ДЗ по линейной алгебре на 21.11.2021

Кожевников Илья 2112-1

21 ноября 2021 г.

№1

a) $|-3i| = \sqrt{9} = 3$
$\cos\varphi = 0, \sin\varphi = -1 \Rightarrow \varphi = \frac{3\pi}{2} + 2\pi k, k \in Z$
$-3i = 3(\cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2})) = -3i$
Ответ: $3(\cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2}))$

b) $|1 + i\frac{1}{\sqrt{3}}| = \sqrt{1 + \frac{1}{3}} = \frac{2}{\sqrt{3}}$
$\cos\varphi = \frac{\sqrt{3}}{2}, \sin\varphi = \frac{1}{2} \Rightarrow \varphi = \frac{\pi}{6} + 2\pi k, k \in Z$
$1 + i\frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = 1 + i\frac{1}{\sqrt{3}}$
Ответ: $\frac{2}{\sqrt{3}}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

c) $\frac{10-6\sqrt{3}i}{2\sqrt{3}-i} = \frac{(10-6\sqrt{3}i)(2\sqrt{3}+i)}{(2\sqrt{3}-i)(2\sqrt{3}+i)} = \frac{26\sqrt{3}-26i}{13} = 2\sqrt{3} - 2i$
$|2\sqrt{3} - 2i| = \sqrt{12+4} = 4$
$\cos\varphi = \frac{\sqrt{3}}{2}, \sin\varphi = -\frac{1}{2} \Rightarrow \varphi = \frac{11\pi}{6} + 2\pi k, k \in Z$
$\frac{10-6\sqrt{3}i}{2\sqrt{3}-i} = 4(\cos(\frac{11\pi}{6}) + i\sin(\frac{11\pi}{6})) = \frac{10-6\sqrt{3}i}{2\sqrt{3}-i}$
Ответ: $4(\cos(\frac{11\pi}{6}) + i\sin(\frac{11\pi}{6}))$

d) $\frac{\cos\varphi + i\sin\varphi}{\cos\psi + i\sin\psi} = \frac{(\cos\varphi + i\sin\varphi)(\cos\psi - i\sin\psi)}{(\cos\psi + i\sin\psi)(\cos\psi - i\sin\psi)} =$
$= (\cos\varphi + i\sin\varphi)(\cos\psi - i\sin\psi) =$
$= \cos\varphi\cos\psi + \sin\varphi\sin\psi + (-i\sin\psi\cos\varphi + i\sin\varphi\cos\psi)^*$

$^* = -i\frac{1}{2}(\sin((\varphi+\psi)) + \sin(\psi-\varphi)) + i\frac{1}{2}(\sin(\psi+\varphi) + \sin(\varphi-\psi)) =$
$= -\frac{i}{2}(\sin(\varphi+\psi) + \sin(\psi-\varphi) - \sin(\psi+\varphi) + \sin(\psi-\varphi)) = -i\sin(\psi-\varphi)$

$(\cos\varphi\cos\psi + \sin\varphi\sin\psi)^* - i\sin(\psi-\varphi)$

$^* = \cos\varphi\cos\psi + \sin\varphi\sin\psi = \frac{1}{2}(\cos(\psi+\varphi) + \cos(\psi-\varphi)) + \frac{1}{2}(\cos(\psi-\varphi) - \cos(\psi+\varphi)) = \frac{2\cos(\psi-\varphi)}{2} = \cos(\psi-\varphi)$

$\cos(\psi-\varphi) - i\sin(\psi-\varphi)$
Пусть α - аргумент. Тогда $|\cos\alpha - i\sin\alpha| = 1$

$\cos\alpha = \cos(\psi-\varphi), \sin\alpha = \sin(\psi-\varphi)$
$\cos\alpha - i\sin\alpha = 1(\cos(\psi-\varphi) + i\sin(\psi-\varphi)) = \cos\alpha - i\sin\alpha$
Ответ: $\cos(\psi-\varphi) + i\sin(\psi-\varphi)$

№2

1)

