On the isomorphism problem for generalized Baumslag-Solitar groups: angles and flexible configurations

Dario Ascari

Department of Mathematics, University of the Basque Country,

Barrio Sarriena, Leioa, 48940, Spain

e-mail: ascari.maths@gmail.com

Montserrat Casals-Ruiz

Ikerbasque - Basque Foundation for Science and Matematika Saila,

UPV/EHU, Sarriena s/n, 48940, Leioa - Bizkaia, Spain

e-mail: montsecasals@gmail.com

Ilya Kazachkov

Ikerbasque - Basque Foundation for Science and Matematika Saila,

UPV/EHU, Sarriena s/n, 48940, Leioa - Bizkaia, Spain

e-mail: ilya.kazachkov@gmail.com

1 Introduction

Limit angles Theorem 5.7.

Theorem A (See Theorem 4.14)**.** *There is an algorithm that, given two GBS graph* $(\Gamma, \psi), (\Delta, \phi)$*, where* (Γ, ψ) *has one vertex and two edges, decides whether the corresponding GBS groups are isomorphic or not. In case they are, the algorithm also computes a sequence of sign-changes, inductions, slides, swaps, connections going from* (Γ, ψ) *to* (Δ, ϕ)*.*

2 GBS graphs and quasi-conjugacy classes

In this section we set up the notation for the rest of the paper. We review the notions of conjugacy and quasi-conjugacy, and we show how they can be described by means of a finite amount of data. We provide explicit algorithms for dealing with them, and examples to illustrate these notions.

2.1 Graphs of groups

We consider graphs as combinatorial objects, following the notation of [Ser77]. A **graph** is a quadruple $\Gamma = (V, E, \bar{\cdot}, \iota)$ consisting of a set $V = V(\Gamma)$ of *vertices*, a set $E = E(\Gamma)$ of *edges*, a map $\bar{\cdot} : E \to E$ called *reverse* and a map $\iota : E \to V$ called *initial endpoint*; we require that, for every edge $e \in E$, we have $\bar{e} \neq e$ and $\bar{\bar{e}} = e$. For an edge $e \in E$, we denote with $\tau(e) = \iota(\bar{e})$ the *terminal endpoint* of e. A **path** in a graph Γ, with *initial endpoint* $v \in V(\Gamma)$ and *terminal endpoint* $v' \in V(\Gamma)$, is a sequence $\sigma = (e_1, ..., e_\ell)$ of edges $e_1, ..., e_\ell \in E(\Gamma)$ for some integer $\ell \geq 0$, with the conditions $\iota(e_1) = v$ and $\tau(e_\ell) = v'$ and $\tau(e_i) = \iota(e_{i+1})$ for $i = 1, ..., \ell - 1$. A graph is **connected** if for every couple of vertices, there is a path going from one to the other. For a connected graph Γ, we define its **rank** $\mathrm{rk}\,\Gamma \in \mathbb{N} \cup \{+\infty\}$ as the rank of its fundamental group (which is a free group).

Definition 2.1. *A* ***graph of groups*** *is a quadruple*

$$\mathcal{G} = (\Gamma, \{G_v\}_{v\in V(\Gamma)}, \{G_e\}_{e\in E(\Gamma)}, \{\psi_e\}_{e\in E(\Gamma)})$$

consisting of a connected graph Γ, a group G_v for each vertex $v \in V(\Gamma)$, a group G_e for every edge $e \in E(\Gamma)$ with the condition $G_e = G_{\bar{e}}$, and an injective homomorphism $\psi_e : G_e \to G_{\tau(e)}$ for every edge $e \in E(\Gamma)$.

Let $\mathcal{G} = (\Gamma, \{G_v\}_{v\in V(\Gamma)}, \{G_e\}_{e\in E(\Gamma)}, \{\psi_e\}_{e\in E(\Gamma)})$ be a graph of groups. Define the **universal group** $\mathrm{FG}(\mathcal{G})$ as the quotient of the free product $(*_{v\in V(\Gamma)} G_v) * F(E(\Gamma))$ by the relations

$$\bar{e} = e^{-1} \qquad\qquad \psi_{\bar{e}}(g) \cdot e = e \cdot \psi_e(g)$$

for $e \in E(\Gamma)$ and $g \in G_e$.

Define the **fundamental group** $\pi_1(\mathcal{G}, \widetilde{v})$ of a graph of group $\mathcal{G}$ with basepoint $\widetilde{v} \in V(\Gamma)$ to be the subgroup of $\mathrm{FG}(\mathcal{G})$ of the elements that can be represented by words such that, when going along the word, we read a path in the graph Γ from $\widetilde{v}$ to $\widetilde{v}$. The fundamental group $\pi_1(\mathcal{G}, \widetilde{v})$ doesn't depend on the chosen basepoint $\widetilde{v}$, up to isomorphism. Given an element g inside a vertex group G_v, we can take a path $(e_1, ..., e_\ell)$ from $\widetilde{v}$ to v: we define the conjugacy class $[g] := [e_1...e_\ell g \bar{e}_\ell ... \bar{e}_1] \in \pi_1(\mathcal{G}, \widetilde{v})$, and notice that this doesn't depend on the chosen path.

2.2 Generalized Baumslag-Solitar groups

Definition 2.2. *A* ***GBS graph of groups*** *is a finite graph of groups*

$$\mathcal{G} = (\Gamma, \{G_v\}_{v\in V(\Gamma)}, \{G_e\}_{e\in E(\Gamma)}, \{\psi_e\}_{e\in E(\Gamma)})$$

such that each vertex group and each edge group is $\mathbb{Z}$.

A **Generalized Baumslag-Solitar group** is a group G isomorphic to the fundamental group of some GBS graph of groups.

Definition 2.3. *A* ***GBS graph*** *is a couple (Γ, ψ) where Γ is a finite graph and $\psi : E(\Gamma) \to \mathbb{Z}\setminus\{0\}$ is a function.*

Given a GBS graph of groups $\mathcal{G} = (\Gamma, \{G_v\}_{v\in V(\Gamma)}, \{G_e\}_{e\in E(\Gamma)}, \{\psi_e\}_{e\in E(\Gamma)})$, the map $\psi_e : G_e \to G_{\tau(e)}$ is an injective homomorphism $\psi_e : \mathbb{Z} \to \mathbb{Z}$, and thus coincides with multiplication by a unique non-zero integer $\psi(e) \in \mathbb{Z} \setminus \{0\}$. We define the GBS graph associated to $\mathcal{G}$ as (Γ, ψ) associating to each edge e the factor $\psi(e)$ characterizing the homomorphism ψ_e, see Figure 1. Giving a GBS graph of groups is equivalent to giving the corresponding GBS graph. In fact, the numbers on the edges are enough to reconstruct the injective homomorphisms and thus the graph of groups.

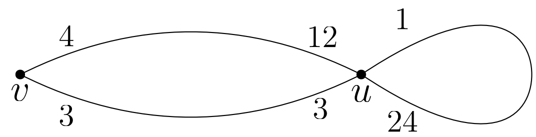


Figure 1: In the figure we can see a GBS graph (Γ, ψ) with two vertices v, u and three edges e_1, e_2, e_3 (and their reverses). The edge e_1 goes from v to u and has $\psi(\overline{e}_1) = 4$ and $\psi(e_1) = 12$. The edge e_2 goes from v to u and has $\psi(\overline{e}_2) = \psi(\overline{e}_2) = 3$. The edge e_3 goes from u to u and has $\psi(\overline{e}_3) = 1$ and $\psi(e_3) = 24$.

Let $\mathcal{G}$ be a GBS graph of groups and let (Γ, ψ) be the corresponding GBS graph. The universal group $\mathrm{FG}(\mathcal{G})$ has a presentation with generators $V(\Gamma) \cup E(\Gamma)$, the generator $v \in V(\Gamma)$ representing the element 1 in $\mathbb{Z} = G_v$. The relations are given by $\overline{e} = e^{-1}$ and $u^{\psi(\overline{e})} e = e v^{\psi(e)}$ for every edge $e \in E(\Gamma)$ with $\iota(e) = u$ and $\tau(e) = v$.

2.3 Reduced affine representation of a GBS graph

Definition 2.4. *For a GBS graph (Γ, ψ), define its* ***set of primes***

$$\mathcal{P}(\Gamma, \psi) := \{r \in \mathbb{N} \text{ prime } : r \mid \psi(e) \text{ for some } e \in E(\Gamma)\}$$

Given a GBS graph (Γ, ψ), consider the finitely generated abelian group

$$\mathbf{A} := \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{r \in \mathcal{P}(\Gamma, \psi)} \mathbb{Z}.$$

We denote with $\mathbf{0} \in \mathbf{A}$ the neutral element. For an element $\mathbf{a} = (a_0, a_r : r \in \mathcal{P}(\Gamma, \psi)) \in \mathbf{A}$ (with $a_0 \in \mathbb{Z}/2\mathbb{Z}$ and $a_r \in \mathbb{Z}$ for $r \in \mathcal{P}(\Gamma, \psi)$), we denote $\mathbf{a} \geq \mathbf{0}$ if $a_r \geq 0$ for all $r \in \mathcal{P}(\Gamma, \psi)$; notice that we aren't requiring any condition on a_0. We define the positive cone $\mathbf{A}^+ := \{\mathbf{a} \in \mathbf{A} : \mathbf{a} \geq \mathbf{0}\}$.

Definition 2.5. *Let (Γ, ψ) be a GBS graph. Define its* ***(reduced) affine representation*** *to be the graph $\Lambda = \Lambda(\Gamma, \psi)$ given by:*

1. *$V(\Lambda) = V(\Gamma) \times \mathbf{A}^+$ is the disjoint union of copies of $\mathbf{A}^+$, one for each vertex of Γ.*
2. *$E(\Lambda) = E(\Gamma)$ is the same set of edges as Γ, and with the same reverse map.*
3. *For an edge $e \in E(\Lambda)$ we write the unique factorization $\psi(e) = (-1)^{a_0} \prod_{r \in \mathcal{P}(\Gamma, \psi)} r^{a_r}$ and we define the terminal endpoint $\tau_\Lambda(e) = (\tau_\Gamma(e), (a_0, a_r, ...))$, see Figure 2.*

For a vertex $v \in V(\Gamma)$ we denote $\mathbf{A}_v^+ := \{v\} \times \mathbf{A}^+$ the corresponding copy of $\mathbf{A}^+$.

If Λ contains an edge going from p to q, then we denote p —— q. If Λ contains edges from p_i to q_i for $i = 1, ..., m$, then we denote

$$\begin{cases} p_1 \text{ —— } q_1 \\ ... \\ p_m \text{ —— } q_m \end{cases}$$

This doesn't mean that p_1 —— $q_1, ..., p_m$ —— q_m are all the edges of Λ, but only that in a certain situation we are focusing on those edges. If we are focusing on a specific copy $\mathbf{A}_v^+$ of $\mathbf{A}^+$, for some $v \in V(\Gamma)$, and we have edges $(v, \mathbf{a}_i)$ —— $(v, \mathbf{b}_i)$ for $i = 1, ..., m$, then we say that $\mathbf{A}_v^+$ contains edges

$$\begin{cases} \mathbf{a}_1 \text{ —— } \mathbf{b}_1 \\ ... \\ \mathbf{a}_m \text{ —— } \mathbf{b}_m \end{cases}$$

omitting the vertex v.

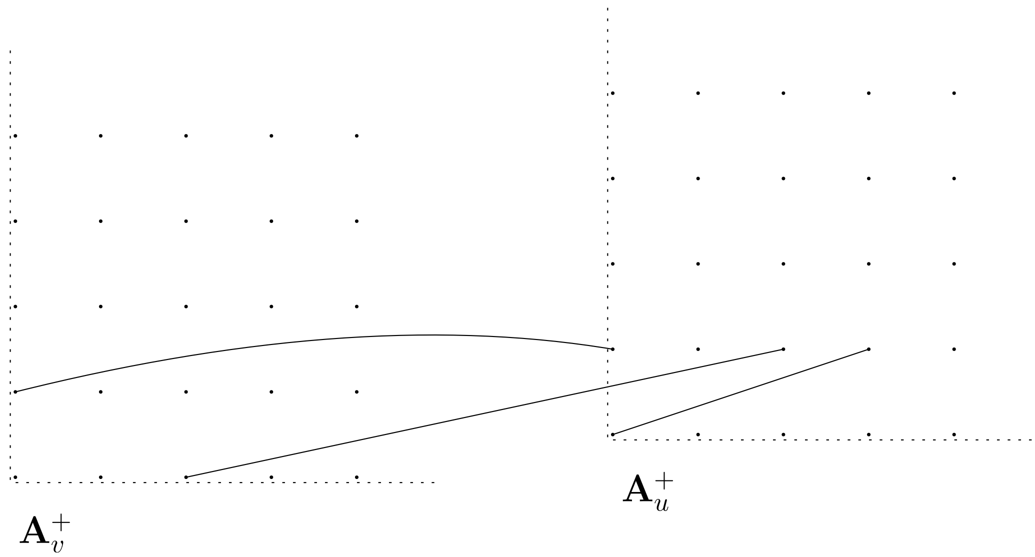


Figure 2: The affine representation Λ of the GBS graph (Γ, ψ) of Figure 1. The set of vertices consists of two copies $\mathbf{A}_v^+$ and $\mathbf{A}_u^+$ of the positive affine cone $\mathbf{A}^+$, associated to the two vertices v and u respectively. The edge e_1 going from v to u was labeled with $(\psi(\overline{e}_1), \psi(e_1)) = (4, 12) = (2^2 3^0, 2^2 3^1)$, and thus now it goes from the point $(2, 0)$ in $\mathbf{A}_v^+$ to the point $(2, 1)$ in $\mathbf{A}_u^+$. Similarly for e_2 and e_3.

2.4 Support and control of vectors

The following notions for elements of $\mathbf{A}$ will be widely used along the paper.

Definition 2.6. *For $\mathbf{x} \in \mathbf{A}$ define its **support** as the set*

$$\operatorname{supp}(\mathbf{x}) := \{r \in \mathcal{P}(\Gamma, \psi) : x_r \neq 0\}$$

Remark 2.7. Note that we omit the $\mathbb{Z}/2\mathbb{Z}$ component from the definition of support.

Definition 2.8. *Let $\mathbf{a}, \mathbf{b}, \mathbf{w} \in \mathbf{A}^+$. We say that $\mathbf{a}, \mathbf{w}$ **controls** $\mathbf{b}$ if any of the following equivalent conditions holds:*

1. *We have $\mathbf{a} \leq \mathbf{b} \leq \mathbf{a} + k\mathbf{w}$ for some $k \in \mathbb{N}$.*
2. *We have $\mathbf{b} - \mathbf{a} \geq \mathbf{0}$ and $\operatorname{supp}(\mathbf{b} - \mathbf{a}) \subseteq \operatorname{supp}(\mathbf{w})$.*

2.5 Affine paths and conjugacy classes

Let (Γ, ψ) be a GBS graph and let Λ be its affine representation. Given a vertex $p = (v, \mathbf{a}) \in V(\Lambda)$ and an element $\mathbf{w} \in \mathbf{A}^+$, we define the vertex $p + \mathbf{w} := (v, \mathbf{a} + \mathbf{w}) \in V(\Lambda)$. For two vertices $p, p' \in V(\Lambda)$ we denote $p' \geq p$ if $p' = p + \mathbf{w}$ for some $\mathbf{w} \in \mathbf{A}^+$; in particular this implies that both p, p' belong to the same $\mathbf{A}_v^+$ for some $v \in V(\Gamma)$.

Definition 2.9. *An **affine path** in Λ, with initial endpoint $p \in V(\Lambda)$ and terminal endpoint $p' \in V(\Lambda)$, is a sequence $(e_1, ..., e_\ell)$ of edges $e_1, ..., e_\ell \in E(\Lambda)$ for some $\ell \geq 0$, such that there exist $\mathbf{w}_1, ..., \mathbf{w}_\ell \in \mathbf{A}^+$ satisfying the conditions $\iota(e_1) + \mathbf{w}_1 = p$ and $\tau(e_\ell) + \mathbf{w}_\ell = p'$ and $\tau(e_i) + \mathbf{w}_i = \iota(e_{i+1}) + \mathbf{w}_{i+1}$ for $i = 1, ..., \ell - 1$.*

The elements $\mathbf{w}_1, ..., \mathbf{w}_\ell$ are called *translation coefficients* of the path; if they exist, then they are uniquely determined by the path and by the endpoints, and they can be computed algorithmically. They mean that an edge $e \in E(\Lambda)$ connecting p to q allows us also to travel from $p + \mathbf{w}$ to $q + \mathbf{w}$ for every $\mathbf{w} \in \mathbf{A}^+$.

Definition 2.10. *Let $p, q \in V(\Lambda)$.*

1. *We denote $p \sim_{\mathrm{c}} q$, and we say that p, q are **conjugate**, if there is an affine path going from p to q.*

2. *We denote $p \preceq_{\mathrm{c}} q$ if $p \leq q'$ for some $q' \sim_{\mathrm{c}} q$.*

3. *We denote $p \sim_{\mathrm{qc}} q$, and we say that p, q are **quasi-conjugate**, if $p \preceq_{\mathrm{c}} q$ and $q \preceq_{\mathrm{c}} p$.*

The relation $\sim_{\mathrm{c}}$ is an equivalence relations on the set $V(\Lambda)$. The relation $\preceq_{\mathrm{c}}$ is a pre-order on $V(\Lambda)$, and $\sim_{\mathrm{qc}}$ is the equivalence relation induced by the pre-order. Note that if $p \sim_{\mathrm{c}} p'$ then $p + \mathbf{w} \sim_{\mathrm{c}} p' + \mathbf{w}$ for all $\mathbf{w} \in \mathbf{A}^+$. Similarly, if $p \sim_{\mathrm{qc}} p'$ then $p + \mathbf{w} \sim_{\mathrm{qc}} p' + \mathbf{w}$ for all $\mathbf{w} \in \mathbf{A}^+$.

3 Moves on GBS graphs

In this section we define a family of moves that, given a GBS graph, produce another GBS graph with isomorphic fundamental group; these are called slide, induction, swap, connection (see [For06] [ACRK25b]). We recall some literature about the topic.

3.1 Sign-change move

Let (Γ, ψ) be a GBS graph. Let $v \in V(\Gamma)$ be a vertex. Define the map $\psi' : E(\Gamma) \to \mathbf{Z} \setminus \{0\}$ such that $\psi'(e) = -\psi(e)$ if $\tau(e) = v$ and $\psi'(e) = \psi(e)$ otherwise. We say that the GBS graph (Γ, ψ') is obtained from (Γ, ψ) by means of a **vertex sign change**. If $\mathcal{G}$ is the GBS graph of groups associated to (Γ, ψ), then the vertex sign change move corresponds to changing the chosen generator for the vertex group G_v; in particular it induces an isomorphism at the level of the universal group $\mathrm{FG}(\mathcal{G})$ and of the fundamental group $\pi_1(\mathcal{G})$.

Let (Γ, ψ) be a GBS graph. Let $d \in E(\Gamma)$ be an edge. Define the map $\psi' : E(\Gamma) \to \mathbf{Z} \setminus \{0\}$ such that $\psi'(e) = -\psi(e)$ if $e = d, \overline{d}$ and $\psi'(e) = \psi(e)$ otherwise. We say that the GBS graph (Γ, ψ') is obtained from (Γ, ψ) by means of an **edge sign change**. If $\mathcal{G}$ is the GBS graph of groups associated to (Γ, ψ), then the vertex sign change move corresponds to changing the chosen generator for the edge group G_d; in particular it induces an isomorphism at the level of the universal group $\mathrm{FG}(\mathcal{G})$ and of the fundamental group $\pi_1(\mathcal{G})$.

3.2 Slide move

Let (Γ, ψ) be a GBS graph. Let d, e be distinct edges with $\tau(d) = \iota(e) = u$ and $\tau(e) = v$; suppose that $\psi(\overline{e}) = n$ and $\psi(e) = m$ and $\psi(d) = \ell n$ for some $n, m, \ell \in \mathbb{Z} \setminus \{0\}$ (see Figure 3). Define the graph Γ' by replacing the edge d with an edge d'; we set $\iota(d') = \iota(d)$ and $\tau(d') = v$; we set $\psi(\overline{d'}) = \psi(\overline{d})$ and $\psi(d') = \ell m$. We say that the GBS graph (Γ', ψ) is obtained from (Γ, ψ) by means of a **slide**.

At the level of the affine representation, we have an edge p —— q and we have another edge with an endpoint at $p + \mathbf{a}$ for some $\mathbf{a} \in \mathbf{A}^+$. The slide has the effect of moving the endpoint from $p + \mathbf{a}$ to $q + \mathbf{a}$ (see Figure 3).

$$\begin{cases} p \text{ —— } q \\ r \text{ —— } p + \mathbf{a} \end{cases} \xrightarrow{\text{slide}} \begin{cases} p \text{ —— } q \\ r \text{ —— } q + \mathbf{a} \end{cases}$$

3.3 Induction move

Let (Γ, ψ) be a GBS graph. Let e be an edge with $\iota(e) = \tau(e) = v$; suppose that $\psi(\overline{e}) = 1$ and $\psi(e) = n$ for some $n \in \mathbb{Z} \setminus \{0\}$, and choose $\ell \in \mathbb{Z} \setminus \{0\}$ and $k \in \mathbb{N}$ such that $\ell \mid n^k$. Define the map ψ' equal to ψ except on the edges $d \neq e, \overline{e}$ with $\tau(e) = v$, where we set $\psi'(d) = \ell \cdot \psi(d)$. We say that the GBS graph (Γ, ψ') is obtained from (Γ, ψ) by means of an **induction**.

At the level of the affine representation, we have an edge $(v, \mathbf{0})$ —— $(v, \mathbf{w})$. We choose $\mathbf{w}_1 \in \mathbf{A}^+$ such that $\mathbf{w}_1 \leq k\mathbf{w}$ for some $k \in \mathbb{N}$; we take all the endpoints of other edges lying in $\mathbf{A}_v^+$, and we translate them up by adding $\mathbf{w}_1$ (see Figure 4).

3.4 Swap move

Let (Γ, ψ) be a GBS graph. Let e_1, e_2 be distinct edges with $\iota(e_1) = \tau(e_1) = \iota(e_2) = \tau(e_2) = v$; suppose that $\psi(\overline{e}_1) = n$ and $\psi(e_1) = \ell_1 n$ and $\psi(\overline{e}_2) = m$ and $\psi(e_2) = \ell_2 m$ and $n \mid m$ and $m \mid \ell_1^{k_1} n$ and $m \mid \ell_2^{k_2} n$ for some $n, m, \ell_1, \ell_2 \in \mathbb{Z} \setminus \{0\}$ and $k_1, k_2 \in \mathbb{N}$ (see Figure 5). Define the graph Γ' by substituting the edges e_1, e_2 with two edges e_1', e_2'; we set $\iota(e_1') = \tau(e_1') = \iota(e_2') = \tau(e_2') = v$; we set $\psi(\overline{e_1'}) = m$ and $\psi(e_1') = \ell_1 m$ and $\psi(\overline{e_2'}) = n$ and $\psi(e_2') = \ell_2 n$. We say that the GBS graph (Γ', ψ) is obtained from (Γ, ψ) by means of a **swap move**.

Let $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{A}^+$ and $p, q \in V(\Lambda)$ be such that $p \leq q \leq p + k_1 \mathbf{w}_1$ and $p \leq q \leq p + k_2 \mathbf{w}_2$ for some $k_1, k_2 \in \mathbb{N}$. At the level of the affine representation, we have an edge e_1 going from p to $p + \mathbf{w}_1$ and an edge e_2 going from q to $q + \mathbf{w}_2$. The swap has the effect of substituting them with e_1' from q to $q + \mathbf{w}_1$ and with e_2' from p to $p + \mathbf{w}_2$ (see Figure 5).

$$\begin{cases} p \text{ --- } p + \mathbf{w}_1 \\ q \text{ --- } q + \mathbf{w}_2 \end{cases} \xrightarrow{\text{swap}} \begin{cases} q \text{ --- } q + \mathbf{w}_1 \\ p \text{ --- } p + \mathbf{w}_2 \end{cases}$$

3.5 Connection move

Let (Γ, ψ) be a GBS graph. Let d, e be distinct edges with $\iota(d) = u$ and $\tau(d) = \iota(e) = \tau(e) = v$; suppose that $\psi(\overline{d}) = m$ and $\psi(d) = \ell_1 n$ and $\psi(\overline{e}) = n$ and $\psi(e) = \ell n$ and $\ell_1 \ell_2 = \ell^k$ for some $m, n, \ell_1, \ell_2, \ell \in \mathbb{Z} \setminus \{0\}$ and $k \in \mathbb{N}$ (see Figure 6). Define the graph Γ' by substituting the edges d, e with two edges d', e'; we set $\iota(d') = v$ and $\tau(d') = \iota(e') = \tau(e') = u$; we set $\psi(\overline{d'}) = n$ and $\psi(d') = \ell_2 m$ and $\psi(\overline{e'}) = m$ and $\psi(e') = \ell m$. We say that the GBS graph (Γ', ψ) is obtained from (Γ, ψ) by means of a **connection move**.

