Nonlinear Decoherence of an Offset Waterbag

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Abstract

1 Introduction

The goal of this work is to provide a model for the centroid motion of bunch undergoing nonlinear decoherence. To begin we assume that the bunch starts out with some displacement Δx along one transverse axis at turn $N = 0$. We will then find a representation of $\hat{x}(N)$ the normalized centroid position $\langle x \rangle / \sigma_x$ as a function of turn number.

We begin by following SSC-N-360 and describe particle motion in the coordinates $a = \sqrt{\beta_x \varepsilon_x}/\sigma_x$ and φ, for the amplitude and initial phase of a particle. This means that the actual particle displacement is then $x = \sigma_x acos(2\pi\nu N + \varphi)$.

Because of the presence of the nonlinear element there will be amplitude-dependence in the particle tune. may be characterized in terms of a unitless strength parameter t and geometric scale factor c, with units of $m^{\frac{1}{2}}$, that describes the location of two singularities in the x-plane. The first few terms of the multipole expansion for the elliptic potential in normalized coordinates $\hat{x}, \hat{y} = x/\sqrt{\beta}, y/\sqrt{\beta}$ are given by

$$U(\hat{x}, \hat{y}) = \frac{-t}{c^2} Im \left\{ (\hat{x} + i\hat{y})^2 + \frac{2}{3c^2}(\hat{x} + i\hat{y})^4 + \frac{8}{15c^4}(\hat{x} + i\hat{y})^6 + ... \right\}. \quad (1)$$

Note: this expansion is only valid in the region $\sqrt{\hat{x}^2 + \hat{y}^2} < c$. Because of the form of the potential we expect to see just the even terms in amplitude effecting the tune.

2 Nonlinear Decoherence and the Centroid

Due to the octupole and higher terms in the potential the tune ν will have an amplitude dependence of the form

$$\nu = \nu_0 - \sum_{i=1} \mu_i a^{2i}, \quad (2)$$

where ν_0 is the unperturbed tune and μ_i are coefficients determined by the octupole, duodecapole, etc. multipole components in the nonlinear element. The calculation of these coefficients becomes cumbersome beyond the lowest order. This is a major obstacle to the use of the formulation developed here. This issue will be discussed further on.

This amplitude-dependent tune will result in particles having a phase shift each turn of

$$\Delta\varphi(a, N) = -2\pi N \sum_{i=1} \mu_i a^{2i}, \tag{3}$$

when compared to the unperturbed phase advance ν_0. From this the centroid motion for a distribution $\rho(a, \varphi)$ as a function of turn number in a lattice with some nonlinearities will be

$$\hat{x}(N) = \int_0^\infty da \int_0^{2\pi} d\varphi \, acos(\varphi)\rho(a, \varphi - 2\pi N\nu). \tag{4}$$

3 Calculation for a Waterbag Distribution

Because all our simulation data we will compare to uses a waterbag distribution we are first interested in the calculation of Eq. 4 for such a distribution. here we define a 'waterbag' according to the definition of Reiser, that is the bunch has a uniform distribution of particle amplitudes from 0 to some cutoff, that is

$$\rho(a, \varphi) = \begin{cases} 1 & a \leq 1 \\ 0 & a > 1 \end{cases} \tag{5}$$

.

We then assume that the distribution starts out with a centroid offset, in our normalized coordinates is, $Z = \Delta x/\sigma_x$. This offset waterbag then takes form

$$\rho(a, \varphi) = \begin{cases} \frac{1+Z^2-2Zcos(\varphi)}{\pi} & 0 < a \leq 1 \\ 0 & a > 1 \ o\ ra < 0 \end{cases} \tag{6}$$

Inserting Eq. 6 in Eq. 4 we can make a convenient change of variable $u = a^2$ and perform the integration in φ, resulting in

$$\hat{x}(N) = \pi Z \int_0^1 du \ cos(2\pi\nu_0 N)cos(\Delta\varphi(u, N)) + sin(2\pi\nu_0 N)sin(\Delta\varphi(u, N)) \tag{7}$$

A second change of variable $\hat{u} = 2\pi N u$ is then made to assist in numerical integration down the road. The general result for the centroid motion is then

$$\hat{x}(N) = \frac{Z}{2N} \int_0^{2\pi N} d\hat{u} \ cos(2\pi\nu_0 N)cos(\Delta\varphi(u, N)) - sin(2\varphi\nu_0 N)sin(\Delta\varphi(u, N)) \tag{8}$$

