1 Definitions

Union and intersection of A and B:

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$
$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Union and intersection for three or more sets:

$$A \cup B \cup C = \{x \mid x \in A \vee x \in B \vee x \in C\}$$
$$A \cap B \cap C = \{x \mid x \in A \wedge x \in B \wedge x \in C\}$$

Disjoint set:

$$A \cap B = \varnothing$$

Union from 1 to n of A sub k

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

Complement of B with respect to A:

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Complement of A:

$$\overline{A} = \{x \mid x \notin A\}$$

Symmetric difference:

$$A \oplus B = \{x \mid (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$$

2 Theorem: Algebraic properties of Set Operations

2.1 Commutative Properties

Theorem 2.1.

$$A \cup B = B \cup A$$

Proof. By definition

$$A \cup B = \{x \mid x \in A \ \vee \ x \in B\}$$

consists of all elements that belong to A or B
if x is a member in either A or B, with commutative property the only thing that changes is order of checking whether $x \in A \vee \ x \in B$ □

Proof. Bad way to prove by bruteforcing

□

$x \in A$	$x \in B$	$(x \in A) \vee (x \in B)$	$(x \in B) \vee (x \in A)$
True	True	True	True
True	False	True	True
False	True	True	True
False	False	False	False

Table 1: Truth Table Demonstrating the Commutativity of $\vee$ in Set Union

Theorem 2.2.
$$A \cap B = B \cap A$$

Proof. By definition

$$A \cap B = \{x \mid x \in A \ \wedge \ x \in B\}$$

Both conditions must be satisfied. If $x \in B$ and $x \in A$ implies that both conditions have to be satisfied.

□

2.2 Associative Properties

Theorem 2.3.
$$A \cup (B \cup C) = (A \cup B) \cup C$$

Proof.
$$\begin{aligned} B \cup C &= \{x \mid x \in B \vee x \in C\} \\ A \cup (B \cup C) &= \{x \mid x \in A \vee (x \in B \vee x \in C)\} \\ A \cup (B \cup C) &= \{x \mid (x \in A \vee x \in B) \vee x \in C\} \\ A \cup B \cup C &= \{x \mid x \in A \vee x \in B \vee x \in C\} \end{aligned}$$

Hence the result follows. □

Theorem 2.4.
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Proof.
$$\begin{aligned} B \cap C &= \{x \mid x \in B \wedge x \in C\} \\ A \cap (B \cap C) &= \{x \mid x \in A \wedge (x \in B \wedge x \in C)\} \\ A \cap (B \cap C) &= \{x \mid (x \in A \wedge x \in B) \wedge x \in C\} \\ A \cap B \cap C &= \{x \mid x \in A \wedge x \in B \wedge x \in C\} \end{aligned}$$

Hence the result follows. □

2.3 Distributive Properties

Theorem 2.5.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof.

$$B \cup C = \{x \mid x \in B \vee x \in C\}$$
$$A \cap (B \cup C) = \{x | (x \in B \vee x \in C) \wedge x \in A\} = (A \cap B) \cup (A \cap C)$$

$$(A \cap B) \cup (A \cap C) = \{x \mid x \in (A \cap B) \vee x \in (A \cap C)\}$$
$$x \in A \cap B = x \in A \cap x \in B$$
$$x \in A \cap C = x \in A \cap x \in C$$

Hence the result follows. □

Theorem 2.6.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

2.4 Idempotent Properties

Theorem 2.7.

$$A \cup A = A$$

Theorem 2.8.

$$A \cap A = A$$

2.5 Properties of the Complement

Theorem 2.9.

$$(\overline{\overline{A}}) = A$$

Theorem 2.10.

$$A \cup \overline{A} = U$$

Theorem 2.11.

$$A \cap \overline{A} = \varnothing$$

Theorem 2.12.

$$\overline{\varnothing} = U$$

Theorem 2.13.

$$\overline{U} = \varnothing$$

Theorem 2.14.

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Theorem 2.15.

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

2.6 Properties of a Universal Set

Theorem 2.16.
$$A \cup U = U$$

Theorem 2.17.
$$A \cap U = A$$

2.7 Properties of the Empty Set

Theorem 2.18.
$$A \cup \varnothing = A \vee A \cup \{\} = A$$

Theorem 2.19.
$$A \cap \varnothing = A \wedge A \cap \{\} = \{\}$$

2.8 Example theorem solving

$$\begin{aligned}
x &\in \overline{A \cup B} \\
x &\notin A \cup B \\
x &\notin A \wedge x \notin B \implies x \in \overline{A} \wedge x \in \overline{B} \\
\overline{A \cup B} &\subseteq \overline{A} \cap \overline{B} \\
\text{conversely } x &\in \overline{A} \cap \overline{B} \implies x \notin A \wedge x \notin B \\
x &\notin A \cup B \implies x \in \overline{A \cup B} \\
\{x \mid x \in \overline{A} \cap \overline{B}\} = \{x \mid x &\in \overline{A \cup B}\} \implies \overline{A \cup B} = \overline{A} \cap \overline{B}
\end{aligned}$$

3 Theorem: Addition principle aka Inclusion-exclusion principle

Theorem 3.1. *If A and B are finite sets.*

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof. If $A \cap B = \varnothing$, then each element of $A \cup B$ is in either A or B, but not both. Hence $|A \cup B| = |A| + |B|$.

On the other hand, if $A \cap B \neq \varnothing$, then elements in $A \cap B$ belong to both sets. Thus $|A| + |B|$ double-counts them, and we must subtract $|A \cap B|$.

Hence the result follows. □

Theorem 3.2. *If A , B and C are finite sets*

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$