \pmod p.$$

If  $\alpha_2 \equiv 0 \pmod p$  then  $\alpha_1 \not\equiv 0 \pmod p$  (since  $\alpha_1 \beta_2 - \alpha_2 \beta_1 \not\equiv 0 \pmod p$ ) and we substitute  $w_2$  with  $w_2 + w_1$ . Thus we can assume

$$\alpha_2 \not\equiv 0 \pmod p.$$

By Lemma B.6 we find  $\lambda \geq 0$  such that  $\gcd(\alpha_1 + \lambda \alpha_3, \alpha_2) = \gcd(\alpha_1, \alpha_2, \alpha_3)$ . We substitute  $w_1$  with  $w_1 + \lambda w_3$  and thus we can assume  $\gcd(\alpha_1, \alpha_2) = \gcd(\alpha_1, \alpha_2, \alpha_3)$ . Since  $\lambda$  can be chosen modulo  $\alpha_2$ , which is coprime with  $p$ , we can also take  $\lambda$  to be a multiple of  $p$ , so that  $\alpha_1 \beta_2 - \alpha_2 \beta_1 \pmod p$  remains the same. Reiterating the same reasoning, we obtain  $\gcd(\alpha_1, \alpha_2) = \gcd(\alpha_1, \dots, \alpha_k)$ , which is 1 since  $[w_1], \dots, [w_k]$  generate  $A$ . Thus we can assume

$$\gcd(\alpha_1, \alpha_2) = 1.$$

By means of the Chinese remainder theorem we find  $C \geq 0$  such that  $\alpha_3 + C\alpha_1 \equiv 1 \pmod{\alpha_2}$  and  $(\alpha_3 \beta_2 - \alpha_2 \beta_3) + C(\alpha_1 \beta_2 - \alpha_2 \beta_1) \not\equiv 0 \pmod p$ ; here we are using the facts that  $\gcd(\alpha_2, p) = 1$  and  $\gcd(\alpha_1, \alpha_2) = 1$  and  $\gcd(\alpha_1 \beta_2 - \alpha_2 \beta_1, p) = 1$ . Since  $\gcd(\alpha_2, p) = 1$  we can find integers  $e \geq 0$  as

big as we want and such that  $p^e \equiv 1 \pmod{\alpha_2}$ . Using the fact that  $\alpha_3 + C\alpha_1 \equiv 1 \pmod{\alpha_2}$  we are thus able to find integers  $D \geq 0$  and  $e \geq 0$  such that  $\alpha_3 + C\alpha_1 + D\alpha_2 = p^e$ . We now have that

$$\begin{aligned} \gcd(p, (\alpha_3\beta_2 - \alpha_2\beta_3) + C(\alpha_1\beta_2 - \alpha_2\beta_1)) &= 1 \\ \gcd(p, (\alpha_3\beta_2 - \alpha_2\beta_3) + C(\alpha_1\beta_2 - \alpha_2\beta_1) - \beta_2(\alpha_3 + C\alpha_1 + D\alpha_2)) &= 1 \\ \gcd(p, -\alpha_2(\beta_3 + C\beta_1 + D\beta_2)) &= 1 \\ \gcd(p, \beta_3 + C\beta_1 + D\beta_2) &= 1 \end{aligned}$$

We substitute  $w_3$  with  $w_3 + Cw_1 + Dw_2$  and thus we can assume that  $\alpha_3 = p^e$  and  $(p, \beta_3) = 1$ . In particular  $(\alpha_3, \beta_3) = 1$ . The element  $[w_3]$  can be a proper power in  $A$ : if  $[w_3] = d[w]$ , then  $\alpha_3 = p^e$  implies that  $d \mid p^e$ , and  $(p, \beta_3) = 1$  implies that  $(p, d) = 1$ ; it follows that  $d = \pm 1$ . To conclude, we permute  $(w_1, w_2, w_3)$  using Lemma B.2 in order to bring  $w_3$  in first position.  $\square$

**Proposition B.9.** *Let  $k \geq 3$  and let  $(w_1, \dots, w_k), (w'_1, \dots, w'_k)$  be  $M$ -big ordered  $k$ -tuples such that  $A(w_1, \dots, w_k; v_1, \dots, v_h) = A(w'_1, \dots, w'_k; v_1, \dots, v_h) = A \cong \mathbb{Z}$ . Then there is a sequence of Nielsen moves and relative Nielsen moves from  $(w_1, \dots, w_k)$  to  $(w'_1, \dots, w'_k)$  such that every  $k$ -tuple that we write along the sequence is  $M$ -big.*

*Proof.* Let  $[w_i], [w'_i]$  be equal to  $\alpha_i, \alpha'_i$  through the isomorphism  $A \cong \mathbb{Z}$ , for  $i = 1, \dots, k$  and  $\alpha_i, \alpha'_i \in \mathbb{Z}$ . If among  $\alpha_i, \alpha'_i$  there is at least one integer  $> 0$  and at least one integer  $< 0$ , then  $\langle v_1, \dots, v_h \rangle$  has to contain a vector whose components are all strictly positive, and thus we are done by Proposition B.4. Thus we assume  $\alpha_i, \alpha'_i > 0$  for all  $i = 1, \dots, k$ .