$(\sqrt{3}-i)^{32} = |\sqrt{3}-i|^{32}(\cos 32\varphi + i\sin 32\varphi)$
$\cos\varphi = \frac{\sqrt{3}}{2}, \sin\varphi = -\frac{1}{2} \Rightarrow \varphi = \frac{11\pi}{6} + 2\pi k, k \in Z$
$|\sqrt{3}-i|^{32}(\cos 32\varphi + i\sin 32\varphi) = 2^{32}(\cos 32\frac{11\pi}{6} + i\sin 32\frac{11\pi}{6}) = 2^{32}(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$

$2^{32}(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) = -2^{31} - i2^{31}\sqrt{3}$
Ответ: $2^{32}(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}), -2^{31} - i2^{31}\sqrt{3}$

2)

$(\frac{\sqrt{3}+i}{1-i})^{30} = (\frac{(\sqrt{3}+i)(1+i)}{(1-i)(1+i)})^{30} = (\frac{\sqrt{3}-1+i(\sqrt{3}+1)}{2})^{30} = (\frac{\sqrt{3}}{2} - \frac{1}{2} + i\frac{(\sqrt{3}+1)}{2})^{30} \Rightarrow$
$\Rightarrow |\frac{\sqrt{3}}{2} - \frac{1}{2} + i\frac{(\sqrt{3}+1)}{2}|^{30}(\cos 30\varphi + i\sin 30\varphi)$

$|\frac{\sqrt{3}}{2} - \frac{1}{2} + i\frac{(\sqrt{3}+1)}{2}| = \frac{\sqrt{3-2\sqrt{3}+1+3+2\sqrt{3}+1}}{2} = \sqrt{2}$
$\cos\varphi = \frac{\sqrt{3}-1}{2\sqrt{2}}, \sin\varphi = \frac{\sqrt{3}+1}{2\sqrt{2}} \Rightarrow \varphi = \arcsin\frac{\sqrt{6}+\sqrt{2}}{4} + 2\pi k, k \in Z$
$\cos 30\varphi = \cos 30(\arcsin\frac{\sqrt{6}+\sqrt{2}}{4}) = 0,\ \sin 30\varphi = \sin 30(\arcsin\frac{\sqrt{6}+\sqrt{2}}{4}) = 1$
$2^{15}(\cos 30(\arcsin\frac{\sqrt{6}+\sqrt{2}}{4}) + i\sin 30(\arcsin\frac{\sqrt{6}+\sqrt{2}}{4})) = 2^{15}i$
Ответ: $2^{15}(\cos 30(\arcsin\frac{\sqrt{6}+\sqrt{2}}{4}) + i\sin 30(\arcsin\frac{\sqrt{6}+\sqrt{2}}{4})),\ 2^{15}i$