As a reminder the phase slip $\Delta\varphi$ has now become

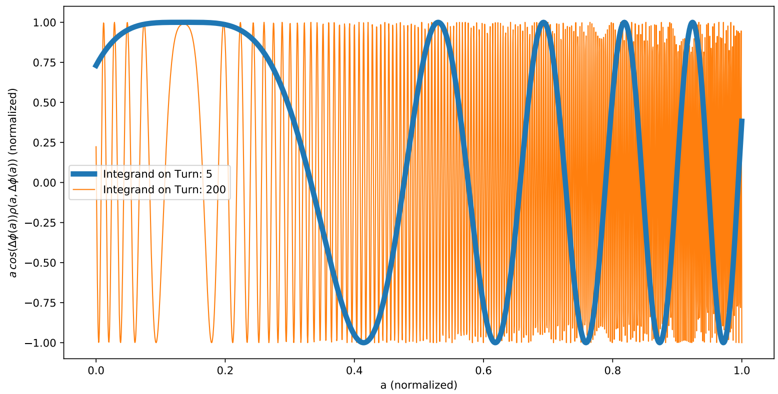

Figure 1: Integrand of Eq. 8 on turn 5 and turn 100.

$$\Delta\varphi(\hat{u}, N) = \sum_{i=1} \frac{\mu_i \hat{u}^i}{(2\pi N)^{i-1}}. \tag{9}$$

We now have a general solution for our original problem, the centroid motion as a function of turn number. Quite nicely, this solution is not iterative, any turn can be calculated independently. This does aid speeding up the turn-by-turn centroid motion, however, this is where the good news stops. We still have a set of unknown coefficients μ_i that will depend on particulars of the nonlinear element in the lattice and must be iteratively derived from a perturbation method or found from numerical fitting. For this second option we can use simulation data from IOTA and fit up to the desired order (as well as ν_0 possibly). However, this numerical method is greatly complicated by the highly-oscillatory nature of the integrand of Eq. 8. As an example the integrand is plotted in Fig. 1 for two different values of N. For the case of $N = 200$ the high-frequency oscillations tend to make the numerical integration both extremely slow and greatly limits accuracy in many cases.

Evaluation of the general waterbag case, Eq. 8, is available through the `rsbeams.rsphysics.decoherence` module. The general integration is performed with `scipy.integrate.quad` which in turns uses the Fortran QUADPACK library. (Note: This uses the Clenshaw-Curtis Quadrature, which has the possibility of introducing a weight function to the integrand that can ease integration of high-oscillatory functions. This has not been explored though.). For evaluating N turns we take advantage of the independent nature of each turn and use the `Pool` function from the `pathos.multiprocessing` package to parallelize the calculations. The standard `multiprocessing` library will not work in this instance as it is not able to be run from within a class instance and evaluate a method belonging to the class. The Pathos implementation does not have this limitation.

3.1 Special Cases

While the general evaluation of Eq. 8 does not appear analytically tractable there are two special cases. If we truncate $\Delta\varphi$ at $\hat{a}$ there is an exact analytic solution, in the case of truncating at $\hat{a}^2$ a solution can be found in terms of Fresnel integrals, which are relatively easy to calculate.

For just $i = 1$ the exact solution is

$$\hat{x}(N) = \frac{Z}{2\pi N} \frac{cos(2\pi\nu_0 N) sin(2\pi N\mu_1) + 2 sin(2\pi\nu_0 N) sin^2(\pi N\mu_1)}{\mu_1}. \tag{10}$$

With both $i = 1, 2$ the solution becomes

$$\hat{x}(\hat{u}, N) = \frac{Z}{\sqrt{8\pi N^2\mu_2}} \left[C\left(\frac{\pi N\mu_1 + \mu_2\hat{u}}{\sqrt{\pi^2 N\mu_2}}\right) cos\left(\frac{\pi N\mu_1^2}{2\mu_2} + 2\pi N\nu_1\right) + S\left(\frac{\pi N\mu_1 + \mu_2\hat{u}}{\sqrt{\pi^2 N\mu_2}}\right) sin\left(\frac{\pi N\mu_1^2}{2\mu_2} + 2\pi N\nu_1\right)\right]\Bigg|_0^{2\pi N}, \tag{11}$$

where C and S are the Fresnel integrals. The `rsbeams.rsphysics.decoherence` package includes both these representations and will automatically use them when appropriate.