Let $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2 \in \mathbf{A}^+$ and $k \in \mathbb{N}$ be such that $\mathbf{w}_1 + \mathbf{w}_2 = k \cdot \mathbf{w}$. At the level of the affine representation, we have a two edges $q \text{ --- } p + \mathbf{w}_1$ and $p \text{ --- } p + \mathbf{w}$. The connection move has the effect of replacing them with two edges $p \text{ --- } p + \mathbf{w}_2$ and $q \text{ --- } q + \mathbf{w}$ (see Figure 6).

$$\begin{cases} q \text{ --- } p + \mathbf{w}_1 \\ p \text{ --- } p + \mathbf{w} \end{cases} \xrightarrow{\text{connection}} \begin{cases} p \text{ --- } q + \mathbf{w}_2 \\ q \text{ --- } q + \mathbf{w} \end{cases}$$

Remark 3.1. In the definition of connection move, we also allow for the two vertices u, v to coincide.

3.6 Self-slide and reverse slide

The following Lemmas 3.2 and 3.3 introduce two additional moves that will be used in the next sections.

Lemma 3.2 (Self-slide)**.** *Let* $\mathbf{a}, \mathbf{b}, \mathbf{w} \in \mathbf{A}^+$ *and* $\mathbf{x} \in \mathbf{A}$. *Suppose* $\mathbf{a}, \mathbf{w}$ *controls* $\mathbf{b}$ *and* $\mathbf{b} + 2\mathbf{x}$. *Then we can change*

$$\begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w} \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x} \end{cases} \quad \textit{into} \quad \begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w} \\ \mathbf{b} + \mathbf{x} \text{ --- } \mathbf{b} + 2\mathbf{x} \end{cases}$$

by means of a sequence of slides and swaps.

Proof. See [ACRK25b]. □

Lemma 3.3 (Reverse slide)**.** *Let* $\mathbf{a}, \mathbf{b}, \mathbf{w} \in \mathbf{A}^+$ *and* $\mathbf{x} \in \mathbf{A}$ *with* $\mathbf{w} + \mathbf{x} \geq \mathbf{0}$. *Suppose* $\mathbf{a}, \mathbf{w}$ *controls* $\mathbf{b}, \mathbf{b} + \mathbf{x}$ *and suppose* $\mathbf{a}, \mathbf{w} + \mathbf{x}$ *controls* $\mathbf{b}, \mathbf{b} + \mathbf{x}$. *If* $\mathbf{A}_v^+$ *contains edges* $\mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w}$ *and* $\mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x}$, *then we can change*

$$\begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w} \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x} \end{cases} \quad \textit{into} \quad \begin{cases} \mathbf{a} \text{ --- } \mathbf{a} + \mathbf{w} + \mathbf{x} \\ \mathbf{b} \text{ --- } \mathbf{b} + \mathbf{x} \end{cases}$$

by means of a sequence of slides and swaps.

Proof. See [ACRK25b]. □

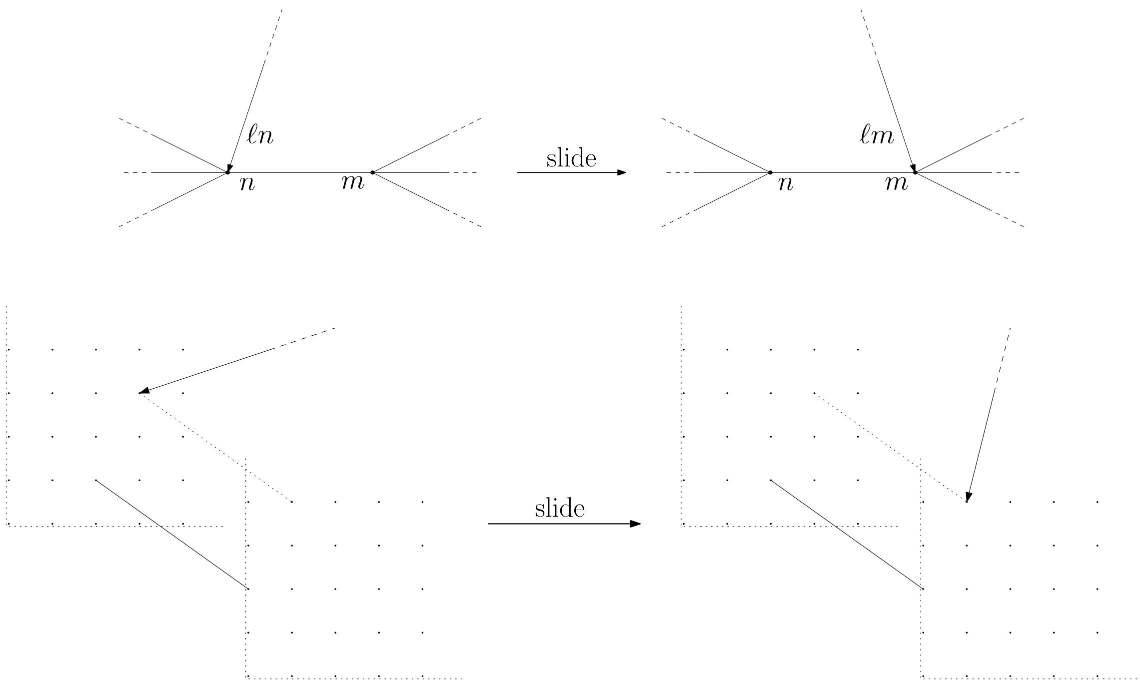


Figure 3: An example of a slide move. Above you can see the GBS graphs. Below you can see the corresponding affine representations.

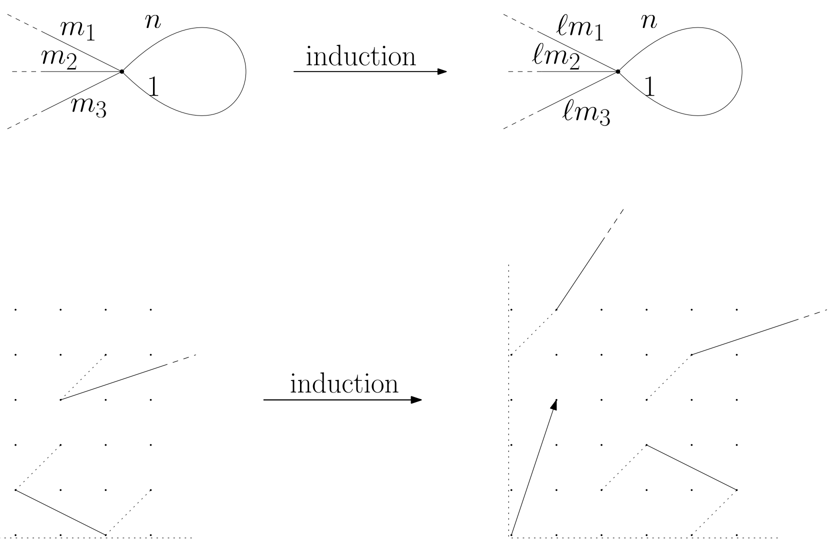


Figure 4: An example of an induction move. Above you can see the GBS graphs; here $\ell \mid n^k$ for some integer $k \geq 0$. Below you can see the corresponding affine representations.

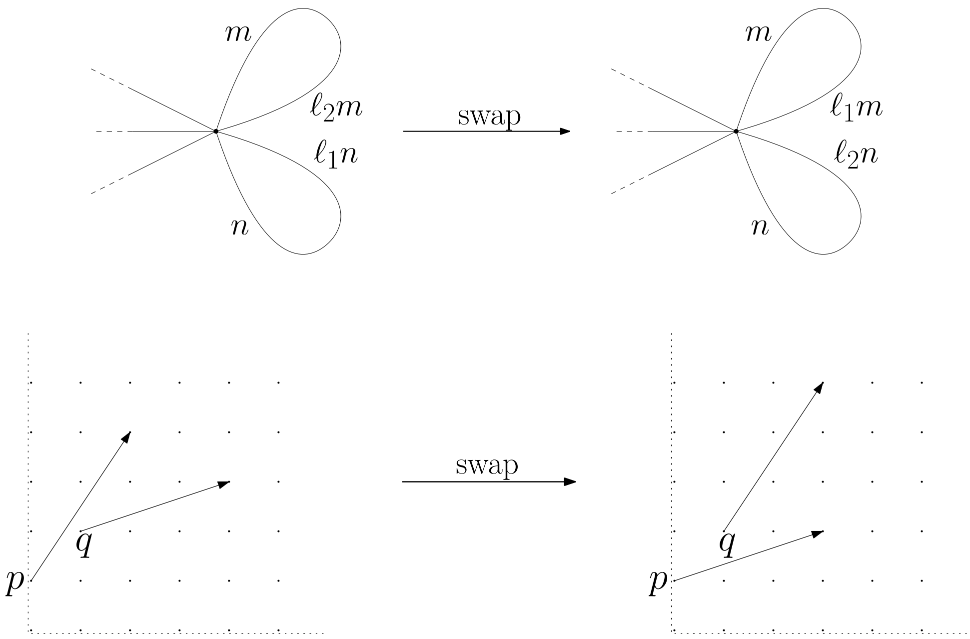


Figure 5: An example of a swap move. Above you can see the GBS graphs; here $n \mid m \mid \ell_1^{k_1} n$ and $n \mid m \mid \ell_2^{k_2} n$ for some integers $k_1, k_2 \geq 0$. Below you can see the corresponding affine representations.

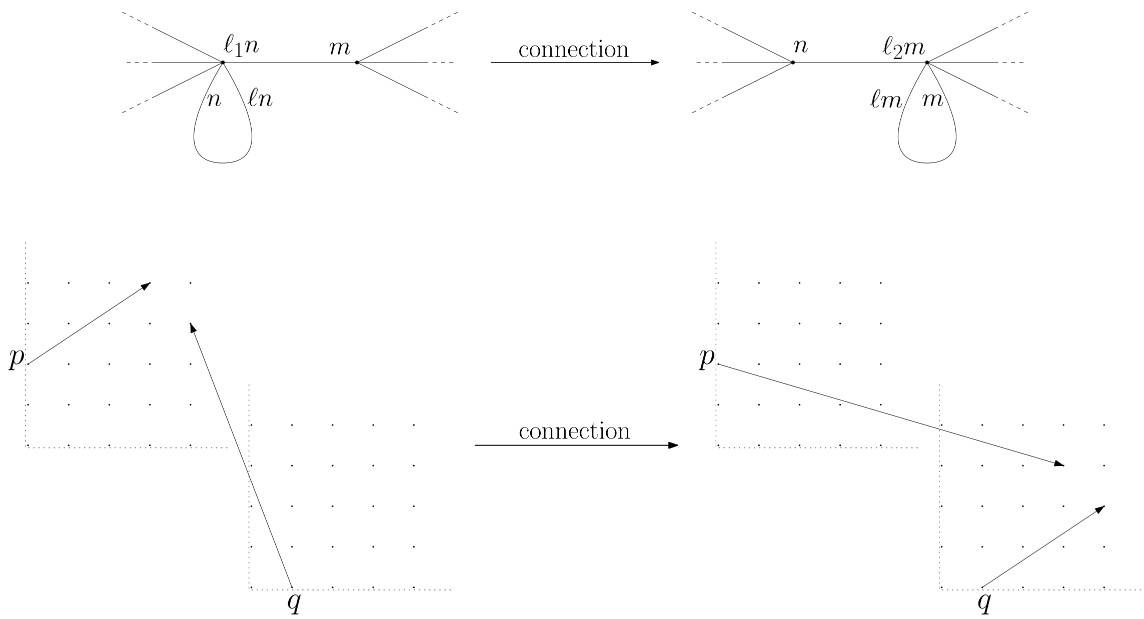


Figure 6: An example of a connection move. Above you can see the GBS graphs; here $\ell_1 \ell_2 = \ell^k$ for some integer $k \geq 0$. Below you can see the corresponding affine representations.

3.7 Sequences of moves

In this section we recall how the isomorphism problem for generalized Baumslag-Solitar groups is reduced to checking whether two GBS graphs can be related by means of a sequence of moves. We need to introduce the notion of *totally reduced* GBS graph (see [ACRK25b]); we will only need the fact that every GBS graph (Γ, ψ) can be algorithmically changed to a totally reduced one.

Definition 3.4. *A GBS graph (Γ, ψ), with affine representation Λ, is called **totally reduced** if the following conditions hold:*

1. *For every vertex $v \in V(\Gamma)$, if $(v, \mathbf{0}) \sim_{\mathrm{c}} (u, \mathbf{b})$ for $u \in V(\Gamma)$ and $\mathbf{b} \in \mathbf{A}^+$, then $u = v$.*
2. *For every vertex $v \in V(\Gamma)$, in Λ there is an edge $(v, \mathbf{0})$ —— $(v, \mathbf{w})$ with $\mathbf{w} \in \mathbf{A}^+$ such that, if $(v, \mathbf{0}) \sim_{\mathrm{c}} (v, \mathbf{b})$ for $\mathbf{b} \in \mathbf{A}^+$, then $\mathbf{0}, \mathbf{w}$ controls $\mathbf{b}$.*

Remark 3.5. Condition 1 in the above Definiton 3.4 coincides with the notion of *fully reduced* GBS graph introduced in [For06].

Proposition 3.6. *There is an algorithm that, given a GBS graph (Γ, ψ), computes a totally reduced GBS graph (Γ', ψ') such that the corresponding graphs of groups have isomorphic fundamental group.*

Proof. See [ACRK25b]. □

Here's one of the main results from [ACRK25b], which will be the starting point for all the results of this paper.

Theorem 3.7. *Let $(\Gamma, \psi), (\Delta, \phi)$ be totally reduced GBS graphs and suppose that the corresponding graphs of groups have isomorphic fundamental group. Then $|V(\Gamma)| = |V(\Delta)|$ and there is a sequence of slides, swaps, connections, sign-changes and inductions going from (Δ, ϕ) to (Γ, ψ). Moreover, all the sign-changes and inductions can be performed at the beginning of the sequence.*

Proof. See [ACRK25b]. □

Thus it's enough to deal with the following problem: given two GBS graphs $(\Gamma, \psi), (\Gamma', \psi')$, determine whether there is a sequence of edge sign-changes, inductions, slides, swaps, connections going from one to the other. Note that a sign-change (resp. a slide, an induction, a swap, a connection) induces a natural bijection between the set of vertices of the graph before and after the move. Of course we can ignore the issue of guessing the bijection among the sets of vertices and the sign-changes at the beginning of the sequence, as these choices can be done only in finitely many ways. In what follows, we will also ignore the issue of guessing the inductions at the beginning of the sequence, as this hopefully represents a marginal issue - even though sometimes there are infinitely many possibilities, and thus this issue should be dealt with. In this paper, we will focus on the following question:

Question 3.8. *Given two totally reduced GBS graphs $(\Gamma, \psi), (\Gamma', \psi')$ and a bijection $b : V(\Gamma') \to V(\Gamma)$, is there a sequence of slides, swaps, connections going from (Γ, ψ) to (Γ', ψ') and inducing the bijection b on the set of vertices?*

In [ACRK25b] we also show that the Question 3.8 can be reduced to the case of one-vertex graphs, and with all edge-labels positive and $\neq 1$. However, the tools we develop in this paper are general and don't make use of these additional assumptions.

4 Limit angles

The aim of this section is to prove Theorem 4.14, solving the isomorphism problem for GBS groups associated to GBS graphs with one vertex and two edges. As we will see, the hard case is the one where the two edges belong to the same quasi-conjugacy class, which has exactly two minimal regions (which in this case are just minimal points). Thus, most of this section will be dedicated to dealing with configurations as in Definition 4.1 below.

Usually, slide and connection moves will be able to change two of the four endpoints of our edges, while the other two are forced to stay fixed at the minimal points (swaps can be applied basically never). In Section 4.2 we show how, when using only slides, one obtains (a subset of) a rooted binary tree of configurations, with a special configuration being the *root* of the binary tree. In Section 4.3 we show that the only relevant connections are the ones applied at the root of the binary tree. In Section 4.4 we describe the set of all possible roots arising from a given one with connection, which turns out to be a union of two arithmetic progressions (plus finitely many configurations). This is already enough to solve the isomorphism problem, and from here the reader could move straight to the proof of the main result Theorem 4.14.

4.1 Configurations

Let (Γ, ψ) be a GBS graph with one vertex, two edges, and such that the two edges belong to a common quasi-conjugacy class with exactly two minimal regions. We denote with Λ the affine representation of (Γ, ψ), which consists of a unique copy of $\mathbf{A}^+$, corresponding to the unique vertex of Γ, and two edges.

Since we only have one quasi-conjugacy classes, the minimal regions in this case are basically just two points (up to the $\mathbb{Z}/2\mathbb{Z}$ component). By **??** we know that every minimal region must contain at least one end-point of one edge. Since we don't have any external slide move available, every edge must have at least one endpoint in a minimal region. Thus we must be in a situation as in the following definition.

Definition 4.1 (Configurations)**.** *A* ***configuration*** *is given by two edges inside* $\mathbf{A}^+$

$$(C) = \begin{cases} \mathbf{a}_1 \;\text{——}\; \mathbf{a}_1 + \mathbf{x}_1 \\ \mathbf{a}_2 \;\text{——}\; \mathbf{a}_2 + \mathbf{x}_2 \end{cases}$$

for some $\mathbf{a}_1, \mathbf{a}_2 \in \mathbf{A}^+$ *and* $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{A}$ *satisfying the following properties:*

1. $\mathbf{a}_1 \not\geq \mathbf{a}_2$ *and* $\mathbf{a}_2 \not\geq \mathbf{a}_1$.
2. $\mathbf{a}_1 + \mathbf{x}_1$ *is bigger or equal than at least one of* $\mathbf{a}_1, \mathbf{a}_2$.
3. $\mathbf{a}_2 + \mathbf{x}_2$ *is bigger or equal than at least one of* $\mathbf{a}_1, \mathbf{a}_2$.

In what follows, we will say that the configuration (C) has *minimal points* $\mathbf{a}_1, \mathbf{a}_2$ and *vectors* $\mathbf{x}_1, \mathbf{x}_2$. When $\mathbf{a}_1, \mathbf{a}_2$ are clear from the context, we will just say that (C) has vectors $\mathbf{x}_1, \mathbf{x}_2$.

Definition 4.2. *Let* (C) *be a configuration with minimal points* $\mathbf{a}_1, \mathbf{a}_2$ *and vectors* $\mathbf{x}_1, \mathbf{x}_2$. *We say that* (C) *is* ***degenerate*** *if it falls into at least one of the following cases:*

1. *At least one of* $\mathbf{x}_1, \mathbf{x}_2$ *lies in* $\mathbb{Z}/2\mathbb{Z} \leq \mathbf{A}$.
2. *The two edges are* $\mathbf{a}_1 \;\text{——}\; \mathbf{a}_2 + \mathbf{e}$ *and* $\mathbf{a}_2 \;\text{——}\; \mathbf{a}_1 + \mathbf{e}'$ *for some* $\mathbf{e}, \mathbf{e}' \in \mathbb{Z}/2\mathbb{Z} \leq \mathbf{A}$.
3. $\mathbf{a}_1 + \mathbf{x}_1 \not\geq \mathbf{a}_2$ *and* $\mathbf{a}_2 + \mathbf{x}_2 \not\geq \mathbf{a}_1$.

A configuration is degenerate if and only if, by applying sequences of slides, swaps, connections involving only the two edges of the configuration, we can reach only finitely many GBS graphs. In that case, in all the GBS graphs that we can reach, the two edges form a degenerate configuration.

In what follows, we will assume that our configurations are non-degenerate. In this case, the only possible moves are slides and connections, and by applying these moves we end up at other configurations. We point out that almost all slide moves only change one of the two endpoints $\mathbf{a}_1 + \mathbf{x}_1, \mathbf{a}_2 + \mathbf{x}_2$, while the endpoints $\mathbf{a}_1, \mathbf{a}_2$ are usually not involved. The only exception is the case of twin roots (Definition 4.4 below), which usually has to be treated separately.

4.2 Sons, roots and binary trees

We are interested in studying the set of all configurations that can be reached from a given one by means of slide moves only. Suppose that we are given a configuration (C) with minimal points $\mathbf{a}_1, \mathbf{a}_2$ and vectors $\mathbf{x}_1, \mathbf{x}_2$. If $\mathbf{a}_1 + \mathbf{x}_1 \geq \mathbf{a}_2$ then we can perform a slide move to obtain the configuration

$$(C1) = \begin{cases} \mathbf{a}_1 \text{ —— } \mathbf{a}_1 + \mathbf{x}_1 + \mathbf{x}_2 \\ \mathbf{a}_2 \text{ —— } \mathbf{a}_2 + \mathbf{x}_2 \end{cases}$$

and if $\mathbf{a}_2 + \mathbf{x}_2 \geq \mathbf{a}_1$ then we can perform a slide move to obtain the configuration

$$(C2) = \begin{cases} \mathbf{a}_1 \text{ —— } \mathbf{a}_1 + \mathbf{x}_1 \\ \mathbf{a}_2 \text{ —— } \mathbf{a}_2 + \mathbf{x}_2 + \mathbf{x}_1 \end{cases}$$

We say that the configurations $(C1)$ and $(C2)$ are the **sons** of the configuration (C). The first son $(C1)$ is the one obtained by changing the first edge and has vectors $\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_2$; the second son $(C2)$ is the one obtained by changing the second edge and has vectors $\mathbf{x}_1, \mathbf{x}_2 + \mathbf{x}_1$ (see Figure 7). Every non-degenerate configuration has at least one son, and at most two.

Denote with $\{1, 2\}^*$ the set of all finite words in the letters $1, 2$. Then we can consider the configurations (Cs) for $s \in \{1, 2\}^*$ obtained by taking iterated sons. We say that the configuration (Cs) *exists* if the sequence of slide moves given by s can be performed, and we say that the configuration (Cs) *doesn't exist* otherwise. For example, if the configuration $(C1222)$ exists, then it has vectors $\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_2 + 4(\mathbf{x}_1 + \mathbf{x}_2)$.

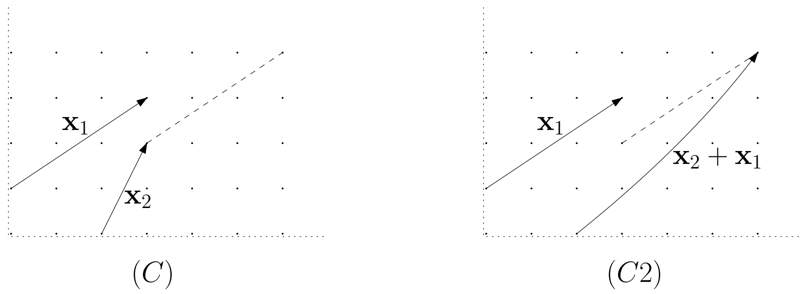


Figure 7: An example of a configuration (C) and of its second son $(C2)$.

Definition 4.3. *A configuration (R) with minimal points $\mathbf{a}_1, \mathbf{a}_2$ and vectors $\mathbf{x}_1, \mathbf{x}_2$ is called **root** if $\mathbf{a}_1 + \mathbf{x}_1 \not\geq \mathbf{a}_2 + \mathbf{x}_2$ and $\mathbf{a}_2 + \mathbf{x}_2 \not\geq \mathbf{a}_1 + \mathbf{x}_1$.*

A configuration is root if and only if it's not the son of any other configuration. Every configuration has a root, and the root is unique in most cases, except in the case of twin roots, as we now explain.