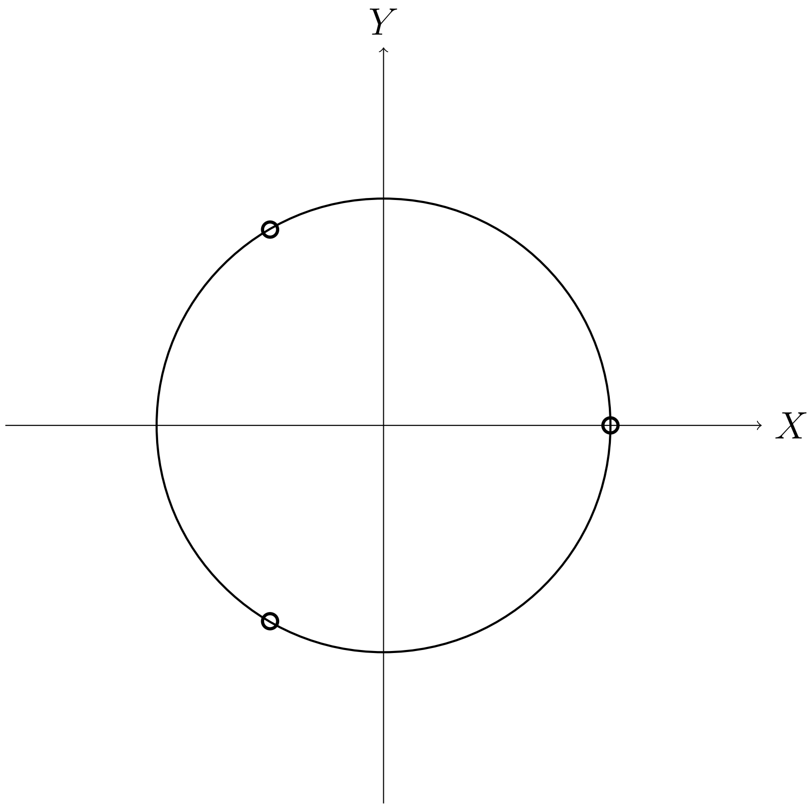
№3

1)

$z^3 = 1$
$z^3 = |z|^3(\cos 3\varphi + i\sin 3\varphi)$
$1 = \cos 3\varphi + i\sin 3\varphi$
$\cos 3\varphi = 1, \sin 3\varphi = 0$
$\varphi = 2\pi k, k \in Z$

$$\begin{cases} |z| = 1 \\ \varphi = \dfrac{2\pi k}{3}, k \in Z \end{cases}$$

$z = \cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}, k \in \{0,1,2\}$
$z = 1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$



r = 1

Ответ: $z = 1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

2)

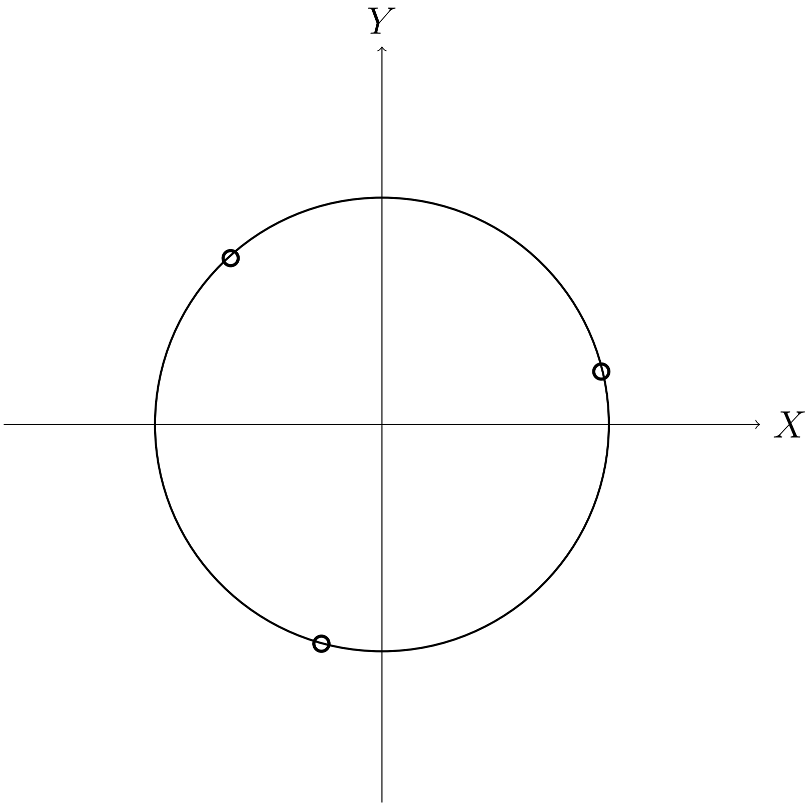
$z^2 = i$
$z^2 = |z|^2(\cos 2\varphi + i \sin 2\varphi)$
$i = i(\cos 2\varphi + i \sin 2\varphi)$
$\cos 2\varphi = 1, \sin 2\varphi = 0$
$\varphi = \pi k, k \in Z$
$$\begin{cases} |z| = \sqrt{i} \\ \varphi = \pi k, k \in Z \end{cases}$$
$z = \sqrt{i}(\cos \pi k + i \sin \pi k), k \in 0, 1$
$z = \pm\sqrt{i}$
Ответ: $z = \pm\sqrt{i}$