By Lemma B.6 we find  $\lambda \geq 0$  such that  $\gcd(\alpha_1 + \lambda\alpha_3, \alpha_2) = \gcd(\alpha_1, \alpha_2, \alpha_3)$ . We substitute  $w_1$  with  $w_1 + \lambda w_3$  and thus we can assume  $\gcd(\alpha_1, \alpha_2) = \gcd(\alpha_1, \alpha_2, \alpha_3)$ . By reiterating the same reasoning, we can assume  $\gcd(\alpha_1, \alpha_2) = \gcd(\alpha_1, \dots, \alpha_k) = 1$ . Now, by substituting  $w_3$  with  $w_3 + \lambda_1 w_1 + \lambda_2 w_2$  for  $\lambda_1, \lambda_2 \geq 0$ , we can set  $\alpha_3$  to be equal to any natural number big enough. Similarly, we can assume  $\gcd(\alpha'_1, \alpha'_2) = 1$ , and we are able to set  $\alpha'_3$  to be any natural number big enough. We choose to set  $\alpha_3 = \alpha'_3$  and such that  $\gcd(\alpha_1, \alpha_3) = \gcd(\alpha'_1, \alpha_3) = 1$ . Now, by substituting  $w_2$  with  $w_2 + \mu_1 w_1 + \mu_3 w_3$  for  $\mu_1, \mu_3 \geq 0$ , we can set  $\alpha_2$  to be any natural number big enough. Similarly, we are able to set  $\alpha'_2$  to be any natural number big enough. We choose to set  $\alpha_2 = \alpha'_2$  and such that  $\gcd(\alpha_2, \alpha_3) = 1$ . Reasoning in the same way, and keeping fixed  $w_2, w_3, w'_2, w'_3$ , we can set  $\alpha_1 = \alpha'_1$  and  $\alpha_i = \alpha'_i$  for  $i \geq 4$ .

To conclude, for  $i = 1, \dots, k$ , we observe that  $\alpha_i = \alpha'_i$  implies that  $w'_i = w_i + \mu_1 v_1 + \dots + \mu_h v_h$  for some  $\mu_1, \dots, \mu_h \in \mathbb{Z}$  and thus by Lemma B.3 we can obtain  $w_i = w'_i$  by means of Nielsen moves and relative Nielsen moves. The thesis follows.  $\square$

*Proof of Theorem B.1.*  $1 \Rightarrow 2$ . Trivial.

$2 \Rightarrow 1$ . If  $A$  is a finite abelian group, then we must have  $d[w_1] = 0$  in  $A$  for some  $d \geq 1$ , and this implies that  $\langle v_1, \dots, v_h \rangle$  contains the vector  $dw_1$ , which has all the components strictly positive; the conclusion follows from Proposition B.4. If  $A$  is isomorphic to  $\mathbb{Z}$ , then we are done by Proposition B.9.

Otherwise we write  $A \cong \mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/d_m\mathbb{Z}$  for integers  $m \geq 2$  and  $d_2, \dots, d_m \geq 0$  with  $1 \neq d_m \mid d_{m-1} \mid \dots \mid d_2$ . By either Proposition B.7 or Proposition B.8, we are able to change  $(w'_1, \dots, w'_k)$  by means of Nielsen moves and relative Nielsen moves in such a way that  $[w'_1]$  isn't a proper power in  $A$ , and that  $[w'_1]$  can be represented by a vector whose first two components are coprime. We now apply Proposition B.5 and we apply a sequence of Nielsen moves and relative Nielsen moves to  $(w_1, \dots, w_k)$ , in order to obtain that  $w'_1 = w_1 - w_2 + \mu_1 v_1 + \dots + \mu_h v_h$ . By means of Proposition B.3 we can easily change  $w_1$  in such a way that  $w_1 = w'_1$ .

If  $m \geq 3$  then we can apply Lemma A.7, since  $[w'_1]$  can be represented by a vector whose first two components are coprime, and we obtain that  $([w_2], \dots, [w_k]), ([w'_2], \dots, [w'_k])$  are Nielsen equivalent  $(k-1)$ -tuples of generators for  $A/\langle [w_1] \rangle$ . If  $m = 2$  then  $[w'_1]$  can be represented by a vector whose components are coprime, and thus we can apply Lemma A.6 to obtain that  $([w_2], \dots, [w_k]), ([w'_2], \dots, [w'_k])$  are Nielsen equivalent  $(k-1)$ -tuples of generators for  $A/\langle [w_1] \rangle$ . In both cases, the conclusion follows from Proposition B.4.  $\square$