Definition 4.4 (Twin roots)**.** *Two non-degenerate roots $(R), (Q)$ are called **twin roots** if they are of the form*

$$(R) = \begin{cases} \mathbf{a}_1 \text{ —— } \mathbf{a}_2 + \mathbf{e} \\ \mathbf{a}_2 \text{ —— } \mathbf{a}_2 + \mathbf{x}_2 \end{cases} \qquad \textit{and} \qquad (Q) = \begin{cases} \mathbf{a}_1 \text{ —— } \mathbf{a}_1 + \mathbf{x}_1 \\ \mathbf{a}_2 \text{ —— } \mathbf{a}_1 + \mathbf{e} \end{cases}$$

with $\mathbf{e} \in \mathbb{Z}/2\mathbb{Z} \leq \mathbf{A}$ and $\mathbf{x}_1, \mathbf{x}_2 \not\geq \mathbf{0}$ satisfying $\mathbf{x}_1 + \lambda\mathbf{a}_1 = \mathbf{x}_2 + \lambda\mathbf{a}_2 + \lambda\mathbf{e}$ for some integer $\lambda \geq 2$.

If a root has a twin, then the twin is uniquely determined. It's easy to see that, given two twin roots, we can go from one to the other by means of a sequence of slide moves.

Proposition 4.5 (Existence and uniqueness of roots)**.** *Every non-degenerate configuration (C) is an iterated son $(C) = (Rs)$ of a root (R). Moreover, exactly one of the following cases takes place:*

1. *There is exactly one possible choice of (R) and of $s \in \{1, 2\}^*$.*

2. *There are exactly two possible choices of (R), and they are twin roots. For each of them, there is a unique possible choice of $s \in \{1,2\}^*$.*

Proof. Take a configuration given by edges $\mathbf{a}_1$ —— $\mathbf{a}_1 + \mathbf{x}_1$ and $\mathbf{a}_2$ —— $\mathbf{a}_2 + \mathbf{x}_2$. This is a first son of some other configuration if and only if $\mathbf{a}_1 + \mathbf{x}_1 \geq \mathbf{a}_2 + \mathbf{x}_2$, and in that case the "first father" is uniquely determined. Similarly, it is a second son of another configuration if and only if $\mathbf{a}_2 + \mathbf{x}_2 \geq \mathbf{a}_1 + \mathbf{x}_1$, and in that case "second father" is uniquely determined. Thus we can find a unique father (either a first father or a second father) of our configuration; unless $\mathbf{a}_2 + \mathbf{x}_2 = \mathbf{a}_1 + \mathbf{x}_1 + \mathbf{e}$ with $\mathbf{e} \in \mathbb{Z}/2\mathbb{Z} \in \mathbf{A}$, in which case we can find exactly one first father and exactly one second father.

CASE OF TWIN ROOTS: suppose that $\mathbf{a}_2 + \mathbf{x}_2 = \mathbf{a}_1 + \mathbf{x}_1 + \mathbf{e}$ with $\mathbf{e} \in \mathbb{Z}/2\mathbb{Z} \in \mathbf{A}$. Then we can go to the first father with edges $\mathbf{a}_1$ —— $\mathbf{a}_2 + \mathbf{e}$ and $\mathbf{a}_2$ —— $\mathbf{a}_2 + \mathbf{x}_2$; from here, we are forced to take iterated second fathers with edges $\mathbf{a}_1$ —— $\mathbf{a}_2 + \mathbf{e}$ and $\mathbf{a}_2$ —— $\mathbf{a}_2 + \mathbf{x}_2 - k(\mathbf{a}_2 - \mathbf{a}_1 + \mathbf{e})$ for $k \geq 0$, and note that at some point this must stop, since $\mathbf{a}_1 - \mathbf{a}_2 \not\geq 0$; thus we have a unique way of reaching a root, which will have edges $\mathbf{a}_1$ —— $\mathbf{a}_2 + \mathbf{e}$ and $\mathbf{a}_2$ —— $\mathbf{a}_2 + \mathbf{x}_2 + \mathbf{e} - \mu(\mathbf{a}_2 - \mathbf{a}_1 + \mathbf{e})$ for some $\mu \in \mathbb{N}$. Similarly, we can go to the second father with edges $\mathbf{a}_1$ —— $\mathbf{a}_1 + \mathbf{x}_1$ and $\mathbf{a}_2$ —— $\mathbf{a}_1 + \mathbf{e}$; from here, we have a unique way of reaching a root, which will have edges $\mathbf{a}_1$ —— $\mathbf{a}_1 + \mathbf{x}_1 - \nu(\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{e})$ and $\mathbf{a}_2$ —— $\mathbf{a}_1 + \mathbf{e}$ for some $\nu \in \mathbb{N}$. It's easy to check that these are twin roots, and are the unique two roots possible for our configuration.

If we are not in the "case of twin roots", then we have a unique father for our configuration, so we can move to this father. Similarly, we keep taking iterated fathers; if at any point we end up in a configuration which has more than one father, then we are in the "case of twin roots", and we are done. Otherwise, we just have to show that the process terminates, and the conclusion will follow.

If $\mathbf{x}_1, \mathbf{x}_2 \geq \mathbf{0}$, then the sum of all of the components of $\mathbf{x}_1 + \mathbf{x}_2$ strictly decreases every time we take a father (since the configuration is non-degenerate). Thus this can't be the case along a whole infinite sequence. Suppose that at some point we end up, say, in a configuration with $\mathbf{x}_1 \geq \mathbf{0}$ and $\mathbf{x}_2 \not\geq \mathbf{0}$; note that in any case $-\mathbf{x}_2 \not\geq \mathbf{0}$. If $\mathbf{a}_2 + \mathbf{x}_2 \geq \mathbf{a}_1 + \mathbf{x}_1$, then take iterated second fathers, changing $\mathbf{x}_2$ into $\mathbf{x}_2 - k\mathbf{x}_1$ for $k \geq 0$, and this must stop since $\mathbf{x}_1 \geq \mathbf{0}$ and the configuration is non-degenerate. If $\mathbf{a}_1 + \mathbf{x}_1 \geq \mathbf{a}_2 + \mathbf{x}_2$, then we take iterated first fathers, changing $\mathbf{x}_1$ into $\mathbf{x}_1 - k\mathbf{x}_2$, and the process must terminate since $-\mathbf{x}_2 \not\geq \mathbf{0}$. The thesis follows. □

Fixed a configuration (C), we are interested in describing the family of its iterated sons (Cs) for $s \in \{1,2\}^*$. By Proposition 4.5, we know that the set of iterated sons of (C) is a subset of a rooted binary tree; in fact, if $(Cs) = (Cs')$ then we write $(C) = (Rt)$ for some root (R), and we deduce that $(Rts) = (Rts')$ and thus $ts = ts'$, yielding $s = s'$. The following Proposition 4.6 gives a precise characterization of which subsets of the binary tree can occur (see Figure 8), and in which cases. We are mainly interested in the particular case when (C) is a root - but the result is proved without this assumption.

Proposition 4.6 (Binary trees)**.** *Suppose that (C) is a non-degenerate configuration with minimal points $\mathbf{a}_1, \mathbf{a}_2$ and vectors $\mathbf{x}_1, \mathbf{x}_2$. Then exactly one of the following cases happens:*

1. $\mathbf{a}_1 + \mathbf{x}_1 \not\geq \mathbf{a}_2$. *In this case, (Cs) exists if and only if $s = 2^i$ for some $i \geq 0$.*

2. $\mathbf{a}_2 + \mathbf{x}_2 \not\geq \mathbf{a}_1$. *In this case, (Cs) exists if and only if $s = 1^i$ for some $i \geq 0$.*

3. $\mathbf{a}_1 + \mathbf{x}_1 \geq \mathbf{a}_2$ *and* $\mathbf{a}_2 + \mathbf{x}_2 \geq \mathbf{a}_1$ *and* $\mathbf{x}_1, \mathbf{x}_2 \geq \mathbf{0}$. *In this case, (Cs) exists for all $s \in \{1,2\}^*$.*

4. $\mathbf{a}_1 + \mathbf{x}_1 \geq \mathbf{a}_2$ *and* $\mathbf{a}_2 + \mathbf{x}_2 \geq \mathbf{a}_1$ *and* $\mathbf{x}_1 \geq \mathbf{0}$ *and* $\mathbf{x}_2 \not\geq \mathbf{0}$. *In this case, call $h \geq 0$ the maximum integer such that $\mathbf{a}_1 + \mathbf{x}_1 + h\mathbf{x}_2 \geq \mathbf{a}_2$. Then, (Rs) exists if and only if either s begins with $1^{h'}2$ for some $0 \leq h' \leq h$, or $s = 1^{h+1}2^i$ for some $i \geq 0$.*

5. $\mathbf{a}_1 + \mathbf{x}_1 \geq \mathbf{a}_2$ *and* $\mathbf{a}_2 + \mathbf{x}_2 \geq \mathbf{a}_1$ *and* $\mathbf{x}_1 \not\geq \mathbf{0}$ *and* $\mathbf{x}_2 \geq \mathbf{0}$. *In this case, call $k \geq 0$ the maximum integer such that $\mathbf{a}_2 + \mathbf{x}_2 + k\mathbf{x}_1 \geq \mathbf{a}_1$. Then, (Cs) exists if and only if either s begins with $2^{k'}1$ for some $0 \leq k' \leq k$, or $s = 2^{k+1}1^i$ for some $i \geq 0$.*

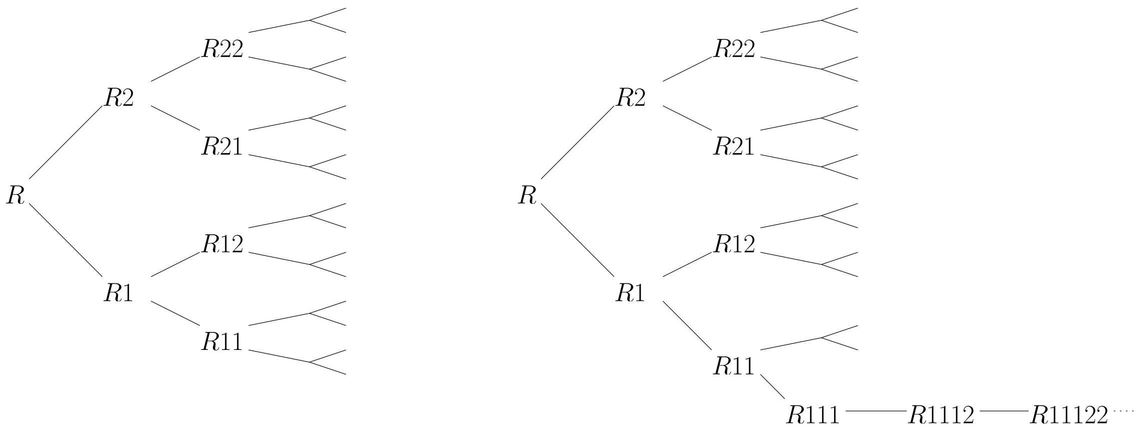
6. $\mathbf{a}_1 + \mathbf{x}_1 \geq \mathbf{a}_2$ *and* $\mathbf{a}_2 + \mathbf{x}_2 \geq \mathbf{a}_1$ *and* $\mathbf{x}_1, \mathbf{x}_2 \not\geq \mathbf{0}$. *In this case, call $h \geq 0$ the maximum integer such that $\mathbf{a}_1 + \mathbf{x}_1 + h\mathbf{x}_2 \geq \mathbf{a}_2$, and call $k \geq 0$ the maximum integer such that $\mathbf{a}_2 + \mathbf{x}_2 + k\mathbf{x}_1 \geq \mathbf{a}_1$. Then (Cs) exists if and only if either s begins with $1^{h'}2$ for some $0 \leq h' \leq h$, or s begins with $2^{k'}1$ for some $0 \leq k' \leq k$, or $s = 1^{h+1}2^i$ for some $i \geq 0$, or $s = 2^{k+1}1^i$ for some $i \geq 0$.*

Proof. Item 1 follows from the fact that the configuration is non-degenerate, and thus $\mathbf{a}_2+\mathbf{x}_1 \geq \mathbf{a}_1$. Similarly for Item 1.

If we are in a configuration satisfying the conditions of Item 3, then the configuration has two sons, that will still satisfy the conditions of Item 3. Thus the full binary tree originating from that configuration exists. In particular we obtain Item 3.

Suppose that we are in the conditions of Item 4. The configuration $(C2)$ exists, and it satisfies the conditions of Item 3, and thus the full binary tree over this configuration exists. For every integer $0 \leq h' \leq h$, the configuration $(C1^{h'}2)$ exists, and it satisfies the conditions of Item 3, and thus the full binary tree over this configuration exists. Finally, the configuration $(C1^{h+1})$ exists, and it satisfies the conditions of Item 1; the conclusion follows.

Item 5 and Item 6 are analogous. $\square$



Complete binary tree

Binary tree truncated on one side

Figure 8: Two examples of subsets of a rooted binary tree. On the left the tree corresponding to Item 3 of Proposition 4.6, while on the right the tree corresponding to Item 4.

In the particular case of twin roots, the behavior of the binary trees is as follows. Let

$$(R) = \begin{cases} \mathbf{a}_1 \text{ ---- } \mathbf{a}_2 + \mathbf{e} \\ \mathbf{a}_2 \text{ ---- } \mathbf{a}_2 + \mathbf{x}_2 \end{cases} \qquad \text{and} \qquad (Q) = \begin{cases} \mathbf{a}_1 \text{ ---- } \mathbf{a}_1 + \mathbf{x}_1 \\ \mathbf{a}_2 \text{ ---- } \mathbf{a}_1 + \mathbf{e} \end{cases}$$

be non-degenerate twin roots with $\mathbf{x}_1 + \lambda\mathbf{a}_1 = \mathbf{x}_2 + \lambda\mathbf{a}_2 + \lambda\mathbf{e}$ for $\lambda \geq 2$ (see Figure 9).

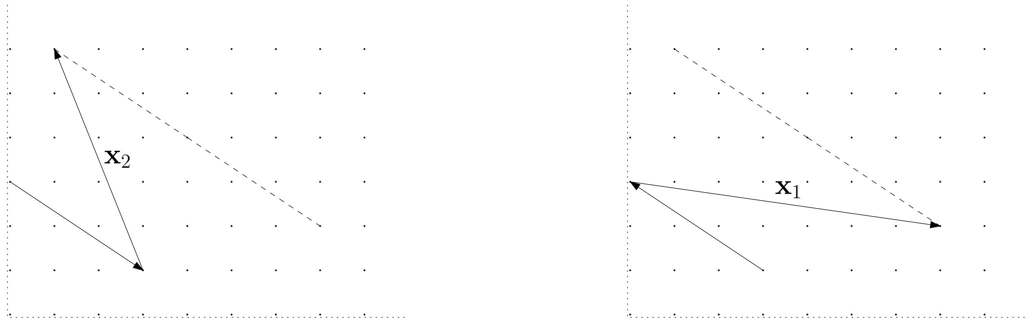
Lemma 4.7 (Binary trees for twin roots)**.** *We have the following:*

1. *For $s \in \{1,2\}^*$, we have that (Rs) exists if and only if either s doesn't begin with $1, 2^{\lambda-1}$, or $s = 12^i, 2^{\lambda-1}1^i$ for some $i \geq 0$.*

2. *For $s \in \{1,2\}^*$, we have that (Qs) exists if and only if either s doesn't begin with $2, 1^{\lambda-1}$, or $s = 21^i, 1^{\lambda-1}2^i$ for some $i \geq 0$.*

3. *A configuration (C) is an iterated son of both $(R), (Q)$ if and only if (C) is an iterated son of $(R2^\ell 1) = (Q1^{\lambda-1-\ell}2)$ for some $0 \leq \ell \leq \lambda - 1$. In that case, ℓ is uniquely determined.*

Proof. Item 1 and Item 2 follow using Proposition 4.6. For Item 3, we observe that the configurations $(R), (R2), \ldots, (R2^{\lambda-1})$ can't be iterated sons of (Q) (since in iterated sons of (Q) the edge $\mathbf{a}_1$ —— $\mathbf{a}_2 + \mathbf{e}$ never appears as first edge), and similarly $(Q), (Q1), \ldots, (Q1^{\lambda-1})$ can't be iterated sons of (R). It's easy to check that $(R2^\ell 1) = (Q1^{\lambda-1-\ell}2)$ for $0 \leq \ell \leq \lambda - 1$. If (C) has more than one root, then by Proposition 4.5 we can write $(C) = (Rs)$ for a unique $s \in \{1,2\}^*$, and since $(C) \neq (R), (R2), \ldots, (R2^{\lambda-1})$ we obtain that (s) must begin with $2^\ell 1$ for a unique $0 \leq \ell \leq \lambda - 1$. The conclusion follows. $\square$



(R) (Q)

Figure 9: An example of two twin roots. The dashed line witnesses the identity $\mathbf{x}_1+3\mathbf{a}_1 = \mathbf{x}_2+3\mathbf{a}_2$, and hints the sequence of slide moves necessary to pass from one twin to the other.

4.3 Connections in a binary tree

Lemma 4.8 tells us that, if we can perform a connection from a configuration (C), then one of the two sons exists, and we can obtain the same result by performing a connection from the son - this means that we can restrict our attention only to connections performed at very deep levels in the trees. On the contrary, Lemma 4.9 tells us that, if we can apply a connection from a son configuration $(C1)$, then we can obtain the same result by performing a connection from (C) - this means that we can restrict our attention only to connections performed at the roots.

Lemma 4.8 (Moving to sons)**.** *If a non-degenerate configuration (C) allows for a connection (with the first edge controlling the endpoint of the second), then $(C2)$ exists and it allows for a connection (with the first edge controlling the endpoint of the second). Moreover, the result of the two connections is the same (for suitable choices of the parameter k of Section 3.5).*

Proof. Immediate from the definitions. □

Lemma 4.9 (Moving to roots)**.** *Let (C) be a non-degenerate configuration and suppose that $(C1)$ exists.*

1. *$(C1)$ allows for a connection (with the first edge controlling the endpoint of the second) if and only if $(C2)$ exists. In that case, the result of the connection is $(C2)$ (for a suitable choice of the parameter k of Section 3.5).*

2. *If $(C1)$ allows for a connection (with the second edge controlling the endpoint of the first), then (C) allows for a connection (with the second edge controlling the endpoint of the first). Moreover, the result of the two connections is the same (for suitable choices of the parameter k of Section 3.5).*

Proof. Let's say that (C) has minimal points $\mathbf{a}_1, \mathbf{a}_2$ and vectors $\mathbf{x}_1, \mathbf{x}_2$. Since $(C1)$ exists, we must have $\mathbf{a}_1 + \mathbf{x}_1 \geq \mathbf{a}_2$.

If $(C1)$ allows for a connection (with the first edge controlling the endpoint of the second) then $\mathbf{a}_2 + \mathbf{x}_2 \geq \mathbf{a}_1$ and thus $(C2)$ exists. If $(C2)$ exists then $\mathbf{a}_2 + \mathbf{x}_2 \geq \mathbf{a}_1$, and thus $\mathbf{a}_1 + \mathbf{x}_1 + \mathbf{x}_2 \geq \mathbf{a}_2 + \mathbf{x}_2 \geq \mathbf{a}_1$, giving that $(C1)$ allows for a connection (with the first edge controlling the endpoint of the second). It's immediate to check that the result of the connection from $(C1)$ is $(C2)$, as desired.

The second item is immediate from the definitions. □

Lemma 4.10 (Jumping between roots)**.** *Let (R) be a non-degenerate root and suppose that we are in the hypothesis to perform a connection (with the first edge controlling the endpoint of the second). If we perform the connection, choosing the parameter k of Section 3.5 to be the minimum possible integer, then the result of the connection is a non-degenerate root.*

Proof. Let's say that (R) has minimal points $\mathbf{a}_1, \mathbf{a}_2$ and vectors $\mathbf{x}_1, \mathbf{x}_2$. Suppose that $\mathbf{x}_1 \geq \mathbf{0}$ and that $\mathbf{a}_1, \mathbf{x}_1$ controls $\mathbf{a}_2 + \mathbf{x}_2$, so that we are in the hypothesis to perform a connection, with the first edge controlling the endpoint of the second. Consider the minimum integer k such that $\mathbf{a}_1 + k\mathbf{x}_1 \geq \mathbf{a}_2 + \mathbf{x}_2$, and observe that $k \geq 2$, otherwise (R) wouldn't be a root. We now perform a connection obtaining the configuration

$$\begin{cases} \mathbf{a}_1 \longrightarrow \mathbf{a}_1 + k\mathbf{x}_1 - \mathbf{x}_2 \\ \mathbf{a}_2 \longrightarrow \mathbf{a}_2 + \mathbf{x}_1 \end{cases}$$

Suppose that $\mathbf{a}_1 + k\mathbf{x}_1 - \mathbf{x}_2 \geq \mathbf{a}_2 + \mathbf{x}_1$: then $\mathbf{a}_1 + (k-1)\mathbf{x}_1 \geq \mathbf{a}_2 + \mathbf{x}_2$ contradicting the minimality of k. Suppose that $\mathbf{a}_1 + k\mathbf{x}_1 - \mathbf{x}_2 \leq \mathbf{a}_2 + \mathbf{x}_1$: then $\mathbf{a}_1 + \mathbf{x}_1 \leq \mathbf{a}_1 + (k-1)\mathbf{x}_1 \leq \mathbf{a}_2 + \mathbf{x}_2$ contradicting the fact that (R) was a root. This shows that the configuration obtained after the connection is a root, as desired. □

4.4 The sequence of roots

Definition 4.11. *Let (R) be a non-degenerate root.*

1. *If (R) allows for a connection (with the first edge controlling the endpoint of the second), define the root (R^-) as we one obtained performing the connection, as in Lemma 4.10.*
2. *If (R) allows for a connection (with the second edge controlling the endpoint of the first), define the root (R^+) as we one obtained performing the connection, as in Lemma 4.10.*

It's immediate from the definitions that, if (R^-) exists, then (R^{-+}) exists and it's equal to (R); and similarly on the other side. Given a non-degenerate root (R), we can consider the sequence of non-degenerate roots

$$\ldots, (R^{--}), (R^-), (R), (R^+), (R^{++}), \ldots$$

obtained by an iterated application of Lemma 4.10 (see Figure 10). On each side, the sequence can be finite or infinite. If an element $\mathbf{v} \in \mathbf{A}^+$ plays the role of $\mathbf{v} = \mathbf{x}_1$ in (R), then we have that the same element plays the role of $\mathbf{v} = \mathbf{x}_2$ in (R^-). Thus, associated with the sequence of roots we have a unique sequence

$$\ldots, \mathbf{v}_{-2}, \mathbf{v}_{-1}, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots$$

with $\mathbf{v}_i \in \mathbf{A}$, and such that (R^{+i}) has vectors $\mathbf{v}_i, \mathbf{v}_{i+1}$. As for the sequence of roots, this sequence of elements of $\mathbf{A}$ can be finite or infinite on each side. Moreover, we have that $\mathbf{v}_i \geq \mathbf{0}$ except possibly for the first/last element of the sequence, if the sequence is finite on the left/right.

Remark 4.12. One might be worried about twin roots appearing in the sequence above. However, it's easy to check that a twin root never allows for any connection. Thus, if we start with a non-twin root, then no root along the sequence will be a twin root.

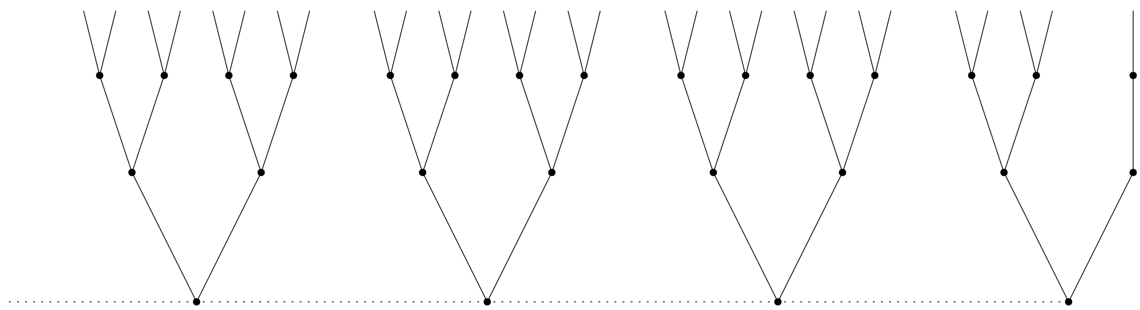


Figure 10: A sequence of binary trees. The roots are related to each other by means of connection moves. On the right, the sequence terminates at a root that allows for connection only on one side. On the left, the sequence is infinite.

The following Proposition 4.13 allows us to effectively compute the sequence $\mathbf{v}_i$ starting from the two initial values $\mathbf{v}_0, \mathbf{v}_1$.