№4

1)

$\sqrt[3]{2 - 2i} = ?$
$z^3 = 2 - 2i$
$|z|^3 = 2\sqrt{2}$
$|z| = \sqrt{2}$
$z^3 = |z|^3(\cos 3\varphi + i \sin 3\varphi)$
$2 - 2i = 2\sqrt{2}(\cos 3\varphi + i \sin 3\varphi)$
$\cos 3\varphi = \frac{1}{\sqrt{2}},\ \sin 3\varphi = \frac{1}{\sqrt{2}}$

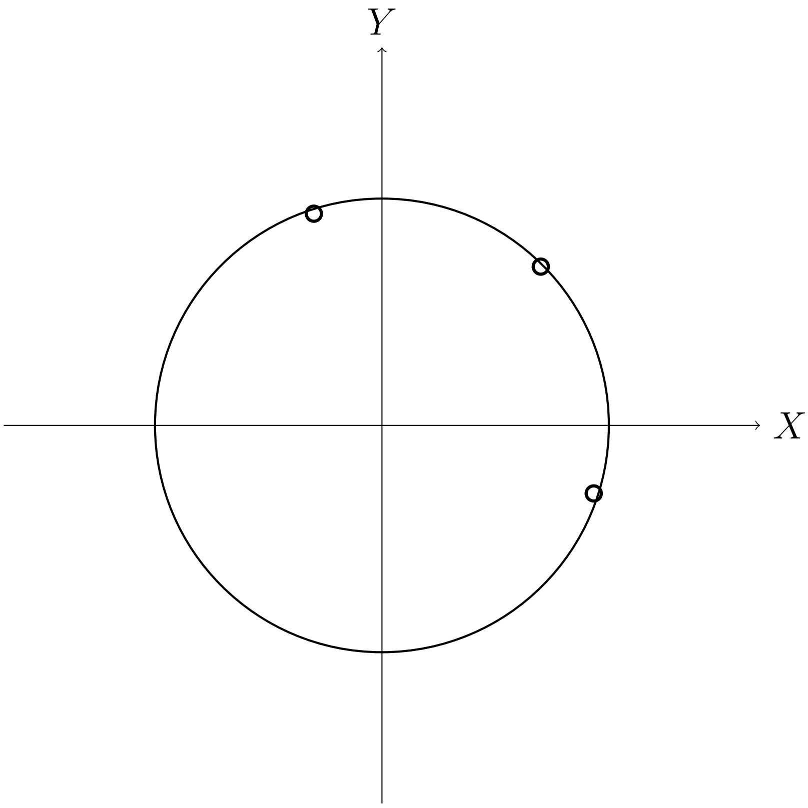
$\varphi = \frac{\pi}{12} + \frac{2\pi k}{3},\ k \in Z$
$z = \sqrt{2}(\cos(\frac{\pi}{12} + \frac{2\pi k}{3}) + i\sin(\frac{\pi}{12} + \frac{2\pi k}{3})),\ k \in \{0, 1, 2\}$
$z = \frac{\sqrt{3}+1}{2} + i\frac{\sqrt{3}-1}{2},\ i - 1,\ \frac{-\sqrt{3}+1}{2} - i\frac{1+\sqrt{3}}{2}$
Ответ: $z = \frac{\sqrt{3}+1}{2} + i\frac{\sqrt{3}-1}{2},\ i - 1,\ \frac{-\sqrt{3}+1}{2} - i\frac{1+\sqrt{3}}{2}$



r = $\sqrt{2}$

2)

