What sets a star's size?

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Kinetic theory of gas. Example: what sets the sound speed of air?

Imagine a square wave of higher pressure and density; at what speed c_s can this sound wave propagate?

Move to a frame where this wave is static; in that frame, as material enters the wave (with cross-section area A), it slows down. This must happen because the total rate of material flow has to be conserved, i.e., ρv is constant, so the velocity is lower inside the high-pressure wave. This slowing down must match the acceleration provided by the pressure.

Let $\delta v \ll v$ be the velocity difference across the boundary, and $\delta P \ll P$ be the pressure difference. The force across the boundary is $F = A\delta P$.

The amount of mass that flows through the boundary in a small time dt is $dm = \rho A v dt$. The acceleration is $a = F/dm$ and must change the velocity by $\delta v = a dt$.

Combining these expressions, we find

$$\delta v = a dt = \frac{F}{dm} dt = \frac{A\delta P}{\rho A v dt} dt = \frac{\delta P}{\rho v},$$

or

$$\delta P = -v\rho\delta v = v^2 \delta\rho,$$

where the last equality follows from the conservation of mass through the flow: ρv =const implies that $\rho\delta v = -v\delta\rho$.

Thus, $v^2 = \frac{\delta P}{\delta \rho}$, or, replacing differences with differentials,

$$c_s^2 = \frac{dP}{d\rho}.$$

But what is that in practice?

This requires us to write down the equation of state of gas:

$$P \propto \rho^\gamma.$$

Here, γ is the adiabatic index, $\gamma = c_p/c_v$, where c_p is the specific heat at constant pressure and c_v is the specific heat at constant volume. These can be computed by considering the entropy (grand canonical ensemble), which we won't do now, except to mention that $\gamma = 5/3$ for a non-relativistic monatomic gas, $\gamma = 7/5$ for a non-relativistic diatomic gas, and $\gamma = 4/3$ for an ultra-relativistic monatomic gas. Then the sound speed is

$$c_s = \sqrt{\frac{dP}{d\rho}} = \sqrt{\gamma \frac{P}{\rho}}.$$

Meanwhile, the ideal gas law is

$$PV = NkT.$$

For simplicity, suppose that all gas particles have the same mass m_p; then the ideal gas law can be re-written as

$$P = \frac{M}{V m_p} kT = \rho \frac{kT}{m_p},$$

or

$$\frac{P}{\rho} = \frac{kT}{m_p},$$

so the sound speed is

$$c_s = \sqrt{\gamma \frac{kT}{m_p}}.$$

I can plug in all the values, or just remember that room temperature is around 1/40 eV, a proton's mass (times c^2) is around a GeV, and air is mostly diatomic nitrogen, so $m_p c^2 \approx 30$ GeV. Thus, $c_s \approx 10^{-6} c \approx 300$ m/s at standard temperature and pressure.

[Note: Newton derived the speed of sound in the Principia, but didn't know about the adiabatic index – i.e., he assumed that temperature stayed constant inside the sound wave, and pressure and density changed proportionately – so he got the sound speed off a bit... not that it makes his achievement any less impressive!]

Finally, the root-mean-square speed of individual particles is given by

$$\frac{m_p v_{rms}^2}{2} = \frac{3}{2} kT,$$

so

$$v_{rms} = \sqrt{\frac{3kT}{m_p}} = \sqrt{\frac{3P}{\rho}} = \sqrt{\frac{3}{\gamma}} c_s.$$

So what about that solar radius?

A main-sequence star like the Sun is in hydrostatic equilibrium. That means that the inward force of gravity on every shell of thickness dr at radius r is balanced by the outward force due to the pressure gradient across that shell. The mass of the shell is $dm = 4\pi r^2 dr \rho$. The force of gravity is

$$F_G = \frac{GM_{\rm enc} dm}{r^2} = \frac{GM_{\rm enc} 4\pi \rho r^2 dr}{r^2},$$

where $M_{\rm enc}$ is the enclosed mass.