Proposition 4.13 (Sequence of roots). *Let (R) be a non-degenerate non-twin root, with vectors $\mathbf{v}_0, \mathbf{v}_1$. Let $\dots, \mathbf{v}_{-1}, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots$ be the corresponding sequence of vectors. Then we have the following:*

1. *If $\mathbf{v}_i, \mathbf{v}_{i+1}, \mathbf{v}_{i+2}$ exist, then*
$$\mathbf{v}_{i+2} = k\mathbf{v}_{i+1} - \mathbf{v}_i$$
where $k \geq 2$ is the minimum natural number such that $k\mathbf{v}_{i+1} - \mathbf{v}_i \geq \mathbf{a}_1 - \mathbf{a}_2$.

2. *If $\mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1}$ exist, then*
$$\mathbf{v}_{i-1} = h\mathbf{v}_i - \mathbf{v}_{i+1}$$
where $h \geq 2$ is the minimum natural number such that $h\mathbf{v}_i - \mathbf{v}_{i+1} \geq \mathbf{a}_2 - \mathbf{a}_1$.

3. *If the sequence $\dots, \mathbf{v}_{-2}, \mathbf{v}_{-1}, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots$ is infinite on the right, then $\mathbf{v}_i \leq \mathbf{v}_{i+1}$ for some $i \in \mathbb{Z}$. For every such i, the subsequence $\mathbf{v}_{i+1}, \mathbf{v}_{i+2}, \mathbf{v}_{i+3}, \dots$ is an arithmetic progression.*

4. *If the sequence $\dots, \mathbf{v}_{-2}, \mathbf{v}_{-1}, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots$ is infinite on the left, then $\mathbf{v}_{i-1} \geq \mathbf{v}_i$ for some $i \in \mathbb{Z}$. For every such i, the subsequence $\dots, \mathbf{v}_{i-3}, \mathbf{v}_{i-2}, \mathbf{v}_{i-1}$ is an arithmetic progression.*

Proof. Item 1 is immediate from the definitions. Note that $k\mathbf{v}_{i+1} - \mathbf{v}_i \geq \mathbf{a}_1 - \mathbf{a}_2$ for $k = 0$ would imply that $\mathbf{a}_2 \geq \mathbf{a}_1 + \mathbf{v}_i$, and thus we would be in the case of a twin root, and no connection would be possible. Note that $k\mathbf{v}_{i+1} - \mathbf{v}_i \geq \mathbf{a}_1 - \mathbf{a}_2$ for $k = 1$ would imply that $\mathbf{a}_2 + \mathbf{v}_{i+1} \geq \mathbf{a}_1 + \mathbf{v}_i$ contradicting the fact that (R^{+i}) is root. Therefore $k \geq 2$. Similarly for Item 2.

Take $r \in \mathcal{P}(\Gamma, \psi)$ with the projection $\pi_r : \mathbf{A} \to \mathbb{Z}$ on the corresponding component, and consider the sequence of integers

$$\dots, \pi_r(\mathbf{v}_i) - \pi_r(\mathbf{v}_{i-1}), \pi_r(\mathbf{v}_{i+1}) - \pi_r(\mathbf{v}_i), \pi_r(\mathbf{v}_{i+2}) - \pi_r(\mathbf{v}_{i+1}), \dots. \tag{1}$$

If $\mathbf{v}_{i-1}, \mathbf{v}_{i+1}$ exist, then $\mathbf{v}_i \geq \mathbf{0}$ and $\mathbf{v}_{i-1} + \mathbf{v}_{i+1} = k\mathbf{v}_i \geq 2\mathbf{v}_i$ for some $k \geq 3$. Therefore the sequence (1) is a non-decreasing sequence of integers.

For Item 3, suppose that the sequence is infinite on the right. For every $r \in \mathcal{P}(\Gamma, \psi)$, there must be an index $i \in \mathbb{Z}$ such that $\pi_r(\mathbf{v}_{i+1}) - \pi_r(\mathbf{v}_i) \geq 0$ (otherwise $\pi_r(\mathbf{v}_i) \to -\infty$ for $i \to +\infty$, contradiction). Since we have finitely many choices of $r \in \mathcal{P}(\Gamma, \psi)$, there must be an index $i \in \mathbb{Z}$ such that $\mathbf{v}_{i+1} \geq \mathbf{v}_i$. If $i \in \mathbb{Z}$ satisfies $\mathbf{v}_i \leq \mathbf{v}_{i+1}$, then $\mathbf{v}_i \leq \mathbf{v}_{i+1} \leq \mathbf{v}_{i+2} \leq \dots$ since (1) is non-decreasing. For every $j \geq i+1$ we must have $\mathbf{a}_2 + \mathbf{v}_{j+1} \geq \mathbf{a}_1$ (since $\mathbf{v}_{j-1}$ exists), and thus $2\mathbf{v}_{j+1} - \mathbf{v}_j \geq \mathbf{a}_1 - \mathbf{a}_2$, yielding that $\mathbf{v}_{j+2} = 2\mathbf{v}_{j+1} - \mathbf{v}_j$. It follows that $\mathbf{v}_{i+1}, \mathbf{v}_{i+2}, \mathbf{v}_{i+3}, \dots$ is an arithmetic progression. Item 4 is analogous. ☐

4.5 Classification of GBS graphs with one vertex and two edges

We are now ready to classify GBS graphs of groups with one vertex and two edges.

Theorem 4.14. *There is an algorithm that, given two totally reduced GBS graph $(\Gamma, \psi), (\Delta, \phi)$, where (Γ, ψ) has one vertex and two edges, decides*

1. *Whether there is a sequence of slides, swaps, connections going from (Γ, ψ) to (Δ, ϕ), and in case there is, computes one such sequence.*

2. *Whether the corresponding GBS groups are isomorphic or not, and in case they are, computes a sequence of sign-changes, inductions, slides, swaps, connections going from (Γ, ψ) to (Δ, ϕ).*

Proof. We can assume that both GBS graphs have one vertex and two edges. We deal with Item 1 first. If both edges of (Γ, ψ) are in different quasi-conjugacy classes, then we are done by **??**; thus we assume that they belong to a common quasi-conjugacy class. Note that the minimal regions of this quasi-conjugacy class consist basically of a point each (modulo the $\mathbb{Z}/2\mathbb{Z}$ component).

CASE 1: Suppose that the quasi-conjugacy class has four minimal regions M_1, M_2, M_3, M_4. By **??** the four endpoints of the two edges must lie one in each, le'ts say that the first edge has endpoints in M_1, M_2 and the other in M_3, M_4. But then $M_1 \cup M_2$ is a quasi-conjugacy class by itself, contradiction.

CASE 2: Suppose that the quasi-conjugacy class has three minimal regions M_1, M_2, M_3. By **??**, three of the four endpoints of the edges must lie in the minimal regions, let's say we have one edge with endpoints in M_1, M_2 and the other with one endpoint in M_3. The other endpoint must lie in one of M_1, M_2, otherwise $M_1 \cup M_2$ would be a quasi-conjugacy class by itself. It follows that we can reach only finitely many GBS graphs, and we are done.

CASE 3: Suppose that the quasi-conjugacy class has two minimal regions M_1, M_2. If one edge has no endpoint in any of them, then by **??** the other endpoint must have one endpoint in each, and thus $M_1 \cup M_2$ is a quasi-conjugacy class by itself, contradiction. Thus each edge must have at least one endpoint in each, and thus we are in a configuration

$$(R) = \begin{cases} \mathbf{a}_1 \text{ --- } \mathbf{a}_1 + \mathbf{v}_1 \\ \mathbf{a}_2 \text{ --- } \mathbf{a}_2 + \mathbf{v}_2 \end{cases}$$

where $\mathbf{a}_1, \mathbf{a}_2$ lie in M_1, M_2 respectively. If the configuration is degenerate, then we can only reach finitely many GBS graphs, and we are done. Otherwise, up to slides, we can assume that (R) is a root.

If (R) isn't a twin root, then every slide and connection will always lead to another configuration, with the same endpoints $\mathbf{a}_1, \mathbf{a}_2$, changing only the two endpoints $\mathbf{a}_1 + \mathbf{v}_1, \mathbf{a}_2 + \mathbf{v}_2$. We consider the sequence of vectors $\dots, \mathbf{v}_{-1}, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots$ as in Section 4.4, and note that by Proposition 4.13 we can algorithmically compute a finite segment of it, until, one each side, the sequence either terminates or becomes an infinite arithmetic progression. Thus (Δ, ϕ) must be a configuration of the same kind, and we can compute a root for (Δ, ϕ), let's say with edges $\mathbf{a}_1$ — $\mathbf{a}_1 + \mathbf{u}_1$ and $\mathbf{a}_2$ — $\mathbf{a}_2 + \mathbf{u}_2$. We algorithmically check whether $\mathbf{u}_1, \mathbf{u}_2$ is a subsegment of our finite sequence, or whether it belongs to one the arithmetic progressions. If it does, then we can explicitly compute a sequence of moves going from (Γ, ψ) to (Δ, ϕ), otherwise such a sequence of moves doesn't exist.

If (R) is a twin root, we call

$$(R) = \begin{cases} \mathbf{a}_1 \text{ --- } \mathbf{a}_2 + \mathbf{e} \\ \mathbf{a}_2 \text{ --- } \mathbf{a}_2 + \mathbf{x}_2 \end{cases} \qquad \text{and} \qquad (Q) = \begin{cases} \mathbf{a}_1 \text{ --- } \mathbf{a}_1 + \mathbf{x}_1 \\ \mathbf{a}_2 \text{ --- } \mathbf{a}_1 + \mathbf{e} \end{cases}$$

with $\mathbf{e} \in \mathbb{Z}/2\mathbb{Z} \leq \mathbf{A}$ and $\mathbf{x}_1, \mathbf{x}_2 \not\geq \mathbf{0}$ satisfying $\mathbf{x}_1 + \lambda \mathbf{a}_1 = \mathbf{x}_2 + \lambda \mathbf{a}_2 + \lambda \mathbf{e}$ for some integer $\lambda \geq 2$. If from (R) we perform a slide changing the endpoint $\mathbf{a}_2$ of the second edge, then we end up with two edges $\mathbf{a}_1$ — $\mathbf{a}_2 + \mathbf{e}$ and $\mathbf{a}_1 + \mathbf{e}$ — $\mathbf{a}_2 + \mathbf{x}_2$, and the root of this configuration is a twin root (Q'), with twin (R'), given by

$$(R') = \begin{cases} \mathbf{a}_1 + \mathbf{e} \text{ --- } \mathbf{a}_2 \\ \mathbf{a}_2 + \mathbf{e} \text{ --- } \mathbf{a}_2 + \mathbf{x}_2 + \mathbf{e} \end{cases} \qquad \text{and} \qquad (Q') = \begin{cases} \mathbf{a}_1 + \mathbf{e} \text{ --- } \mathbf{a}_1 + \mathbf{x}_1 + \mathbf{e} \\ \mathbf{a}_2 + \mathbf{e} \text{ --- } \mathbf{a}_1 \end{cases}$$

and note that these coincide with $(R), (Q)$ is $\mathbf{e}$ is trivial. In any case, the configurations that we can reach from (R) by means of moves are exactly the iterated sons of $(R), (Q), (R'), (Q')$. Thus (Δ, ϕ) must be a configuration of the same kind, and we can compute a root for (Δ, ϕ), and check whether it coincides with one of $(R), (Q), (R'), (Q')$. If it does, then we can explicitly compute a sequence of moves going from (Γ, ψ) to (Δ, ϕ), otherwise such a sequence of moves doesn't exist.

CASE 4: Suppose that the quasi-conjugacy class has one minimal region M. We check whether the two GBS graphs have the same minimal region, the same linear algebra, and the same number of edges in each conjugacy class - this can all be done algorithmically. If they don't, then by **??** there is no sequence of moves going from one to the other. We now prove that, if they do, then they are related to each other by a sequence of moves.

Up to performing some slide an swap, we can assume that (Γ, ψ) has edges $\mathbf{a}$ — $\mathbf{a} + \mathbf{w}$ and $\mathbf{b}$ — $\mathbf{b} + \mathbf{x}$, such that $\mathbf{a}, \mathbf{w}$ controls $\mathbf{b}, \mathbf{b} + \mathbf{x}$. Similarly, we can assume that (Δ, ϕ) has edges $\mathbf{a}'$ — $\mathbf{a}' + \mathbf{w}'$ and $\mathbf{b}'$ — $\mathbf{b}' + \mathbf{x}'$, such that $\mathbf{a}', \mathbf{w}'$ controls $\mathbf{b}', \mathbf{b}' + \mathbf{x}'$. We must have that $\mathbf{a}' = \mathbf{a} + \mathbf{e}$ for some $\mathbf{e} \in \mathbb{Z}/2\mathbb{Z} \leq \mathbf{A}$, and $\langle \mathbf{w}, \mathbf{x} \rangle = \langle \mathbf{w}', \mathbf{x}' \rangle \leq \mathbf{A}$.

If $\mathbf{e}$ is trivial then we must also have $\mathbf{b} - \mathbf{b}' \in \langle \mathbf{w}, \mathbf{x} \rangle \leq \mathbf{A}$; in this case we can use the results of [ACRK25b] to find a sequence of moves that goes from (Γ, ψ) to (Δ, ϕ), and we are done.

If $\mathbf{e}$ is non-trivial, then we must have $\mathbf{b} \sim_{\mathrm{c}} \mathbf{a}'$ and $\mathbf{b}' \sim_{\mathrm{c}} \mathbf{a}$. Up to performing a sequence of slides and swap using the results of [ACRK25b], we can assume that $\mathbf{b} = \mathbf{a}'$ and $\mathbf{x} \geq \mathbf{0}$, and that $\mathbf{b}' = \mathbf{a}$ and $\mathbf{x}' \geq \mathbf{0}$. Now the conclusion follows using the results of [ACRK25b] once again.

CONCLUSION: The above discussion proves Item 1. For Item 2, we only need to deal with the case where we have induction (since sign-changes can be guessed in finitely many ways, see Theorem 3.7 in Section 3). Thus we can assume that (Γ, ψ) has edges $\mathbf{0}$ —— $\mathbf{w}$ and $\mathbf{b}$ —— $\mathbf{b}+\mathbf{x}$, and that (Δ, ϕ) has edges $\mathbf{0}$ —— $\mathbf{w}'$ and $\mathbf{b}'$ —— $\mathbf{b}'+\mathbf{x}'$.

If the two edges lie in different quasi-conjugacy classes, then induction will translate the conjugacy class of $\mathbf{b}$ —— $\mathbf{b}+\mathbf{x}$, and after that only external equivalence is possible. Thus we just have to check whether $\mathbf{w} = \mathbf{w}'$ (so that the lower quasi-conjugacy class is dealt with) and $\mathrm{supp}_{\mathrm{qc}}(\mathbf{b}-\mathbf{b}') \subseteq \mathrm{supp}_{\mathrm{qc}}(\mathbf{w})$ (so that, up to induction, we can set $\mathbf{b} = \mathbf{b}'$) and $\mathbf{x}-\mathbf{x}' \in \langle \mathbf{w} \rangle$ (so that, after the induction, we can make the two edges equal by means of external equivalence). All of this can be done algorithmically.

If the two edges lie in the same quasi-conjugacy class, then we can perform an induction and change $\mathbf{b}$ in such a way that it lies in the same conjugacy class as $\mathbf{b}'$, and we are done as in case 4 above. The conclusion follows. □

5 Angles

This section is dedicated to giving an interpretation of the previous results in terms of angles in the Euclidean plane $\mathbb{R}^2$. Each configuration determines an angle, and moving away from the root in a binary tree makes the angle narrower. The root of the binary tree, having the largest angle, is able to "see" more configurations. A connection move has the effect of rotating the angle: you can choose to rotate to the left or to the right. This allows us to reach new configurations that before we would not see. If you keep performing connections to rotate the angle always in the same direction, you will converge to a direction, which we call *limit direction.* The two limit directions (on the left and on the right) determine an angle, which we call the *limit angle* of the GBS group. This limit angle is an isomorphism invariant of the GBS group: limit angles for non-isomorphic GBS groups are disjoint. We give a description of the set of all possible limit angles that can appear, showing that this is a discrete set (it can only accumulate at the boundary of the positive cone). See also Figure 14 and Figure 15 from the example at the end of the section.

5.1 Limit directions

We are now interested in giving an interpretation of the results in terms of *angles.* In this section we give the technical definition of *limit direction* (see Definition 5.2 below), which will be crucial in what follows.

Definition 5.1. *Let (C) be a configuration with minimal points $\mathbf{a}_1, \mathbf{a}_2$ and vectors $\mathbf{x}_1, \mathbf{x}_2$. We say that (C) is* ***full-tree*** *if $\mathbf{a}_1+\mathbf{x}_1, \mathbf{a}_2+\mathbf{x}_2 \geq \mathbf{a}_1, \mathbf{a}_2$.*

This means that (C) is non-degenerate and falls into the case of Item 3 of Proposition 4.6; equivalently, the iterated son (Cs) exists for all $s \in \{1,2\}^*$. Every (iterated) son of a full-tree configuration is again a full-tree configuration.

Let (R) be a non-degenerate non-twin root, with vectors $\mathbf{v}_0, \mathbf{v}_1$. Let $\ldots, \mathbf{v}_{-1}, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots$ be the corresponding sequence of vectors, as in Section 4.4. We want to characterize all the full-tree configurations that can be obtained from (R) by means of slides and connections. As note before, every root in the sequence is full-tree (and thus all the iterated sons are full-tree too), except possibly for the first/last root of the sequence (when the sequence is finite on the left/right). If the sequence of roots terminates on the left/right, then the first/last root will fall into exactly one of the cases of Proposition 4.6. In each of those cases, it's quite easy to characterize which of the iterated sons are full-tree.

It's always possible to go from any full-tree configuration to every other full-tree configuration (possibly in a different tree), by means of slides and connections, passing only through full-tree configurations. Thus, in the same way as in Section 4.4 we defined the sequence of roots, one could define a "sequence of minimal full-tree configurations" (most of them would be roots, except near the beginning/end of the sequence). However, we are only interested in the first and last elements of the sequence (or in the limits, when the sequence is infinite), as we now explain.

Define an order $\preceq_2$ on $\{1,2\}^*$ as follows. For $s, t \in \{1,2\}^*$, we set $s \preceq_2 t$ if the first different digit is a 1 in s and a 2 in t (or if s is an initial segment of t). For a root (R) with at least one full-tree iterated son, we take the minimum $s \in \{1,2\}^*$ (with respect to $\preceq_2$) such that (Rs) is full-tree (note that this always exists by Proposition 4.6, and can be explicitly computed); we say that (Rs) is the *right-most full-tree iterated son* of (R).

Definition 5.2 (Limit directions)**.** *Let (R) be a non-degenerate non-twin root, with vectors $\mathbf{v}_0, \mathbf{v}_1$. Let $\ldots, \mathbf{v}_{-1}, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots$ be the corresponding sequence of vectors as in Section 4.4. Define the* ***(right) limit direction*** *of (R) as the element $\mathbf{l}^+ \in \mathbf{A}$ obtained as follows:*

1. *If the sequence is infinite on the right: then we set $\mathbf{l}^+ = \mathbf{v}_{i+1} - \mathbf{v}_i$ for $i \in \mathbb{N}$ big enough.*
2. *If the sequence terminates on the right, and some root in the sequence has some full-tree iterated son: then we take the maximum $i \in \mathbb{Z}$ such that (R^{+i}) has a full-tree iterated son, we take the right-most full tree iterated son $(R^{+i}s)$ of (R^{+i}), and we set $\mathbf{l}^+$ to be the second vector of $(R^{+i}s)$.*
3. *If no root in the sequence has any full-tree iterated son: then we say that $\mathbf{l}^+$ is not defined.*

Similarly, we can define an order $\preceq_1$ on $\{1,2\}^*$, and the left-most full-tree iterated son of a root (R). In the same way as in Definition 5.2, we can define the **(left) limit direction** of (R), which will be an element $\mathbf{l}^- \in \mathbf{A}$. The two limit directions satisfy several properties:

- We have that $\mathbf{l}^+$ is defined if and only if $\mathbf{l}^-$ is defined, and if and only if there is at least one full-tree configuration that can be reached from (R) by means of slides and connections.
- If $\mathbf{l}^-, \mathbf{l}^+$ are defined, then we have that $\mathbf{l}^-, \mathbf{l}^+ \geq \mathbf{0}$.
- If $\mathbf{l}^-, \mathbf{l}^+$ are defined, then each of them is part of some basis for $\langle \mathbf{v}_0, \mathbf{v}_1 \rangle$. In general, $\mathbf{l}^-$ and $\mathbf{l}^+$ together don't generate $\langle \mathbf{v}_0, \mathbf{v}_1 \rangle$ - i.e. they are part of two different bases.

Remark 5.3. It's natural to wonder for which sequences of roots the limit directions don't exist. Such a sequence contains either one or two roots. It can be a single root, falling into the case of Item 6 of Proposition 4.6 with parameters $h = k = 1$. It can consist of a single root, falling into the case of Item 2 or Item 1 of Proposition 4.6. It can consist of two roots, the first falling into the case of Item 2 and the second falling into the case of Item 1 of Proposition 4.6.

5.2 Angles

Definition 5.4. *A configuration (C) with vectors $\mathbf{x}_1, \mathbf{x}_2$ is called* ***full-rank*** *if $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle \cong \mathbb{Z}^2$.*

Let (R) be a non-degenerate root with vectors $\mathbf{v}_0, \mathbf{v}_1$, and suppose that (R) is full-rank. Then the results about the roots can be reinterpreted in terms of angles in the Euclidean plane $\mathbb{R}^2 = \langle \mathbf{v}_0, \mathbf{v}_1 \rangle \otimes \mathbb{R}$. With Proposition 5.5 we show that (C) belongs to the binary tree of (R) if and only if (C) lies inside the *angle* defined by (R) - up to a few exceptional cases (the non-full-tree configurations). However, by means of connection moves, it's possible to escape from this angle; with the subsequent Theorem 5.7 we show that (C) can be obtained from (R) by means of slides and connections if and only if (C) belongs to a wider *limit angle* induced by (R) - again, up to a few exceptional cases (the non-full-tree configurations). In other words, for GBS groups corresponding to full-rank configurations, we have a complete set of invariants classifying them (i.e. minimal points, subgroup generated, limit angle) - once again, up to the few exceptional cases where the limit angles are not defined. With Proposition 5.9 we show that distinct limit angles don't intersect, except possibly at their boundary.

Proposition 5.5 (Angles)**.** *Let (C) be a full-rank full-tree configuration with vectors $\mathbf{x}_1, \mathbf{x}_2$. Then, a configuration (D) with vectors $\mathbf{y}_1, \mathbf{y}_2$ is an iterated son of (C) if and only if the following conditions hold:*

1. *$\langle \mathbf{y}_1, \mathbf{y}_2 \rangle = \langle \mathbf{x}_1, \mathbf{x}_2 \rangle$ and $\mathbf{y}_1, \mathbf{y}_2$ is Nielsen equivalent to $\mathbf{x}_1, \mathbf{x}_2$.*
2. *We have that $\mathbf{y}_i = \lambda_i \mathbf{x}_1 + \mu_i \mathbf{x}_2$ for some integers $\lambda_i, \mu_i \geq 0$, for $i = 1, 2$.*

Remark 5.6. Item 2 of the above Proposition 5.5 means that the vectors $\mathbf{y}_1, \mathbf{y}_2$ are internal to the positive cone determined by $\mathbf{x}_1, \mathbf{x}_2$. We also refer to this positive cone as the *angle* determined by $\mathbf{x}_1, \mathbf{x}_2$.

Proof. If (D) is an iterated son of (C), then the two conditions hold (by induction on the number of iterations of taking the son). Suppose now that the two conditions hold.

Since $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle \cong \mathbb{Z}^2$, the condition $\langle \mathbf{y}_1, \mathbf{y}_2 \rangle = \langle \mathbf{x}_1, \mathbf{x}_2 \rangle$ implies that $\mathbf{y}_1 = \lambda \mathbf{x}_1 + \mu \mathbf{x}_2$ for some unique coprime integers λ, μ, and by hypothesis we must have $\lambda, \mu \geq 0$. We now start with the configuration (C) with vectors $\mathbf{x}_1, \mathbf{x}_2$, and with the numbers λ, μ; we take iterated sons to perform the Euclidean algorithm, as follows.