$\sqrt[6]{(2-2i)^2}$
$(2-2i)^2 = -8i$
$\sqrt[6]{-8i} = ?$
$z^6 = -8i$
$z^6 = |z|^6(\cos 6\varphi + i\sin 6\varphi)$
$|z|^6 = 8$
$|z| = \sqrt[6]{8} = \sqrt{2}$
$-8i = 8(\cos 6\varphi + i\sin 6\varphi)$
$\sin 6\varphi = -1, \cos 6\varphi = 0$
$\varphi = \frac{\pi}{4} + \frac{\pi k}{3},\ k \in Z$
$z = \sqrt{2}(\cos(\frac{\pi}{4} + \frac{\pi k}{3}) + i\sin(\frac{\pi}{4} + \frac{\pi k}{3})),\ k \in \{0, 1, 2\}$
$z = 1 + i, \frac{1-\sqrt{3}}{2} + i\frac{1+\sqrt{3}}{2}, -\frac{1+\sqrt{3}}{2} + i\frac{-1+\sqrt{3}}{2}$
Ответ: $z = 1 + i, \frac{1-\sqrt{3}}{2} + i\frac{1+\sqrt{3}}{2}, -\frac{1+\sqrt{3}}{2} + i\frac{-1+\sqrt{3}}{2}$



r $= \sqrt{2}$

№5

$\sqrt[4]{\frac{-18}{1+i\sqrt{3}}} = \sqrt[4]{\frac{-18(1-i\sqrt{3})}{4}} = \sqrt[4]{\frac{-9+i9\sqrt{3}}{2}} =?$

$z^4 = \frac{-9+i9\sqrt{3}}{2}$

$z^4 = |z|^4(\cos 4\varphi + i\sin 4\varphi)$

$|z^4| = 9 \Rightarrow |z| = \sqrt{3}$

$\frac{-9+i9\sqrt{3}}{2} = 9(\cos 4\varphi + i\sin 4\varphi)$

$\frac{-1+i\sqrt{3}}{2} = \cos 4\varphi + i\sin 4\varphi$

$\cos 4\varphi = -\frac{1}{2},\ \sin 4\varphi = \frac{\sqrt{3}}{2}$

$4\varphi = \frac{2\pi}{3} + 2\pi k,\ k \in Z$

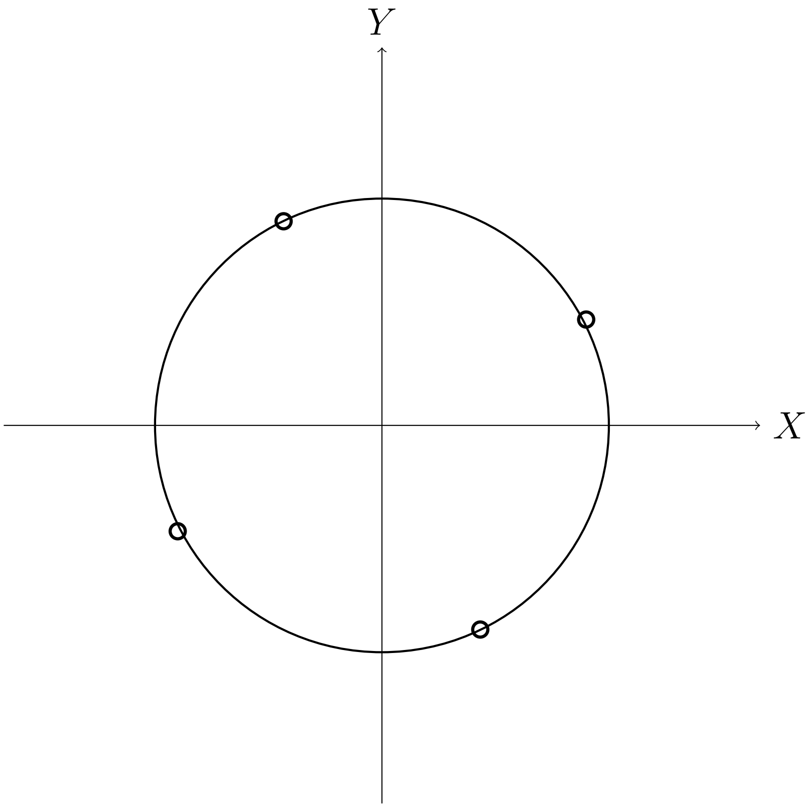
$\varphi = \frac{\pi}{6} + \frac{\pi k}{2},\ k \in Z$