Meanwhile, the difference in pressure across the shell is $dP = (dP/dr)dr$, and the outward force due to pressure is

$$F_P = 4\pi r^2 \frac{dP}{dr} dr.$$

Assuming the star is made of non-relativistic monatomic gas (hydrogen atoms) and the pressure is mostly due to the matter, not radiation, we can use the ideal gas law:

$$\frac{P}{\rho} = \frac{kT}{m_p}$$

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At the centre of the Sun, hydrogen fusion happens through the p-p process. This process is very sensitive to temperature: increasing the temperature by a factor of 2 would increase the fusion rate by a factor of 64. That means only a narrow range of temperatures are possible, around 1.5×10^7 degrees Kelvin: if the temperature gets just a bit lower, the heat input and hence pressure support drop precipitously, the star contracts, heats up, and returns to something very close to the previous fusion temperature (and similarly for temperature increases).

Now, we will make some pretty crude approximations to keep the problem tractable. We will approximate $dP/dr \sim P/R$, where R is the stellar radius and P is the central pressure (the pressure is zero outside the star). We will also approximate $M_{\rm enc} = M$, the total stellar mass; and $\rho = M/R^3$ throughout the star. Then, equating the force of gravity and the force of pressure, we find

$$\frac{GM 4\pi \rho R^2 dr}{R^2} \sim 4\pi R^2 dr \frac{P}{R}$$

or

$$\frac{GMR}{R^2} \sim \frac{P}{\rho} = \frac{kT}{m_p},$$

so that we can solve for the star's radius:

$$R = \frac{GMm_p}{kT}.$$

This is hardly a surprise, of course: the statement that the thermal energy per particle must be of the same order of magnitude as the gravitational energy per particle, $kT/m_p \sim GM/R$, should be familiar as the virial theorem.

But this does allow us to compute the solar radius. Or it would if we could remember G, $M_\odot$, m_p and k. I don't, but I remember that a solar mass is 1.5 km (or 5 microseconds), i.e., $GM_\odot/c^2 = 1.5$ km; that the proton's mass is $m_p c^2 \approx 1$ GeV; and that the thermal energy at room temperature, ≈ 300 K, is about 1/40 eV – so the Sun's central temperature of $1.5e7$ K is $50,000 \times (1/40) \approx 1000$ eV. Armed with this knowledge, we solve for the solar radius:

$$R_\odot = \frac{GM}{c^2}\frac{m_p c^2}{kT} \approx 1.5 \text{ km} \frac{1 \text{ GeV}}{1000 \text{ eV}} \approx 1.5 \times 10^6 \text{ km}.$$

The actual radius of the Sun is a factor of 2 smaller – so we've done pretty well with this approximation.

That said, it's not entirely satisfactory. We apparently concluded that the stellar radius is proportional to to the mass. But it's not – it scales as $M^{0.8}$ for pp-chain burning stars lighter than the Sun, and as $M^{0.6}$ for CNO-cycle powered stars more massive than the Sun. Part of this is presumably explained by the changes in central temperature – can we derive this?

What about radiation-pressure supported stars? For these, since the outward photon impulse flux through a shell is L/c, where L is the luminosity, and a fraction $\kappa\rho dr$ of photons transfer their energy to the atoms in the shell, where κ is the opacity, it is the radiation pressure support

$$F_\gamma = \frac{L}{c}\kappa\rho dr$$

that must balance gravity – but this seemingly leads to nonsensical requirements, except for stars approaching the Eddington limit.

And what happens during the red giant phase, when the Sun expands by two orders of magnitude after the end of the main sequence? The standard mirror principle argument says that since the total thermal and gravitational energy must be conserved, a contraction of the stellar core (which ran out of hydrogen, but isn't yet hot enough to fuse helium) must be accompanied by the expansion of the envelope – but can we quantitatively approximate the amount of this expansion? These are all questions for next time.