If $\mu \geq \lambda > 0$, then we take the first son $(C1)$ with vectors $\mathbf{x}'_1 = \mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}'_2 = \mathbf{x}_2$, and we take the integers $\lambda' = \lambda, \mu' = \mu - \lambda$ such that $\mathbf{y}_1 = \lambda' \mathbf{x}'_1 + \mu' \mathbf{x}_2$. If $\lambda > \mu > 0$, then we take the second son $(C2)$ with vectors $\mathbf{x}'_1 = \mathbf{x}_1, \mathbf{x}'_2 = \mathbf{x}_2 + \mathbf{x}_1$), and we take the integers $\lambda' = \lambda - \mu, \mu' = \mu$ such that $\mathbf{y}_1 = \lambda' \mathbf{x}'_1 + \mu' \mathbf{x}'_2$. Note that, in both cases, the couple $\mathbf{x}'_1, \mathbf{x}'_2$ is Nielsen equivalent to $\mathbf{x}_1, \mathbf{x}_2$, and $\lambda', \mu' \geq 0$ are coprime. We then reiterate the procedure with $(C1)$ or $(C2)$ respectively; and so on.

The procedure will stop with an iterated son (Cs), for some $s \in \{1,2\}^*$, associated with vectors $(\mathbf{u}_1, \mathbf{u}_2)$ and coprime non-negative integers (η, θ), such that (i) the couple $\mathbf{u}_1, \mathbf{u}_2$ is Nielsen equivalent to $\mathbf{x}_1, \mathbf{x}_2$ and (ii) $\mathbf{y}_1 = \eta \mathbf{u}_1 + \theta \mathbf{u}_2$ and (iii) either $\eta = 0$ or $\theta = 0$.

CASE 1: Suppose that $\theta = 0$. Then $\eta = 1$ (since they are coprime) and thus $\mathbf{y}_1 = \mathbf{u}_1$. Since $\mathbf{u}_1, \mathbf{u}_2$ is Nielsen equivalent to $\mathbf{y}_1, \mathbf{y}_2$ (and they generate a $\mathbb{Z}^2$), we must have that $\mathbf{y}_2 = \mathbf{u}_2 + k\mathbf{u}_1$ for some integer $k \in \mathbb{Z}$. If $k \geq 0$ then $(D) = (Cs2^k)$ and we are done. Otherwise $(D2^{|k|}) = (Cs)$ with $k < 0$: we take $|k|$ to be the smallest such that there is such an s, and in particular s must terminate with 1. Let's say $s = t1$, and we get that (Ct) has vectors $\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_2$ and $(D2^{|k|-1})$ has vectors $\mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1$. But then $\mathbf{u}_1 - \mathbf{u}_2$ and $\mathbf{u}_2 - \mathbf{u}_1$ are both combinations with non-negative coefficients of $\mathbf{x}_1, \mathbf{x}_2 \geq \mathbf{0}$, contradiction.

CASE 2: Suppose that $\eta = 0$. Then $\theta = 1$ (since they are coprime) and thus $\mathbf{y}_1 = \mathbf{u}_2$. Since $\mathbf{u}_1, \mathbf{u}_2$ is Nielsen equivalent to $\mathbf{y}_1, \mathbf{y}_2$, we must have that $\mathbf{y}_2 = -\mathbf{u}_1 + k\mathbf{u}_2$ for some integer $k \in \mathbb{Z}$. This means that we can jump from some iterated son of (C) to (D) by means of a connection. By Lemma 4.9 we obtain that either (D) is an iterated son of (C) (in which case we are done) or (D) is obtained from (C) itself by means of a connection. But if we obtain (C) from (D) by means of a connection, then $\mathbf{y}_2$ can't be written as a linear combination of $\mathbf{x}_1, \mathbf{x}_2$ with non-negative coefficients, contradiction. □

Let (R) be a non-degenerate non-twin full-rank root with vectors $\mathbf{v}_0, \mathbf{v}_1$. Let $\ldots, \mathbf{v}_{-1}, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots$ be the corresponding sequence of vectors as in Section 4.4. We want to characterize all full-rank configurations that can be obtained from (R) by means of slides and connections.

Theorem 5.7 (Limit angles)**.** *Let (R) be a non-degenerate non-twin full-rank root with vectors $\mathbf{v}_0, \mathbf{v}_1$, and with limit directions $\mathbf{l}^-, \mathbf{l}^+$. Then, a full-tree configuration (D) with vectors $\mathbf{y}_1, \mathbf{y}_2$ can be obtained from (R) by means of slides and connections if and only if the following conditions hold:*

1. *$\langle \mathbf{y}_1, \mathbf{y}_2 \rangle = \langle \mathbf{v}_0, \mathbf{v}_1 \rangle$ and $\mathbf{y}_1, \mathbf{y}_2$ is Nielsen equivalent to $\mathbf{v}_0, \mathbf{v}_1$.*
2. *We have that $\nu_i \mathbf{y}_i = \lambda_i \mathbf{l}^- + \mu_i \mathbf{l}^+$ for some integers $\lambda_i, \mu_i \geq 0$ and $\nu_i > 0$, for $i = 1, 2$.*

Remark 5.8. Item 2 of the above Theorem 5.7 means that the vectors $\mathbf{y}_1, \mathbf{y}_2$ are internal to the (rational) positive cone induced by $\mathbf{l}^-, \mathbf{l}^+$. We also refer to this positive cone as the *limit angle* determined by $\mathbf{l}^-, \mathbf{l}^+$.

Proof. OBSERVATION 1: Let $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s} \geq \mathbf{0}$ be non-zero. Suppose that $\mathbf{q}$ is a combination with strictly positive rational coefficients of $\mathbf{p}, \mathbf{r}$, and $\mathbf{r}$ is a combination with strictly positive rational coefficients of $\mathbf{q}, \mathbf{s}$. Then $\mathbf{q}, \mathbf{r}$ are combinations with strictly positive rational coefficients of $\mathbf{p}, \mathbf{s}$.

In fact, write $\mathbf{q} = \lambda \mathbf{p} + \mu \mathbf{r}$ and $\mathbf{r} = \eta \mathbf{q} + \theta \mathbf{s}$ with $\lambda, \mu, \eta, \theta > 0$. We deduce that $(1 - \mu\eta)\mathbf{q} = \lambda \mathbf{p} + \mu\theta \mathbf{s}$; since $\mathbf{p}, \mathbf{s} \geq \mathbf{0}$ are non-zero, we deduce that $1 - \mu\eta > 0$ and the conclusion follows.

OBSERVATION 2: Let $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{x} \geq \mathbf{0}$ be non-zero. Suppose that $\mathbf{q}, \mathbf{x}$ are combinations with non-negative rational coefficients of $\mathbf{p}, \mathbf{r}$. Then $\mathbf{x}$ is a combination with non-negative rational coefficients of either $\mathbf{p}, \mathbf{q}$ or $\mathbf{q}, \mathbf{r}$.

In fact, write $\mathbf{q} = \lambda\mathbf{p} + \mu\mathbf{r}$ and $\mathbf{x} = \eta\mathbf{p} + \theta\mathbf{r}$ and observe that $\lambda\mathbf{x} = \eta\mathbf{q} + (\lambda\theta - \mu\eta)\mathbf{r}$ and $\mu\mathbf{x} = \theta\mathbf{q} + (\mu\eta - \lambda\theta)\mathbf{p}$. If $\lambda = 0$ or $\mu = 0$ we are done; otherwise, we use one or the other identity, depending on the sign of $\lambda\theta - \mu\eta$, and we are done.

STEP 1: We prove that for all full-tree configurations, which are iterated sons of some root in the sequence, has vectors that can be written as combinations of $\mathbf{l}^-, \mathbf{l}^+$ with non-negative rational coefficients.

We note that, in the sequence $\ldots, \mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1}, \ldots$, every term can be written as a combination of the two adjacent ones with strictly positive rational coefficients.

We note that if the sequence is infinite on the right, then for $j \in \mathbb{Z}$ big enough we have that $\mathbf{l}^+ = \mathbf{v}_j - \mathbf{v}_{j-1}$ and thus $\mathbf{v}_j$ can be written as a linear combination of $\mathbf{v}_{j-1}$ and $\mathbf{l}^+$ with strictly positive rational coefficients.

We note that if $\mathbf{v}_{j-1}, \mathbf{v}_j, \mathbf{v}_{j+1}$ exist, and if $\mathbf{l}^+$ is a vector in some iterated son of $\mathbf{v}_j, \mathbf{v}_{j+1}$, then we can write $\mathbf{l}^+ = \lambda\mathbf{v}_j + \mu\mathbf{v}_{j+1}$ for some integers $\lambda \geq 0$ and $\mu > 0$, and $\mathbf{v}_{j-1} + \mathbf{v}_{j+1} = k\mathbf{v}_j$ for some integer $k \geq 2$. But then $\mathbf{v}_j = \frac{\mu}{k\mu+\lambda}\mathbf{v}_{j-1} + \frac{1}{k\mu+1}\mathbf{l}^+$ is a combination of $\mathbf{v}_{j-1}, \mathbf{l}^+$ with strictly positive rational coefficients.

Putting all of this together, and using observation 1, we deduce that, for every full-tree root in the sequence, its vectors are combinations of $\mathbf{l}^-, \mathbf{l}^+$ with non-negative rational coefficients. If it's the case, the first/last tree in the sequence containing full-tree configurations is dealt with by hand. The conclusion follows.

STEP 2: Suppose that (D) is a full-tree configuration with vectors $\mathbf{y}_1, \mathbf{y}_2$ that satisfies the two conditions. We observe that $\mathbf{l}^-, \mathbf{l}^+$ are linearly independent, since they generate $\mathbf{y}_1, \mathbf{y}_2$. For simplicity, we assume that the sequence of vectors is finite on the left and infinite on the right, the other cases being similar.

Since $\mathbf{y}_1, \mathbf{y}_2$ are linearly independent, one of them can be written as a combination of $\mathbf{l}^-, \mathbf{l}^+$ with non-zero coefficient of $\mathbf{l}^+$. Let's say $\mathbf{y}_1 = \lambda\mathbf{l}^- + \mu\mathbf{l}^+$ with $\lambda \geq 0$ and $\mu > 0$ rationals, the case with $\mathbf{y}_2$ being analogous.

We have that $\frac{\mathbf{v}_j}{j} \to \mathbf{l}^+$ for $j \to +\infty$, and thus for all j big enough we can write $\mathbf{y}_1 = \lambda_j\mathbf{l}^- + \mu_j\frac{\mathbf{v}_j}{j}$, and we must have that $\lambda_j \to \lambda$ and $\mu_j \to \mu$. In particular $\mathbf{y}_1$ must be a combination of $\mathbf{l}^-, \mathbf{v}_j$ with non-negative rational coefficients for all j big enough.

By step 1 we know that $\ldots, \mathbf{v}_{j-1}, \mathbf{v}_j$ are all combinations with non-negative rational coefficients of $\mathbf{l}^-, \mathbf{v}_j$ (except possibly for the one/two initial terms of the sequence). By an iterated application of observation 2, we deduce that $\mathbf{y}_1$ is a combination with non-negative rational coefficients of some full-tree configuration (C) with vectors $\mathbf{x}_1, \mathbf{x}_2$, obtained from (R) with slides and connections.

We write $\mathbf{y}_1 = \eta\mathbf{x}_1 + \theta\mathbf{x}_2$ with $\eta, \theta \geq 0$ rationals and $\mathbf{y}_2 = \sigma\mathbf{x}_1 + \tau\mathbf{x}_2$ with σ, τ rationals. We observe that, since $\mathbf{y}_1, \mathbf{y}_2$ is Nielsen equivalent to $\mathbf{x}_1, \mathbf{x}_2$, and since they generate a group isomorphic to $\mathbb{Z}^2$, we have that $\eta, \theta, \sigma, \tau$ are uniquely determined, and thus they must be integers. If $\eta = 0$ then we must have that $\theta = 1$ and $\mathbf{y}_1, \mathbf{y}_2$ is related to $\mathbf{x}_1, \mathbf{x}_2$ by slides and possibly a connection, and we are done. Similarly, if $\theta = 0$ then we are done. If $\eta, \theta > 0$, then up to changing $\mathbf{y}_2$ to $\mathbf{y}_2 + h\mathbf{y}_1$ by means of slide moves, for h very big, we can assume that $\sigma, \tau > 0$. The conclusion follows by Proposition 5.5. □

Proposition 5.9 (Different limit angles have disjoint interiors)**.** *Let $(R), (S)$ be non-degenerate non-twin full-rank roots with vectors $\mathbf{v}_0, \mathbf{v}_1$ and $\mathbf{u}_0, \mathbf{u}_1$ respectively, and with limit directions $\mathbf{l}^+, \mathbf{l}^-$ and $\mathbf{m}^+, \mathbf{m}^-$ respectively. Suppose that $\langle \mathbf{v}_0, \mathbf{v}_1 \rangle = \langle \mathbf{u}_0, \mathbf{u}_1 \rangle$ and $\mathbf{v}_0, \mathbf{v}_1$ is Nielsen equivalent to $\mathbf{u}_0, \mathbf{u}_1$. Then the (closed, rational) positive cones determined by $\mathbf{l}^+, \mathbf{l}^-$ and by $\mathbf{m}^+, \mathbf{m}^-$ satisfy exactly one of the following possibilities:*

1. *The cones coincide. In this case $\mathbf{l}^+ = \mathbf{m}^+$ and $\mathbf{l}^- = \mathbf{m}^-$ and $(R), (S)$ belong to the same sequence of roots.*
2. *The cones don't coincide, but they intersect non-trivially. In this case either $\mathbf{l}^+ = \mathbf{m}^-$ (and the intersection is exactly the line $\langle \mathbf{l}^+ \rangle = \langle \mathbf{m}^- \rangle$) or $\mathbf{l}^- = \mathbf{m}^+$ (and the intersection is exactly the line $\langle \mathbf{l}^- \rangle = \langle \mathbf{m}^+ \rangle$).*
3. *The cones are disjoint.*

Proof. Suppose that the interiors of the cones intersect. Then we must have that one of $\mathbf{m}^+, \mathbf{m}^-$, let's say $\mathbf{m}^-$, is a combination of $\mathbf{l}^+, \mathbf{l}^-$ with strictly positive rational coefficients.

Suppose that, by performing slides and connections on (R'), we can obtain a full-tree configuration (C') with vectors $\mathbf{m}^-$ and some other vector. Then by Theorem 5.7 we obtain that (C') can be obtained also from (R) by performing slides and connections. Thus in this case (R) and (S) can be obtained from each other by means of connections, and the two cones coincide.

Suppose that the sequence of vectors $\dots, \mathbf{u}_{-1}, \mathbf{u}_0, \mathbf{u}_1, \dots$ is infinite on the left. Then we have that $\frac{\mathbf{u}_j}{|j|} \to \mathbf{m}_-$ for $j \to -\infty$ and thus for all but finitely many integers $j < 0$ we have that $\frac{\mathbf{u}_j}{|j|}$ is a combination of $\mathbf{l}^+, \mathbf{l}^-$ with strictly positive rational coefficients. But then some full-tree root in the sequence of (S) has both vectors $\mathbf{u}_{j-1}, \mathbf{u}_j$ which are combinations of $\mathbf{l}^+, \mathbf{l}^-$ with strictly positive rational coefficients. By Theorem 5.7, this root must also belong to the sequence of roots associated with (R). Therefore (R) and (S) can be obtained from each other by means of connections, and the two cones coincide. $\square$

5.3 The space of limit angles

Suppose that we fix two minimal points $\mathbf{a}_1, \mathbf{a}_2 \in \mathbf{A}^+$ and two vectors $\mathbf{h}_1, \mathbf{h}_2 \in \mathbf{A}$ generating a subgroup $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle \cong \mathbb{Z}^2$. We are interested in characterizing all the isomorphism classes of configurations (C) with vectors $\mathbf{x}_1, \mathbf{x}_2$ such that $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle = \langle \mathbf{h}_1, \mathbf{h}_2 \rangle$ and $\mathbf{x}_1, \mathbf{x}_2$ is Nielsen equivalent to $\mathbf{h}_1, \mathbf{h}_2$.

In order to do this, the main step is characterizing which directions can be realized as limit directions; this is done with the following Proposition 5.10. As we had already observed, a limit direction $\mathbf{l}^+$ always satisfies $\mathbf{l}^+ \geq \mathbf{0}$ and is always part of some basis for $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle$.

Proposition 5.10 (Realizing limit directions)**.** *Let $\mathbf{h}_1, \mathbf{h}_2 \in \mathbf{A}$ be such that $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle \cong \mathbb{Z}^2$. Let $\mathbf{l} \in \langle \mathbf{h}_1, \mathbf{h}_2 \rangle$ be an element which is part of some basis (but not together with $\mathbf{a}_1 - \mathbf{a}_2$), and such that all components of $\mathbf{l}$ are strictly positive. Then exactly one of the following cases takes place.*

1. *$\mathbf{a}_2 + \mathbf{l} \not\geq \mathbf{a}_1$. In this case, there is a full-tree configuration (C), with vectors Nielsen equivalent to $\mathbf{h}_1, \mathbf{h}_2$, with limit angle $\mathbf{l}^+ = \mathbf{l}$. The sequence of roots is uniquely determined, and is infinite on the right.*

2. *$\mathbf{a}_2 + \mathbf{l} \geq \mathbf{a}_1$ and $\mathbf{a}_1 + \mathbf{l} \not\geq \mathbf{a}_2$. In this case there is a full-tree configuration (C), with vectors Nielsen equivalent to $\mathbf{h}_1, \mathbf{h}_2$, and with limit angle $\mathbf{l}^+ = \mathbf{l}$. The sequence of roots is uniquely determined, and is finite on the right.*

3. *$\mathbf{a}_2 + \mathbf{l} \geq \mathbf{a}_1$ and $\mathbf{a}_1 + \mathbf{l} \geq \mathbf{a}_2$. In this case, there is no full-tree configuration (C), with vectors Nielsen equivalent to $\mathbf{h}_1, \mathbf{h}_2$, with limit angle $\mathbf{l}^+ = \mathbf{l}$.*

Remark 5.11. Analogous conclusions holds with $\mathbf{l}^-$ instead of $\mathbf{l}^+$. In particular, if all components of $\mathbf{l}$ are strictly positive, then $\mathbf{l}$ can be realized as a right limit direction if and only if $\mathbf{l}$ can be realized as a left limit direction, and if and only if $\mathbf{l} \not\geq |\mathbf{a}_1 - \mathbf{a}_2|$ (the absolute value taken componentwise).

Remark 5.12. If $\mathbf{l}$ is part of a basis together with $\mathbf{a}_1 - \mathbf{a}_2$, then $\mathbf{l}$ appears in some full-tree configuration which has two twin roots.

Remark 5.13. In the case in which the results are applied to a quasi-conjugacy class inside a bigger GBS, the components which are required to be strictly positive are only the ones in $\mathrm{supp}_{\mathrm{qc}}(\mathbf{a}_1 - \mathbf{a}_2)$. The requirement doesn't include the component $\mathbb{Z}/2\mathbb{Z}$.

Proof. We denote with $\sim$ the relation of Nielsen equivalence. Take $\mathbf{p} \in \mathbf{A}$ such that $\mathbf{p}, \mathbf{l} \sim \mathbf{h}_1, \mathbf{h}_2$. Note that $\mathbf{p}$ is uniquely determined up to adding multiples of $\mathbf{l}$.

Suppose that $\mathbf{l}$ is the limit angle of a sequence of roots $\dots, \mathbf{v}_i, \mathbf{v}_{i+1}, \dots$ infinite on the right, with $\mathbf{v}_i, \mathbf{v}_{i+1}$ Nielsen equivalent to $\mathbf{h}_1, \mathbf{h}_2$. Then, for $i \in \mathbb{Z}$ big enough, we have that $\mathbf{v}_i, \mathbf{l} = \mathbf{v}_i, \mathbf{v}_{i+1} - \mathbf{v}_i \sim \mathbf{v}_i, \mathbf{v}_{i+1} \sim \mathbf{h}_1, \mathbf{h}_2$ and thus $\mathbf{v}_i = \mathbf{p} + A\mathbf{l}$ and $\mathbf{v}_{i+1} = \mathbf{p} + (A+1)\mathbf{l}$ for some integer $A \in \mathbb{Z}$. But since $\mathbf{v}_i, \mathbf{v}_{i+1}$ is a root, we get that $\mathbf{a}_2 + \mathbf{p} + (A+1)\mathbf{l} \not\geq \mathbf{a}_1 + \mathbf{p} + A\mathbf{l}$ and thus $\mathbf{a}_2 + \mathbf{l} \not\geq \mathbf{a}_1$ as desired.

Note that the sequence of roots is uniquely determined in this case (up to shift).

Conversely, suppose that $\mathbf{a}_2 + \mathbf{l} \not\geq \mathbf{a}_1$. Since all components of $\mathbf{l}$ are strictly positive, it's easy to check that for $A \in \mathbb{N}$ big enough the elements $\mathbf{v}_0 = \mathbf{p} + A\mathbf{l}$ and $\mathbf{v}_1 = \mathbf{p} + (A+1)\mathbf{l}$ define a full-tree root, whose sequence is infinite on the right and has limit angle $\mathbf{l}^+ = \mathbf{l}$.

Suppose that $\mathbf{l}$ is the limit angle of a sequence of roots finite on the right. Then there must be a full-tree configuration (C) with vectors $\mathbf{x}, \mathbf{l}$, which is an iterated son of some root of the sequence, and with the following additional property: if we perform a connection (with the second edge controlling the endpoint of the first) we don't get a full-tree configuration. Since (C) is full-tree, we deduce that $\mathbf{a}_2 + \mathbf{l} \geq \mathbf{a}_1$ and that we can perform a connection (with the first edge controlling the endpoint of the first). When we perform the connection we get $\mathbf{l}, -\mathbf{x} + B\mathbf{l}$, and, for $B \in \mathbb{N}$ big enough, we have that $-\mathbf{x} + B\mathbf{l} \geq \mathbf{0}$ and $\mathbf{a}_2 - \mathbf{x} + B\mathbf{l} \geq \mathbf{a}_1$ (since all the components of $\mathbf{l}$ are strictly positive); if this newly obtained configuration isn't full-tree, it must be because $\mathbf{a}_1 + \mathbf{l} \not\geq \mathbf{a}_2$, as desired.

Note that the sequence of roots is uniquely determined in this case, since $\mathbf{x}, \mathbf{l} \sim \mathbf{h}_1, \mathbf{h}_2$ and being full-tree uniquely determines $\mathbf{x}$ up to slide moves.

Conversely, suppose that $\mathbf{a}_2 + \mathbf{l} \geq \mathbf{a}_1$ and $\mathbf{a}_1 + \mathbf{l} \not\geq \mathbf{a}_2$. Since all components of $\mathbf{l}$ are strictly positive, we can choose $\mathbf{p} \in \mathbf{A}$ in such a way that $\mathbf{p}, \mathbf{l}$ is a full-tree configuration (and $\mathbf{p}, \mathbf{l} \sim \mathbf{h}_1, \mathbf{h}_2$). Note that, by performing a connection (with the second edge controlling the endpoint of the first) we obtain the configurations $\mathbf{l}, -\mathbf{p} + B\mathbf{l}$, which will never be full-tree since $\mathbf{a}_1 + \mathbf{l} \not\geq \mathbf{a}_2$. Thus the root of this configuration will give the desired sequence. □

In order to get a full-tree configuration, a necessary condition is that the subgroup $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle$ contains at least one element with all components > 0 (or at least, the ones in $\mathrm{supp}_{\mathrm{qc}}(\mathbf{a}_1 - \mathbf{a}_2)$). In that case, the Euclidean plane $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle \otimes \mathbb{R}$ will contain a non-empty open cone P, generated by linear combinations with positive coefficients of elements with positive components. Note that the complement P^c has non-empty interior. The two lines in the boundary ∂P correspond to directions of $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle \otimes \mathbb{R}$ where some component is equal to 0; note that such directions are not necessarily realized in $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle$.