$z = \sqrt{3}(\cos\frac{\pi}{6} + \frac{\pi k}{2} + i\sin\frac{\pi}{6} + \frac{\pi k}{2}),\ k \in \{0,1,2,3\}$

$z = \frac{3}{2} + i\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2} + i\frac{3}{2}, -\frac{3}{2} - i\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} - i\frac{3}{2}$

Ответ: $z = \sqrt{3}(\cos\frac{\pi}{6} + \frac{\pi k}{2} + i\sin\frac{\pi}{6} + \frac{\pi k}{2}),\ k \in \{0,1,2,3\}$

$z = \frac{3}{2} + i\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2} + i\frac{3}{2}, -\frac{3}{2} - i\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} - i\frac{3}{2},$



r = $\sqrt{3}$

№6

$(2\sqrt{3}-i)z^4 = 10-6\sqrt{3}i$
$z^4 = \frac{10-6\sqrt{3}i}{2\sqrt{3}-i}$
$z^4 = \frac{(10-6\sqrt{3}i)(2\sqrt{3}+i)}{(2\sqrt{3}-i)(2\sqrt{3}+i)}$
$z^4 = \frac{26\sqrt{3}-26i}{13}$
$z^4 = 2\sqrt{3}-2i$
$|z^4| = 4 \Rightarrow |z| = \sqrt{2}$
$2\sqrt{3}-2i = 4(\cos 4\varphi + i\sin 4\varphi)$
$\frac{\sqrt{3}}{2} - \frac{1}{2}i = \cos 4\varphi + i\sin 4\varphi$
$\cos 4\varphi = \frac{\sqrt{3}}{2},\ \sin 4\varphi = -\frac{1}{2}$
$4\varphi = \frac{11\pi}{6} + 2\pi k,\ k \in Z$
$\varphi = \frac{11\pi}{24} + \frac{\pi k}{2},\ k \in \{0,1,2,3\}$
$z = \sqrt{2}(\cos(\frac{11\pi}{24} + \frac{\pi k}{2}) + i\sin(\frac{11\pi}{24} + \frac{\pi k}{2})),\ k \in \{0,1,2,3\}$
Заметим, что к заданному промежутку относятся лишь числа аргументы при k = 1.
$z = \sqrt{2}(\cos(\frac{11\pi}{24} + \frac{\pi k}{2}) + i\sin(\frac{11\pi}{24} + \frac{\pi k}{2}))$
$z = \sqrt{2}(\cos(\frac{23\pi}{24}) + i\sin(\frac{23\pi}{24}))$
Ответ: $z = \sqrt{2}(\cos(\frac{23\pi}{24}) + i\sin(\frac{23\pi}{24}))$

№7

$(\cos x + i\sin x)^3 = \cos x^3 + 3i\cos x^2 \sin x - 3\cos x \sin x^2 - i\sin x^3$

$(\cos x + i \sin x)^3 = |(\cos x + i \sin x)^3|(\cos 3\varphi + i \sin 3\varphi) = \cos 3\varphi + i \sin 3\varphi$
Отсюда,
$\cos 3\varphi = \cos x^3 + 3i \cos x^2 \sin x - 3 \cos x \sin x^2 - i \sin x^3 - i \sin 3\varphi,$
$\sin 3\varphi = \frac{\cos x^3 + 3i \cos x^2 \sin x - 3 \cos x \sin x^2 - i \sin x^3 - \cos 3\varphi}{i}$

Ответ: $\cos 3\varphi = \cos x^3 + 3i \cos x^2 \sin x - 3 \cos x \sin x^2 - i \sin x^3 - i \sin 3\varphi,$
$\sin 3\varphi = \frac{\cos x^3 + 3i \cos x^2 \sin x - 3 \cos x \sin x^2 - i \sin x^3 - \cos 3\varphi}{i}$