Theorem 5.14. *Let $\mathbf{h}_1, \mathbf{h}_2 \in \mathbf{A}$ be such that $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle \cong \mathbb{Z}^2$. Suppose that $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle$ contains at least one vector whose components are all strictly positive. Then we have the following:*

1. *The set of directions that can be realized as limit directions is a finite union of arithmetic progressions in $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle$.*
2. *This finite set of arithmetic progressions can be algorithmically computed from $\mathbf{a}_1, \mathbf{a}_2, \mathbf{h}_1, \mathbf{h}_2$.*
3. *If $P \subseteq \langle \mathbf{h}_1, \mathbf{h}_2 \rangle$ is the open cone defined above, then the set of limit directions has no accumulation point in the interior of P (i.e. it's finite inside any closed sub-cone of P).*

Proof. Let $\mathbf{l}$ be a limit direction. There are at most two possibilities for $\mathbf{l}$ with some component equal to zero; thus we assume that all components of $\mathbf{l}$ are strictly positive. According to Proposition 5.10 at least one of $\mathbf{l} \not\geq \mathbf{a}_1 - \mathbf{a}_2$ or $\mathbf{l} \not\geq \mathbf{a}_2 - \mathbf{a}_1$ holds, so let's assume that $\mathbf{l} \not\geq \mathbf{a}_1 - \mathbf{a}_2$. The inequality can fail only on finitely many components, so let's assume that it fails on the p_0-th component for some $p_0 \in \mathcal{P}(\Gamma, \psi)$, and let's c the p_0-th component of $\mathbf{a}_1 - \mathbf{a}_2$. We can write $\mathbf{l} = x\mathbf{h}_1 + y\mathbf{h}_2$ for some $x, y \in \mathbb{Z}$ coprime. Let's call a, b the p_0-th components of $\mathbf{h}_1, \mathbf{h}_2$ respectively. Then we must have $0 < xa + yb < c$ and thus $xa + yb$ can only assume finitely many values. Let's assume that $xa + yb = d$ for some fixed $d \in \mathbb{N}$.

If the equation $xa + yb = d$ has a solution $x_0 a + y_0 b = d$, then all the other solutions are given by $(x_0 + \lambda \frac{b}{(a,b)})a + (y_0 - \lambda \frac{a}{(a,b)})b = d$ for $\lambda \in \mathbb{Z}$. We observe that $(x_0 + \lambda \frac{b}{(a,b)}, y_0 - \lambda \frac{a}{(a,b)})$ divides $(ax_0 + \lambda \frac{ab}{(a,b)}, by_0 - \lambda \frac{ba}{(a,b)}) = (ax_0 + by_0, by_0 - \lambda \frac{ba}{(a,b)})$ which divides $ax_0 + by_0 = d$. In particular,

$$(x_0 + \lambda \frac{b}{(a,b)}, y_0 - \lambda \frac{a}{(a,b)}) = (x_0 + \lambda' \frac{b}{(a,b)}, y_0 - \lambda' \frac{a}{(a,b)})$$

whenever $\lambda' - \lambda$ is multiple of d. Thus we can define the set $\mathcal{S} = \{(x,y) \in \mathbb{Z}^2 : ax + by = d$ and x, y coprime$\}$, and we have that $\mathcal{S}$ is a finite union of arithmetic progressions. We now intersect the set $\mathcal{S}$ with the conditions that $x\mathbf{h}_1 + y\mathbf{h}_2$ has all components > 0 and we get a finite union of arithmetic progressions (possibly truncated on one or both sides). This describes the set of all possible values for $\mathbf{l} = x\mathbf{h}_1 + y\mathbf{h}_2$ as a finite union of arithmetic progressions, yielding Item 1.

All the above steps can be performed algorithmically, and thus we get Item 2.

For Item 3, if we have a sequence of limit directions $\mathbf{l}_n$ converging to some direction, then up to taking a subsequence we can assume that they all belong to a common arithmetic progression, and thus $\mathbf{l}_n = \mathbf{t} + k_n\mathbf{r}$ for some $\mathbf{t}, \mathbf{r} \in \langle \mathbf{h}_1, \mathbf{h}_2 \rangle$ and for $k_n \to +\infty$ integers, and thus the limit of the sequence of directions must be direction of $\mathbf{r} \in \langle \mathbf{h}_1, \mathbf{h}_2 \rangle$. Without loss of generality, we can also assume that $\mathbf{r}$ is not a proper power (as we only care about the direction of $\mathbf{r}$).

But if all components of $\mathbf{r}$ are strictly positive, then $\mathbf{r}$ can be realized as limit direction (by Proposition 5.10), and thus all $\mathbf{l}_n$ for n big enough must belong to some full-tree configuration with limit direction $\mathbf{r}$, contradiction. $\square$

5.4 Examples

As usual, in the examples we use $\mathbf{A} = \mathbb{Z}^{\mathcal{P}(\Gamma,\psi)}$, omitting the $\mathbb{Z}/2\mathbb{Z}$ summand.

Example 5.15. Consider the GBS graph (Γ, ψ) with one vertex and two edges, as in Figure 11, and call Λ its affine representation. We have that $\mathcal{P}(\Gamma, \psi) = \{2, 3\}$. The only vertex of Γ corresponds to a single copy of $\mathbf{A}^+$ in Λ. The two edges belong to a common quasi-conjugacy class, with two minimal regions, corresponding to the points $\mathbf{a}_1, \mathbf{a}_2 \in \mathbf{A}^+$ with $\mathbf{a}_1 = (0, 3)$ and $\mathbf{a}_2 = (3, 0)$. The affine representation has edges $(0, 3)$ —— $(11, 13)$ and $(3, 0)$ —— $(18, 13)$.

This gives us a configuration (C) with vectors $\mathbf{x}_1 = (11, 10), \mathbf{x}_2 = (15, 13)$. It's fairly easy to compute the root (R) of this configuration, which has vectors $\mathbf{v}_1 = (3, 4), \mathbf{v}_2 = (4, 3)$. One readily checks that $(C) = (R112)$, see Figure 12.

$$(R) = \begin{cases} (0,3) \text{ —— } (0,3) + (3,4) \\ (3,0) \text{ —— } (3,0) + (4,3) \end{cases} \qquad (C) = (R112) = \begin{cases} (0,3) \text{ —— } (0,3) + (11,10) \\ (3,0) \text{ —— } (3,0) + (15,13) \end{cases}$$

In the subsequent Figure 13, we can observe concretely the effect of Proposition 5.5. The picture is NOT the affine representation; instead, we place all the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}_1, \mathbf{x}_2$ at the origin of the Euclidean plane $\mathbb{R}^2$, and we put in evidence the directions to which the vectors are pointing, and the *angles* spanned by two of them. This shows explicitly how taking sons corresponds to making the angle narrower. The roots can "see" more configurations, as they correspond to wider angles.

For the root (R) we can explicitly compute the sequence $\ldots, \mathbf{v}_{-1}, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ as in Section 4.4. This is infinite in both directions, and it's summarized in the following table:

vector	...	$\mathbf{v}_{-2}$	$\mathbf{v}_{-1}$	$\mathbf{v}_0$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	...
component $p = 2$	...	9	7	5	3	4	9	14	19	...
component $p = 3$	...	19	14	9	4	3	5	7	9	...
k of the connection	...	$k = 2$	$k = 2$	$k = 2$	$k = 3$	$k = 3$	$k = 2$	$k = 2$	$k = 2$	...

In the first three rows we can see the list of the vectors, and their respective first components and second components. In the fourth row, we can see the parameter $k \geq 2$ for the connection, satisfying $\mathbf{v}_{i-1} + \mathbf{v}_{i+1} = k\mathbf{v}_i$ for $i \in \mathbb{Z}$. In order to compute the next term $\mathbf{v}_{i+1}$ in the sequence, we just choose the minimum value of $k \geq 2$ such that the next edge $\mathbf{v}_{i+1}$ satisfies $\mathbf{v}_{i+1} \geq \mathbf{a}_1 - \mathbf{a}_2$; and similarly to extend the sequence on the left. The sequence is an arithmetic progression, except at the (finitely many) singular points where $k \neq 2$ (and in fact, the reader can see the arithmetic progressions $4, 9, 14, 19, \ldots$ and $3, 5, 7, 9, \ldots$ in the table); the limit directions are $\mathbf{l}^- = (2, 5)$ and $\mathbf{l}^+ = (5, 2)$. This is enough to describe the isomorphism problem for the given GBS group: the configurations that can be reached by means of slides, swaps, connections are exactly the iterated sons of the roots in the sequence above.

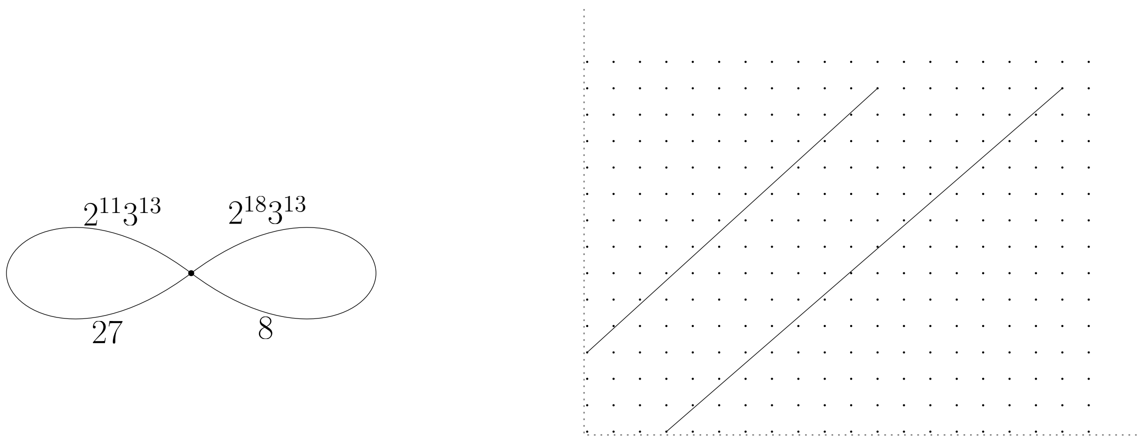
We are now interested about all configurations (C) with minimal points $\mathbf{a}_1, \mathbf{a}_2$ and edges $\mathbf{x}_1, \mathbf{x}_2$ such that $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle = \langle (3, 4), (4, 3) \rangle$ and $\mathbf{x}_1, \mathbf{x}_2$ is Nielsen equivalent to $(3, 4), (4, 3)$. By Proposition 5.10 we know that all $\mathbf{l} \in \langle (3, 4), (4, 3) \rangle$ which are part of a basis, all of whose components are strictly positive, and with at least one component < 3 (because $|\mathbf{a}_1 - \mathbf{a}_2| = (3, 3)$) can be realized as limit direction of some configuration. The values of $\mathbf{l}$ satisfying these conditions are $(7\ell + 6, 1), (14\ell + 5, 2), (2, 14\ell + 5), (1, 7\ell + 6)$ for $\ell \geq 0$ integer (see Theorem 5.14). In Figure 14 we can see all of these directions represented in the plane; as long as we stay far away from the axes, this is a discrete set. Between each consecutive pair of directions, lies exactly one isomorphism class of GBS groups, corresponding to the gray regions represented in Figure 15. We point out that each gray region contains most of the configurations in a certain isomorphism class (to be precise, the full-tree ones) - but a few exceptional ones can fall outside. We also point out that there are a few isomorphism classes which don't contain any full-tree configuration, and thus are not represented by any region; these are described in Remark 5.3. We note that $\mathbf{a}_1 - \mathbf{a}_2 = (3, -3)$ belongs to $\langle (3, 4), (4, 3)\rangle$ but it's not part of a basis; thus in this case no twin roots can appear. For example, the following sequences of vectors (giving sequences of roots) correspond to different isomorphism classes of GBS groups.

vector	$\mathbf{l}^-$		$\mathbf{v}_0$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	...		$\mathbf{l}^+$
component $p = 2$	5		4	5	11	17	23	...		6
component $p = 3$	2		3	2	3	4	5	...		1

vector	$\mathbf{l}^-$		$\mathbf{v}_0$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	...		$\mathbf{l}^+$
component $p = 2$	6		5	6	25	44	63	...		19
component $p = 3$	1		2	1	3	5	7	...		2

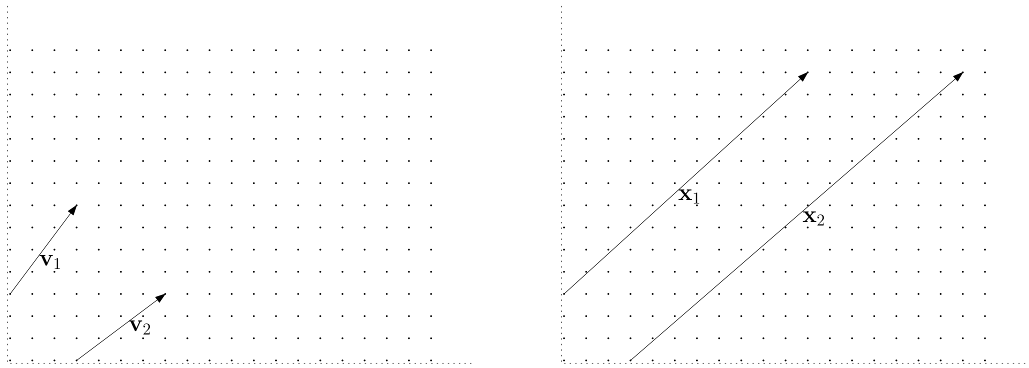
vector	$\mathbf{l}^-$		$\mathbf{v}_0$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	...		$\mathbf{l}^+$
component $p = 2$	19		6	19	32	54	58	...		13
component $p = 3$	2		1	2	3	4	5	...		1

vector	$\mathbf{l}^-$		$\mathbf{v}_0$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	...		$\mathbf{l}^+$
component $p = 2$	13		6	13	46	79	112	...		33
component $p = 3$	1		1	1	3	5	7	...		2



(Γ, ψ) Λ

Figure 11: The GBS graph (Γ, ψ) of Example 5.15 and the corresponding affine representation Λ.



(R) $\quad\quad$ $(C) = (R122)$

Figure 12: The root (R) (corresponding to vectors $\mathbf{v}_1, \mathbf{v}_2$) for the configuration $(C) = (R112)$ (corresponding to vectors $\mathbf{x}_1, \mathbf{x}_2$) of our initial GBS graph.

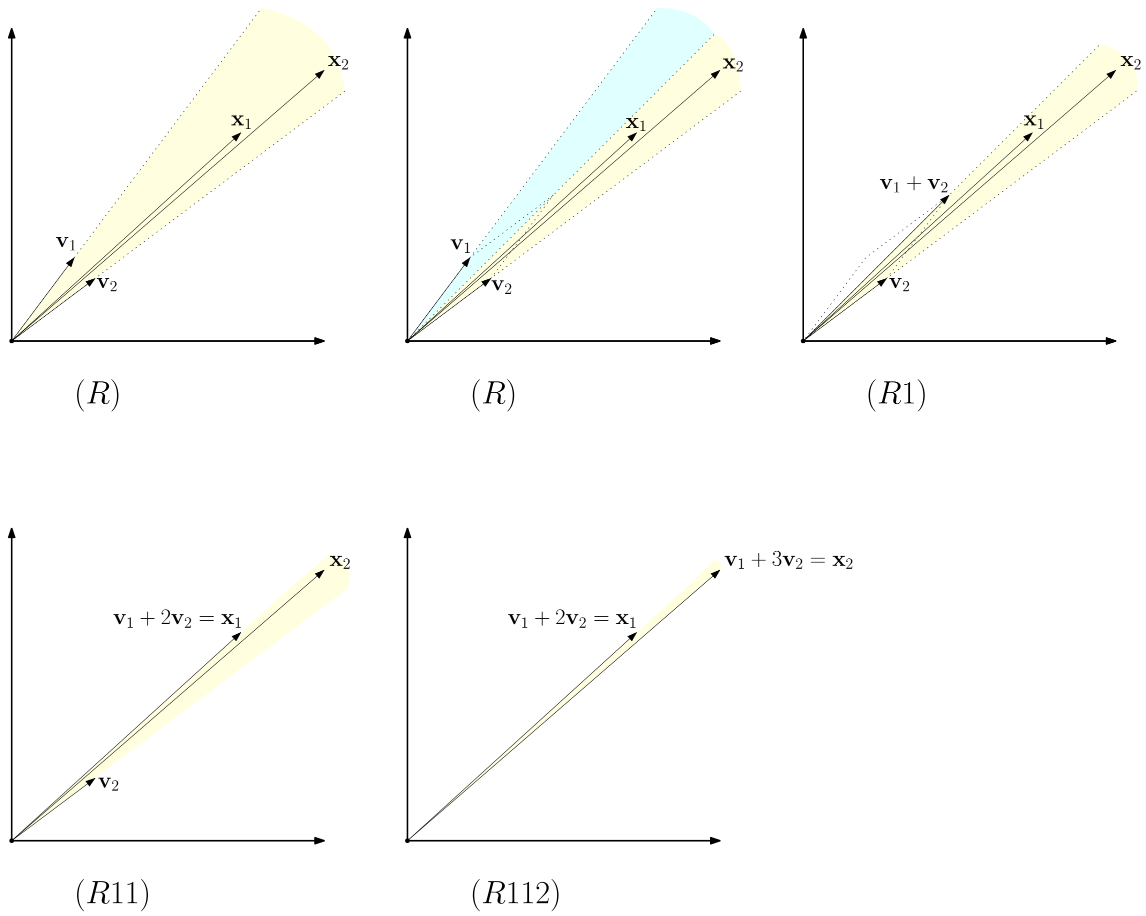


Figure 13: In the picture, we see the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}_1, \mathbf{x}_2$ inside the Euclidean plane $\mathbb{R}^2$. We point out that this is NOT the affine representation; here we place the initial points of the vectors at the origin of the Euclidean plane, and we are mainly interested in the directions in which the vectors point. This should be thought as a graphical representation of the procedure described in the proof of Proposition 5.5.
In the first image (top left), we see our initial vectors $\mathbf{v}_1, \mathbf{v}_2$ defining a certain angle, and the two "target vectors" $\mathbf{x}_1, \mathbf{x}_2$ are contained inside this angle. In the second image (top middle), we draw a parallelogram, whose diagonal corresponds to the vector $\mathbf{v}_1 + \mathbf{v}_2$, which divides the angle into two parts; the target vectors $\mathbf{x}_1, \mathbf{x}_2$ are contained in one of the two halves. In the third image (top right), we choose, among the two halves of the angle, the one containing the two target vectors $\mathbf{x}_1, \mathbf{x}_2$; this means that we have to choose the first son, changing $\mathbf{v}_1$ into $\mathbf{v}_1 + \mathbf{v}_2$ and keeping $\mathbf{v}_2$ as it is. The fourth and the fifth image (bottom) show the subsequent iteration of the same procedure; at each step, the angle becomes narrower, but it keeps containing the two target vectors $\mathbf{x}_1, \mathbf{x}_2$. At the end of the procedure, we obtain exactly the desired configuration.

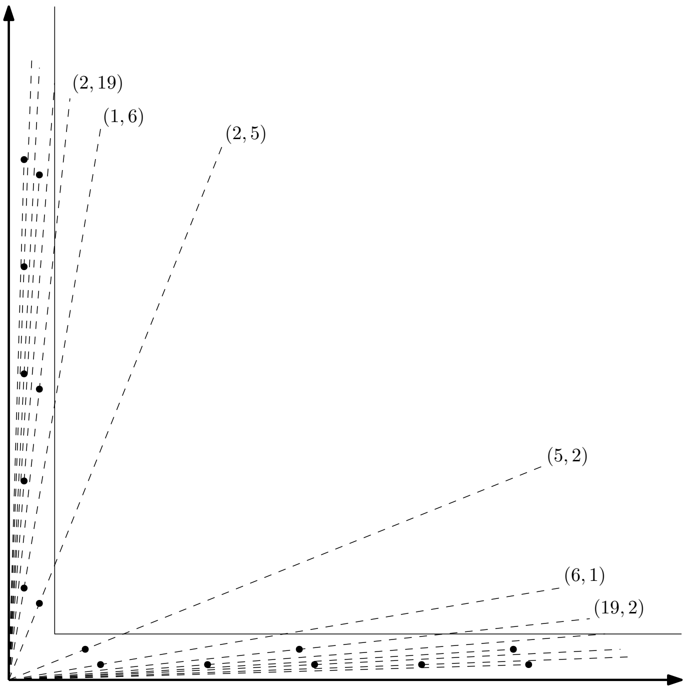


Figure 14: The set of all possible limit directions for the subgroup $H = \langle(4,3),(3,4)\rangle$ as in Example 5.15. The dots represents all the elements which are part of a basis for H, and which lie near enough to the two axes - we only consider elements which lie left/below of the line in the figure. For each dot, we draw the corresponding direction (the dashed lines), which can always be realized as limit direction of some GBS group. From the picture we can see that the points form a finite union of arithmetic progressions, accumulating only in the horizontal and vertical directions.

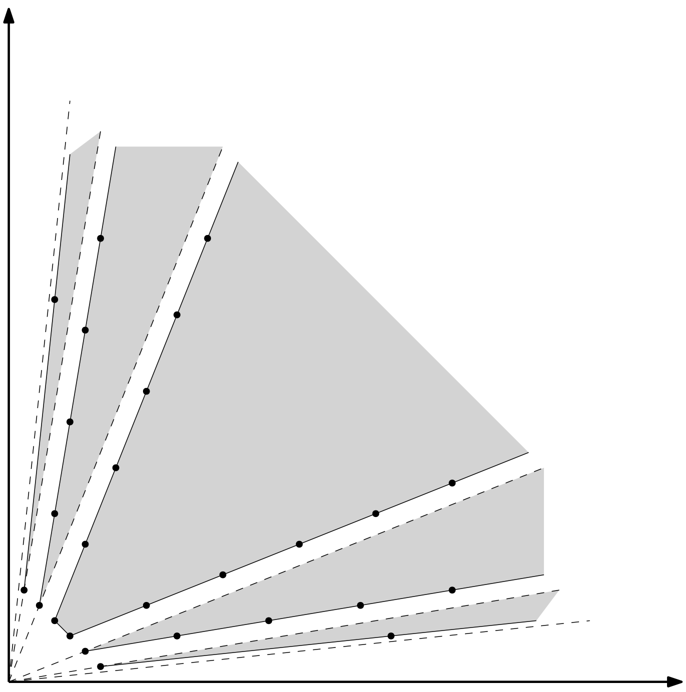


Figure 15: The limit angles obtained from the limit directions represented in Figure 14. Each gray region is an isomorphism class of GBS groups. Two consecutive points on the boundary of a gray region give us one of the roots for the binary trees - consecutive couples of points being roots related by a connection move.

References

[ACRK25a] D. Ascari, M. Casals-Ruiz, and I. Kazachkov. Automorphisms of graphs of groups, 2025. preprint.

[ACRK25b] D. Ascari, M. Casals-Ruiz, and I. Kazachkov. On the isomorphism problem for cyclic JSJ decompositions: vertex elimination, 2025. preprint.

[ACRK25c] D. Ascari, M. Casals-Ruiz, and I. Kazachkov. On the isomorphism problem for generalized Baumslag-Solitar groups: angles and flexible configurations, 2025. preprint.

[Bar18] B. Barrett. Computing JSJ decompositions of hyperbolic groups. *Journal of Topology*, 11(2):527–558, 2018.

[Bau77] A. Baudisch. Kommutationsgleichungen in semifreien Gruppen. *Acta Math. Acad. Sci. Hungar.*, 29(3-4):235–249, 1977.

[Bee15] B. Beeker. Multiple conjugacy problem in graphs of free abelian groups. *Groups Geom. Dyn.*, 9(1):1–27, 2015.

[BF91] M. Bestvina and M. Feighn. Bounding the complexity of simplicial group actions on trees. *Inventiones mathematicae*, 103(3):449–470, 1991.

[BJ96] Hyman Bass and Renfang Jiang. Automorphism groups of tree actions and of graphs of groups. *J. Pure Appl. Algebra*, 112(2):109–155, 1996.

[Bla65] Norman Blackburn. Conjugacy in nilpotent groups. *Proc. Amer. Math. Soc.*, 16:143–148, 1965.

[Bri63] John L. Britton. The word problem. *Ann. of Math. (2)*, 77:16–32, 1963.

[Bum04] Inna Bumagin. The conjugacy problem for relatively hyperbolic groups. *Algebr. Geom. Topol.*, 4:1013–1040, 2004.

[Cas16] C. H. Cashen. Splitting line patterns in free groups. *Algebraic & Geometric Topology*, 16(2):621–673, 2016.

[CF08] M. Clay and M. Forester. On the isomorphism problem for generalized Baumslag-Solitar groups. *Algebraic & Geometric Topology*, 8:2289–2322, 2008.

[CGW09] John Crisp, Eddy Godelle, and Bert Wiest. The conjugacy problem in subgroups of right-angled artin groups. *Journal of Topology*, 2, 2009.

[Cla07] M. Clay. Deformation Spaces of G–trees and Automorphisms of Baumslag–Solitar groups. *Groups, Geometry, and Dynamics*, 3, 2007.

[Cla14] Matt Clay. When does a right-angled Artin group split over $\mathbb{Z}$? *Internat. J. Algebra Comput.*, 24(6):815–825, 2014.

[CM11] C. H. Cashen and N. Macura. Line patterns in free groups. *Geometry & Topology*, 15(3):1419–1475, 2011.

[CRKZ21] M. Casals-Ruiz, I. Kazachkov, and A. Zakharov. Commensurability of Baumslag-Solitar groups. *Indiana Univ. Math. J.*, 70:2527–2555, 2021.

[CV86] M. Culler and K. Vogtmann. Moduli of graphs and automorphisms of free groups. *Inventiones mathematicae*, 84(1):91–119, 1986.

[Day14] Matthew B. Day. Full-featured peak reduction in right-angled Artin groups. *Algebr. Geom. Topol.*, 14(3):1677–1743, 2014.

[DF05] G. Diao and M. Feighn. The Grushko decomposition of a finite graph of finite rank free groups: an algorithm. *Geometry & Topology*, 9(4):1835–1880, 2005.

[DG08a] F. Dahmani and D. Groves. The isomorphism problem for toral relatively hyperbolic groups. *Publications Mathématiques de l'IHÉS*, 107:211–290, 2008.

[DG08b] François Dahmani and Daniel Groves. Detecting free splittings in relatively hyperbolic groups. *Trans. Amer. Math. Soc.*, 360(12):6303–6318, 2008.

[DG11] F. Dahmani and V. Guirardel. The isomorphism problem for all hyperbolic groups. *Geometric & Functional Analysis*, 21(2):223–300, apr 2011.

[DS99] M. J. Dunwoody and M. E. Sageev. JSJ-splittings for finitely presented groups over slender groups. *Inventiones mathematicae*, 135(1):25–44, 1999.

[DT19] François Dahmani and Nicholas Touikan. Deciding isomorphy using Dehn fillings, the splitting case. *Invent. Math.*, 215(1):81–169, 2019.

[Dud17] F. Dudkin. The isomorphism problem for generalized Baumslag–Solitar groups with one mobile edge. *Algebra and Logic*, 56:1–13, 2017.

[Dun85] M. J. Dunwoody. The accessibility of finitely presented groups. *Inventiones mathematicae*, 81(3):449–457, 1985.

[Dun93] M. J. Dunwoody. *An inaccessible group*, volume 1 of *London Mathematical Society Lecture Note Series*, pages 75–78. Cambridge University Press, Cambridge, 1993.

[Eme58] V. A. Emelichev. Commutative semigroups with one defining relation. *Shuya Gosudarstvennyi Pedagogicheskii Institut Uchenye Zapiski*, 5:227–242, 1958.

[ES69] Samuel Eilenberg and M. P. Schützenberger. Rational sets in commutative monoids. *J. Algebra*, 13:173–191, 1969.

[FM98] B. Farb and L. Mosher. A rigidity theorem for the solvable Baumslag-Solitar groups. *Inventiones mathematicae*, 131(2):419–451, 1998.

[For02] M. Forester. Deformation and rigidity of simplicial group actions on trees. *Geometry & Topology*, 6(1):219 – 267, 2002.

[For03] M. Forester. On uniqueness of JSJ decompositions of finitely generated groups. *Commentarii Mathematici Helvetici*, 78(4):740–751, 2003.

[For06] M. Forester. Splittings of generalized Baumslag-Solitar groups. *Geometriae Dedicata*, 121(1):43–59, 2006.

[FP06] K. Fujiwara and P. Papasoglu. JSJ-decompositions of finitely presented groups and complexes of groups. *Geometric & Functional Analysis*, 16:70–125, 02 2006.

[GL11] V. Guirardel and G. Levitt. Trees of cylinders and canonical splittings. *Geometry & Topology*, 15(2):977–1012, 2011.

[GL17] V. Guirardel and G. Levitt. *JSJ decompositions of groups*, volume 395 of *Astérisque*. Société Mathématique de France, 2017.

[Gru40] I. A. Grushko. On the bases of a free product of groups. *Matematicheskii Sbornik*, 8:169–182, 1940.

[GS66] Seymour Ginsburg and Edwin H. Spanier. Semigroups, Presburger formulas, and languages. *Pacific J. Math.*, 16:285–296, 1966.

[GW09] Daniel Groves and Henry Wilton. Enumerating limit groups. *Groups Geom. Dyn.*, 3(3):389–399, 2009.

[Hat] A. Hatcher. Notes on basic 3-manifold topology. https://pi.math.cornell.edu/ hatcher/3M/3Mfds.pdf.

[IJCR91] Oscar H. Ibarra, Tao Jiang, Jik H. Chang, and Bala Ravikumar. Some classes of languages in NC^1. *Inform. and Comput.*, 90(1):86–106, 1991.

[KM79] M. I. Kargapolov and Ju.Ĩ. Merzljakov. *Fundamentals of the theory of groups*, volume 62 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, russian edition, 1979.

[KM05] O. Kharlampovich and A. Myasnikov. Effective JSJ decompositions, 2005.

[Kro90] P. H. Kropholler. Baumslag-solitar groups and some other groups of cohomological dimension two. *Commentarii mathematici Helvetici*, 65(4):547–558, 1990.

[Lev05] G. Levitt. On the automorphism group of generalized Baumslag-Solitar groups. *Geometry & Topology*, 11:473 – 515, 2005.

[Lev07] G. Levitt. On the automorphism group of generalized baumslag–solitar groups. *Geometry & Topology*, 11(1):473–515, 2007.

[Lev08] G. Levitt. Unsolvability of the isomorphism problem for [free abelian]-by-free groups. *arXiv: Group Theory*, 2008.

[Loc92] Jody Meyer Lockhart. The conjugacy problem for graph products with infinite cyclic edge groups. *Proceedings of the American Mathematical Society*, 114(3):603–606, 1992.

[LS77] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory*, volume Band 89 of *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*. Springer-Verlag, Berlin-New York, 1977.

[Mal58] A. I. Mal'cev. Defining relations in categories. *Ivanovo Gosudarstvennyi Pedagogicheskii Institut Uchenye Zapiski*, 18:49–60, 1958.

[McC91] J. McCool. A class of one-relator groups with centre. *Bulletin of the Australian Mathematical Society*, 44(2):245–252, 1991.

[MMNV22] Jeremy Macdonald, Alexei Myasnikov, Andrey Nikolaev, and Svetla Vassileva. Logspace and compressed-word computations in nilpotent groups. *Trans. Amer. Math. Soc.*, 375(8):5425–5459, 2022.

[MS84] John W. Morgan and Peter B. Shalen. Valuations, trees, and degenerations of hyperbolic structures. I. *Ann. of Math. (2)*, 120(3):401–476, 1984.

[Osi06] Denis V. Osin. Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. *Mem. Amer. Math. Soc.*, 179(843):vi+100, 2006.

[Pap05] P. Papasoglu. Quasi-isometry invariance of group splittings. *Annals of Mathematics*, 161(2):759–830, 2005.

[RS97] E. Rips and Z. Sela. Cyclic splittings of finitely presented groups and the canonical JSJ decomposition. *Annals of Mathematics*, 146(1):53–109, 1997.

[Sel95] Z. Sela. The isomorphism problem for hyperbolic groups I. *Annals of Mathematics*, 141(2):217–283, 1995.

[Sel01] Z. Sela. Diophantine geometry over groups I: Makanin-Razborov diagrams. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 93(1):31–106, 2001.

[Ser77] J.P. Serre. *Arbres, Amalgames, Sl2.* Asterisque, tome 46. Soc. Math. De France, 1977.

[Sta68] J. R. Stallings. On torsion-free groups with infinitely many ends. *Annals of Mathematics*, 88(2):312–334, 1968.

[Tai68] M. A. Taiclin. Algorithmic problems for commutative semigroups. *Dokl. Akad. Nauk SSSR*, 178:786–789, 1968.

[Tou18] Nicholas W. M. Touikan. Detecting geometric splittings in finitely presented groups. *Trans. Amer. Math. Soc.*, 370(8):5635–5704, 2018.

[Wan23] D. Wang. The isomorphism problem for small-rose generalized Baumslag-Solitar groups, 2023.

[Wei16] Armin Weiss. A logspace solution to the word and conjugacy problem of generalized baumslag-solitar groups. In Delaram Kahrobaei, Bren Cavallo, and David Garber, editors, *Algebra and computer science*. American Mathematical Society, 2016.

[Whi36] J. H. C. Whitehead. On equivalent sets of elements in a free group. *Ann. of Math. (2)*, 37(4):782–800, 1936.

[Why01] K. Whyte. The large scale geometry of the higher Baumslag-Solitar groups. *Geometric & Functional Analysis*, 11(6):1327–1343, 2001.

[Zim90] B. P. Zimmermann. Finite abelian group actions on surfaces. *Yokohama Mathematical Journal*, 38, 1990.

A Nielsen equivalence in finitely generated abelian groups

We consider elements of $\mathbb{Z}^n$ as column vectors. We denote with $\mathcal{M}_{n\times k}(\mathbb{Z})$ the set of matrices with n rows and k columns and integer coefficients. For $M \in \mathcal{M}_{n\times k}(\mathbb{Z})$ we denote with M_i^j the integer number in the i-th row and j-th column of M, for $i = 1, ..., n$ and $j = 1, ..., k$; we denote with $\mathrm{span}(M)$ the subgroup of $\mathbb{Z}^n$ generated by the columns of M. A **column operation** on a matrix consists of choosing a column and adding or subtracting it from another column. A **(column) reordering operation** consists of interchanging two columns of the matrix. Notice that column and column reordering operations don't change $\mathrm{span}(M)$. A **row operation** consists of choosing a row and adding or subtracting it from another row. Row operations correspond to changing basis for the free abelian group $\mathbb{Z}^n$.

Smith normal form

Definition A.1. *A matrix $S \in \mathcal{M}_{n\times k}(\mathbb{Z})$ is in **Smith normal form** if it satisfies the following conditions:*

1. *$S_i^j = 0$ whenever $i \neq j$.*
2. *$S_i^i \geq 0$ for $i = 1, ..., m$ where $m = \min\{n, k\}$.*
3. *We have $S_{i+1}^{i+1} \mid S_i^i$ for all $i = 1, ..., m-1$.*

Every matrix is equivalent, up to row and column operations and reordering operations, to a unique matrix in Smith normal form. An interesting feature is that the integers S_i^i can be computed directly from the initial matrix, as follows. For $M \in \mathcal{M}_{n\times k}(\mathbb{Z})$, define $D_\ell(M) \geq 0$ as the greatest common divisor of all the determinants of the $\ell \times \ell$ minors of M, for $1 \leq \ell \leq m$ where $m = \min\{n, k\}$. Notice that, if M has rank r, then $D_\ell(M) = 0$ if and only if $\ell \geq r + 1$.

Lemma A.2. *Let $M \in \mathcal{M}_{n\times k}(\mathbb{Z})$ be a matrix of rank r. Let S be the unique matrix in Smith normal form obtained from M by means of row and column operations. Then we have the following:*

1. *$D_1(M) \mid D_2(M) \mid ... \mid D_m(M)$ where $m = \min\{n, k\}$.*
2. *$S_i^i = D_{m+1-i}(M) = 0$ for $i = 1, ..., m-r$.*
3. *$S_i^i = D_{m+1-i}(M)/D_{m-i}(M)$ for $i = m-r+1, ..., m-1$.*
4. *$S_m^m = D_1(M)$.*

Proof. It's immediate to notice that $D_\ell(M)$ is invariant under row and column operations, as well as column swap operation. But for a matrix in Smith normal the thesis holds, so it holds for all matrices. □

Finitely generated abelian groups

Every finitely generated abelian group is isomorphic to

$$\mathbb{Z}/d_1\mathbb{Z} \oplus ... \oplus \mathbb{Z}/d_n\mathbb{Z}$$

for unique integers numbers $n, d_1, ..., d_n \geq 0$ satisfying $1 \neq d_n \mid d_{n-1} \mid ... \mid d_1$. Notice that some d_i can be equal to 0, producing $\mathbb{Z}$ summands.

Lemma A.3. *Let $M \in \mathcal{M}_{n\times k}(\mathbb{Z})$. Let $S \in \mathcal{M}_{n\times k}(\mathbb{Z})$ be the Smith normal form of M. Then $\mathbb{Z}^n/span(M)$ is isomorphic to $\mathbb{Z}/d_1\mathbb{Z} \oplus ... \oplus \mathbb{Z}/d_{n'}\mathbb{Z}$ where*

1. *$n' \leq \min\{n, k\}$ is the maximum integer such that $S_{n'}^{n'} \neq 1$.*
2. *$d_i = S_i^i$ for $i = 1, ..., n'$.*

Proof. Column and reordering operations don't change $\mathrm{span}(M)$, and in particular they don't change the quotient $\mathbb{Z}^n/\mathrm{span}(M)$. Row operations correspond to changing basis for $\mathbb{Z}^n$, so they don't change the isomorphism type of $\mathbf{Z}^n/\mathrm{span}(M)$. Thus we have that $\mathbb{Z}^n/\mathrm{span}(M)$ is isomorphic to $\mathbb{Z}^n/\mathrm{span}(S)$. The conclusion follows. □

Nielsen equivalence in finitely generated abelian groups

Let A be a finitely generated abelian group. Let $(w_1, ..., w_k)$ be an ordered k-tuple of elements of A: a **Nielsen move** on $(w_1, ..., w_k)$ is any operation that consists of the substitution w_i with $w_i + w_j$ or with $w_i - w_j$, for some $j \neq i$. Two ordered k-tuples are **Nielsen equivalent** if there is a sequence of Nielsen moves going from one to the other. Our aim in this section is to classify the k-tuples of generators for A up to Nielsen equivalence (we ignore k-tuples that don't generate the whole group A, as they would require us to change the ambient group we are working in).

Fix an isomorphism

$$A = \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus ... \oplus \mathbb{Z}/d_n\mathbb{Z}$$

for integers $n, d_1, ..., d_n \geq 0$ with $1 \neq d_n \mid d_{n-1} \mid ... \mid d_1$. Then we can represent elements of A as column vectors with n entries, where the i-th entry is an element of $\mathbb{Z}/d_i\mathbb{Z}$ for $i = 1, ..., n$. We represent an ordered k-tuple $(w_1, ..., w_k)$ as a matrix $M = M(w_1, ..., w_k)$ with n rows and k columns. Nielsen moves correspond to column operations on the matrix M.

Lemma A.4. *The group $A = \mathbb{Z}/d_1\mathbb{Z} \oplus ... \oplus \mathbb{Z}/d_n\mathbb{Z}$ can't be generated by less than n elements.*

Proof. Take a prime $p \mid d_n$. Consider the surjective projection map from $\mathbb{Z}/d_1\mathbb{Z} \oplus ... \oplus \mathbb{Z}/d_n\mathbb{Z}$ to $(\mathbb{Z}/p\mathbb{Z})^n$. But $(\mathbb{Z}/p\mathbb{Z})^n$ can't be generated by less than n elements. □

Proposition A.5. *For every $k \geq n+1$, every two ordered k-tuples of generators for A are Nielsen equivalent. Two ordered n-tuples of generators for A are Nielsen equivalent if and only if the corresponding matrices have the same determinant modulo d_n.*

Proof. Let $(w_1, ..., w_k)$ and $(u_1, ..., u_k)$ be k-tuples of generators for A and let $M, N \in \mathcal{M}_{n\times k}(\mathbb{Z})$ be the matrices whose columns represent $w_1, ..., w_k$ and $u_1, ...u_k$ respectively. If $k = n$ and $w_1, ..., w_k$ and $u_1, ..., u_k$ are Nielsen equivalent, then there must be a sequence of columns operations going from M to N, and in particular the two matrices must have the same determinant.

We look at the first row of M, and using column operations we perform the Euclidean algorithm; we obtain a new matrix $M_1^1 \neq 0$ and $M_1^j = 0$ for $j = 2, ..., k$. But since $w_1, ..., w_k$ generate the whole group, we must have that M_1^1 is coprime with d_1. We add d_1 to M_1^2 (we can, since elements of the first row are elements of $\mathbb{Z}/d_1\mathbb{Z}$), and then we perform the Euclidean algorithm again, obtaining a new matrix with $M_1^1 = 1$ and $M_1^j = 0$ for $j = 2, ..., k$.

We reiterate the same reasoning by induction. If $k \geq n+1$ we obtain a matrix with $M_i^i = 1$ for $i = 1, ..., n$ and $M_i^j = 0$ for $j \neq i$. Performing the same procedureon N we obtain the same matrix, and in particular there is a sequence of column operations going from M to N, and thus $w_1, ..., w_k$ is Nielsen equivalent to $u_1, ..., u_k$. If $k = n$, we obtain a matrix with $M_i^i = 1$ for $i = 1, ..., n-1$ and M_n^n invertible modulo d_n and $M_i^j = 0$ for $j \neq n$. Notice that the determinant of M modulo d_n is an invariant under column operations, and for the matrix reduced in this form it's equal to M_n^n. If the determinant of M was equal to the one of N, then performing the same procedure on N we obtain the same matrix, and in particular there is a sequence of column operations going from M to N, and thus $(w_1, ..., w_k)$ is Nielsen equivalent to $(u_1, ..., u_k)$. □

We will also need the following two lemmas.

Lemma A.6. *Let $a \in A$ be an element represented by a vector $(a_1, ..., a_n)$, and suppose that* $\gcd(a_1, ..., a_{n-1}, a_n) = 1$. *Let $(a, w_2, ..., w_k)$, $(a, u_2, ..., u_k)$ be Nielsen equivalent k-tuples of generators for A with $k \geq n+1$. Then $([w_2], ..., [w_k])$, $([u_2], ..., [u_k])$ are Nielsen equivalent $(k-1)$-tuples of generators for $A/\langle a\rangle$.*

Proof. Let $R \in \mathcal{M}_{n\times n}(\mathbb{Z})$ be the matrix given by $R_i^i = d_i$ for $i = 1, ..., n$ and $R_i^j = 0$ for $j \neq i$. We have that $A = \mathbb{Z}^n/\mathrm{span}(R)$. Let R' be the matrix obtained from R by adding an extra column, equal to $(a_1, ..., a_n)$. Then $A/\langle a\rangle = \mathbb{Z}^n/\mathrm{span}(R')$. Since $\gcd(a_1, ..., a_{n-1}) = 1$, we see that $D_1(R') = 1$ and $D_2(R') = d_n \neq 1$. In particular, by Lemmas A.2, A.3 and A.4 we have that the minimum number of generators for $A/\langle a\rangle$ is $n-1$. But then by Proposition A.5 the $(k-1)$-tuples of generators $([w_2], ..., [w_k])$, $([u_2], ..., [u_k])$ for $A/\langle a\rangle$ are Nielsen equivalent, since $k - 1 \geq n$. □

Lemma A.7. *Let $a \in A$ be an element represented by a vector $(a_1, ..., a_n)$, and suppose that $\gcd(a_1, ..., a_{n-1}) = 1$. Let $(a, w_2, ..., w_n)$, $(a, u_2, ..., u_n)$ be Nielsen equivalent n-tuples of generators for A. Then $([w_2], ..., [w_n])$, $([u_2], ..., [u_n])$ are Nielsen equivalent $(n-1)$-tuples of generators for $A/\langle a\rangle$.*

Proof. Let $R \in \mathcal{M}_{n\times n}(\mathbb{Z})$ be the matrix given by $R_i^i = d_i$ for $i = 1, ..., n$ and $R_i^j = 0$ for $j \neq i$. We have that $A = \mathbb{Z}^n/\text{span}(R)$. Let also $M, N \in \mathcal{M}_{n\times k}(\mathbb{Z})$ be matrices whose columns represent $a, w_2, ..., w_n$ and $a, u_2, ..., u_n$ respectively. By Proposition A.5 M and N must have the same determinant modulo d_n.

Let R' be the matrix obtained from R by adding an extra column, equal to the first column of M and of N. We have that $A/\langle a\rangle = \mathbb{Z}^n/\text{span}(R')$ and $D_1(R') = 1$ and $D_2(R') = d_n$ (since $\gcd(a_1, ..., a_{n-1}) = 1$). Thus, by Proposition A.5, we only need to check that the matrices representing $([w_2], ..., [w_n])$ and $([u_2], ..., [u_n])$ have the same determinant modulo d_n.

We perform row operations at the same time on R, M, N, running the Euclidean algorithm on the first column of M (which is also the first column on N). We obtain matrices $\overline{R}, \overline{M}, \overline{N}$ such that the first column of $\overline{M}$ (which is also the first column of $\overline{N}$) is equal to $(1, 0, ..., 0, 0)$. Notice that the determinant of $\overline{M}$ is the same as the one of $\overline{N}$ modulo d_n, since row operations don't change the determinant. Since row operations correspond to changing basis of $\mathbb{Z}^n$, we have an isomorphism $A \cong \mathbb{Z}^n/\text{span}(\overline{R})$, with the two matrices $\overline{M}$ and $\overline{N}$ representing the two k-tuples $a, w_2, ..., w_n$ and $a, u_2, ..., u_n$. Let $\overline{M}''$ be the minor of $\overline{M}$ obtained by cancelling the first row and the first column of $\overline{M}$, and define $\overline{N}''$ analogously; let also $\overline{R}''$ be the matrix obtained from $\overline{R}$ by removing the first row. The isomorphism $A \cong \mathbb{Z}^n/\text{span}(\overline{R})$ induces an isomorphism $A/\langle a\rangle \cong \mathbb{Z}^{n-1}/\text{span}(\overline{R}'')$ and the matrices $\overline{M}''$ and $\overline{N}''$ represent the $(n-1)$-tuples $([w_2], ..., [w_n])$ and $([u_2], ..., [u_n])$. But since the first column of $\overline{M}$ and of $\overline{N}$ is $(1, 0, ..., 0)$ and $\overline{M}, \overline{N}$ have the same determinant modulo d_n, it follows that $\overline{M}''$ and $\overline{N}''$ have the same determinant modulo d_n. The conclusion follows. □

B Nielsen equivalence for big vectors

Fix a free abelian group $\mathbb{Z}^n$. Let $(w_1, ..., w_k)$ be an ordered k-tuple of vectors of $\mathbb{Z}^n$. For an integer $M \geq 1$ we say that an ordered k-tuple $(w_1, ..., w_k)$ is ***M*-big** if each component of each of $w_1, ..., w_k$ is $\geq M$. Our aim is to classify the M-big ordered k-tuples of vectors up to Nielsen moves. This means that, given two M-big ordered k-tuples, we want to know whether there is a sequence of Nielsen moves that goes from one to the other, and such that all the k-tuples that we write along the sequence are M-big.

We are going to need also a relative version of the same problem, that we now introduce. Let $v_1, ..., v_h$ be vectors of $\mathbb{Z}^n$ and let $(w_1, ..., w_k)$ be an ordered k-tuple of vectors of $\mathbb{Z}^n$. A **Nielsen move relative to** $v_1, ..., v_h$, also called **relative Nielsen move** when there is no ambiguity, is any operation that consists of substituting w_i with $w_i + v_j$ or $w_i - v_j$ for some $i \in \{1, ..., k\}$ and $j \in \{1, ..., h\}$. Fixed vectors $v_1, ..., v_h$, we want to classify the M-big ordered k-tuples up to Nielsen moves and relative Nielsen moves. This means that, given two M-big ordered k-tuples, we want to know whether there is a sequence of Nielsen moves and relative Nielsen moves that goes from one to the other, and such that all the k-tuples that we write along the sequence are M-big.

Given vectors $v_1, ..., v_h \in \mathbb{Z}^n$ and an ordered k-tuple $(w_1, ..., w_k)$, we define the finitely generated abelian group

$$A(w_1, ..., w_k; v_1, ..., v_h) = \langle v_1, ..., v_h, w_1, ..., w_k\rangle/\langle v_1, ..., v_h\rangle$$

It is immediate to see that A is invariant if we change the k-tuple by Nielsen moves and relative Nielsen moves. Moreover, we can look at the ordered k-tuple of generators $([w_1], ..., [w_k])$ for the abelian group A, and we notice that Nielsen moves on $(w_1, ..., w_k)$ correspond to Nielsen moves on $([w_1], ..., [w_k])$, while relative Nielsen moves on $(w_1, ..., w_k)$ correspond to the identity on $([w_1], ..., [w_k])$.

The problem of classification of M-big ordered k-tuples of vectors up to Nielsen moves and relative Nielsen moves, and the problem of classification of ordered k-tuples of generators for A up to Nielsen equivalence, are strictly related. In fact, we will prove that they are equivalent for $k \geq 3$, regardless of M. The main result of Appendix B is the following theorem:

Theorem B.1. *Let $v_1, ..., v_h \in \mathbb{Z}^n$ and let $M \geq 1$. Let $k \geq 3$ and let $(w_1, ..., w_k), (w'_1, ..., w'_k)$ be M-big ordered k-tuples. Then the following are equivalent:*

1. *There is a sequence of Nielsen moves and Nielsen moves relative to $v_1, ..., v_h$ going from $(w_1, ..., w_k)$ to $(w'_1, ..., w'_k)$ such that all the k-tuples that we write along the sequence are M-big.*
2. *We have that $A(w_1, ..., w_k; v_1, ..., v_h) = A(w'_1, ..., w'_k; v_1, ..., v_h) = A$ and the two ordered k-tuples of generators $([w_1], ..., [w_k]), ([w'_1], ..., [w'_k])$ are Nielsen equivalent in A.*

Preliminary results

We provide a couple of preliminary lemmas, that will be useful in what follows. Fix $n \geq 1$ and vectors $v_1, ..., v_h$ in $\mathbb{Z}^n$; fix also an integer $M \geq 1$.

Lemma B.2. *Let $k \geq 1$ and let $(w_1, ..., w_k)$ be an M-big ordered k-tuple. Then for every even permutation σ of $\{1, ..., k\}$ there is a sequence of Nielsen moves going from $(w_1, ..., w_k)$ to $(w_{\sigma(1)}, ..., w_{\sigma(k)})$ such that all the k-tuples that we write along the sequence are M-big.*

Proof. Consider the sequence of Nielsen moves

$$(w_1, w_2, w_3) \to (w_1, w_2 + w_3, w_3) \to (w_1, w_2 + w_3, w_3 + w_1) \to (w_1 + w_2 + w_3, w_2 + w_3, w_3 + w_1) \to$$
$$\to (w_2, w_2 + w_3, w_3 + w_1) \to (w_2, w_3, w_3 + w_1) \to (w_2, w_3, w_1)$$

and notice that in the same way we can obtain any permutation which is a 3-cycle. But 3-cycles generate all even permutations; the conclusion follows. □

Lemma B.3. *Let $k \geq 2$ and let $(w_1, w_2, ..., w_k), (w'_1, w_2, ..., w_k)$ be M-big ordered couples. Suppose that $w'_1 = w_1 + \lambda_2 w_2 + ... + \lambda_k w_k + \mu_1 v_1 + ... + \mu_h v_h$ for some $\lambda_2, ..., \lambda_k, \mu_1, ..., \mu_h \in \mathbb{Z}$. Then there is a sequence of Nielsen moves and relative Nielsen moves going from $(w_1, w_2, ..., w_k)$ to $(w'_1, w_2, ..., w_k)$ such that all the k-tuples that we write along the sequence are M-big.*

Proof. We choose an arbitrary positive integer C which is much bigger than all the components of $w_2, ..., w_k, v_1, ..., v_h$ and than all of $\lambda_2, ..., \lambda_k, \mu_1, ..., \mu_h$. We proceed as follows

$$(w_1, w_2, ..., w_k) \xrightarrow{*} (w_1 + Cw_2, w_2, ..., w_k) \xrightarrow{*} (w'_1 + Cw_2, w_2, ..., w_k) \xrightarrow{*} (w'_1, w_2, ..., w_k)$$

where $\xrightarrow{*}$ denotes a finite sequence of Nielsen moves and relative Nielsen moves. It's evident that the first and the third $\xrightarrow{*}$ can be performed in such a way that every k-tuple that we write along the sequence is M-big. All the k-tuples that we write during the second $\xrightarrow{*}$ are M-big, provided that C had been chosen big enough. □

Proposition B.4. *Let $k \geq 1$ and let $(w_1, ..., w_k), (w'_1, ..., w'_k)$ be M-big ordered k-tuples such that $A(w_1, ..., w_k; v_1, ..., v_h) = A(w'_1, ..., w'_k; v_1, ..., v_h) = A$. Suppose there is a vector in $\langle v_1, ..., v_h \rangle$ such that all of its components are strictly positive. Then the following are equivalent:*

1. *There is a sequence of Nielsen moves and relative Nielsen moves going from $(w_1, ..., w_k)$ to $(w'_1, ..., w'_k)$ such that all the k-tuples that we write along the sequence are M-big.*
2. *The ordered k-uples of generators $([w_1], ..., [w_k]), ([w'_1], ..., [w'_k])$ are Nielsen equivalent in A.*

Proof. $1 \Rightarrow 2$. Trivial.

$2 \Rightarrow 1$. Let v be a vector in $\langle v_1, ..., v_h \rangle$ with all components strictly positive. For every element $[w] \in A$ we can construct a lifting $w \in \langle w_1, ..., w_k, v_1, ..., v_h \rangle$ such that all the components of w are $\geq M$; in fact, we can just pick any lifting and add v repeatedly until the components become all $\geq M$.

Suppose there is a sequence of Nielsen moves going from $([w_1], ..., [w_k])$ to $([w'_1], ..., [w'_k])$ in A. Let $([w_1^r], ..., [w_k^r])$ be such a sequence, for $r = 1, ..., R$, so that $[w_i^1] = [w_i]$ and $[w_i^R] = [w'_i]$ for $i = 1, ..., k$. Let w_i^r be an M-big lifting of $[w_i^r]$ for $i = 1, ..., k$ and $r = 1, ..., R$, and we can take $w_i^1 = w_i$ and $w_i^R = w'_i$ for $i = 1, ..., k$. By Lemma B.3, it's possible to pass from $(w_1^r, ..., w_k^r)$ to $(w_1^{r+1}, ..., w_k^{r+1})$ with a sequence of Nielsen moves and relative Nielsen moves. The conclusion follows. □

Nielsen equivalence for three or more big vectors

Fix $n \geq 1$ and vectors $v_1, ..., v_h$ in $\mathbb{Z}^n$; fix also an integer $M \geq 1$. This section is dedicated to the proof of Theorem B.1. The idea is to try and apply Proposition B.4. When we don't have a positive vector in $\langle v_1, ..., v_h \rangle$, we essentially use one of the w_is instead. This means that, given two M-big ordered k-tuples $(w_1, ..., w_k), (w'_1, ..., w'_k)$, we want to manipulate them until they end up with a common vector. The following Proposition B.5 explains how to manipulate one of the two k-tuples in order to get a vector "almost" in common with the other, up to an integer multiple; the next Propositions B.7 and B.8 are aimed at removing the integer multiple, and to make the vector "really" in common to the k-tuples.

Proposition B.5. *Let $k \geq 3$ and let $(w_1, ..., w_k), (w'_1, ..., w'_k)$ be M-big ordered k-tuples such that $A(w_1, ..., w_k; v_1, ..., v_h) = A(w'_1, ..., w'_k; v_1, ..., v_h) = A$. Then there is an M-big ordered k-tuple $(w''_1, ..., w''_k)$ such that:*

1. *There is a sequence of Nielsen moves and relative Nielsen moves going from $(w_1, ..., w_k)$ to $(w''_1, ..., w''_k)$ such that all the k-tuples that we write along the sequence are M-big.*
2. *We have $w'_1 = d(w''_1 - w''_2) + \mu_1 v_1 + ... + \mu_h v_h$ for some $d, \mu_1, ..., \mu_h \in \mathbb{Z}$.*

Proof. Let $w'_1 = \lambda_1 w_1 + ... + \lambda_k w_k + \mu_1 v_1 + ... + \mu_h v_h$ for $\lambda_1, ..., \lambda_k, \mu_1, ..., \mu_h \in \mathbb{Z}$. We notice that for $i \neq j$ we have the identity

$$\lambda_i w_i + \lambda_j w_j = (\lambda_i - \lambda_j) w_i + \lambda_j (w_j + w_i)$$

For an ordered k-tuple of integers $(\lambda_1, ..., \lambda_k)$ we define a *subtraction* to be any operation that consists of substituting λ_i with $\lambda_i - \lambda_j$ for some $i \neq j$. We can perform any sequence of subtractions to the k-tuple of integers $(\lambda_1, ..., \lambda_k)$, as these are achieved by means of Nielsen moves on the k-tuple of vectors $(w_1, ..., w_k)$ that keep it M-big.

We now work on the ordered k-tuple of integers $(\lambda_1, ..., \lambda_k)$. As $k \geq 3$, it's easy to see that by means of subtractions we can assume that $\lambda_k < 0$. We now subtract λ_k from each of $\lambda_1, ..., \lambda_{k-1}$ enough times, in order to get $\lambda_1, ..., \lambda_{k-1} > 0$. By means of Lemma B.6, we can subtract λ_k from λ_1 the right amount of times, in order to get $\gcd(\lambda_1, ..., \lambda_{k-1}) = \gcd(\lambda_1, ..., \lambda_k)$. We now use subtractions on the non-negative integers $\lambda_1, ..., \lambda_{k-1}$ in order to perform the Euclidean algorithm (and here we need $k \geq 3$): we reduce to a situation of the type $d, 0, ..., 0, -cd$ for some integers $c, d \geq 1$. From this we obtain $d, -d, 0, ..., 0, -cd$ and finally $d, -d, 0, ..., 0$. The conclusion follows. $\square$

Lemma B.6. *Let $a, b, c \in \mathbb{Z}$ with $b \neq 0$. Then there is $\lambda \in \mathbb{Z}$ such that*

$$\gcd(a, b, c) = \gcd(a + \lambda c, b)$$

Moreover, the same holds with λ' instead of λ, where λ' is any element of the coset $\lambda + b\mathbb{Z}$.

Proof. Let $p \mid b$ be a prime. If $v_p(c) > v_p(a)$ then we have $v_p(a + \lambda c) = v_p(a) = \min\{v_p(a), v_p(c)\}$. If $v_p(c) < v_p(a)$ then we impose the condition $\lambda \not\equiv 0 \bmod p$; this implies that $v_p(a + \lambda c) = v_p(c) = \min\{v_p(a), v_p(c)\}$. If $v_p(c) = v_p(a)$ then we write $a = p^{v_p(a)} a'$ and $c = p^{v_p(c)} c'$ with a', c' coprime with p, and we impose the condition $\lambda \not\equiv -a'/c' \bmod p$; this implies that $v_p(a + \lambda c) = v_p(a) = v_p(c)$.

By the Chinese reminder theorem, we can find λ satisfying the above conditions for all primes $p \mid b$. Now for every prime p that divides b we have $v_p(\gcd(a + \lambda c, b)) = \min\{v_p(a + \lambda c), v_p(b)\} = \min\{v_p(a), v_p(b), v_p(c)\} = v_p(\gcd(a, b, c))$, and for every prime p that doesn't divide b we have $v_p(\gcd(a + \lambda c, b)) = 0 = v_p(\gcd(a, b, c))$. It follows that $\gcd(a + \lambda c, b) = \gcd(a, b, c)$, as desired. $\square$

Proposition B.7. *Let $k \geq 2$ and let $(w_1, ..., w_k)$ be an M-big ordered k-tuple. Suppose $A = A(w_1, ..., w_k; v_1, ..., v_h) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus A'$ for some finitely generated abelian group A'. Then there is an M-big ordered k-tuple $(w'_1, ..., w'_k)$ such that:*

1. *There is a sequence of Nielsen moves and relative Nielsen moves going from $(w_1, ..., w_k)$ to $(w'_1, ..., w'_k)$ such that each k-tuple that we write along the sequence is M-big.*
2. *The first two components of $[w'_1] \in A \cong \mathbb{Z} \oplus \mathbb{Z} \oplus A'$ are coprime, and $[w'_1]$ is not a proper power.*

Proof. Let $[w_i]$ be equal to the vector $(\alpha_i, \beta_i, \gamma_i)$ through the isomorphism $A \cong \mathbb{Z} \oplus \mathbb{Z} \oplus A'$, for $i = 1, ..., k$ and $\alpha_i, \beta_i \in \mathbb{Z}$ and $\gamma_i \in A'$. Since $[w_1], ..., [w_k]$ generate A, we must have $\alpha_i\beta_j - \alpha_j\beta_i \neq 0$ for some $i, j \in \{1, ..., k\}$. By Lemma B.2 we can reorder the vectors with an arbitrary even permutation. Thus we can assume

$$\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0.$$

If $\alpha_2 = 0$ then $\alpha_1 \neq 0$ (since $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$) and we substitute w_2 with $w_2 + w_1$. Thus we can assume

$$\alpha_2 \neq 0.$$

By Lemma B.6 we find $\lambda \geq 0$ such that $\gcd(\alpha_1 + \lambda\alpha_3, \alpha_2) = \gcd(\alpha_1, \alpha_2, \alpha_3)$. We substitute w_1 with $w_1 + \lambda w_3$ and thus we can assume $\gcd(\alpha_1, \alpha_2) = \gcd(\alpha_1, \alpha_2, \alpha_3)$. Since λ can be chosen modulo α_2, we can also take λ in such a way that $\alpha_1\beta_2 - \alpha_2\beta_1$ remains $\neq 0$. Reiterating the same reasoning, we obtain $\gcd(\alpha_1, \alpha_2) = \gcd(\alpha_1, ..., \alpha_k)$, which is 1 since $[w_1], ..., [w_k]$ generate A. Thus we can assume

$$\gcd(\alpha_1, \alpha_2) = 1.$$

By Lemma B.6 we find $C \geq 0$ such that $\gcd(\alpha_1 + C\alpha_2, \alpha_1\beta_2 - \alpha_2\beta_1) = \gcd(\alpha_1, \alpha_2, \alpha_1\beta_2 - \alpha_2\beta_1) = 1$. We now have that

$$\gcd(\alpha_1 + C\alpha_2, \alpha_1\beta_2 - \alpha_2\beta_1 - \beta_2(\alpha_1 + C\alpha_2)) = 1$$

$$\gcd(\alpha_1 + C\alpha_2, -\alpha_2(\beta_1 + C\beta_2)) = 1$$

$$\gcd(\alpha_1 + C\alpha_2, \beta_1 + C\beta_2) = 1$$

We substitute w_1 with $w_1 + Cw_2$ and thus obtain that $(\alpha_1, \beta_1) = 1$. This also implies that the element $[w_1]$ can't be a proper power in A. ☐

Proposition B.8. *Let $k \geq 3$ and let $(w_1, ..., w_k)$ be an M-big ordered k-tuple. Suppose that $A = A(w_1, ..., w_k; v_1, ..., v_h) \cong \mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus A'$ for some $d_2 \geq 2$ and A' finitely generated abelian group. Then there is an M-big ordered k-tuple $(w'_1, ..., w'_k)$ such that:*

1. *There is a sequence of Nielsen moves and relative Nielsen moves going from $(w_1, ..., w_k)$ to $(w'_1, ..., w'_k)$ such that each k-tuple that we write along the sequence is M-big.*
2. *The first two components of $[w'_1] \in A \cong \mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus A'$ are coprime, and $[w'_1]$ is not a proper power.*

Proof. Let $[w_i]$ be equal to the vector $(\alpha_i, \beta_i, \gamma_i)$ through the isomorphism $A \cong \mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus A'$, for $i = 1, ..., k$ and $\alpha_i \in \mathbb{Z}$ and $\beta_i \in \mathbb{Z}/d_2\mathbb{Z}$ and $\gamma_i \in A'$. Let also p be a prime that divides d_2.

Since $[w_1], ..., [w_k]$ generate A, we must have $\alpha_i\beta_j - \alpha_j\beta_i \not\equiv 0 \mod p$ for some $i, j \in \{1, ..., k\}$. By Lemma B.2 we can reorder the vectors with an arbitrary even permutation. Thus we can assume

$$\alpha_1\beta_2 - \alpha_2\beta_1 \not\equiv 0 \mod p.$$

If $\alpha_2 \equiv 0 \mod p$ then $\alpha_1 \not\equiv 0 \mod p$ (since $\alpha_1\beta_2 - \alpha_2\beta_1 \not\equiv 0 \mod p$) and we substitute w_2 with $w_2 + w_1$. Thus we can assume

$$\alpha_2 \not\equiv 0 \mod p.$$

By Lemma B.6 we find $\lambda \geq 0$ such that $\gcd(\alpha_1 + \lambda\alpha_3, \alpha_2) = \gcd(\alpha_1, \alpha_2, \alpha_3)$. We substitute w_1 with $w_1 + \lambda w_3$ and thus we can assume $\gcd(\alpha_1, \alpha_2) = \gcd(\alpha_1, \alpha_2, \alpha_3)$. Since λ can be chosen modulo α_2, which is coprime with p, we can also take λ to be a multiple of p, so that $\alpha_1\beta_2 - \alpha_2\beta_1 \mod p$ remains the same. Reiterating the same reasoning, we obtain $\gcd(\alpha_1, \alpha_2) = \gcd(\alpha_1, ..., \alpha_k)$, which is 1 since $[w_1], ..., [w_k]$ generate A. Thus we can assume

$$\gcd(\alpha_1, \alpha_2) = 1.$$

By means of the Chinese reminder theorem we find $C \geq 0$ such that $\alpha_3 + C\alpha_1 \equiv 1 \mod \alpha_2$ and $(\alpha_3\beta_2 - \alpha_2\beta_3) + C(\alpha_1\beta_2 - \alpha_2\beta_1) \not\equiv 0 \mod p$; here we are using the facts that $\gcd(\alpha_2, p) = 1$ and $\gcd(\alpha_1, \alpha_2) = 1$ and $\gcd(\alpha_1\beta_2 - \alpha_2\beta_1, p) = 1$. Since $\gcd(\alpha_2, p) = 1$ we can find integers $e \geq 0$ as big as we want and such that $p^e \equiv 1 \mod \alpha_2$. Using the fact that $\alpha_3 + C\alpha_1 \equiv 1 \mod \alpha_2$ we are thus able to find integers $D \geq 0$ and $e \geq 0$ such that $\alpha_3 + C\alpha_1 + D\alpha_2 = p^e$. We now have that

$$\gcd(p, (\alpha_3\beta_2 - \alpha_2\beta_3) + C(\alpha_1\beta_2 - \alpha_2\beta_1)) = 1$$

$$\gcd(p, (\alpha_3\beta_2 - \alpha_2\beta_3) + C(\alpha_1\beta_2 - \alpha_2\beta_1) - \beta_2(\alpha_3 + C\alpha_1 + D\alpha_2)) = 1$$

$$\gcd(p, -\alpha_2(\beta_3 + C\beta_1 + D\beta_2)) = 1$$

$$\gcd(p, \beta_3 + C\beta_1 + D\beta_2) = 1$$

We substitute w_3 with $w_3 + Cw_1 + Dw_2$ and thus we can assume that $\alpha_3 = p^e$ and $(p, \beta_3) = 1$. In particular $(\alpha_3, \beta_3) = 1$. The element $[w_3]$ can be a proper power in A: if $[w_3] = d[w]$, then $\alpha_3 = p^e$ implies that $d \mid p^e$, and $(p, \beta_3) = 1$ implies that $(p, d) = 1$; it follows that $d = \pm 1$. To conclude, we permute (w_1, w_2, w_3) using Lemma B.2 in order to bring w_3 in first position. $\square$

Proposition B.9. *Let $k \geq 3$ and let $(w_1, ..., w_k), (w'_1, ..., w'_k)$ be M-big ordered k-tuples such that $A(w_1, ..., w_k; v_1, ..., v_h) = A(w'_1, ..., w'_k; v_1, ..., v_h) = A \cong \mathbb{Z}$. Then there is a sequence of Nielsen moves and relative Nielsen moves from $(w_1, ..., w_k)$ to $(w'_1, ..., w'_k)$ such that every k-tuple that we write along the sequence is M-big.*

Proof. Let $[w_i], [w'_i]$ be equal to α_i, α'_i through the isomorphism $A \cong \mathbb{Z}$, for $i = 1, ..., k$ and $\alpha_i, \alpha'_i \in \mathbb{Z}$. If among α_i, α'_i there is at least one integer > 0 and at least one integer < 0, then $\langle v_1, ..., v_h \rangle$ has to contain a vector whose components are all strictly positive, and thus we are done by Proposition B.4. Thus we assume $\alpha_i, \alpha'_i > 0$ for all $i = 1, ..., k$.

By Lemma B.6 we find $\lambda \geq 0$ such that $\gcd(\alpha_1 + \lambda\alpha_3, \alpha_2) = \gcd(\alpha_1, \alpha_2, \alpha_3)$. We substitute w_1 with $w_1 + \lambda w_3$ and thus we can assume $\gcd(\alpha_1, \alpha_2) = \gcd(\alpha_1, \alpha_2, \alpha_3)$. By reiterating the same reasoning, we can assume $\gcd(\alpha_1, \alpha_2) = \gcd(\alpha_1, ..., \alpha_k) = 1$. Now, by substituting w_3 with $w_3 + \lambda_1 w_1 + \lambda_2 w_2$ for $\lambda_1, \lambda_2 \geq 0$, we can set α_3 to be equal to any natural number big enough. Similarly, we can assume $\gcd(\alpha'_1, \alpha'_2) = 1$, and we are able to set α'_3 to be any natural number big enough. We choose to set $\alpha_3 = \alpha'_3$ and such that $\gcd(\alpha_1, \alpha_3) = \gcd(\alpha'_1, \alpha_3) = 1$. Now, by substituting w_2 with $w_2 + \mu_1 w_1 + \mu_3 w_3$ for $\mu_1, \mu_3 \geq 0$, we can set α_2 to be any natural number big enough. Similarly, we are able to set α'_2 to be any natural number big enough. We choose to set $\alpha_2 = \alpha'_2$ and such that $\gcd(\alpha_2, \alpha_3) = 1$. Reasoning in the same way, and keeping fixed w_2, w_3, w'_2, w'_3, we can set $\alpha_1 = \alpha'_1$ and $\alpha_i = \alpha'_i$ for $i \geq 4$.

To conclude, for $i = 1, ..., k$, we observe that $\alpha_i = \alpha'_i$ implies that $w'_i = w_i + \mu_1 v_1 + ... + \mu_h v_h$ for some $\mu_1, ..., \mu_h \in \mathbb{Z}$ and thus by Lemma B.3 we can obtain $w_i = w'_i$ by means of Nielsen moves and relative Nielsen moves. The thesis follows. $\square$

Proof of Theorem B.1. $1 \Rightarrow 2$. Trivial.

$2 \Rightarrow 1$. If A is a finite abelian group, then we must have $d[w_1] = 0$ in A for some $d \geq 1$, and this implies that $\langle v_1, ..., v_h \rangle$ contains the vector dw_1, which has all the components strictly positive; the conclusion follows from Proposition B.4. If A is isomorphic to $\mathbb{Z}$, then we are done by Proposition B.9.

Otherwise we write $A \cong \mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus ... \oplus \mathbb{Z}/d_m\mathbb{Z}$ for integers $m \geq 2$ and $d_2, ..., d_m \geq 0$ with $1 \neq d_m \mid d_{m-1} \mid ... \mid d_2$. By either Proposition B.7 or Proposition B.8, we are able to change $(w'_1, ..., w'_k)$ by means of Nielsen moves and relative Nielsen moves in such a way that $[w'_1]$ isn't a proper power in A, and that $[w'_1]$ can be represented by a vector whose first two components are coprime. We now apply Proposition B.5 and we apply a sequence of Nielsen moves and relative Nielsen moves to $(w_1, ..., w_k)$, in order to obtain that $w'_1 = w_1 - w_2 + \mu_1 v_1 + ... + \mu_h v_h$. By means of Proposition B.3 we can easily change w_1 in such a way that $w_1 = w'_1$.

If $m \geq 3$ then we can apply Lemma A.7, since $[w'_1]$ can be represented by a vector whose first two components are coprime, and we obtain that $([w_2], ..., [w_k]), ([w'_2], ..., [w'_k])$ are Nielsen equivalent $(k-1)$-tuples of generators for $A/\langle [w_1] \rangle$. If $m = 2$ then $[w'_1]$ can be represented by a vector whose components are coprime, and thus we can apply Lemma A.6 to obtain that $([w_2], ..., [w_k]), ([w'_2], ..., [w'_k])$ are Nielsen equivalent $(k-1)$-tuples of generators for $A/\langle [w_1] \rangle$. In both cases, the conclusion follows from Proposition B.4. $\square$